WEIGHTED INEQUALITIES FOR MULTILINEAR OPERATORS IN DUNKL SETTING

By

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Suman Mukherjee

Suman Mukherjee

Dedicated

to

my

mother

Smt. Malati Mukherjee

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ABSTRACT

In modern Fourier analysis, the study of multilinear (m-linear) operators focuses on establishing their boundedness from the m-fold product of normed spaces to the appropriate normed spaces. For instance, Lebesgue spaces are commonly considered as the normed spaces in this context. These operators find applications in various fields, including partial differential equations, complex analysis, and quantum mechanics. A more comprehensive investigation consists of exploring weighted inequalities, which involve determining the boundedness of these operators on weighted Lebesgue spaces. Weighted inequalities hold significance in a broader scope, impacting areas such as vector-valued operators, operator extrapolation, and the theory of Laplace's equation boundary value problems on Lipschitz domains.

Over the past three decades, a parallel theory to classical Fourier analysis, associated with root systems and reflection groups, has emerged in Euclidean harmonic analysis. This theory, known as Fourier analysis in the Dunkl setting, serves as a generalization of classical Fourier analysis. Within this context, significant progress has been made, particularly in understanding singular integrals, Fourier multiplier operators, and potential-type operators. However, exploration of multilinear operators or weighted inequalities within the Dunkl setting has been relatively limited. The primary aim of this thesis is to delve into the weighted boundedness of some multilinear operators in the Dunkl framework.

The first result of this thesis is one and two-weight estimates for multilinear Calderón-Zygmund type singular integral operators in the Dunkl setting, along with the associated maximal operators. Importantly, these operators distinguish themselves from classical Calderón-Zygmund singular integral operators by incorporating both the 'Dunkl metric' and the usual metric in their definition. In the subsequent chapter, we initially establish Littlewood-Paley theory in the Dunkl framework, utilizing it to prove a Coifman-Meyer type bilinear multiplier theorem associated with the Dunkl transform. Additionally, we show that these multiplier operators are examples of multilinear Dunkl-Calderón-Zygmund operators and derive weighted estimates for them. In the final chapter, we study similar weighted inequalities for a different type of operators known as multilinear Dunkl fractional integral operators and multilinear fractional maximal operators.

Contents

Summary

A generalization of classical Fourier analysis is Fourier analysis in the Dunkl setting. Indeed, following the discovery of Dunkl operators as a generalization of partial derivative operators, analysis in the Dunkl setup has been explored in various directions. There has been a particular emphasis on singular integrals, multiplier operators, and potential-type operators. However, not much investigation has been conducted concerning multilinear operators or weighted inequalities within this context. The aim of this thesis is to bridge this gap by addressing and exploring the weighted boundedness of certain multilinear operators in the Dunkl setup.

We consider a fixed root system R on $\mathbb{R}^d$, a fixed non-negative valued multiplicity function k defined on R, G as the associated reflection group, and $d\mu_k$ as the associated Dunkl measure. Through the well-established connection between the Fourier transform and the partial derivative operator, Dunkl operators introduce a new operator that generalizes the classical Fourier transform, called the Dunkl transform. It is defined for all $f \in L^1(\mathbb{R}^d, d\mu_k)$, by

$$\mathcal{F}_k f(\xi) = \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x)\, d\mu_k(x),$$

where E_k is called the Dunkl kernel, which can be thought of as a generalization of the exponential function. $\mathcal{F}_k$ enjoys a many properties similar to those of the classical Fourier transform.

Generalizing the notion of classical multilinear Calderón-Zygmund operators operators introduced by Grafakos and Torres [38], we define the multilinear Dunkl-Calderón-Zygmund operators as follows.

$$\mathcal{T}(f_1, f_2, \cdots, f_m)(x) = \int_{(\mathbb{R}^d)^m} K(x, y_1, y_2, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j)\, d\mu_k(y_j),$$

for all $f_j \in C_c^\infty(\mathbb{R}^d)$ with $\sigma(x) \notin \bigcap_{j=1}^{m} supp\, f_j$ for all $\sigma \in G$, where K is a function defined

away from the set $\mathcal{O}(\triangle_{m+1})$

$$=: \big\{(x, y_1, y_2, \cdots, y_m) \in (\mathbb{R}^d)^{m+1} : x = \sigma_j(y_j) \text{ for some } \sigma_j \in G, \text{ for all } 1 \leq j \leq m\big\},$$

which satisfies the some suitable size estimate and smoothness estimates. We derive one and two-weight estimates for the operator $\mathcal{T}$, extending the results established in the classical framework by Lerner et al. [48]. We also prove a multilinear Cotlar-type inequality and weighted boundedness for the maximal operators associated with $\mathcal{T}$, given by

$$\mathcal{T}^*(f_1, f_2, \cdots, f_m)(x) = \sup_{\delta>0} |\mathcal{T}_\delta(f_1, f_2, \cdots, f_m)(x)|,$$

where for any $\delta > 0$, $\mathcal{T}_\delta$ represents the appropriately truncated operator obtained from $\mathcal{T}$.

Next, we delve into multiplier operators associated with the Dunkl transform. In analogy to the classical case, for a bounded function $\mathbf{m}$ on $\mathbb{R}^d \times \mathbb{R}^d$ define the bilinear Dunkl multiplier operator $\mathcal{T}_{\mathbf{m}}$ as

$$\mathcal{T}_{\mathbf{m}}(f_1, f_2)(x) = \int_{\mathbb{R}^{2d}} \mathbf{m}(\xi, \eta)\mathcal{F}_k f_1(\xi)\mathcal{F}_k f_2(\eta) E_k(ix, \xi) E_k(ix, \eta)\, d\mu_k(\xi) d\mu_k(\eta)$$

for all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$, the space of all Schwartz class functions on $\mathbb{R}^d$. By establishing a Littlewood-Paley type theory, we initially derive a Coifman-Meyer [16] type bilinear multiplier theorem for $\mathcal{T}_{\mathbf{m}}$. Subsequently, utilizing this theorem along with results for the multilinear Dunkl-Calderón-Zygmund operators, we obtain weighted estimates for the operator $\mathcal{T}_{\mathbf{m}}$.

Lastly, we study multilinear fractional integral operators $\mathcal{I}_\alpha^k$ and multilinear fractional maximal operators $\mathcal{M}_\alpha^k$ in the Dunkl setup. For $\overrightarrow{f} = (f_1, f_2, ..., f_m) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times ... \times \mathcal{S}(\mathbb{R}^d)$, they are defined as follows

$$\mathcal{I}_\alpha^k \overrightarrow{f}(x) \;:=\; \int_{(\mathbb{R}^d)^m} \frac{\tau_x^k f_1(-y_1)\tau_x^k f_2(-y_2)...\tau_x^k f_m(-y_m)}{(|y_1|^2 + |y_2|^2 + ... + |y_m|^2)^{(md_k-\alpha)/2}} d\mu_k(y_1) d\mu_k(y_2)...d\mu_k(y_m),$$

where $0 < \alpha < md_k$ and τ_x^k is the Dunkl translation operator;

$$\mathcal{M}_\alpha^k \overrightarrow{f}(x) \;:=\; \sup_{r>0} \prod_{j=1}^m r^{\frac{\alpha}{m}-d_k} \left| \int_{\mathbb{R}^d} f_j(y_j)\tau_x^k \chi_{B(0,r)}(-y_j)\, d\mu_k(y_j) \right|,$$

where $0 \leq \alpha < md_k$. We establish both one and two-weight inequalities for these two operators which extends the results of Moen [51] in the classical setting.

Chapter 1

Introduction

In harmonic analysis, the study of bilinear or m-linear operators represents an extension of the concept of the multiplication of two functions or m-functions. A significant aspect of the study of multilinear operators involves demonstrating that certain integral operations on specific types of functions are not excessively irregular. In simpler terms, when dealing with an m-linear integral operator $\mathbf{T}$, the main aim is to establish an inequality of the form

$$\|\mathbf{T}(f_1, f_2, \cdots, f_m)\|_{X_0} \leq C\, \|f_1\|_{X_1} \|f_2\|_{X_2} \cdots \|f_m\|_{X_m}, \tag{1.0.1}$$

where X_j's are appropriate normed function spaces with norms $\|\cdot\|_{X_j}$ for $j = 0, 1, \cdots, m$. These operators naturally arise not only in harmonic analysis but also in different areas of analysis, such as partial differential equations, complex analysis, potential theory, and quantum mechanics, etc. The study of such operators helps us understand non-linear problems where the product of more than one function is considered. The basis of these studies goes back to the contributions of Coifman and Meyer [14–16]. After the pioneering works [46,47] of Lacey and Thiele, numerous developments have been made in the field of multilinear operators. Some of those that we are interested in are [8,29,34,36–38,42,48–51,73], where the boundedness of the form (1.0.1) for certain multilinear integral operators are studied on the Lebesgue spaces.

Now, given the inequality (1.0.1), a natural question is whether we can derive conditions on the functions $w_0, w_1, \cdots, w_m$ such that we have

$$\|\mathbf{T}(f_1, f_2, \cdots, f_m)\, w_0\|_{X_0} \leq C\, \|f_1\, w_1\|_{X_1} \|f_2\, w_2\|_{X_2} \cdots \|f_m\, w_m\|_{X_m}? \tag{1.0.2}$$

This type of inequalities are known as *weighted inequalities* and the functions w_j's are called *weights*. If there is a relation between w_0 and the product $w_1 w_2 \cdots w_m$, then inequality (1.0.2) is called a *one-weight inequality*; otherwise, it is known as a *two-weight inequality*. Weighted inequalities, with their broad implications, are not merely generalizations. Applications of these inequalities extend to areas such as vector-valued operators and the extrapolation of operators (for example, see [5, 18]). They also play a significant role in the theory of boundary value problems for Laplace's equation on Lipschitz domains (for example, see [67]), showcasing their importance.

The initial response in this direction was provided by Muckenhoupt [52] for the linear Hardy-Littlewood maximal operators in the context of Lebesgue spaces. He gave a characterization stating that the Hardy-Littlewood maximal operator M is bounded on the weighted Lebesgue space $L^p(\mathbb{R}^d, w(x)dx)$ for $1 < p < \infty$, if and only if the weight w belongs to the class A_p, that is w satisfies

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q w\,dx\Big)\Big(\int_Q w^{1-p'}\,dx\Big)^{p-1} < \infty, \tag{1.0.3}$$

where Q denotes a cube in $\mathbb{R}^d$ and p' is the Hölder conjugate of p.

Later, Sawyer [64] characterized the two-weight inequality for M, proving that $M : L^p(\mathbb{R}^d, u(x)dx) \to L^p(\mathbb{R}^d, v(x)dx)$ if and only if

$$\sup_Q \frac{\int_Q M(\chi_Q v^{1-p'})(x)^p u(x)\,dx}{\int_Q v(x)^{1-p'}\,dx} < \infty. \tag{1.0.4}$$

These results led to the investigation of similar weighted inequalities for various operators including the Hilbert transform [43], Calderón-Zygmund operators [13], and fractional operators [53].

In 2009, Lerner et al. [48] presented a suitable adaptation of the Muckenhoupt A_p classes from the linear setting to the multilinear setting:

Definition 1.0.1. Let $1 \leq p_1, p_2, \cdots, p_m < \infty$, $\overrightarrow{P} = (p_1, p_2, \cdots, p_m)$ and p be the number given by $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$. Furthermore, let $w_1, w_2, \cdots, w_m$ be non negative, locally integrable functions on $\mathbb{R}^d$ and $\overrightarrow{w} = (w_1, w_2, \cdots, w_m)$. We say that the vector weight $\overrightarrow{w}$ is in the class $A_{\overrightarrow{P}}$, if it satisfies

$$\sup_{Q\subset\mathbb{R}^d} \Big(\frac{1}{|Q|}\int_Q \prod_{j=1}^{m} \big(w_j(x)\big)^{p/p_j} dx\Big)^{1/p} \prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q w_j(x)^{1-p_j'} dx\Big)^{1/p_j'} < \infty,$$

when $p_j = 1$, $\left(\frac{1}{|Q|}\int_Q w_j(x)^{1-p_j'} dx\right)^{1/p_j'}$ is understood as $\left(\inf_Q w_j\right)^{-1}$.

They characterized that the multilinear Hardy-Littlewood maximal operators are bounded from the product of the Lebesgue spaces $L^{p_1}(\mathbb{R}^d, w_1(x)dx) \times L^{p_2}(\mathbb{R}^d, w_2(x)dx) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m(x)dx)$ to $L^p(\mathbb{R}^d, \prod_{j=1}^{m} \big(w_j(x)\big)^{p/p_j} dx)$ (weak L^p if one of the p_j's is 1) if and only if $\overrightarrow{w} \in A_{\overrightarrow{P}}$. The natural progression involved exploring how weighted inequalities extend to the multilinear context for other operators. We will delve into a few of these topics in more detail in the next sections.

In 1989, C.F. Dunkl introduced the Dunkl operators [22] by adding a rational part to the standard directional derivatives. These operators are linked to root systems and reflection groups in Euclidean spaces and serve as a broadened perspective on partial derivatives. This concept originates from the examination of root systems, which are fundamental tools in the theory of Lie groups and Lie algebras, consisting of configurations of vectors in Euclidean spaces satisfying certain geometrical properties. By exploiting the well-established correlation between the Fourier transform and the partial derivative operator, Dunkl operators unveil a new operator known as the Dunkl transform. This extension of the classical Fourier transform marks the initiation of the analytical aspect of Dunkl theory – a comprehensive endeavor to extend the core findings of classical Fourier analysis and special function theory to the realm of root systems and reflection groups.

Over the past three decades, counterparts of many classical harmonic analysis theories related to singular integrals, multiplier operators, and potential-type operators have been studied in the Dunkl setting. However, there has not been much exploration regarding multilinear operators or weighted inequalities in this context. The purpose of this thesis is to address this gap and investigate weighted boundedness for certain multilinear operators within the Dunkl setup. Specifically, we are interested in studying one and two-weight inequalities for multilinear Calderón-Zygmund-type singular integral operators and maximal operators associated with them, bilinear multiplier operators, multilinear fractional integral operators, and multilinear fractional maximal operators in the Dunkl setup. Next, we briefly discuss a few of the theories in the classical setup relevant to our results.

1.1 Multilinear Calderón-Zygmund Operators

Multilinear Calderón-Zygmund theory in the unweighted case was systematically studied by Grafakos and Torres [38]. Let us use the notation $\mathcal{S}(\mathbb{R}^d)$ to denote the class of all Schwartz functions on $\mathbb{R}^d$. Let $\mathcal{S}'(\mathbb{R}^d)$ be the corresponding space of all tempered distributions, and let $\overrightarrow{f}$ denote the vector consisting of an m-tuple of functions $(f_1, f_2, \ldots, f_m)$. The following definition of multilinear Calderón-Zygmund operators was given in [38].

Definition 1.1.1. A function $\mathbf{T} : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is called an m-linear Calderón-Zygmund operator, if for all $f_1, f_2, \cdots, f_m \in C_c^\infty(\mathbb{R}^d)$ with $x \notin \bigcap_{j=1}^{m} supp\, f_j$, $\mathbf{T}$ has the integral representation

$$\mathbf{T}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^d)^m} K(x, y_1, y_2, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j)\, dy_j,$$

and the kernel K is defined on the compliment of the set $\triangle_{m+1}$

$$=: \left\{(x, y_1, y_2, \cdots, y_m) \in (\mathbb{R}^d)^{m+1} : x = y_1 = \cdots = y_m\right\}$$

and K fulfils the following size and smoothness conditions for some $\epsilon > 0$:

$$|K(y_0, y_1, y_2, \cdots, y_m)| \leq C \frac{1}{\big(|y_0 - y_1| + |y_0 - y_2| + \cdots |y_0 - y_m|\big)^{md}}, \tag{1.1.1}$$

for all $(y_0, y_1, y_2, \cdots, y_m) \in (\mathbb{R}^d)^{m+1} \setminus \triangle_{m+1}$;

$$\begin{aligned} & |K(y_0, y_1, y_2, \cdots, y_n, \cdots, y_m) - K(y_0, y_1, y_2, \cdots, y_n', \cdots, y_m)| \\ \leq \;& C \frac{|y_n - y_n'|^\epsilon}{\big(|y_0 - y_1| + |y_0 - y_2| + \cdots |y_0 - y_m|\big)^{md+\epsilon}}, \end{aligned} \tag{1.1.2}$$

whenever $|y_n - y_n'| \leq \max\limits_{1\leq j\leq m} |y_0 - y_j|/2$, for all $n \in \{0, 1, \cdots, m\}$.

In the same paper, the boundedness of **T** in the Lebesgue spaces was proved in the unweighted case. Later in [48], a theory of weighted boundedness results was presented. We state it below.

Theorem 1.1.2. *Let $1 \leq p_1, p_2, \cdots, p_m < \infty$, $\overrightarrow{P} = (p_1, p_2, \cdots, p_m)$, p be the number given by $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$ and the vector weight $\overrightarrow{w} \in A_{\overrightarrow{P}}$. Furthermore, let* **T** *maps from $L^{q_1}(\mathbb{R}^d, dx) \times L^{q_2}(\mathbb{R}^d, dx) \times \cdots \times L^{q_m}(\mathbb{R}^d, dx)$ to $L^q(\mathbb{R}^d, dx)$ for some $q, q_1, q_2, \cdots, q_m$ satisfying $1 \leq q_1, q_2, \cdots, q_m < \infty$ with $1/q = 1/q_1 + 1/q_2 + \cdots + 1/q_m$. Then the following hold:*

(i) if $p_j = 1$ for some $1 \leq j \leq m$, then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1(x)\,dx) \times L^{p_2}(\mathbb{R}^d, w_2(x)\,dx) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m(x)\,dx)$, the following boundedness holds:

$$\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\, |\mathbf{T}\overrightarrow{f}(y)|>t\}} \prod_{j=1}^m w_j(x)^{p/p_j}\,dx\Big)^{1/p} \;\leq\; C\prod_{j=1}^m\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)dx\Big)^{1/p_j};$$

(ii) if $p_j > 1$ for all $1 \leq j \leq m$, then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1(x)\,dx) \times L^{p_2}(\mathbb{R}^d, w_2(x)\,dx) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m(x)\,dx)$, the following boundedness holds:

$$\Big(\int_{\mathbb{R}^d} |\mathbf{T}\overrightarrow{f}(x)|^p \prod_{j=1}^m w_j(x)^{p/p_j}\,dx\Big)^{1/p} \;\leq\; C\prod_{j=1}^m\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\,dx\Big)^{1/p_j}.$$

In a similar manner, a two-weight inequality for the operator **T** can also be established, provided certain additional assumptions on the weights are satisfied. However, we are not including the details here.

Also one can consider the corresponding maximal operator $\mathbf{T}^*$, given by

$$\mathbf{T}^*(\overrightarrow{f})(x) = \sup_{\delta>0} |\mathbf{T}_\delta(\overrightarrow{f})(x)|,$$

where for $\delta > 0$,

$$\mathbf{T}_\delta(\overrightarrow{f})(x) := \int\limits_{\sum_{j=1}^{m} |x-y_j|^2 \geq \delta^2} K(x, y_1, y_2, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j)\, dy_j,$$

and K is the m-linear Calderón-Zygmund kernel as in Definition 1.1.1. In the paper [37], the authors examined the boundedness properties of the operator $\mathbf{T}^*$ by utilizing the boundedness results for **T**. Specifically, they established the boundedness of this operator in L^p-spaces in both weighted and unweighted cases. A crucial tool in proving this result is the following multilinear Cotlar-type inequality:

$$|\mathbf{T}^*\overrightarrow{f}(x)| \leq C\Big(\big[M(|\mathbf{T}\overrightarrow{f}|^\nu)(x)\big]^{1/\nu} + \prod_{j=1}^{m} Mf_j(x)\Big). \tag{1.1.3}$$

Our objective is to formulate an appropriate analogue of multilinear Calderón-Zygmund operators in the Dunkl setup and to present one and two-weight inequalities resembling those stated in Theorem 1.1.2. We also want to derive a multilinear Cotlar-type inequality in this setup and explore the relevant literature for maximal operators associated with multilinear Dunkl-Calderón-Zygmund operators.

1.2 Bilinear Multiplier Operators

To keep things simple and avoid the complexity of the large expressions, we will focus on Fourier multipliers for the bilinear case, i.e., when $m = 2$. Let, $\mathcal{F}$ denote the classical Fourier transform on the Euclidean space $\mathbb{R}^d$. A priori, for a function $\mathbf{m} \in L^\infty(\mathbb{R}^{2d})$, the bilinear Fourier multiplier operator $\mathbf{T_m}$ associated with $\mathbf{m}$ is defined on the product of Schwartz spaces by

$$\mathbf{T_m}(f_1, f_2)(x) = \int_{\mathbb{R}^{2d}} \mathbf{m}(\xi, \eta)\mathcal{F}f_1(\xi)\mathcal{F}f_2(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

With certain smoothness assumptions on the multiplier $\mathbf{m}$, Coifman and Meyer [16] established the boundedness of the operator $\mathbf{T_m}$ in the L^p-spaces. Afterward, a significant amount of literature (for example, see [8, 29, 38, 42, 45, 49, 73]) has developed concerning the study of bilinear multipliers. In this context, we will specifically highlight weighted inequalities involving multiple weights, which bear relevance to this thesis. The following result was proved in [8].

Theorem 1.2.1. *Let* $\mathbf{m} \in C^L\left(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}\right)$ *satisfies the condition*

$$|\partial_\xi^\alpha \partial_\eta^\beta \mathbf{m}(\xi, \eta)| \leq C_{\alpha,\beta}\left(|\xi| + |\eta|\right)^{-(|\alpha|+|\beta|)} \tag{1.2.1}$$

for all multi-indices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$ *such that* $|\alpha| + |\beta| \leq L$ *and for all* $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}$*, where* L *is an integer with* $d+1 \leq L \leq 2d$*. Also, let* p, p_1, p_2 *be exponents satisfying* $1/p = 1/p_1 + 1/p_2$ *and* $p_0 := 2d/L < p_1, p_2 < \infty$ *and the vector weight* $(w_1, w_2) \in A_{(\frac{p_1}{p_0}, \frac{p_2}{p_0})}$*. Then for all* $f_1 \in L^{p_1}(\mathbb{R}^d, w_1(x)dx)$ *and* $f_2 \in L^{p_2}(\mathbb{R}^d, w_2(x)dx)$*, the following boundedness holds:*

$$\left(\int_{\mathbb{R}^d} |\mathbf{T_m}\overrightarrow{f}(x)|^p w_1^{p/p_1}(x) w_2^{p/p_2}(x)\, dx\right)^{1/p} \leq C \prod_{j=1}^{2}\left(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\, dx\right)^{1/p_j}.$$

We aim to study the weighted boundedness properties of bilinear multiplier operators associated with the Dunkl transform in the place of the Fourier transform.

1.3 Multilinear Fractional Operators

In this section, we will discuss the multilinear fractional integral operator and the associated multilinear fractional maximal operator. These operators were introduced by Moen [51].

For $\overrightarrow{f} = (f_1, f_2, \cdots, f_m) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$, the multilinear fractional integral operator $\mathcal{I}_\alpha$ is defined by

$$\mathcal{I}_\alpha \overrightarrow{f}(x) = \int_{(\mathbb{R}^d)^m} \frac{f_1(y_1)f_2(y_2)...f_m(y_m)}{(|x-y_1|+|x-y_2|+...+|x-y_m|))^{md-\alpha}} dy_1 dy_2 ... dy_m,$$

$$\text{where } 0<\alpha<md;$$

and for all locally integrable functions $f_1, f_2, \cdots, f_m$ in $\mathbb{R}^d$, the multilinear fractional maximal operator is given by

$$\mathcal{M}_\alpha \overrightarrow{f}(x) = \sup_{x\in Q} \prod_{j=1}^m \frac{l(Q)^{\alpha/m}}{|Q|} \int_Q |f_j(y_j)|\, dy_j, \qquad \text{where } 0 \le \alpha < md.$$

These two operators are closely interrelated. In [51], a distinct class of weights $A_{\overrightarrow{P},q}$ was introduced to characterize the weighted boundedness of these multilinear operators. This extension builds upon Muckenhoupt and Wheeden's [53] $A_{p,q}$ weight classes from the linear case. We present the following one-weight inequality form [51].

Theorem 1.3.1. *Suppose that* $1 < p_1, p_2, \cdots, p_m < \infty$*,* $0 < \alpha < md$*,* $1/m < p < d/\alpha$ *and* q *be a number defined by* $1/q = 1/p - \alpha/d$*. Furthermore, let the vector weight* $\overrightarrow{w} = (w_1, w_2, \cdots, w_m) \in A_{\overrightarrow{P},q}$*, i.e.,*

$$\sup_{Q\subset\mathbb{R}^d} \left(\frac{1}{|Q|}\int_Q \Big(\prod_{j=1}^m w_j(y)\Big)^q dy\right)^{1/q} \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j(y)^{-p_j'} dy\right)^{1/p_j'} < \infty.$$

Then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1^{p_1}(x)dx) \times L^{p_2}(\mathbb{R}^d, w_2^{p_2}(x)dx) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m^{p_m}(x)dx)$*, the following inequality holds:*

$$\left(\int_{\mathbb{R}^d} \Big(\Big|\mathcal{N}_\alpha \overrightarrow{f}(x)\Big| \prod_{j=1}^m w_j(x)\Big)^q dx\right)^{1/q} \le C \prod_{j=1}^m \left(\int_{\mathbb{R}^d} (|f_j(x)| w_j(x))^{p_j} dx\right)^{1/p_j},$$

where $\mathcal{N}_\alpha$ *is either* $\mathcal{I}_\alpha$ *or* $\mathcal{M}_\alpha$*.*

For $\alpha = 0$, the operator $\mathcal{M}_\alpha$ coincides with the multilinear Hardy-Littlewood maximal operator. Therefore, based on the results in [48], the above inequality holds true for $\mathcal{M}_\alpha$, even when $\alpha = 0$.

In this case as well, a two-weight inequality for these operators was established in the same paper, in a comparable fashion, provided that specific additional conditions on the weights are met. However, we choose not to present the details here.

Our goal is to devise suitable counterparts for these operators in the Dunkl setup. We intend to investigate the weighted boundedness characteristics of the multilinear Dunkl fractional integral operators and the multilinear Dunkl fractional maximal operators.

Chapter 2

Preliminaries

This chapter serves as the prerequisite chapter for the thesis. In the next section, we present several notations, definitions, and preliminary results of Dunkl theory, which are essential for this thesis. We also extend these concepts to the multilinear Dunkl setting, as outlined in Section 2.2. Moving on to Section 2.3, we discuss the topics of spaces of homogeneous type according to Coifman and Weiss [17], Muckenhoupt weight classes, and the boundedness of Hardy-Littlewood maximal operators for the spaces of homogeneous type.

2.1 Preliminaries of Dunkl Theory

Let us consider the usual inner product $\langle \cdot\, , \cdot \rangle$ and the usual norm $|\cdot| := \sqrt{\langle \cdot\, , \cdot \rangle}$ on $\mathbb{R}^d$. For any $\lambda \in \mathbb{R}^d \setminus \{0\}$, the *reflection* σ_λ with respect to the hyperplane $\lambda^\perp$ orthogonal to λ is given by

$$\sigma_\lambda(x) = x - 2\frac{\langle x, \lambda \rangle}{|\lambda|^2}\lambda.$$

Let R be a finite subset of $\mathbb{R}^d$ which does not contain 0. If R satisfies $R \cap \mathbb{R}\lambda = \{\pm\lambda\}$ for all $\lambda \in R$ and $\sigma_\lambda(R) = R$ for all $\lambda \in R$, then R is called a *root system*. Throughout this thesis we will consider a fixed normalized root system R, that is $|\lambda| = \sqrt{2}$, $\forall \lambda \in R$. The subgroup G generated by reflections $\{\sigma_\lambda : \lambda \in R\}$ is called the *reflection group* (or *Coxeter group*) associated with R and a G-invariant function $k : R \to \mathbb{C}$, is known as a *multiplicity function*. In this paper, we take a fixed multiplicity function $k \geq 0$. Let h_k be the G-invariant homogeneous weight function given by $h_k(x) = \prod\limits_{\lambda \in R} |\langle x, \lambda \rangle|^{k(\lambda)}$ and

$d\mu_k(x)$ be the normalized measure $c_k h_k(x)dx$, where

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2}\, h_k(x)\, dx,$$

$d_k = d + \gamma_k$ and $\gamma_k = \sum_{\lambda \in R} k(\lambda)$.

Let $x \in \mathbb{R}^d$, $r > 0$ and $B(x,r)$ be the ball with centre at x and radius r. Then the volume $\mu_k(B(x,r))$ of $B(x,r)$ is given by [1]

$$\mu_k(B(x,r)) \sim\ r^d \prod_{\lambda \in R} (|\langle x, \lambda\rangle| + r)^{k(\lambda)}\,. \tag{2.1.1}$$

It is immediate from above that if $r_2 > r_1 > 0$ then

$$C\left(\frac{r_1}{r_2}\right)^{d_k} \leq \frac{\mu_k(B(x,r_1))}{\mu_k(B(x,r_2))} \leq C^{-1}\left(\frac{r_1}{r_2}\right)^{d}. \tag{2.1.2}$$

In fact, if $x_1, x_2 \in \mathbb{R}^d$ and $B(x_1,r_1) \subseteq B(x_2,r_2)$, then

$$C\left(\frac{r_1}{r_2}\right)^{d_k} \leq \frac{\mu_k(B(x_1,r_1))}{\mu_k(B(x_2,r_2))} \leq C^{-1}\left(\frac{r_1}{r_2}\right)^{d}. \tag{2.1.3}$$

Let $d_G(x,y)$ denote the distance between the G-orbits of x and y that is $d_G(x,y) = \min_{\sigma \in G} |\sigma(x) - y|$ and for any $r > 0$ we write $V_G(x,y,r) = \max\,\{\mu_k(B(x,r)), \mu_k(B(y,r))\}$. Then from the expression for volume of a ball, it follows that

$$V_G(x,y,d_G(x,y)) \sim \mu_k(B(x,d_G(x,y))) \sim \mu_k(B(y,d_G(x,y))). \tag{2.1.4}$$

Let us define the orbit $\mathcal{O}(B)$ of the ball B by

$$\mathcal{O}(B) = \left\{y \in \mathbb{R}^d : d_G(c_B, y) \leq r(B)\right\} = \bigcup_{\sigma \in G} \sigma(B),$$

where c_B denotes the centre and $r(B)$ denotes the radius of the ball B. It then follows that

$$\mu_k(B) \leq \mu_k(\mathcal{O}(B)) \leq |G|\, \mu_k(B). \tag{2.1.5}$$

[1]The symbol $\sim$ between two positive expressions means that their ratio remains between two positive constants.

Although d_G satisfies the triangle inequality, it is *not* a metric on $\mathbb{R}^d$. Also note that for any $x, y \in \mathbb{R}^d$ we have $d_G(x, y) \leq |x - y|$.

The *differential-difference operators* or the *Dunkl operators* T_ξ introduced by C.F. Dunkl [22] is given by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\lambda \in R} \frac{k(\lambda)}{2} \langle \lambda, \xi \rangle \frac{f(x) - f(\sigma_\lambda x)}{\langle \lambda, x \rangle}.$$

The Dunkl operators T_ξ are the k-deformations of the directional derivative operators ∂_ξ and coincides with them in the case $k = 0$. For a fixed $y \in \mathbb{R}^d$, there is a unique real analytic solution for the system $T_\xi f = \langle y, \xi \rangle f$ satisfying $f(0) = 1$. The solution $f(x) = E_k(x, y)$ is known as the *Dunkl kernel*. For two reasonable functions f and g, the following integration by parts formula is well known:

$$\int_{\mathbb{R}^d} T_\xi\, f(x) g(x)\, d\mu_k(x) = - \int_{\mathbb{R}^d} f(x) T_\xi\, g(x)\, d\mu_k(x).$$

Also, if at least one of f or g is G-invariant then the Leibniz-type rule holds:

$$T_\xi(fg)(x) = T_\xi f(x)\, g(x) + f(x) T_\xi g(x).$$

Let us consider the canonical orthonormal basis $\{e_j : j = 1, 2, \cdots, d\}$ in $\mathbb{R}^d$ and set $T_j = T_{e_j}$ and $\partial_j = \partial_{e_j}$. For any multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$, we use the following notations.

- $|\alpha| = (\alpha_1 + \alpha_2 + \cdots + \alpha_d)$,
- $\partial_j^0 = I, \partial^\alpha = \partial_1^{\alpha_1} \circ \partial_2^{\alpha_2} \circ \cdots \circ \partial_d^{\alpha_d}$,
- $T_j^0 = I, T^\alpha = T_1^{\alpha_1} \circ T_2^{\alpha_2} \circ \cdots \circ T_d^{\alpha_d}$.

Sometimes we will write ∂_ξ^α to indicate that the partial derivatives are taken with respect to the variable ξ.

The Dunkl kernel $E_k(x, y)$ which actually generalizes the exponential functions $e^{<x,y>}$, has a unique extension to a holomorphic function on $\mathbb{C}^d \times \mathbb{C}^d$. We list below few properties of the Dunkl kernel (see [60–62] for details).

- $E_k(x,y) = E_k(y,x)$ for any $x,\ y \in \mathbb{C}^d$,
- $E_k(tx,y) = E_k(x,ty)$ for any $x,\ y \in \mathbb{C}^d$ and $t \in \mathbb{C}$,
- $|\partial_z^\alpha E_k(ix,z)| \le |x|^{|\alpha|}$ for any $x,\ z \in \mathbb{R}^d$ and $\alpha \in (\mathbb{N} \cup \{0\})^d$.

Let $L^p(\mathbb{R}^d, d\mu_k)$ denote the space of complex valued measurable functions f such that

$$||f||_{L^p(d\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p \, d\mu_k(x)\Big)^{1/p} < \infty$$

and $L^{p,\infty}(\mathbb{R}^d, d\mu_k)$ be the corresponding weak space with norm

$$||f||_{L^{p,\infty}(d\mu_k)} := \sup_{t>0} t \left[\mu_k\big(\{x \in \mathbb{R}^d : |f(x)| > t\}\big)\right]^{1/p} < \infty.$$

For any $f \in L^1(\mathbb{R}^d, d\mu_k)$, the *Dunkl transform* of f is defined by

$$\mathcal{F}_k f(\xi) = \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) \, d\mu_k(x).$$

The following properties of Dunkl transform are known in the literature [21, 23].

- $\mathcal{F}_k$ preserves the space $\mathcal{S}(\mathbb{R}^d)$,
- $\mathcal{F}_k$ extends to an isometry on $L^2(\mathbb{R}^d, d\mu_k)$ (Plancherel formula) that is,

$$||\mathcal{F}_k f||_{L^2(d\mu_k)} = ||f||_{L^2(d\mu_k)},$$

- If both f and $\mathcal{F}_k f$ are in $L^1(\mathbb{R}^d, d\mu_k)$, then the following Dunkl inversion formula holds

$$f(x) = \mathcal{F}_k^{-1}(\mathcal{F}_k f)(x) =: \int_{\mathbb{R}^d} \mathcal{F}_k f(\xi) E_k(i\xi, x) \, d\mu_k(\xi),$$

- From definition of the Dunkl kernel, for any $f \in \mathcal{S}(\mathbb{R}^d)$, the following relations holds :

$$T_j \mathcal{F}_k f(\xi) = -\mathcal{F}_k(i(\cdot)_j f)(\xi) \text{ and } \mathcal{F}_k(T_j f)(\xi) = i\xi_j \mathcal{F}_k f(\xi).$$

The *Dunkl translation* $\tau_x^k f$ of a function $f \in L^2(\mathbb{R}^d, d\mu_k)$ is defined in [71] in terms of Dunkl transform by

$$\mathcal{F}_k(\tau_x^k f)(y) = E_k(ix, y) \mathcal{F}_k f(y).$$

Since $E_k(ix, y)$ is bounded, the above formula defines τ_x^k as a bounded operator on $L^2(\mathbb{R}^d, d\mu_k)$. We collect few properties of the Dunkl translations which will be used later.

• For $f \in \mathcal{S}(\mathbb{R}^d)$, τ_x^k can be pointwise defined as

$$\tau_x^k f(y) = \int_{\mathbb{R}^d} E_k(ix, \xi) E_k(iy, \xi) \mathcal{F}_k f(\xi)\, d\mu_k(\xi),$$

• $\tau_y^k f(x) = \tau_x^k f(y)$ for any f in $\mathcal{S}(\mathbb{R}^d)$,

• $\int_{\mathbb{R}^d} \tau_x^k f(y) g(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) \tau_{-x}^k g(y) d\mu_k(y)$, for any $f \in \mathcal{S}(\mathbb{R}^d)$ and any bounded function $g \in L^1(\mathbb{R}^d, d\mu_k)$;

• $\tau_x^k(f_t) = (\tau_{t^{-1}x}^k f)_t$, $\forall x \in \mathbb{R}^d$ and $\forall f \in \mathcal{S}(\mathbb{R}^d)$, where $f_t(x) = t^{-d_k} f(t^{-1}x)$ and $t > 0$;

• $\tau_x^k f \geq 0$ for all bounded, radial functions $f \in L^1(\mathbb{R}^d, d\mu_k)$ such that $f \geq 0$.

Also the following specific formula for the Dunkl translation of Schwartz class radial functions $f(x) = f_0(|x|)$ was obtained by Rösler [62].

$$\tau_x^k f(-y) = \int_{\mathbb{R}^d} (f_0 \circ \mathcal{A})(x, y, \eta)\, d\mu_x(\eta), \tag{2.1.6}$$

where $\mathcal{A}(x, y, \eta) = \sqrt{|x|^2 + |y|^2 - 2\langle y, \eta \rangle}$ and μ_x is a probability measure supported in the convex hull of the set $\{\sigma(x) : \sigma \in G\}$.

A useful formula that for any $\eta \in \text{conv}\, \{\sigma(x) : \sigma \in G\}$,

$$d_G(x, y) \leq \mathcal{A}(x, y, \eta) \leq \max_{\sigma \in G} |\sigma(x) - y|. \tag{2.1.7}$$

Thangavelu and Xu [71, Proposition 3.3] observed that the formula (2.1.6) also holds for radial functions f such that both f and $\mathcal{F}_k f \in L^1(\mathbb{R}^d, d\mu_k)$. Later Dai and Wang [19, Lemma 3.4] extended it to all continuous radial functions in $L^2(\mathbb{R}^d, d\mu_k)$.

Although τ_x^k is bounded operator for radial functions in $L^p(\mathbb{R}^d, d\mu_k)$ (see [31]), it is not known whether Dunkl translation is bounded operator or not on whole $L^p(\mathbb{R}^d, d\mu_k)$ for $p \neq 2$.

For $f, g \in L^2(\mathbb{R}^d, d\mu_k)$, the *Dunkl convolution* $f *_k g$ of f and g is defined by

$$f *_k g(x) = \int_{\mathbb{R}^d} f(y) \tau_x^k g(-y)\, d\mu_k(y).$$

$*_k$ has the following basic properties (see [71] for details).

• $f *_k g(x) = g *_k f(x)$ for any $f, g \in L^2(\mathbb{R}^d, d\mu_k)$;

• $\mathcal{F}_k(f *_k g)(\xi) = \mathcal{F}_k f(\xi)\mathcal{F}_k g(\xi)$ for any $f, g \in L^2(\mathbb{R}^d, d\mu_k)$.

As mentioned for the Dunkl operators case, for $k = 0$ Dunkl transform becomes the Euclidean Fourier transform and the Dunkl translation operator becomes the usual translation. In this sense, Dunkl transform is a generalization of Euclidean Fourier transform and putting $k = 0$, we can recover the corresponding results in the classical setting from our results.

2.2 A Multilinear Dunkl setup

We start this section by extending the Dunkl theory to a multilinear setup. Let R be the root system and k be the multiplicity function as in the last Section. Then

$$R^m := (R \times (0)_{m-1}) \cup \big((0)_1 \times R \times (0)_{m-2}\big) \cup \cdots \cup \big((0)_{m-1} \times R\big),$$

where $(0)_j = \{(0, 0, \cdots, 0)\} \subset (\mathbb{R}^d)^j$, defines a root system in $\big(\mathbb{R}^d\big)^m$. The reflection group acting on $(\mathbb{R}^d)^m$ is isomorphic to the m-fold product $G \times G \times \cdots \times G$. Let $k^m : R^m \to \mathbb{C}$ be defined by

$$k^m((0, 0, \cdots, \lambda, \cdots, 0)) = k(\lambda) \text{ for any } \lambda \in R.$$

Then it follows that k^m is a non-negative normalized multiplicity function on R^m. Due to this choice of the Root system and multiplicity function, Dunkl objects on $(\mathbb{R}^d)^m$ splits into product of the corresponding objects in $\mathbb{R}^d$. In fact, using the notations as described in Section 2.1, it follows that for any $x_1, y_1, x_2, y_2, \cdots, x_m, y_m \in \mathbb{R}^d$, we have

$$d\mu_{k^m}\big((x_1, x_2, \cdots, x_m)\big) = d\mu_k(x_1)d\mu_k(x_2)\cdots d\mu_k(x_m).$$

Again the structures of the root system and the multiplicity function allow us to write

$$E_{k^m}\big((x_1, x_2, \cdots, x_m), (y_1, y_2, \cdots, y_m)\big) = E_k(x_1, y_1)E_k(x_2, y_2)\cdots E_k(x_m, y_m).$$

This at once implies that for reasonable functions $f_1, f_2, \cdots, f_m$;

$$\mathcal{F}_{k^m}(f_1 \otimes f_2 \otimes \cdots \otimes f_m)\big((x_1, x_2, \cdots, x_m)\big) = \mathcal{F}_k f_1(x_1)\mathcal{F}_k f_2(x_2)\cdots \mathcal{F}_k f_m(x_m)$$

$$\text{and } \tau^{k^m}_{(x_1,x_2,\cdots,x_m)}(f_1 \otimes \cdots \otimes f_m)\Big((y_1, y_2, \cdots, y_m)\Big) = \tau^k_{x_1} f_1(y_1)\cdots \tau^k_{x_m} f_m(y_m).$$

Also the m-fold counterparts of all the properties mentioned in Section 2.1 hold in this case.

2.3 Spaces of Homogeneous type and Muckenhoupt Weights

Definition 2.3.1. A *space of homogeneous type* $(X, \rho, d\mu)$ is a topological space equipped with a quasi metric ρ and a Borel measure $d\mu$ such that

(i) ρ is continuous on $X \times X$ and the balls $B_\rho(x, r)) := \{y \in X : \rho(x, y) < r\}$ are open in X;

(ii) the measure μ fulfills the doubling condition :

$$\mu(B_\rho(x, 2r)) \leq C\mu(B_\rho(x, r)),\ \forall x \in X,\ \forall r > 0;$$

(iii) $0 < \mu(B_\rho(x, r)) < \infty$ for every $x \in X$ and $r > 0$.

Note that the Dunkl measure μ_k is a Borel measure on $\mathbb{R}^d$ and satisfies the doubling condition, and hence $(\mathbb{R}^d, |x - y|, d\mu_k)$ is a space of homogeneous type.

For any $m \in \mathbb{N}$ and given any $\overrightarrow{f} = (f_1, f_2, \cdots, f_m)$ with each f_j being a locally integrable function on $(X, \rho, d\mu)$, we define the *multi(sub)linear Hardy-Littlewood maximal function* $\mathcal{M}^X_{HL}$ by

$$\mathcal{M}^X_{HL}\overrightarrow{f}(x) = \sup_{\substack{B_\rho \subset X\\ x \in B_\rho}} \prod_{j=1}^{m} \frac{1}{\mu(B_\rho)} \int_{B_\rho} |f_j(y)|\, d\mu(y),$$

where supremum is taken over all balls B_ρ in X which contains x.

We will use the notation $\mathcal{M}_{HL}^k$ for the multi(sub)linear Hardy-Littlewood maximal function on the space $(\mathbb{R}^d, |x-y|, d\mu_k)$, i.e.,

$$\mathcal{M}_{HL}^k \overrightarrow{f}(x) = \sup_{\substack{B\subset\mathbb{R}^d \\ x\in B}} \prod_{j=1}^m \frac{1}{\mu_k(B)} \int_B |f_j(y)|\, d\mu_k(y).$$

When $m = 1$, we will write

$$M_{HL}^k f(x) := \sup_{\substack{B\subset\mathbb{R}^d \\ x\in B}} \frac{1}{\mu_k(B)} \int_B |f(y)|\, d\mu_k(y).$$

For any locally integrable function f, we also define the *sharp maximal function* $M_{HL}^{k,\#}$, given by

$$M_{HL}^{k,\#} f(x) = \sup_{\substack{B\subset\mathbb{R}^d \\ x\in B}} \frac{1}{\mu_k(B)} \int_B |f(y) - f_B|\, d\mu_k(y),$$

where

$$f_B = \frac{1}{\mu_k(B)} \int_B f(y)\, d\mu_k(y).$$

For $\epsilon > 0$ set

$$M_{HL,\epsilon}^{k,\#} f(x) = \left(M_{HL}^{k,\#}(|f|^\epsilon)(x)\right)^{1/\epsilon}.$$

Observe that for $\epsilon, a, b > 0$, the inequalities

$$\min\{1, 2^{\epsilon-1}\}(a^\epsilon + b^\epsilon) \le (a+b)^\epsilon \le \max\{1, 2^{\epsilon-1}\}(a^\epsilon + b^\epsilon)$$

yield

$$M_{HL,\epsilon}^{k,\#} f(x) \sim \sup_{\substack{B\subset\mathbb{R}^d \\ x\in B}} \inf_{c\in\mathbb{C}} \left[\frac{1}{\mu_k(B)} \int_B \big|\,|f(y)|^\epsilon - |c|^\epsilon\big|\, d\mu_k(y)\right]^{1/\epsilon}.$$

For $t > 1$, we also define the *average maximal operator* $M_{t,HL}^k$, given by $M_{t,HL}^k f = (M_{HL}^k |f|^t)^{1/t}$. From the well known result in homogeneous space type [17], we can state the following theorem for the above maximal type operators.

Theorem 2.3.2. *Let $m \in \mathbb{N}$, then* *(i)* $\mathcal{M}_{HL}^k$ *is weak type* $L^1(\mathbb{R}^d, d\mu_k) \times L^1(\mathbb{R}^d, d\mu_k) \times \cdots \times L^1(\mathbb{R}^d, d\mu_k)$ *to* $L^{1/m}(\mathbb{R}^d, d\mu_k)$*;*

(ii) for $1 < p < \infty$*,* M_{HL}^k *is strong type* $L^p(\mathbb{R}^d, d\mu_k)$ *to* $L^p(\mathbb{R}^d, d\mu_k)$*;*

(iii) for $t > 1$*,* $M_{(tp)',HL}^k$ *is strong type* $L^{p'}(\mathbb{R}^d, d\mu_k)$ *to* $L^{p'}(\mathbb{R}^d, d\mu_k)$*.*

A non-negative locally integrable function in a homogeneous space is called a *weight*. Next we define Muckenhoupt class in homogeneous space setting as follows.

Definition 2.3.3. Let $1 \le p < \infty$ and w be a weight. The weight w is said to belong to the class $A_p(X, \rho, d\mu)$, if it satisfies

$$\sup_{B_\rho \subset X} \Big(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w(y) d\mu(y)\Big)\Big(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w(y)^{1-p'} d\mu(y)\Big)^{p-1} < \infty,$$

when $p = 1$, $\Big(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w(y)^{1-p'} d\mu(y)\Big)^{p-1}$ is understood as $\left(\inf_{B_\rho} w\right)^{-1}$.

Set $A_\infty(X, \rho, d\mu) := \bigcup_{1 \le p < \infty} A_p(X, \rho, d\mu)$.

Definition 2.3.4. Let $1 < p \le q < \infty$ and w be a weight. We say that the weight w is in the class $A_{p,q}(X, \rho, d\mu)$, if it satisfies

$$\sup_{B_\rho \subset X} \left(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w(y)^q d\mu(y)\right)^{1/q} \left(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w(y)^{-p'} d\mu(y)\right)^{1/p'} < \infty.$$

We also have the multilinear analogue of the above classes as follows.

Definition 2.3.5. Let $1 \le p_1, p_2, \cdots, p_m < \infty$, $\overrightarrow{P} = (p_1, p_2, \cdots, p_m)$ and p be the number given by $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$. Furthermore, let $v, w_1, w_2, \cdots, w_m$ be weights and $\overrightarrow{w} = (w_1, w_2, \cdots, w_m)$. We say that the vector weight $(v, \overrightarrow{w})$ is in the class $A_{\overrightarrow{P}}(X, \rho, d\mu)$, if it satisfies

$$\sup_{B_\rho \subset X} \Big(\frac{1}{\mu(B_\rho)} \int_{B_\rho} v(y) d\mu(y)\Big)^{1/p} \prod_{j=1}^m \Big(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w_j(y)^{1-p_j'} d\mu(y)\Big)^{1/p_j'} < \infty,$$

when $p_j = 1$, $\Big(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w_j(y)^{1-p_j'} d\mu(y)\Big)^{1/p_j'}$ is understood as $\left(\inf_{B_\rho} w_j\right)^{-1}$.

In particular when $v = \prod_{j=1}^{m} w_j^{p/p_j}$, we will simply say that $\overrightarrow{w}$ is in the class $A_{\overrightarrow{P}}(X, \rho, d\mu)$.

Definition 2.3.6. Let $1 < p_1, p_2, \cdots, p_m < \infty$, q be a number such that $1/m < p \leq q < \infty$ and $w_1, w_2, \cdots, w_m$ be weights. We say that the vector weight $\overrightarrow{w} = (w_1, w_2, \cdots, w_m)$ is in the class $A_{\overrightarrow{P},q}(X, \rho, d\mu)$, if it satisfies

$$\sup_{B_\rho \subset X} \left(\frac{1}{\mu(B_\rho)} \int_{B_\rho} \Big(\prod_{j=1}^{m} w_j(y) \Big)^q d\mu(y) \right)^{1/q} \prod_{j=1}^{m} \left(\frac{1}{\mu(B_\rho)} \int_{B_\rho} w_j(y)^{-p'_j} d\mu(y) \right)^{1/p'_j} < \infty.$$

For the homogeneous space type$(\mathbb{R}^d, |x-y|, d\mu_k)$, we will simply write A_p^k, A_∞^k, $A_{p,q}^k$, $A_{\overrightarrow{P}}^k$ and $A_{\overrightarrow{P},q}^k$ in place of $A_p(\mathbb{R}^d, |x-y|, d\mu_k)$, $A_\infty(\mathbb{R}^d, |x-y|, d\mu_k)$, $A_{p,q}(\mathbb{R}^d, |x-y|, d\mu_k)$, $A_{\overrightarrow{P}}(\mathbb{R}^d, |x-y|, d\mu_k)$ and $A_{\overrightarrow{P},q}(\mathbb{R}^d, |x-y|, d\mu_k)$, respectively. A_p^k weights are recently studied in [41] without using the results of spaces of homogeneous type. However in this article will use the known results for weights in a general space of homogeneous type. In our main theorems we will use G-invariant weights in Dunkl setting, i.e. weights w in $\mathbb{R}^d$ that satisfies $w(\sigma(x)) = w(x),\ \forall x \in \mathbb{R}^d$ and $\forall \sigma \in G$. It is quite natural to restrict to the weights which are G-invariant as the Dunkl measure $d\mu_k$ itself is G-invariant. From G-invariance of a weight w, it follows that for a reasonable function f on $\mathbb{R}^d$, for any $\sigma \in G$ and for any exponent p,

$$\int_{\mathbb{R}^d} f \circ \sigma(x) w(x)\, d\mu_k(x) = \int_{\mathbb{R}^d} f(x) w(x)\, d\mu_k(x),$$

which will be used many times in our proofs.

As in the classical setting [7, Remark 2.4] the following property of $A_{\overrightarrow{P},q}^k$ follows immediately from the definition.

Proposition 2.3.7. *Let $1 < p_1, p_2, \cdots, p_m < \infty$, q be such that $1/m < p \leq q < \infty$ and $\overrightarrow{w} \in A_{\overrightarrow{P},q}^k$, then there exist $t > 1$ such that for any ball $B \in \mathbb{R}^d$,*

$$\left(\frac{1}{\mu_k(B)} \int_B w_j(y)^{-tp'_j} d\mu_k(y) \right)^{1/t} \leq C \frac{1}{\mu_k(B)} \int_B w_j(y)^{-p'_j} d\mu_k(y).$$

Proof. We only give an outline of the proof as the same argument as in the classical case works here also. From the definition one can find that $\vec{w} \in A^k_{\vec{P},q}$ implies each $w_j^{-p'_j}$ satisfies the A^k_∞ condition (see [51, Theorem 3.4], for the classical case). Then the rest of the proof is a direct consequence of the fact that A_∞ weights satisfy the reverse Hölder condition. □

We also have the following improving property of $A^k_{\vec{P},q}$ weights.

Proposition 2.3.8. *Suppose that* $1 < p_1, p_2, \cdots, p_m < \infty$, $0 < \alpha < md_k$, $1/m < p < d_k/\alpha$ *and* q *be a number defined by* $1/q = 1/p - \alpha/d_k$. *Furthermore, let the vector weight* $\vec{w} = (w_1, w_2, \cdots, w_m) \in A^k_{\vec{P},q}$, *then there exists* $\epsilon > 0$ *such that*

$$\vec{w} \in A^k_{\vec{P},q_\epsilon} \cap A^k_{\vec{P},\widetilde{q}_\epsilon},$$

where $1/q_\epsilon = 1/p - (\alpha + \epsilon)/d_k$ *and* $1/\widetilde{q}_\epsilon = 1/p - (\alpha - \epsilon)/d_k$. *Also* ϵ *satisfies* $\epsilon < \min\{\alpha, md_k - \alpha\}$; $1/p > (\alpha + \epsilon)/d_k$ *and* $1/q < (md_k - \epsilon)/d_k$.

Proof. Let $1 < r < \infty$ and $w \in A^k_r$, then we have the following.

(i) for $r \le s < \infty$, $A^k_r \subseteq A^k_s$;

(ii) for any $0 < \delta < 1$, $w^\delta \in A^k_r$;

(iii) there exists $\delta > 0$ with $r - \delta > 1$ such that $w \in A^k_{r-\delta}$;

(iv) there exists $\delta > 0$ such that $w^{1+\delta} \in A^k_r$.

In fact (i) and (ii) follows easily by applying Hölder's inequality. For any homogeneous space with doubling measure, proof of (iii) can be found in [68, Lemma 8]. Finally (iv) follows by using the fact that $w \in A^k_r$ implies $w^{1-r'} \in A^k_{r'}$, using the result in [68, Theorem 15] for homogeneous spaces together with decreasing property of reverse Hölder classes.

On the other hand, let $1 < s_1, s_2, \cdots, s_m < \infty$, $1/s = 1/s_1 + 1/s_2 + \cdots + 1/s_m$ and $0 < r < \infty$. Then $w \in A^k_{\vec{S},r}$ if and only if

(i) $\Big(\prod_{j=1}^{m} w_j\Big)^r \in A^k_{1+r(m-1/s)}$;

(ii) for all $j \in \{1, 2, \cdots, m\}$, $w_j^{-s'_j} \in A^k_{1+s'_j t_j}$, where $t_j = 1/r + m - 1/s - 1/s'_j$.

This actually follows by repeating the arguments used in the proof by Iida [44, Theorem 2] for the classical case with obvious modifications of the parameters involved.

Thus we have acquired all the ingredients used in the proof the corresponding result in classical setting [76, Lemma 3.3]. Hence the proof follows by arguing in the same way as in the classical case. □

We end this section by stating two theorems which follow from well known results for general space of homogeneous type [34, Theorem 4.4, Theorem 4.6, and Theorem 4.7].

Theorem 2.3.9. *Let $1 \le p_1, p_2, \cdots, p_m < \infty$, $\overrightarrow{P} = (p_1, p_2, \cdots, p_m)$, p be the number given by $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$ and $v, w_1, w_2, \cdots, w_m$ be weights. Then the following hold:*

(i) if $p_j = 1$ for some $1 \le j \le m$ and the vector weight $(v, \overrightarrow{w}) \in A^k_{\overrightarrow{P}}$, then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m\, d\mu_k)$, the following boundedness holds:

$$\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\, M^k_{HL}\overrightarrow{f}(y)>t\}} v(x)\, d\mu_k(x)\Big)^{1/p} \le C \prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x) d\mu_k(x)\Big)^{1/p_j};$$

(ii) if $p_j > 1$ for all $1 \le j \le m$ and for some $t > 1$ the vector weight $(v, \overrightarrow{w})$ satisfies the bump condition

$$\sup_{B\subset\mathbb{R}^d} \Big(\frac{1}{\mu_k(B)}\int_B v(y) d\mu_k(y)\Big)^{1/p} \prod_{j=1}^{m}\Big(\frac{1}{\mu_k(B)}\int_B w_j(y)^{-tp'_j/p_j} d\mu_k(y)\Big)^{1/tp'_j} < \infty, \tag{2.3.1}$$

then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m\, d\mu_k)$*, the following boundedness holds:*

$$\Big(\int_{\mathbb{R}^d} \big(\mathcal{M}_{HL}^k \overrightarrow{f}(x)\big)^p v(x)\, d\mu_k(x)\Big)^{1/p} \le C \prod_{j=1}^m \Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\, d\mu_k(x)\Big)^{1/p_j}.$$

Theorem 2.3.10. *Let* $1 < p_1, p_2, \cdots, p_m < \infty$*,* $\overrightarrow{P} = (p_1, p_2, \cdots, p_m)$*,* p *be the number given by* $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$ *and* $w_1, w_2, \cdots, w_m$ *be weights such that the vector weight* $\overrightarrow{w} \in A^k_{\overrightarrow{P}}$*; then for all* $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m\, d\mu_k)$*, the following boundedness holds:*

$$\Big(\int_{\mathbb{R}^d} \big(\mathcal{M}_{HL}^k \overrightarrow{f}(x)\big)^p \prod_{j=1}^m w_j(x)^{p/p_j}\, d\mu_k(x)\Big)^{1/p} \le C \prod_{j=1}^m \Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\, d\mu_k(x)\Big)^{1/p_j}.$$

Chapter 3

Multilinear Dunkl- Calderón- Zygmund Operators

In this chapter, we introduce multilinear Calderón-Zygmund operators that incorporate the actions of reflection groups and orbit distances, aligning appropriately with the Dunkl setup. Our focus lies in establishing weighted bounds for these operators, extending the corresponding conclusions from the classical setup to the Dunkl setup. Assuming the initial boundedness condition, we first prove an end-point weak-type boundedness result in Section 3.2. Then, in Section 3.3, by combining these weak-type end-point estimates with known bounds for the maximal operators, we establish one and two-weight estimates for multilinear Dunkl-Calderón-Zygmund operators. In Section 3.4, we delve into the study of maximal operators associated with multilinear Dunkl-Calderón-Zygmund operators. Specifically, we establish a multilinear Cotlar-type inequality as a means of attaining the weighted boundedness of these maximal operators. The content of this chapter is based on a portion of the work [56] and on the work [54].

3.1 Introduction

In the 1950's, Calderón and Zygmund [9–11] made significant progress in laying the groundwork for studying a broad category of singular integral operators. These operators later became known as Calderón-Zygmund operators. Much later, in the 2000's, a considerable portion of these works was extended to the multilinear setting (see e.g. [14,15,25,36–38,45, 48]). Our primary focus among these references lies in the multilinear Calderón-Zygmund operators, as introduced by Grafakos and Torres [38].

The study of singular integral operators has been well-explored in the Dunkl setup [4, 27, 69]. Remarkably, Tan et al. [69] recently introduced a category of singular integrals that bears resemblance to classical Calderón-Zygmund operators. However, currently, there is no known theory for multilinear singular integrals in this framework. Thus, it is crucial to investigate this unexplored domain. In this chapter, our objective is to explore m-linear Dunkl-Calderón-Zygmund operators, which can be treated as Dunkl counter part of the multilinear Calderón-Zygmund operators in classical setting introduced by Grafakos and Torres [38].

In the theory of Dunkl analysis, our findings on multilinear singular integral operators pave the way for studying various multilinear operators within the Dunkl framework. Notably, in the next chapter, we have derived that bilinear multipliers represent specific class of such operators. Furthermore, we anticipate the extension of these results to encompass other type of operators, including multilinear Dunkl-pseudo-differential operators belonging to some particular symbol classes, much like in the classical case ([12, 38]). Also, this, in turn, prompts us to explore other types of multilinear singular integrals, such as rough singular integrals and singular integrals with Dini-type conditions, along with their associated commutators.

Motivated by the Definition 1.1.1 of classical multilinear Calderón-Zygmund operators, we provide the following definition for multilinear Calderón-Zygmund operators in the Dunkl setting.

Definition 3.1.1. An m-linear *Dunkl-Calderón-Zygmund operator* is a function $\mathcal{T}$ defined on the m-fold product $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$ and taking values on $\mathcal{S}'(\mathbb{R}^d)$ such that for all $f_j \in C_c^\infty(\mathbb{R}^d)$ with $\sigma(x) \notin \bigcap_{j=1}^{m} supp\, f_j$ for all $\sigma \in G$, $\mathcal{T}$ can be represented as

$$\mathcal{T}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^d)^m} K(x, y_1, y_2, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j)\, d\mu_k(y_j),$$

where K is a function defined away from the set $\mathcal{O}(\triangle_{m+1})$

$$=: \{(x, y_1, y_2, \cdots, y_m) \in (\mathbb{R}^d)^{m+1} : x = \sigma_j(y_j) \text{ for some } \sigma_j \in G, \text{ for all } 1 \leq j \leq m\}$$

which satisfies the following size estimate and smoothness estimates for some $0 < \epsilon \leq 1$:

$$|K(y_0, y_1, y_2, \cdots, y_m)| \leq C_K \Big[\sum_{j=1}^{m} \mu_k\big(B(y_0, d_G(y_0, y_j))\big)\Big]^{-m} \left[\frac{\sum\limits_{j=1}^{m} d_G(y_0, y_j)}{\sum\limits_{j=1}^{m} |y_0 - y_j|}\right]^{\epsilon}, \tag{3.1.1}$$

for all $(y_0, y_1, y_2, \cdots, y_m) \in (\mathbb{R}^d)^{m+1} \setminus \mathcal{O}(\triangle_{m+1})$;

$$\begin{aligned} & |K(y_0, y_1, y_2, \cdots, y_n, \cdots, y_m) - K(y_0, y_1, y_2, \cdots, y'_n, \cdots, y_m)| \\ \leq\ & C_K \Big[\sum_{j=1}^{m} \mu_k\big(B(y_0, d_G(y_0, y_j))\big)\Big]^{-m} \left[\frac{|y_n - y'_n|}{\max\limits_{1\leq j\leq m} |y_0 - y_j|}\right]^{\epsilon}, \end{aligned} \tag{3.1.2}$$

whenever $|y_n - y'_n| \leq \max\limits_{1\leq j\leq m} d_G(y_0, y_j)/2$, for all $n \in \{0, 1, \cdots, m\}$.

Note that the size condition (3.1.1) guarantees that the above integral is convergent and hence pointwise $\mathcal{T}(\overrightarrow{f})$ makes sense. Also this definition of multilinear Calderón-Zygmund operators in Dunkl setting matches with the definition of multilinear Calderón-Zygmund operators in classical setting [38] as well as with the definition of linear Calderón-Zygmund operators in Dunkl setting [69]. In this context, it is worth noting that the regularity conditions (3.1.2) assumed on the kernel are much weaker than those required for kernels of the multilinear Calderón-Zygmund type operators defined in [34] for spaces of homogeneous type. This contrast is consistent with the situation in the linear case, as shown in [69, p. 10].

Our main results regarding Dunkl-Calderón-Zygmund operators are one-weight inequalities Theorem 3.3.4 and and two-weight inequalities Theorem 3.3.3. In the proofs of these theorems, we closely follow the scheme used in [48] (see also [34]). However, due to involvement of the action of the reflection group in Definition 3.1.1, some new ideas regarding G-orbits, the Dunkl metric d_G, and results for spaces of homogeneous type are required to complete the proofs. Here, we also mention that by arguing as in [69, p. 10], we can see that the smoothness conditions (3.1.2) assumed on the kernel are weaker than that of the Calderón-Zygmund singular integral operators given in spaces of homogeneous type [34, p. 20]. So, the results for singular integrals in spaces of homogeneous type do not imply our results.

3.2 Weak Boundedness for Multilinear Dunkl-Calderón-Zygmund Operators

Before going into the weak-type estimates for $\mathcal{T}$, we require the following lemma.

Lemma 3.2.1. *For any $\epsilon > 0$ and $N \in \mathbb{N}$ there exists a constant $C > 0$ such that for any $s \in \{1, 2, \cdots, N\}$ and $x, y_n \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \Big[\sum_{n=1}^{N} \mu_k\big(B(x, d_G(x, y_n))\big)\Big]^{-1} \big[\max_{1\le n\le N} d_G(x, y_n)\big]^{-\epsilon} \, d\mu_k(y_s) \le C \big[\max_{\substack{1\le n\le N \\ n\neq s}} d_G(x, y_n)\big]^{-\epsilon}.$$

Proof. Take $t = \max\limits_{\substack{1\le n\le N \\ n\neq s}} d_G(x, y_n)$.

Estimate for $d_G(x, y_s) < 2t$.

In this case using (2.1.2), it is not hard to see that

$$\sum_{n=1}^{N} \mu_k\big(B(x, d_G(x, y_n))\big) \sim \mu_k(B(x, t)).$$

Then from (2.1.5),

$$\int_{d_G(x,y_s)<2t} \Big[\sum_{n=1}^{N} \mu_k\big(B(x, d_G(x, y_n))\big)\Big]^{-1} \big[\max_{1\le n\le N} d_G(x, y_n)\big]^{-\epsilon} \, d\mu_k(y_s)$$

$$\begin{aligned}
&\leq C\,[\max_{\substack{1\leq n\leq N\\ n\neq s}} d_G(x,y_n)]^{-\epsilon} \int_{d_G(x,y_s)<2t} \frac{1}{\mu_k(B(x,t))}\, d\mu_k(y_s)\\
&\leq C\,[\max_{\substack{1\leq n\leq N\\ n\neq s}} d_G(x,y_n)]^{-\epsilon}.
\end{aligned}$$

Estimate for $d_G(x,y_s)\geq 2t$.

In this case we have

$$\sum_{n=1}^{N} \mu_k\big(B(x,d_G(x,y_n))\big) \sim \mu_k(B(x,d_G(x,y_s))).$$

Hence, similarly applying (2.1.5) again,

$$\begin{aligned}
&\int_{d_G(x,y_s)\geq 2t} \Big[\sum_{n=1}^{N} \mu_k\big(B(x,d_G(x,y_n))\big)\Big]^{-1} [\max_{1\leq n\leq N} d_G(x,y_n)]^{-\epsilon}\, d\mu_k(y_s)\\
&\leq C \int_{d_G(x,y_s)\geq 2t} \frac{1}{\mu_k\big(B(x,d_G(x,y_s))\big)} [d_G(x,y_s)]^{-\epsilon}\, d\mu_k(y_s)\\
&\leq C \sum_{r=1}^{\infty} \int_{2^r t\leq d_G(x,y_s)<2^{r+1}t} \frac{1}{\mu_k\big(B(x,d_G(x,y_s))\big)} [d_G(x,y_s)]^{-\epsilon}\, d\mu_k(y_s)\\
&\leq C \sum_{r=1}^{\infty} \frac{1}{(2^r t)^{\epsilon}} \frac{|G|\,\mu_k(B(x,2^r t))}{\mu_k(B(x,2^r t))} \leq C\,t^{-\epsilon} = C\,[\max_{\substack{1\leq n\leq N\\ n\neq s}} d_G(x,y_n)]^{-\epsilon}.
\end{aligned}$$

This completes the proof of the lemma. □

Throughout this section we will assume that $\mathcal{T}$ is an m-linear Dunkl-Calderón-Zygmund operator and $\mathcal{T}$ maps from $L^{q_1}(\mathbb{R}^d,\, d\mu_k)\times L^{q_2}(\mathbb{R}^d,\, d\mu_k)\times\cdots\times L^{q_m}(\mathbb{R}^d,\, d\mu_k)$ to $L^{q,\infty}(\mathbb{R}^d,\, d\mu_k)$ with norm A for some $q, q_1, q_2, \cdots, q_m$ satisfying $1\leq q_1, q_2, \cdots, q_m<\infty$ with $1/q = 1/q_1 + 1/q_2 + \cdots + 1/q_m$. From this a priori boundedness condition we can prove the following weak-type end-point estimates.

Theorem 3.2.2. *Let $\mathcal{T}$ be a multilinear operator as described above. Then $\mathcal{T}$ extends to a bounded operator from the m-fold product $L^1(\mathbb{R}^d,\, d\mu_k)\times L^1(\mathbb{R}^d,\, d\mu_k)\times\cdots\times L^1(\mathbb{R}^d,\, d\mu_k)$ to $L^{1/m,\infty}(\mathbb{R}^d,\, d\mu_k)$ with norm $\leq C(C_K + A)$.*

Proof. By density argument, it is enough to show the result for functions in $\mathcal{S}(\mathbb{R}^d)$. Take $f_1, f_2, \cdots, f_m \in \mathcal{S}(\mathbb{R}^d)$ and fix $\alpha > 0$. Define

$$E_\alpha = \big\{x \in \mathbb{R}^d : |\mathcal{T}(f_1, f_2, \cdots, f_m)(x)| > \alpha\big\}.$$

Also, without loss of generality we can take $\|f_j\|_{L^1(d\mu_k)} = 1$. It then suffices to prove that $\mu_k(E_\alpha) \leq C(C_K + A)^{1/m}\,\alpha^{-1/m}$.

Let $\nu > 0$ be constant to be defined later. Applying Calderón-Zygmund decomposition to each of the functions f_j at height $(\alpha\nu)^{1/m}$, we obtain m number of good functions g_j, m number of bad functions b_j and m families I_j of balls $\big\{B_{j,n} : n \in I_j\big\}$ such that $f_j = g_j + b_j$, $b_j = \sum\limits_{n\in I_j} b_{j,n}$ with the properties that for all $n \in I_j$ and $s \in [1, \infty)$,

(i) $supp\, b_{j,n} \subseteq B_{j,n}$ and $\int_{\mathbb{R}^d} b_{j,n}(y)\, d\mu_k(y) = 0$;

(ii) $\|b_{j,n}\|_{L^1(d\mu_k)} \leq C(\alpha\nu)^{1/m}\, \mu_k(B_{j,n})$;

(iii) $\sum\limits_{n\in I_j} \mu_k(B_{j,n}) \leq C(\alpha\nu)^{-1/m}$;

(iv) $\|g_j\|_{L^\infty} \leq C(\alpha\nu)^{1/m}$, $\|g_j\|_{L^s(d\mu_k)} \leq C(\alpha\nu)^{1/m(1-1/s)}$ and $\|b_j\|_{L^1(d\mu_k)} \leq C$.

Now let

$$\begin{aligned}
E_1 &= \{x \in \mathbb{R}^d : |\mathcal{T}(g_1, g_2, \cdots, g_m)(x)| > \alpha/2^m\},\\
E_2 &= \{x \in \mathbb{R}^d : |\mathcal{T}(b_1, g_2, \cdots, g_m)(x)| > \alpha/2^m\},\\
E_3 &= \{x \in \mathbb{R}^d : |\mathcal{T}(g_1, b_2, \cdots, g_m)(x)| > \alpha/2^m\}\\
&\vdots\\
\text{and } E_{2^m} &= \{x \in \mathbb{R}^d : |\mathcal{T}(b_1, b_2, \cdots, b_m)(x)| > \alpha/2^m\}.
\end{aligned}$$

Then $\mu_k\big(\big\{x \in \mathbb{R}^d : |\mathcal{T}(f_1, f_2, \cdots, f_m)(x)| > \alpha\big\}\big) \leq \sum\limits_{n=1}^{2^m} \mu_k(E_n)$. It is now enough to show that for all $n \in \{1, 2, \cdots, 2^m\}$,

$$\mu_k(E_n) \leq C(C_K + A)^{1/m}\alpha^{-1/m}. \tag{3.2.1}$$

Applying the given hypothesis on $\mathcal{T}$, we get

$$\begin{aligned}\mu_k(E_1) &\leq C(2^m A/\alpha)^q ||g_1||^q_{L^{q_1}(d\mu_k)} ||g_2||^q_{L^{q_2}(d\mu_k)} \cdots ||g_m||^q_{L^{q_m}(d\mu_k)} \\ &\leq C\, A^q \alpha^{-q} (\alpha\nu)^{(q/m)(m-1/q)} \leq C\, A^q \alpha^{-1/m} \nu^{q-1/m}. \end{aligned} \tag{3.2.2}$$

Next let us take E_n, where $2 \leq n \leq 2^m$. Consider the case where there are exactly l bad functions appearing in $\mathcal{T}(h_1, h_2, \cdots, h_m)$ where h_j is either g_j or b_j and also let $j_1, j_2, \cdots, j_l$ are the indices which corresponds to the bad functions. We will prove that

$$\mu_k(E_n) \leq C\alpha^{-1/m}[\nu^{-1/m} + \nu^{-1/m}(\nu\, C_K)^{1/l}]. \tag{3.2.3}$$

Let $r_{j,n}$ be the radius and $c_{j,n}$ be the centre of the ball $B_{j,n}$. Define $(B_{j,n})^* = B(c_{j,n},\, 2r_{j,n})$ and $(B_{j,n})^{**} = B(c_{j,n},\, 5r_{j,n})$. Now

$$\begin{aligned}\mu_k\Big(\bigcup_{j=1}^m \bigcup_{n\in I_j} \mathcal{O}\big((B_{j,n})^{**}\big)\Big) &\leq \sum_{j=1}^m \sum_{n\in I_j} \mu_k\big(\mathcal{O}\big((B_{j,n})^{**}\big)\big) \\ &\leq |G| \sum_{j=1}^m \sum_{n\in I_j} \mu_k\big((B_{j,n})^{**}\big) \\ &\leq C \sum_{j=1}^m \sum_{n\in I_j} \mu_k\big(B_{j,n}\big) \leq C(\alpha\nu)^{-1/m}.\end{aligned}$$

Thus in view of the above inequality, to prove (3.2.3), we only need to show that

$$\begin{aligned}&\mu_k\Big(\Big\{x \notin \bigcup_{j=1}^m \bigcup_{n\in I_j} \mathcal{O}\big((B_{j,n})^{**}\big) : |T(h_1, h_2, \cdots, h_m)(x)| > \alpha/2^m\Big\}\Big) \\ &\leq C\,(\alpha\nu)^{-1/m}(C_K\,\nu)^{1/l}.\end{aligned} \tag{3.2.4}$$

Fix $x \notin \bigcup_{j=1}^m \bigcup_{n\in I_j} \mathcal{O}\big((B_{j,n})^{**}\big)$. Then

$$\begin{aligned}|T(h_1, h_2, \cdots, h_m)(x)| \leq& \sum_{n_1\in I_{j_1}} \cdots \sum_{n_l\in I_{j_l}} \Big| \int_{(\mathbb{R}^d)^m} K(x, y_1, y_2, \cdots, y_m) \\ &\times \prod_{s\notin\{j_1,\cdots,j_l\}} g_s(y_s) \prod_{s=1}^l b_{j_s,n_s}(y_{j_s})\, d\mu_k(y_1)\cdots d\mu_k(y_m)\Big|\end{aligned}$$

$$=: \sum_{n_1\in I_{j_1}} \cdots \sum_{n_l\in I_{j_l}} H_{n_1,n_2,\cdots,n_l}. \tag{3.2.5}$$

Let us fix balls $B_{j_1,n_1}, B_{j_2,n_2}, \cdots, B_{j_l,n_l}$ and without loss of generality let us take

$$r_{j_1,n_1} = \min_{1\le s\le l} r_{j_s,n_s}.$$

Then using the smoothness condition (3.1.2) we have

$$\begin{aligned}
& \Big|\int_{(B_{j_1,n_1})^*} K(x,y_1,y_2,\cdots,y_m)\, b_{j_1,n_1}(y_{j_1})\, d\mu_k(y_{j_1})\Big| \\
= & \Big|\int_{(B_{j_1,n_1})^*} [K(x,y_1,\cdots,y_{j_1},\cdots,y_m) - K(x,y_1,\cdots,c_{j_1,n_1},\cdots,y_m)] \\
& \times b_{j_1,n_1}(y_{j_1})\, d\mu_k(y_{j_1})\Big| \\
\le & C_K \int_{(B_{j_1,n_1})^*} \Big[\sum_{n=1}^m \mu_k\big(B(x,d_G(x,y_n))\big)\Big]^{-m} \Big[\frac{|y_{j_1}-c_{j_1,n_1}|}{\max\limits_{1\le n\le m}|x-y_n|}\Big]^{\epsilon} \\
& \times |b_{j_1,n_1}(y_{j_1})|\, d\mu_k(y_{j_1}).
\end{aligned} \tag{3.2.6}$$

To complete the proof, taking integration on both sides of (3.2.6) with respect to $y_s \in \{1,2,\cdots,m\}\setminus\{j_1,j_2,\cdots,j_l\}$ and using Lemma 3.2.1 $(m-l)$ times, we get

$$\begin{aligned}
& \int_{(\mathbb{R}^d)^{m-l}} \Big|\int_{(B_{j_1,n_1})^*} K(x,y_1,y_2,\cdots,y_m)\, b_{j_1,n_1}(y_{j_1})\, d\mu_k(y_{j_1})\Big| \prod_{s\notin\{j_1,j_2,\cdots,j_l\}} d\mu_k(y_s) \\
\le & C_K \int_{(B_{j_1,n_1})^*} |b_{j_1,n_1}(y_{j_1})| \Big[\int_{(\mathbb{R}^d)^{m-l}} \Big[\sum_{n=1}^m \mu_k\big(B(x,d_G(x,y_n))\big)\Big]^{-m} \Big[\frac{|y_{j_1}-c_{j_1,n_1}|}{\max\limits_{1\le n\le m} d_G(x,y_n)}\Big]^{\epsilon} \\
& \times \prod_{s\notin\{j_1,j_2,\cdots,j_l\}} d\mu_k(y_s)\Big]\, d\mu_k(y_{j_1}) \\
\le & CC_K \int_{(B_{j_1,n_1})^*} |b_{j_1,n_1}(y_{j_1})| \Big[\sum_{s=1}^l \mu_k\big(B(x,d_G(x,y_{j_s}))\big)\Big]^{-l} |y_{j_1}-c_{j_1,n_1}|^{\epsilon} \\
& \times \big[\max_{1\le s\le l} d_G(x,y_{j_s})\big]^{-\epsilon}\, d\mu_k(y_{j_1}) \\
\le & CC_K \int_{(B_{j_1,n_1})^*} \Big[\frac{r_{j_1,n_1}}{\max\limits_{1\le s\le l} d_G(x,y_{j_s})}\Big]^{\epsilon} \Big[\sum_{s=1}^l \mu_k\big(B(x,d_G(x,y_{j_s}))\big)\Big]^{-l} |b_{j_1,n_1}(y_{j_1})|\, d\mu_k(y_{j_1}).
\end{aligned} \tag{3.2.7}$$

Now $y_{j_s} \in B_{j_s,n_s}$ and $x \notin \bigcup\limits_{j=1}^m \bigcup\limits_{n\in I_j} \mathcal{O}\big((B_{j,n})^{**}\big)$ together implies

$$d_G(x,y_{j_s}) \sim d_G(x,c_{j_s,n_s}) \text{ and hence } \mu_k\big(B(x,d_G(x,y_{j_s}))\big) \sim \mu_k\big(B(x,d_G(x,c_{j_s,n_s}))\big).$$

Also the minimality of r_{j_1,n_1} implies

$$r_{j_1,n_1} \leq \prod_{s=1}^{l} (r_{j_s,n_s})^{1/l}.$$

Similarly,

$$\max_{1\leq s\leq l} d_G(x, c_{j_s,n_s}) \geq \prod_{s=1}^{l} [d_G(x, c_{j_s,n_s})]^{1/l}$$

$$\text{and } \sum_{s=1}^{l} \mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big) \geq \prod_{s=1}^{l} [\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)]^{1/l}.$$

Now, taking the above discussions into account, from (3.2.7) we can write

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{m-l}} \Big| \int_{(B_{j_1,n_1})^*} K(x, y_1, y_2, \cdots, y_m)\, b_{j_1,n_1}(y_{j_1})\, d\mu_k(y_{j_1}) \Big| \prod_{s\notin\{j_1,j_2,\cdots,j_l\}} d\mu_k(y_s) \\
\leq\; & CC_K \Big[\frac{r_{j_1,n_1}}{\max_{1\leq s\leq l} d_G(x, c_{j_s,n_s})} \Big]^{\epsilon} \Big[\sum_{s=1}^{l} \mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big) \Big]^{-l} \|b_{j_1,n_1}\|_{L^1(d\mu_k)} \\
\leq\; & CC_K \|b_{j_1,n_1}\|_{L^1(d\mu_k)} \prod_{s=1}^{l} \Big[\frac{r_{j_s,n_s}}{d_G(x, c_{j_s,n_s})} \Big]^{\epsilon/l} \frac{1}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)}.
\end{aligned}$$

So using properties of Calderón-Zygmund decomposition, from (3.2.5) we write

$$\begin{aligned}
H_{n_1,n_2,\cdots,n_l} \leq\; & \int_{(\mathbb{R}^d)^{m-1}} \Big| \int_{(B_{j_1,n_1})^*} K(x, y_1, y_2, \cdots, y_m)\, b_{j_1,n_1}\, d\mu_k(y_{j_1}) \Big| \\
& \times \prod_{s\notin\{j_1,\cdots,j_l\}} |g_s(y_s)|\, d\mu_k(y_s) \prod_{s=2}^{l} |b_{j_s,n_s}(y_{j_s})|\, d\mu_k(y_{j_s}) \\
\leq\; & CC_K\, (\alpha\nu)^{(m-l)/m} \|b_{j_1,n_1}\|_{L^1(d\mu_k)} \int_{(\mathbb{R}^d)^{l-1}} \prod_{j=2}^{l} |b_{j_s,n_s}(y_{j_s})|\, d\mu_k(y_{j_s}) \\
& \times \prod_{s=1}^{l} \Big[\frac{r_{j_1,n_1}}{d_G(x, c_{j_s,n_s})} \Big]^{\epsilon/l} \frac{1}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)} \\
\leq\; & CC_K\, (\alpha\nu)^{(m-l)/m} \prod_{s=1}^{l} \Big[\frac{r_{j_s,n_s}}{d_G(x, c_{j_s,n_s})} \Big]^{\epsilon/l} \frac{\|b_{j_s,n_s}\|_{L^1(d\mu_k)}}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)} \\
\leq\; & CC_K\, \alpha\nu \prod_{s=1}^{l} \Big[\frac{r_{j_s,n_s}}{d_G(x, c_{j_s,n_s})} \Big]^{\epsilon/l} \frac{\mu_k(B_{j_s,n_s})}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)}.
\end{aligned}$$

Thus for any $x \notin \bigcup_{j=1}^{m} \bigcup_{n\in I_j} \mathcal{O}\big((B_{j,n})^{**}\big)$, substituting the above inequality in (3.2.5), we have

$$\begin{aligned}
&|T(h_1, h_2, \cdots, h_m)(x)| \\
\leq\ & CC_K\, \alpha\nu \sum_{n_1\in I_{j_1}} \cdots \sum_{n_l \in I_{j_l}} \prod_{s=1}^{l} \Big[\frac{r_{j_s,n_s}}{d_G(x, c_{j_s,n_s})}\Big]^{\epsilon/l} \frac{\mu_k(B_{j_s,n_s})}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)} \\
\leq\ & CC_K\, \alpha\nu \prod_{s=1}^{l} \Bigg[\sum_{n_s\in I_{j_s}} \Big[\frac{r_{j_s,n_s}}{d_G(x, c_{j_s,n_s})}\Big]^{\epsilon/l} \frac{\mu_k(B_{j_s,n_s})}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)}\Bigg]. \qquad (3.2.8)
\end{aligned}$$

Now using the facts that $d_G(x, c_{j_s,n_s}) \sim d_G(x, c_{j_s,n_s}) + r_{j_s,n_s}$, $\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big) \sim \mu_k\big(B(x, d_G(x, c_{j_s,n_s}) + r_{j_s,n_s})\big)$ and the condition (2.1.2), we get

$$\begin{aligned}
&\Big[\frac{r_{j_s,n_s}}{d_G(x, c_{j_s,n_s})}\Big]^{\epsilon/l} \frac{\mu_k(B_{j_s,n_s})}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}))\big)} \\
\leq\ & C \Big[\frac{\mu_k\big(B_{j_s,n_s}\big)}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}) + r_{j_s,n_s})\big)}\Big]^{\epsilon/(l\, d_k)} \frac{\mu_k(B_{j_s,n_s})}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}) + r_{j_s,n_s})\big)} \\
\leq\ & C \Bigg[\frac{1}{\mu_k\big(B(x, d_G(x, c_{j_s,n_s}) + r_{j_s,n_s})\big)} \int_{\mathbb{R}^d} \chi_{B_{j_s,n_s}}(y)\, d\mu_k(y)\Bigg]^{1+\epsilon/(l\, d_k)} \\
\leq\ & C \Bigg[\frac{1}{\mu_k\big(B(x,\ d_G(x, c_{j_s,n_s}) + r_{j_s,n_s})\big)} \\
& \times \int_{\mathcal{O}\left(B(x, d_G(x, c_{j_s,n_s}) + r_{j_s,n_s})\right)} \chi_{B_{j_s,n_s}}(y)\, d\mu_k(y)\Bigg]^{1+\epsilon/(l\, d_k)} \\
\leq\ & C \Bigg[\sum_{\sigma\in G} \frac{1}{\mu_k\big(B(\sigma(x), d_G(x,\ c_{j_s,n_s}) + r_{j_s,n_s})\big)} \\
& \times \int_{B(\sigma(x), d_G(x, c_{j_s,n_s}) + r_{j_s,n_s})} \chi_{B_{j_s,n_s}}(y)\, d\mu_k(y)\Bigg]^{1+\epsilon/(l\, d_k)} \\
\leq\ & C \Big[\sum_{\sigma\in G} M_{HL}^{k}\big(\chi_{B_{j_s,n_s}}\big)(\sigma(x))\Big]^{1+\epsilon/(l\, d_k)}. \qquad (3.2.9)
\end{aligned}$$

Finally, using Chebyshev's inequality, (3.2.8), (3.2.9), Hölder's inequality, $L^{1+\epsilon/(l\, d_k)}$ boundedness of M_{HL}^{k} and properties of Calderón-Zygmund decomposition, we obtain

$$\mu_k\big(\{x \notin \bigcup_{j=1}^{m} \bigcup_{n\in I_j} \mathcal{O}\big((B_{j,n})^{**}\big) : |T(h_1, h_2, \cdots, h_m)(x)| > \alpha/2^m\}\big)$$

$$
\begin{aligned}
&\leq C\,\alpha^{-1/l} \int_{\mathbb{R}^d \setminus \bigcup_{j=1}^{m} \bigcup_{n\in I_j} \mathcal{O}\left((B_{j,n})^{**}\right)} |T(h_1,h_2,\cdots,h_m)(x)|^{1/l}\, d\mu_k(x) \\
&\leq C\,(C_K\nu)^{1/l} \int_{\mathbb{R}^d} \left[\prod_{s=1}^{l} \sum_{n_s\in I_{j_s}} \left[\sum_{\sigma\in G} M_{HL}^k\big(\chi_{B_{j_s,n_s}}\big)(\sigma(x)) \right]^{1+\epsilon/(l\,d_k)} \right]^{1/l} d\mu_k(x) \\
&\leq C\,(C_K\nu)^{1/l} \prod_{s=1}^{l} \left[\int_{\mathbb{R}^d} \sum_{n_s\in I_{j_s}} \left[\sum_{\sigma\in G} M_{HL}^k\big(\chi_{B_{j_s,n_s}}\big)(\sigma(x)) \right]^{1+\epsilon/(l\,d_k)} d\mu_k(x) \right]^{1/l} \\
&\leq C\,(C_K\nu)^{1/l} \prod_{s=1}^{l} \left[\sum_{n_s\in I_{j_s}} \sum_{\sigma\in G} \int_{\mathbb{R}^d} \left[M_{HL}^k\big(\chi_{B_{j_s,n_s}}\big)(\sigma(x)) \right]^{1+\epsilon/(l\,d_k)} d\mu_k(x) \right]^{1/l} \\
&= C\,(C_K\nu)^{1/l} \prod_{s=1}^{l} \left[\sum_{n_s\in I_{j_s}} \sum_{\sigma\in G} \int_{\mathbb{R}^d} \left[M_{HL}^k\big(\chi_{B_{j_s,n_s}}\big)(x) \right]^{1+\epsilon/(l\,d_k)} d\mu_k(x) \right]^{1/l} \\
&\leq C\,(C_K\nu)^{1/l} \prod_{s=1}^{l} \Big[\sum_{n_s\in I_{j_s}} |G|\, \mu_k\big(B_{j_s,n_s}\big) \Big]^{1/l} \leq C\,(C_K\nu)^{1/l} (\alpha\nu)^{-1/m}.
\end{aligned}
$$

This completes the proof of the inequality (3.2.4). Finally choosing $\nu = 1/(C_K + A)$, from (3.2.2) and (3.2.3), we see that (3.2.1) holds and hence the proof is concluded. □

3.3 Weighted Inequalities for Multilinear Dunkl-Calderón-Zygmund Operators

In this section, we discuss our main results, namely Theorem 3.3.3 and Theorem 3.3.4, regrading weighted estimates for multilinear Dunkl-Calderón-Zygmund Operators. We will now prove two propositions that will be very useful in the proofs of these results.

Proposition 3.3.1. *Let* $1 \leq p_1, p_2, \cdots, p_m < \infty$ *and* $\nu \in (0, 1/m)$. *Then there is a constant* $C > 0$ *depending only on* ν, m, ϵ *and* p_j*'s such that for all* $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, d\mu_k) \times L^{p_2}(\mathbb{R}^d, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, d\mu_k)$,

$$
M_{HL,\nu}^{k,\#}(\mathcal{T}\overrightarrow{f})(x) \leq C\,(C_K + A) \sum_{\substack{(n_1,n_2,\cdots,n_m) \\ \sigma_{n_s}\in G}} \mathcal{M}_{HL}^k\left(f_1\circ\sigma_{n_1}, f_2\circ\sigma_{n_2}, \cdots, f_m\circ\sigma_{n_m}\right)(x).
$$

Proof. Fix a ball B such that $x \in B$. From the definition given in Section 2.3, it suffices to prove that there is a $\mathfrak{c}_B \in \mathbb{C}$ depending only on B such that

$$\begin{aligned} &\Big[\frac{1}{\mu_k(B)}\int_B \big|\mathcal{T}\overrightarrow{f}(z)-\mathfrak{c}_B\big|^\nu\, d\mu_k(z)\Big]^{1/\nu} \\ \leq\ & C\,(C_K+A)\sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \mathcal{M}^k_{HL}\left(f_1\circ\sigma_{n_1}, f_2\circ\sigma_{n_2},\cdots,f_m\circ\sigma_{n_m}\right)(x). \end{aligned} \tag{3.3.1}$$

Let c_B denotes the centre and $r(B)$ denote the radius of the ball B and set $B^{**}=\big(B(c_B,5r(B))\big)$. Also define $f_j^0=f_j\chi_{\mathcal{O}(B^{**})}$ and $f_j^\infty=f_j-f_j^0$. Then

$$\begin{aligned} \prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m \left[f_j^0(y_j)+f_j^\infty(y_j)\right] \\ &= \sum_{\alpha_1,\alpha_2,\cdots,\alpha_m\in\{0,\infty\}} f_1^{\alpha_1}(y_1)f_2^{\alpha_2}(y_2)\cdots f_m^{\alpha_m}(y_m) \\ &= \prod_{j=1}^m f_j^0(y_j)+\sum_{\text{at least one }\alpha_n\neq 0} f_1^{\alpha_1}(y_1)f_2^{\alpha_2}(y_2)\cdots f_m^{\alpha_m}(y_m) \end{aligned}$$

Denote $(f_1^0,f_2^0,\cdots,f_m^0)$ by $\overrightarrow{f^0}$, then we have

$$\mathcal{T}\overrightarrow{f}(z)=\mathcal{T}\overrightarrow{f^0}(z)+\sum_{\text{at least one }\alpha_n\neq 0}\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(z). \tag{3.3.2}$$

Set $N=\prod_{j=1}^m \frac{1}{\mu_k(B)}\|f_j^0\|_{L^1(d\mu_k)}$ and $\mathfrak{c}_B=\sum_{\text{at lest one }\alpha_n\neq 0}\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(x)$.

Then from (3.3.2) we can write

$$\begin{aligned} &\Big[\frac{1}{\mu_k(B)}\int_B \big|\mathcal{T}\overrightarrow{f}(z)-\mathfrak{c}_B\big|^\nu\, d\mu_k(z)\Big]^{1/\nu} \\ \leq\ & C\Big[\frac{1}{\mu_k(B)}\int_B \big|\mathcal{T}\overrightarrow{f^0}(z)\big|^\nu\, d\mu_k(z)\Big]^{1/\nu}+C\Big[\frac{1}{\mu_k(B)}\int_B \sum_{\text{at least one }\alpha_n\neq 0} \\ &\times\big|\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(z)-\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(x)\big|^\nu\, d\mu_k(z)\Big]^{1/\nu} \end{aligned} \tag{3.3.3}$$

Now we consider the first term. It follows from Theorem 3.2.2 that

$$\Big[\frac{1}{\mu_k(B)}\int_B \big|\mathcal{T}\overrightarrow{f^0}(z)\big|^\nu\, d\mu_k(z)\Big]^{1/\nu}$$

$$
\begin{aligned}
&= \Big[\frac{1}{\mu_k(B)}\int_0^{N(C_K+A)} \nu\, t^{\nu-1}\mu_k\big(\{z\in B: |\mathcal{T}\overrightarrow{f^0}(z)|>t\}\big)\,dt \\
&\quad+\frac{1}{\mu_k(B)}\int_{N(C_K+A)}^{\infty} \nu\, t^{\nu-1}\mu_k\big(\{z\in B: |\mathcal{T}\overrightarrow{f^0}(z)|>t\}\big)\,dt\Big]^{1/\nu}\\
&\le C\Big[N^{\nu}(C_K+A)^{\nu}+(C_K+A)^{1/m}N^{1/m}\int_{N(C_K+A)}^{\infty}\nu\, t^{\nu-1-1/m}\,dt\Big]^{1/\nu}\\
&\le C(C_K+A)N\le C\,(C_K+A)\mathcal{M}_{HL}^{k}\overrightarrow{f}(x)\\
&\le C\,(C_K+A)\sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}}\mathcal{M}_{HL}^{k}\left(f_1\circ\sigma_{n_1},f_2\circ\sigma_{n_2},\cdots,f_m\circ\sigma_{n_m}\right)(x). \qquad (3.3.4)
\end{aligned}
$$

For the second term in (3.3.3), we take $\alpha_{j_1}=\alpha_{j_2}=\cdots=\alpha_{j_l}=0$, where for $0\le s\le l,\ j_s\in\{1,2,\cdots,m\}$ and $0\le l\le m$ with the convention that $\{j_1,j_2,\cdots,j_l\}=\emptyset$ if $l=0$. Then keeping in mind that $m-l\ge 1$, $x\in B$ and *supp* $f_j^{\infty}\subset\mathbb{R}^d\setminus\mathcal{O}(B^{**})$, for any $z\in B$ we can apply smoothness condition (3.1.2) to obtain

$$
\begin{aligned}
&\big|\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(z)-\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(x)\big|\\
\le\ & C_K\int_{(\mathbb{R}^d)^m}\big|K(z,y_1,y_2,\cdots,y_m)-K(x,y_1,y_2,\cdots,y_m)\big|\prod_{j=1}^{m}\big|f_j^{\alpha_j}(y_j)\big|\,d\mu_k(y_j)\\
\le\ & C_K\int_{(\mathbb{R}^d)^m}\Big[\sum_{n=1}^{m}\mu_k\big(B(z,d_G(z,y_n))\big)\Big]^{-m}\left[\frac{|z-x|}{\max\limits_{1\le n\le m}|z-y_n|}\right]^{\epsilon}\prod_{j=1}^{m}\big|f_j^{\alpha_j}(y_j)\big|\,d\mu_k(y_j)\\
=\ & C_K\int_{\left(\mathcal{O}(B^{**})\right)^{l}}\prod_{s=1}^{l}\big|f_{j_s}^{0}(y_{j_s})\big|\int_{\left(\mathbb{R}^d\setminus\mathcal{O}(B^{**})\right)^{m-l}}\Big[\sum_{n=1}^{m}\mu_k\big(B(z,d_G(z,y_n))\big)\Big]^{-m}\\
&\times\left[\frac{2r(B)}{\max\limits_{1\le n\le m}|z-y_n|}\right]^{\epsilon}\prod_{j\notin\{j_1,j_2,\cdots,j_l\}}|f_j(y_j)|\prod_{s=1}^{l}d\mu_k(y_{j_s})\prod_{j\notin\{j_1,j_2,\cdots,j_l\}}d\mu_k(y_j),\\
\le\ & CC_K\sum_{n=1}^{\infty}\int_{\left(\mathcal{O}(B^{**})\right)^{l}}\prod_{s=1}^{l}\big|f_{j_s}^{0}(y_{j_s})\big|\int_{\left(\mathcal{O}(3^nB^{**})\right)^{m-l}\setminus\left(\mathcal{O}(3^{n-1}B^{**})\right)^{m-l}}\Big[\sum_{s=1}^{m}\mu_k\big(B(z,d_G(z,y_s))\big)\Big]^{-m}\\
&\times\left[\frac{r(B)}{\max\limits_{1\le s\le m}d_G(z,y_s)}\right]^{\epsilon}\prod_{j\notin\{j_1,j_2,\cdots,j_l\}}|f_j(y_j)|\prod_{s=1}^{l}d\mu_k(y_{j_s})\prod_{j\notin\{j_1,j_2,\cdots,j_l\}}d\mu_k(y_j),
\end{aligned}
$$

where in the last step, we have used the fact that $(\mathbb{R}^d\backslash\mathcal{O}(B^{**}))^{m-l} \subseteq \big(\mathbb{R}^d\big)^{m-l}\backslash\big(\mathcal{O}(B^{**})\big)^{m-l}$.

Using the inequalities $\max\limits_{1\le s\le m} d_G(z,y_s) \ge C\,3^n r(B)$ and $\sum\limits_{s=1}^{m}\mu_k\big(B(z,d_G(z,y_s))\big) \ge C\,\mu_k(3^nB)$, from above we write

$$\begin{aligned}
&\big|\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(z)-\mathcal{T}\big(f_1^{\alpha_1},f_2^{\alpha_2},\cdots,f_m^{\alpha_m}\big)(x)\big|\\
\le\;& CC_K\sum_{n=1}^{\infty}3^{-n\epsilon}\int\limits_{\left(\mathcal{O}(B^{**})\right)^{l}}\prod_{s=1}^{l}\big|f_{j_s}^0(y_{j_s})\big|\,d\mu_k(y_{j_s})\\
&\times\int\limits_{\left(\mathcal{O}(3^nB^{**})\right)^{m-l}\backslash\left(\mathcal{O}(3^{n-1}B^{**})\right)^{m-l}}\big(\mu_k(3^nB)\big)^{-m}\prod_{j\notin\{j_1,j_2,\cdots,j_l\}}\big|f_j(y_j)\big|\,d\mu_k(y_j)\\
\le\;& CC_K\sum_{n=1}^{\infty}3^{-n\epsilon}\prod_{j=1}^{m}\frac{1}{\mu_k(3^nB)}\int\limits_{\mathcal{O}(3^nB^{**})}\big|f_j(y_j)\big|\,d\mu_k(y_j)\\
\le\;& CC_K\sum_{n=1}^{\infty}3^{-n\epsilon}\prod_{j=1}^{m}\Big[\sum_{\sigma\in G}\frac{1}{\mu_k(3^nB)}\int\limits_{3^nB^{**}}\big|f_j\circ\sigma(y_j)\big|\,d\mu_k(y_j)\Big]\\
=\;& CC_K\sum_{n=1}^{\infty}3^{-n\epsilon}\sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}}\Big[\prod_{j=1}^{m}\frac{1}{\mu_k(3^nB^{**})}\int\limits_{3^nB^{**}}\big|f_j\circ\sigma_{n_j}(y_j)\big|\,d\mu_k(y_j)\Big]\\
\le\;& C\,(C_K+A)\sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}}\mathcal{M}_{HL}^{k}\left(f_1\circ\sigma_{n_1},f_2\circ\sigma_{n_2},\cdots,f_m\circ\sigma_{n_m}\right)(x).
\end{aligned}$$

Now making use of (3.3.4) and the last inequality in (3.3.3), we conclude the proof of (3.3.1). □

Proposition 3.3.2. *Let w be a weight in the class A_∞^k and $p\in[1/m,\infty)$. Then there exists a constant $C>0$ such that if $f_1,f_2,\cdots,f_m$ are bounded functions with compact support*

(i) if $p>1/m$, then

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^d}\big|\mathcal{T}\vec{f}(x)\big|^p w(x)\,d\mu_k(x)\Big)^{1/p}\\
\le\;& C(C_K+A)\sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}}\Big(\int_{\mathbb{R}^d}\big(\mathcal{M}_{HL}^{k}\vec{f_\sigma}(x)\big)^p w(x)\,d\mu_k(x)\Big)^{1/p};
\end{aligned}$$

(ii) if $p \geq 1/m$, then

$$\begin{aligned}
&\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\, |\mathcal{T}\overrightarrow{f}(y)|>t\}} w(x)\, d\mu_k(x)\Big)^{1/p} \\
\leq\ & C(C_K+A) \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\, \mathcal{M}^k_{HL}\overrightarrow{f_\sigma}(y)>t\}} w(x)\, d\mu_k(x)\Big)^{1/p},
\end{aligned}$$

where we have used the notation $\overrightarrow{f_\sigma} = (f_1\circ\sigma_{n_1}, \cdots, f_m\circ\sigma_{n_m})$.

Proof. We only prove (i) as (ii) follows by similar arguments. For any $N \in \mathbb{N}$, define $w_N(x) = \min\{w(x), N\}$. Then the weight $w_N \in A^k_\infty$ (cf. [18, p. 215]) and the constant $[w_N]_{A^k_\infty}$ in the A^k_∞ condition, does not depend on N.

Also by Fatou's lemma

$$\int_{\mathbb{R}^d} |\mathcal{T}\overrightarrow{f}(x)|^p\, w(x)\, d\mu_k(x) \leq \lim_{N\to\infty} \int_{\mathbb{R}^d} |\mathcal{T}\overrightarrow{f}(x)|^p\, w_N(x)\, d\mu_k(x).$$

Since $f_1, f_2, \cdots, f_m$ are bounded functions with compact support, by our hypothesis $\mathcal{T}\overrightarrow{f} \in L^{q,\infty}(\mathbb{R}^d, d\mu_k)$ which implies for any $\nu \in (0, 1/m)$, $|\mathcal{T}\overrightarrow{f}|^\nu$ is locally integrable on $\mathbb{R}^d$ (by similar method as in (3.3.4)). Next we claim that

$$\sup_{t>0} t\Big(\int_{\big\{y\in\mathbb{R}^d:\, \big(M^k_{HL}(|\mathcal{T}\overrightarrow{f}|^\nu)(y)\big)^{1/\nu}>t\big\}} w_N(x)\, d\mu_k(x)\Big)^{1/p_0} < \infty, \tag{3.3.5}$$

for some $0 < p_0 < p$. Taking $p_0 = 1/m$ and using the fact that M^k_{HL} is bounded on $L^{r,\infty}(\mathbb{R}^d, d\mu_k)$ for $1 \leq r < \infty$ (cf. [33, p. 103]), we have

$$\begin{aligned}
&\sup_{t>0} t\Big(\int_{\big\{y\in\mathbb{R}^d:\, \big(M^k_{HL}(|\mathcal{T}\overrightarrow{f}|^\nu)(y)\big)^{1/\nu}>t\big\}} w_N(x)\, d\mu_k(x)\Big)^{m} \\
\leq\ & \|w_N\|^m_{L^\infty} \|M^k_{HL}(|\mathcal{T}\overrightarrow{f}|^\nu)\|^{1/\nu}_{L^{1/m\nu,\infty}(d\mu_k)} \\
\leq\ & \|w_N\|^m_{L^\infty} \|M^k_{HL}\|^{1/\nu}_{L^{1/m\nu,\infty}(d\mu_k)\to L^{1/m\nu,\infty}(d\mu_k)} \||\mathcal{T}\overrightarrow{f}|^\nu\|^{1/\nu}_{L^{1/m\nu,\infty}(d\mu_k)} \\
=\ & \|w_N\|^m_{L^\infty} \|M^k_{HL}\|^{1/\nu}_{L^{1/m\nu,\infty}(d\mu_k)\to L^{1/m\nu,\infty}(d\mu_k)} \|\mathcal{T}\overrightarrow{f}\|_{L^{1/m,\infty}(d\mu_k)} < \infty.
\end{aligned}$$

This completes the proof of the claim (3.3.5).

Therefore, by arguing as [34, Lemma 4.11 (i)] for the space of homogeneous type $(\mathbb{R}^d, |x-y|, d\mu_k)$ and using Proposition 3.3.1, we finally write

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^d} |\mathcal{T}\overrightarrow{f}(x)|^p w_N(x)\, d\mu_k(x)\Big)^{1/p} \\
\leq\ &\Big(\int_{\mathbb{R}^d} (M_{HL}^k(|\mathcal{T}\overrightarrow{f}|^\nu)(x))^{p/\nu} w_N(x)\, d\mu_k(x)\Big)^{1/p} \\
\leq\ & C_{[w_N]_{A_\infty^k}} \Big(\int_{\mathbb{R}^d} (M_{HL,\nu}^{k,\#}(\mathcal{T}\overrightarrow{f})(x))^p w_N(x)\, d\mu_k(x)\Big)^{1/p} \\
\leq\ & C(C_K+A) \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \Big(\int_{\mathbb{R}^d} \big(\mathcal{M}_{HL}^k\left(f_1\circ\sigma_{n_1},\cdots,f_m\circ\sigma_{n_m}\right)(x)\big)^p w_N(x)\, d\mu_k(x)\Big)^{1/p} \\
\leq\ & C(C_K+A) \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \Big(\int_{\mathbb{R}^d} \big(\mathcal{M}_{HL}^k\left(f_1\circ\sigma_{n_1},\cdots,f_m\circ\sigma_{n_m}\right)(x)\big)^p w(x)\, d\mu_k(x)\Big)^{1/p}.
\end{aligned}$$

Recalling the fact that $[w_N]_{A_\infty^k}$ is independent of N, we get the required result by taking $N\to\infty$. □

Finally, our main results concerning multilinear Dunkl-Calderón-Zygmund operators consist of the following two-weight and one-weight inequalities:

Theorem 3.3.3. *Let $1\leq p_1,p_2,\cdots,p_m<\infty$, $\overrightarrow{P}=(p_1,p_2,\cdots,p_m)$, p be the number given by $1/p=1/p_1+1/p_2+\cdots+1/p_m$ and $v,w_1,w_2,\cdots,w_m$ be G-invariant weights with $v\in A_\infty^k$. Furthermore let $\mathcal{T}$ maps from $L^{q_1}(\mathbb{R}^d,\, d\mu_k)\times L^{q_2}(\mathbb{R}^d,\, d\mu_k)\times\cdots\times L^{q_m}(\mathbb{R}^d,\, d\mu_k)$ to $L^{q,\infty}(\mathbb{R}^d,\, d\mu_k)$ with norm A for some $q,q_1,q_2,\cdots,q_m$ satisfying $1\leq q_1,q_2,\cdots,q_m<\infty$ with $1/q=1/q_1+1/q_2+\cdots+1/q_m$. Then the following hold:*

(i) if $p_j=1$ for some $1\leq j\leq m$ and the vector weight $(v,\overrightarrow{w})\in A_{\overrightarrow{P}}^k$, then for all $\overrightarrow{f}\in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k)\times L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k)\times\cdots\times L^{p_m}(\mathbb{R}^d, w_m\, d\mu_k)$, the following boundedness holds:

$$\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\, |\mathcal{T}\overrightarrow{f}(y)|>t\}} v(x)\, d\mu_k(x)\Big)^{1/p}$$

$$\leq \quad C(C_K + A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x) d\mu_k(x)\Big)^{1/p_j};$$

(ii) if $p_j > 1$ for all $1 \leq j \leq m$ and the vector weight $(v, \overrightarrow{w})$ satisfies the bump condition (2.3.1) for some $t > 1$, then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m\, d\mu_k)$, the following boundedness holds:

$$\Big(\int_{\mathbb{R}^d} |\mathcal{T}\overrightarrow{f}(x)|^p v(x)\, d\mu_k(x)\Big)^{1/p} \leq C(C_K + A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\, d\mu_k(x)\Big)^{1/p_j}.$$

Proof of (i). Proof follows at once from Proposition 3.3.2 (ii), Theorem 2.3.9 (i) and the G-invariance of the weights. □

Proof of (ii). Similarly the proof follows form Proposition 3.3.2 (i), Theorem 2.3.9 (ii) and the G-invariance of the weights. □

Theorem 3.3.4. *Let $1 \leq p_1, p_2, \cdots, p_m < \infty$, $\overrightarrow{P} = (p_1, p_2, \cdots, p_m)$, p be the number given by $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$ and $w_1, w_2, \cdots, w_m$ be G-invariant weights and the vector weight $\overrightarrow{w} \in A^k_{\overrightarrow{P}}$. Furthermore let $\mathcal{T}$ maps from $L^{q_1}(\mathbb{R}^d,\, d\mu_k) \times L^{q_2}(\mathbb{R}^d,\, d\mu_k) \times \cdots \times L^{q_m}(\mathbb{R}^d,\, d\mu_k)$ to $L^{q,\infty}(\mathbb{R}^d,\, d\mu_k)$ with norm A for some $q, q_1, q_2, \cdots, q_m$ satisfying $1 \leq q_1, q_2, \cdots, q_m < \infty$ with $1/q = 1/q_1 + 1/q_2 + \cdots + 1/q_m$. Then the following hold:*

(i) if $p_j = 1$ for some $1 \leq j \leq m$, then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m\, d\mu_k)$, the following boundedness holds:

$$\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\, |\mathcal{T}\overrightarrow{f}(y)|>t\}} \prod_{j=1}^{m} w_j(x)^{p/p_j}\, d\mu_k(x)\Big)^{1/p}$$
$$\leq \quad C(C_K + A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x) d\mu_k(x)\Big)^{1/p_j};$$

(ii) if $p_j > 1$ for all $1 \leq j \leq m$, then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m\, d\mu_k)$, the following boundedness holds:

$$\Big(\int_{\mathbb{R}^d} |\mathcal{T}\overrightarrow{f}(x)|^p \prod_{j=1}^{m} w_j(x)^{p/p_j}\, d\mu_k(x)\Big)^{1/p}$$

$$\leq \quad C(C_K + A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\, d\mu_k(x)\Big)^{1/p_j}.$$

Proof of (i). Note that $\overrightarrow{w} \in A^k_{\overrightarrow{P}}$ implies that $\prod_{j=1}^{d} w_j^{p/p_j} \in A^k_\infty$ (see [34, Proposition 4.3]). Hence, in this case also proof follows from Proposition 3.3.2 (ii), Theorem 2.3.9 (i) with $v = \prod_{j=1}^{d} w_j^{p/p_j}$ and the G-invariance of the weights. □

Proof of (ii). The proof can be completed by using Proposition 3.3.2 (i), Theorem 2.3.10 and the G-invariance of the weights together with the property of the $A^k_{\overrightarrow{P}}$ weights as used in the last proof. □

Remark 3.3.5. In the classical setting, for $n = 1, 2, \cdots, d$, the m-linear n-th Riesz tranas-form is defined by

$$\mathcal{R}_n(\overrightarrow{f})(x) = \text{p.v.} \int_{\mathbb{R}^{md}} \frac{\sum_{j=1}^{m}(x_n - (y_j)_n)}{\big(\sum_{j=1}^{m} |x - y_j|^2\big)^{(md+1)/2}} f_1(y_1)\cdots f_m(y_m)\, dy_1 \cdots dy_m,$$

where $(y_j)_n$ denotes the n-th coordinate of y_j. These multilinear Riesz transforms are examples of classical multilinear Calderón–Zygmund operators. It is shown in [48] that if Theorem 1.1.2 holds for each of the operators $\mathcal{R}_n$, then $\overrightarrow{w} \in A_{\overrightarrow{P}}$. However, since there is not much information available about Dunkl translations, we cannot provide such necessary conditions for weighted boundedness results in this setup.

3.4 Maximal Multilinear Dunkl-Calderón-Zygmund Operators

In [70], the authors investigated boundedness results in the non-weighted scenario for maximal operators associated with linear Dunkl-Calderón–Zygmund singular integrals introduced in [69]. We aim to extend these findings to multilinear operators, addressing not only the multilinear case but also exploring weighted scenarios. Therefore, proving our theorems is much more complicated than in the linear case. This complexity arises from dealing with multilinear situations and considering weighted scenarios, adding extra layers of difficulty.

Following the classical case (see Section 1.1) as an analogy, we define maximal multilinear truncated operators $\mathcal{T}^*$ by

$$\mathcal{T}^*(\overrightarrow{f})(x) = \sup_{\delta>0} |\mathcal{T}_\delta(\overrightarrow{f})(x)|,$$

where we set, for any $\delta > 0$,

$$\mathcal{T}_\delta(\overrightarrow{f})(x) = \int\limits_{\sum_{j=1}^{m} d_G(x,y_j)\geq\delta} K(x, y_1, y_2, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j)\, d\mu_k(y_j)$$

and K is the kernel as in Definition 3.1.1.

If for all $j = 1, 2, \cdots, m$, $f_j \in L^{q_j}(\mathbb{R}^d, d\mu_k)$, then $\mathcal{T}_\delta(\overrightarrow{f})$ is well defined. In fact, the size condition (3.1.1) on the kernel K implies

$$|\mathcal{T}_\delta(\overrightarrow{f})(x)| \leq C_K \int\limits_{\sum_{j=1}^{m} d_G(x,y_j)\geq\delta} \frac{\prod_{j=1}^{m} |f_j(y_j)|\, d\mu_k(y_j)}{\left[\sum_{j=1}^{m} \mu_k\big(B(x, d_G(x, y_j))\big)\right]^m}. \tag{3.4.1}$$

Let $d_G(x, y_2) = \max_{2\leq j\leq m} d_G(x, y_j)$, then $\sum_{j=2}^{m} \mu_k\big(B(x, d_G(x, y_j))\big) \sim \mu_k\big(B(x, d_G(x, y_2)\big)$ and hence, using (2.1.5) together with this relation, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{d\mu_k(y_1)}{\left[\sum_{j=1}^{m} \mu_k\big(B(x, d_G(x, y_j))\big)\right]^{mq_1'}} \\
\leq& \int\limits_{d_G(x,y_1)<d_G(x,y_2)} \frac{d\mu_k(y_1)}{\left[\mu_k\big(B(x, d_G(x, y_2))\big)\right]^{mq_1'}} \\
&+ \sum_{l=1}^{\infty} \int\limits_{2^{l-1}d_G(x,y_2)\leq d_G(x,y_1)<2^l d_G(x,y_2)} \frac{d\mu_k(y_1)}{\left[\mu_k\big(B(x, d_G(x, y_1))\big)\right]^{mq_1'}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\frac{|G|\mu_k\big(B(x,d_G(x,y_2))\big)}{\big[\mu_k\big(B(x,d_G(x,y_2))\big)\big]^{mq_1'}}+C\sum_{l=1}^{\infty}\frac{|G|\mu_k\big(B(x,2^l d_G(x,y_2))\big)}{\big[\mu_k\big(B(x,2^l d_G(x,y_2))\big)\big]^{mq_1'}}\\
&\leq C\,\frac{1}{\big[\mu_k\big(B(x,d_G(x,y_2))\big)\big]^{mq_1'-1}}\\
&\leq C\,\frac{1}{\big[\sum_{j=2}^{m}\mu_k\big(B(x,d_G(x,y_j))\big)\big]^{mq_1'-1}}. \qquad (3.4.2)
\end{aligned}
$$

Now, without loss of generality we can take $d_G(x,y_m)\geq\delta/m$. Then applying Hölders inequality repeatedly and using (3.4.2), from (3.4.1) we get

$$
\begin{aligned}
|\mathcal{T}_\delta(\overrightarrow{f})(x)| &\leq C_K\,||f_1||_{L^{q_1}(d\mu_k)}\int\limits_{(\mathbb{R}^d)^{m-2}}\int\limits_{d_G(x,y_m)\geq\delta/m}\\
&\quad\times\Big(\int_{\mathbb{R}^d}\frac{d\mu_k(y_1)}{\big[\sum_{j=1}^{m}\mu_k\big(B(x,d_G(x,y_j))\big)\big]^{mq_1'}}\Big)^{1/q_1'}\prod_{j=2}^{m}|f_j(y_j)|\,d\mu_k(y_j)\\
&\leq C\,C_K\,||f_1||_{L^{q_1}(d\mu_k)}\int\limits_{(\mathbb{R}^d)^{m-2}}\int\limits_{d_G(x,y_m)\geq\delta/m}\frac{\prod_{j=2}^{m}|f_j(y_j)|\,d\mu_k(y_j)}{\big[\sum_{j=2}^{m}\mu_k\big(B(x,d_G(x,y_j))\big)\big]^{m-1/q_1'}}\\
&\vdots\\
&\leq C\,C_K\,||f_1||_{L^{q_1}(d\mu_k)}||f_2||_{L^{q_2}(d\mu_k)}\cdots||f_m||_{L^{q_m}(d\mu_k)}\\
&\quad\times\sum_{l=1}^{\infty}\frac{1}{\big[\mu_k\big(B(x,2^l\delta/m)\big)\big]^{m-(1/q_1'+\cdots+1/qm')}}<\infty.
\end{aligned}
$$

Our main results related to maximal multilinear Dunkl-Calderón-Zygmund singular integrals are two-weight and one-weight inequalities (Theorem 3.4.2 and Theorem 3.4.3 respectively). In the classical scenario, the Cotlar-type inequality plays a pivotal role in establishing these results. Here, we prove a variant of a multilinear Cotlar-type inequality (see Lemma 3.4.1) within this framework, which encompasses the action of the involved reflection group. To prove this inequality, we closely follow classical ideas from [25, 34, 36, 37]. However, as the integral representation provided in Definition 3.1.1 holds only when the support of f_j's is outside the orbit of x and due to the involvement of both metrics-'the

Dunkl metric' and the usual metric in the regularity conditions (eq. (3.1.1) and (3.1.2)) on the kernel, some new arguments are essentially required to successfully conclude the proof. Also, as a direct application of the Cotlar inequality, we obtain pointwise convergence of principal value integrals, as stated in Theorem 3.4.4, much like the classical case.

3.4.1 Multilinear Cotlar-type Inequality

To prove the boundedness results, we first prove an analogue of the multilinear Cotlar inequality involving action of the reflection group.

Lemma 3.4.1. *(Multilinear Cotlar-type inequality in Dunkl setting) For $0 < \nu < 1/m$, there exists a constant C depending on m, ϵ, ν such that for all $f_j \in L^{q_j}(\mathbb{R}^d, d\mu_k)$; $j = 1, 2, \cdots, m$; and for all $x \in \mathbb{R}^d$, we have*

$$\begin{aligned} |\mathcal{T}^* \overrightarrow{f}(x)| \ &\leq \ C\Big(\big[M^k_{HL}(|\mathcal{T}\overrightarrow{f}|^\nu)(x)\big]^{1/\nu} \\ &+(C_K + A) \sum_{\substack{(n_1,n_2,\cdots,n_m) \\ \sigma_{n_s}\in G}} \mathcal{M}^k_{HL}\left(f_1 \circ \sigma_{n_1}, f_2 \circ \sigma_{n_2}, \cdots, f_m \circ \sigma_{n_m}\right)(x)\Big), \end{aligned}$$

where M^k_{HL} and $\mathcal{M}^k_{HL}$ are respectively linear and m-linear Hardy-Littlewood maximal operators defined as

$$M^k_{HL} f(x) := \sup_{\substack{B\subset\mathbb{R}^d \\ x\in B}} \frac{1}{\mu_k(B)} \int_B |f(y)|\, d\mu_k(y)$$

$$\text{and } \mathcal{M}^k_{HL} \overrightarrow{f}(x) := \sup_{\substack{B\subset\mathbb{R}^d \\ x\in B}} \prod_{j=1}^m \frac{1}{\mu_k(B)} \int_B |f_j(y)|\, d\mu_k(y).$$

Proof. Let us consider $x \in \mathbb{R}^d$ and $\nu \in (0, 1/m)$. We define two sets

$$S_\delta = \Big\{(y_1, y_2, \cdots, y_m) \in \mathbb{R}^{md} : \sup_{1\leq j\leq m} d_G(x, y_j) < \delta\Big\}$$

$$\text{and } U_\delta = \Big\{(y_1, y_2, \cdots, y_m) \in S_\delta : \sum_{1\leq j\leq m} d_G(x, y_j) \geq \delta\Big\}$$

Take any $(y_1, y_2, \cdots, y_m) \in U_\delta$. Then there exists $y_{j_0} \in \mathbb{R}^d$ satisfying $d_G(x, y_{j_0}) \geq \delta/m$. So

$$\begin{aligned}
\mu_k(B(x,\delta)) &\leq \sum_{j=1}^{m} \mu_k\big(B(x, m\, d_G(x,y_j))\big) \\
&\leq C_m \sum_{j=1}^{m} \mu_k\big(B(x, d_G(x,y_j))\big).
\end{aligned}$$

Therefore, using the size condition (3.1.1) on K, we can compute

$$\begin{aligned}
&\sup_{\delta>0}\Big|\int_{U_\delta} K(x,y_1,\cdots,y_m)\, f_1(y_1)\cdots f_m(y_m)\, d\mu_k(y_1)\cdots d\mu_k(y_m)\Big| \\
\leq\ & \sup_{\delta>0}\int_{U_\delta} \frac{C_K\,|f_1(y_1)\cdots f_m(y_m)|}{\big[\sum_{j=1}^{m}\mu_k\big(B(x, m\, d_G(x,y_j))\big)\big]^m}\, d\mu_k(y_1)\cdots d\mu_k(y_m) \\
\leq\ & C\, C_K \sup_{\delta>0} \frac{1}{\big[\mu_k(B(x,\delta))\big]^m}\int_{U_\delta}\prod_{j=1}^{m}|f_j(y_j)|\, d\mu_k(y_j) \\
\leq\ & C\, C_K \sup_{\delta>0}\prod_{j=1}^{m}\frac{1}{\mu_k(B(x,\delta))}\int_{\mathcal{O}(B(x,\delta))}|f_j(y_j)|\, d\mu_k(y_j) \\
\leq\ & C\, C_K \sup_{\delta>0}\prod_{j=1}^{m}\Big[\sum_{\sigma\in G}\frac{1}{\mu_k\big(B(\sigma(x),\delta)\big)}\int_{B\left(\sigma(x),\delta\right)}|f_j(y_j)|\, d\mu_k(y_j)\Big] \\
\leq\ & C\, C_K \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \mathcal{M}_{HL}^k\,(f_1\circ\sigma_{n_1}, f_2\circ\sigma_{n_2},\cdots, f_m\circ\sigma_{n_m})\,(x). \qquad (3.4.3)
\end{aligned}$$

Now, let us consider

$$\widetilde{\mathcal{T}}_\delta \overrightarrow{f}(x) := \int_{S_\delta^c} K(x,y_1,\cdots,y_m)\, f_1(y_1)\cdots f_m(y_m)\, d\mu_k(y_1)\cdots d\mu_k(y_m)$$

$$\text{and } \widetilde{\mathcal{T}}^* \overrightarrow{f}(x) := \sup_{\delta>0}|\widetilde{\mathcal{T}}_\delta \overrightarrow{f}(x)|.$$

Also for $z \in \mathbb{R}^d$, define

$$G_\delta \overrightarrow{f}(x,z) = \int_{S_\delta^c} K(z,y_1,\cdots,y_m)\, f_1(y_1)\cdots f_m(y_m)\, d\mu_k(y_1)\cdots d\mu_k(y_m).$$

Then for $z \in B(x, \delta/2)$, we have

$$\begin{aligned}
& \widetilde{\mathcal{T}_\delta} \overrightarrow{f}(x) \\
= \ & \widetilde{\mathcal{T}_\delta} \overrightarrow{f}(x) - G_\delta \overrightarrow{f}(x,z) + \int_{S_\delta^c} K(z, y_1, \cdots, y_m)\, f_1(y_1) \cdots f_m(y_m)\, d\mu_k(y_1) \cdots d\mu_k(y_m) \\
= \ & \widetilde{\mathcal{T}_\delta} \overrightarrow{f}(x) - G_\delta \overrightarrow{f}(x,z) + \int_{\mathbb{R}^{md}} K(z, y_1, \cdots, y_m)\, f_1(y_1) \cdots f_m(y_m)\, d\mu_k(y_1) \cdots d\mu_k(y_m) \\
& - \int_{S_\delta} K(z, y_1, \cdots, y_m)\, f_1(y_1) \cdots f_m(y_m)\, d\mu_k(y_1) \cdots d\mu_k(y_m) \\
= \ & \widetilde{\mathcal{T}_\delta} \overrightarrow{f}(x) - G_\delta \overrightarrow{f}(x,z) + \mathcal{T} \overrightarrow{f}(z) \\
& - \int_{\mathbb{R}^{md}} K(z, y_1, \cdots, y_m)\, f_1 \chi_{\mathcal{O}(B(x,\delta))}(y_1) \cdots f_m \chi_{\mathcal{O}(B(x,\delta))}(y_m)\, d\mu_k(y_1) \cdots d\mu_k(y_m) \\
= \ & \widetilde{\mathcal{T}_\delta} \overrightarrow{f}(x) - G_\delta \overrightarrow{f}(x,z) + \mathcal{T} \overrightarrow{f}(z) - \mathcal{T}(\tilde{f}_1, \cdots, \tilde{f}_m)(z), \qquad (3.4.4)
\end{aligned}$$

where $\tilde{f}_j(y) = f_j(y)\chi_{\mathcal{O}(B(x,\delta))}(y)$ for all $y \in \mathbb{R}^d$ and $j = 1, 2, \cdots, m$.

Again from the smoothness condition (3.1.2) on K,

$$\begin{aligned}
|\widetilde{\mathcal{T}_\delta} \overrightarrow{f}(x) - G_\delta \overrightarrow{f}(x,z)| \ \leq \ & \int_{S_\delta^c} |K(x, y_1, \cdots, y_m) - K(z, y_1, \cdots, y_m)|\, |f_1(y_1) \cdots f_m(y_m)| \\
& \times d\mu_k(y_1) \cdots d\mu_k(y_m) \\
\leq \ & C_K \int_{S_\delta^c} \Big[\frac{|x-z|}{\max\limits_{1 \leq j \leq m} |x - y_j|} \Big]^\epsilon \frac{\prod\limits_{j=1}^m |f_j(y_j)|\, d\mu_k(y_j)}{\big[\sum\limits_{j=1}^m \mu_k\big(B(x, d_G(x, y_j))\big)\big]^m}.
\end{aligned}$$

Now we can express the above integral as a sum of integrals over $R_{j_1, j_2, \cdots, j_l} \subseteq \mathbb{R}^{md}$ for some $\{j_1, j_2, \cdots, j_l\} \subsetneq \{1, 2, \cdots, m\}$ so that, for $j = 1, 2, \cdots, m$; $d_G(x, y_j) < \delta$ if and only if $j \in \{j_1, j_2, \cdots, j_l\}$ for all $(y_1, y_2, \cdots, y_m) \in R_{j_1, j_2, \cdots, j_l}$, where $l < m$. Set

$$\{s_1, s_2, \cdots, s_l\} = \{1, 2, \cdots, m\} \setminus \{j_1, j_2, \cdots, j_l\}.$$

Then the last inequality can be rewritten as

$$\begin{aligned}
& |\widetilde{\mathcal{T}_\delta} \overrightarrow{f}(x) - G_\delta \overrightarrow{f}(x,z)| \\
\leq \ & C_K\, \delta^\epsilon \prod_{j \in \{j_1, j_2, \cdots, j_l\}} \int_{d_G(x, y_j) < \delta} |f_j(y_j)|\, d\mu_k(y_j)
\end{aligned}$$

$$\times \int\limits_{\left(\mathbb{R}^d-\mathcal{O}(B(x,\delta))\right)^{m-l}} \frac{1}{\left[\max\limits_{1\le j\le m} d_G(x,y_j)\right]^{\epsilon}} \frac{\prod\limits_{j=1}^{m-l} |f_{s_j}(y_{s_j})|\, d\mu_k(y_{s_j})}{\left[\sum\limits_{j=1}^{m} \mu_k\big(B(x,d_G(x,y_j))\big)\right]^m}$$

$$\le C\, C_K\, \delta^{\epsilon} \prod_{j\in\{j_1,j_2,\cdots,j_l\}} \int\limits_{d_G(x,y_j)<\delta} |f_j(y_j)|\, d\mu_k(y_j)$$

$$\times \int\limits_{\sum\limits_{j=1}^{m-l} d_G(x,y_{s_j})\ge\delta} \frac{1}{\left[\sum\limits_{j=1}^{m-l} d_G(x,y_{s_j})\right]^{\epsilon}} \frac{\prod\limits_{j=1}^{m-l} |f_{s_j}(y_{s_j})|\, d\mu_k(y_{s_j})}{\left[\sum\limits_{j=1}^{m-l} \mu_k\big(B(x,d_G(x,y_{s_j}))\big)\right]^m}$$

$$\le C\, C_K \prod_{j\in\{j_1,j_2,\cdots,j_l\}} \int\limits_{d_G(x,y_j)<\delta} |f_j(y_j)|\, d\mu_k(y_j)$$

$$\times \left[\sum_{r=0}^{\infty} \int\limits_{2^r\delta\le\sum\limits_{j=1}^{m-l} d_G(x,y_{s_j})<2^{r+1}\delta} \cdots\right]$$

$$\le C\, C_K \sum_{r=0}^{\infty} \frac{1}{2^{r\epsilon}} \frac{1}{\left[\mu_k\big(B(x,2^{r+1}\delta)\big)\right]^m} \prod_{j=1}^{m} \int\limits_{\mathcal{O}(B(x,2^{r+1}\delta))} |f_j(y_j)|\, d\mu_k(y_j)$$

$$\le C\, C_K \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \mathcal{M}^k_{HL}\left(f_1\circ\sigma_{n_1}, f_2\circ\sigma_{n_2},\cdots,f_m\circ\sigma_{n_m}\right)(x). \tag{3.4.5}$$

Substituting (3.4.5) in (3.4.4), we obtain

$$|\widetilde{\mathcal{T}_\delta}\overrightarrow{f}(x)| \le C\Big(C_K \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \mathcal{M}^k_{HL}\left(f_1\circ\sigma_{n_1}, f_2\circ\sigma_{n_2},\cdots,f_m\circ\sigma_{n_m}\right)(x) + |\mathcal{T}\overrightarrow{f}(z)| + |\mathcal{T}(\tilde{f}_1,\cdots,\tilde{f}_m)(z)|\Big). \tag{3.4.6}$$

Taking νth power and averaging over $B(x,\delta/2)$ with respect to the variable z, we get

$$|\widetilde{\mathcal{T}_\delta}\overrightarrow{f}(x)|^{\nu} \le C\bigg(\Big[C_K \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \mathcal{M}^k_{HL}\left(f_1\circ\sigma_{n_1}, f_2\circ\sigma_{n_2},\cdots,f_m\circ\sigma_{n_m}\right)(x)\Big]^{\nu} + M^k_{HL}(|\mathcal{T}\overrightarrow{f}|^{\nu})(x) + \frac{1}{\mu_k(B(x,\delta/2))} \int\limits_{B(x,\delta/2)} |\mathcal{T}(\tilde{f}_1,\cdots,\tilde{f}_m)(z)|^{\nu}\, d\mu_k(z)\bigg). \tag{3.4.7}$$

Now let $\alpha = \frac{1}{\mu_k(B(x,\delta))} A^{1/m} \prod_{j=1}^{m} \|f_j \chi_{\mathcal{O}(B(x,\delta))}\|_{L^1(d\mu_k)}^{1/m}$. Then

$$\begin{aligned}
&\int_{B(x,\delta/2)} |\mathcal{T}(\tilde{f}_1, \cdots, \tilde{f}_m)(z)|^{\nu} \, d\mu_k(z) \\
= \;& m\nu \int_0^\infty t^{m\nu-1} \mu_k\big(\{z \in B(x,\delta/2) : |\mathcal{T}(\tilde{f}_1, \cdots, \tilde{f}_m)(z)|^{1/m} > t\}\big) \, dt \\
\leq \;& m\nu \int_0^\infty t^{m\nu-1} \min\Big\{\mu_k(B(x,\delta/2)), \frac{A^{1/m}}{t} \prod_{j=1}^{m} \|f_j \chi_{\mathcal{O}(B(x,\delta))}\|_{L^1(d\mu_k)}^{1/m}\Big\} \, dt \\
= \;& m\nu \int_0^\alpha t^{m\nu-1} \mu_k\big(B(x,\delta/2)\big) \, dt + m\nu \int_\alpha^\infty t^{m\nu-2} \alpha \, \mu_k\big(B(x,\delta/2)\big) \, dt \\
\leq \;& C \, A^{\nu} \mu_k\big(B(x,\delta)\big)^{1-m\nu} \prod_{j=1}^{m} \|f_j \chi_{\mathcal{O}(B(x,\delta))}\|_{L^1(d\mu_k)}^{\nu}
\end{aligned}$$

Therefore, by (2.1.2),

$$\begin{aligned}
&\Big[\frac{1}{\mu_k(B(x,\delta/2))} \int_{B(x,\delta/2)} |\mathcal{T}(\tilde{f}_1, \cdots, \tilde{f}_m)(z)|^{\nu} \, d\mu_k(z)\Big]^{1/\nu} \\
\leq \;& C \, A \prod_{j=1}^{m} \frac{1}{\mu_k(B(x,\delta))} \int_{\mathcal{O}(B(x,\delta))} |f_j(y_j)| \, d\mu_k(y_j) \\
\leq \;& C \, A \sum_{\substack{(n_1,n_2,\cdots,n_m) \\ \sigma_{n_s} \in G}} \mathcal{M}_{HL}^{k} \left(f_1 \circ \sigma_{n_1}, f_2 \circ \sigma_{n_2}, \cdots, f_m \circ \sigma_{n_m}\right)(x). \qquad (3.4.8)
\end{aligned}$$

Hence, (3.4.3), (3.4.7) and (3.4.8) together conclude the proof. □

3.4.2 Weighted Boundedness

Now we are ready to state our main results regarding weighted boundedness for maximal multilinear Dunkl singular integrals.

Theorem 3.4.2. *Under the assumptions of Theorem 3.3.3, the following boundedness results hold:*

(i) if $p_j = 1$ for at least one $j \in \{1, 2, \cdots, m\}$ and the vector weight $(v, \overrightarrow{w}) \in A_{\overrightarrow{P}}^{k}$, then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1 \, d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2 \, d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m \, d\mu_k)$, the

following boundedness holds:

$$\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\,|\mathcal{T}^*\vec{f}(y)|>t\}} v(x)\,d\mu_k(x)\Big)^{1/p}$$
$$\leq C(C_K+A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d}|f_j(x)|^{p_j}w_j(x)d\mu_k(x)\Big)^{1/p_j};$$

(ii) *if for any $j\in\{1,2,\cdots,m\}$, $p_j>1$ and the weight $(v,\vec{w})$ fulfills the bumped-$A^k_{\vec{P}}$ property (2.3.1) for some $t>1$, then for all $\vec{f}\in L^{p_1}(\mathbb{R}^d,w_1\,d\mu_k)\times L^{p_2}(\mathbb{R}^d,w_2\,d\mu_k)\times\cdots\times L^{p_m}(\mathbb{R}^d,w_m\,d\mu_k)$, the following boundedness holds:*

$$\Big(\int_{\mathbb{R}^d}|\mathcal{T}^*\vec{f}(x)|^p v(x)\,d\mu_k(x)\Big)^{1/p}\leq C(C_K+A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d}|f_j(x)|^{p_j}w_j(x)\,d\mu_k(x)\Big)^{1/p_j}.$$

Theorem 3.4.3. *Under the assumptions of Theorem 3.3.4, the following boundedness results hold:*

(i) *if $p_j=1$ for at least one $j\in\{1,2,\cdots,m\}$, then for all $\vec{f}\in L^{p_1}(\mathbb{R}^d,w_1\,d\mu_k)\times L^{p_2}(\mathbb{R}^d,w_2\,d\mu_k)\times\cdots\times L^{p_m}(\mathbb{R}^d,w_m\,d\mu_k)$, the following boundedness holds:*

$$\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\,|\mathcal{T}^*\vec{f}(y)|>t\}} \prod_{j=1}^{m} w_j(x)^{p/p_j}\,d\mu_k(x)\Big)^{1/p}$$
$$\leq\ C(C_K+A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d}|f_j(x)|^{p_j}w_j(x)d\mu_k(x)\Big)^{1/p_j};$$

(ii) *if for any $j\in\{1,2,\cdots,m\}$, $p_j>1$, then for all $\vec{f}\in L^{p_1}(\mathbb{R}^d,w_1\,d\mu_k)\times L^{p_2}(\mathbb{R}^d,w_2\,d\mu_k)\times\cdots\times L^{p_m}(\mathbb{R}^d,w_m\,d\mu_k)$, the following boundedness holds:*

$$\Big(\int_{\mathbb{R}^d}|\mathcal{T}^*\vec{f}(x)|^p\prod_{j=1}^{m} w_j(x)^{p/p_j}\,d\mu_k(x)\Big)^{1/p}$$
$$\leq\ C(C_K+A)\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^d}|f_j(x)|^{p_j}w_j(x)\,d\mu_k(x)\Big)^{1/p_j}.$$

Once Cotlar type inequality is proved, the proofs of the weighted inequalities for $\mathcal{T}^*$ follow in the same way as in the classical case [37]. Hence, we omit the details.

Proof of Theorem 3.4.2 and Theorem 3.4.3. In view of the weighted boundedness of $\mathcal{T}$ (Theorem 3.3.3 and Theorem 3.3.4) and the weighted boundedness of Hardy-Littlewood maximal functions in spaces of homogeneous type [34], the proofs can be done in the same way as in the proofs of [34, Theorem 4.17 and Theorem 4.16]. Only change is that the term $\mathcal{M}_{HL}^{k}\overrightarrow{f}$ needs to be replaced by $\sum\limits_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \mathcal{M}_{HL}^{k}\left(f_1\circ\sigma_{n_1}, f_2\circ\sigma_{n_2},\cdots,f_m\circ\sigma_{n_m}\right)$ whose boundedness follows from the quasi-triangle inequalities, applying change of variables and the G-invariance of the weights. $\square$

Next, we state the pointwise convergence result for principal value integrals associated with the multilinear Dunkl-Calderón-Zygmund kernels, which is a direct consequence of Lemma 3.4.1.

Theorem 3.4.4. *For $f_j \in \mathcal{S}(\mathbb{R}^d)$, $j = 1, 2, \cdots, m$; if*

$$\widetilde{\mathcal{T}}(\overrightarrow{f})(x) = \lim_{\delta\to 0} \int\limits_{\sum_{j=1}^{m} d_G(x,y_j)\geq\delta} K(x, y_1, y_2, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j)\, d\mu_k(y_j),$$

where K is the kernel as in Definition 3.1.1. Then the above integral is convergent almost everywhere for all $f_j \in L^{q_j}(\mathbb{R}^d, d\mu_k)$.

Chapter 4

Bilinear Multiplier Operators for Dunkl Transform

The theme of this chapter is to explore bilinear multiplier operators associated with the Dunkl transform. In Section 4.1, we recall the theory for Fourier multiplier operators and introduce multiplier operators associated with the Dunkl transform. Next, in Section 4.2, we present new Littlewood-Paley type theorems for the Dunkl transform, which are essential ingredients for the analysis of multiplier operators. We establish a generalization of the Coifman-Meyer bilinear multiplier theorem in Section 4.3, extending it from the classical setting to the Dunkl setting. Finally, in Section 4.4, we prove weighted estimates for bilinear Dunkl multiplier operators, utilizing the results from Section 4.3 and the theory of singular integrals presented in Chapter 3. This chapter is built upon a part from the work [56].

4.1 Introduction

One of the well studied and trending topics in modern Harmonic analysis is the Fourier multipliers and its multilinear versions. For $f_1,\ f_2 \in \mathcal{S}(\mathbb{R}^d)$, the bilinear Fourier multiplier operators is defined as

$$\mathbf{T_m}(f_1, f_2)(x) = \int_{\mathbb{R}^{2d}} \mathbf{m}(\xi, \eta)\mathcal{F}f_1(\xi)\mathcal{F}f_2(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta,$$

where $\mathbf{m}$ is some reasonable function on $\mathbb{R}^{2d}$ and $\mathcal{F}$ is the classical Fourier transform on $\mathbb{R}^d$. The classical Coifman-Meyer [16] (bilinear) multiplier theorem from 1970's states that if $\mathbf{m}$ is a bounded function on $\mathbb{R}^{2d}$, which is smooth away from the origin and satisfies the decay condition:

$$|\partial_\xi^\alpha \partial_\eta^\beta \mathbf{m}(\xi,\eta)| \le C_{\alpha,\beta}\,(|\xi|+|\eta|)^{-(|\alpha|+|\beta|)} \tag{4.1.1}$$

for all multi-indices $\alpha, \beta \in (\mathbb{N}\cup\{0\})^d$, then the operator $\mathbf{T}_\mathbf{m}$ is bounded from $L^{p_1}(\mathbb{R}^d, dx)\times L^{p_2}(\mathbb{R}^d, dx)$ to $L^p(\mathbb{R}^d, dx)$ for $1 < p, p_1, p_2 < \infty$ with the relation $1/p = 1/p_1 + 1/p_2$. Later, significant improvements has been done to this result by improving the range of p [38, 45] and by reducing the smoothness condition on $\mathbf{m}$ [73]. In 2010's many authors were concerned with weighted inequalities for the bilinear multipliers. In this direction, weighted inequalities with classical A_p weights were proved by Fujita and Tomita [29] and Hu et. al [42] under Hörmander condition which is weaker than the condition (4.1.1). Also, Bui and Duong [8] and Li and Sun [49] presented similar results but with multiple weights introduced by Lerner et. al [48] in place of the classical weights.

In parallel with the classical scenario, for a bounded function $\mathbf{m}$ on $\mathbb{R}^d\times\mathbb{R}^d$ define the *bilinear Dunkl multiplier operator* $\mathcal{T}_\mathbf{m}$ as

$$\mathcal{T}_\mathbf{m}(f_1, f_2)(x) = \int_{\mathbb{R}^{2d}} \mathbf{m}(\xi,\eta)\mathcal{F}_k f_1(\xi)\mathcal{F}_k f_2(\eta)E_k(ix,\xi)E_k(ix,\eta)\,d\mu_k(\xi)d\mu_k(\eta)$$

for all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$.

For the linear Dunkl multiplier operators, there are analogous results [3, 26] to classical Fourier multiplier operators for the non-weighted cases. However, for bilinear multipliers there is a lack of proper analogue to the classical setting even for the non-weighted case. In fact, in Dunkl setting boundedness of bilinear multiplier operators are known only in two special cases. The first case is due to Wróbel [75], where he assumes that the multiplier $\mathbf{m}$ is radial in both the variables, that is there exists a function $\mathbf{m}_0$ on $(0,\infty)\times(0,\infty)$ such that $\mathbf{m}(\xi,\eta) = \mathbf{m}_0(|\xi|, |\eta|)$ and the second one is obtained by Amri et al. [3], where they restricted themselves to the one-dimensional case only. This motivates us to address the gap and acquire a suitable counterpart to the classical results for bilinear multipliers in the Dunkl setting.

Our first aim in this chapter is to prove a Coifman-Meyer type multiplier theorem in Dunkl setting (Theorem 4.3.1) without those extra assumptions mentioned above. The main obstacle that comes in obtaining such results is due to lack of appropriate Littlewood-Paley type theorems in Dunkl setting. Thanks to recent results [28] on pointwise estimates of multipliers, which allows us to overcome such difficulties and present a Littlewood-Paley type theory. Once such tools are available, the rest of our work lies in properly adapting some classical techniques ([57, pp. 67-71], [75, Theorem 4.1]) in Dunkl setup along with some new ideas.

Moving forward to the next step, we want to prove one and two-weight inequalities for the bilinear Dunkl-multiplier operators (Theorem 4.4.2 and Theorem 4.4.1) with multiple weights and also for the exponents beyond the Banach range $1 < p < \infty$. In our results, the smoothness condition on the multiplier $\mathbf{m}$ and weight classes are not the same to that of the corresponding results in the classical case (see [49, Theorem 1.2] and [8, Theorem 4.2]). To prove the boundedness results, the approaches used in the classical setting highly depend on the fact that the two transposes of the operator $\mathbf{T_m}$ are also bilinear multiplier operators with multipliers $\mathbf{m}(-\xi-\eta,\eta)$ and $\mathbf{m}(\xi,-\xi-\eta)$. In Dunkl case, no mechanism is known to find the multipliers of the so called transposes of the Dunkl bilinear multiplier operator. Moreover, our results are different than the classical case as we have also included the two-weight case and the end-point cases. Therefore, merely adapting classical techniques in a routine manner is insufficient to attain these results. In this context, we will adopt a different approach. To achieve these weighted inequalities, we will mainly rely on the estimates for Dunkl translations established in [28] and apply the theory of multilinear Calderón-Zygmund type singular integrals in the Dunkl setting.

These results concerning bilinear multipliers may have many potential applications, namely in establishing fractional Leibniz-type rules for the Dunkl Laplacian and various Kato-Ponce-type inequalities. For instance, in [75], a fractional Leibniz-type rules for the

Dunkl Laplacian has been proved for the group $\mathbb{Z}_2^d$ using bilinear multiplier theorem with radial multipliers.

4.2 Littlewood-Paley type Theorems

In this section, we prove two different Littlewood-Paley type theorems which are the main ingredients in the proof of Theorem 4.3.1. We start with the following theorem. A particular case [20, Theorem 5.2] of this theorem is known for the group $\mathbb{Z}_2^d$ with Muckenhoupt weights.

Theorem 4.2.1. *Let $u \in \mathbb{R}^d, 1 < p < \infty$. Let ψ be a smooth function on $\mathbb{R}^d$ such that $supp\, \psi \subset \{\xi \in \mathbb{R}^d : 1/r \leq |\xi| \leq r\}$ for some $r > 1$. For $j \in \mathbb{Z}$, define $\psi_j(\xi) = \psi(\xi/2^j)$ and for $f \in \mathcal{S}(\mathbb{R}^d)$ define*

$$\psi(u, D_k/2^j)f(x) = \int_{\mathbb{R}^d} \psi_j(\xi)\, e^{i\langle u,\xi\rangle/2^j} \mathcal{F}_k f(\xi) E_k(ix, \xi)\, d\mu_k(\xi).$$

Then

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |\psi(u, D_k/2^j)f|^2 \Big)^{1/2} \Big\|_{L^p(d\mu_k)} \leq C\,(1 + |u|)^n ||f||_{L^p(d\mu_k)},$$

where $n = \lfloor d_k \rfloor + 2$ and C is independent of u.

Proof. We will use the theory of Banach-valued singular integral operators [4, Theorem 3.1] to prove the above theorem. The L^2-case follows in similar way to the classical case [24, p. 160]. Using Plancherel formula for Dunkl transform, we get

$$\begin{aligned}
\Big\| \Big(\sum_{j\in\mathbb{Z}} |\psi(u, D_k/2^j)f|^2 \Big)^{1/2} \Big\|_{L^2(d\mu_k)}^2 &= \int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}} |\psi_j(x)\, e^{i\langle u,\xi\rangle/2^j}|^2 |\mathcal{F}_k f(x)|^2\, d\mu_k(x) \\
&\leq \int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}} |\psi_j(x)|^2 |\mathcal{F}_k f(x)|^2\, d\mu_k(x)
\end{aligned}$$

$$\leq \; C_r\,||f||^2_{L^2(d\mu_k)},$$

where in the last step we have used the fact that for any x only a fixed finite number of j's (depending on r) will contribute in the sum. This concludes the L^2-case.

Let $\Psi^u \in \mathcal{S}(\mathbb{R}^d)$ be such that $\mathcal{F}_k\Psi^u(\xi) = \psi(\xi)e^{i\langle u,\,\xi\rangle}$ and define $\Psi^u_j(\xi) = 2^{jd_k}\Psi^u(2^j\xi)$. Then $\mathcal{F}_k\Psi^u_j(\xi) = \psi_j(\xi)e^{i\langle u,\,\xi\rangle/2^j}$ and we also have that

$$\Big(\sum_{j\in\mathbb{Z}}|\psi(u, D_k/2^j)f(x)|^2\Big)^{1/2} = \Big(\sum_{j\in\mathbb{Z}}|f *_k \Psi^u_j(x)|^2\Big)^{1/2}.$$

Thus to apply the above mentioned Theorem, we only need to show that

$$\int\limits_{|y-y'|\leq d_G(x,y)/2} \|\tau^k_x\Psi^u_j(-y) - \tau^k_x\Psi^u_j(-y')\|_{\ell^2(\mathbb{Z})}\,d\mu_k(x) \leq C\,(1+|u|)^n$$

and $$\int\limits_{|y-y'|\leq d_G(x,y)/2} \|\tau^k_y\Psi^u_j(-x) - \tau^k_{y'}\Psi^u_j(-x)\|_{\ell^2(\mathbb{Z})}\,d\mu_k(x) \leq C\,(1+|u|)^n.$$

Again to prove the above two inequalities, it is enough to show that for $x, y, y' \in \mathbb{R}^d$ with $|y-y'| \leq d_G(x,y)/2$,

$$\|\tau^k_x\Psi^u_j(-y) - \tau^k_x\Psi^u_j(-y')\|_{\ell^2(\mathbb{Z})} \leq \frac{C\,(|1+|u|)^n}{V_G\,(x,y,d_G(x,y))}\,\frac{|y-y'|}{d_G(x,y)} \tag{4.2.1}$$

$$\|\tau^k_y\Psi^u_j(-x) - \tau^k_{y'}\Psi^u_j(-x)\|_{\ell^2(\mathbb{Z})} \leq \frac{C\,(|1+|u|)^n}{V_G\,(x,y,d_G(x,y))}\,\frac{|y-y'|}{d_G(x,y)}. \tag{4.2.2}$$

We will only proof (4.2.1), as proof of (4.2.2) follows from (4.2.1) by symmetry.

Now using the formula for Dunkl translation and the definition of Ψ^u_j, we have

$$\tau^k_x\Psi^u_j(-y) = 2^{jd_k}\int_{\mathbb{R}^d}\psi(\xi)\,e^{i\langle u,\,\xi\rangle}E_k(i\xi, 2^jx)E_k(-i\xi, 2^jy)\,d\mu_k(\xi). \tag{4.2.3}$$

Next, we calculate $\|\psi(\cdot)\,e^{i\langle u,\,\cdot\rangle}\|_{C^n(\mathbb{R}^d)}$. Using usual Leibniz rule for any multi-index α we have

$$\partial^\alpha\big(\psi(\xi)\,e^{i\langle u,\,\xi\rangle}\big) \;=\; \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta e^{i\langle u,\,\xi\rangle}\,\partial^{\alpha-\beta}\psi(\xi),$$

where the summation ranges over all multi-indices β such that $\beta_j \leq \alpha_j$ for all $1 \leq j \leq d$.

Hence,

$$\begin{aligned} \|\psi(\cdot)\, e^{i\langle u,\,\cdot\rangle}\|_{C^n(\mathbb{R}^d)} &= \sup_{\xi\in\mathbb{R}^d,\,|\alpha|\leq n} |\partial^\alpha \psi(\xi)\, e^{i\langle u,\,\xi\rangle}| \\ &\leq C\,(1+|u|)^n \sup_{|\alpha|\leq n} \|\partial^\alpha\psi\|_{L^\infty}. \end{aligned} \tag{4.2.4}$$

Next, we estimate $|\tau_x^k \Psi_j^u(-y) - \tau_x^k \Psi_j^u(-y')|$. We calculate the estimate in two parts:

<ins>If $|2^j y - 2^j y'| \leq 1$:</ins>

In view of (4.2.3), applying [28, eq.(4.31)] and using the inequalities (4.2.4) and (2.1.2), we have

$$\begin{aligned} &|\tau_x^k \Psi_j^u(-y) - \tau_x^k \Psi_j^u(-y')| \\ \leq\;& C\,(1+|u|)^n \frac{2^{jd_k}|2^j y - 2^j y'|}{(\mu_k(B(2^j x,1))\,\mu_k(B(2^j y,1)))^{1/2}} \frac{1}{1+2^j|x-y|} \frac{1}{(1+2^j d_G(x,y))^{n-1}} \\ \leq\;& C(1+|u|)^n \frac{|y-y'|}{|x-y|} \frac{1}{(\mu_k(B(x,2^{-j}))\,\mu_k(B(y,2^{-j})))^{1/2}} \\ &\times \frac{1}{(1+2^j d_G(x,y))^{n-1}}. \end{aligned} \tag{4.2.5}$$

Now, when $2^j d_G(x,y) \leq 1$, from (2.1.2) and (2.1.4), we get

$$\begin{aligned} \frac{1}{\mu_k(B(x,2^{-j}))} &\leq C\,\big(2^j d_G(x,y)\big)^d \frac{1}{\mu_k(B(x,d_G(x,y)))} \\ &\leq C\,\big(2^j d_G(x,y)\big)^d \frac{1}{V_G(x,y,d_G(x,y))} \\ &\leq C\,\frac{\big(2^j d_G(x,y)\big)^d + \big(2^j d_G(x,y)\big)^{d_k}}{V_G(x,y,d_G(x,y))}. \end{aligned}$$

Similarly, when $2^j d_G(x,y) > 1$, from (2.1.2) and (2.1.4), we get

$$\frac{1}{\mu_k(B(x,2^{-j}))} \leq C\,\frac{\big(2^j d_G(x,y)\big)^d + \big(2^j d_G(x,y)\big)^{d_k}}{V_G(x,y,d_G(x,y))}.$$

Thus, in any case

$$\frac{1}{\mu_k(B(x,2^{-j}))} \leq C\,\frac{\big(2^j d_G(x,y)\big)^d + \big(2^j d_G(x,y)\big)^{d_k}}{V_G(x,y,d_G(x,y))}.$$

In similar manner we can deduce

$$\frac{1}{\mu_k(B(y,2^{-j}))} \leq C\,\frac{\big(2^j d_G(x,y)\big)^d + \big(2^j d_G(x,y)\big)^{d_k}}{V_G(x,y,d_G(x,y))}.$$

Therefore, if $|2^j y - 2^j y'| \leq 1$, using above two estimates, from (4.2.5) we write

$$\begin{aligned}
&|\tau_x^k \Psi_j^u(-y) - \tau_x^k \Psi_j^u(-y')| \\
\leq\ & C(1+|u|)^n \frac{|y-y'|}{|x-y|}\frac{\big(2^j d_G(x,y)\big)^d + \big(2^j d_G(x,y)\big)^{d_k}}{V_G(x,y,d_G(x,y))} \\
&\times \frac{1}{(1+2^j d_G(x,y))^{n-1}}.
\end{aligned} \tag{4.2.6}$$

<u>If $|2^j y - 2^j y'| > 1$:</u>

Again, in view of (4.2.3), applying [28, eq.(4.30)] and using the inequalities (4.2.4), (2.1.2) and (2.1.4) in similar manner as in the last case, we get

$$\begin{aligned}
&|\tau_x^k \Psi_j^u(-y) - \tau_x^k \Psi_j^u(-y')| \\
\leq\ & C\,(1+|u|)^n \Bigg[\frac{2^{jd_k}}{\big(\mu_k(B(2^j x,1))\,\mu_k(B(2^j y,1))\big)^{1/2}}\frac{1}{1+2^j|x-y|}\frac{1}{(1+2^j d_G(x,y))^{n-1}} \\
&+\frac{2^{jd_k}}{\big(\mu_k(B(2^j x,1))\,\mu_k(B(2^j y',1))\big)^{1/2}}\frac{1}{1+2^j|x-y'|}\frac{1}{(1+2^j d_G(x,y'))^{n-1}}\Bigg] \\
\leq\ & C\,(1+|u|)^n \Bigg[\frac{\big(2^j d_G(x,y)\big)^d + \big(2^j d_G(x,y)\big)^{d_k}}{V_G(x,y,d_G(x,y))}\frac{1}{1+2^j|x-y|}\frac{1}{(1+2^j d_G(x,y))^{n-1}} \\
&+\frac{\big(2^j d_G(x,y')\big)^d + \big(2^j d_G(x,y')\big)^{d_k}}{V_G(x,y',d_G(x,y'))}\frac{1}{1+2^j|x-y'|}\frac{1}{(1+2^j d_G(x,y'))^{n-1}}\Bigg]
\end{aligned} \tag{4.2.7}$$

It is easy to see that the condition $|y-y'| \leq d_G(x,y)/2$ implies that

$$d_G(x,y) \sim d_G(x,y'),\ |x-y| \sim |x-y'| \text{ and } V_G(x,y,d_G(x,y)) \sim V_G(x,y',d_G(x,y')).$$

So, if $|2^jy - 2^jy'| > 1$, applying the above estimates in (4.2.7), we write

$$\begin{aligned}
&|\tau_x^k\Psi_j^u(-y) - \tau_x^k\Psi_j^u(-y')| \\
\leq\ & C\,(1+|u|)^n \frac{\left(2^jd_G(x,y)\right)^d + \left(2^jd_G(x,y)\right)^{d_k}}{V_G(x,y,d_G(x,y))} \frac{1}{1+2^j|x-y|} \frac{1}{(1+2^jd_G(x,y))^{n-1}} \\
\leq\ & C\,(1+|u|)^n \frac{\left(2^jd_G(x,y)\right)^d + \left(2^jd_G(x,y)\right)^{d_k}}{V_G(x,y,d_G(x,y))} \frac{|2^jy-2^jy'|}{1+2^j|x-y|} \frac{1}{(1+2^jd_G(x,y))^{n-1}} \\
\leq\ & C\,(1+|u|)^n \frac{|y-y'|}{|x-y|} \frac{\left(2^jd_G(x,y)\right)^d + \left(2^jd_G(x,y)\right)^{d_k}}{V_G(x,y,d_G(x,y))} \frac{1}{(1+2^jd_G(x,y))^{n-1}}. \quad (4.2.8)
\end{aligned}$$

Now taking (4.2.6) and (4.2.8) into account and using $|x-y| \geq d_G(x,y)$ together with the condition $n = \lfloor d_k \rfloor + 2$, we have

$$\begin{aligned}
&\|\tau_x^k\Psi_j^u(-y) - \tau_x^k\Psi_j^u(-y')\|_{\ell^2(\mathbb{Z})} \\
\leq\ & \sum_{j\in\mathbb{Z}} |\tau_x^k\Psi_j^u(-y) - \tau_x^k\Psi_j^u(-y')| \\
\leq\ & \frac{C\,(1+|u|)^n}{V_G(x,y,d_G(x,y))} \frac{|y-y'|}{d_G(x,y)} \sum_{j\in\mathbb{Z}} \frac{\left(2^jd_G(x,y)\right)^d + \left(2^jd_G(x,y)\right)^{d_k}}{(1+2^jd_G(x,y))^{n-1}}. \\
\leq\ & \frac{C\,(1+|u|)^n}{V_G(x,y,d_G(x,y))} \frac{|y-y'|}{d_G(x,y)} \left(\sum_{j\in\mathbb{Z}:\, 2^jd_G(x,y)\leq 1} \cdots + \sum_{j\in\mathbb{Z}:\, 2^jd_G(x,y)>1} \cdots \right) \\
\leq\ & \frac{C\,(1+|u|)^n}{V_G(x,y,d_G(x,y))} \frac{|y-y'|}{d_G(x,y)} \left(\sum_{j\in\mathbb{Z}:\, 2^jd_G(x,y)\leq 1} \left(2^jd_G(x,y)\right)^d \right. \\
& \left. + \sum_{j\in\mathbb{Z}:\, 2^jd_G(x,y)>1} \frac{(2^jd_G(x,y))^{d_k}}{(2^jd_G(x,y))^{n-1}} \right) \\
\leq\ & \frac{C\,(1+|u|)^n}{V_G(x,y,d_G(x,y))} \frac{|y-y'|}{d_G(x,y)}.
\end{aligned}$$

This completes the proof of (4.2.1) and hence the proof of the theorem. □

Remark 4.2.2. In [4, Theorem 3.1] the explicit constant for the boundedness of Banach-valued singular integrals is not calculated. However, a close observation (see [4, Theorem 3.1] and [35, Theorem 1.1]) assures that the constant in our proof will vary as $(1+|u|)^n$.

Let $\mathbf{m}$ be a bounded function on $\mathbb{R}^d$. For any $t > 0$ and for any $f \in \mathcal{S}(\mathbb{R}^d)$, we define a Dunkl-multiplier operator $\mathbf{m}_t(D_k)$ as

$$\mathbf{m}_t(D_k)f(x) = \int_{\mathbb{R}^d} \mathbf{m}(t\xi)\mathcal{F}_k f(\xi)E_k(ix,\xi)\, d\mu_k(\xi).$$

Then, we have the following boundedness result.

Proposition 4.2.3. *Let* $\mathbf{m}$ *be a function on* $\mathbb{R}^d$ *such that*

$$|\mathbf{m}(x)| \leq C_{\mathbf{m}}/(1+|x|) \text{ and } |\nabla \mathbf{m}(x)| \leq C_{\mathbf{m}'}/(1+|x|) \text{ for all } x \in \mathbb{R}^d,$$

where ∇ *is the usual gradient on* $\mathbb{R}^d$*. Then*

$$\Big\| \sup_{t>0} |\mathbf{m}_t(D_k)f| \Big\|_{L^2(d\mu_k)} \leq C(C_{\mathbf{m}} + C_{\mathbf{m}'})||f||_{L^2(d\mu_k)}$$

Proof. The proof follows by repeating the proof in the classical case [63, pp. 397-398] with the classical objects replaced by their Dunkl-counterparts. □

The next main result of this section is a different variant of the Littlewood-Paley theorem, stated as follows.

Theorem 4.2.4. *Let* $u \in \mathbb{R}^d$, $1 < p < \infty$*. Let* ψ *be a compactly supported smooth function on* $\mathbb{R}^d$*. For* $j \in \mathbb{Z}$*, define* $\psi_j(\xi) = \psi(\xi/2^j)$ *and for* $f \in \mathcal{S}(\mathbb{R}^d)$ *define*

$$\psi(u, D_k/2^j)f(x) = \int_{\mathbb{R}^d} \psi_j(\xi)\, e^{i\langle u,\,\xi\rangle/2^j}\, \mathcal{F}_k f(\xi)E_k(ix,\xi)\, d\mu_k(\xi).$$

Then

$$\Big\| \sup_{j\in\mathbb{Z}} |\psi(u, D_k/2^j)f| \Big\|_{L^p(d\mu_k)} \leq C\,(1+|u|)^n ||f||_{L^p(d\mu_k)},$$

where $n = \lfloor d_k \rfloor + 2$ *and* C *is independent of* u*.*

Proof. The proof follows in the same scheme as the proof of Theorem 4.2.1. We will only provide a outline of the proof.

Since ψ is a smooth function with compact support, $\partial_j \psi$ is also so and $\partial_j e^{i\langle u,\xi\rangle} = iu_j e^{i\langle u,\xi\rangle}$ for any $1 \leq j \leq d$. Therefore, the following estimates hold for all $\xi \in \mathbb{R}^d$:

$$\left|\psi(\xi)e^{i\langle u,\xi\rangle}\right| \leq \frac{C}{1+|\xi|} \text{ and } \left|\nabla\left(\psi(\xi)e^{i\langle u,\xi\rangle}\right)\right| \leq \frac{C(1+|u|)}{1+|\xi|},$$

where C does not depend on u.

Hence by Proposition 4.2.3,

$$\left\|\sup_{j\in\mathbb{Z}} |\psi(u, D_k/2^j)f|\right\|_{L^2(d\mu_k)} \leq C(1+|u|)||f||_{L^2(d\mu_k)}.$$

Let Ψ^u be as in the proof of Theorem 4.2.1. Thus, to complete the proof, we only need to prove that for $x, y, y' \in \mathbb{R}^d$ with $|y-y'| \leq d_G(x,y)/2$,

$$\sup_{j\in\mathbb{Z}} |\tau_x^k \Psi_j^u(-y) - \tau_x^k \Psi_j^u(-y')| \leq \frac{C\,(1+|u|)^n}{V_G\,(x, y, d_G(x,y))}\,\frac{|y-y'|}{d_G(x,y)}, \tag{4.2.9}$$

which follows by repeating the arguments used in the proof of Theorem 4.2.1. □

4.3 Coifman-Meyer type Bilinear Multiplier Theorem

The first main result of this chapter is Coifman-Meyer [16] type multiplier theorem in Dunkl setting. This theorem extends the boundedness of Fourier multiplier operators to include multiplier operators associated with the Dunkl transform.

Theorem 4.3.1. *Let $1 < p, p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $L \in \mathbb{N}$ be such that $L > 2d + 2\lfloor d_k \rfloor + 4$. If $\mathbf{m} \in C^L\left(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}\right)$ be a function satisfying*

$$|\partial_\xi^\alpha \partial_\eta^\beta \mathbf{m}(\xi,\eta)| \leq C_{\alpha,\beta}\left(|\xi|+|\eta|\right)^{-(|\alpha|+|\beta|)}$$

for all multi-indices $\alpha, \beta \in (\mathbb{N}\cup\{0\})^d$ such that $|\alpha|+|\beta| \leq L$ and for all $(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}$; then for all $f_1 \in L^{p_1}(\mathbb{R}^d, d\mu_k)$ and $f_2 \in L^{p_2}(\mathbb{R}^d, d\mu_k)$, the following boundedness holds:

$$||\mathcal{T}_{\mathbf{m}}(f_1, f_2)||_{L^p(d\mu_k)} \leq C\,||f_1||_{L^{p_1}(d\mu_k)}\,||f_2||_{L^{p_2}(d\mu_k)}.$$

Proof. Let $\psi \in C^\infty(\mathbb{R}^d)$ be such that $supp\,\psi \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1 \text{ for all } \xi \neq 0,$$

where $\psi_j(\xi) = \psi(\xi/2^j)$ for all $\xi \in \mathbb{R}^d$. Then

$$\begin{aligned}
\mathcal{T}_{\mathbf{m}}(f_1, f_2)(x) &= \int_{\mathbb{R}^{2d}} \sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} \psi_{j_1}(\xi)\psi_{j_2}(\eta)\mathbf{m}(\xi,\eta)\mathcal{F}_k f_1(\xi)\mathcal{F}_k f_2(\eta)E_k(ix,\xi) \\
&\quad \times E_k(ix,\eta)\, d\mu_k(\xi)d\mu_k(\eta) \\
&= \int_{\mathbb{R}^{2d}} \sum_{|j_1-j_2|\leq 4} \cdots + \int_{\mathbb{R}^{2d}} \sum_{j_1>j_2+4} \cdots + \int_{\mathbb{R}^{2d}} \sum_{j_2>j_1+4} \cdots \\
&=: \mathcal{T}_1(f_1,f_2)(x) + \mathcal{T}_2(f_1,f_2)(x) + \mathcal{T}_3(f_1,f_2)(x)
\end{aligned}$$

We will calculate the estimates for $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ separately.

<u>Estimate of $\mathcal{T}_1$:</u>

$$\begin{aligned}
\text{For any } j \in \mathbb{Z} \text{ define } \mathbf{m}_j(\xi,\eta) &= \psi_j(\xi) \sum_{j_2:|j-j_2|\leq 4} \psi_{j_2}(\eta)\mathbf{m}(\xi,\eta) \\
&=: \psi_j(\xi)\phi_j(\eta)\mathbf{m}(\xi,\eta),
\end{aligned}$$

where $\phi(\eta) = \sum_{|j|\leq 4} \psi_j(\eta)$ and $\phi_j(\eta) = \phi(\eta/2^j)$.

Then $supp\,\phi \subset \{\xi \in \mathbb{R}^d : 2^{-5} \leq |\xi| \leq 2^5\}$.

Let $\tilde{\psi} \in C^\infty(\mathbb{R}^d)$ be another function such that $0 \leq \tilde{\psi} \leq 1$ and

$$\tilde{\psi}(\xi) = \begin{cases} 0 & \text{if } |\xi| \notin [2^{-6}, 2^6], \\ 1 & \text{if } |\xi| \in [2^{-5}, 2^5]. \end{cases}$$

Then we have for any $j \in \mathbb{Z}$,

$$\begin{aligned}
\mathbf{m}_j(\xi,\eta) &= \psi_j(\xi)\phi_j(\eta)\mathbf{m}(\xi,\eta) \\
&= \tilde{\psi}_j(\xi)\psi_j(\xi)\tilde{\psi}_j(\eta)\phi_j(\eta)\mathbf{m}(\xi,\eta) \\
&= \tilde{\psi}_j(\xi)\tilde{\psi}_j(\eta)\mathbf{m}_j(\xi,\eta).
\end{aligned}$$

Now $supp\,\mathbf{m}_j \subset \{(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d : 2^{j-6} \leq |\xi| \leq 2^{j+6},\ 2^{j-6} \leq |\eta| \leq 2^{j+6}\}$.

Define for all $j \in \mathbb{Z}$,

$$a_j(\xi,\eta) = \mathbf{m}_j(2^j\xi, 2^j\eta).$$

Then $supp\,a_j \subset \{(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d : 2^{-6} \leq |\xi| \leq 2^6,\ 2^{-6} \leq |\eta| \leq 2^6\}$. Now using support of a_j and smoothness assumption on $\mathbf{m}$, expanding a_j in terms Fourier series over $[2^{-6}, 2^6]^d \times [2^{-6}, 2^6]^d$ we write

$$a_j(\xi,\eta) = \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d} c_j(\mathbf{n}_1,\mathbf{n}_2)\, e^{2\pi i(\langle\xi,\,\mathbf{n}_1\rangle+\langle\eta,\,\mathbf{n}_2\rangle)}, \tag{4.3.1}$$

where the Fourier coefficients $c_j(\mathbf{n}_1,\mathbf{n}_2)$ are given by

$$c_j(\mathbf{n}_1,\mathbf{n}_2) = \iint\limits_{[2^{-6},\,2^6]^d\times[2^{-6},\,2^6]^d} a_j(y,z)\, e^{-2\pi i(\langle y,\,\mathbf{n}_1\rangle+\langle z,\,\mathbf{n}_2\rangle)}\, dydz.$$

Thus we get

$$\begin{aligned}
\mathbf{m}_j(\xi,\eta) &= \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d} c_j(\mathbf{n}_1,\mathbf{n}_2)\, e^{2\pi i(\langle\xi,\,\mathbf{n}_1\rangle+\langle\eta,\,\mathbf{n}_2\rangle)/2^j} \\
&= \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d} c_j(\mathbf{n}_1,\mathbf{n}_2)\, e^{2\pi i(\langle\xi,\,\mathbf{n}_1\rangle+\langle\eta,\,\mathbf{n}_2\rangle)/2^j}\, \tilde{\psi}_j(\xi)\tilde{\psi}_j(\eta).
\end{aligned}$$

Substituting this in the expression for $\mathcal{T}_1$ and interchanging sum and integration, we obtain

$$\begin{aligned}
&\mathcal{T}_1(f_1,f_2)(x) \\
=& \int_{\mathbb{R}^{2d}} \sum_{j\in\mathbb{Z}} \mathbf{m}_j(\xi,\eta)\mathcal{F}_k f_1(\xi)\mathcal{F}_k f_2(\eta) E_k(ix,\xi)E_k(ix,\eta)\, d\mu_k(\xi)d\mu_k(\eta) \\
=& \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}} c_j(\mathbf{n}_1,\mathbf{n}_2)\left(\int_{\mathbb{R}^d} \tilde{\psi}_j(\xi)\, e^{2\pi i\langle\mathbf{n}_1,\,\xi\rangle/2^j}\mathcal{F}_k f_1(\xi)E_k(ix,\xi)\, d\mu_k(\xi)\right) \\
&\times\left(\int_{\mathbb{R}^d} \tilde{\psi}_j(\eta)\, e^{2\pi i\langle\mathbf{n}_2,\,\eta\rangle/2^j}\mathcal{F}_k f_2(\eta)E_k(ix,\eta)\, d\mu_k(\eta)\right) \\
=& \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}} c_j(\mathbf{n}_1,\mathbf{n}_2)\, \tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j)f_1(x)\, \tilde{\psi}(2\pi\mathbf{n}_2, D_k/2^j)f_2(x), \qquad (4.3.2)
\end{aligned}$$

where $\tilde{\psi}(2\pi\mathbf{n}_1, D/2^j)f_1(x)$ and $\tilde{\psi}(2\pi\mathbf{n}_2, D/2^j)f_2(x)$ are as in Section 4.2. Now as ψ and ϕ are supported compactly away from origin and $\mathbf{m}$ satisfies (4.4.1), by Leibniz rule we have that for all $j \in \mathbb{Z}$,

$$|\partial_\xi^\alpha \partial_\eta^\beta \left(\mathbf{m}(2^j\xi, 2^j\eta)\psi(\xi)\phi(\eta)\right)| \leq C_{\alpha,\beta}\left(|\xi| + |\eta|\right)^{-(|\alpha|+|\beta|)} \tag{4.3.3}$$

for all $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$ such that $|\alpha| + |\beta| \leq L$ and for all $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}$.

Now for any $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$, we have

$$\begin{aligned} c_j(\mathbf{n}_1, \mathbf{n}_2) &= \iint\limits_{[2^{-6},2^6]^d \times [2^{-6},2^6]^d} \mathbf{m}(2^j y, 2^j z)\psi(y)\phi(z)\, e^{-2\pi i(\langle y, \mathbf{n}_1\rangle + \langle z, \mathbf{n}_2\rangle)}\, dydz \\ &= \frac{C}{\mathbf{n}_1^\alpha \mathbf{n}_2^\beta} \iint\limits_{[2^{-6},2^6]^d \times [2^{-6},2^6]^d} \mathbf{m}(2^j y, 2^j z)\psi(y)\phi(z)\, \partial_y^\alpha \partial_z^\beta e^{-2\pi i(\langle y, \mathbf{n}_1\rangle + \langle z, \mathbf{n}_2\rangle)}\, dydz. \end{aligned}$$

Hence, applying integration by parts formula and using (4.3.3), we get

$$|c_j(\mathbf{n}_1, \mathbf{n}_2)| \leq C_L \frac{1}{(1 + |\mathbf{n}_1| + |\mathbf{n}_2|)^L} \tag{4.3.4}$$

Now using (4.3.4) and applying Cauchy-Schwarz inequality, from (4.3.2) we have

$$\begin{aligned} |\mathcal{T}_1(f_1, f_2)(x)| &\leq \sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n}_2 \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} \left| c_j(\mathbf{n}_1, \mathbf{n}_2)\, \tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j) f_1(x)\, \tilde{\psi}(2\pi\mathbf{n}_2, D_k/2^j) f_2(x) \right| \\ &\leq C \sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n}_2 \in \mathbb{Z}^d} \frac{C}{(1 + |\mathbf{n}_1| + |\mathbf{n}_2|)^L} \Big(\sum_{j \in \mathbb{Z}} |\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j) f_1(x)|^2 \Big)^{1/2} \\ &\quad \times \Big(\sum_{j \in \mathbb{Z}} |\tilde{\psi}(2\pi\mathbf{n}_2, D_k/2^j) f_2(x)|^2 \Big)^{1/2} \end{aligned}$$

Finally, from Hölder's inequality and Theorem 4.2.1 and using the facts that $L > 2d + 2\lfloor d_k \rfloor + 4$ and $n = \lfloor d_k \rfloor + 2$, we obtain

$$\begin{aligned} &\Big(\int_{\mathbb{R}^d} |\mathcal{T}_1(f_1, f_2)(x)|^p\, d\mu_k(x) \Big)^{1/p} \\ &\leq C \sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n}_2 \in \mathbb{Z}^d} \frac{1}{(1 + |\mathbf{n}_1| + |\mathbf{n}_2|)^L} \Big\| \Big(\sum_{j \in \mathbb{Z}} |\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j) f_1|^2 \Big)^{1/2} \Big\|_{L^{p_1}(d\mu_k)} \\ &\quad \times \Big\| \Big(\sum_{j \in \mathbb{Z}} |\tilde{\psi}(2\pi\mathbf{n}_2, D_k/2^j) f_2|^2 \Big)^{1/2} \Big\|_{L^{p_2}(d\mu_k)} \\ &\leq C\, ||f_1||_{L^{p_1}(d\mu_k)}\, ||f_2||_{L^{p_2}(d\mu_k)} \sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n}_2 \in \mathbb{Z}^d} \frac{(1 + |\mathbf{n}_1|)^n (1 + |\mathbf{n}_2|)^n}{(1 + |\mathbf{n}_1| + |\mathbf{n}_2|)^L} \end{aligned}$$

$$\leq \ C\,||f_1||_{L^{p_1}(d\mu_k)}\,||f_2||_{L^{p_2}(d\mu_k)}$$

This concludes the proof for $\mathcal{T}_1$.

Estimate of $\mathcal{T}_2$:

$$\begin{aligned}\text{For any } j \in \mathbb{Z},\ \text{define } \mathbf{m}_j(\xi,\eta) &= \psi_j(\xi) \sum_{j_2 : j_2 < j-4} \psi_{j_2}(\eta)\mathbf{m}(\xi,\eta) \\ &=: \psi_j(\xi)\phi_j(\eta)\mathbf{m}(\xi,\eta),\end{aligned}$$

where $\phi(\eta) = \sum_{j<-4} \psi_j(\eta)$ and $\phi_j(\eta) = \phi(\eta/2^j)$.

Then $supp\,\phi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2^{-3}\}$.

Let $\tilde{\psi},\ \tilde{\phi} \in C^\infty(\mathbb{R}^d)$ be another two functions such that $0 \leq \tilde{\psi},\ \tilde{\phi} \leq 1$ and

$$\tilde{\psi}(\xi) = \begin{cases} 0 & \text{if } |\xi| \notin [\frac{2}{5}, \frac{5}{2}], \\ 1 & \text{if } |\xi| \in [2^{-1}, 2]; \end{cases}$$

$$\tilde{\phi}(\xi) = \begin{cases} 0 & \text{if } |\xi| \notin [0, 2^{-2}], \\ 1 & \text{if } |\xi| \in [0, 2^{-3}]. \end{cases}$$

Then, we have for any $j \in \mathbb{Z}$,

$$\begin{aligned}\mathbf{m}_j(\xi,\eta) &= \psi_j(\xi)\phi_j(\eta)\mathbf{m}(\xi,\eta) \\ &= \tilde{\psi}_j(\xi)\psi_j(\xi)\tilde{\phi}_j(\eta)\phi_j(\eta)\mathbf{m}(\xi,\eta) \\ &= \tilde{\psi}_j(\xi)\tilde{\phi}_j(\eta)\mathbf{m}_j(\xi,\eta).\end{aligned}$$

Now $supp\,\mathbf{m}_j \subset \big\{(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d : 2^{j-1} \leq |\xi| \leq 2^{j+1},\ |\eta| \leq 2^{j-3}\big\}$.

Define for all $j \in \mathbb{Z}$,

$$a_j(\xi,\eta) = \mathbf{m}_j(2^j\xi, 2^j\eta).$$

Then $supp\,a_j \subset \big\{(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d : 2^{-1} \leq |\xi| \leq 2,\ |\eta| \leq 2^{-3}\big\}$.

Although $\psi(\xi)\,\phi(\eta)$ does not vanish at $\eta = 0$ but it is clear that it vanishes near $\xi = 0$. Hence, we can repeat the arguments as in the case of $\mathcal{T}_1$ to obtain that

$$\mathcal{T}_2(f_1, f_2)(x)$$

$$= \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}} c_j(\mathbf{n}_1,\mathbf{n}_2)\,\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j)f_1(x) \times \tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j)f_2(x) \tag{4.3.5}$$

and also (4.3.4) holds.

To complete the proof in this case, we need the following lemma. A version of this lemma can be found in [75], however for the sake of correctness and completeness, we provide a proof here.

Lemma 4.3.2. *Let Φ be a smooth function on $\mathbb{R}^d$ such that $0\le\Phi\le 1$, supp $\Phi\subset\{\xi\in\mathbb{R}^d : 2^{-6}\le|\xi|\le 2^6\}$ and $\Phi(\xi)=1$ for $2^{-5}\le|\xi|\le 2^5$. For $j\in\mathbb{Z}$, define $\Phi_j(\xi)=\Phi(\xi/2^j)$ and for $f\in\mathcal{S}(\mathbb{R}^d)$ define*

$$\Phi(0, D_k/2^j)f(x) = \int_{\mathbb{R}^d}\Phi_j(\xi)\mathcal{F}_kf(\xi)E_k(ix,\xi)\,d\mu_k(\xi).$$

Then for any $j\in\mathbb{Z}$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}&\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j)f_1(x)\,\tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j)f_2(x)\\ =\;&\Phi(0,D_k/2^j)\left(\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j)f_1(\cdot)\,\tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j)f_2(\cdot)\right)(x).\end{aligned}$$

Proof. It is enough to prove that

$$\begin{aligned}&\mathcal{F}_k\left(\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j)f_1(\cdot)\,\tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j)f_2(\cdot)\right)(\xi)\\ =\;&\Phi_j(\xi)\mathcal{F}_k\left(\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j)f_1(\cdot)\,\tilde{\phi}(2\pi\mathbf{n}_1, D_k/2^j)f_2(\cdot)\right)(\xi),\end{aligned}$$

for all $\xi\in\mathbb{R}^d$.

Applying properties of Dunkl convolution the above equality is equivalent to

$$\begin{aligned}&\left(\tilde{\psi}_j(\cdot)e^{\langle 2\pi i\mathbf{n}_1,\cdot\rangle/2^j}\mathcal{F}_kf_1(\cdot)\;*_k\;\tilde{\phi}_j(\cdot)e^{\langle 2\pi i\mathbf{n}_2,\cdot\rangle/2^j}\mathcal{F}_kf_2(\cdot)\right)(\xi)\\ =\;&\Phi_j(\xi)\left(\tilde{\psi}_j(\cdot)e^{\langle 2\pi i\mathbf{n}_1,\cdot\rangle/2^j}\mathcal{F}_kf_1(\cdot)\;*_k\;\tilde{\phi}_j(\cdot)e^{\langle 2\pi i\mathbf{n}_2,\cdot\rangle/2^j}\mathcal{F}_kf_2(\cdot)\right)(\xi),\end{aligned}$$

for all $\xi \in \mathbb{R}^d$. Again, to prove that the above equality holds, from definition of $\tilde{\psi}$, $\tilde{\phi}$ and Φ, it suffices to show that for any $f,\, g \in \mathcal{S}(\mathbb{R}^d)$ with $supp\, f \subset \left\{\xi \in \mathbb{R}^d : \frac{2^{j+1}}{5} \leq |\xi| \leq 5.2^{j-1}\right\}$, $supp\, g \subset \left\{\eta \in \mathbb{R}^d : |\eta| \leq 2^{j-2}\right\}$, we have that

$$supp\,(f *_k g) \subset \left\{\xi \in \mathbb{R}^d : 2^{j-5} \leq |\xi| \leq 2^{j+5}\right\}.$$

Take $x \in \mathbb{R}^d$ such that $|x| \notin [2^{j-5},\ 2^{j+5}]$. Now

$$f *_k g(x) = \int\limits_{\frac{2^{j+1}}{5} \leq |y| \leq 5.2^{j-1}} f(y)\tau_x^k g(-y)\, d\mu_k(y)$$

Now from [26, Theorem 1.7] or [2, Theorem 5.1], we have

$$supp\, \tau_x^k g(-\cdot) \subset \left\{y \in \mathbb{R}^d : |2^{j-2} - |x|\,| \leq |y| \leq |x| + 2^{j-2}\right\}.$$

But $|x| \notin [2^{j-5},\ 2^{j+5}]$ and $y \in supp\, \tau_x^k g(-\cdot)$ together implies either $|y| < \frac{2^{j+1}}{5}$ or $|y| > 5.2^{j-1}$ and which implies $f *_k g(x) = 0$. This completes the proof of the Lemma. □

Coming back to the proof for $\mathcal{T}_2$, by Lemma 4.3.2, from (4.3.5) we get

$$\begin{aligned} &\mathcal{T}_2(f_1, f_2)(x) \\ = &\sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n}_2 \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} c_j(\mathbf{n}_1, \mathbf{n}_2)\, \Phi(0, D_k/2^j) \left(\tilde{\psi}(2\pi \mathbf{n}_1, D_k/2^j) f_1(\cdot)\, \tilde{\phi}(2\pi \mathbf{n}_2, D_k/2^j) f_2(\cdot)\right)(x). \end{aligned}$$

To prove L^p boundedness of $\mathcal{T}_2$, take $g \in \mathcal{S}(\mathbb{R}^d)$ with $||g||_{L^{p'}(d\mu_k)} = 1$ where $1/p + 1/p' = 1$, then using Plancherel formula for Dunkl transform, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathcal{T}_2(f_1, f_2)(x)\, g(x)\, d\mu_k(x) \\ = &\int_{\mathbb{R}^d} \sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n}_2 \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} c_j(\mathbf{n}_1, \mathbf{n}_2) \\ &\times \Phi(0, D_k/2^j) \left(\tilde{\psi}(2\pi \mathbf{n}_1, D_k/2^j) f_1(\cdot)\, \tilde{\phi}(2\pi \mathbf{n}_2, D_k/2^j) f_2(\cdot)\right)(x)\, g(x)\, d\mu_k(x) \\ = &\sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n}_2 \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} c_j(\mathbf{n}_1, \mathbf{n}_2) \end{aligned}$$

$$\times\Big(\int_{\mathbb{R}^d} \Phi(0, D_k/2^j)\left(\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j) f_1(\cdot)\, \tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j) f_2(\cdot)\right)(x) g(x)\, d\mu_k(x)\Big)$$

$$= \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}} c_j(\mathbf{n}_1,\mathbf{n}_2)\Big(\int_{\mathbb{R}^d} \tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j) f_1(x)\, \tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j) f_2(x)$$

$$\times\Phi(0, D_k/2^j) g(x)\, d\mu_k(x)\Big).$$

Again using Cauchy-Schwarz inequality, Hölder's inequality, decay condition (4.3.4), Theorem 4.2.1 and Theorem 4.2.4 and using the facts that $L > 2d + 2\lfloor d_k\rfloor + 4$ and $n = \lfloor d_k\rfloor + 2$, we have

$$\Big|\int_{\mathbb{R}^d} \mathcal{T}_2(f_1,f_2)(x)\, g(x)\, d\mu_k(x)\Big|$$

$$= C\sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d}\frac{1}{(1+|\mathbf{n}_1|+|\mathbf{n}_2|)^L}\Big\|\Big(\sum_{j\in\mathbb{Z}}|\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j) f_1\, \tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j) f_2|^2\Big)^{1/2}\Big\|_{L^p(d\mu_k)}$$

$$\times\Big\|\Big(\sum_{j\in\mathbb{Z}}|\Phi(0, D_k/2^j) g|^2\Big)^{1/2}\Big\|_{L^{p'}(d\mu_k)}$$

$$\leq C\,||g||_{L^{p'}(d\mu_k)}\sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d}\frac{1}{(1+|\mathbf{n}_1|+|\mathbf{n}_2|)^L}\Big\|\Big(\sum_{j\in\mathbb{Z}}|\tilde{\psi}(2\pi\mathbf{n}_1, D_k/2^j) f_1|^2\Big)^{1/2}\Big\|_{L^{p_1}(d\mu_k)}$$

$$\times\Big\|\sup_{j\in\mathbb{Z}}|\tilde{\phi}(2\pi\mathbf{n}_2, D_k/2^j) f_2|\Big\|_{L^{p_2}(d\mu_k)}$$

$$\leq ||f_1||_{L^{p_1}(d\mu_k)}\,||f_2||_{L^{p_2}(d\mu_k)}\sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{n}_2\in\mathbb{Z}^d}\frac{(1+|\mathbf{n}_1|)^n(1+|\mathbf{n}_2|)^n}{(1+|\mathbf{n}_1|+|\mathbf{n}_2|)^L}$$

$$\leq C\,||f_1||_{L^{p_1}(d\mu_k)}\,||f_2||_{L^{p_2}(d\mu_k)}$$

Hence the proof for $\mathcal{T}_2$ is completed.

Estimate of $\mathcal{T}_3$:

The estimate for $\mathcal{T}_3$ follows exactly in the same as in the case of $\mathcal{T}_2$, hence it is omitted. □

Remark 4.3.3. In our result smoothness condition on **m** is more than what one may expect from the viewpoint of the classical case [73, Corollary 1.2]. We do not know whether the above result can be proven by assuming less number of derivatives on **m**.

4.4 Weighted Inequalities for Bilinear Multiplier Operators

Next, we state and prove our main results regarding one and two-weight estimates the for bilinear Dunkl multiplier operators.

Theorem 4.4.1. *Let $1 \leq p_1, p_2 < \infty$, p be the number given by $1/p = 1/p_1 + 1/p_2$ and $L \in \mathbb{N}$ be such that $L > 2d + 2\lfloor d_k \rfloor + 4$. If $\mathbf{m} \in C^L\left(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}\right)$ be a function satisfying*

$$|\partial_\xi^\alpha \partial_\eta^\beta \mathbf{m}(\xi, \eta)| \leq C_{\alpha,\beta}\left(|\xi| + |\eta|\right)^{-(|\alpha|+|\beta|)} \tag{4.4.1}$$

for all multi-indices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$ such that $|\alpha| + |\beta| \leq L$ and for all $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}$ and v, w_1, w_2 be G-invariant weights with $v \in A_\infty^k$; then the following hold:

(i) if at least one of p_1 or p_2 is 1 and the vector weight $(v, (w_1, w_2)) \in A^k_{(p_1,p_2)}$, then for all $f_1 \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k)$ and $f_2 \in L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k)$, the following boundedness holds:

$$\sup_{t>0} t\Big(\int_{\{y \in \mathbb{R}^d :\, |\mathcal{T}_{\mathbf{m}} \overrightarrow{f}(y)| > t\}} v(x)\, d\mu_k(x)\Big)^{1/p} \leq C \prod_{j=1}^{2} \Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x) d\mu_k(x)\Big)^{1/p_j};$$

(ii) if both $p_1, p_2 > 1$ and the vector weight $(v, \overrightarrow{w})$ satisfies the bump condition (2.3.1) with $m = 2$ for some $t > 1$, then for all $f_1 \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k)$ and $f_2 \in L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k)$, the following boundedness holds:

$$\Big(\int_{\mathbb{R}^d} |\mathcal{T}_{\mathbf{m}} \overrightarrow{f}(x)|^p v(x)\, d\mu_k(x)\Big)^{1/p} \leq C \prod_{j=1}^{2} \Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\, d\mu_k(x)\Big)^{1/p_j}.$$

Theorem 4.4.2. *Let $1 \leq p_1, p_2 < \infty$ and p be the number given by $1/p = 1/p_1 + 1/p_2$ and $L \in \mathbb{N}$ be such that $L > 2d + 2\lfloor d_k \rfloor + 4$. If $\mathbf{m} \in C^L\left(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}\right)$ be a function which satisfies the condition (4.4.1) for all multi-indices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$ such* *that* $|\alpha|+|\beta| \le L$ *and for all* $(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{(0,0)\}$ *and* w_1, w_2 *be G-invariant weights with* $(w_1, w_2) \in A^k_{(p_1,p_2)}$*; then the following hold:*

(i) if at least one of p_1 *or* p_2 *is* 1*, then for all* $f_1 \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k)$ *and* $f_2 \in L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k)$*, the following boundedness holds:*

$$\sup_{t>0} t\Big(\int_{\{y\in\mathbb{R}^d:\, |\mathcal{T}_{\mathbf{m}}\vec{f}(y)|>t\}} \prod_{j=1}^{2} w_j^{p/p_j}(x)\, d\mu_k(x)\Big)^{1/p} \le C\prod_{j=1}^{2}\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x) d\mu_k(x)\Big)^{1/p_j};$$

(ii) if both $p_1, p_2 > 1$*, then for all* $f_1 \in L^{p_1}(\mathbb{R}^d, w_1\, d\mu_k)$ *and* $f_2 \in L^{p_2}(\mathbb{R}^d, w_2\, d\mu_k)$*, the following boundedness holds:*

$$\Big(\int_{\mathbb{R}^d} |\mathcal{T}_{\mathbf{m}}\vec{f}(x)|^p w_1^{p/p_1}(x) w_2^{p/p_2}(x)\, d\mu_k(x)\Big)^{1/p} \le C\prod_{j=1}^{2}\Big(\int_{\mathbb{R}^d} |f_j(x)|^{p_j} w_j(x)\, d\mu_k(x)\Big)^{1/p_j}.$$

Proofs of Theorem 4.4.1 and Theorem 4.4.2. Let $\phi \in C^\infty(\mathbb{R}^{2d})$ be such that $supp\,\phi \subset \{(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d : 1/4 \le \big(|\xi|^2 + |\eta|^2\big)^{1/2} \le 4\}$ and

$$\sum_{j\in\mathbb{Z}} \phi(\xi/2^j, \eta/2^j) = 1 \text{ for all } (\xi,\eta) \ne (0,0)$$

where we recall that the notation $|\cdot|$ stands for the usual norm on $\mathbb{R}^d$. Then

$$\begin{aligned}\mathbf{m}(\xi,\eta) &= \sum_{j\in\mathbb{Z}} \mathbf{m}(\xi,\eta)\,\phi(\xi/2^j, \eta/2^j)\\ &=: \sum_{j\in\mathbb{Z}} \mathbf{m}_j(\xi/2^j, \eta/2^j).\end{aligned}$$

In view of the multilinear Dunkl setting defined in Section 2.2, for $x, y_1, y_2 \in \mathbb{R}^d$, let us define

$$K_j(x,y_1,y_2) = \tau^{k^2}_{(x,x)} \mathcal{F}^{-1}_{k^2}\big(\mathbf{m}(\cdot,\cdot)\,\phi(\cdot/2^j, \cdot/2^j)\big)\big((-y_1,-y_2)\big)$$
$$\text{and } \widetilde{K}_j(x,y_1,y_2) = \tau^{k^2}_{(x,x)} \mathcal{F}^{-1}_{k^2}\,\mathbf{m}_j\big((-y_1,-y_2)\big).$$

Then $K_j(x,y_1,y_2) = 2^{2jd_k}\widetilde{K}_j(2^j x, 2^j y_1, 2^j y_2)$ for all $x, y_1, y_2 \in \mathbb{R}^d$ and for all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{T}_{\mathbf{m}}(f_1,f_2)(x) = \int_{\mathbb{R}^{2d}} \mathbf{m}(\xi,\eta)\mathcal{F}_k f_1(\xi)\mathcal{F}_k f_2(\eta) E_k(ix,\xi)E_k(ix,\eta)\, d\mu_k(\xi) d\mu_k(\eta)$$

$$\begin{aligned}
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2d}} \mathbf{m}_j(\xi/2^j, \eta/2^j)\mathcal{F}_{k^2}\big(f_1 \otimes f_2\big)\big((\xi,\eta)\big)\, E_{k^2}\big(i(x,x),(\xi,\eta)\big) \\
&\quad \times d\mu_{k^2}\big((\xi,\eta)\big) \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2d}} K_j(x,y_1,y_2)\, f_1(y_1) f_2(y_2)\, d\mu_k(y_1) d\mu_k(y_2) \\
&=: \int_{\mathbb{R}^{2d}} K(x,y_1,y_2)\, f_1(y_1) f_2(y_2)\, d\mu_k(y_1) d\mu_k(y_2).
\end{aligned}$$

Having Theorem 4.3.1 already proved, proofs of Theorem 4.4.1 and Theorem 4.4.2 will follow directly from Theorem 3.3.3 and Theorem 3.3.4, if we can show that integral kernel K of $\mathcal{T}_{\mathbf{m}}$, satisfies the size estimate (3.1.1) and smoothness estimates (3.1.2) for $m = 2$. For that, we need to show that for any $x, x', y_1, y_2 \in \mathbb{R}^d$,

(i)
$$\begin{aligned}
&|K(x,y_1,y_2)| \\
\leq\ & C\Big[\mu_k\big(B(x,d_G(x,y_1))\big) + \mu_k\big(B(x,d_G(x,y_2))\big)\Big]^{-2} \frac{d_G(x,y_1)+d_G(x,y_2)}{|x-y_1|+|x-y_2|}
\end{aligned} \tag{4.4.2}$$

for $d_G(x,y_1) + d_G(x,y_2) > 0$;

(ii)
$$\begin{aligned}
&|K(x,y_1,y_2) - K(x,y_1,y_2')| \\
\leq\ & C\Big[\mu_k\big(B(x,d_G(x,y_1))\big) + \mu_k\big(B(x,d_G(x,y_2))\big)\Big]^{-2} \frac{|y_2-y_2'|}{\max\{|x-y_1|,\ |x-y_2|\}}
\end{aligned} \tag{4.4.3}$$

for $|y_2 - y_2'| < \max\{d_G(x,y_1)/2,\ d_G(x,y_2)/2\}$;

(iii)
$$\begin{aligned}
&|K(x,y_1,y_2) - K(x,y_1',y_2)| \\
\leq\ & C\Big[\mu_k\big(B(x,d_G(x,y_1))\big) + \mu_k\big(B(x,d_G(x,y_2))\big)\Big]^{-2} \frac{|y_1-y_1'|}{\max\{|x-y_1|,\ |x-y_2|\}}
\end{aligned} \tag{4.4.4}$$

for $|y_1 - y_1'| < \max\{d_G(x,y_1)/2,\ d_G(x,y_2)/2\}$

and

(iv)
$$\begin{aligned} & |K(x,y_1,y_2)-K(x',y_1,y_2)| \\ \leq\ & C\Big[\mu_k\big(B(x,d_G(x,y_1))\big)+\mu_k\big(B(x,d_G(x,y_2))\big)\Big]^{-2}\frac{|x-x'|}{\max\{|x-y_1|,\ |x-y_2|\}} \end{aligned} \tag{4.4.5}$$

for $|x-x'|<\max\{d_G(x,y_1)/2,\ d_G(x,y_2)/2\}$.

Proof of the inequality (4.4.2)

The condition (4.4.1) assures that

$$\sup_{j\in\mathbb{Z}}\|\mathbf{m}_j\|_{C^L(\mathbb{R}^{2d})}\leq C. \tag{4.4.6}$$

Since

$$\widetilde{K}_j(x,y_1,y_2)=\int_{\mathbb{R}^{2d}}\mathbf{m}_j(\xi,\eta)\,E_{k^2}\big(i(\xi,\eta),\ (x,x)\big)E_{k^2}\big(-i(\xi,\eta),\ (y_1,y_2)\big)\,d\mu_{k^2}\big((\xi,\eta)\big),$$

by applying [28, eq.(4.30)] for $\mathbb{R}^{2d}$, we write

$$\begin{aligned} |\widetilde{K}_j(x,y_1,y_2)| \ \leq\ & \frac{C}{\big[\mu_{k^2}\big(B((x,x),1)\big)\mu_{k^2}\big(B((y_1,y_2),1)\big)\big]^{1/2}} \\ & \times\frac{1}{1+\big(|x-y_1|^2+|x-y_2|^2\big)^{1/2}}\ \frac{1}{\big[1+d_{G\times G}\big((x,x),\ (y_1,y_2)\big)\big]^{L-1}}. \end{aligned}$$

Therefore, using (2.1.2) for $\mathbb{R}^{2d}$ we get

$$\begin{aligned} & |K(x,y_1,y_2)| \\ \leq\ & \sum_{j\in\mathbb{Z}}|K_j(x,y_1,y_2)| \\ =\ & \sum_{j\in\mathbb{Z}}2^{2jd_k}\,|\widetilde{K}_j(2^jx,2^jy_1,2^jy_2)| \\ \leq\ & \sum_{j\in\mathbb{Z}}\frac{C\,2^{2jd_k}}{\big[\mu_{k^2}\big(B((2^jx,2^jx),1)\big)\mu_{k^2}\big(B((2^jy_1,2^jy_2),1)\big)\big]^{1/2}} \end{aligned}$$

$$
\begin{aligned}
&\times \frac{1}{1+2^j\left(|x-y_1|^2+|x-y_2|^2\right)^{1/2}} \frac{1}{\left[1+2^j\, d_{G\times G}\big((x,x),\,(y_1,y_2)\big)\right]^{L-1}} \\
\leq\ & C \sum_{j\in\mathbb{Z}} \frac{1}{\left[\mu_{k^2}\big(B((x,x),2^{-j})\big)\mu_{k^2}\big(B((y_1,y_2),2^{-j})\big)\right]^{1/2}} \\
&\times \frac{1}{1+2^j\left(|x-y_1|^2+|x-y_2|^2\right)^{1/2}} \frac{1}{\left[1+2^j\, d_{G\times G}\big((x,x),\,(y_1,y_2)\big)\right]^{L-1}} \\
=\ & C \sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))\leq 1} \cdots + \sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))>1} \cdots . \qquad (4.4.7)
\end{aligned}
$$

Again from the discussion in Section 2.2, using the product nature of the root system, it is not too hard to see that

$$
\begin{aligned}
\mu_{k^2}\big(B((z_1,z_2),r)\big) &\sim \mu_k\big(B(z_1,r)\big)\,\mu_k\big(B(z_2,r)\big), \\
d_{G\times G}\big((z,z),\,(z_1,z_2)\big) &\sim d_G(z,z_1)+d_G(z,z_2) \qquad (4.4.8)\\
\text{and } \mu_k\big(B(z,r_1+r_2)\big) &\geq C\left[\mu_k\big(B(z,r_1)\big)+\mu_k\big(B(z,r_2)\big)\right]
\end{aligned}
$$

for all $z, z_1, z_2 \in \mathbb{R}^d$ and $r, r_1, r_2 > 0$.

Now, if $2^j\, d_{G\times G}((x,x),\,(y_1,y_2)) \leq 1$, by applying (2.1.2) for $\mathbb{R}^{2d}$ and the relations (4.4.8), we deduce

$$
\begin{aligned}
\frac{1}{\mu_{k^2}\big(B((x,x),2^{-j})\big)} &\leq C\,\frac{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d}}{\mu_{k^2}\big(B((x,x),d_{G\times G}((x,x),\,(y_1,y_2)))\big)} \\
&\leq C\,\frac{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d}}{[\mu_k(B(x,d_G(x,y_1)))+\mu_k(B(x,d_G(x,y_2)))]^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\mu_{k^2}\big(B((y_1,y_2),2^{-j})\big)} &\leq C\,\frac{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d}}{\mu_{k^2}\big(B((y_1,y_2),d_{G\times G}((x,x),\,(y_1,y_2)))\big)} \\
&\leq C\,\frac{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d}}{\mu_{k^2}\big(B((x,x),d_{G\times G}((x,x),\,(y_1,y_2)))\big)} \\
&\leq C\,\frac{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d}}{[\mu_k(B(x,d_G(x,y_1)))+\mu_k(B(x,d_G(x,y_2)))]^2},
\end{aligned}
$$

where in the second last inequality we used (2.1.4) for $\mathbb{R}^{2d}$.

Using the above two inequalities in (4.4.7), we have

$$
\sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))\leq 1} \frac{1}{\left[\mu_{k^2}\big(B((x,x),2^{-j})\big)\mu_{k^2}\big(B((y_1,y_2),2^{-j})\big)\right]^{1/2}}
$$

$$
\begin{aligned}
&\times \frac{1}{1+2^j\left(|x-y_1|^2+|x-y_2|^2\right)^{1/2}} \frac{1}{\left[1+2^j\, d_{G\times G}\big((x,x),\,(y_1,y_2)\big)\right]^{L-1}} \\
\leq\ & C \sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))\leq 1} \frac{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d}}{[\mu_k(B(x,d_G(x,y_1)))+\mu_k(B(x,d_G(x,y_2)))]^2} \\
&\times \frac{1}{2^j(|x-y_1|+|x-y_2|)} \\
\leq\ & C\left[\mu_k\big(B(x,d_G(x,y_1))\big)+\mu_k\big(B(x,d_G(x,y_2))\big)\right]^{-2} \frac{d_G(x,y_1)+d_G(x,y_2)}{|x-y_1|+|x-y_2|} \\
&\times \sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))\leq 1} [2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d-1} \\
\leq\ & C\left[\mu_k\big(B(x,d_G(x,y_1))\big)+\mu_k\big(B(x,d_G(x,y_2))\big)\right]^{-2} \frac{d_G(x,y_1)+d_G(x,y_2)}{|x-y_1|+|x-y_2|}.
\end{aligned}
\tag{4.4.9}
$$

Similarly for the second sum in (4.4.7), where $2^j\, d_{G\times G}((x,x),\,(y_1,y_2))>1$, we get

$$
\begin{aligned}
& \sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))>1} \frac{1}{\left[\mu_{k^2}\big(B((x,x),2^{-j})\big)\mu_{k^2}\big(B((y_1,y_2),2^{-j})\big)\right]^{1/2}} \\
&\times \frac{1}{1+2^j\left(|x-y_1|^2+|x-y_2|^2\right)^{1/2}} \frac{1}{\left[1+2^j\, d_{G\times G}\big((x,x),\,(y_1,y_2)\big)\right]^{L-1}} \\
\leq\ & C \sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))>1} \frac{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{2d_k}}{[\mu_k(B(x,d_G(x,y_1)))+\mu_k(B(x,d_G(x,y_2)))]^2} \\
&\times \frac{1}{2^j(|x-y_1|+|x-y_2|)} \frac{1}{\left[2^j\, d_{G\times G}\big((x,x),\,(y_1,y_2)\big)\right]^{L-1}} \\
\leq\ & C\left[\mu_k\big(B(x,d_G(x,y_1))\big)+\mu_k\big(B(x,d_G(x,y_2))\big)\right]^{-2} \frac{d_G(x,y_1)+d_G(x,y_2)}{|x-y_1|+|x-y_2|} \\
&\times \sum_{j\in\mathbb{Z}:\, 2^j\, d_{G\times G}((x,x),\,(y_1,y_2))>1} \frac{1}{[2^j\, d_{G\times G}((x,x),\,(y_1,y_2))]^{L-2d_k}} \\
\leq\ & C\left[\mu_k\big(B(x,d_G(x,y_1))\big)+\mu_k\big(B(x,d_G(x,y_2))\big)\right]^{-2} \frac{d_G(x,y_1)+d_G(x,y_2)}{|x-y_1|+|x-y_2|}.
\end{aligned}
\tag{4.4.10}
$$

Substituting (4.4.9) and (4.4.10) in (4.4.7), we complete the proof of (4.4.2).

Proof of the inequality (4.4.3)

It is easy to see that

$$d_G(x,y_1)\le d_{G\times G}((x,x),\,(y_1,y_2)) \text{ and } d_G(x,y_2)\le d_{G\times G}((x,x),\,(y_1,y_2)).$$

So the condition $|y_2-y_2'|<\max\{d_G(x,y_1)/2,\ d_G(x,y_2)/2\}$ implies that the $2d$-norm [1]

$$|(y_1,y_2)-(y_1,y_2')| \quad\le\quad d_{G\times G}((x,x),\,(y_1,y_2))/2,$$

which further implies

$$d_{G\times G}\big((x,x),(y_1,y_2)\big)\sim d_{G\times G}\big((x,x),(y_1,y_2')\big),\ |(x,x)-(y_1,y_2)|\sim|(x,x)-(y_1,y_2')| \text{ and}$$

$$V_{G\times G}\big((x,x),(y_1,y_2),d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)\sim V_{G\times G}\big((x,x),(y_1,y_2'),d_{G\times G}\big((x,x),(y_1,y_2')\big)\big).$$

By applying the techniques used in the proof of Theorem 4.2.1 in $\mathbb{R}^{2d}$ and using (4.4.6) in place of (4.2.4), we get

$$\begin{aligned}
&2^{2jd_k}|\widetilde{K}_j(2^jx,2^jy_1,2^jy_2)-\widetilde{K}_j(2^jx,2^jy_1,2^jy_2')|\\
\le\ & C\,\frac{|(y_1,y_2)-(y_1,y_2')|}{\big|(x,x)-(y_1,y_2)\big|}\,\frac{\big(2^jd_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d}+\big(2^jd_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d_k}}{V_{G\times G}\big((x,x),(y_1,y_2),d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)}\\
&\times\frac{1}{\big(1+2^jd_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{L-1}}.
\end{aligned}\tag{4.4.11}$$

Hence, using (2.1.4) for $\mathbb{R}^{2d}$, and (4.4.8) repeatedly, from (4.4.11) we have

$$\begin{aligned}
&|K(x,y_1,y_2)-K(x,y_1,y_2')|\\
\le\ &\sum_{j\in\mathbb{Z}}2^{2jd_k}|\widetilde{K}_j(2^jx,2^jy_1,2^jy_2)-\widetilde{K}_j(2^jx,2^jy_1,2^jy_2')|\\
\le\ & C\,\frac{|(y_1,y_2)-(y_1,y_2')|}{\big|(x,x)-(y_1,y_2)\big|}\,\frac{1}{V_{G\times G}\big((x,x),(y_1,y_2),d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)}\\
&\times\sum_{j\in\mathbb{Z}}\frac{\big(2^jd_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d}+\big(2^jd_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d_k}}{\big(1+2^jd_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{L-1}}
\end{aligned}$$

[1] We have used the same notation $|\cdot|$ for norms on $\mathbb{R}^d$ and $\mathbb{R}^{2d}$.

$$\begin{aligned}
&\le C\,\frac{|y_2-y_2'|}{|x-y_1|+|x-y_2|}\,\frac{1}{\mu_{k^2}\big(B((x,x),d_{G\times G}((x,x),(y_1,y_2)))\big)}\\
&\quad\times\sum_{j\in\mathbb{Z}}\frac{\big(2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d}+\big(2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d_k}}{\big(1+2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{L-1}}\\
&\le C\,\frac{|y_2-y_2'|}{|x-y_1|+|x-y_2|}\,\frac{1}{\big[\mu_k\big(B(x,d_{G\times G}((x,x),(y_1,y_2)))\big)\big]^2}\\
&\quad\times\sum_{j\in\mathbb{Z}}\frac{\big(2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d}+\big(2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d_k}}{\big(1+2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{L-1}}\\
&\le C\,\frac{|y_2-y_2'|}{\max\{|x-y_1|,\,|x-y_2|\}}\,\frac{1}{\big[\mu_k\big(B(x,d_G(x,y_1))\big)+\mu_k\big(B(x,d_G(x,y_2))\big)\big]^2}\\
&\quad\times\sum_{j\in\mathbb{Z}}\frac{\big(2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d}+\big(2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{2d_k}}{\big(1+2^j d_{G\times G}\big((x,x),(y_1,y_2)\big)\big)^{L-1}}\\
&\le C\,\big[\mu_k\big(B(x,d_G(x,y_1))\big)+\mu_k\big(B(x,d_G(x,y_2))\big)\big]^{-2}\frac{|y_2-y_2'|}{\max\{|x-y_1|,\,|x-y_2|\}},
\end{aligned}$$

where the convergence of the last sum can be shown in exact same way as in the proof of Theorem 4.2.1.

Proof of the inequality (4.4.4)

The proof of (4.4.4) is exactly the same as the proof of (4.4.3) with interchange of the roles of y_1 and y_2.

Proof of the inequality (4.4.5)

Note that $\widetilde{K}_j$ can be written as

$$\widetilde{K}_j(x,y_1,y_2)=\int_{\mathbb{R}^{2d}}\mathbf{m}_j(\xi,\eta)\,E_{k^2}\big(i(\xi,\eta),(-y_1,-y_2)\big)E_{k^2}\big(-i(\xi,\eta),(-x,-x)\big)\,d\mu_{k^2}\big((\xi,\eta)\big),$$

Now the condition $|x-x'|<\max\{d_G(x,y_1)/2,\ d_G(x,y_2)/2\}$ implies that the $2d$-norm

$$|(x,x)-(x',x')|\ \le\ \sqrt{2}\,d_{G\times G}((x,x),\ (y_1,y_2))/2,$$

which further implies

$d_{G\times G}\big((x,x),(y_1,y_2)\big) \sim d_{G\times G}\big((x',x'),(y_1,y_2)\big)$, $|(x,x)-(y_1,y_2)| \sim |(x',x')-(y_1,y_2)|$ and

$V_{G\times G}\big((x,x),(y_1,y_2),d_{G\times G}\big((x,x),(y_1,y_2)\big)\big) \sim V_{G\times G}\big((x',x'),(y_1,y_2),d_{G\times G}\big((x',x'),(y_1,y_2)\big)\big).$

Thus in this case also, rest of the proof can be carried forward in the same way as in the proof of (4.4.3). □

Chapter 5

Multilinear Fractional Operators in Dunkl Setting

This chapter is devoted to studying weighted estimates for the multilinear Dunkl fractional integral operator $\mathcal{I}_\alpha^k$ and multilinear Dunkl fractional maximal operator $\mathcal{M}_\alpha^k$. It extends the results in the classical setting mentioned in Section 1.3 to the Dunkl setting. After a concise introduction and a brief overview of the history pertaining to these operators, we formally define them in the next section. In Section 5.2, we use Rösler's formula for Dunkl translations of radial functions to dominate $\mathcal{I}_\alpha^k$ by an operator similar to the classical multilinear fractional integral operator, which involves the Dunkl metric instead of the usual metric. Subsequently, we investigate the two-weight inequalities for $\mathcal{I}_\alpha^k$. Similarly, in Section 5.3, we control $\mathcal{M}_\alpha^k$ by a finite sum of classical multilinear fractional maximal operators and study two-weight inequalities for them. Utilizing two-weight inequalities, we establish corresponding one-weight inequalities for $\mathcal{M}_\alpha^k$ in Section 5.4. This enables us to prove one-weight inequalities for $\mathcal{I}_\alpha^k$ in Section 5.5, depending on the one-weight results for $\mathcal{M}_\alpha^k$. This whole chapter is based on the work [55].

5.1 Introduction

After the groundbreaking work of Muckenhoupt [52] regarding the characterization of weighted inequalities for the Hardy-Littlewood maximal operator M, the study of weighted norm inequalities in harmonic analysis has garnered significant attention. The operator M is closely related to the fractional integral operator I_α, defined for all functions f in the

Schwartz class $\mathcal{S}(\mathbb{R}^d)$, by

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy, \qquad 0 < \alpha < d$$

and to the fractional maximal operator M_α, defined for locally integrable functions f, by

$$M_\alpha f(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |f(y)|\, dy, \qquad 0 \le \alpha < d.$$

Along with the maximal function, a parallel weight-theory for M_α and I_α has been studied as well by several authors. A characterization of the one-weight inequality for these two operators has been given by Muckenhoupt and Wheeden [53]. In particular, they proved that for $1 < p < d/\alpha$ and q given by $1/q = 1/p - \alpha/d$; I_α or $M_\alpha : L^p(\mathbb{R}^d, w(x)^p dx) \to L^q(\mathbb{R}^d, w(x)^q dx)$ if and only if w satisfy the $A_{p,q}$ condition, i.e.,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w^q\, dy\right)^{1/q} \left(\frac{1}{|Q|}\int_Q w^{-p'}\, dy\right)^{1/p'} < \infty. \tag{5.1.1}$$

On the other hand, Sawyer [65] gave a similar characterization as in (1.0.4) for the two-weight $L^p - L^q$ inequalities for these two operators. Another characterization of the two-weight inequalities for I_α was given by Sawyer and Wheeden [66] using "power bump" conditions on the weights.

In 2009, Moen [51] presented a multilinear analogue of the above weighted inequalities for multilinear fractional integrals $\mathcal{I}_\alpha$ and multilinear fractional maximal operators $\mathcal{M}_\alpha$, where $\mathcal{I}_\alpha$ is defined for all $\overrightarrow{f} = (f_1, f_2, \cdots, f_m) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$, by

$$\mathcal{I}_\alpha \overrightarrow{f}(x) = \int_{(\mathbb{R}^d)^m} \frac{f_1(y_1) f_2(y_2) \ldots f_m(y_m)}{(|x-y_1| + |x-y_2| + \ldots + |x-y_m|))^{md-\alpha}}\, dy_1 dy_2 \ldots dy_m,$$

$$0 < \alpha < md$$

and $\mathcal{M}_\alpha$ is defined for all locally integrable functions f_i in $\mathbb{R}^d$, by

$$\mathcal{M}_\alpha \overrightarrow{f}(x) = \sup_{x\in Q} \prod_{j=1}^m \frac{l(Q)^{\alpha/m}}{|Q|} \int_Q |f_j(y_j)|\, dy_j, \qquad 0 \le \alpha < md.$$

Regarding one-weight inequality, the author in [51] proved that $\mathcal{I}_\alpha$ or $\mathcal{M}_\alpha$ is a bounded operator from $L^{p_1}(\mathbb{R}^d, w_1(x)^{p_1}dx) \times L^{p_1}(\mathbb{R}^d, w_2(x)^{p_2}dx) \times \cdots \times L^{p_1}(\mathbb{R}^d, w_m(x)^{p_m}dx)$ to $L^q(\mathbb{R}^d, (\prod_{j=1}^m w_j(x))^q dx)$, where $1/q = 1/p_1 + 1/p_2 + \cdots + 1/p_m - \alpha/n$ and $1 < p_j < \infty$, if and only if $\overrightarrow{w} = (w_1, w_2, \cdots, w_m)$ satisfy the $A_{\overrightarrow{P},q}$ condition:

$$\sup_Q \left(\frac{1}{|Q|}\int_Q (\prod_{j=1}^m w_j)^q\, dy\right)^{1/q} \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j^{-p_j'}\, dy\right)^{1/p_j'} < \infty, \tag{5.1.2}$$

which is multilinear version of the $A_{p,q}$ condition (5.1.1). In the same paper two-weight inequalities for these operators were also studied using “bump conditions” on the weights which are similar to the linear case.

In analogy to the classical case, Thangavelu and Xu [72] defined the fractional integral operator I_α^k associated to Dunkl operator by

$$I_\alpha^k f(x) = \int_{\mathbb{R}^d} \tau_{-y}^k f(x)|y|^{\alpha - d_k}\, d\mu_k(y),$$

where $0 < \alpha < d_k$.

Also the associated fractional maximal function M_α^k was introduced by Gorbachev et al. [32], given by

$$M_\alpha^k f(x) = \sup_{r>0} r^{\alpha - d_k}\left|\int_{\mathbb{R}^d} f(y)\tau_x^k \chi_{B(0,r)}(-y)\, d\mu_k(y)\right|,$$

where $0 \le \alpha < d_k$ and $\chi_{B(0,r)}$ denotes the characteristic function of the ball $B(0,r)$.

For the reflection group $\mathbb{Z}_2^d$, the $L^p - L^q$ inequalities for Dunkl fractional integral and Dunkl fractional maximal function in the non-weighted case were proved by Thangavelu and Xu [72] and it was further generalized for any reflection group G by Hassani et al. [39]. Regarding weighted boundedness, weighted inequalities for I_α^k and M_α^k with radial power weights are there in the literature [1, 32].

Motivated by the work of Moen [51], in this chapter we define the multilinear fractional integral operator in Dunkl setting $\mathcal{I}_\alpha^k$ and the corresponding multilinear fractional operator

$\mathcal{M}_\alpha^k$, which could be seen as multilinear analogues of I_α^k and M_α^k respectively. For $\overrightarrow{f} = (f_1, f_2, \cdots, f_m) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$, we define the *multilinear Dunkl fractional integral operator* $\mathcal{I}_\alpha^k$ as

$$\mathcal{I}_\alpha^k \overrightarrow{f}(x) = \int_{(\mathbb{R}^d)^m} \frac{\tau^{k^m}_{(-y_1,-y_2,\cdots,-y_m)} f_1(x) f_2(x) \cdots f_m(x)}{(|y_1|^2 + |y_2|^2 + \cdots + |y_m|^2)^{(md_k - \alpha)/2}} \, d\mu_{k^m}(y_1, y_2, \cdots, y_m),$$

$$\text{where } 0 < \alpha < md_k$$

and the associated *multilinear Dunkl fractional maximal operator* $\mathcal{M}_\alpha^k$ as

$$\mathcal{M}_\alpha^k \overrightarrow{f}(x) = \sup_{r>0} \prod_{j=1}^m r^{\frac{\alpha}{m} - d_k} \left| \int_{\mathbb{R}^d} f_j(y_j) \tau_x^k \chi_{B(0,r)}(-y_j) \, d\mu_k(y_j) \right|, \text{ where } 0 \le \alpha < md_k.$$

Clearly, $\mathcal{I}_\alpha^k$ and $\mathcal{M}_\alpha^k$ represent the m-linear extensions of the Dunkl fractional integral operators and the Dunkl fractional maximal operators.

Our main purpose here is to study both one and two-weight inequalities for $\mathcal{I}_\alpha^k$ and $\mathcal{M}_\alpha^k$ for weights which are counter parts of those in classical setting. Our results extend the literature in the Dunkl setting from the linear case to the multilinear case. For general space of homogeneous type multiple-weighted inequalities for fractional integrals (which are not equivalent to Dunkl fractional integrals) were proved by Maldonado et al. [50] using the techniques in [58,66] (see also [51,59]). But we can not directly use their result here as none of the conditions are satisfied by kernel associated with $\mathcal{I}_\alpha^k$. Here similar techniques will be used to prove two-weight inequalities. On the other hand to prove one-weight inequalities we will make use of the two-weight inequalities and follow the similar approaches which are used to prove weighted inequalities for multilinear maximal function (in the classical setting by Lerner et al. [48] and Moen [51] and for the homogeneous spaces by Grafakos et al. [34]). The main essence of our results lies in some new tricks (Lemma 5.2.1 and Lemma 5.3.3) that enable us to overcome the challenges inherent in this setup, transitioning from Dunkl-characteristic to the characteristics similar to those of spaces of homogeneous type.

5.2 Two-weight Inequalities for Multilinear Fractional Integral Operators

Before delving into the main theorem of this section, we first state the following lemma, which will be employed in proving the main theorem.

Lemma 5.2.1. *For* $\overrightarrow{f} \in \mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)\times\cdots\times\mathcal{S}(\mathbb{R}^d)$ *with* $f_j \geq 0$ *for all* $j = 1, 2, \cdots, m$*; we have*

$$\begin{aligned} & |\mathcal{I}_\alpha^k \overrightarrow{f}(x)| \\ \leq\ & C \int_{(\mathbb{R}^d)^m} \frac{f_1(y_1)f_2(y_2)\cdots f_m(y_m)}{\Big(d_G(x,y_1)+d_G(x,y_2)+\cdots+d_G(x,y_m)\Big)^{md_k-\alpha}}\, d\mu_k(y_1)\cdots d\mu_k(y_m) \end{aligned}$$

Proof. From the identity

$$\begin{aligned} & \frac{1}{(|y_1|^2+|y_2|^2+\cdots+|y_m|^2)^{(md_k-\alpha)/2}} \\ &= \frac{1}{\Gamma(\frac{md_k-\alpha}{2})}\int_0^\infty s^{(md_k-\alpha)/2-1} e^{-s(|y_1|^2+|y_2|^2+\cdots+|y_m|^2)}\, ds \end{aligned}$$

and using the properties of Dunkl convolution, we have

$$\begin{aligned} \mathcal{I}_\alpha^k \overrightarrow{f}(x) &= \frac{1}{\Gamma(\frac{md_k-\alpha}{2})}\int_{(\mathbb{R}^d)^m} \tau_x^k f_1(-y_1)\tau_x^k f_2(-y_2)\cdots\tau_x^k f_m(-y_m) \\ &\quad \times \int_0^\infty s^{(md_k-\alpha)/2-1} e^{-s(|y_1|^2+|y_2|^2+\cdots+|y_m|^2)}\, ds\, d\mu_k(y_1)d\mu_k(y_2)\cdots d\mu_k(y_m) \\ &= \frac{1}{\Gamma(\frac{md_k-\alpha}{2})}\int_0^\infty s^{(md_k-\alpha)/2-1}\prod_{j=1}^m \left(\int_{\mathbb{R}^d} \tau_x^k f_j(-y_j) e^{-s|y_j|^2}\, d\mu_k(y_j)\right) ds \\ &= \frac{1}{\Gamma(\frac{md_k-\alpha}{2})}\int_0^\infty s^{(md_k-\alpha)/2-1}\prod_{j=1}^m \left(\int_{\mathbb{R}^d} f_j(y_j)\tau_x^k e^{-s|-y_j|^2}\, d\mu_k(y_j)\right) ds. \end{aligned}$$

Now using (2.1.6), applying the inequality (2.1.7) and reversing the above process, we get

$$\begin{aligned} & |\mathcal{I}_\alpha^k \overrightarrow{f}(x)| \\ =\ & \frac{1}{\Gamma(\frac{md_k-\alpha}{2})}\int_0^\infty s^{(md_k-\alpha)/2-1}\prod_{j=1}^m \left(\int_{\mathbb{R}^d} f_j(y_j)\Big(\int_{\mathbb{R}^d} e^{-s(\mathcal{A}(x,y_j,\eta))^2} d\mu_x(\eta)\Big)\, d\mu_k(y_j)\right) ds \end{aligned}$$

$$\leq \; C \int_{(\mathbb{R}^d)^m} \frac{f_1(y_1) f_2(y_2) \cdots f_m(y_m)}{\Big(d_G(x,y_1) + d_G(x,y_2) + \cdots + d_G(x,y_m)\Big)^{md_k - \alpha}} \, d\mu_k(y_1) \cdots d\mu_k(y_m)$$

□

The following is the main theorem in this section.

Theorem 5.2.2. *Suppose that* $1 < p_1, p_2, \cdots, p_m < \infty$*,* q *be such that* $1/m < p \leq q < \infty$ *and* $m\gamma_k < \alpha < md_k$*. Furthermore, let* $u, v_1, v_2, \cdots, v_m$ *be G-invariant weights such that the following two-weight conditions hold:*

(i) if $q > 1$*,*

$$\sup_{B \subset \mathbb{R}^d} r(B)^{\alpha - md_k} \; \mu_k(B)^{\frac{1}{q} + \frac{1}{p_1'} + \cdots + \frac{1}{p_m'}} \left(\frac{1}{\mu_k(B)} \int_B u^{tq} \, d\mu_k \right)^{1/tq}$$
$$\times \prod_{j=1}^m \left(\frac{1}{\mu_k(B)} \int_B v_j^{-tp_j'} d\mu_k \right)^{1/tp_j'} < \infty,$$

for some $t > 1$*;*

(ii) if $q \leq 1$*,*

$$\sup_{B \subset \mathbb{R}^d} r(B)^{\alpha - md_k} \; \mu_k(B)^{\frac{1}{q} + \frac{1}{p_1'} + \cdots + \frac{1}{p_m'}} \left(\frac{1}{\mu_k(B)} \int_B u^{q} \, d\mu_k \right)^{1/q}$$
$$\times \prod_{j=1}^m \left(\frac{1}{\mu_k(B)} \int_B v_j^{-tp_j'} d\mu_k \right)^{1/tp_j'} < \infty,$$

for some $t > 1$*.*

Then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, v_1^{p_1} d\mu_k) \times L^{p_2}(\mathbb{R}^d, v_2^{p_2} d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, v_m^{p_m} d\mu_k)$*, the following inequality holds:*

$$\left(\int_{\mathbb{R}^d} \left(\left| \mathcal{I}_\alpha^k \overrightarrow{f}(x) \right| u(x) \right)^q d\mu_k(x) \right)^{1/q} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^d} \left(|f_j(x)| v_j(x) \right)^{p_j} d\mu_k(x) \right)^{1/p_j}.$$

Proof. We start with the set of all dyadic cubes $\mathcal{D}$ in $\mathbb{R}^d$, i.e.,

$$\mathcal{D} = \Big\{ \prod_{j=1}^{d} \big[m_j 2^l, (m_j+1)2^l\big) : m_1, m_2, \cdots, m_d, l \in \mathbb{Z} \Big\}.$$

Let us define

$$\mathcal{D}^l = \Big\{ Q \in \mathcal{D} : \text{ side length of } Q = 2^l \Big\}.$$

Then the following properties hold.

(i) for any $l \in \mathbb{Z}$, $\mathbb{R}^d = \bigcup_{Q \in \mathcal{D}^l} Q$;

(ii) given $Q \in \mathcal{D}^l$ and $Q' \in \mathcal{D}^{l'}$ with $l < l'$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$;

(iii) for each $Q \in \mathcal{D}^l$ and each $l' > l$, there exists a unique $Q' \in \mathcal{D}^{l'}$ such that $Q \subset Q'$;

(iv) if $Q \in \mathcal{D}^l$, then diameter of $Q \leq 2^l\sqrt{d}$;

(v) for a given $Q \in \mathcal{D}^l$, let x_Q denotes the centre of Q, then $B(x_Q, 2^{l-1}) \subset Q$.

Clearly by the construction, the cubes in $\mathcal{D}^l$ are pairwise disjoint. For any $Q \in \mathcal{D}^l$, we define $B(Q) = B\big(x_Q, 2^{l+1}\sqrt{d}\big)$. It then follows that if $Q \in \mathcal{D}^l$, $Q' \in \mathcal{D}^{l'}$ and $Q \subset Q'$, then we have $B(Q) \subseteq B(Q')$. In fact, by property (ii), we get $l < l'$ and hence $y \in B(Q)$ implies that

$$|y - x_{Q'}| \leq |y - x_Q| + |x_Q - x_{Q'}| \leq 2^{l+1}\sqrt{d} + 2^{l'}\sqrt{d} \leq 2^{l'+1}\sqrt{d}.$$

Also observe that by property (iii), for every $Q \in \mathcal{D}^l$ we can find a unique $Q^* \in \mathcal{D}^{l+1}$ such that $Q \subset Q^*$. Since μ_k satisfies doubling condition and

$$B(x_Q, 2^{l-1}) \subseteq Q \subseteq B(Q) \subseteq B(Q^*) \subseteq 3B(Q),$$

so we have $\mu_k(Q) \sim \mu_k(B(Q)) \sim \mu_k(B(Q^*))$. It is also important to notice that if $Q \in \mathcal{D}^l$, there exists $s > l$ such that Q^{*s} is the cube which contains Q (by property (iii)), then $Q \subset Q^{*s}$.

Now we return to our operator $\mathcal{I}_\alpha^k$. By density argument it is enough to prove the Theorem for $\overrightarrow{f} \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$ with $f_j \geq 0$ for all $j = 1, 2, \cdots, m$.

As usual, our next aim is to discretize $\mathcal{I}_\alpha^k$. For that we take $(x, y_1, y_2, \cdots y_m) \in \left(\mathbb{R}^d\right)^{m+1}$ and $l \in \mathbb{Z}$ such that

$$2^{l-1} \leq d_G(x, y_1) + d_G(x, y_2) + \cdots d_G(x, y_m) \leq 2^l.$$

Then we can find a $Q \in \mathcal{D}^l$ with $x \in Q$. Now as diameter of $Q \leq 2^l\sqrt{d}$,

$$d_G(x_Q, y_j) \leq d_G(x, x_Q) + d_G(x, y_j) \leq 2^l\sqrt{d} + 2^l \leq 2^{l+1}\sqrt{d}.$$

This implies $(y_1, y_2, \cdots y_m) \in (\mathcal{O}(B(Q)))^m$ with

$$d_G(x, y_1) + d_G(x, y_2) + \cdots d_G(x, y_m) \geq 2^{l-1} = r(B(Q))/4\sqrt{d}.$$

Hence, we get for $x \in Q$ and $(y_1, y_2, \cdots, y_m) \in \mathcal{O}(B(Q)) \times \mathcal{O}(B(Q)) \times \cdots \times \mathcal{O}(B(Q))$,

$$\begin{aligned}
& \frac{1}{\Big(d_G(x, y_1) + d_G(x, y_2) + \cdots + d_G(x, y_m)\Big)^{md_k - \alpha}} \\
\leq\ & Cr(B(Q))^{\alpha - md_k} \chi_Q(x) \prod_{j=1}^m \chi_{\mathcal{O}(B(Q))}(y_j) \\
\leq\ & C \sum_{Q \in \mathcal{D}} r(B(Q))^{\alpha - md_k} \chi_Q(x) \prod_{j=1}^m \chi_{\mathcal{O}(B(Q))}(y_j)
\end{aligned}$$

Then from Lemma 5.2.1, we have

$$\mathcal{I}_\alpha^k \overrightarrow{f}(x) \leq C \sum_{Q \in \mathcal{D}} r(B(Q))^{\alpha - md_k} \left(\prod_{j=1}^m \int_{\mathcal{O}(B(Q))} f_j(y_j)\, d\mu_k(y_j) \right) \chi_Q(x)$$

Case 1. $q > 1$

As $\mathcal{I}_\alpha^k$ is a positive operator, it is enough to prove for all $g \in L^{q'}(\mathbb{R}^d, d\mu_k)$, with $g \geq 0$,

$$\int_{\mathbb{R}^d} \mathcal{I}_\alpha^k \overrightarrow{f}(x) u(x) g(x) d\mu_k(x)$$

$$\leq C \left(\int_{\mathbb{R}^d} g(x)^{q'} d\mu_k(x) \right)^{1/q'} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^d} (f_j(x)v_j(x))^{p_j} \, d\mu_k(x) \right)^{1/p_j}.$$

From above, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathcal{I}_\alpha^k \overrightarrow{f}(x) u(x) g(x) d\mu_k(x) \\
\leq\ & C \sum_{Q\in\mathcal{D}} r(B(Q))^{\alpha - md_k} \int_Q g(x)u(x)\, d\mu_k(x) \left(\prod_{j=1}^{m} \int_{\bigcup_{\sigma\in G} \sigma(B(Q))} f_j(y_j)\, d\mu_k(y_j) \right) \\
\leq\ & C \sum_{\substack{(n_1,n_2,\cdots,n_m) \\ \sigma_{n_i}\in G}} \sum_{Q\in\mathcal{D}} r(B(Q))^{\alpha - md_k} \int_Q g(x)u(x)\, d\mu_k(x) \left(\prod_{j=1}^{m} \int_{\sigma_{n_j}(B(Q))} f_j(y_j)\, d\mu_k(y_j) \right) \\
\leq\ & C \sum_{\substack{(n_1,n_2,\cdots,n_m) \\ \sigma_{n_i}\in G}} \sum_{Q\in\mathcal{D}} r(B(Q))^{\alpha - md_k} \int_Q g(x)u(x)\, d\mu_k(x) \left(\prod_{j=1}^{m} \int_{B(Q)} f_j \circ \sigma_{n_j}(y_j)\, d\mu_k(y_j) \right).
\end{aligned}$$

Since the first sum is over finite indices, we only show that for any $(\sigma_{n_1}, \sigma_{n_2}, \cdots, \sigma_{n_m}) \in G \times G \times \cdots \times G$,

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}} r(B(Q))^{\alpha - md_k} \int_Q g(x)u(x)\, d\mu_k(x) \left(\prod_{j=1}^{m} \int_{B(Q)} f_{\sigma_j}(y_j)\, d\mu_k(y_j) \right) \\
&\leq\ C \left(\int_{\mathbb{R}^d} g(x)^{q'} d\mu_k(x) \right)^{1/q'} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^d} (f_j(x)v_j(x))^{p_j}\, d\mu_k(x) \right)^{1/p_j}, \qquad (5.2.1)
\end{aligned}$$

where we have written f_{σ_j} in place of $f_j \circ \sigma_{n_j}$ for simplicity of notation.

Further, for $x \in \bigcup_{Q\in\mathcal{D}} Q$ we define

$$\mathcal{M}_{B(\mathcal{D})}^k \overrightarrow{h}(x) = \sup_{\substack{Q\in\mathcal{D} \\ x\in Q}} \prod_{j=1}^{m} \frac{1}{\mu_k(B(Q))} \int_{B(Q)} |h_j(y)|\, d\mu_k(y).$$

Let $a > 1$ be a constant to be defined later. Define

$$\mathcal{S}^l = \{ x \in \bigcup_{Q\in\mathcal{D}} Q : \mathcal{M}_{B(\mathcal{D})}^k \overrightarrow{f_\sigma}(x) > a^l \},$$

where

$$\overrightarrow{f_\sigma} = (f_{\sigma_1}, f_{\sigma_2}, \cdots, f_{\sigma_m}). \tag{5.2.2}$$

If $\mathcal{S}^l \neq \emptyset$, then there exists a cube $Q \in \mathcal{D}$ with $x \in Q$ and

$$\prod_{j=1}^{m} \frac{1}{\mu_k(B(Q))} \int_{B(Q)} f_{\sigma_j}(y)\, d\mu_k(y) > a^l. \tag{5.2.3}$$

Thus, we get $Q \subseteq \mathcal{S}^l$. Now using the facts that dyadic cubes in $\mathcal{D}$ are nested and $\int\limits_{\mathbb{R}^d} f_{\sigma_j}(y) d\mu_k(y) < \infty$, we can write $\mathcal{S}^l = \bigcup\limits_s Q_{l,s}$, where for each l the cubes $Q_{l,s} \in \mathcal{D}$ are maximal, disjoint that satisfy (5.2.3). Now if we take a to be sufficiently large, then by maximality of $Q_{l,s}$

$$\begin{aligned} a^l &< \prod_{j=1}^{m} \frac{1}{\mu_k(B(Q_{l,s}))} \int_{B(Q_{l,s})} f_{\sigma_j}(y)\, d\mu_k(y) \\ &\leq C \prod_{j=1}^{m} \frac{1}{\mu_k(B(Q^*_{l,s}))} \int_{B(Q^*_{l,s})} f_{\sigma_j}(y)\, d\mu_k(y) \\ &\leq Ca^l \leq a^{l+1}. \end{aligned} \tag{5.2.4}$$

Next we compute the part of $Q_{l,s}$ inside $\mathcal{S}^{l+1}$. Take $x \in Q_{l,s} \cap \mathcal{S}^{l+1}$, then

$$\mathcal{M}^k_{B(\mathcal{D})} \overrightarrow{f_\sigma}(x) = \sup_{\substack{P \in \mathcal{D} \\ x \in P}} \prod_{j=1}^{m} \frac{1}{\mu_k(B(P))} \int_{B(P)} f_{\sigma_j}(y)\, d\mu_k(y) > a^{l+1}.$$

But the nested property of dyadic cubes and maximality of $Q_{l,s}$ implies that

$$\prod_{j=1}^{m} \frac{1}{\mu_k(B(P))} \int_{B(P)} f_{\sigma_j}(y)\, d\mu_k(y) \leq a^l,\ \forall P \supset Q_{l,s} \text{ such that } x \in P.$$

Consequently, we have

$$\begin{aligned} a^{l+1} < \mathcal{M}^k_{B(\mathcal{D})} \overrightarrow{f_\sigma}(x) &= \sup_{\substack{P \in \mathcal{D} \\ x \in P \subseteq Q_{l,s}}} \prod_{j=1}^{m} \frac{1}{\mu_k(B(P))} \int_{B(P)} f_{\sigma_j}(y)\, d\mu_k(y) \\ &\leq \sup_{\substack{P \in \mathcal{D} \\ x \in P}} \prod_{j=1}^{m} \frac{1}{\mu_k(B(P))} \int_{B(P)} f_{\sigma_j} \chi_{B(Q_{l,s})}(y)\, d\mu_k(y). \end{aligned}$$

From this, it follows that if $x \in Q_{l,s}$ and $\mathcal{M}^k_{B(\mathcal{D})} \overrightarrow{f_\sigma}(x) > a^{l+1}$, then

$$\mathcal{M}^k_{B(\mathcal{D})} \left(f_{\sigma_1} \chi_{B(Q_{l,s})}, f_{\sigma_2} \chi_{B(Q_{l,s})}, \cdots, f_{\sigma_m} \chi_{B(Q_{l,s})} \right)(x) > a^{l+1}.$$

Then from Theorem 2.3.2 and (5.2.4), we have

$$\begin{aligned}
&\mu_k(Q_{l,s}\cap\mathcal{S}^{l+1})\\
=\;&\mu_k\left(\{x\in Q_{l,s}:\mathcal{M}^k_{B(\mathcal{D})}\overrightarrow{f_\sigma}(x)>a^{l+1}\}\right)\\
\leq\;&\mu_k\left(\{x\in Q_{l,s}:\mathcal{M}^k_{HL}\left(f_{\sigma_1}\chi_{B(Q_{l,s})},f_{\sigma_2}\chi_{B(Q_{l,s})},\cdots,f_{\sigma_m}\chi_{B(Q_{l,s})}\right)(x)>a^{l+1}\}\right)\\
\leq\;&\left(\frac{C_{\mathcal{M}}}{a^{l+1}}\prod_{j=1}^m\int_{B(Q_{l,s})}f_{\sigma_j}(y)\,d\mu_k(y)\right)^{1/m}\\
\leq\;&\mu_k(B(Q_{l,s}))\left(\frac{C_{\mathcal{M}}}{a^{l+1}}\prod_{j=1}^m\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}f_{\sigma_j}(y)\,d\mu_k(y)\right)^{1/m}\\
\leq\;&C_{\mathcal{M}}\mu_k(B(Q_{l,s}))\left(\frac{1}{a}\right)^{1/m}\leq C_{\mathcal{M}}\mu_k(Q_{l,s})\left(\frac{1}{a}\right)^{1/m}\leq\theta\mu_k(Q_{l,s}),
\end{aligned}$$

where the constant θ can be taken less than one by choosing a large enough.

Let $E_{l,s}=Q_{l,s}\setminus\mathcal{S}^{l+1}$, then $\{E_{l,s}\}_{l,s}$ is a disjoint family of sets such that $\mu_k(E_{l,s})\geq\gamma\mu_k(Q_{l,s})$, for some $0<\gamma<1$. Next define

$$\mathcal{C}^l=\{Q\in\mathcal{D}:a^l<\prod_{j=1}^m\frac{1}{\mu_k(B(Q))}\int_{B(Q)}f_{\sigma_j}(y)d\mu(y)\leq a^{l+1}\}.$$

Notice that for any s, $Q_{l,s}\in\mathcal{C}^l$ and by maximality of $Q_{l,s}$ if $Q\in\mathcal{C}^l$ then $Q\subseteq Q_{l,s}$ for some s. Then

$$\begin{aligned}
\text{LHS of (5.2.1)}\;\leq\;&\sum_{l\in\mathbb{Z}}\sum_{Q\in\mathcal{C}^l}\mu_k(B(Q))^m r(B(Q))^{\alpha-md_k}\int_Q g(x)u(x)\,d\mu_k(x)\\
&\times\left(\prod_{j=1}^m\frac{1}{\mu_k(B(Q))}\int_{B(Q)}f_{\sigma_j}(y_j)\,d\mu_k(y_j)\right)\\
\leq\;&\sum_{l\in\mathbb{Z}}a^{l+1}\sum_s\sum_{\substack{Q\in\mathcal{C}^l\\ Q\subseteq Q_{l,s}}}\mu_k(B(Q))^m r(B(Q))^{\alpha-md_k}\int_Q g(x)u(x)\,d\mu_k(x).
\end{aligned}$$

Now for any $Q,\ Q_0\in\mathcal{D}$ with $Q\subseteq Q_0$, by (2.1.3), we get that $\frac{\mu_k(B(Q))}{\mu_k(B((Q_0))}\leq C\left(\frac{r(B(Q))}{r(B(Q_0))}\right)^d$, hence

$$\frac{\mu_k(B(Q))^m r(B(Q))^{\alpha-md_k}}{\mu_k(B(Q_0))^m r(B(Q_0))^{\alpha-md_k}} \leq C\left(\frac{r(B(Q))}{r(B(Q_0))}\right)^{\alpha-m\gamma_k}.$$

Since $\alpha - m\gamma_k > 0$, from above, (5.2.4) and applying Hölder's inequality, we have

$$\begin{aligned}
&\text{LHS of (5.2.1)}\\
\leq\ & a\sum_{l\in\mathbb{Z}} a^l \sum_s \mu_k(B(Q_{l,s}))^m r(B(Q_{l,s}))^{\alpha-md_k}\int_{Q_{l,s}} g(x)u(x)\,d\mu_k(x)\\
\leq\ & C\sum_{l,s}\mu_k(B(Q_{l,s}))^m r(B(Q_{l,s}))^{\alpha-md_k}\\
&\times\left(\prod_{j=1}^m \frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})} f_{\sigma_j}(y)v_j(y)v_j^{-1}(y)d\mu_k(y)\right)\\
&\times\left(\frac{1}{\mu_k(Q_{l,s})}\int_{Q_{l,s}} g(x)u(x)\,d\mu_k(x)\right)\mu_k(Q_{l,s})\\
\leq\ & C\sum_{l,s}\mu_k(B(Q_{l,s}))^m r(B(Q_{l,s}))^{\alpha-md_k}\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})} v_j^{-tp_j'}d\mu_k\right)^{1/tp_j'}\\
&\times\left(\frac{1}{\mu_k(Q_{l,s})}\int_{Q_{l,s}} u^{tq}\,d\mu_k\right)^{1/tq}\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}(f_{\sigma_j}v_j)^{(tp_j')'}d\mu_k\right)^{1/(tp_j')'}\\
&\times\left(\frac{1}{\mu_k(Q_{l,s})}\int_{Q_{l,s}} g^{(tq)'}d\mu_k\right)^{1/(tq)'}\mu_k(Q_{l,s}).
\end{aligned}$$

Using the two-weight condition and replacing $Q_{l,s}$ with the disjoint sets $E_{l,s}$,

$$\begin{aligned}
&\text{LHS of (5.2.1)}\\
\leq\ & C\,[u,\overrightarrow{v}]\sum_{l,s}\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}(f_{\sigma_j}v_j)^{(tp_j')'}d\mu_k\right)^{1/(tp_j')'}\\
&\times\left(\frac{1}{\mu_k(Q_{l,s})}\int_{Q_{l,s}} g^{(tq)'}d\mu_k\right)^{1/(tq)'}\mu_k(Q_{l,s})^{1/q'+1/p}\\
\leq\ & C\,[u,\overrightarrow{v}]\left(\sum_{l,s}\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}(f_{\sigma_j}v_j)^{(tp_j')'}d\mu_k\right)^{q/(tp_j')'}\mu_k(Q_{l,s})^{q/p}\right)^{1/q}\\
&\times\left(\sum_{l,s}\left(\frac{1}{\mu_k(Q_{l,s})}\int_{Q_{l,s}} g^{(tq)'}d\mu_k\right)^{q'/(tq)'}\mu_k(Q_{l,s})\right)^{1/q'}
\end{aligned}$$

$$\leq \; C\,[u,\overrightarrow{v}]\left(\sum_{l,s}\prod_{j=1}^{m}\Big(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}(f_{\sigma_j}v_j)^{(tp_j')'}d\mu_k\Big)^{p/(tp_j')'}\mu_k(E_{l,s})\right)^{1/p}$$
$$\times\left(\sum_{l,s}\Big(\frac{1}{\mu_k(Q_{l,s})}\int_{Q_{l,s}}g^{(tq)'}d\mu_k\Big)^{q'/(tq)'}\mu_k(E_{l,s})\right)^{1/q'},$$

where $[u,\overrightarrow{v}]$ denotes the smallest constant in two-weight condition.

Since $x\in E_{l,s}$ implies that $x\in Q_{l,s}$, so from above we have

$$\begin{aligned}
&\text{LHS of (5.2.1)}\\
&\leq \; C\,[u,\overrightarrow{v}]\prod_{j=1}^{m}\left(\sum_{l,s}\int_{E_{l,s}}M^k_{(tp_j')',HL}(f_{\sigma_j}v_j)(x)^{p_j}\,d\mu_k(x)\right)^{1/p_j}\\
&\quad\times\left(\sum_{l,s}\int_{E_{l,r}}M^k_{(tq)',HL}(g)(x)^{q'}\,d\mu_k(x)\right)^{1/q'}\\
&\leq \; C\,[u,\overrightarrow{v}]\prod_{j=1}^{m}\left(\int_{\mathbb{R}^d}M^k_{(tp_j')',HL}(f_{\sigma_j}v_j)(x)^{p_j}\,d\mu_k(x)\right)^{1/p_j}\\
&\quad\times\left(\int_{\mathbb{R}^d}M^k_{(tq)',HL}(g)(x)^{q'}\,d\mu_k(x)\right)^{1/q'}.
\end{aligned}$$

Now, Theorem 2.3.2 and G-invariance of the weights concludes the proof of (5.2.1).

Case 2. $q\leq 1$

By same estimates as in previous case and using $q\leq 1$, we get

$$\begin{aligned}
\mathcal{I}^k_\alpha\overrightarrow{f}(x)^q \;&\leq\; C\sum_{Q\in\mathcal{D}}\left(r(B(Q))^{\alpha-md_k}\prod_{j=1}^{m}\int_{\mathcal{O}(B(Q))}f_j(y_j)\,d\mu_k(y_j)\right)^q\chi_Q(x)\\
&\leq\; C\sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_i}\in G}}\sum_{Q\in\mathcal{D}}\left(r(B(Q))^{\alpha-md_k}\prod_{j=1}^{m}\int_{B(Q)}f_j\circ\sigma_{n_j}(y_j)\,d\mu_k(y_j)\right)^q\chi_Q(x)
\end{aligned}$$

Hence integrating,

$$\int_{\mathbb{R}^d}\left(\mathcal{I}^k_\alpha\overrightarrow{f}u\right)^q d\mu_k \;\leq\; C\sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_i}\in G}}\sum_{Q\in\mathcal{D}}\left(r(B(Q))^{\alpha-md_k}\prod_{j=1}^{m}\int_{B(Q)}f_j\circ\sigma_{n_j}(y_j)\,d\mu_k(y_j)\right)^q$$

$$\times \int_Q u(x)^q \, d\mu_k(x).$$

As before, since the first sum is over finite indices, we only show that for any $(\sigma_1, \sigma_2, \cdots, \sigma_m) \in G \times G \times \cdots \times G$

$$\begin{aligned} & \sum_{Q \in \mathcal{D}} \left(r(B(Q))^{\alpha - md_k} \prod_{j=1}^m \int_{B(Q)} f_{\sigma_j}(y_j) \, d\mu_k(y_j) \right)^q \int_Q u(x)^q \, d\mu_k(x) \\ \leq \; & C \prod_{j=1}^m \left(\int_{\mathbb{R}^d} (f_j(x) v_j(x))^{p_j} \, d\mu_k(x) \right)^{\frac{q}{p_j}}, \end{aligned} \tag{5.2.5}$$

where f_{σ_j} is as before. Now

$$\begin{aligned} & \text{LHS of (5.2.5)} \\ \leq \; & \sum_{Q \in \mathcal{D}} \left(\mu_k(B(Q))^m r(B(Q))^{\alpha - md_k} \prod_{j=1}^m \frac{1}{\mu_k(B(Q))} \int_{B(Q)} f_{\sigma_j}(y_j) \, d\mu_k(y_j) \right)^q \\ & \times \int_Q u(x)^q \, d\mu_k(x) \\ \leq \; & \sum_l a^{(l+1)q} \sum_s \sum_{\substack{Q \in \mathcal{C}^l \\ Q \subseteq Q_{l,s}}} \mu_k(B(Q))^{mq} r(B(Q))^{\alpha q - md_k q} \int_Q u(x)^q \, d\mu_k(x) \\ \leq \; & C \sum_{l,s} \mu_k(B(Q_{l,s}))^{mq} r(B(Q_{l,s}))^{\alpha q - md_k q} \left(\prod_{j=1}^m \frac{1}{\mu_k(B(Q_{l,s}))} \int_{B(Q_{l,s})} f_{\sigma_j} \, d\mu_k \right)^q \\ & \times \int_{Q_{l,s}} u(x)^q \, d\mu_k(x) \\ \leq \; & C \sum_{l,s} \left\{ \mu_k(B(Q_{l,s}))^m r(B(Q_{l,s}))^{\alpha - md_k} \left(\prod_{j=1}^m \frac{1}{\mu_k(B(Q_{l,s}))} \int_{B(Q_{l,s})} f_{\sigma_j} v_j v_j^{-1} \, d\mu_k \right) \right. \\ & \left. \times \left(\frac{1}{\mu_k(Q_{l,s})} \int_{Q_{l,s}} u^q \, d\mu_k \right)^{1/q} \right\}^q \times \mu_k(Q_{l,s}). \end{aligned}$$

Once again applying Hölder's inequality, using the two-weight condition and replacing $Q_{l,s}$ with the disjoint sets $E_{l,s}$, we have

LHS of (5.2.5)

$$
\begin{aligned}
&\le C\sum_{l,s}\Bigg\{\mu_k(B(Q_{l,s}))^m r(B(Q_{l,s}))^{\alpha-md_k}\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})} v^{-tp_j'}d\mu_k\right)^{1/tp_j'}\\
&\quad\times\left(\frac{1}{\mu_k(Q_{l,s})}\int_{Q_{l,s}} u^q\,d\mu_k\right)^{1/q}\Bigg\}^q\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}(f_{\sigma_j}v_j)^{(tp_j')'}d\mu_k\right)^{q/(tp_j')'}\\
&\quad\times\mu_k(Q_{l,s})\\
&\le C\,[u,\vec{v}]^q\sum_{l,s}\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}(f_{\sigma_j}v_j)^{(tp_j')'}d\mu_k\right)^{q/(tp_j')'}\mu_k(Q_{l,s})^{q/p}\\
&\le C\,[u,\vec{v}]^q\Bigg\{\sum_{l,s}\prod_{j=1}^m\left(\frac{1}{\mu_k(B(Q_{l,s}))}\int_{B(Q_{l,s})}(f_{\sigma_j}v_j)^{(tp_j')'}d\mu_k\right)^{p/(tp_j')'}\mu_k(E_{l,s})\Bigg\}^{q/p}.
\end{aligned}
$$

As $E_{l,s}\subseteq Q_{l,s}$, applying multilinear Hölder's inequality, using Theorem 2.3.2 and using the G-invariance of the weights, we get

$$
\begin{aligned}
\text{LHS of (5.2.5)} &\le C\,[u,\vec{v}]^q\prod_{j=1}^m\left(\int_{\mathbb{R}^d} M^k_{(tp_j')',HL}(f_{\sigma_j}v_j)(x)^{p_j}\right)^{q/p_j}\\
&\le C\,[u,\vec{v}]^q\prod_{j=1}^m\left(\int_{\mathbb{R}^d}(f_j(x)v_j(x))^{p_j}\,d\mu_k(x)\right)^{q/p_j}.
\end{aligned}
$$

This completes the proof. □

Remark 5.2.3. In the two-weight case for $\mathcal{I}^k_\alpha$, we could not get the expected range $0<\alpha<md_k$, we only got the range $m\gamma_k<\alpha<md_k$. However, in the one-weight case we have the full range.

5.3 Two-weight Inequalities for Multilinear Fractional Maximal Operators

We first prove the following proposition for Dunkl translation of the balls $B(0,r)$.

Proposition 5.3.1. *[26, 30] For any $x,y\in\mathbb{R}^d$ and $r>0$, $\tau^k_x\chi_{B(0,r)}$ satisfies*

$$\tau^k_x\chi_{B(0,r)}(y)\le\frac{Cr^{d_k}}{\mu_k(B(x,r))}.$$

Proof. Let ϕ be a continuous radial function such that $0 \leq \phi \leq 1$ and $\phi(y) = \begin{cases} 1 & \text{if } |y| \leq r, \\ 0 & \text{if } |y| \geq 2r. \end{cases}$ Let $f(\cdot) = \phi(r\cdot)$. Then f satisfies

$$f(y) \leq \frac{C}{(1+|y|)^M}, \text{ where } M > d_k.$$

Recall the definition $f_t(x) = t^{-d_k} f(t^{-1}x)$. Then clearly f_r is continuous and is in $L^2(\mathbb{R}^d, d\mu_k)$. So applying the formula (2.1.6) and using [6, Proposition 3.1] with $t = r$, we get

$$\tau_x^k f_r(y) \leq \frac{C}{\mu_k(B(x,r))}.$$

Now applying the scaling $\tau_x^k(f_t) = (\tau_{t^{-1}x}^k f)_t$, we get

$$\tau_x^k \phi(y) \leq \frac{C r^{d_k}}{\mu_k(B(x,r))}.$$

Since τ_x^k is positive on bounded, radial functions in $L^1(\mathbb{R}^d, d\mu_k)$ and $\chi_{B(0,r)}(\cdot) \leq \phi(\cdot)$, we have

$$\tau_x^k \chi_{B(0,r)}(y) \leq \tau_x^k \phi(y) \leq \frac{C r^{d_k}}{\mu_k(B(x,r))}.$$

□

Remark 5.3.2. Here we have used [26, Theorem 1.7] for the estimate of the support of $\tau_x^k \chi_{B(0,r)}(\cdot)$. This can also be obtained using the formula (2.1.6) together with (2.1.7) (see [26, Remark 2.11] for details).

For locally integrable functions f_j on $\mathbb{R}^d$, define the multilinear homogeneous factional maximal function $\widetilde{\mathcal{M}}_\alpha^k$ by

$$\widetilde{\mathcal{M}}_\alpha^k \overrightarrow{f}(x) = \sup_{\substack{B \subset \mathbb{R}^d \\ x \in B}} \prod_{j=1}^m \frac{1}{\mu_k(B)^{1-\alpha/md_k}} \int_B |f_j(y)| \, d\mu_k(y), \text{ where } 0 \leq \alpha < md_k.$$

In the next lemma, we establish a relationship between the two types of fractional maximal operators $\mathcal{M}_\alpha^k$ and $\widetilde{\mathcal{M}}_\alpha^k$.

Lemma 5.3.3. *For* $\overrightarrow{f} \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{M}_\alpha^k \overrightarrow{f}(x) \leq C \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \widetilde{\mathcal{M}}_\alpha^k \left(f_1 \circ \sigma_{n_1}, f_2 \circ \sigma_{n_2}, \cdots, f_m \circ \sigma_{n_m}\right)(x).$$

Proof. Fix $x \in \mathbb{R}^d$ and $r > 0$. Then using [26, Theorem 1.7] and from Proposition 5.3.1 and the facts that $\alpha < md_k$ and $r^{d_k} \leq C\mu_k(B(x,r))$; for any $1 \leq j \leq m$,

$$\begin{aligned}
& r^{\alpha/m-d_k} \left| \int_{\mathbb{R}^d} f_j(y_j)\tau_x^k \chi_{B(0,r)}(-y_j) d\mu_k(y_j) \right| \\
\leq\ & r^{\alpha/m-d_k} \int_{\mathbb{R}^d} |f_j(y_j)| \tau_x^k \chi_{B(0,r)}(-y_j)\, d\mu_k(y_j) \\
\leq\ & C\, r^{\alpha/m-d_k} \int_{\bigcup_{\sigma\in G} \sigma(B(x,r))} |f_j(y_j)| \left(\frac{r^{d_k}}{\mu_k(B(x,r))}\right)^{1-\alpha/md_k} d\mu_k(y_j) \\
\leq\ & C \sum_{\sigma\in G} \frac{1}{(\mu_k(B(x,r)))^{1-\alpha/md_k}} \int_{B(x,r)} |f_j \circ \sigma(y_j)|\, d\mu_k(y_j).
\end{aligned}$$

Thus, from above we get

$$\begin{aligned}
& \prod_{j=1}^m r^{\frac{\alpha}{m}-d_k} \left| \int_{\mathbb{R}^d} f_j(y_j)\tau_x^k \chi_{B(0,r)}(-y_j)\, d\mu_k(y_j) \right| \\
\leq\ & C \prod_{j=1}^m \left(\sum_{\sigma\in G} \frac{1}{(\mu_k(B(x,r)))^{1-\alpha/md_k}} \int_{B(x,r)} |f_j \circ \sigma(y_j)|\, d\mu_k(y_j) \right) \\
=\ & C \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \left(\prod_{j=1}^m \frac{1}{(\mu_k(B(x,r)))^{1-\alpha/md_k}} \int_{B(x,r)} |f_j \circ \sigma_{n_j}(y_j)|\, d\mu_k(y_j) \right) \\
\leq\ & C \sum_{\substack{(n_1,n_2,\cdots,n_m)\\ \sigma_{n_s}\in G}} \widetilde{\mathcal{M}}_\alpha^k \left(f_1 \circ \sigma_{n_1}, f_2 \circ \sigma_{n_2}, \cdots, f_m \circ \sigma_{n_m}\right)(x).
\end{aligned}$$

Taking supremum over all $r > 0$ completes the proof. □

In this section, the main result regarding two-weight estimates for $\mathcal{M}_\alpha^k$ is the following theorem.

Theorem 5.3.4. *Suppose that $1 < p_1, p_2, \cdots, p_m < \infty$, q be such that $1/m < p \le q < \infty$ and $0 \le \alpha < md_k$. Furthermore, let $u, v_1, v_2, \cdots, v_m$ be G-invariant weights such that the following two-weight condition holds:*

$$\sup_{B\subset\mathbb{R}^d} r(B)^{\alpha-md_k}\, \mu_k(B)^{\frac{1}{q}+\frac{1}{p_1'}+\cdots+\frac{1}{p_m'}} \left(\frac{1}{\mu_k(B)}\int_B u^q\, d\mu_k\right)^{1/q} \times \prod_{j=1}^{m}\left(\frac{1}{\mu_k(B)}\int_B v_j^{-tp_j'}\, d\mu_k\right)^{1/tp_j'} < \infty,$$

for some $t > 1$. Then for all $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, v_1^{p_1} d\mu_k)\times L^{p_2}(\mathbb{R}^d, v_2^{p_2} d\mu_k)\times\cdots\times L^{p_m}(\mathbb{R}^d, v_m^{p_m} d\mu_k)$, the following inequality holds:

$$\left(\int_{\mathbb{R}^d}\left(\mathcal{M}_\alpha^k\overrightarrow{f}(x)u(x)\right)^q d\mu_k(x)\right)^{1/q} \le C\prod_{j=1}^{m}\left(\int_{\mathbb{R}^d}\left(|f_j(x)|v_j(x)\right)^{p_j} d\mu_k(x)\right)^{1/p_j}.$$

Proof. In view of Lemma 5.3.3 to prove Theorem 5.3.4, it is enough to prove that for any $(\sigma_{n_1}, \sigma_{n_2}, \cdots, \sigma_{n_m}) \in G\times G\times\cdots\times G$,

$$\left(\int_{\mathbb{R}^d}\left(\widetilde{\mathcal{M}}_\alpha^k\overrightarrow{f_\sigma}(x)u(x)\right)^q d\mu_k(x)\right)^{1/q} \le C\prod_{j=1}^{m}\left(\int_{\mathbb{R}^d}\left(|f_j(x)|v_j(x)\right)^{p_j} d\mu_k(x)\right)^{1/p_j}, \quad (5.3.1)$$

where $\overrightarrow{f_\sigma}$ is as in (5.2.2).

For any $N \in \mathbb{N}$, define

$$\widetilde{\mathcal{M}}_{\alpha,N}^k\overrightarrow{f}(x) = \sup_{\substack{B\subset\mathbb{R}^d\\ x\in B,\ r(B)\le N}} \prod_{j=1}^{m}\frac{1}{\mu_k(B)^{1-\alpha/md_k}}\int_B |f_j(y)|\, d\mu_k(y).$$

Then it suffices to prove (5.3.1) holds for $\widetilde{\mathcal{M}}_{\alpha,N}^k$ (independent of N) instead of $\widetilde{\mathcal{M}}_\alpha^k$ (by using monotone convergence theorem as $N\to\infty$). Further, we may assume that each f_j is non-negative bounded function with compact support.

For $l \in \mathbb{Z}$, let

$$\Omega_l = \left\{x\in\mathbb{R}^d : \widetilde{\mathcal{M}}_{\alpha,N}^k\overrightarrow{f_\sigma} > 2^l\right\}.$$

Take $x \in \Omega_l \setminus \Omega_{l+1}$, then we can find a ball B_x containing x with $r(B_x) < N$ such that

$$2^l < \prod_{j=1}^{m} \frac{1}{\mu_k(B_x)^{1-\alpha/md_k}} \int_{B_x} f_{\sigma_j}(y)\, d\mu_k(y) \le 2^{l+1}.$$

Note that here $B_x \subseteq \Omega_l \setminus \Omega_{l+1}$ so that we can write $\Omega_l \setminus \Omega_{l+1} = \bigcup_{x\in\Omega_l\setminus\Omega_{l+1}} B_x$. Then using basic covering lemma for metric spaces [40, p. 2] for the family $\Big\{B_x : x \in \Omega_l \setminus \Omega_{l+1}\Big\}$, we get a pairwise disjoint family $\{B^l_\beta\}_{\beta\in\mathcal{B}}$ of balls each inside $\Omega_l \setminus \Omega_{l+1}$ such that

$$\Omega_l \setminus \Omega_{l+1} \subseteq \bigcup_{\beta\in\mathcal{B}} 5B^l_\beta.$$

Also, we have

$$2^l < \prod_{j=1}^{m} \frac{1}{\mu_k(B^l_\beta)^{1-\alpha/md_k}} \int_{B^l_\beta} f_{\sigma_j}(y)\, d\mu_k(y) \le 2^{l+1}. \tag{5.3.2}$$

Again, note that $\{B^l_\beta : \beta \in \mathcal{B},\ l \in \mathbb{Z}\}$ is also a pairwise disjoint family.

Let $r(B)$ denotes the radius of the ball B. Now from the fact that $\mu_k(B) \ge C\, r(B)^{d_k}$ and using the estimates above, we get

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} \Big(\widetilde{\mathcal{M}}^k_{\alpha,N}\overrightarrow{f_\sigma}(x)u(x)\Big)^q d\mu_k(x)\right)^{1/q} \\
= &\left(\sum_{l\in\mathbb{Z}} \int_{\Omega_l\setminus\Omega_{l+1}} \Big(\widetilde{\mathcal{M}}^k_{\alpha,N}\overrightarrow{f_\sigma}(x)u(x)\Big)^q d\mu_k(x)\right)^{1/q} \\
\le &\left(\sum_{l\in\mathbb{Z}} 2^{(l+1)q} \int_{\Omega_l\setminus\Omega_{l+1}} u(x)^q\, d\mu_k(x)\right)^{1/q} \\
\le\ & 2\Bigg\{\sum_{l\in\mathbb{Z}}\sum_{\beta\in\mathcal{B}} \Big(\int_{5B^l_\beta} u(x)^q\, d\mu_k(x)\Big) \\
&\times\Big(\prod_{j=1}^{m} \frac{1}{\mu_k(B^l_\beta)^{1-\alpha/md_k}} \int_{B^l_\beta} f_{\sigma_j}(y)\, d\mu_k(y)\Big)^q\Bigg\}^{1/q} \qquad (5.3.3)\\
\le\ & C\Bigg\{\sum_{l\in\mathbb{Z}}\sum_{\beta\in\mathcal{B}} \Big(\int_{5B^l_\beta} u(x)^q\, d\mu_k(x)\Big) r\left(B^l_\beta\right)^{q\alpha-qmd_k} \\
&\times\Big(\prod_{j=1}^{m} \int_{B^l_\beta} f_{\sigma_j}(y)\, d\mu_k(y)\Big)^q\Bigg\}^{1/q}. \qquad (5.3.4)
\end{aligned}$$

Using Hölder's inequality and the two-weight condition, we have

$$\begin{aligned}
&\Big(\int_{5B^l_\beta} u(x)^q\, d\mu_k(x)\Big) r\left(B^l_\beta\right)^{q\alpha - qmd_k}\Big(\prod_{j=1}^m \int_{B^l_\beta} f_{\sigma_j}(y)\, d\mu_k(y)\Big)^q \\
\leq\ & \left(\int\limits_{5B^l_\beta} u^q\, d\mu_k\right) r\left(B^l_\beta\right)^{q\alpha - qmd_k} \prod_{j=1}^m \left(\int_{B^l_\beta} (f_{\sigma_j} v_j)^{(tp'_j)'}\, d\mu_k\right)^{q/(tp'_j)'} \\
& \times \prod_{j=1}^m \left(\int_{B^l_\beta} v_j^{-tp'_j}\, d\mu_k\right)^{q/tp'_j} \\
\leq\ & C\,[u, \overrightarrow{v}]^q \mu_k\left(B^l_\beta\right)^{q/p} \prod_{j=1}^m \left(\frac{1}{\mu_k\left(B^l_\beta\right)} \int_{B^l_\beta} (f_{\sigma_j} v_j)^{(tp'_j)'} d\mu_k\right)^{q/(tp'_j)'} . \qquad (5.3.5)
\end{aligned}$$

Now substituting (5.3.5) in (5.3.4) and using the fact that $p \leq q$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} \left(\widetilde{\mathcal{M}}^k_{\alpha,N} \overrightarrow{f_\sigma}(x) u(x)\right)^q d\mu_k(x)\right)^{1/q} \\
\leq\ & C\,[u, \overrightarrow{v}] \left\{ \sum_{l\in\mathbb{Z}} \sum_{\beta\in\mathcal{B}} \left(\mu_k\left(B^l_\beta\right)^{1/p} \prod_{j=1}^m \Big(\frac{1}{\mu_k\left(B^l_\beta\right)} \int_{B^l_\beta} (f_{\sigma_j} v_j)^{(tp'_j)'} d\mu_k\Big)^{1/(tp'_j)'} \right)^q \right\}^{1/q} \\
\leq\ & C\,[u, \overrightarrow{v}] \left\{ \sum_{l\in\mathbb{Z}} \sum_{\beta\in\mathcal{B}} \prod_{j=1}^m \Big(\frac{1}{\mu_k\left(B^l_\beta\right)} \int_{B^l_\beta} (f_{\sigma_j} v_j)^{(tp'_j)'} d\mu_k\Big)^{p/(tp'_j)'} \mu_k\left(B^l_\beta\right) \right\}^{1/p} . \qquad (5.3.6)
\end{aligned}$$

Recalling the definition of M^k_{HL} and applying multilinear Hölder's inequality, we get

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} \left(\widetilde{\mathcal{M}}^k_{\alpha,N} \overrightarrow{f_\sigma}(x) u(x)\right)^q d\mu_k(x)\right)^{1/q} \\
\leq\ & C\,[u, \overrightarrow{v}] \left\{ \sum_{l\in\mathbb{Z}} \sum_{\beta\in\mathcal{B}} \int_{B^l_\beta} \prod_{j=1}^m \left(M^k_{HL}(f_{\sigma_j} v_j)^{(rp'_j)'}(x)\right)^{p/(tp'_j)'} d\mu_k(x) \right\}^{1/p} \\
\leq\ & C\,[u, \overrightarrow{v}] \left\{ \int_{\mathbb{R}^d} \prod_{j=1}^m \left(M^k_{HL}(f_{\sigma_j} v_j)^{(rp'_j)'}(x)\right)^{p/(tp'_j)'} d\mu_k(x) \right\}^{1/p} \\
\leq\ & C\,[u, \overrightarrow{v}] \prod_{j=1}^m \left\{ \int_{\mathbb{R}^d} \left(M^k_{HL}(f_{\sigma_j} v_j)^{(rp'_j)'}(x)\right)^{p_j/(tp'_j)'} d\mu_k(x) \right\}^{1/p_j} .
\end{aligned}$$

Since $p_j/(tp'_j)' > 1$, Theorem 2.3.2 yields

$$\left(\int_{\mathbb{R}^d} \left(\widetilde{\mathcal{M}}^k_{\alpha,N} \overrightarrow{f_\sigma}(x) u(x)\right)^q d\mu_k(x)\right)^{1/q} \leq C\,[u, \overrightarrow{v}] \prod_{j=1}^m \left(\int_{\mathbb{R}^d} \left(f_{\sigma_j}(x) v_j(x)\right)^{p_j} d\mu_k(x)\right)^{1/p_j} .$$

Finally G-invariance of the weights concludes the proof. $\square$

Remark 5.3.5. The conditions assumed in Theorem 5.2.2 and Theorem 5.3.4, may seem to be inappropriate as they involve both the volume and radius of the ball, but we want to point out that such conditions for two-weight inequalities is not new in the context of spaces of homogeneous type. This type of condition first appeared for the study of fractional type operators in spaces of homogeneous type in the paper of Sawyer and Wheeden [66, Theorem 3]. Later, other authors [50, 58] have also used analogous conditions involving both radius and volume of balls. These are our main motivation for proposing such type of conditions on the weights for the two-weight inequalities.

5.4 One-weight Inequalities for Multilinear Fractional Maximal Operators

In this section, we state and prove one-weight estimates for the operator $\mathcal{M}_\alpha^k$.

Theorem 5.4.1. *Suppose that* $1 < p_1, p_2, \cdots, p_m < \infty$, $0 \le \alpha < md_k$, $1/m < p < d_k/\alpha$ *and* q *be a number defined by* $1/q = 1/p - \alpha/d_k$. *Furthermore, let the vector weight* $\overrightarrow{w} = (w_1, w_2, \cdots, w_m) \in A^k_{\overrightarrow{P},q}$ *and each* w_j *is* G*-invariant. Then for all* $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1^{p_1} d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2^{p_2} d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m^{p_m} d\mu_k)$*, the following inequality holds:*

$$\left(\int_{\mathbb{R}^d} \left(\mathcal{M}_\alpha^k \overrightarrow{f}(x) \prod_{j=1}^m w_j(x) \right)^q d\mu_k(x) \right)^{1/q} \le C \prod_{j=1}^m \left(\int_{\mathbb{R}^d} (|f_j(x)| w_j(x))^{p_j} \, d\mu_k(x) \right)^{1/p_j}.$$

Proof. By same arguments as in the proof of Theorem 5.3.4 with $u = \prod_{j=1}^m w_j$ and $v_j = w_j$, in place of (5.3.3) we have

$$\left(\int_{\mathbb{R}^d} \left(\widetilde{\mathcal{M}}^k_{\alpha,N} \overrightarrow{f_\sigma}(x) \prod_{j=1}^m w_j(x) \right)^q d\mu_k(x) \right)^{1/q}$$

$$\leq C\Big\{\sum_{l\in\mathbb{Z}}\sum_{\beta\in\mathcal{B}}\Big(\int_{5B^l_\beta}\Big(\prod_{j=1}^m w_j(x)\Big)^q d\mu_k(x)\Big) \times\Big(\prod_{j=1}^m \frac{1}{\mu_k(B^l_\beta)^{1-\alpha/md_k}}\int_{B^l_\beta} f_{\sigma_j}(y)\,d\mu(y)\Big)^q\Big\}^{1/q}. \tag{5.4.1}$$

Using Hölder's inequality, Proposition 2.3.7 and $A^k_{\overrightarrow{P},q}$ condition, we have

$$\begin{aligned}
&\left(\int_{5B^l_\beta}\Big(\prod_{j=1}^m w_j(x)\Big)^q d\mu_k(x)\right)\mu_k\left(B^l_\beta\right)^{q\alpha/d_k-qm}\left(\prod_{j=1}^m\int_{B^l_\beta} f_{\sigma_j}(y)\,d\mu(y)\right)^q\\
\leq\;& \left(\int_{5B^l_\beta}\Big(\prod_{j=1}^m w_j\Big)^q d\mu_k\right)\mu_k\left(B^l_\beta\right)^{q\alpha/d_k-qm}\prod_{j=1}^m\left(\int_{B^l_\beta}(f_{\sigma_j}w_j)_j^{(tp'_j)'}d\mu_k\right)^{q/(tp'_j)'}\\
&\times\prod_{j=1}^m\left(\int_{B^l_\beta} w_j^{-tp'_j}d\mu_k\right)^{q/tp'_j}\\
\leq\;& C\,[\overrightarrow{w}]^q\mu_k\left(B^l_\beta\right)^{q/p}\prod_{j=1}^m\left(\frac{1}{\mu_k\left(B^l_\beta\right)}\int_{B^l_\beta}(f_{\sigma_j}w_j)_j^{(tp'_j)'}d\mu_k\right)^{q/(tp'_j)'},
\end{aligned}\tag{5.4.2}$$

where $[\overrightarrow{w}]$ denotes the smallest constant in one-weight condition.

Now substituting (5.4.2) in (5.4.1) and using the fact that $p\leq q$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d}\left(\widetilde{\mathcal{M}}^k_{\alpha,N}\overrightarrow{f_\sigma}(x)\prod_{j=1}^m w_j(x)\right)^q d\mu_k(x)\right)^{1/q}\\
\leq\;& C\,[\overrightarrow{w}]\left\{\sum_{l\in\mathbb{Z}}\sum_{\beta\in\mathcal{B}}\left(\mu_k\left(B^l_\beta\right)^{1/p}\prod_{j=1}^m\Big(\frac{1}{\mu_k\left(B^l_\beta\right)}\int_{B^l_\beta}(f_{\sigma_j}w_j)_j^{(tp'_j)'}d\mu_k\Big)^{1/(tp'_j)'}\right)^q\right\}^{1/q}\\
\leq\;& C\,[\overrightarrow{w}]\left\{\sum_{l\in\mathbb{Z}}\sum_{\beta\in\mathcal{B}}\prod_{j=1}^m\Big(\frac{1}{\mu_k\left(B^l_\beta\right)}\int_{B^l_\beta}(f_{\sigma_j}w_j)_j^{(tp'_j)'}d\mu_k\Big)^{p/(tp'_j)'}\mu_k\left(B^l_\beta\right)\right\}^{1/p},
\end{aligned}$$

which is same as (5.3.6) with $u=\prod_{j=1}^m w_j$ and $v_j=w_j$. Hence, the rest of the proof follows from the proof of Theorem 5.3.4. □

5.5 One-weight Inequalities for Multilinear Fractional Integral Operators

Before addressing one-weight inequalities for $\mathcal{I}_\alpha^k$, we establish a Welland-type Lemma [74] in the Dunkl setting, which will be useful here.

Lemma 5.5.1. *For $0 < \epsilon < \min\{\alpha, md_k - \alpha\}$ and $\overrightarrow{f} = (f_1, f_2, \cdots, f_m)$ with $f_j \geq 0$, there is constant C_ϵ such that,*

$$|\mathcal{I}_\alpha^k \overrightarrow{f}(x)| = \mathcal{I}_\alpha^k \overrightarrow{f}(x) \leq C_\epsilon \left(\mathcal{M}_{\alpha+\epsilon}^k \overrightarrow{f}(x) \mathcal{M}_{\alpha-\epsilon}^k \overrightarrow{f}(x) \right)^{1/2}.$$

Proof. Since $\mathcal{I}_\alpha^k$ is a positive operator and using properties of Dunkl convolution together with the fact that τ_x^k is positive on radial bounded functions in $L^1(\mathbb{R}^d, d\mu_k)$,

$$\begin{aligned}
\mathcal{I}_\alpha^k \overrightarrow{f}(x) &= \int_{(\mathbb{R}^d)^m} \frac{\tau_{-y_1}^k f_1(x) \tau_{-y_2}^k f_2(x) \cdots \tau_{-y_m}^k f_m(x)}{(|y_1|^2 + |y_2|^2 + \cdots + |y_m|^2)^{(md_k-\alpha)/2}} \, d\mu_k(y_1) d\mu_k(y_2) \cdots d\mu_k(y_m) \\
&= \int_{B(0,r)^m} \frac{\tau_{-y_1}^k f_1(x) \tau_{-y_2}^k f_2(x) \cdots \tau_{-y_m}^k f_m(x)}{(|y_1|^2 + |y_2|^2 + \cdots + |y_m|^2)^{(md_k-\alpha)/2}} \, d\mu_k(y_1) d\mu_k(y_2) \cdots d\mu_k(y_m) \\
&\quad + \int_{(\mathbb{R}^d)^m \setminus B(0,r)^m} \frac{\tau_{-y_1}^k f_1(x) \tau_{-y_2}^k f_2(x) \cdots \tau_{-y_m}^k f_m(x)}{(|y_1|^2 + |y_2|^2 + \cdots + |y_m|^2)^{(md_k-\alpha)/2}} \, d\mu_k(y_1) \cdots d\mu_k(y_m) \\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

Since, $0 < \epsilon < \alpha$, we have

$$\begin{aligned}
\mathrm{I} &= \sum_{j=0}^{\infty} \int_{B(2^{-j}r)^m \setminus B(2^{-j-1}r)^m} \frac{\tau_{-y_1}^k f_1(x) \tau_{-y_2}^k f_2(x) \cdots \tau_{-y_m}^k f_m(x)}{(|y_1|^2 + |y_2|^2 + \cdots + |y_m|^2)^{(md_k-\alpha)/2}} \\
&\quad \times d\mu_k(y_1) d\mu_k(y_2) \cdots d\mu_k(y_m) \\
&\leq C \sum_{j=0}^{\infty} (2^{-j}r)^{\alpha - md_k} \int_{B(2^{-j}r)^m} \tau_{-y_1}^k f_1(x) \tau_{-y_2}^k f_2(x) \cdots \tau_{-y_m}^k f_m(x) \\
&\quad \times d\mu_k(y_1) d\mu_k(y_2) \cdots d\mu_k(y_m) \\
&= C_\epsilon r^\epsilon \mathcal{M}_{\alpha-\epsilon}^k \overrightarrow{f}(x).
\end{aligned}$$

On the other hand, using $0 < \epsilon < md_k - \alpha$, we get

$$\begin{aligned} \mathrm{II} &= \sum_{j=0}^{\infty} \int\limits_{B(2^{j+1}r)^m \setminus B(2^j r)^m} \frac{\tau^k_{-y_1} f_1(x) \tau^k_{-y_2} f_2(x) \cdots \tau^k_{-y_m} f_m(x)}{(|y_1|^2 + |y_2|^2 + \cdots + |y_m|^2)^{(md_k - \alpha)/2}} \\ &\quad \times d\mu_k(y_1)\, d\mu_k(y_2) \cdots d\mu_k(y_m) \\ &\leq C \sum_{j=0}^{\infty} (2^j r)^{\alpha - md_k} \int\limits_{B(2^{j+1}r)^m} \tau^k_{-y_1} f_1(x) \tau^k_{-y_2} f_2(x) \cdots \tau^k_{-y_m} f_m(x) \\ &\quad \times d\mu_k(y_1) d\mu_k(y_2) \cdots d\mu_k(y_m) \\ &= C_\epsilon r^{-\epsilon} \mathcal{M}^k_{\alpha+\epsilon} \overrightarrow{f}(x). \end{aligned}$$

Thus combining I and II, we have

$$\mathcal{I}^k_\alpha \overrightarrow{f}(x) \leq C_\epsilon \left(r^{-\epsilon} \mathcal{M}^k_{\alpha+\epsilon} \overrightarrow{f}(x) + r^\epsilon \mathcal{M}^k_{\alpha-\epsilon} \overrightarrow{f}(x) \right).$$

Putting $r^\epsilon = \left(\mathcal{M}^k_{\alpha+\epsilon} \overrightarrow{f}(x) / \mathcal{M}^k_{\alpha-\epsilon} \overrightarrow{f}(x) \right)^{1/2}$, we conclude the proof of the Lemma. □

Finally, we prove one-weight estimates for the operator $\mathcal{I}^k_\alpha$.

Theorem 5.5.2. *Suppose that* $1 < p_1, p_2, \cdots, p_m < \infty$, $0 < \alpha < md_k$, $1/m < p < d_k/\alpha$ *and* q *be a number defined by* $1/q = 1/p - \alpha/d_k$. *Furthermore, let the vector weight* $\overrightarrow{w} = (w_1, w_2, \cdots, w_m) \in A^k_{\overrightarrow{P},q}$ *and each* w_j *is G-invariant. Then for all* $\overrightarrow{f} \in L^{p_1}(\mathbb{R}^d, w_1^{p_1} d\mu_k) \times L^{p_2}(\mathbb{R}^d, w_2^{p_2} d\mu_k) \times \cdots \times L^{p_m}(\mathbb{R}^d, w_m^{p_m} d\mu_k)$*, the following inequality holds:*

$$\left(\int_{\mathbb{R}^d} \left(\left| \mathcal{I}^k_\alpha \overrightarrow{f}(x) \right| \prod_{j=1}^m w_j(x) \right)^q d\mu_k(x) \right)^{1/q} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^d} (|f_j(x)| w_j(x))^{p_j} \, d\mu_k(x) \right)^{1/p_j}.$$

Proof. For $0 < \epsilon < \min(\alpha, md_k - \alpha)$, define

$$\frac{1}{q_\epsilon} = \frac{1}{p} - \frac{\alpha+\epsilon}{d_k}$$

$$\text{and } \frac{1}{\widetilde{q}_\epsilon} = \frac{1}{p} - \frac{\alpha-\epsilon}{d_k}.$$

Taking ϵ sufficiently small, from Proposition 2.3.8 it follows that $\overrightarrow{w} \in A^k_{\overrightarrow{P},q_\epsilon}$ and $\overrightarrow{w} \in A^k_{\overrightarrow{P},\widetilde{q_\epsilon}}$. Let $q_1 = 2q_\epsilon/q$ and $q_2 = 2\widetilde{q_\epsilon}/q$. Then q_1 and q_2 satisfies

$$\frac{1}{q_1} + \frac{1}{q_2} = 1.$$

Taking $\overrightarrow{f} = (f_1, f_2, \cdots, f_m) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$, Lemma 5.5.1 together with Hölder's inequality implies

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(|\mathcal{I}^k_\alpha \overrightarrow{f}(x)| w(x)\right)^q d\mu_k(x) \\
\leq\ & \int_{\mathbb{R}^d} (\mathcal{I}^k_\alpha |\overrightarrow{f}|)(x)^q w(x)^q \, d\mu_k(x) \\
\leq\ & C_\epsilon \int_{\mathbb{R}^d} \left((\mathcal{M}^k_{\alpha+\epsilon}|\overrightarrow{f}|)w\right)^{q/2} \left((\mathcal{M}^k_{\alpha-\epsilon}|\overrightarrow{f}|)w\right)^{q/2} d\mu_k \\
\leq\ & C_\epsilon \left(\int_{\mathbb{R}^d} \left((\mathcal{M}^k_{\alpha+\epsilon}|\overrightarrow{f}|)w\right)^{q_\epsilon} d\mu_k\right)^{1/q_1} \left(\int_{\mathbb{R}^d} \left((\mathcal{M}^k_{\alpha-\epsilon}|\overrightarrow{f}|)w\right)^{\widetilde{q_\epsilon}} d\mu_k\right)^{1/q_2},
\end{aligned}$$

where $|\overrightarrow{f}| = (|f_1|, |f_2|, \cdots, |f_m|)$.

Since p also satisfies $p < d_k/(\alpha + \epsilon)$ and $p < d_k/(\alpha - \epsilon)$, from Theorem 5.4.1 we conclude the proof. □

Remark 5.5.3. We do not know whether these conditions are also necessary for the boundedness results. Even in the classical setting the two-weight conditions with "power-bump" on the weights are not known to be necessary for the two-weight boundedness for the fractional maximal function and fractional integral operators. Although the two-weight conditions without "power-bumps" on the weights can be obtained as necessary condition for the two-weight boundedness for these operators in the classical setting. On the other hand in the classical setting $A_{\overrightarrow{P},q}$ condition (5.1.2) is necessary and sufficient for the one-weight inequalities. However, in the Dunkl setting the lack of information about the Dunkl translation prevents us from obtaining necessary conditions for both one and two-weight inequalities.

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