NONVANISHING OF L-FUNCTIONS AND DIFFERENTIAL OPERATORS FOR JACOBI FORMS

By

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Enrolment Number: MATH11201904008

National Institute of Science Education and Research, Bhubaneswar

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

February, 2024

Homi Bhabha National Institute

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List of Publications arising from the thesis

Journal

1. S. Pandey and B. Sahu, *Nonvanishing of kernel functions and Poincaré series for Jacobi forms*, **J. Math. Anal. Appl.**, Vol. 515, 12pp , 2022.

2. S. Pandey, *Construction of Jacobi cusp forms using adjoint operator of certain differential operator*, accepted for publication in **Rocky Mountain J. Math.**.

3. S. Pandey and A. K. Jha, *Rankin–Cohen brackets of Jacobi forms and Jacobi Poincaré series*, accepted for publication in **Acta Arith.**.

4. S. Pandey, A. K. Jha and B. Sahu, *L-functions associated to Jacobi forms of half-integral weight and a converse theorem*, **J. Math. Anal. Appl.**, Vol. 534, 17pp , 2024.

Preprints

1. S. Pandey and B. Sahu, *Nonvanishing of L-functions and Poincaré series for Jacobi forms of matrix index* (submitted).

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Dedicated to My Family

ACKNOWLEDGEMENTS

I wish to convey my deep gratitude to my thesis supervisor Prof. Brundaban Sahu for his continuous support and constant encouragement throughout my Ph.D. journey. Without his guidance, this thesis would not have been possible. His knowledge and expertise have been invaluable assets during my research. The extent of gratitude I owe to him is immeasurable.

I am sincerely thankful to members of my doctoral committee Dr. B. K. Sahoo, Dr. J. Meher, Dr. K. Senthil Kumar, and Dr. G. Kasi Vishwanadham for their invaluable guidance throughout my tenure at NISER. I extend my deepest appreciation to all the faculty at NISER who imparted various courses during my coursework. I express my gratitude to Dr. A. K. Jha for being my co-author and offering invaluable assistance at various stages of my Ph.D. journey.

I would like to acknowledge the CSIR, India for providing financial support through the CSIR-NET fellowship, and NISER for infrastructure and providing a nice environment for research.

I am thankful to my seniors and friends, Anup, Mithun, Mohit, Anshu, Shubham, Kiran, Gorekh, Diptesh, Rajeeb, Pushpendu, Dinesh, Mrityunjay, Sanjay, Suman, Nilima, Snehal, Arindam, Sayan, Anjaneya, Ajith, Saurabh, Satyajyoti, Devjyoti, Raveena, Sahanawaj and Subhadeep with whom I had many beautiful memories at NISER.

I want to express my deep and sincere gratitude to my father and mother, for their love, encouragement, assistance, and support. I am thankful to my brother for always being there for me as my best friend. I would also like to express my gratitude to my sister-in-law and love to my nephew.

ABSTRACT

Nonvanishing of L-functions has many consequences in analytic number theory, for example, the nonvanishing of the Riemann zeta function is the key point in proving the prime number theorem. The generalized Riemann hypothesis states that all the zeros of L-functions associated with a Hecke eigenform of weight k lie only on the critical line $Re(s) = k/2$. Another interesting problem is to study the equivalence of modular properties of an automorphic form and analytic properties of L-functions attached to it, i.e., to derive the transformation properties from the functional equation of L-functions and vice versa.

Jacobi forms are natural generalizations of modular forms and they appear as Fourier-Jacobi coefficients of Siegel modular forms. Jacobi forms play a crucial role in the proof of the Saito-Kurokawa conjecture.

In this thesis, we study the nonvanishing of L-functions and the Poincaré series for Jacobi forms of integer index and matrix index as well. More precisely, we prove that given certain points inside the critical strip, L-functions attached to the Jacobi form do not vanish for large weights. We also study the analytic continuation of L-functions and a converse theorem for Jacobi forms of half-integral weight. Then, we study certain properties of Rankin-Cohen brackets and their relation with the Poincaré series for Jacobi forms. Finally, we construct Jacobi cusp forms involving special values of certain Dirichlet series as their Fourier coefficients.

Contents

Summary 1

Chapter 1 Introduction 3

1.1 Notations 3
1.2 Modular forms for $SL_2(\mathbb{Z})$ 4
1.2.1 Hecke operators 8
1.2.2 L-functions associated with a modular form 8
1.3 Modular forms for $\Gamma_0(N)$ 9
1.4 Modular forms of half-integral weight 11
1.5 Jacobi forms 12
1.5.1 Jacobi forms of matrix index 14
1.5.2 Theta decomposition of a Jacobi form 18
1.5.3 Dirichlet Series associated with Jacobi forms 18
1.5.4 Hecke operators 19
1.6 Jacobi forms of half-integral weights 20
1.6.1 Theta decomposition 22
1.7 Differential operators 24
1.7.1 Rankin-Cohen brackets for modular forms 24
1.7.2 Serre derivative 25
1.7.3 Heat operators 25
1.7.4 Heat operators for Jacobi forms of degree g 26
1.7.5 Rankin-Cohen brackets for Jacobi forms 27

Chapter 2 Nonvanishing of L-functions associated with Jacobi forms 28

2.1 Introduction 28
2.2 Statements of results 32
2.3 Kernel Functions 33
2.4 Nonvanishing of L-functions 43
2.5 Nonvanishing of Poincaré series 52

Chapter 3 A converse theorem for Jacobi forms of half-integral weight 53

3.1 Introduction 53
3.2 Statement of results 54
3.3 Twist and Fricke involution for Jacobi forms of half-integral weight 59
3.4 Proof of results 68
3.4.1 Proof of Theorem 3.2.4 68
3.4.2 Proof of Theorem 3.2.5 69

Chapter 4 Differential operators and Poincaré series for Jacobi forms 74

4.1 Introduction 74
4.2 Statment of results 77
4.3 Proof of results 78
4.3.1 Proof of Theorem 4.2.1 80
4.3.2 Proof of Theorem 4.2.2 88
4.3.3 Proof of Theorem 4.2.4 90
4.4 Applications 93

References 96

Summary

This thesis deals with the study of L-functions and differential operators associated with Jacobi forms and examines their analytic properties. Modular forms which played a crucial role in the proof of Fermat's last theorem, have applications beyond number theory. L-functions attached to modular forms have interesting analytic properties and play a crucial role in analytic number theory.

The nonvanishing property of L-functions holds huge significance, serving as a key point in various analytic results such as the prime number theorem, where, the nonvanishing of the Riemann zeta function assumes a pivotal role. One of the problems in the theory of L-functions is to find the zero-free region for L-functions attached to modular forms. The Generalized Riemann Hypothesis remains an unsolved conjecture for a long time. It posits a stringent constraint on the zeros of L-functions, claiming that L-functions associated with Hecke eigenforms of weight k admit zeros only on the critical line $Re(s) = k/2$. Towards this direction, Kohnen [24] proved the nonvanishing of L-functions attached to Hecke cusp forms on average. The approach of Kohnen has been adopted for various kinds of automorphic forms to prove the nonvanishing of associated L-functions.

The reciprocal relationship between the modular properties of an automorphic form and the analytic properties of its associated L-function forms an interesting topic of exploration. The derivation of transformation properties from the functional equation of L-functions and vice versa reveals the underlying relation between these two mathematical entities. The study of such a relationship is known as the converse theorem for automorphic forms. Hecke studied the converse theorem for modular forms for $SL_2(\mathbb{Z})$. In particular, Hecke [16] proved that the transformation properties and the analytic properties of associated L-functions are equivalent. Later, Weil [48] generalized Hecke's work for congruence

subgroups.

Differential operators on modular forms form a crucial aspect of mathematical analysis within the realm of number theory. By applying differential operators to modular forms, one can study the arithmetic behavior of many interesting number theoretic functions that appear as Fourier coefficients of modular forms. Using differential operators, one can obtain interesting congruences between Fourier coefficients of modular forms.

Jacobi forms emerge as natural extensions of modular forms in two variables. They appear as Fourier-Jacobi coefficients of Siegel modular forms. Jacobi forms assume a crucial role in the proof of the Saito-Kurokawa conjecture. In this thesis, we investigate the nonvanishing of L-functions associated with Jacobi forms of integer and matrix index and Jacobi Poincaré series. The converse theorem for Jacobi forms for congruence subgroups has been studied by Martin and Osses [36] for Jacobi forms with respect to congruence subgroups. We also investigate L-functions attached to Jacobi forms of half-integral weight. More precisely, we associate L-functions to a Jacobi form of half-integral weight using theta decomposition. Then we study the analytic continuation of these L-functions and prove a converse theorem. We also investigate Rankin-Cohen brackets and their relation with the Jacobi Poincaré series. More precisely, we prove that Rankin-Cohen brackets and Jacobi Poincaré series commute in a certain sense analogous to [49]. We also give some applications as a consequence of the above property. Finally, we construct cusp forms whose Fourier coefficients involve special values of convolution-type Dirichlet series by constructing the adjoint of certain differential operator on the spaces of Jacobi forms.

Chapter 1

Introduction

This chapter introduces basic notions in the theory of modular forms and Jacobi forms of integral weight and half-integral weight. We also introduce L-functions and differential operators associated with modular forms and Jacobi forms.

1.1 Notations

Let $\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, rational numbers, real numbers, and complex numbers, respectively. We denote the real part and imaginary part of $z \in \mathbb{C}$ by $Re(z)$ and $Im(z)$, respectively. We denote $e^{2\pi i z \frac{m}{n}}$ by $e_n^m(z)$ where $n \neq 0$ and m are real numbers. We also denote $e_1^1(z)$ by $e(z)$. Let $\mathcal{H} = \{\tau \in \mathbb{C} : Im\,\tau > 0\}$ be the complex upper half-plane. We denote the variable in the complex upper half-plane by τ and the variable in $\mathbb{C}$ or $\mathbb{C}^g$ by z. We denote $q = e(\tau)$, for $\tau \in \mathcal{H}$ and $\zeta = e(z)$, for $z \in \mathbb{C}$. For a complex number z, the square root is defined as follows:

$$\sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i}{2} arg(z)}, \text{ with } -\pi < arg(z) \leqslant \pi.$$

We set $z^{\frac{k}{2}} = (\sqrt{z})^k$ for any $k \in \mathbb{Z}$. We denote $A[X] = X^t AX$ where A and X are matrices of suitable orders. For a ring R, we denote R^g as the set of all row vectors with g columns and $R^{g,1}$ as the set of all column vectors with g rows.

The full modular group $\Gamma = SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

For a positive integer N, we denote the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ as follows:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

1.2 Modular forms for $SL_2(\mathbb{Z})$

The group $GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc > 0 \right\}$ acts on $\mathcal{H}$ via fractional linear transformations, i.e., for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and $\tau \in \mathcal{H}$,

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

Let $k \in \mathbb{Z}$. The group $GL_2^+(\mathbb{R})$ acts on set of all complex-valued holomorphic functions on $\mathcal{H}$ via the action defined by:

$$(f \mid_k \gamma)(\tau) := (det\, \gamma)^{\frac{k}{2}} (c\tau + d)^{-k} f(\gamma \cdot \tau),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and f is a complex-valued holomorphic function on $\mathcal{H}$.

Definition 1.2.1. *A modular form of weight k with respect to the group $SL_2(\mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ satisfying*

1. *$f\mid_k \gamma = f,\ \forall\gamma \in SL_2(\mathbb{Z})$, i.e.,*
   $$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau),\ \forall\,\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } \forall\tau\in\mathcal{H}.$$
2. *f is holomorphic at the cusp infinity, i.e., f has a Fourier series expansion of the form $f(\tau) = \sum\limits_{n=0}^{\infty} a(n)q^n$.*

   Moreover, if $a(0) = 0$, then f is called a cusp form.

The set of all modular forms of weight k forms a vector space over $\mathbb{C}$. The set of all modular forms of weight k for $SL_2(\mathbb{Z})$ is denoted by M_k and that for cusp forms by S_k, respectively.

Let $f,\ g \in M_k$ be such that either f or g is a cusp form. We define the Petersson inner product of f and g as:

$$\langle f, g\rangle \ = \int\limits_{SL_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\overline{g(\tau)}(Im(\tau))^k dV,$$

where $SL_2(\mathbb{Z}) \setminus \mathcal{H}$ is a fundamental domain, $\tau = u + iv$ and $dV = \dfrac{dudv}{v^2}$ is an invariant measure under the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$. The inner product is independent of the choice of fundamental domain.

Example 1.2.1. (Eisenstein series): *Let $k > 2$ be an even integer. The normalized Eisenstein series E_k of weight k for $SL_2(\mathbb{Z})$ is defined by:*

$$E_k(\tau) := \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\} \\ (m,n)=1}} \frac{1}{(mz+n)^k}.$$

Then E_k is a modular form of weight k for $SL_2(\mathbb{Z})$ with Fourier expansion

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ *and* B_k*'s are Bernoulli numbers defined by*

$$\frac{x}{e^x - 1} = \sum_{k=o}^{\infty} B_k \frac{x^k}{k!}.$$

The Fourier expansions of E_k *for* $k = 4, 6, 8, 10$ *and* 12 *are as follows:*

$$\begin{aligned}
E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\
E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \\
E_8(\tau) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \\
E_{10}(\tau) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n, \\
E_{12}(\tau) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n.
\end{aligned}$$

Example 1.2.2. (Ramanujan delta function): *The Ramanujan delta function is defined as*

$$\Delta(\tau) := \frac{(E_4(\tau)^3 - E_6(\tau)^2)}{1728}.$$

Δ *is a cusp form of weight* 12 *for* $SL_2(\mathbb{Z})$ *with Fourier expansion*

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $\tau(n)$ *is called the Ramanujan tau function. Ramanujan delta function has a product expansion*

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Example 1.2.3. (Poincaré series): *Let* n *be a positive integer. The* n*-th Poincaré series of* *weight k for $SL_2(\mathbb{Z})$ is defined by*

$$P_{k,n}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} e^{2\pi i n\tau}|_k\gamma, \tag{1.1}$$

where $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{Z} \right\}$. *$P_{k,n}$ is a cusp form of weight $k > 2$ for $SL_2(\mathbb{Z})$ with Fourier expansion*

$$P_{k,n}(\tau) = \sum_{m=1}^{\infty} g_n(m) q^m,$$

where

$$g_n(m) = \delta_{n,m} + (-1)^{\frac{k}{2}+1} \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \pi \sum_{c=1}^{\infty} K_c(n,m) J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right),$$

$\delta_{n,m}$ is Kronecker delta symbol and $K_c(n,m)$ is the Kloosterman sum defined by

$$\frac{1}{c} \sum_{\substack{d \pmod c \\ dd^{-1} \equiv 1 \pmod c}} e^{2\pi i \left(\frac{md + nd^{-1}}{c}\right)},$$

and $J_{k-1}(x)$ is the Bessel function of order $k-1$.

The Poincaré series has the following property:

Lemma 1.2.2. *If $f \in S_k$ with Fourier expansion $f(\tau) = \sum_{m=1}^{\infty} a(m) q^m$, then*

$$\langle f, P_{k,n} \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a(n), \tag{1.2}$$

where $\Gamma(x)$ is the usual gamma function.

1.2.1 Hecke operators

We now define linear operators on the space of modular forms, called the Hecke operators.

Definition 1.2.3. *Let n be a natural number. For $f(\tau) = \sum_{m \geq 0} a(m)q^m \in M_k$, the n-th Hecke operator is defined by*

$$T_n f(\tau) := \sum_{m \geq 0} a_n(m) q^m,$$

where $a_n(m) = \sum_{d|(m,n)} d^{k-1} a(\frac{mn}{d^2})$. Then $T_n f$ is a modular form of weight k. Moreover, if f is a cusp form then $T_n f$ is also a cusp form.

The set $\{T_n : n \in \mathbb{N}\}$ consists of self-adjoint, commuting operators on the space of cusp forms.

Definition 1.2.4. *A cusp form is said to be an eigenform if $T_n f = \lambda_n f$ for all $n \in \mathbb{N}$.*

The space of cusp forms S_k is a finite-dimensional Hilbert space with respect to the Petersson inner product. Hence, there exists an orthonormal basis consisting of eigenforms of all the Hecke operators T_n.

1.2.2 L-functions associated with a modular form

Let $f(z) \in S_k$, with Fourier expansion $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n$. The L-function associated to f is defined by

$$L(f, s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{1.3}$$

Since the Fourier coefficients of a cusp form of weight k satisfy $a(n) = O(n^{\frac{k}{2}})$, the above series converges for $Re(s) > c + 1$, where $c = \frac{k}{2}$. The completed L-function is defined by

$$L^*(f, s) \ := \ \frac{1}{(2\pi)^s}\Gamma(s)L(f, s),$$

$L^*(f, s)$ can be extended to an entire function of $s \in \mathbb{C}$, and satisfies the functional equation

$$L^*(f, s) = (-1)^{\frac{k}{2}} L^*(f, k - s).$$

Further, if f is an eigenform, then $L(f, s)$ has an Euler product

$$L(f, s) = \prod_{p\text{-prime}} (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}.$$

1.3 Modular forms for $\Gamma_0(N)$

Definition 1.3.1. *Let k be an integer and χ be a Dirichlet character modulo N. A holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight k, with level N and character χ if*

1. $(f \mid_k \gamma)(\tau) \ = \ \chi(d)f(\tau), \ \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, *i.e.,*

   $$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau), \ \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

2. *f is holomorphic at all the cusps of $\Gamma_0(N)$, i.e., $f \mid_k \gamma$ has Fourier expansion of the form*

   $$(f \mid_k \gamma)(\tau) = \sum_{n=0}^{\infty} a_\gamma(n) q^{\frac{n}{d}}$$

   for every $\gamma \in SL_2(\mathbb{Z})$.

Further, f is called a cusp form if f vanishes at all the cusps of $\Gamma_0(N)$, i.e. $a_\gamma(0) = 0$ for every $\gamma \in SL_2(\mathbb{Z})$.

Denote the space of all modular forms and the subspace of all cusp forms of weight k, level N with character χ on $\Gamma_0(N)$ by $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, respectively. If χ is the trivial character, then we denote the spaces as $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$, respectively.

If $f,\ g \in M_k(\Gamma_0(N), \chi)$ are such that either f or g is a cusp form, then the Petersson inner product of f and g is defined as:

$$\langle f, g\rangle = \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)\overline{g(\tau)}(Im(\tau))^k dV,$$

where $\Gamma_0(N) \setminus \mathcal{H}$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$ and $[SL_2(\mathbb{Z}) : \Gamma_0(N)]$ is the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$.

We state a lemma on the bounds of Fourier coefficients of a modular form.

Lemma 1.3.2. *If $f \in M_k(\Gamma_0(N), \chi)$ with Fourier coefficients $a(n)$, then*

$$a(n) \ll |n|^{k-1+\epsilon}.$$

Moreover, if f is a cusp form, then the coefficients satisfy the following Ramanujan-Petersson bounds:

$$a(n) \ll |n|^{\frac{k}{2}-\frac{1}{2}+\epsilon}.$$

For more details on the theory of modular forms of integral weight, we refer to [18, 23].

1.4 Modular forms of half-integral weight

Let $\Gamma = \Gamma_0(4)$. For an odd integer k and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, define the slash operator as follows:

$$\left(f\widetilde{|_{\frac{k}{2}}}\gamma\right)(\tau) := \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\frac{k}{2}}(c\tau + d)^{-\frac{k}{2}} f(\gamma \cdot \tau),$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol and f is a complex-valued holomorphic function on $\mathcal{H}$.

Definition 1.4.1. *Let k be an odd integer and χ be a Dirichlet character modulo 4. A holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight $\frac{k}{2}$ and character χ for the group Γ if*

1. $\left(f\widetilde{|_{\frac{k}{2}}}\gamma\right)(\tau) \; = \; \chi(d)f(\tau), \; \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$

2. *f is holomorphic at all the cusps of Γ, i.e., $f\widetilde{|_{\frac{k}{2}}}\gamma$ has Fourier expansion of the form*

$$\left(f\widetilde{|_{\frac{k}{2}}}\gamma\right)(\tau) = \sum_{n=0}^{\infty} a_\gamma(n) q^{\frac{n}{d}}$$

   for every $\gamma \in SL_2(\mathbb{Z})$.

   Moreover, we say f is a cusp form if f vanishes at all the cusps of $\Gamma_0(4)$, i.e., $a_\gamma(0) = 0$ for every $\gamma \in SL_2(\mathbb{Z})$.

Let $M_{\frac{k}{2}}(\Gamma, \chi)$ and $S_{\frac{k}{2}}(\Gamma, \chi)$ denote the spaces of modular forms and cusp forms respectively of weight $\frac{k}{2}$ and group $\Gamma = \Gamma_0(4)$. The space $S_{\frac{k}{2}}(\Gamma, \chi)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product defined by:

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathcal{H}} f(\tau)\overline{g(\tau)}(Im(\tau))^{\frac{k}{2}} dV,$$

for $f,\ g \in S_{\frac{k}{2}}(\Gamma, \chi)$. Moreover, the inner product is well defined if at least one of f and g is a cusp form. For more details on modular forms of half-integral weight, we refer to [23].

1.5 Jacobi forms

Consider the Jacobi group Γ^J defined by

$$\Gamma^J := SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 = \left\{ (M, X) : M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), X = (\lambda, \nu) \in \mathbb{Z}^2 \right\}.$$

The set Γ^J has a binary group operation defined by

$$(M, X)(M', X') = (MM', XM' + X').$$

The group Γ^J acts on $\mathcal{H} \times \mathbb{C}$ via

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \nu) \right) \cdot (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \nu}{c\tau + d} \right).$$

Let k, m be any fixed positive integers. Let ϕ be a complex-valued holomorphic function on $\mathcal{H} \times \mathbb{C}$. Let $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma^J$. Define the slash operator

$$(\phi|_{k,m}\gamma)(\tau, z) := (c\tau + d)^{-k} e^m \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z \right) \phi(\gamma \cdot (\tau, z)).$$

Definition 1.5.1. *Let $\phi : \mathcal{H} \times \mathbb{C} \to$ be a holomorphic function. Then ϕ is said to be a Jacobi form of weight k and index m for Γ^J, if ϕ is invariant under the slash operator with respect to the Jacobi group, i.e.,*

$$\phi|_{k,m}\gamma = \phi, \ \forall \gamma \in \Gamma^J$$

and has a Fourier series expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ r^2 \leq 4nm}} c(n,r) q^n \zeta^r \qquad (q = e^{2\pi i \tau}, \ \zeta = e^{2\pi i z}). \tag{1.4}$$

Moreover, ϕ is called a cusp form if $c(n,r) \neq 0$ implies $r^2 < 4nm$.

Remark 1.5.1. The property $\phi|_{k,m}\gamma = \phi$ for every $\gamma \in \Gamma^J$ is equivalent to

$$\phi|_{k,m}[M, (0,0)] = \phi, \quad \text{for every } M \in SL_2(\mathbb{Z}) \tag{1.5}$$

and

$$\phi|_{k,m}[Id, (\lambda, \mu)] = \phi, \quad \text{for every } (\lambda, \mu) \in \mathbb{Z}^2. \tag{1.6}$$

Denote the space of all Jacobi forms and the subspace of all Jacobi cusp forms by $J_{k,m}$ and $J_{k,m}^{cusp}$, respectively.

Let $\phi, \ \psi \in J_{k,m}$ such that at least one of them is cusp form. The Petersson inner product of ϕ and ψ is defined by:

$$\langle \phi, \psi \rangle = \int\limits_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} \phi(\tau, z) \overline{\psi(\tau, z)} v^k e^{\frac{-4\pi m y^2}{v}} dV_J,$$

where $\tau = u + iv, z = x + iy$ and $dV_J = \dfrac{dudvdxdy}{v^3}$ is an invariant measure under the action on Γ^J on $\mathcal{H} \times \mathbb{C}$. The space of Jacobi cusp forms of weight k and index m is a finite-dimensional Hilbert space with respect to the Petersson inner product.

Example 1.5.1. *Let $k \geqslant 4$ be an even integer. The Jacobi Eisenstein series of weight k and index m is defined as*

$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e^m \left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right),$$

where a and b are such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then, $E_{k,m} \in J_{k,m}$.

For more details on Jacobi forms of integer weight, one can refer to [13].

1.5.1 Jacobi forms of matrix index

Let g be a positive integer. Consider the Jacobi group Γ_g^J of degree g defined by $\Gamma_g^J = \{(M, X) : M \in SL_2(\mathbb{Z}),\ X = (\lambda, \mu) \in \mathbb{Z}^{g,1} \times \mathbb{Z}^{g,1}\}$ with the group law defined by

$$(M_1, X_1) \cdot (M_2, X_2) = (M_1 M_2, X_1 M_2 + X_2)$$

for $(M_1, X_1),\ (M_2, X_2) \in \Gamma_g^J$. The Jacobi group acts on the space $\mathcal{H} \times \mathbb{C}^{g,1}$ via

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Let k be a positive integer, $\mathcal{M}$ be a positive definite symmetric half-integral matrix of order $g \times g$ and $h = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma_g^J$. We define the automorphic factor with respect to h by

$$j_{h,k,\mathcal{M}}(\tau,z) := (c\tau+d)^{-k}e\left(\frac{-c}{c\tau+d}\mathcal{M}[z+\lambda\tau+\mu]+\mathcal{M}[\lambda]\tau+2\lambda^t\mathcal{M}z\right).$$

Jacobi group Γ_g^J acts on the set of all holomorphic functions $\phi : \mathcal{H}\times\mathbb{C}^{g,1}\to\mathbb{C}$ via the action:

$$(\phi|_{k,\mathcal{M}}h)(\tau,z) = j_{h,k,\mathcal{M}}(\tau,z)\phi(h\cdot(\tau,z)).$$

Definition 1.5.2. *Let $\phi : \mathcal{H}\times\mathbb{C}^{g,1}\to$ be a holomorphic function. The function ϕ is said to be a Jacobi form of weight k and index $\mathcal{M}$ for Γ^J if ϕ is invariant under the slash operator with respect to the Jacobi group Γ_g^J, i.e.,*

$$\phi|_{k,\mathcal{M}}\gamma = \phi,\ \forall\gamma\in\Gamma_g^J$$

and satisfies the cuspidality condition, i.e.,

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{N},r\in\mathbb{Z}^g,\\ n\geqslant\frac{1}{4}\mathcal{M}^{-1}[r^t]}} c(n,r)q^n e(r\cdot z).$$

Moreover, ϕ is called a cusp form if $c(n,r)\neq 0$ implies $n>\frac{1}{4}\mathcal{M}^{-1}[r^t]$.

Denote the space of all Jacobi forms and Jacobi cusp forms of weight k and index $\mathcal{M}$ by $J_{k,\mathcal{M}}$ and $J_{k,\mathcal{M}}^{cusp}$, respectively. Define the Petersson inner product on $J_{k,\mathcal{M}}^{cusp}$ by

$$\langle\phi,\psi\rangle = \int\limits_{\Gamma_g^J\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{\psi(\tau,z)}y^k e(-4\pi\mathcal{M}[v]y^{-1})dV_g^J,$$

where $\tau = x+iy,\ z = u+iv$ and $dV_g^J = v^{-g-2}dxdydudv$. Define $z = p\tau+q$ with $p,q\in\mathbb{R}^{g,1}$ and $\mu_{k,\mathcal{M}}(\tau,z) := y^{\frac{k}{2}}e(iy\mathcal{M}[p])$. One can rewrite the above Petersson inner product as

$$\langle f, g\rangle = \int\limits_{\Gamma_g^J\backslash\mathcal{H}\times\mathbb{C}^{g,1}} f(\tau,z)\overline{g(\tau,z)}y^k e(2iy\mathcal{M}[p])y^{-2}dxdydpdq.$$

The space $J^{cusp}_{k,\mathcal{M}}$ is a finite-dimensional Hilbert space with respect to the Petersson Inner product.

Example 1.5.2. (Poincaré series): *Let k be a positive integer, $\mathcal{M}$ be a symmetric, positive definite, half-integral $g\times g$ matrix. Let $n\in\mathbb{Z}$ and $R\in\mathbb{Z}^g$ such that $n-\frac{1}{4}\mathcal{M}^{-1}[R^t]>0$. We define (n,R)-th Poincaré series by*

$$P_{k,\mathcal{M};n,R}(\tau,z) = \sum_{\gamma\in\Gamma^J_{g,\infty}\backslash\Gamma^J_g} e(n\tau)e(Rz)|_{k,\mathcal{M}}\gamma(\tau,z), \tag{1.7}$$

where $(\tau,z)\in\mathcal{H}\times\mathbb{C}^{g,1}$ and $\Gamma^J_{g,\infty} = \left\{\left(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix},(0,\mu)\right): n\in\mathbb{Z},\ \mu\in\mathbb{Z}^{g,1}\right\}.$

The Poincaré series have the following properties:

Theorem 1.5.3. *[3] The Poincaré series $P_{k,\mathcal{M};n,R}\in J^{cusp}_{k,\mathcal{M}}$ and the set of all Poincaré series generates the space $J^{cusp}_{k,\mathcal{M}}$. For a Jacobi form $\phi(\tau,z)=\sum\limits_{\substack{n'\in\mathbb{Z},\ R'\in\mathbb{Z}^g\\ n'\geqslant\frac{1}{4}\mathcal{M}^{-1}[R'^t]}} c(n',R')e(n'\tau)e(R'z)$ we have,*

$$\langle\phi, P_{k,\mathcal{M};n,R}\rangle = 2^{(g-1)(k-\frac{g}{2}-1)}\pi^{-k+\frac{g}{2}+1}|\mathcal{M}|^{k-(\frac{g+3}{2})}D^{-k+\frac{g}{2}+1}\Gamma(k-\frac{g}{2}-1)c(n,R).$$

The Fourier expansion of the Poincaré series $P_{k,\mathcal{M};n,R}$ is given by

$$P_{k,\mathcal{M};n,R}(\tau,z) = \sum_{\substack{n'\in\mathbb{Z},\ R'\in\mathbb{Z}^g\\ n'\geqslant\frac{1}{4}\mathcal{M}^{-1}[R'^t]}} p_{k,\mathcal{M};n,R}(n',R')e(n'\tau+R'z),$$

where

$$\begin{aligned} p_{k,\mathcal{M};n,R}(n',R') &= \delta_{\mathcal{M}}(n,R,n',R') + (-1)^k \delta_{\mathcal{M}}(n,R,n',-R') \\ &+ i^k \pi 2^{1-\frac{g}{2}} |\mathcal{M}|^{-\frac{1}{2}} \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{g}{2}-\frac{1}{2}} \sum_{c \geqslant 1} \Big(H_{\mathcal{M}}(n,R,n',R') \\ &+ (-1)^k H_{\mathcal{M}}(n,R,n',-R') \Big) J_{k-\frac{g}{2}-1} \left(\frac{\pi\sqrt{DD'}}{2^{g-1}|\mathcal{M}|c} \right), \end{aligned}$$

where $D = det\left(2\begin{pmatrix} n & \frac{1}{2}R \\ \frac{1}{2}R^t & M \end{pmatrix}\right)$, $D' = det\left(2\begin{pmatrix} n' & \frac{1}{2}R' \\ \frac{1}{2}R'^t & M \end{pmatrix}\right)$ *and*

$$\delta_{\mathcal{M}}(n,R,n',R') = \begin{cases} 1, & \textit{if } D = D', R' \equiv R(\mathbb{Z}^g\ 2\mathcal{M}), \\ 0, & \textit{otherwise} \end{cases}$$

and

$$H_{\mathcal{M}}(n,R,n',R') = c^{-\frac{g}{2}-1} \sum_{x(c)y(c^*)} e_c((\mathcal{M}[x] + Rx + n)\overline{y} + n'y + R'x) e_{2c}(R'\mathcal{M}^{-1}R^t)$$

is the generalized Kloosterman sum. Here y *runs over* $(\mathbb{Z}/c\mathbb{Z})^*$ *with* $\overline{y}y \equiv 1(c)$ *and* x *runs over* $(\mathbb{Z}^{g,1}/c\mathbb{Z}^{g,1})$.

For more details on the theory of Jacobi forms of matrix index, we refer to [52].

For any $\mu \in \mathbb{Z}^g \setminus \mathbb{Z}^g(2\mathcal{M})$ define the μ-th theta series of weight $\frac{g}{2}$ and index $\mathcal{M}$ by

$$\Theta_{\mathcal{M},\mu}(\tau,z) = \sum_{\substack{R \in \mathbb{Z}^g \\ R \equiv \mu \pmod{2\mathcal{M}}}} e(\frac{1}{4}\mathcal{M}^{-1}[R^t]\tau) e(Rz).$$

1.5.2 Theta decomposition of a Jacobi form

Let $\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\ R\in\mathbb{Z}^g \\ n\geqslant\frac{1}{4}\mathcal{M}^{-1}[R^t]}} c(n, R)e(n\tau)e(Rz)$ be a Jacobi form. The Fourier coefficients of ϕ satisfy the property $c(n, R) = c(n', R')$ whenever $n - \frac{1}{4}\mathcal{M}^{-1}[R^t] = n' - \frac{1}{4}\mathcal{M}^{-1}[R'^t]$ and $R \equiv R' \pmod{2\mathcal{M}}$. Hence one can define $c_R(N) = c(n, R)$ whenever $N = 4n - \mathcal{M}^{-1}[R^t]$. Then ϕ can be represented as follows:

$$\phi(\tau, z) = \sum_{R\in\mathbb{Z}^g \pmod{2\mathcal{M}}} \phi_R(\tau)\Theta_{\mathcal{M},R}(\tau, z), \tag{1.8}$$

where $\phi_R(\tau) = \sum_{N=0}^{\infty} c_R(N)e(\frac{N}{4}\tau)$. The holomorphic functions $\{\phi_R(\tau)\}$ behave like a vector-valued modular form of half-integral weight. The expression (1.8) is called the theta decomposition of ϕ. Using the theta decomposition (1.8), one can define L-functions associated with Jacobi forms.

1.5.3 Dirichlet Series associated with Jacobi forms

Let ϕ be a Jacobi cusp form with theta decomposition (1.8). For every $R \in \mathbb{Z}^g \setminus \mathbb{Z}^g(2\mathcal{M})$ define the Dirichlet series

$$L_R(\phi, s) = \sum_{D=1}^{\infty} c_R(D)\left(\frac{D}{4|\mathcal{M}|}\right)^{-s} \tag{1.9}$$

and the completed Dirichlet series by

$$\Lambda_R(\phi, s) = (2\pi)^{-s}\Gamma(s)L_R(\phi, s). \tag{1.10}$$

Martin studied the analytic properties of these Dirichlet series and established a set of functional equations.

Theorem 1.5.4. *[35] Let k be a positive even integer and $\mathcal{M}$ be symmetric, positive definite, half-integral matrix of order $g \times g$. Let $\phi : \mathcal{H} \times \mathbb{C}^{g,1} \to \mathbb{C}$ be a Jacobi form of weight k and index $\mathcal{M}$. Then for any $R \in \mathbb{Z}^g \setminus \mathbb{Z}^g(2\mathcal{M})$, the completed Dirichlet series $\Lambda_R(\phi, s)$ has analytic continuation to whole complex plane and they satisfy*

$$\frac{1}{\sqrt{2^g|\mathcal{M}|}} \sum_{R' \pmod{2\mathcal{M}}} (e(-R'(2\mathcal{M})^{-1}R^t)+e(R'(2\mathcal{M})^{-1}R^t))\Lambda_{R'}(\phi, s) = i^k \Lambda_R(\phi, k-s-\frac{g}{2}). \tag{1.11}$$

1.5.4 Hecke operators

Let $M_2(\mathbb{Z})$ denote the set of all 2×2 matrices with integer entries and $l > 0$ be a positive integer.

Definition 1.5.5. *Let $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function. The l-th Hecke operator T_l is defined by*

$$(T_l\phi)(\tau, z) = l^{k-4} \sum_{\substack{M \in \Gamma \setminus M_2(\mathbb{Z}), \\ det(M)=l^2, g.c.d.(M)=\square}} \sum_{X \in \mathbb{Z}^2/l\mathbb{Z}^2} \phi \mid_{k,m} M \mid_m X,$$

where $g.c.d.(M) = \square$ means that the greatest common divisor of all the entries of M is a square number.

Theorem 1.5.6. *The Hecke operators T_l are well defined linear operators on the space $J_{k,m}$.*

1.6 Jacobi forms of half-integral weights

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ and $\widetilde{\gamma} = (\gamma, \phi(\tau))$, with $\phi(\tau)$ a complex-valued holomorphic function on $\mathcal{H}$ such that $\phi^2(\tau) = t\frac{c\tau + d}{\sqrt{det(\gamma)}}$ with $t \in \{1, -1\}$. Then the set

$$G := \left\{ \widetilde{\gamma} = (\gamma, \varphi(\tau)) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}),\ \varphi^2(\tau) = t\frac{c\tau + d}{\sqrt{det(\gamma)}},\ t = \pm 1 \right\},$$

forms a group with the following operation

$$(\gamma_1, \varphi_1(\tau)) \cdot (\gamma_2, \varphi_2(\tau)) := (\gamma_1\gamma_2, \varphi_1(\gamma_2\tau)\varphi_2(\tau)).$$

The association $\gamma \mapsto \widetilde{\gamma} = (\gamma, j(\gamma, \tau))$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, and $j(\gamma, \tau) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(c\tau + d)^{1/2}$, is an injective map from $\Gamma_0(4)$ into G.

Let

$$\widetilde{G^J} = \{(\widetilde{\gamma}, X, s) :\ \gamma \in SL_2(\mathbb{R}),\ X \in \mathbb{R}^2,\ s \in S^1\}.$$

Then $\widetilde{G^J}$ is a group with the group law

$$(\widetilde{\gamma}_1, X, s)(\widetilde{\gamma}_2, Y, s') = \left(\widetilde{\gamma}_1\widetilde{\gamma}_2,\ X\gamma_2 + Y,\ ss' \cdot \det\begin{pmatrix} X\gamma_2 \\ Y \end{pmatrix}\right).$$

and it acts on $\mathcal{H} \times \mathbb{C}$ as follows:

$$h \cdot (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right),$$

where $(\tau, z) \in \mathcal{H} \times \mathbb{C},\;\; h = (\gamma, X, \zeta) \in G^J$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let k and m be fixed positive integers with k odd. For a function $\phi : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ and $h = (\widetilde{\gamma}, X, s) \in \widetilde{G^J}$ with $X = (\lambda, \mu) \in \mathbb{R}^2$, the slash operator $|_{\frac{k}{2},m}$ is defined by

$$\left(\phi|_{\frac{k}{2},m}h\right)(\tau, z) := s^m \varphi(\tau)^{-k} e^m\left(\tfrac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + 2\lambda^2\tau + 2\lambda z + \lambda\mu\right)\phi\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

For $h = (\widetilde{\gamma}, (0,0), 1)$ with $\widetilde{\gamma} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j(\gamma, \tau)\right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), 4 \mid N$ the above definition reduces to

$$\left(\phi|_{\frac{k}{2},m}h\right)(\tau, z) := j(\gamma, \tau)^{-k} e^m\left(\tfrac{-cz^2}{c\tau+d}\right)\phi\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}\right).$$

Denote $\phi|_{\frac{k}{2},m}h$ by $\phi|_{\frac{k}{2},m}\widetilde{\gamma}$ whenever $h = (\widetilde{\gamma}, (0,0), 1)$ with $\widetilde{\gamma} = (\gamma, j(\gamma, \tau)), \gamma \in \Gamma_0(N)$. For positive integer N with $4|N$, consider the subgroup $\Gamma^J(N)$ of $\widetilde{G^J}$ defined by $\Gamma^J(N) := \widetilde{\Gamma_0(N)} \ltimes (\mathbb{Z} \times \mathbb{Z})$ i.e.,

Definition 1.6.1. *Let k, N, m be positive integers such that k is odd and $4|N$. Let χ be a Dirichlet character modulo N. A Jacobi form of weight $\frac{k}{2}$ and index m with character χ for the group $\Gamma^J(N)$ is a complex-valued holomorphic function ϕ defined on $\mathcal{H} \times \mathbb{C}$ satisfying the following conditions:*

1. *$\phi \mid_{\frac{k}{2},m} h = \chi(d)\phi$, for all $h = (\widetilde{\gamma}, X, s) \in \Gamma^J(N)$ with $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$,*

2. *for each $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q})$, there exists an integer d_σ such that the function*

$\phi\mid_{\frac{k}{2},m} h$, where $h = (\sigma^{-1}, (0,0,1))$ has a Fourier expansion of the form

$$\phi\mid_{\frac{k}{2},m} h = \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\leq 4nmd_\sigma}} c_{\phi,\sigma}(n,r)e\left(\frac{n}{d_\sigma}\tau + \frac{r}{d_\sigma}z\right).$$

Further, if the inequality in the above expression is strict for every $\sigma \in SL_2(\mathbb{Q})$, then ϕ is said to be a Jacobi cusp form.

Denote the space of all Jacobi forms and Jacobi cusp forms of weight $\frac{k}{2}$ and index m with character χ for the group $\Gamma^J(N)$ by $J_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ and $J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$, respectively. For more details on Jacobi forms of half-integral weight, we refer to [46].

1.6.1 Theta decomposition

Let $\phi \in J_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ with the Fourier series expansion given by

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\leq 4nm}} c_\phi(n,r)e(n\tau + rz). \tag{1.12}$$

For $D \geq 0$ and $r \pmod{2m}$, we define a sequence $\{c_\mu(D)\}$ of complex numbers as follows:

$$c_\mu(D) := \begin{cases} c_\phi\left(\frac{D+r^2}{4m}, r\right), & \text{if } D \equiv -r^2 \pmod{4m}, r \equiv \mu \pmod{2m}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.13}$$

Set

$$h_\mu(\tau) := \sum_{D=0}^{\infty} c_\mu(D)e_{4m}(D\tau), \tag{1.14}$$

and for a natural number l, consider the Jacobi theta function defined by

$$\theta_{l,\mu}(\tau, z) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu (\mathrm{mod}\ 2l)}} e\left(\frac{r^2}{4l}\tau + rz\right). \tag{1.15}$$

The equations (1.12), (1.14) and (1.15) imply the following decomposition of the $\phi(\tau, z)$:

$$\phi(\tau, z) = \sum_{\mu=1}^{2m} h_\mu(\tau)\theta_{m,\mu}(\tau, z). \tag{1.16}$$

The above representation is called the theta decomposition of ϕ. The transformation properties of the Jacobi form ϕ imply certain transformation properties of h_μ. For more details on the transformation properties satisfied by the function h_μ, we refer to [41].

Lemma 1.6.2. *Let $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ be a Jacobi form with the Fourier series expansion as given in* (1.12). *Then there exists a positive real number C_0 such that $|c_\phi(n, r)| \le C_0 D^{\frac{k}{4}}$, where $D = 4mn - r^2$.*

The above estimate for the Fourier coefficients has nice analytic consequences as given in the following lemma:

Lemma 1.6.3. *[36] Let m be a positive integer and $\{c_\mu(D)\}, \mu = 1, \cdots, 2m,$* where $D > 0$ *be a sequence as defined in* (1.13). *Let $h_\mu(\tau), \theta_{m,\mu}(\tau, z)$ and $\phi(\tau, z)$ be the power series given by* (1.14), (1.15) *and* (1.16)*, respectively. If $c_\mu(D) = O(D^\delta)$ for some $\delta > 0$, then each of the series $h_\mu(\tau)$ (respectively, $h_\mu(\tau)\theta_{m,\mu}(\tau, z)$) converges absolutely and uniformly on any compact subset of $\mathcal{H}$ (respectively, $\mathcal{H} \times \mathbb{C}$). In particular they define holomorphic functions on $\mathcal{H}$ (respectively, $\mathcal{H} \times \mathbb{C}$). Moreover*

$$\begin{aligned} h_\mu(\tau)\theta_{m,\mu}(\tau, z)e^m(pz) &= O(y^{-\delta-\frac{3}{2}}) \ \text{ as } \ y \to 0, \\ h_\mu(\tau)\theta_{m,\mu}(\tau, z)e^m(pz) &= O\left(e\left(\frac{iy}{4m}\right)\right) \ \text{ as } \ y \to \infty \end{aligned}$$

hold uniformly with respect to x, where $\tau = x + iy$ and $z = p\tau + q$.

Lemma 1.6.4. *Let $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying part (i) of the definition 1.6.1. Assume that the estimate $e^m(pz)\phi(\tau, z) = O(y^{-\delta})$ as $y \to 0$ holds uniformly with respect to $\Re(\tau)$ for some positive real number δ. Then, $\phi \in J_{\frac{k}{2},m}(\Gamma^J(N), \chi)$. Moreover, if $\delta < \frac{k-1}{2}$, then $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$.*

Proof. The proof is similar to that of Lemma 3 in [36]. □

1.7 Differential operators

Differential operators on the spaces of automorphic forms are weight-increasing linear operators and they give rise to many interesting identities between Fourier coefficients of automorphic forms. The derivative of a modular form need not be a modular form. However, one can construct differential operators by taking an appropriate linear combination of higher-order derivatives. Rankin [42, 43] studied a general description of the differential operators on the space of modular forms. Cohen [10] explicitly constructed certain bilinear operators using differential operators and obtained cusp forms with interesting Fourier coefficients. Zagier [50, 51] studied the algebraic properties of these operators and called them Rankin–Cohen brackets.

1.7.1 Rankin-Cohen brackets for modular forms

Let k and l be positive integers and $\nu \geq 0$ be an integer. Let f and g be two complex-valued holomorphic functions on $\mathcal{H}$. The ν-th Rankin-Cohen bracket of f and g is defined by

$$[f, g]_\nu := \sum_{r=0}^{\nu} (-1)^{\nu-r} \binom{\nu}{r} \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)} D^r f D^{\nu-r} g, \tag{1.17}$$

where $D^r f = \dfrac{1}{(2\pi i)^r}\dfrac{d^r f}{d\tau^r}$.

Remark 1.7.1. Note that the 0-th Rankin-Cohen bracket [42, 43] is the usual product, i.e., $[f,g]_0 = fg$ and $[-,-]_\nu$ has the following property:

$$[f|_k\gamma, g|_l\gamma]_\nu = [f,g]|_{k+l+2\nu}\gamma, \;\; \forall\gamma \in SL_2(\mathbb{Z}). \tag{1.18}$$

Theorem 1.7.1 ([10]). *Let $\nu \geq 0$, be an integer. If $f \in M_k$ and $g \in M_l$, then $[f,g]_\nu \in M_{k+l+2\nu}$. Moreover, if $\nu > 0$, then $[f,g]_\nu \in S_{k+l+2\nu}$. In fact, $[\ ,\ ]_\nu$ is a bilinear map from $M_k \times M_l$ to $M_{k+l+2\nu}$.*

1.7.2 Serre derivative

Let k be a positive integer and f be a complex-valued holomorphic function on $\mathcal{H}$. Define the Serre derivative by

$$\vartheta(f)(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}f(\tau) - \frac{k}{12}E_2(\tau)f(\tau), \tag{1.19}$$

where $E_2 = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$ is the Eisenstein series of weight 2.

Theorem 1.7.2. *Let f be a modular form (cusp form) of weight k. Then $\vartheta(f)$ is a modular form (cusp form) of weight $k+2$.*

1.7.3 Heat operators

Let m be a positive integer. Define the heat operator by

$$L_m := \frac{1}{(2\pi i)^2}\left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right).$$

The heat operator does not necessarily map a Jacobi form to another Jacobi form. For k and m positive integers, define the modified heat operator by

$$L_{k,m} := L_m - \frac{2k-1}{6} m E_2. \tag{1.20}$$

Theorem 1.7.3. *Let k and m be positive integers and $\phi \in J_{k,m}$. Then $L_{k,m}(\phi) \in J_{k+2,m}$. Moreover, if ϕ is a cusp form, then $L_{k,m}(\phi)$ is also a Jacobi cusp form.*

1.7.4 Heat operators for Jacobi forms of degree g

Let $\mathcal{M}$ be a positive definite, symmetric, half-integral $g \times g$. Define the heat operator by

$$L_{\mathcal{M}} := \frac{1}{(2\pi i)^2} \left(8\pi i |\mathcal{M}| \frac{\partial}{\partial \tau} - \sum_{1 \le i,j \le g} \mathcal{M}_{ij} \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \right), \tag{1.21}$$

where $\tau \in \mathcal{H}$ and $z^t = (z_1, z_2, \cdots, z_g) \in \mathbb{C}^g$ and $\mathcal{M}_{ij}$ is the (i,j)-th cofactor of the matrix $\mathcal{M}$. $L_{\mathcal{M}}$ acts on $e(n\tau)e(Rz)$ by

$$L_{\mathcal{M}}(e(n\tau)e(Rz)) = (4n|\mathcal{M}| - \widetilde{\mathcal{M}}[r^t])e(n\tau)e(Rz),$$

where $\widetilde{\mathcal{M}}$ denotes the matrix of cofactors $\mathcal{M}_{ij}$ of the matrix $\mathcal{M}$.

Lemma 1.7.4. *[41] Let $\phi \in J_{k,\mathcal{M}}$. Then for $k \in \mathbb{Z}^+$, $\nu \geq 0$ and $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$(L_{\mathcal{M}}\phi)|_{k+2,\mathcal{M}} A = L_{\mathcal{M}}(\phi|_{k,\mathcal{M}} A) + \frac{2|\mathcal{M}|(k - \frac{g}{2})}{\pi i} \left(\frac{c}{c\tau + d} \right) (\phi|_{k,\mathcal{M}} A). \tag{1.22}$$

The heat operator commutes with the lattice action of the Jacobi group.

Define the modified heat operator which maps Jacobi forms to Jacobi forms as

$$L_{k,\mathcal{M}} := L_{\mathcal{M}} - \frac{(k-\frac{g}{2})|\mathcal{M}|}{3} E_2. \tag{1.23}$$

Lemma 1.7.5. *The operator $L_{k,\mathcal{M}}$ maps a Jacobi form (resp. Jacobi cusp form) of weight k and index $\mathcal{M}$ to a Jacobi form (resp. Jacobi cusp form) of weight $k+2$ and index $\mathcal{M}$.*

1.7.5 Rankin-Cohen brackets for Jacobi forms

Let k_1, k_2, m_1 and m_2 be positive integers and $\nu \geq 0$ be an integer. Let ϕ and ψ be complex-valued holomorphic functions defined on $\mathcal{H} \times \mathbb{C}$. The ν-th Rankin-Cohen bracket of ϕ and ψ is defined by

$$[\phi, \psi]_\nu := \sum_{l=0}^{\nu} (-1)^l \binom{k_1+\nu-\frac{3}{2}}{\nu-l} \binom{k_2+\nu-\frac{3}{2}}{l} m_1^{\nu-l} m_2^l L_{m_1}^l(\phi) L_{m_2}^{\nu-l}(\psi).$$

We note that here $x! = \Gamma(x+1)$.

Remark 1.7.2. One can easily verify that

$$[\phi|_{k_1,m_1}\gamma, \psi|_{k_2,m_2}\gamma]_\nu = [\phi,\psi]|_{k_1+k_2+2\nu, m_1+m_2}\gamma, \ \ \forall \gamma \in \Gamma^J. \tag{1.24}$$

Theorem 1.7.6. *[6] Let $\nu \geq 0$ be integer. If $\phi \in J_{k_1,m_1}$ and $\psi \in J_{k_2,m_2}$, then $[\phi,\psi]_\nu$ is a Jacobi form of weight $k_1+k_2+2\nu$ and index m_1+m_2.*

Chapter 2

Nonvanishing of L-functions associated with Jacobi forms

2.1 Introduction

Let $f(\tau) = \sum_{n\geqslant 1} a(n)e^{2\pi i n\tau}$ be a normalized Hecke eigenform of weight k for the group $SL_2(\mathbb{Z})$. Let $L^*(f,s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geqslant 1} a(n)n^{-s}$ be the completed L-function associated with f. The completed L-function $L^*(f,s)$ has an Euler product for $Re(s) \geq \frac{k+1}{2}$ and all the zeros of $L^*(f,s)$ can exist only inside the critical strip $\frac{k-1}{2} \leq Re(s) \leq \frac{k+1}{2}$. According to the generalized Riemann hypothesis, all the zeros of $L^*(f,s)$ can occur only on the line $Re(s) = \frac{k}{2}$. Towards this direction, Kohnen proved the following:

Theorem 2.1.1. *[24] Let $\mathcal{B}_k = \{f_1, f_2, ..., f_{dim(S_k)}\}$ be a basis of normalized Hecke eigenforms for S_k. Let $\epsilon > 0$ and t_0 be given real numbers. Then there exists a constant $C(t_0,\epsilon)$ such that for $k > C(t_0,\epsilon)$, the function*

$$\sum_{i=1}^{dim\ S_k} \frac{L^*(f_i,s)}{\langle f_i, f_i\rangle} \tag{2.1}$$

does not vanish on any point of the line segments $Im(s) = t_0$ with $\frac{k-1}{2} < Re(s) < \frac{k}{2} - \epsilon$ and $\frac{k}{2} + \epsilon < Re(s) < \frac{k+1}{2}$.

As a corollary Kohnen obtained the following result:

Corollary 2.1.2. *[24] Let $\epsilon > 0$ and t_0 be given real numbers. For $k > C(t_0, \epsilon)$ and any $s = Re(s) + it_0$ with $\frac{k-1}{2} < Re(s) < \frac{k}{2} - \epsilon$ and $\frac{k}{2} + \epsilon < Re(s) < \frac{k+1}{2}$, $\exists$ a cusp form $f \in \mathcal{B}_k$ such that $L^*(f, s) \neq 0$.*

To prove Theorem 2.1.1 and Corollary 2.1.2, Kohnen constructed the following kernel functions:

$$R_{k,s}(\tau) = \gamma_{k,s} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (c\tau + d)^{-k} \left(\frac{a\tau + b}{c\tau + d} \right)^{-s},$$

where $\tau \in \mathcal{H}$ and $1 < Re(s) < k - 1$ and $\gamma_{k,s} = \frac{1}{2} e^{\pi i s/2} \Gamma(s) \Gamma(k - s)$. These kernel functions relate a cusp form to its L-values.

Theorem 2.1.3. *[24] For a given positive integer k and a complex number $s = \sigma + it$ with $1 < \sigma < k - 1$, the function $R_{k,s} \in S_k$. Moreover, if $f \in S_k$, then we have*

$$\langle f, R_{k,s} \rangle = \frac{(-1)^{\frac{k}{2}} \pi (k-2)!}{2^{k-2}} L^*(f, s).$$

To prove Theorem 2.1.1 it is sufficient to prove the nonvanishing of the first Fourier coefficient of $R_{k,s}$ for large weights.

Kohnen's work is generalized to other kinds of automorphic forms like half-integral weight modular forms [28], Siegel modular forms [12], Hilbert modular forms [40]. Also, there are some results on the nonvanishing of derivatives of L-functions [15, 27] and products of L-functions [9].

One of the key points to prove Theorem 2.1.3 is to express the kernel functions $R_{k,s}$ as a linear combination of Poincaré series for modular forms, i.e.

$$R_{k,s}(\tau) = (2\pi)^s \Gamma(k - s) \sum_{n \geq 1} n^{s-1} P_{k,n}(\tau). \tag{2.2}$$

Nonvanishing of Poincaré series $P_{k,n}$ is an interesting problem in number theory. It is not known whether $P_{k,n}$ vanishes identically or not for general k and n. From (2.2), one can observe that the nonvanishing of kernel functions implies the nonvanishing of the Poincaré series. Rankin [44] studied the nonvanishing of Poincaré series using analytic tools and proved the following:

Theorem 2.1.4. *[44] There exists positive constants k_0 and B with $B > 4log\ 2$, such that for all $k \geq k_0$ and all positive integers $n \leq k^2 e^{-B\frac{\log(k)}{\log\log(k)}}$ the Poincaré series $P_{k,n}$ does not vanish identicaly.*

The Jacobi Poincaré series $P_{k,m;n,r}$ (1.7) are a natural generalization of $P_{k,n}$ to several variables. Following the work of Rankin, Das [11] obtained the following result for the nonvanishing of Jacobi Poincaré series:

Theorem 2.1.5. *[11] Let $m,\ n \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $D = 4nm - r^2 > 0$ and $\pi D > 2m$. Then the Jacobi Poincaré series $P_{k,m;n,r} \neq 0$ whenever*

$$M\left(\frac{\pi D}{m}\right)\sigma_0(D)D < \frac{m^{\frac{8}{7}}}{2^{\frac{2}{9}}\pi}\left(\frac{2}{6^{2/3}} + \frac{54}{2^{5/6}} + \frac{16}{2^{3/4}}\right)^{-\frac{3}{2}},$$

where $\sigma_0(D)$ denotes the number of divisors of d, $M(x) = e^{\frac{Blog\ x}{loglog\ 2x}}$ and B is as in Theorem 2.1.4.

In the same paper, Das also obtained the nonvanishing of Jacobi Poincaré series of matrix index.

Theorem 2.1.6. *[11] Let $2R \equiv 0\ (mod\ \mathbb{Z}^g\ 2\mathcal{M})$. Then there exists an integer k_0 and a constant $B \geq 3log2$ such that for all even $k \geqslant k_0$, $P_{k,\mathcal{M};(n,R)}$ does not vanish identically whenever*

$$k' \leq \frac{\pi D}{|2\mathcal{M}|} \leq k'^{1+\alpha(g)} exp\left(-\frac{B\ \log(k')}{\log\log(k')}\right),$$

where $k' = k - \frac{g}{2} - 1$ *and* $\alpha(g) = \begin{cases} \frac{2}{3(g+2)} & if\ 1 \leqslant g \leqslant 4, \\ \frac{2}{3g} & if\ g \geqslant 5. \end{cases}$

Jacobi Poincaré series for the congruence subgroups are defined below:

Definition 2.1.7. *Let* $n \in \mathbb{Z}$ *and* $R \in \mathbb{Z}^g$ *with* $4n > \mathcal{M}^{-1}[R^t]$. *For* $k \geq g+2$, *define the Poincaré series*

$$P^N_{k,\mathcal{M};n,R}(\tau, z) = \sum_{\gamma \in \Gamma^J_{g,\infty} \backslash \gamma \in \Gamma^J_g(N)} e(n\tau + Rz)|_{k,m}\gamma(\tau, z),$$

where $\Gamma^J_g(N) = \Gamma_0(N) \ltimes (\mathbb{Z}^{g,1} \times \mathbb{Z}^{g,1})$.

It is well known that $P^N_{k,\mathcal{M};(n,R)}$ is a Jacobi cusp form of weight k and index $\mathcal{M}$ with respect to the group $\Gamma^J_g(N)$.

Shankhadhar [45] generalized the work of Das and obtained nonvanishing of Poincaré series for congruence subgroups.

Theorem 2.1.8. *[45] For any* $\epsilon > 0$ *there exists a positive integer* $k_0(\epsilon, \mathcal{M}, N)$ *such that* $P^N_{k,\mathcal{M};n,R}(\tau, z)$ *does not vanish identically if* $k > k_0$ *and*

$$D^\epsilon \left(\frac{\pi D}{det(2\mathcal{M})}\right) (D, N)^{\frac{2}{g}} \ll_\epsilon (det(2\mathcal{M}))^{\frac{1}{g}} \left(\frac{N}{\sigma_0(N)}\right)^{\frac{2}{g}} k'^{1+\alpha(g)},$$

where $k' = k - \frac{g}{2} - 1,\ \alpha(g) = \begin{cases} \frac{2}{3(g+2)} & \text{if } 1 \leqslant g \leqslant 4, \\ \frac{2}{3g} & \text{if } g \geqslant 5. \end{cases}$

In this chapter, we generalize the work of Kohnen in the context of Jacobi forms of integer index [38] and matrix index [39] as well. We also obtain nonvanishing of Jacobi Poincaré series in both cases.

2.2 Statements of results

First we state our main results in the case $g = 1$. For any Jacobi cusp form ϕ we write $\overline{\phi}(\tau, z) = \overline{\phi(-\overline{\tau}, -\overline{z})}$. Then $\overline{\phi}$ has Fourier coefficients $\overline{c_\mu(N)}$ in the corresponding theta decomposition.

Theorem 2.2.1. *[38] Let $m \in \mathbb{Z}$, $\epsilon > 0$ and t' be real numbers. Then for any given $s = \sigma + it'$ with $\frac{k}{2} - \frac{3}{4} < \sigma < \frac{k}{2} - \frac{1}{4} - \epsilon$ or $\frac{k}{2} - \frac{1}{4} + \epsilon < \sigma < \frac{k}{2} + \frac{1}{4}$ there exists $k_0 = k_0(t', \epsilon)$ and a Hecke eigenform ϕ of weight $k > k_0$ and index m such that the vector-valued function $\Lambda(\overline{\phi}, s) = (\Lambda_i(\overline{\phi}, s))_{i=0,1,\ldots,2m-1} \neq 0$.*

In [39] we generalize the above result to Jacobi forms of matrix index ($g \in \mathbb{N}$).

Theorem 2.2.2. *[39] Let $\mathcal{M}$ be a $g \times g$ symmetric positive-definite half-integral matrix, $\epsilon > 0$ and t' be real numbers. Then for any given $s = \sigma + it'$ with $\frac{k}{2} - \frac{g}{4} - \frac{1}{2} < \sigma < \frac{k}{2} - \frac{g}{4} - \epsilon$ or $\frac{k}{2} - \frac{g}{4} + \epsilon < \sigma < \frac{k}{2} - \frac{g}{4} + \frac{1}{2}$ there exists $k_0 = k_0(t', \epsilon)$ and a Hecke eigenform ϕ of weight $k > k_0$ and index $\mathcal{M}$ such that the vector-valued function $\Lambda(\overline{\phi}, s) = (\Lambda_i(\overline{\phi}, s))_{i \in \mathbb{Z}^g \backslash \mathbb{Z}^g 2\mathcal{M}} \neq 0$.*

We have the following nonvanishing of the Jacobi Poincaré series.

Theorem 2.2.3. *[39] Let $\mathcal{M}$ be a $g \times g$ symmetric positive-definite half-integral matrix, $n \in \mathbb{N}$ and R be as in Theorem 2.4.2 and $0 < \delta < \frac{1}{2}$. Then for any positive integer $k > k_0$ we have $P_{k,\mathcal{M};n,R} \neq 0$ where*

$$k_0 = max \left\{ 8\pi D + 2, 2\left(\frac{(2\pi D)^{2\delta} 2^{\frac{g}{2}+1}\sqrt{\mathcal{M}}}{\pi(e(Rt_0) + e(-Rt_0))}\right)^{\frac{1}{2\delta}} + g + 1, \frac{2log\left(2^2\pi^3(2\pi D)^{1+\delta+\frac{g}{4}}\right)}{log2} + g + 2 \right\}.$$

The approach of the proof is similar to the work of Kohnen. We first define kernel functions, calculate the Fourier coefficients of these kernel functions, and obtain their nonvanishing. We only give a detailed proof of Theorem 2.2.2. The case $g = 1$ [38] can be deduced as a particular case of Theorem 2.2.2.

2.3 Kernel Functions

Let $k > 2g + 4$ be a positive even integer and $\mathcal{M}$ be a symmetric positive definite half-integral $g \times g$ matrix. For $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^{g,1}$ and $s \in \mathbb{C}$ with $1 < Re(s) < k - 2g - 1$ define the kernel functions

$$\Omega^{k,\mathcal{M}}_{t_0,s}(\tau, z) = \sum_{h \in H^J_g \backslash \Gamma^J_g} \phi_{t_0,s}(\tau, z)|_{k,\mathcal{M}} h(\tau, z), \tag{2.3}$$

where $\phi_{t_0,s}(\tau, z) = \frac{1}{\tau^s} e(-\dfrac{1}{\tau}\mathcal{M}[z - t_0])$ and $H^J_g = \{(Id, (\lambda, 0)) : \lambda \in \mathbb{Z}^{g,1}\}$. A set of all coset representatives for $H^J_g \setminus \Gamma^J_g$ is given by $\{(Id, (0, \nu))(M, (0, 0)) : M \in \Gamma,\ \nu \in \mathbb{Z}^{g,1}\}$.

Theorem 2.3.1. *[39] Let k be a positive integer, $\mathcal{M}$ be a positive definite symmetric matrix of order g with $k > 2g + 4$ and $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^{g,1}$. If $1 < Re(s) < k - 2g - 1$ then*

$$\Omega^{k,\mathcal{M}}_{t_0,s} \in J^{cusp}_{k,\mathcal{M}}.$$

To prove Theorem 2.3.1, it is sufficient to prove the absolute and uniform convergence of the kernel functions $\Omega^{k,\mathcal{M}}_{t_0,s}$ as the required transformation properties for $\Omega^{k,\mathcal{M}}_{t_0,s}$ to be a Jacobi form are easy to observe from (2.3). The required Fourier expansion will be computed later in Theorem 2.4.1 and the cuspidality will be deduced. First, we state a fact that will be used in the proof of the above theorem.

Lemma 2.3.2. *For every $(\tau, z) \in \mathcal{H} \times \mathbb{C}^{g,1}$, there exists $r = r(\tau, z) \geq 0$ such that the image of $B(\tau, \frac{1}{2}) \times D(z, \frac{1}{2})$ (where $B(\tau, \frac{1}{2})$ denotes for hyperbolic ball) under any $M \in \Gamma$ is contained in $B(M(\tau), \frac{1}{2}) \times D(0, r)$.*

Now we prove Theorem 2.3.1.

Proof. Let $(\tau, z) \in \mathcal{H} \times \mathbb{C}^{g,1}$. Using the fact $B(\tau, \frac{1}{2}) = D(\tau_0, r_0)$ for some τ_0 and holomorphicity of the functions $\phi_{t_0,s}$ and $\phi_{t_0,s}|_{k,\mathcal{M}}h$ for any $h \in H_g^J \setminus \Gamma_g^J$, we have

$$|\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau, z)| \leqslant \frac{2^g\Gamma(1+\frac{g}{2})}{\pi^{1+\frac{g}{2}}r_0^2} \int\limits_{D(\tau_0,r_0)\times D(z,\frac{1}{2})} |\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau', z'))|dx'dy'du'dv'.$$

The map $(\tau', z') \mapsto \mu_{k,\mathcal{M}}(\tau', z')y'^{-g-2}$ is continuous and hence there exists a positive real number $m_{(\tau,z)}$ such that

$$1 \leqslant \frac{\mu_{k,\mathcal{M}}(\tau', z')y'^{-g-2}}{m_{(\tau,z)}}$$

for all $(\tau', z') \in D(\tau_0, r_0) \times D(z, \frac{1}{2})$. Hence rewriting the above equation we get

$$|\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau, z)| \leqslant \frac{2^g\Gamma(1+\frac{g}{2})}{\pi^{1+\frac{g}{2}}r_0^2 m_{\tau,z}} \int\limits_{B(\tau,\frac{1}{2})\times D(z,\frac{1}{2})} |\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau', z')|\mu_{k,\mathcal{M}}(\tau', z')dV(\tau', z').$$

Summing over all the elements of the coset $\mathcal{H}_g^J \setminus \Gamma_g^J$, we have

$$\begin{aligned}
& 2^{-g}\pi^{1+\frac{g}{2}}\frac{r_0^2}{\Gamma(1+\frac{g}{2})}m_{\tau,z}\sum_{h\in\mathcal{H}_g^J\setminus\Gamma_g^J}|\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau, z)| \\
\leqslant\ & \sum_{h\in\mathcal{H}_g^J\setminus\Gamma_g^J}\ \int\limits_{B(\tau,\frac{1}{2})\times D(z,\frac{1}{2})} |\phi_{t_0,s}(h(\tau', z'))|\mu_{k,\mathcal{M}}(h(\tau', z'))dV(\tau', z') \\
=\ & \sum_{h\in\mathcal{H}_g^J\setminus\Gamma_g^J}\ \int\limits_{h(B(\tau,\frac{1}{2})\times D(z,\frac{1}{2}))} |\phi_{t_0,s}(\tau', z')|\mu_{k,\mathcal{M}}(\tau', z')dV(\tau' z') \\
=\ & \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}}\ \int\limits_{[Id,0,\nu].M(B(\tau,\frac{1}{2})\times D(z,\frac{1}{2}))} |\phi_{t_0,s}(\tau', z')|\mu_{k,\mathcal{M}}(\tau', z')dV(\tau', z') \\
\leqslant\ & \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}}\ \int\limits_{[Id,0,\nu].(B(M(\tau),\frac{1}{2})\times D(0,r))} |\phi_{t_0,s}(\tau', z')|\mu_{k,\mathcal{M}}(\tau', z')dV(\tau', z')
\end{aligned}$$

$$= \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{B(M(\tau),\frac{1}{2})}\int_{D(\nu,r)} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dV(\tau',z').$$

Estimating the integral we have

$$\begin{aligned}&\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{D(\nu,r)} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dp'dq'\\ &\leqslant 2rg\int_{\cup_{\nu\in\mathbb{Z}^{g,1}}D(\nu,r)} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')y'^{-g}du'dv'dp'dq'.\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}&\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{D(\nu,r)} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dp'dq'\\ &\leqslant \sqrt{\frac{2^{3g}}{|\mathcal{M}|}}R^{2g}|\frac{1}{\tau'^{s-g}}|y'^{\frac{k-3g}{2}}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}&2^{-g}\pi^{1+\frac{g}{2}}\frac{r_0^2}{\Gamma(1+\frac{g}{2})}m_{\tau,z}\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J}|\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))|\\ &\leqslant \sqrt{\frac{2^{3g}}{|\mathcal{M}|}}R^{2g}\sum_{M\in\Gamma}\int_{B(M(\tau),\frac{1}{2})}|\frac{1}{\tau'^{s-g}}|y'^{\frac{k-3g-4}{2}}dx'dy'.\end{aligned}$$

Now estimate $m_{\tau,z}$ whenever $k > 2g+4$ to get

$$\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J}|\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))| \ll \frac{(1+y^2)^g}{y^{(\frac{k}{2}+g)}}\sum_{M\in\Gamma}\int_{B(M(\tau),\frac{1}{2})}|\frac{1}{\tau'^{s-g}}|y'^{\frac{k-3g-4}{2}}dx'dy'.$$

Simplifying, for $1 < r_0 < \sigma$ one gets

$$\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J} |\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))| \ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{g+1}e^{\frac{c_1}{y}}\int\limits_{B'}\frac{{y'}^{\frac{k-3g-4}{2}}}{|\tau'|^{\sigma-g-r}|\tau'|^r}dx'dy',$$

where $B' = \{\tau' \in \mathcal{H} \mid y' < T(\tau,\Gamma) = 2cosh(\frac{1}{2})c_\Gamma(y+\frac{1}{y}), \frac{1}{|\tau'|^2} \ll \frac{y+\frac{1}{y}}{y'}\}$. Proceeding as in [34], for any $1 < r_0 < \sigma$ one gets

$$\begin{aligned}
\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J} |\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))| &\ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{g+1}e^{\frac{c_1}{y}}\int\limits_{y'=0}^{T(\tau,\Gamma)}\int\limits_{x'=-\infty}^{\infty}\frac{{y'}^{\frac{k-3g-4}{2}}}{(x'^2+y'^2)^{\frac{r_0}{2}}}\left(\frac{y+\frac{1}{y}}{y'}\right)^{\frac{\sigma-r_0-g}{2}}dx'dy' \\
&= y^{-\frac{k}{2}}(y+\frac{1}{y})^{\frac{\sigma-r_0+g+2}{2}}e^{\frac{c_1}{y}}\int\limits_{y'=0}^{T(\tau,\Gamma)}\int\limits_{x'=-\infty}^{\infty}\frac{{y'}^{\frac{k-\sigma-2g+r_0-4}{2}}}{(x'^2+y'^2)^{\frac{r_0}{2}}}dx'dy' \\
&\ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{\frac{\sigma-r_0+g+2}{2}}e^{\frac{c_1}{y}}\frac{\Gamma(\frac{r-1}{2})}{\Gamma(\frac{r}{2})}\int\limits_{y'=0}^{T(\tau,\Gamma)}{y'}^{\frac{k-\sigma-2g-r_0-2}{2}}dy \\
&\ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{\frac{\sigma-r_0+g+2}{2}}e^{\frac{c_1}{y}}\frac{\Gamma(\frac{r-1}{2})}{\Gamma(\frac{r}{2})}T(\tau,\Gamma)^{\frac{k-\sigma-r_0-2g}{2}},
\end{aligned}$$

whenever $1 < \sigma < k-2g-1$. Hence $\Omega_{t_0,s}^{k,\mathcal{M}}$ converges absolutely and uniformly on compact subsets of $\mathcal{H}\times\mathbb{C}^{g,1}$. □

Theorem 2.3.3. *[39] Let k and $\mathcal{M}$ be as before. For any $f \in J_{k,\mathcal{M}}^{cusp}$, the inner product $\langle\Omega_{t_0,s}^{k,\mathcal{M}}, f\rangle$ is a holomorphic function on the vertical strip $1+\frac{g}{2} < Re(s) < k-2g-1$.*

Proof. Rewrite the definition of kernel functions as

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) = \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}}\phi_{t_0,s}|_{k,\mathcal{M}}[Id,0,\nu]|_{k,\mathcal{M}}M(\tau,z).$$

Putting $t_0 = (2\mathcal{M})^{-1}\beta^t$ with $\beta \in \mathbb{Z}^g$ one has

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) = \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}} \phi_{0,s}|_{k,\mathcal{M}}[Id,0,\nu]|_{k,\mathcal{M}}[Id,0,(2\mathcal{M})^{-1}\beta^t]|_{k,\mathcal{M}}M(\tau,z).$$

Theta inversion formula is given by

$$\sum_{\nu\in\mathbb{Z}^{g,1}} \phi_{0,s}|_{k,\mathcal{M}}[Id,0,\nu] = \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}\Theta_{\mathcal{M},R}(\tau,z).$$

Hence one gets

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) = \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\sum_{M\in\Gamma}\left(\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z)\right)|_{k,\mathcal{M}}M.$$

Consequently one has

$$\begin{aligned}\langle\Omega_{t_0,s}^{k,\mathcal{M}},f\rangle &= \int\limits_{\Gamma\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z)\overline{f(\tau,z)}\mu_{k,\mathcal{M}}^2 dV\\ &= \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\int\limits_{\Gamma\backslash\mathcal{H}\times\mathbb{C}^{g,1}}\sum_{M\in\Gamma}\left(\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z)\right)|_{k,\mathcal{M}}M\\ &\times\quad\overline{f(\tau,z)|_{k,\mathcal{M}}M}\mu_{k,\mathcal{M}}^2 dV.\end{aligned}$$

The transformation formula for $\mu_{k,\mathcal{M}}$ and the usual unfolding argument implies

$$\begin{aligned}\sqrt{(2i)^g|\mathcal{M}|}\langle\Omega_{t_0,s}^{k,\mathcal{M}},f\rangle &= \int\limits_{\mathcal{H}}\int\limits_{\mathbb{Z}^{g,1}\tau+\mathbb{Z}^{g,1}\backslash\mathbb{C}^{g,1}}\left(\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z)\right)\\ &\times\ \overline{f(\tau,z)}\mu_{k,\mathcal{M}}^2 dV.\end{aligned}$$

Putting the theta decomposition of f in the above equation one gets

$$\sqrt{(2i)^g|\mathcal{M}|}\langle \Omega_{t_0,s}^{k,\mathcal{M}}, f\rangle = \frac{1}{2^g\sqrt{|\mathcal{M}|}}\sum_{R\in\mathbb{Z}^g\setminus\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\int\limits_{\mathcal{H}} \frac{1}{\tau^{s-\frac{g}{2}}}\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy.$$

Now consider the inner integral

$$\begin{aligned}
&\int\limits_{\mathcal{H}} \frac{1}{\tau^{s-\frac{g}{2}}}\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy \\
=&\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1}\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^{s-\frac{g}{2}}}e(\frac{n}{4}M^{-1}[R^t])\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy \\
=&\sum_{n_0 \pmod{4|\mathcal{M}|}} e(\frac{n_0}{4}M^{-1}[R^t])\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1}\zeta_{4|\mathcal{M}|}(\tau+n_0, s-\frac{g}{2})\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy,
\end{aligned}$$

where $\zeta_{m\mathbb{Z}}(\tau,z)=\sum\limits_{l\in\mathbb{Z}}(\tau+4|\mathcal{M}|l)^{-s}$. Hence it is sufficient to show that integral

$$\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1}\zeta_{4|\mathcal{M}|}(\tau+n_0, s-\frac{g}{2})\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy \tag{2.4}$$

defines a holomorphic function of s on the given region. Note that $f_R(\tau)=O(e^{-\pi\frac{y}{2|\mathcal{M}|}})$ as $y\to\infty$ uniformly on x and for $\sigma=Re(s)>1+\frac{g}{2}$ one has

$$\zeta_{4|\mathcal{M}|\mathbb{Z}}\left(\tau+n_0, s-\frac{g}{2}\right) \ll \frac{e^{-\pi\frac{y}{2|\mathcal{M}|}}}{(4|\mathcal{M}|)^{-\sigma+\frac{g}{2}}}(1+y^{-\sigma+\frac{g}{2}}).$$

This implies

$$\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1}|\zeta_{4|\mathcal{M}|}\left(\tau+n_0, s-\frac{g}{2}\right)\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}|dxdy$$

$$\ll \int_{y=0}^{\infty} e^{-\pi\frac{y}{|\mathcal{M}|}}(y^{k-\frac{g}{2}-2}+y^{k-2-\sigma})dy$$
$$\ll \left(\frac{\pi}{|\mathcal{M}|}\right)^{-k+\frac{g}{2}+1}\Gamma\left(k-\frac{g}{2}-1\right)+\left(\frac{\pi}{|\mathcal{M}|}\right)^{-k+\sigma+1}\Gamma(k-\sigma-1).$$

From this relation, one deduces that the integral (2.4) is absolutely and uniformly convergent on $1+\frac{g}{2}<Re(s)<k-2g-1$. Hence the theorem follows. □

Theorem 2.3.4. *[39] Let $k>2g+4$ and $\mathcal{M}$ be as above and $t_0\in(2\mathcal{M})^{-1}\mathbb{Z}^g$. If $s\in\mathbb{C}$ such that $1+\frac{g}{2}<Re(s)<k-2g-1$ then we have*

$$\begin{aligned}\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z) &= \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{R\in z^g\backslash z^g(2\mathcal{M})}e(-R(2\mathcal{M})^{-1}\beta^t)\\ &\times \sum_{D=1}^{\infty}\left(\frac{D}{4|\mathcal{M}|}\right)^{s-\frac{g}{2}-1}P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau,z).\end{aligned}$$

Proof. Rewriting the definition of $\Omega^{k,\mathcal{M}}_{t_0,s}$ as in the proof of Theorem 2.3.3, $\sqrt{(2i)^g|\mathcal{M}|}\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z)$ equals

$$=\sum_{M\in\Gamma\backslash\Gamma_\infty}(\sum_{M'\in\Gamma_\infty}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z))|_{k,\mathcal{M}}M'|_{k,\mathcal{M}}M.$$

Now $\Gamma_\infty=\left\{\pm\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}: l\in\mathbb{Z}\right\}$. Then

$$\begin{aligned}&\sum_{M'\in\Gamma_\infty}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z))|_{k,\mathcal{M}}M'\\ =&\sum_{l\in\mathbb{Z}}\frac{1}{(\tau+l)^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}(e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t))\Theta_{\mathcal{M},R}(\tau+l,z).\end{aligned}$$

Rewrite the above equation to get

$$\begin{aligned}
&\sum_{M'\in\Gamma_\infty}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z))|_{k,\mathcal{M}}M'\\
=&\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}(e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t))\Theta_{\mathcal{M},R}(\tau,z)\\
\times&\sum_{l_0=1}^{4|\mathcal{M}|}\sum_{l\in\mathbb{Z}}\frac{e(\frac{1}{4}\mathcal{M}^{-1}[R^t]l_0)}{(\tau+l_0+4|\mathcal{M}|l)^{s-\frac{g}{2}}}\\
=&\sum_{l_0=1}^{4|\mathcal{M}|}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}\left(e\left(\frac{l_0}{4}\mathcal{M}^{-1}[R^t]-R(2\mathcal{M})^{-1}\beta^t\right)+e\left(\frac{l_0}{4}\mathcal{M}^{-1}[R^t]+R(2\mathcal{M})^{-1}\beta^t\right)\right)\\
\times&\ \Theta_{\mathcal{M},R}(\tau,z)\zeta_{4|\mathcal{M}|\mathbb{Z}}(\tau+l_0,s-\frac{g}{2}).
\end{aligned}$$

Inserting the Fourier expansion of $\zeta_{4|\mathcal{M}|\mathbb{Z}}(\tau,s)=\frac{1}{(4|\mathcal{M}|)}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}e(\frac{D(\tau+l_0)}{4|\mathcal{M}|})$ one obtain

$$\begin{aligned}
&\sqrt{(2i)^g|\mathcal{M}|}\,\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z)\\
=&\ \frac{1}{(4|\mathcal{M}|)}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{l_0=1}^{4|\mathcal{M}|}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}\Bigg[e\left(\frac{l_0}{4}\mathcal{M}^{-1}[R^t]-R(2\mathcal{M})^{-1}\beta^t\right)\\
+&\ e\left(\frac{l_0}{4}\mathcal{M}^{-1}[R^t]+R(2\mathcal{M})^{-1}\beta^t\right)\Bigg]\sum_{D=1}^{\infty}\sum_{M\in\Gamma}\left(\frac{D}{4|\mathcal{M}|}\right)^{s-\frac{g}{2}-1}\Theta_{\mathcal{M},R}(\tau,z)e(\frac{D(\tau+l_0)}{4|\mathcal{M}|})|_{k,\mathcal{M}}M\\
=&\ \frac{1}{(4|\mathcal{M}|)}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{l_0=1}^{4|\mathcal{M}|}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}\Bigg[e\left(\frac{l_0}{4}\mathcal{M}^{-1}[R^t]-R(2\mathcal{M})^{-1}\beta^t\right)\\
+&\ e\left(\frac{l_0}{4}\mathcal{M}^{-1}[R^t]+R(2\mathcal{M})^{-1}\beta^t\right)\Bigg]\sum_{D=1}^{\infty}\left(\frac{D}{4|\mathcal{M}|}\right)^{s-\frac{g}{2}-1}e\left(\frac{Dl_0}{4|\mathcal{M}|}\right)\\
\times&\sum_{M\in\Gamma}\ \sum_{\substack{\mu\in\mathbb{Z}^g\\ \mu\equiv R\pmod{\mathbb{Z}^g(2\mathcal{M})}}} e\left(\left(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t]\right)\tau+\mu z\right)|_{k,\mathcal{M}}M.
\end{aligned}$$

For every $D,\ l_0$ and R one has

$$\begin{aligned}
&\sum_{M\in\Gamma}\ \sum_{\substack{\mu\in\mathbb{Z}^g\\ \mu\equiv R \pmod{\mathbb{Z}^g(2\mathcal{M})}}} e\left(\left(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t]\right)\tau+\mu z\right)|_{k,\mathcal{M}}M\\
=\ &\frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z},\ (c,d)=1,\ \mu\in\mathbb{Z}^g\\ \mu\equiv R \pmod{\mathbb{Z}^g(2\mathcal{M})}}}(c\tau+d)^{-k}e\left(\frac{-c}{c\tau+d}M[z^t]\right)e\left(\left(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t]\right)\frac{a\tau+b}{c\tau+d}+\frac{\mu z}{c\tau+d}\right),
\end{aligned}$$

where a and b are chosen such that $ad-bc=1$. Thus

$$\begin{aligned}
&\sum_{M\in\Gamma}\ \sum_{\substack{\mu\in\mathbb{Z}^g\\ \mu\equiv R \pmod{\mathbb{Z}^g(2\mathcal{M})}}} e\left(\left(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t]\right)\tau+\mu z\right)|_{k,\mathcal{M}}M\\
=\ &\frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z},\ (c,d)=1,\\ \mu\in\mathbb{Z}^g}}(c\tau+d)^{-k}e\left(-\frac{c}{c\tau+d}\mathcal{M}[z^t]+\frac{a\tau+b}{c\tau+d}\mathcal{M}[\mu^t]+2\mu\mathcal{M}\frac{z^t}{c\tau+d}\right)\\
\times\ &e\left(\left(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]\right)\frac{a\tau+b}{c\tau+d}\right)e\left(R\frac{z}{c\tau+d}+R\mu\frac{a\tau+b}{c\tau+d}\right)\\
=\ &\frac{1}{2}P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau,z).
\end{aligned}$$

This implies

$$\begin{aligned}
&\sqrt{(2i)^g|\mathcal{M}|}\,\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z)\\
=\ &\frac{1}{(8|\mathcal{M}|)}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{D=1}^{\infty}\left(\frac{D}{4|\mathcal{M}|}\right)^{s-\frac{g}{2}-1}\\
\times\ &\sum_{l_0=1}^{4|\mathcal{M}|}\ \sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e\left(\left(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]\right)l_0\right)\left(e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t)\right)\\
\times\ &P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau,z).
\end{aligned}$$

We have the following identity:

$$\sum_{l_0=1}^{4|\mathcal{M}|} e\left(\left(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]\right)l_0\right) = \begin{cases} 4|\mathcal{M}|, & \text{if } \frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]\in\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Using the above identity, we see that

$$\begin{aligned} & \sqrt{(2i)^g|\mathcal{M}|}\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) \\ = \; & \frac{(2\pi)^{s-\frac{g}{2}}}{2e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}(e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t)) \\ \times \; & \sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau,z) \\ = \; & \frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}e(-R(2\mathcal{M})^{-1}\beta^t) \\ \times \; & \sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau,z), \end{aligned}$$

where in the last line we have used the relation $P_{k,\mathcal{M};n,R} = P_{k,\mathcal{M};n,-R}$ for k even. Hence the theorem follows. □

Corollary 2.3.5. *Let* $k > 2g+4$ *and* $\mathcal{M}$ *be as above and* $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^g$. *If* $s \in \mathbb{C}$ *such that* $1+\frac{g}{2} < Re(s) < k-2g-1$ *and* $f \in J_{k,\mathcal{M}}^{cusp}$ *then we have*

$$\langle \Omega_{t_0,s}^{k,\mathcal{M}}, f\rangle = \frac{\pi}{2^{k-2}e^{\pi i\frac{s}{2}}}\frac{\Gamma(k-\frac{g}{2}-1)}{\Gamma(s-\frac{g}{2})\Gamma(k-s)}\sum_{R\in\mathbb{Z}^g\backslash(\mathbb{Z}^g(2\mathcal{M}))}e(-Rt_0)\Lambda_R(\overline{f},k-s).$$

Proof. It follows from Theorem 1.5.3, (1.10), and Theorem 2.3.4. □

2.4 Nonvanishing of L-functions

Theorem 2.4.1. *[39] Let k be a positive integer, $\mathcal{M}$ be a positive definite symmetric matrix of order g with $k > 2g+4$ and $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^g$. If $1 < Re(s) < k-2g-1$, then $\Omega_{t_0,s}^{k,\mathcal{M}}$ has Fourier series expansion*

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\ R\in\mathbb{Z}^g,\\ 4n>M^{-1}[R^t]}} \omega_k(n,R)e(n\tau + Rz),$$

where

$$\begin{aligned}\omega_k(n,R) =& \frac{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}} \frac{(e(-Rt_0)+e(Rt_0))}{\Gamma(s-\frac{g}{2})} \\ &+ (-i)^{k-s-1}\frac{(2\pi D)^{k-s-1}}{\Gamma(k-s)}\{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R) + e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)\} \\ &+ \frac{(2\pi)^{k-\frac{g}{2}} D^{k-\frac{g}{2}-1} i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})} \sum_{\substack{(a,c)=1,\ c'c\equiv 1 (mod\ a),\\ ac>0}} (\frac{a}{c})^{k-s} a^{-k} \\ &\sum_{\nu(a\mathbb{Z}^{g,1})} e(R\frac{\nu - t_0}{a})\Big[e(-\frac{c}{a}\mathcal{M}[\nu - t_o])e(\frac{nc'}{a})\ {}_1F_1(k-s, k-\frac{g}{2}; -\frac{2\pi D i}{ac}) \\ &+ \ e(\frac{c}{a}\mathcal{M}[\nu - t_o])e(-\frac{nc'}{a})\ {}_1F_1(k-s, k-\frac{g}{2}; \frac{2\pi D i}{ac})\Big],\end{aligned}$$

where

$$\mathcal{I}_X(a) = \begin{cases} 1 & \text{if } a \in X, \\ 0 & \text{otherwise} \end{cases}$$

and ${}_1F_1$ is Kummer's hypergeometric function.

Proof. Rewriting the definition of $\Omega_{t_0,s}^{k,\mathcal{M}}$ as

$$\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z) = \sum_{\substack{\nu\in\mathbb{Z}^g \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma}} (c\tau+d)^{-k} e(-\frac{c}{c\tau+d}\mathcal{M}[z])(\frac{a\tau+b}{c\tau+d})^{-s} e\left(\frac{-\mathcal{M}[\frac{z}{c\tau+d}+\nu-t_0]}{\frac{a\tau+b}{c\tau+d}}\right).$$

Break the sum into three parts corresponding to the matrices with $c = 0,\ a = 0$, and $ac \neq 0$ and compute the Fourier expansion of each part. Sum C_0 corresponding to matrices satisfying $c = 0$ i.e., $\left\{ \pm \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}$ is given by

$$C_0 = \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^{g,1}} \left[(\tau+l)^{-s} e\left(-\frac{\mathcal{M}[z+\nu-t_0]}{\tau+l}\right) + (\tau+l)^{-s} e\left(-\frac{\mathcal{M}[-z+\nu-t_0]}{\tau+l}\right)\right].$$

The contribution of the first part of the sum to $(n,R)^{th}$ Fourier coefficient $c_{01}(n,R)$ is given by

$$c_{01}(n,R) = \int\limits_{ic-\infty}^{ic+\infty} (\int\limits_{ic_1-\infty}^{ic_1+\infty} \ldots \int\limits_{ic_g-\infty}^{ic_g+\infty} \tau^{-s} e\left(-\frac{\mathcal{M}[z-t_0]}{\tau}\right) e(-R.z)dz) e(-n\tau)d\tau,$$

where $c, c_i \in \mathbb{R}$ for $i = 1, 2, ..., g$ and $c > 0$. We have

$$\begin{aligned} c_{01}(n,R) &= e(-Rt_0) \int\limits_{ic-\infty}^{ic+\infty} (\int\limits_{ic_1-\infty}^{ic_1+\infty} \ldots \int\limits_{ic_g-\infty}^{ic_g+\infty} \tau^{-s} e(-\frac{\mathcal{M}[z]}{\tau} - Rz)dz) e(-n\tau)d\tau \\ &= e(-Rt_0) \frac{1}{(2i)^{\frac{g}{2}}\sqrt{|\mathcal{M}|}} \int\limits_{ic-\infty}^{ic+\infty} \tau^{-s} e(\frac{1}{4}\mathcal{M}^{-1}[R^t]\tau)\tau^{\frac{g}{2}} e(-n\tau)d\tau \\ &= \frac{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4})(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}} \frac{e(-Rt_0)}{\Gamma(s-\frac{g}{2})}. \end{aligned}$$

Similarly one can compute the contribution of the second part to get the Fourier expansion of C_0

$$C_0(n,R) = \frac{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4})(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}} \frac{(e(-Rt_0)+e(Rt_0))}{\Gamma(s-\frac{g}{2})}. \tag{2.5}$$

The sum A_0 corresponding to matrices satisfying $a=0$, i.e., $\left\{ \pm \begin{pmatrix} 0 & -1 \\ l & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}$ is given by

$$\begin{aligned} A_0 &= \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^g} \Bigg[e(-\frac{s}{2})(\tau+l)^{-k+s} e\Bigg(-\frac{\mathcal{M}[z]}{\tau+l}\Big) e(-\frac{\mathcal{M}[\frac{z}{\tau+l}+\nu-t_0]}{\frac{-1}{\tau+l}}\Bigg) \\ &+ e(\frac{s}{2})(\tau+l)^{-k+s} e(-\frac{\mathcal{M}[z]}{\tau+l}) e\Bigg(-\frac{\mathcal{M}[\frac{-z}{\tau+l}+\nu-t_0]}{\frac{-1}{\tau+l}}\Bigg)\Bigg] \\ &= \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^g} \Bigg[e(-\frac{s}{2})(\tau+l)^{-k+s} e\Bigg(-\frac{\mathcal{M}[z]}{\tau+l}\Big) e(\frac{\mathcal{M}[\frac{z}{\tau+l}+\nu-t_0]}{\frac{1}{\tau+l}}\Bigg) \\ &+ e(\frac{s}{2})(\tau+l)^{-k+s} e(-\frac{\mathcal{M}[z]}{\tau+l}) e\Bigg(\frac{\mathcal{M}[\frac{z}{\tau+l}+\nu+t_0]}{\frac{1}{\tau+l}}\Bigg)\Bigg] \\ &= \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^g} \Bigg[e(-\frac{s}{2})(\tau+l)^{-k+s} e(2z\mathcal{M}(\nu-t_0)+(\tau+l)\mathcal{M}[\nu-t_0]) \\ &+ e(\frac{s}{2})(\tau+l)^{-k+s} e(2z\mathcal{M}(\nu+t_0)+(\tau+l)\mathcal{M}[\nu+t_0])\Bigg]. \end{aligned}$$

Similar to the case of $c=0$ one gets the Fourier coefficients corresponding to $a=0$

$$A_0(n,R) = \begin{cases} \dfrac{(2\pi)^{k-s} e(-\frac{s}{2})(-i)^{k-s-1} e(-\frac{s}{2}) D^{k-s-1}}{\Gamma(k-s)} & \text{if } R+2\mathcal{M}t_0 \in 2\mathcal{M}\mathbb{Z}^g, \\ \dfrac{(2\pi)^{k-s} e(\frac{s}{2})(-i)^{k-s-1} e(-\frac{s}{2}) D^{k-s-1}}{\Gamma(k-s)} & \text{if } R-2\mathcal{M}t_0 \in 2\mathcal{M}\mathbb{Z}^g. \end{cases} \tag{2.6}$$

Now the sum corresponding to the matrices with $ac \neq 0$ is given by

$$\begin{aligned}
B_0 &= \sum_{\substack{ac\neq 0,\ (a,c)=1,\\ \nu\in\mathbb{Z}^{g,1}}} (c\tau+d)^{-k} e(-\frac{c}{c\tau+d}\mathcal{M}[z])(\frac{a\tau+b}{c\tau+d})^{-s} e\left(\frac{-\mathcal{M}[\frac{z}{c\tau+d}+\nu-t_0]}{\frac{a\tau+b}{c\tau=d}}\right) \\
&= \sum_{\substack{ac\neq 0,\ (a,c)=1,\\ \nu\in\mathbb{Z}^{g,1}}} \left(\frac{c\tau+d}{a\tau+b}\right)^{-k+s} (a\tau+b)^{-k} e(-\frac{c}{a}\mathcal{M}[\nu-t_0]) e\left(-\frac{a}{a\tau+b}\mathcal{M}[z+\frac{\nu-t_0}{a}]\right) \\
&= \sum_{\substack{ac\neq 0,\ (a,c)=1,\\ \nu\in\mathbb{Z}^{g,1}}} a^{-k}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\frac{b}{a})}\right)^{-k+s} (\tau+\frac{b}{a})^{-k} e(-\frac{c}{a}\mathcal{M}[\nu-t_0]) \\
&\times\ e\left(-\frac{1}{\tau+\frac{b}{a}}\mathcal{M}[z+\frac{\nu-t_0}{a}]\right) \\
&= \sum_{\substack{ac\neq 0,\ (a,c)=1,\ \nu(\mathbb{Z}^{g,1}a),\\ bc\equiv 1(a),\ \alpha\in\mathbb{Z},\ \beta\in\mathbb{Z}^{g,1}}} a^{-k}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\beta+\frac{b}{a})}\right)^{-k+s} (\tau+\beta+\frac{b}{a})^{-k} e(-\frac{c}{a}\mathcal{M}[\nu-t_0]) \\
&\times\ e\left(-\frac{1}{\tau+\beta+\frac{b}{a}}\mathcal{M}[z+\alpha+\frac{\nu-t_0}{a}]\right) \\
&= \sum_{\substack{ac\neq 0,\ (a,c)=1,\ \nu(\mathbb{Z}^{g,1}a),\\ bc\equiv 1(a),\ \alpha\in\mathbb{Z},\ \beta\in\mathbb{Z}^{g,1}}} a^{-k} e(-\frac{c}{a}\mathcal{M}[\nu-t_0]) F_{a,c}(\tau+\frac{b}{a}, z+\frac{\nu-t_0}{a}),
\end{aligned}$$

where $F_{a,c}(\tau,z) = \sum_{\alpha\in\mathbb{Z}^{g,1},\beta\in\mathbb{Z}} \left(\frac{c}{a}-\frac{1}{a^2(\tau+\beta)}\right)^{-k+s} (\tau+\beta)^{-k} e\left(-\frac{1}{\tau+\beta}\mathcal{M}[z+\alpha]\right)$. Now for $ac>0$, the contribution to (n,R)-th Fourier coefficient is given by

$$\begin{aligned}
F^{+}_{a,c}(n,R) &= \int_{ic-\infty}^{ic+\infty} \left(\frac{c}{a}-\frac{1}{a^2\tau}\right)^{-k+s} \tau^{-k} \left(\int_{ic_1-\infty}^{ic_1+\infty} \dots \int_{ic_g-\infty}^{ic_g+\infty} e\left(-\frac{1}{\tau}\mathcal{M}[z]-Rz\right)dz\right) e(-n\tau)d\tau \\
&= \frac{1}{(2i)^{\frac{g}{2}}\sqrt{|\mathcal{M}|}} \int_{ic-\infty}^{ic+\infty} \left(\frac{c}{a}-\frac{1}{a^2\tau}\right)^{-k+s} \tau^{-k} e\left(-(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])\tau\right)d\tau.
\end{aligned}$$

Now the change of variables $\tau \to \frac{a}{c}it$ implies

$$F^{+}_{a,c}(n,R) = \frac{1}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}}(\frac{a}{c})^{-(s-\frac{g}{2}-1)}i^{1-s-\frac{g}{2}}\int_{c-i\infty}^{c+i\infty}\left(t+\frac{i}{a^2}\right)^{-k+s}t^{-(s-\frac{g}{2})}$$
$$\times \quad e\left(2\pi(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])\frac{a}{c}t\right)d\tau.$$

Using the integral representation of Kummer's hypergeometric functions one gets that

$$F^{+}_{a,c}(n,R) \quad = \quad \frac{(2\pi)^{k-\frac{g}{2}}D^{k-\frac{g}{2}-1}i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})}(\frac{a}{c})^{k-s}\,{}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac}).$$

Similarly one can compute the contribution of the terms with $ac<0$ to obtain

$$B_0(n,R) \quad = \quad \frac{(2\pi)^{k-\frac{g}{2}}D^{k-\frac{g}{2}-1}i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})}\sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}}(\frac{a}{c})^{k-s}a^{-k}$$
$$\sum_{\nu(a\mathbb{Z}^{g,1})}e(R\frac{\nu-t_0}{a})[e(-\frac{c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\,{}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac})$$
$$+\quad e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\,{}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})].$$

□

Theorem 2.4.2. *[39] Let $\mathcal{M}$ and $\omega_k(n,R)$ be as in Theorem 2.4.1 for fixed (n,R) with $2Rt_0 \notin \mathbb{Z}+\frac{1}{2}$. Then there exists k_0 such that for all $k>k_0$ the Fourier coefficient $\omega_k(n,R)\neq 0$ for $s=\frac{k}{2}+\frac{g}{4}-\delta-it',\ 0<\delta<\frac{1}{2}$.*

Proof. Let us assume that for given $\mathcal{M},\ n$ and $R,$ there does not exist any such k_0, i.e., there are infinitely many large k such that $\omega_k(n,R)=0$. Then

$$\begin{aligned}
0 \;=\;& \frac{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}} \frac{(e(-Rt_0)+e(Rt_0))}{\Gamma(s-\frac{g}{2})} \\
+\;& (-i)^{k-s-1}\frac{(2\pi D)^{k-s-1}}{\Gamma(k-s)}\{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)+e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)\} \\
+\;& \frac{(2\pi)^{k-\frac{g}{2}} D^{k-\frac{g}{2}-1} i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})} \sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}} (\frac{a}{c})^{k-s} a^{-k} \\
& \sum_{\nu(a\mathbb{Z}^{g,1})} e(R\frac{\nu-t_0}{a})\Big[e(-\frac{c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\,{}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac}) \\
+\;& e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\,{}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})\Big].
\end{aligned}$$

Rewrite the above equation as

$$\begin{aligned}
-1 =\;& \frac{2^{g-s}\sqrt{|\mathcal{M}|}(-i)^{k-s-1}(2\pi D)^{k-s-1}\Gamma(s-\frac{g}{2})}{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}\Gamma(k-s)} \\
\times\;& \frac{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)+e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)}{(e(-Rt_0)+e(Rt_0))} \\
+\;& \frac{(2\pi)^{k-\frac{g}{2}} D^{k-\frac{g}{2}-1} i^{1-s-\frac{g}{2}} 2^{g-s}\sqrt{|\mathcal{M}|}\Gamma(s-\frac{g}{2})}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}} \frac{1}{(e(-Rt_0)+e(Rt_0))} \\
\times & \sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}} (\frac{a}{c})^{k-s} a^{-k} \sum_{\nu(a\mathbb{Z}^{g,1})} e(R\frac{\nu-t_0}{a}) \\
\times\;& \Big[e(\frac{-c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\,{}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac}) \\
+\;& e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\,{}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})\Big].
\end{aligned}$$

Applying modulus on both the sides, one has

$$\begin{aligned}
1 \quad \leqslant \quad & \left|\frac{2^{g-s}\sqrt{|\mathcal{M}|}(-i)^{k-s-1}(2\pi D)^{k-s-1}\Gamma(s-\frac{g}{2})}{\pi^{s-\frac{g}{2}}ie(-\frac{s}{4})D^{s-\frac{g}{2}-1}\Gamma(k-s)}\right| \\
\times \quad & \left|\frac{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)+e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)}{(e(-Rt_0)+e(Rt_0))}\right| \\
+ & \left|\frac{(2\pi)^{k-\frac{g}{2}}D^{k-\frac{g}{2}-1}i^{1-s-\frac{g}{2}}2^{g-s}\sqrt{|\mathcal{M}|}\Gamma(s-\frac{g}{2})}{\left(2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})\pi^{s-\frac{g}{2}}ie(-\frac{s}{4})D^{s-\frac{g}{2}-1}\right)(e(-Rt_0)+e(Rt_0))}\right| \\
\times \quad & \sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}}\left|(\frac{a}{c})^{k-s}a^{-k}\right|\sum_{\nu(a\mathbb{Z}^g)}\left|e(R\frac{\nu-t_0}{a})\right| \\
\times \quad & \left[\left|e(-\frac{c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac})\right|\right. \\
+ \quad & \left.\left|e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})\right|\right].
\end{aligned}$$

For $s=\frac{k}{2}+\frac{g}{4}-\delta-it'$, one has

$$\begin{aligned}
1 \leqslant & 2^{\frac{g}{2}+2\delta}\sqrt{|\mathcal{M}|}(\pi)^{2\delta-1}D^{2\delta}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{2}-\delta-it')}{\Gamma(\frac{k}{2}-\frac{g}{2}+\delta+it')}\right|\frac{1}{\left|(e(-Rt_0)+e(Rt_0))\right|} \\
+ & \quad (2\pi D)^{\frac{k}{2}-\frac{g}{4}+\delta}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{4}-\delta-it')}{\Gamma(k-\frac{g}{2})}\frac{1}{(e(-Rt_0)+e(Rt_0))}\right|\sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}}\left|(\frac{a}{c})^{\frac{k}{2}-\frac{g}{4}+\delta}a^{-k}\right| \\
\times & \sum_{\nu(a\mathbb{Z}^{g,1})}\left|\ {}_1F_1(\frac{k}{2}-\frac{g}{4}+\delta+it',k-\frac{g}{2};-\frac{2\pi Di}{ac})\right|+\left[\left|\ {}_1F_1(\frac{k}{2}-\frac{g}{4}+\delta+it',k-\frac{g}{2};\frac{2\pi Di}{ac})\right|\right].
\end{aligned} \tag{2.7}$$

Using the integral representation of ${}_1F_1$ and estimating, one observes that the infinite series in the sum is convergent and bounded by a constant, say L, for large k. Hence the above inequality is reduced to

$$\begin{aligned} 1 \quad &\leqslant \quad 2^{\frac{g}{2}+2\delta}\sqrt{|\mathcal{M}|}(\pi)^{2\delta-1}D^{2\delta}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{2}-\delta-it')}{\Gamma(\frac{k}{2}-\frac{g}{2}+\delta+it')}\right|\frac{1}{\left|(e(-Rt_0)+e(Rt_0))\right|} \\ &+ \quad \frac{(2\pi D)^{\frac{k}{2}-\frac{g}{4}+\delta}}{(k-\frac{g}{2}-1)(k-\frac{g}{2}-2)\cdots([\frac{k}{2}])}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{4}-\delta-it')}{\Gamma([\frac{k}{2}])}\right|\left|\frac{L}{(e(-Rt_0)+e(Rt_0))}\right|. \end{aligned}$$

Using the fact that $z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)} \to 1$ as $z \to \infty$ we observe that both the terms on the right-hand side tend to zero and hence we get a contradiction.

Remark 2.4.1. In the above inequality (2.7) one can compute k_0 explicitly such that both the summands are strictly less than $\frac{1}{2}$ for $k > k_0$. Let's assume

$$2^{\frac{g}{2}+2\delta}\sqrt{|\mathcal{M}|}(\pi)^{2\delta-1}D^{2\delta}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{2}-\delta-it')}{\Gamma(\frac{k}{2}-\frac{g}{2}+\delta+it')}\right|\frac{1}{\left|(e(-Rt_0)+e(Rt_0))\right|} < \frac{1}{2} \tag{2.8}$$

and estimate the lower bound for k. Estimating the ratio of gamma functions one obtain

$$\frac{2^{\frac{g}{2}+2\delta}\sqrt{|\mathcal{M}|}(\pi)^{2\delta-1}D^{2\delta}}{\left|(e(-Rt_0)+e(Rt_0))\right|}\frac{1}{(\frac{k}{2}-\frac{g}{2}-\frac{1}{2})^{2\delta}} < \frac{1}{2}.$$

Hence (2.8) is satisfied whenever

$$k > 2\left(\frac{2^{\frac{g}{2}+2\delta+1}\sqrt{|\mathcal{M}|\pi^{2\delta-1}D^{2\delta}}}{(e(-Rt_0)+e(Rt_0))}\right)^{\frac{1}{2\delta}} + g + 1.$$

Similarly one can estimate k from the second term of (2.7) to obtain that second term is less than $\frac{1}{2}$ whenever $k > 8\pi D + 2$ and

$$\left(\frac{2\pi D}{\frac{k}{2}-1}\right)^{\frac{k}{2}-\frac{g}{2}-1} (2\pi D)^{\delta+1+\frac{g}{4}} \frac{2\pi^4}{g} < \frac{1}{2}.$$

Hence one obtains the lower bound for k

$$k > 2\frac{\log\left(2^2\pi^3(2\pi D)^{1+\delta+\frac{g}{4}}\right)}{\log 2} + g + 2.$$

$\square$

Now we prove Theorem 2.2.2.

Proof. First, we prove the theorem for s in the right part of the critical strip, i.e., for $s = \sigma + it'$ with $\frac{k}{2} - \frac{g}{4} + \epsilon < \sigma < \frac{k}{2} - \frac{g}{4} + \frac{1}{2}$. Let $\delta > 0$ and $s = \frac{k}{2} + \frac{g}{4} - \delta - it'$ be as in Theorem 2.4.2 and $\mathcal{B}_{k,\mathcal{M}}$ be basis of eigenforms of weight k and index $\mathcal{M}$. As $\Omega^{k,\mathcal{M}}_{t_0,s} \in J^{cusp}_{k,\mathcal{M}}$, we can express the kernel function as

$$\Omega^{k,\mathcal{M}}_{t_0,s}(\tau, z) = \sum_{f_i \in \mathcal{B}_{k,\mathcal{M}}} \frac{\langle \Omega^{k,\mathcal{M}}_{t_0,s}, f_i\rangle}{\langle f_i, f_i\rangle} f_i.$$

Now compare the (n, R)-th Fourier coefficient of both the sides with $2Rt_0 \notin \mathbb{Z} + \frac{1}{2}$ and use Theorem 2.4.2 to obtain k_0 and a Hecke eigenform $f_i \in J^{cusp}_{k,\mathcal{M}}$ with $k > k_0$ such that

$$\langle \Omega^{k,\mathcal{M}}_{t_0,s}, f_i\rangle = \frac{\pi}{2^{k-2}e^{\pi i \frac{s}{2}}} \frac{\Gamma(k - \frac{g}{2} - 1)}{\Gamma(s - \frac{g}{2})\Gamma(k - s)} \sum_{N \in \mathbb{Z}^g \setminus (\mathbb{Z}^g(2\mathcal{M}))} e(-Rt_0)\Lambda_N(\overline{f_i}, k - s) \neq 0.$$

Hence for some $N \in \mathbb{Z}^g \setminus (\mathbb{Z}^g(2\mathcal{M})$, $\Lambda_N(\overline{f_i}, \frac{k}{2} - \frac{g}{4} + \delta + it') \neq 0$. Now using functional equations (1.11) we have the theorem. $\square$

2.5 Nonvanishing of Poincaré series

Using Theorem 2.4.2 and the fact that kernel functions can be written as linear sums of Poincaré series (Theorem 2.3.4), we prove Theorem 2.2.3. Using Theorem 2.3.4 one can write the kernel functions as

$$\begin{aligned}
&\sqrt{(2i)^g|\mathcal{M}|}\Omega_{t_0,s}^{k,\mathcal{M}} \\
=\ &\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{R\in z^g\backslash z^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}.
\end{aligned}$$

Now choose (n', R') as in Theorem 2.4.2 and fix $t' = 0$. Compare the (n', R')-th Fourier coefficient on both sides of the above equation.

$$\begin{aligned}
&\sqrt{(2i)^g|\mathcal{M}|}\omega_{t_0,s}^{k,\mathcal{M}}(n', R') \\
=\ &\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{R\in z^g\backslash z^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t) \\
\times\ &\sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}p_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(n', R'). \\
=\ &\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{R\in z^g\backslash z^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t) \\
\times\ &\sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}p_{k,\mathcal{M};(n',R')}(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t], R).
\end{aligned}$$

Hence for $k > k_0$ we have

$$p_{k,\mathcal{M};(n',R')}(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t], R) \neq 0.$$

In particular, $P_{k,\mathcal{M};(n,R)} \neq 0$ for $k > k_0$.

Chapter 3

A converse theorem for Jacobi forms of half-integral weight

3.1 Introduction

A converse theorem in the context of automorphic forms studies the equivalence of the automorphic properties of a power series and the analytic properties of the Dirichlet series associated with the power series. For example, Hecke [16] proved the following converse theorem:

Theorem 3.1.1. *[16] Let $k \geqslant 2$ be a positive integer. Let $\{a(n)\}_{n\geqslant 1}$ be a sequence of complex numbers such that $a(n) = O(n^{\sigma})$ for some $\sigma > 0$. The function $f(\tau) = \sum\limits_{n\geqslant 1} a(n)e^{2\pi i n\tau}$ defines a cusp form of weight k for full modular group $SL_2(\mathbb{Z})$ if and only if the completed Dirichlet series $L^*(f, s)$ admits a holomorphic continuation to the whole complex-plane $\mathbb{C}$ which is bounded on any vertical strip and satisfies the functional equation $L^*(f, s) = (-1)^{\frac{k}{2}} L^*(f, k - s)$.*

Weil [48] generalized Hecke's work for modular forms on the congruence subgroups. The converse theorem has been studied for various kinds of automorphic forms, for example, modular forms of half-integral weight [5] and Siegel modular forms [17].

Martin studied the analytic continuation of L-functions and a converse theorem for Jacobi forms with respect to Γ^J.

Theorem 3.1.2. *[33] Let k and m be positive integers. Let $\phi(\tau, z) = \sum\limits_{r^2<4nm} c_\phi(n, r)e^{2\pi i(n\tau+rz)}$ be a holomorphic function satisfying*

(i) $\phi(\tau, z)$ converges absolutely and uniformly on compact subsets of $\mathcal{H} \times \mathbb{C}$,

(ii) there exists $\nu > 0$ such that $\phi(\tau, z)e^{2\pi ipz} = O(\Im(\tau)^{-\nu})$ as $\Im(\tau) \to 0$,

(iii) for each λ, we have $c(n,\ r) = c(n + \lambda r + \lambda^2 m, r + 2\lambda m)$.

Then the following statements are equivalent:

1. *the function $\phi(\tau, z)$ is a Jacobi form of weight k and index m.*
2. *each completed L-function $\Lambda_\mu(\phi, s), 0 \leqslant \mu \leqslant 2m - 1$ associated to $\phi(\tau, z)$ can be analytically continued to a holomorphic function on the s-plane. These functions are bounded on any vertical strip and satisfy the functional equations*

$$(2m)^{-\frac{1}{2}} \sum_{\mu=0}^{2m-1} e^{-\frac{\pi i a\mu}{m}} \Lambda_\mu(\phi, s) = i^k \Lambda_a \left(\phi, k - \frac{1}{2} - s\right), \ \ 0 \leqslant a \leqslant 2m - 1.$$

Later, Martin and Osses [36] generalized the above theorem for congruence subgroups of the Jacobi group. In this chapter, we study the analytic properties of L-functions and a converse theorem for Jacobi forms of half-integral weight. Our approach is similar to the work of Bruinier [5] in the case of modular forms of half-integral weight. The content of this chapter is based on [30].

3.2 Statement of results

Following the work of Martin and Osses [36] we define analogous series of type J and the associated twisted Dirichlet series.

Definition 3.2.1. *For a fixed positive integer m, we call $\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2<4nm}} c_\phi(n, r)e(n\tau+rz)$ to be a series of type J, if the following properties hold:*

1. *The series $\phi(\tau, z)$ converges absolutely and uniformly on every compact subset of $\mathcal{H} \times \mathbb{C}$.*

2. *There exist positive real numbers C and δ such that $|c_\phi(n, r)| < C(4mn - r^2)^\delta$ for all n, r such that $r^2 < 4nm$.*

3. *The Fourier coefficients of $c_\phi(n, r)$ satisfy $c_\phi(n, r) = c_\phi(n + \lambda r + \lambda^2 m, r + 2m\lambda)$ for every $\lambda \in \mathbb{Z}$.*

From the condition (1), $\phi : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ is a holomorphic function. The relation between Fourier coefficients implies that ϕ has a theta decomposition similar to (1.16). Note that the Fourier series expansion of a Jacobi cusp form $\phi(\tau, z) \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ represents a series of type J.

Definition 3.2.2. *Let N and M be positive integers with $4|N$ and $(N, M) = 1$. Let $\phi(\tau, z)$ be a series of type J and χ_1 be a primitive Dirichlet character modulo M. Then for each $\mu \in \{0, 1, 2, \cdots, 2mM - 1\}$, we define a Dirichlet series using the theta decomposition of ϕ as follows:*

$$L_\mu(\phi_{\chi_1}; s) = \sum_{D=1}^{\infty} \chi_1\left(\frac{D + \mu^2}{4m}\right) c_\mu(D) \left(\frac{D}{4m}\right)^{-s}. \tag{3.1}$$

The completed Dirichlet series is defined by

$$\Lambda_\mu(\phi_{\chi_1}; s) = \left(\frac{2\pi}{M\sqrt{N}}\right)^{-s} \Gamma(s)L_\mu(\phi_{\chi_1}; s). \tag{3.2}$$

Note that bounds on Fourier coefficients (condition (2) of the definition 3.2.1) imply the $L_\mu(\phi_{\chi_1}; s)$ is absolutely convergent on the complex half-plane $\Re(s) > 1 + \delta$ for every $\mu \in \{0, 1, 2, \cdots, 2mM - 1\}$.

Now, we define the analogous series of type J_N and the associated twisted Dirichlet series.

Definition 3.2.3. *Let m and N be fixed positive integers with $4|N$. We call a series $\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn>Nr^2}} c_\phi(n, r)e(n\tau + rNz)$ to be a series of type J_N, if the following properties hold:*

1. *The series $\phi(\tau, z)$ converges absolutely and uniformly on every compact subset of $\mathcal{H} \times \mathbb{C}$.*

2. *There exist positive real numbers C and δ such that $|c_\phi(n, r)| < C(4mn - Nr^2)^\delta$ for all n, r such that $Nr^2 < 4nm$.*

3. *The Fourier coefficients of $c_\phi(n, r)$ satisfy $c_\phi(n, r) = c_\phi(n+\lambda rN+\lambda^2 mN, r+2m\lambda)$ for every $\lambda \in \mathbb{Z}$.*

A series of type J_N has a theta decomposition given by

$$\phi(\tau, z) = \sum_{\mu=1}^{2m} g_\mu(\tau)\theta_{m,\mu}(N\tau, Nz), \tag{3.3}$$

where $g_\mu(\tau) = \sum_{D=1}^{\infty} d_\mu(D)e\left(\frac{D}{4m}\tau\right)$ with $d_\mu(D) = c_\phi(n, r)$ and $D = 4nm - Nr^2$.
If $\phi(\tau, z) \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$, then $\psi(\tau, z) := \phi|_{\frac{k}{2},m}W_N(\tau, z) \in J_{\frac{k}{2},mN}(\Gamma^J_{1,N}(N), \overline{\chi})$ (see (3.9) for the definition of W_N) and hence $\psi(\tau, z)$ represents a series of type J_N. Here we have used the notation i.e.,

$$\Gamma^J_{\alpha,\beta}(N) = \{(\widetilde{\gamma}, (\lambda, \mu), s) : \gamma \in \Gamma_0(N), \lambda \in \alpha\mathbb{Z}, \mu \in \beta^{-1}\mathbb{Z}, s \in \langle\zeta_\beta\rangle\},$$

where $\langle\zeta_\beta\rangle$ is the cyclic group generated by the primitive β-th roots of unity.
For each $\mu \in \{0, 1, 2, \cdots, 2mM - 1\}$, we define the Dirichlet series $L_\mu(\psi_{\chi_1}; s)$ and the

corresponding completed Dirichlet series $\Lambda_\mu(\psi_{\chi_1}; s)$ associated to ψ as follows:

$$L_\mu(\psi_{\chi_1}; s) = \sum_{D=1}^{\infty} \chi_1\left(\frac{D+N\mu^2}{4m}\right) d_\mu(D) \left(\frac{D}{4m}\right)^{-s}, \tag{3.4}$$

$$\Lambda_\mu(\psi_{\chi_1}; s) = \left(\frac{2\pi}{M\sqrt{N}}\right)^{-s} \Gamma(s) L_\mu(\psi_{\chi_1}; s). \tag{3.5}$$

Now we state our main results.

Theorem 3.2.4. *[30] Let m, N and M be positive integers such that $4|N$ and $(N, M) = 1$. Let χ be a Dirichlet character modulo N, χ_1 be a primitive Dirichlet character modulo M, and χ_2 be a Dirichlet character defined by $\chi_2(\cdot) = \left(\frac{\cdot}{M}\right)$, where $\left(\frac{\cdot}{M}\right)$ denotes the Jacobi symbol. If $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ is a Jacobi cusp form with $W_N(\phi) = \psi$, then for each $\mu = 0, 1, \cdots, 2mM-1$, the completed Dirichlet series $\Lambda_\mu(\phi_{\chi_1}; s)$ associated to ϕ admits a holomorphic continuation to the whole complex plane. Moreover, they are bounded in any vertical strip and satisfy the functional equation:*

- *for $\chi_1 \neq \chi_2$*

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C^{(1)}_{\chi_1} \Lambda_a\left(\psi_{\overline{\chi_1}\chi_2}; \frac{k}{2} - s - \frac{1}{2}\right),$$

  where $C^{(1)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}} \chi(M) (\frac{N}{M}) \chi_1(-N) \mathcal{G}_{\chi_1\chi_2} \mathcal{G}^{-1}_{\overline{\chi_1}}$, for every $a = 0, 1, ...2mM-1$.

- *for $\chi_1 = \chi_2$*

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C^{(2)}_{\chi_1} \Lambda_a\left(M^{\frac{1}{2}} B_M(\psi) - M^{-\frac{1}{2}}\psi; \frac{k}{2} - s - \frac{1}{2}\right),$$

  for every $a = 0, 1, ...2mM-1$, where $C^{(2)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}} \chi(M)$, $\mathcal{G}_\chi$ is the Gauss sum with character χ and $B_M(\psi) := \frac{1}{M} \sum_{u \pmod{M}} \psi|_{\frac{k}{2},m} \widetilde{T_{\frac{y}{M}}}$ and $T_{\frac{y}{M}} = \begin{pmatrix} 1 & y/M \\ 0 & 1 \end{pmatrix}$.

We now state the converse of the above theorem. For a positive integer N, let $\mathcal{M}_N$ be the set of all prime numbers p such that $(p, N) = 1$ and the set $\mathcal{M}_N \cap \{aL + b | L \in \mathbb{Z}\}$ is non-empty for all $a,\ b \in \mathbb{Z} \setminus \{0\}$ with $(a, b) = 1$.

Theorem 3.2.5. *[30] Let m, N, M be positive integers such that $4|N, (M, N) = 1$, χ be a Dirichlet character modulo N and $\chi_2(\cdot) = (\frac{\cdot}{M})$. Let $\{c_\phi(n, r)\}$ and $\{c_\psi(n, r)\}$ be sequences of complex numbers such that the series*

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>r^2}} c_\phi(n, r)e(n\tau + rz)$$

and

$$\psi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>Nr^2}} c_\psi(n, r)e(n\tau + rNz)$$

are of type J and J_N, respectively, and $\psi(\tau, z) = (-1)^{\frac{k}{2}}\overline{\chi}(-1)\psi(\tau, -z)$. Assume that for every primitive Dirichlet character χ_1 of conductor $M \in \mathcal{M}_N \cup \{1\}$, $\Lambda_\mu(\phi_{\chi_1}, s)$ is entire and bounded in every vertical strip and satisfies the following conditions:

(i) if $\chi_1 \neq \chi_2$, then

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C^{(1)}_{\chi_1}\Lambda_a\left(\psi_{\overline{\chi_1}\chi_2}; \frac{k}{2} - s - \frac{1}{2}\right),$$

where $C^{(1)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)(\frac{N}{M})\chi_1(-N)\mathcal{G}_{\chi_1\chi_2}\mathcal{G}^{-1}_{\overline{\chi}_1}$, for every $a = 0, 1, ..., 2mM - 1$.

(ii) if $\chi_1 = \chi_2$, then

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(-\frac{a\mu}{2mM}\right)\Lambda_\mu(\phi_{\chi_1}; s) = C^{(2)}_{\chi_1}\Lambda_a\left(M^{\frac{1}{2}}B_M(\psi) - M^{-\frac{1}{2}}\psi; \frac{k}{2} - s - \frac{1}{2}\right),$$

where $C^{(2)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)$, for every $a = 0, 1, ..., 2mM - 1$.

If for every $\mu \in \{0, 1, 2, \cdots, 2mM-1\}$ *the Dirichlet series* $L_\mu(\phi; s)$ *converges absolutely for* $\frac{k}{2} - 1 - \epsilon$ *for any* $\epsilon > 0$, *then*

$$\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi) \text{ and } \psi = W_N(\phi).$$

We study twists of Jacobi forms of half-integral weight, involution operator, and twisted L-functions to prove Theorem 3.2.4 and Theorem 3.2.5.

3.3 Twist and Fricke involution for Jacobi forms of half-integral weight

Let m, M, N be positive integers and k be an odd integer such that $4|N$. Let χ be a Dirichlet character modulo N and $T_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, where λ is any real number. Let Id denote the identity matrix of order 2. Define ϵ_M by

$$\epsilon_M = \begin{cases} 1, & M \equiv 1 \pmod 4 \\ i, & M \equiv 3 \pmod 4). \end{cases}$$

Definition 3.3.1. *Let* ϕ *be a series of type* J *or* J_N. *Let* χ_1 *be a primitive Dirichlet character modulo* M, *where* $(N, M) = 1$. *The twist of* $\phi(\tau, z)$ *by the character* χ_1 *is defined by*

$$\phi_{\chi_1}(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4nm}} \chi_1(n) c_\phi(n, r) e(n\tau + rz). \tag{3.6}$$

Lemma 3.3.2. *Let* $\phi \in J_{\frac{k}{2},m}(\Gamma^J(N), \chi)$. *Let* χ_1 *be a primitive Dirichlet character modulo*

M, where $(N,M)=1$. Then

$$\phi_{\chi_1}(\tau,z)\in J_{\frac{k}{2},m}(\Gamma^J_{M,1}(NM^2),\chi\chi_1^2),$$

where $\Gamma^J_{M,1}(N):=\widetilde{\Gamma_0(NM^2)}\ltimes(M\mathbb{Z}\times\mathbb{Z})$. Further, if ϕ is a Jacobi cusp form, then ϕ_{χ_1} is also a Jacobi cusp form.

Proof. Let $u\in\mathbb{Z}$. Consider $\widetilde{T_{\frac{u}{M}}}=\left(\begin{pmatrix}1 & \frac{u}{M}\\ 0 & 1\end{pmatrix},1\right)\in G$. Now

$$\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}}(\tau,z)=\phi\left(\tau+\frac{u}{M},z\right)=\sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4nm}}e\left(\frac{un}{M}\right)c_\phi(n,r)e(n\tau+rz).$$

Multiplying by $\overline{\chi}_1(u)$ and summing over all $u \pmod M$, we obtain

$$\begin{aligned}\sum_{u=0}^{M-1}\overline{\chi}_1(u)\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}}(\tau,z) &= \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4nm}}\left(\sum_{u=0}^{M-1}\overline{\chi}_1(u)e\left(\frac{un}{M}\right)\right)c_\phi(n,r)e(n\tau+rz),\\ &= \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4nm}}G_{n,\overline{\chi}_1}c_\phi(n,r)e(n\tau+rz),\end{aligned}$$

where $\mathcal{G}_{n,\overline{\chi}_1}$ is the Gauss sum associated with the primitive Dirichlet character $\overline{\chi}_1$ defined by $\mathcal{G}_{n,\overline{\chi}_1}=\sum\limits_{u=0}^{M-1}\overline{\chi}_1(u)e(\frac{un}{M})$. Note that $G_{n,\overline{\chi}_1}=0$ if $(n,M)>1$. Therefore,

$$\sum_{u=0}^{M-1}\overline{\chi}_1(u)\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}}(\tau,z)=\mathcal{G}_{\overline{\chi}_1}\phi_{\chi_1}(\tau,z), \tag{3.7}$$

where $G_{1,\overline{\chi}_1}=G_{\overline{\chi}_1}$. Let $L=NM^2$ and $\widetilde{\gamma}=(\gamma,j(\gamma,\tau))$, where $\gamma=\begin{pmatrix}a & b\\ cL & d\end{pmatrix}\in\Gamma_0(L)$.

Note that

$$\gamma' := T_{\frac{u}{M}}\gamma T^{-1}_{\frac{d^2u}{M}} \in \Gamma_0(L) \subset \Gamma_0(N), \chi(\gamma) = \chi(\gamma'),\ \widetilde{\gamma'} = (\gamma', j(\gamma',\tau)) \in \widetilde{\Gamma_0(L)} \subset \widetilde{\Gamma_0(N)}$$

and $\widetilde{\gamma'} = (\widetilde{T_{\frac{u}{M}}}, (0,0), 1)\widetilde{\gamma}(\widetilde{T^{-1}_{\frac{ud^2}{M}}}, (0,0), 1)$. Hence for any $[\widetilde{\gamma}, (\lambda,\nu), 1] \in \Gamma^J_{M,1}(L)$, we have

$$\begin{aligned}
&\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}}|_{\frac{k}{2},m}(\widetilde{\gamma}, (\lambda,\nu), 1)(\tau,z)\\
&= \phi|_{\frac{k}{2},m}\left(\widetilde{\gamma'}, \left(\lambda, \nu - \frac{\lambda d^2 u}{M}\right), 1\right)|_{\frac{k}{2},m}\widetilde{T_{\frac{ud^2}{M}}}(\tau,z)\\
&= \chi(\gamma)\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{ud^2}{M}}}(\tau,z).
\end{aligned}$$

From (3.7) and the above equation, we obtain

$$\mathcal{G}_{\overline{\chi_1}}\phi_{\chi_1}|_{\frac{k}{2},m}(\widetilde{\gamma}, (\lambda,\nu), 1)(\tau,z) = \chi(\gamma)\sum_{u=0}^{M-1}\overline{\chi}_1(u)\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{ud^2}{M}}}(\tau,z). \tag{3.8}$$

As $(d,M)=1$, replacing d^2u by u in (3.8), we obtain

$$\begin{aligned}
\mathcal{G}_{\overline{\chi_1}}\phi_{\chi_1}|_{\frac{k}{2},m}(\widetilde{\gamma}, (\lambda,\nu), 1)(\tau,z) &= \chi(\gamma)\sum_{u=0}^{M-1}\overline{\chi}_1(ud^{-2})\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}}(\tau,z)\\
&= \chi(\gamma)\chi_1(d^2)\sum_{u=0}^{M-1}\overline{\chi}_1(u))\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}}(\tau,z).
\end{aligned}$$

Now as $\chi_1(d) = \chi_1(\gamma)$, from (3.7) we get

$$\phi_{\chi_1}|_{\frac{k}{2},m}(\widetilde{\gamma}, (\lambda,\nu), 1)(\tau,z) = \chi\chi_1^2(\gamma)\phi_{\chi_1}(\tau,z).$$

Hence ϕ_{χ_1} satisfies the transformation properties of Jacobi forms. From the Fourier expansion of ϕ, it is easy to see that ϕ_{χ_1} has the required Fourier expansion. □

Definition 3.3.3. *Let k be an odd integer, m be any positive integer and ϕ be a complex-*

valued holomorphic function defined on $\mathcal{H} \times \mathbb{C}$. For a positive integer L, we define the following Fricke involution type operator by

$$W_L^{k,m}(\phi) := (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}h, \tag{3.9}$$

where $h = (\widetilde{\gamma}, (0,0), 1) \in \widetilde{G^J}, \widetilde{\gamma} = \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}} \right) \in G$, and the operator U_L is defined as

$$U_L\phi(\tau, z) := \phi(\tau, Lz).$$

We have the following form of (3.9)

$$W_L^{k,m}(\phi)(\tau, z) = i^{\frac{k}{2}} L^{-\frac{k}{4}} \tau^{-\frac{k}{2}} e^{mL}\left(-\frac{z^2}{\tau}\right) \phi\left(-\frac{1}{L\tau}, \frac{z}{\tau}\right). \tag{3.10}$$

We write W_L instead of $W_L^{k,m}$ when k and m are clear from the context.

Lemma 3.3.4. *Let L be a positive integer with $4|L$ and χ a Dirichlet character modulo L. If $\phi \in J_{\frac{k}{2},m}(\Gamma^J(L), \chi)$, then*

$$W_L(\phi) \in J_{\frac{k}{2},mL}(\Gamma^J_{1,L}(L), \chi^*),$$

where $\chi^*(d) = \overline{\chi(d)} \left(\frac{N}{d}\right)$. Further, if ϕ is a Jacobi cusp form, then $W_L(\phi)$ is also a Jacobi cusp form.

Proof. From the equation (3.10), it is easy to see that $W_L(\phi)$ is holomorphic. For matrices

$\gamma = \begin{pmatrix} a & b \\ cL & d \end{pmatrix}$, $\gamma' = \begin{pmatrix} d & -c \\ -bL & a \end{pmatrix} \in \Gamma_0(L)$, we have

$$\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) \widetilde{\gamma} \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right)^{-1} = \left(\gamma', \left(\frac{N}{d}\right) j(\gamma', \tau)\right).$$

The definition of W_L and the above identity implies

$$\begin{aligned}
&W_L(\phi)|_{\frac{k}{2},mL}(\gamma, j(\gamma,\tau))(\tau,z) \\
&= (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(\gamma', \left(\frac{N}{d}\right) j(\gamma',\tau)\right)\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) \\
= \;& \left(\frac{N}{d}\right) U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}\widetilde{\gamma'})|_{\frac{k}{2},mL}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) \\
= \;& \left(\frac{N}{d}\right) \overline{\chi(d)}(U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) \\
= \;& \left(\frac{N}{d}\right) \overline{\chi(d)} W_L\phi.
\end{aligned}$$

Now, we consider $(Id, (\lambda,\nu), \zeta_L^j) \in \Gamma_{1,L}^J(L)$. We have

$$\begin{aligned}
&W_L(\phi)|_{\frac{k}{2},mL}(I_d, (\lambda,\nu), \zeta_L^j)(\tau,z) \\
&= (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(Id, \left(-\sqrt{L}\nu, \frac{\lambda}{\sqrt{L}}\right), \zeta_L^j\right)\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right).
\end{aligned}$$

Note that

$$(U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(Id,\left(-\sqrt{L}\nu,\frac{\lambda}{\sqrt{L}}\right),\zeta_L^j\right)=U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}(Id,(-L\nu,\lambda),1)).$$

As $\phi\in J_{\frac{k}{2},m}(\Gamma^J(L),\chi)$, we obtain

$$(U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(Id,\left(-\sqrt{L}\nu,\frac{\lambda}{\sqrt{L}}\right),\zeta_L^j\right)=U_{\sqrt{L}}(\phi).$$

Hence we have $W_L(\phi)|_{\frac{k}{2},mL}(Id,(\lambda,\nu),\zeta_L^j)=W_L(\phi)$. It is easy to check that $W_L(\phi)$ has required Fourier expansion and the proof is similar to that of Lemma 5, p. 166 in [36]. □

Lemma 3.3.5. *Let $\phi\in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ be a Jacobi cusp form and χ_1 be a primitive Dirichlet character modulo M, where $(N,M)=1$. Denote $\psi=W_N(\phi)$. Then*

$$(W_{NM^2}(\phi_{\chi_1}))(\tau,z)=C_{\chi_1}\psi^*(\tau,Mz),$$

where

$$C_{\chi_1}=\left(\frac{-1}{M}\right)^{\frac{k-1}{2}}\chi(M)\left(\frac{N}{M}\right)\chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\overline{\chi_1}}^{-1}$$

and

$$\psi^*(\tau,z)=\sum_{u=0}^{M-1}\chi_1(u)\left(\frac{u}{M}\right)\psi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}},(0,0),1).$$

Proof. Let u be an integer such that $(u,M)=1$. Then there exist integers $x,\ y$ such that $xM-yuN=1$. Then $\gamma=\begin{pmatrix}M & -y\\ -uN & x\end{pmatrix}\in\Gamma_0(N)$. Observe that

$$(\widetilde{T_{\frac{u}{M}}},(0,0),1))\left(\left(\begin{pmatrix}0 & -\frac{1}{\sqrt{NM^2}}\\ \sqrt{NM^2} & 0\end{pmatrix},(NM^2)^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}},(0,0),1\right)\right.$$

$$= \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1 \right) \widetilde{\gamma}(\widetilde{T_{\frac{y}{M}}}, (0,0), 1))(Id, \left(\frac{y}{M}\right)\epsilon_M, (0,0), 1).$$

Also, note that $U_{M\sqrt{N}}(\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}})(\tau, z) = (U_{M\sqrt{N}}\phi)|_{\frac{k}{2},mNM^2}\widetilde{T_{\frac{u}{M}}}(\tau, z)$. Therefore

$$\begin{aligned}
&(W_{NM^2}(\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}})(\tau, z) \\
&= U_{M\sqrt{N}}(\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}})|_{\frac{k}{2},mNM^2} \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{NM^2}} \\ \sqrt{NM^2} & 0 \end{pmatrix}, (NM^2)^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1 \right)(\tau, z) \\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}(U_{M\sqrt{N}}\phi)|_{\frac{k}{2},mNM^2} \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1 \right) \widetilde{\gamma}\widetilde{T_{\frac{y}{M}}}(\tau, z) \\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}U_M\left(U_{\sqrt{N}}\phi|_{\frac{k}{2},mN} \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1 \right) \widetilde{\gamma}\widetilde{T_{\frac{y}{M}}} \right)(\tau, z) \\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}U_M\left(W_N(\phi)|_{\frac{k}{2},mN}\widetilde{\gamma}|_{\frac{k}{2},mN}\widetilde{T_{\frac{y}{M}}} \right)(\tau, z) \\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}U_M\left(\psi|_{\frac{k}{2},mN}\widetilde{\gamma}|_{\frac{k}{2},mN}\widetilde{T_{\frac{y}{M}}} \right)(\tau, z).
\end{aligned}$$

Using Lemma 3.3.4, we obtain

$$W_{NM^2}(\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}})(\tau, z) = \left(\frac{y}{M}\right)\epsilon_M^{-k}\chi(M)\left(\frac{N}{x}\right)\psi|_{\frac{k}{2},mN}\widetilde{T_{\frac{y}{M}}}(\tau, Mz).$$

Now multiplying the above equation by $\overline{\chi}_1(u)$ and summing over all $u \pmod M$ as in (3.7), we obtain

$$\begin{aligned}
(W_{NM^2}(\mathcal{G}_{\overline{\chi}_1}\phi_{\chi_1}))(\tau, z) &= \epsilon_M^{-k}\chi(M)\left(\frac{N}{M}\right)\sum_{u=0}^{M-1}\overline{\chi_1}(u)\left(\frac{y}{M}\right)\psi|_{\frac{k}{2},mN}\widetilde{T_{\frac{y}{M}}}(\tau, Mz) \\
&= \epsilon_M^{-k}\chi(M)\left(\frac{N}{M}\right)\overline{\chi_1}(-N)\sum_{u=0}^{M-1}\overline{\chi_1}(y)\left(\frac{y}{M}\right)\psi|_{\frac{k}{2},mN}\widetilde{T_{\frac{y}{M}}}(\tau, Mz)
\end{aligned}$$

$$= \epsilon_M^{-k}\chi(M)\left(\frac{N}{M}\right)\overline{\chi_1}(-N)\psi^*(\tau, Mz).$$

Hence the result follows. □

Lemma 3.3.6. *Let $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ be a Jacobi cusp form, where χ is a Dirichlet character modulo N. Let M be a prime with $(N,M)=1$. Then*

$$B_M(\phi) \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J_{M,1}(NM^2),\chi),$$

where $B_M(\phi)$ is defined by

$$B_M(\phi) := \frac{1}{M}\sum_{u \bmod (M)} \phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}}.$$

Proof. Let $M' = NM^2$. Consider the matrix $\gamma = \begin{pmatrix} a & b \\ cM' & d \end{pmatrix} \in \Gamma_0(M')$. Then we have

$$(\widetilde{T_{\frac{u}{M}}},(0,0),1)\left(\begin{pmatrix} a & b \\ cM' & d \end{pmatrix},(0,0),1\right) = \widetilde{\gamma}'(\widetilde{T_{\frac{ud^2}{M}}},(0,0),1),$$

where $\gamma' = \begin{pmatrix} a' & b' \\ cM' & d' \end{pmatrix} \in \Gamma_0(M')$ with $d' = d - cd^2\frac{uM'}{M}$. We have

$$\begin{aligned} B_M(\phi)|_{\frac{k}{2},m}\widetilde{\gamma} &= \frac{1}{M}\sum_{u \bmod (M)} (\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}})|_{\frac{k}{2},m}\widetilde{\gamma} \\ &= \frac{1}{M}\sum_{u \bmod (M)} (\phi|_{\frac{k}{2},m}\widetilde{\gamma}')|_{\frac{k}{2},m}\widetilde{T_{\frac{ud^2}{M}}} \\ &= \chi(d')\frac{1}{M}\sum_{u \bmod (M)} \phi|_{\frac{k}{2},m}\widetilde{T_{\frac{ud^2}{M}}} \end{aligned}$$

$$\begin{aligned} &= \chi(d')\frac{1}{M}\sum_{u \bmod (M)} \phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}} \\ &= \chi(d)B_M(\phi), \end{aligned}$$

where we have used that $(d, M) = 1$ and $d' \equiv d \pmod N$ to obtain $\chi(d) = \chi(d')$. Other transformation properties and required Fourier expansion follow similarly as in the proof of Lemma 3.3.2.

□

Lemma 3.3.7. *Let M be an odd prime and χ_1 be a primitive Dirichlet character modulo M. For a complex-valued holomorphic function ψ defined on $\mathcal{H} \times \mathbb{C}$, consider the function ψ^* as defined in Lemma 3.3.5. Then*

(i) If $\chi_1 \neq \chi_2$ then $C_{\chi_1}\psi^* = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)(\frac{N}{M})\chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\chi_1\chi_2}\mathcal{G}_{\overline{\chi_1}}^{-1}\psi_{\overline{\chi_1\chi_2}}$.

(ii) If $\chi_1 = \chi_2$, then $C_{\chi_1}\psi^* = (\frac{-1}{M})^{\frac{k-1}{2}}\chi_1(M)(M^{\frac{1}{2}}B_M(\psi) - M^{-\frac{1}{2}}\psi)$.

Here C_{χ_1} is as in Lemma 3.3.5 and $\chi_2(u) = \left(\frac{u}{M}\right)$.

Proof. If $\chi_1 \neq \chi_2$, then $\chi_1\chi_2$ is primitive character modulo M, and the proof follows from Lemma 3.3.2.

If $\chi_1 = \chi_2$, then

$$\psi^* = \sum_{u=1}^{M} \psi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}}} - \psi = MB_M(\psi) - \psi$$

and $C_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)\epsilon_M^{-1}\left(\frac{N}{M}\right)\left(\frac{-N}{M}\right) = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)$. □

3.4 Proof of results

In this section, we present the proofs of Theorem 3.2.4 and Theorem 3.2.5.

3.4.1 Proof of Theorem 3.2.4

We need the following half-integral weight version of Proposition 1 in [36] to prove Theorem 3.2.4.

Lemma 3.4.1. *Let k, m, and N be positive integers with k odd and $4|N$. Let χ_1 be a character mod M with $(M, N) = 1$. If $\phi(\tau, z)$ and $\psi(\tau, z)$ are Fourier series of type J and J_N, respectively. Then the following are equivalent:*

a) There exists a constant C such that

$$(W_{NM^2}(\phi_{\chi_1}))(\tau, z) = C\psi^*(\tau, Mz).$$

b) The functions $\Lambda_\mu(\phi_{\chi_1,s)}$ and $\Lambda_\mu(\psi^*, s)$ $(1 \leqslant \mu \leqslant 2mM)$ have a holomorphic continuation to the whole complex plane. Moreover, they are bounded in any vertical strip and satisfy functional equations

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C\Lambda_a\left(\psi^*; \frac{k}{2} - s - \frac{1}{2}\right),$$

where $1 \leqslant a \leqslant 2mM$.

Proof. Since the definitions of Fricke involution in the case of integral weight ([36], p. 166) and half-integral weight (3.9) differs just by a constant, the lemma follows just by replacing k with $\frac{k}{2}$ in the proof of Proposition 1 in [36]. □

We now give a proof of the Theorem 3.2.4. Since $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ is a Jacobi cusp form with $W_N(\phi) = \psi$, from Lemma 3.3.5, it is easy to see that ϕ and ψ are series of type J and type J_N, respectively satisfying condition (a) of Lemma 3.4.1. Hence, by Lemma 3.4.1, we deduce that for every $\mu = 0, 1, \cdots, 2mM - 1$ the completed Dirichlet series $\Lambda_\mu(\phi_{\chi_1}; s)$ have holomorphic continuation to whole complex plane, are bounded on every vertical strip and satisfy the function equation

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\Big(-\frac{a\mu}{2mM}\Big)\, \Lambda_\mu(\phi_{\chi_1}; s) = C\Lambda_a\left(\psi^*; \frac{k}{2} - s - \frac{1}{2}\right), \text{ where } 1 \leqslant a \leqslant 2mM.$$

Now the result follows from Lemma 3.3.7.

3.4.2 Proof of Theorem 3.2.5

We first state two lemmas which will be used to prove Theorem 3.2.5. To state these lemmas, we need the following notation: for a complex-valued holomorphic function ψ defined on $\mathcal{H} \times \mathbb{C}$, we define $\Omega_\psi = \{\sigma \in \mathbb{C}[G] : \psi|_{\frac{k}{2},m}\sigma = 0\}$, where $\mathbb{C}[G]$ is the group ring. Then Ω_ψ is a right ideal in $\mathbb{C}[G]$.

Lemma 3.4.2. *Let m, N be positive integers and M be prime such that $4|N$ and $(N, M) = 1$. Let χ be a Dirichlet character modulo N, χ_1 be a primitive Dirichlet character modulo M. Let $\phi(\tau, z)$ and $\psi(\tau, z)$ be series of type J and type J_N, respectively. Assume that ϕ and ψ satisfy the following:*

$$W_N(\phi) = C_{\chi_1}\psi^* \text{ with } C_{\chi_1} = \left(\frac{-1}{M}\right)^{\frac{k-1}{2}} \chi(M) \left(\frac{N}{M}\right) \chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\chi_1}^{-1}$$

and

$$\psi^*(\tau, z) = \sum_{u=0}^{M-1} \chi_1(u) \left(\frac{u}{M}\right) \psi|_{\frac{k}{2},m} T_{\frac{u}{M}}.$$

Then, for $u,\ v \in \mathbb{Z}$ *with* $(u, M) = (v, M) = 1$*, we have*

$$\left(\frac{v}{M}\right)\left(\widetilde{\gamma}(M, v) - \chi(M)\left(\frac{N}{M}\right)\right) T_{\frac{v}{M}} \equiv \left(\frac{u}{M}\right)\left(\widetilde{\gamma}(M, u) - \chi(M)\left(\frac{N}{M}\right)\right) T_{\frac{u}{M}} \pmod{\Omega_\psi}.$$

Proof. The proof uses a similar method as given in Lemma 2.17 of [5]. □

Lemma 3.4.3. *Let* N *be a positive integer, and* M_1, M_2 *are prime numbers with* $(M_1, N) = 1 = (M_2, N)$*. Let* χ_1 *be a primitive Dirichlet character with conductor* M_1 *or* M_2*. Let* $\phi(\tau, z)$ *and* $\psi(\tau, z)$ *be series of type* J *and type* J_N*, respectively. Suppose that* ϕ *and* ψ *satisfy the assumptions given in Lemma 3.4.2. Then*

$$\psi|_{\frac{k}{2}, mN}\gamma = \chi(M_1)\left(\frac{N}{M_1}\right)\psi \text{ for all } \gamma = \begin{pmatrix} M_1 & -v \\ -uN & M_2 \end{pmatrix} \in \Gamma_0(N).$$

Proof. The proof is a straightforward adaptation of the Lemma 2.18 of [5]. □

We now give a proof of Theorem 3.2.5. It is easy to observe that $\phi(\tau, z)$ and $\psi(\tau, z)$ are holomorphic functions on $\mathcal{H} \times \mathbb{C}$. From the functional equation for $M = 1$ in Lemma 3.4.1 (χ_1 will be the trivial character), it follows that $\psi = W_N(\phi)$. Let M be a prime number and χ_1 be a primitive Dirichlet character modulo M. Then from the given conditions (i), (ii) and Lemma 3.4.1, it follows that

$$(W_N(\phi_{\chi_1}))(\tau, z) = C_{\chi_1}\psi^*(\tau, Mz).$$

Next, we prove that

$$\psi|_{\frac{k}{2}, mN}\widetilde{\gamma} = \overline{\chi}(M_2)\left(\frac{N}{M_2}\right)\psi \text{ for all } \gamma = \begin{pmatrix} M_1 & -v \\ -uN & M_2 \end{pmatrix} \in \Gamma_0(N).$$

If $c = 0$, then $\gamma = \begin{pmatrix} \pm 1 & v \\ 0 & \pm 1 \end{pmatrix}$ and it is easy to check the required transformation property for ψ. Now, assume that $c \neq 0$ and $\gamma = \begin{pmatrix} a & -b \\ cN & d \end{pmatrix}$. Since $(a, cN) = 1 = (d, cN)$, there exist integers s and t such that both $a + tcN, d + scN \in \mathcal{M}_N$. Put $a' = a + tcN,\ d' = d + scN,\ c' = -c$ and $b' = -(b + as + stcN + dt)$. Then we have

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & -b' \\ -c'N & d, \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}.$$

From the above computation, we obtain

$$\begin{aligned} \psi|_{\frac{k}{2},mN}\widetilde{\gamma} &= \psi|_{\frac{k}{2},mN} \widetilde{\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}} \widetilde{\begin{pmatrix} a' & -b' \\ -c'N & d' \end{pmatrix}}^{\sim} \widetilde{\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}} \\ &= \psi|_{\frac{k}{2},mN} \widetilde{\begin{pmatrix} a' & -b' \\ -c'N & d' \end{pmatrix}} \widetilde{\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}}. \end{aligned}$$

Using Lemma 3.4.3, we obtain

$$\psi|_{\frac{k}{2},mN}\widetilde{\gamma}(\tau, z) = \chi(a') \left(\frac{N}{a'}\right) \psi(\tau, z).$$

Since $a'd' \equiv 1$ and $4 \mid N$, we have

$$\psi|_{\frac{k}{2},mN}\widetilde{\gamma}(\tau, z) = \overline{\chi}(d') \left(\frac{N}{d'}\right) \psi(\tau, z).$$

Also $d' = d + scN$. Thus, we have

$$\psi|_{\frac{k}{2},mN}\widetilde{\gamma}(\tau, z) = \overline{\chi}(d)\left(\frac{N}{d}\right)\psi(\tau, z). \tag{3.11}$$

The invariance of $\psi(\tau, z)$ under the group $(\mathbb{Z} \times N^{-1}\mathbb{Z})\langle\zeta_N\rangle$ follows from the theta decomposition of $\psi(\tau, z)$. Hence

$$\psi|_{\frac{k}{2},mN}h(\tau, z) = \overline{\chi}(d)\left(\frac{N}{d}\right)\psi(\tau, z) \text{ for every } h \in \Gamma^J_{1,N}(N).$$

For matrices $\gamma = \begin{pmatrix} d & -c \\ -bL & a \end{pmatrix}$, $\gamma' = \begin{pmatrix} a & b \\ cL & d \end{pmatrix} \in \Gamma_0(L)$, we have

$$\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right)\widetilde{\gamma}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right)^{-1} = \left(\gamma', \left(\frac{N}{d}\right)j(\gamma', \tau)\right).$$

Thus by the definition of W_L and the above identity, we have

$$\begin{aligned}
&W_L(\phi)|_{\frac{k}{2},mL}(\gamma, j(\gamma, \tau))(\tau, z) \\
&= (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(\gamma', \left(\frac{N}{a}\right)j(\gamma', \tau)\right)\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) \\
&= \left(\frac{N}{a}\right)U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}\widetilde{\gamma'})|_{\frac{k}{2},mL}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right).
\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\left(\frac{N}{a}\right) U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}\widetilde{\gamma}' - \overline{\chi}(a)\phi)|_{\frac{k}{2},mL}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) = 0,$$

for every $(\tau, z) \in \mathcal{H} \times \mathbb{C}$. Hence we have $\phi|_{\frac{k}{2},m}\widetilde{\gamma}' = \overline{\chi}(a)\phi = \chi(d)\phi$ for every $\widetilde{\gamma}' \in \widetilde{\Gamma_0(N)}$.

To check the cuspidality, we need to estimate $e^m(pz)h_\mu(\tau)\theta_{m,\mu}(\tau, z)$. For this, consider $d_\mu(n)$ defined by $d_\mu(n) := \sum_{N=1}^{n} |c_\mu(N)|$. Then, we have

$$d_\mu(n) \leq n^{\frac{k}{2}-1-\epsilon}(\sum_{N=1}^{\infty} |c_\mu(N)N^{-\frac{k}{2}+1+\epsilon}|).$$

Thus, we obtain $d_\mu(n) = O(n^{\frac{k}{2}-1-\epsilon})$ and $\sum_{n=0}^{\infty} d_\mu(n)e^{-2\pi y} = O(y^{-\frac{k}{2}+\epsilon})$. A straightforward calculation shows that $e^m(z)\phi_\mu(\tau)\theta_{m,\mu}(\tau, z) = O(y^{-\frac{k}{2}+\frac{1}{2}+\epsilon})$ and hence $e^m(z)\phi(\tau, z) = O(y^{-\frac{k}{2}+\frac{1}{2}+\epsilon})$. Finally, Lemma 1.6.4 together with the above observation implies that $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$. This completes the proof.

Chapter 4

Differential operators and Poincaré series for Jacobi forms

4.1 Introduction

Let f and g be modular forms of weight k_1 and k_2, respectively. It is well known that $[f,g]_\nu$ is a modular form of weight $k_1+k_2+2\nu$. One can consider the converse question: if the Rankin-Cohen bracket of two holomorphic functions is a modular form, is it necessary that one of the functions is a modular form? In this direction, Choie and Lee [8] proved the following result:

Theorem 4.1.1. *[8] Let $k_1,\ k_2$ and ν be positive integers. Let f and h be non-constant modular forms of weight k_1 and $k_1+k_2+2\nu$, respectively for the group $SL_2(\mathbb{Z})$. Consider the following differential equation*

$$\sum_{r=0}^{\nu}(-1)^r\binom{k_1+\nu-1}{\nu-r}\binom{k_2+\nu-1}{r}f^{(r)}g^{(\nu-r)}=h. \tag{4.1}$$

Then

1. *each solution g of* (4.1) *is a meromorphic modular form of weight k_2 for $SL_2(\mathbb{Z})$ which may have poles in $\mathcal{H}\cup\{\infty\}$;*

2. *if any solution g of* (4.1) *is holomorphic on $\mathcal{H}\cup\{\infty\}$, then it is a holomorphic*

modular form of weight k_2 for $SL_2(\mathbb{Z})$.

Rankin-Cohen brackets have interesting relations with the Poincaré series. Williams studied the following properties:

Theorem 4.1.2. *[49] Let $k_1, k_2(\geq 4)$ be even integers and ν, n be positive integers. For a cusp form f of weight k_1, consider the function $\tilde{f}$ defined by*

$$\tilde{f}(\tau) := q^n \sum_{r=0}^{\nu} (-1)^r \binom{k_1+\nu-1}{\nu-r} \binom{k_2+\nu-1}{r} n^{\nu-r} f^{(r)}.$$

Then

$$[f, P_{k_2,n}]_\nu = \mathbb{P}_{k_1+k_2+2\nu}(\tilde{f}),$$

where $\mathbb{P}_{k_1+k_2+2\nu}(\tilde{f}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f|_k \gamma$ is the generalized Poincaré series defined in [49].

As a corollary Williams proved that Poincaré series and Rankin-Cohen brackets commute in the following sense:

Corollary 4.1.3. *[49] Let f be a modular form of weight k_1 and ϕ be a q-series whose coefficients grow sufficiently slow enough that $\mathbb{P}_l(\phi)$ is well-defined and denote $[f, \phi]_\nu$ to be the formal result of the ν-th Rankin–Cohen bracket, where ϕ is treated like a modular form of weight k_2 (where $k_2 \geqslant k_1 + 2$ if f is not a cusp form). Then*

$$[f, \mathbb{P}_{k_2}(\phi)]_\nu = \mathbb{P}_{k_1+k_2+2\nu}([f, \phi]_\nu).$$

One can construct interesting modular forms using differential operators by computing the adjoint maps. In this direction, Kohnen [25] constructed cusp forms whose Fourier coefficients involve special values of certain convolution type Dirichlet series by computing the adjoint of the linear map $f \mapsto fg$, for a fixed modular form g, between spaces of cusp forms. Kohnen proved:

Theorem 4.1.4. *[24] Let k_1 and k_2 be positive integers with $k_1 > k_2 + 2$. Let $f \in S_{k_1+k_2}$ and $g \in S_{k_2}$ with Fourier expansions*

$$f(\tau) = \sum_{m=1}^{\infty} a(m)q^m \text{ and } g(\tau) = \sum_{m=1}^{\infty} b(m)q^m.$$

Then the function

$$T_g^*(f)(\tau) := \sum_{n=1}^{\infty} n^{k_1-1} L_{f,g;n}(k_1 + k_2 - 1)q^n,$$

where

$$L_{f,g;n}(s) := \sum_{m=1}^{\infty} \frac{a(m+n)\overline{b(m)}}{(m+n)^s}, \tag{4.2}$$

is a cusp form of weight k_1 for $SL_2(\mathbb{Z})$. In fact, the map $S_{k_1+k_2} \to S_{k_1}$ defined by $f \mapsto \dfrac{\Gamma(k_1+k_2-1)}{\Gamma(k_1-1)(4\pi)^{k_2}} T_g^*(f)$ is the adjoint of the map $T_g : S_{k_1} \longrightarrow S_{k_1+k_2},\ h \mapsto gh$, with respect to the Petersson scalar product.

Herrero [19] generalized the work of Kohnen and constructed cusp forms by computing the adjoint of certain maps constructed using Rankin-Cohen brackets. The work of Herrero has been generalized by several authors for various automorphic forms (see [20, 21, 22, 32]).

The map $\vartheta_k : f \mapsto \dfrac{1}{2\pi i}\dfrac{df}{d\tau} - \frac{k}{12}E_2(\tau)f(\tau)$ is a linear map from M_k to M_{k+2} called the Serre derivative. Kumar [31] generalized the work of Kohnen by computing the adjoint map of ϑ_k, and obtained interesting identities involving special values of convolution-type Dirichlet series.

In this chapter, we prove the analogous results of Theorem 4.1.1, Theorem 4.1.2 and Theorem 4.1.4. This chapter is based on our works [29] and [37].

4.2 Statment of results

Following the work of Choie and Lee [8], we answer the analogous question in the context of Jacobi forms.

Theorem 4.2.1. *[29] Let k_1, k_2, m_1 and m_2 be positive integers. Let $\phi \in J_{k_1,m_1}$ and $h \in J_{k_1+k_2+2\nu,m_1+m_2}$ be non-constant Jacobi forms. Then each solution ψ of the following differential equation*

$$\sum_{r=0}^{\nu}(-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r L_{m_1}^r(\phi) L_{m_2}^{\nu-r}(\psi) = h, \tag{4.3}$$

satisfies the transformation properties (1.5) *and* (1.6) *with weight k_2 and index m_2. Moreover, if ψ has a Fourier series expansion similar* (1.4)*, then $\psi \in J_{k_2,m_2}$.*

We have the following analogue of Theorem 4.1.2.

Theorem 4.2.2. *[29] Let $k_1, k_2(\geq 11),\ m_1,\ m_2$ and ν be positive integers. Let $N, R \in \mathbb{Z}$ be such that $4Nm_2 - R^2 > 0$. Consider the function $f(\tau, z)$ defined by*

$$\begin{aligned} f(\tau, z) &= q^N \zeta^R \sum_{r=0}^{\nu}(-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r \\ &\times\ (4Nm_2 - R^2)^{\nu-r} L_{m_1}^r(\phi), \end{aligned}$$

where $\phi \in J_{k_1,m_1}$ (with $k_2 \geqslant k_1 + 10$ when ϕ is not a cusp form). Then we have

$$[\phi, P_{k_2,m_2;N,R}]_\nu = \mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}(f).$$

As an immediate consequence of the above theorem, we obtain the following corollary:

Corollary 4.2.3. *[29] Let ϕ be a Jacobi form of weight k_1, index m_1, and f be a formal (q, ζ)-series such that $\mathbb{P}_{k_2,m_2}(f)$ is well defined. Assume that $k_2 \geqslant k_1 + 2$ when ϕ is not a* *cusp form. Then*

$$[\phi, \mathbb{P}_{k_2,m_2}(f)]_\nu = \mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}([\phi, f]_\nu).$$

We recall that the modified heat operator $L_{k,\mathcal{M}}$ (1.23) is a $\mathbb{C}$-linear map between finite-dimensional Hilbert spaces $J^{cusp}_{k,\mathcal{M}}$, and $J^{cusp}_{k+2,\mathcal{M}}$. Therefore it has an adjoint map $L^*_{k,\mathcal{M}} : J^{cusp}_{k+2,\mathcal{M}} \to J^{cusp}_{k,\mathcal{M}}$ such that

$$\langle L^*_{k,\mathcal{M}}(\phi), \psi\rangle = \langle \phi, L_{k,\mathcal{M}}(\psi)\rangle, \quad \forall \phi \in J^{cusp}_{k+2,\mathcal{M}}, \quad \text{and} \quad \psi \in J^{cusp}_{k,\mathcal{M}}.$$

We explicitly compute the adjoint map, i.e., we calculate the Fourier coefficients of the image of a Jacobi cusp form under the map $L^*_{k,\mathcal{M}}$.

Theorem 4.2.4. *[37] Let $k > 4$, and $\mathcal{M}$. Let $\phi \in J^{cusp}_{k+2,\mathcal{M}}$ with Fourier expansion $\phi(\tau, z) = \sum\limits_{\substack{n,r\in\mathbb{Z}^g,\\ 4n>\mathcal{M}^{-1}[rt]>0}} c_\phi(n, r)q^N\zeta^R$. Then the image of ϕ under $L^*_{k,\mathcal{M}}$ is given by*

$$L^*_{k,\mathcal{M}}(\phi)(\tau, z) = \sum_{\substack{N,R\in\mathbb{Z}^g,\\ 4N-\mathcal{M}^{-1}[R^t]>0}} a(N, R)q^n\zeta^r,$$

where

$$\begin{aligned} a(N, R) = \frac{|\mathcal{M}|^{\frac{5-g}{2}}(\mathcal{K}+1)(\mathcal{K})(4N|\mathcal{M}| - \tilde{M}[R])^{\mathcal{K}}}{\pi^2 2^{(g-1)(k-\frac{g}{2}-1)}} &\Bigg[\frac{\left(4N|\mathcal{M}| - \tilde{\mathcal{M}}[R] - \frac{\mathcal{K}|\mathcal{M}|}{3}\right)}{(4N|\mathcal{M}| - \tilde{\mathcal{M}}[R])^{\mathcal{K}+2}} c_\phi(N, R) \\ + \quad 8(\mathcal{K}+1)|\mathcal{M}| \sum_{n\geqslant 1} &\frac{c_\phi(n+N, R)\sigma(n)}{(4(n+N)|\mathcal{M}| - \tilde{\mathcal{M}}[R])^{\mathcal{K}+2}}\Bigg], \end{aligned}$$

where $\mathcal{K} = k - \frac{g}{2} - 1$.

4.3 Proof of results

The following lemma gives the bound of Fourier coefficients of Jacobi forms. We have

Lemma 4.3.1. *[7] Let $k > 3$ and $\phi = \sum\limits_{4nm-r^2\geqslant 0} c_\phi(n,r)q^n\zeta^r \in J_{k,m}$. Then*

$$c_\phi(n,r) \ll (4nm - r^2)^{k-\frac{3}{2}}.$$

Moreover, if $\phi \in J_{k,m}^{cusp}$, then

$$c_\phi(n,r) \ll (4nm - r^2)^{\frac{k}{2}-\frac{1}{2}}.$$

Let $f(\tau, z)$ be a holomorphic function defined on $\mathcal{H}\times\mathbb{C}$ with Fourier expansion $f(\tau,z) = \sum\limits_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} a_f(n,r)q^n\zeta^r$. We define the generalized Poincaré series with the base function f by

$$\mathbb{P}_{k,m}(f)(\tau,z) = \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} f|_{k,m}\gamma. \tag{4.4}$$

To observe the absolute convergence of $\mathbb{P}_{k,m}(f)$, consider the series

$$\tilde{\mathbb{P}}_{k,m}(f) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} a_f(n,r)P_{k,m;n,r}. \tag{4.5}$$

Since $J_{k,m}$ is a finite-dimensional vector space, in view of the Theorem 1.5.3, the convergence of the above series is equivalent to the convergence of the series

$$\sum_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} a_f(n,r)\langle\psi, P_{k,m;n,r}\rangle = \frac{m^{k-2}\Gamma(k-\frac{3}{2})}{2\pi^{k-\frac{3}{2}}} \sum_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} \frac{c_\psi(n,r)a_f(n,r)}{(4nm-r^2)^{k-\frac{3}{2}}},$$

where $\psi \in J_{k,m}$ with the Fourier expansion given by $\psi(\tau,z) = \sum\limits_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} c_\psi(n,r)q^n\zeta^r$. Now using Lemma 4.3.1, the convergence of the series (4.5) follows immediately provided the coefficient $a_f(n,r)$ of f satisfies the bound $a_f(n,r) = O((4nm-r^2)^{\frac{k}{2}-6-\epsilon})$ for any $\epsilon > 0$.

Thus, $\mathbb{P}_{k,m}(f) = \tilde{\mathbb{P}}_{k,m}(f)$ whenever the later series converges.

We need the following two lemmas.

Lemma 4.3.2. *[7] Let ϕ be a complex-valued holomorphic function defined on $\mathcal{H} \times \mathbb{C}$. Then for a non-negative integer ν and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$L_m^\nu(\phi)|_{k+2\nu,m}M = \sum_{l=0}^{\nu} \binom{\nu}{l} \left(\frac{2mc}{\pi i}\right)^{\nu-l} \frac{(k+\nu-\frac{3}{2})!}{(k+l-\frac{3}{2})!} \frac{L_m^l(\phi|_{k,m}M)}{(c\tau+d)^{\nu-l}}.$$

Lemma 4.3.3. *Let ϕ be a Jacobi form of weight k and index m. Then for a non-negative integer ν and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$L_m^\nu(\phi|_{k,m}M) = \sum_{l=0}^{\nu} (-1)^{\nu-l} \binom{\nu}{l} \left(\frac{2mc}{\pi i}\right)^{\nu-l} \frac{(k+\nu-\frac{3}{2})!}{(k+l-\frac{3}{2})!} \frac{L_m^l(\phi)|_{k+2l,m}M}{(c\tau+d)^{\nu-l}}.$$

Proof. The proof is similar to the proof of Lemma 4.3.2 and it uses a simple induction argument. □

4.3.1 Proof of Theorem 4.2.1

Proof. Let ψ be a solution of (4.3). We prove the transformation properties (1.5) and (1.6). Consider

$$h = \sum_{r=0}^{\nu} (-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r} \binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r L_{m_1}^r(\phi) L_{m_2}^{\nu-r}(\psi). \tag{4.6}$$

Now for any matrix $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, apply the slash operator of weight $k_1 + k_2 + 2\nu$ and index $m_1 + m_2$ to the above equation. Using the fact that h is a Jacobi form of weight $k = k_1 + k_2 + 2\nu$ and index $m = m_1 + m_2$, we obtain

$$\begin{aligned} h &= \left(\sum_{r=0}^{\nu} (-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r} \binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r L_{m_1}^r(\phi) L_{m_2}^{\nu-r}(\psi) \right)|_{k,m}\gamma \\ &= \sum_{r=0}^{\nu} (-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r} \binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r L_{m_1}^r(\phi)|_{k_1+2r,m_1}\gamma \\ &\times \quad L_{m_2}^{\nu-r}(\psi)|_{k_2+2(\nu-r),m_2}\gamma. \end{aligned}$$

Now, Lemma 4.3.2 implies

$$\begin{aligned} h(\tau,z) &= \sum_{r=0}^{\nu} (-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r} \binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r L_{m_1}^r(\phi)|_{k_1+2r,m_1}\gamma \\ &\times \sum_{l=0}^{\nu-r} \binom{\nu-r}{l} \left(\frac{2m_2c}{\pi i}\right)^{\nu-r-l} \frac{(k_2+\nu-r-\frac{3}{2})!}{(k_2+l-\frac{3}{2})!} \frac{L_{m_2}^l(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r-l}}. \end{aligned} \tag{4.7}$$

Applying the change of variable $l \mapsto l-r$ in (4.7), we obtain

$$\begin{aligned} h(\tau,z) &= \sum_{r=0}^{\nu} (-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r} \binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r L_{m_1}^r(\phi)|_{k_1+2r,m_1}\gamma \\ &\times \sum_{l=r}^{\nu} \binom{\nu-r}{l-r} \left(\frac{2m_2c}{\pi i}\right)^{\nu-l} \frac{(k_2+\nu-r-\frac{3}{2})!}{(k_2+l-r-\frac{3}{2})!} \frac{L_{m_2}^{l-r}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-l}}, \\ &= \sum_{r=0}^{\nu} \sum_{l=r}^{\nu} (-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r} \binom{k_2+\nu-\frac{3}{2}}{r} \binom{\nu-r}{l-r} \left(\frac{2m_2c}{\pi i}\right)^{\nu-l} \\ &\times m_1^{\nu-r} m_2^r \frac{(k_2+\nu-r-\frac{3}{2})!}{(k_2+l-r-\frac{3}{2})!} L_{m_1}^r(\phi)|_{k_1+2r,m_1}\gamma \frac{L_{m_2}^{l-r}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-l}}, \\ &= \sum_{l=0}^{\nu} \sum_{r=0}^{l} (-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r} \binom{k_2+\nu-\frac{3}{2}}{r} \binom{\nu-r}{l-r} \left(\frac{2m_2c}{\pi i}\right)^{\nu-l} \\ &\times m_1^{\nu-r} m_2^r \frac{(k_2+\nu-r-\frac{3}{2})!}{(k_2+l-r-\frac{3}{2})!} L_{m_1}^r(\phi)|_{k_1+2r,m_1}\gamma \frac{L_{m_2}^{l-r}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-l}}. \end{aligned}$$

Next, interchange the variables r and l to obtain

$$\begin{aligned} h(\tau,z) &= \sum_{r=0}^{\nu}\sum_{l=0}^{r}(-1)^l\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}\binom{\nu-l}{r-l}\left(\frac{2m_2c}{\pi i}\right)^{\nu-r} \\ &\times\ m_1^{\nu-l}m_2^l\frac{(k_2+\nu-l-\frac{3}{2})!}{(k_2+r-l-\frac{3}{2})!}L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma\frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}}. \end{aligned}$$

A simple calculation yields

$$\begin{aligned} h(\tau,z) &= \sum_{r=0}^{\nu}\sum_{l=0}^{r}\frac{(-1)^l(k_1+\nu-\frac{3}{2})!(k_2+\nu-\frac{3}{2})!m_1^{\nu-l}m_2^l}{(k_1+l-\frac{3}{2})!(k_2+r-l-\frac{3}{2})!l!(r-l)!(\nu-r)!} \\ &\times\ \left(\frac{2m_2c}{\pi i}\right)^{\nu-r}L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma\frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}}. \end{aligned} \tag{4.8}$$

Also, we can rewrite the equation (4.6) as follows

$$h(\tau,z)=\sum_{r=0}^{\nu}(-1)^r\binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r}m_1^{\nu-r}m_2^rL_{m_1}^r(\phi|_{k_1,m_1}\gamma)L_{m_2}^{\nu-r}(\psi),$$

where we have used the fact that ϕ is a Jacobi form of weight k_1, index m_1 and hence $\phi|_{k_1,m_1}\gamma=\phi$. Next, using Lemma 4.3.3 we get

$$\begin{aligned} h(\tau,z) &= \sum_{r=0}^{\nu}(-1)^r\binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r}m_1^{\nu-r}m_2^rL_{m_2}^{\nu-r}(\psi) \\ &\times\ \sum_{l=0}^{r}(-1)^{r-l}\binom{r}{l}\frac{(k_1+r-\frac{3}{2})!}{(k_1+l-\frac{3}{2})!}\left(\frac{2m_1c}{\pi i}\right)^{r-l}\frac{L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma}{(c\tau+d)^{r-l}}. \\ &= \sum_{r=0}^{\nu}\sum_{l=0}^{r}\frac{(-1)^l(k_1+\nu-\frac{3}{2})!(k_2+\nu-\frac{3}{2})!m_1^{\nu-r}m_2^r}{(k_2+\nu-r-\frac{3}{2})!(k_1+l-\frac{3}{2})!(\nu-r)!l!(r-l)!} \\ &\times\ \left(\frac{2m_1c}{\pi i}\right)^{r-l}\frac{L_{m_2}^{\nu-r}(\psi)}{(c\tau+d)^{r-l}}L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma. \end{aligned} \tag{4.9}$$

Now subtracting (4.9) from (4.8), we have

$$\begin{aligned}
0 &= \sum_{r=0}^{\nu}\sum_{l=0}^{r} \frac{(-1)^l m_1^{\nu-l} m_2^l}{(k_1+l-\frac{3}{2})!(k_2+r-l-\frac{3}{2})!l!(r-l)!(\nu-r)!}\left(\frac{2m_2c}{\pi i}\right)^{\nu-r} \\
&\times\ L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma \frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}} \\
&-\ \sum_{r=0}^{\nu}\sum_{l=0}^{r} \frac{(-1)^l m_1^{\nu-r} m_2^r}{(k_2+\nu-r-\frac{3}{2})!(k_1+l-\frac{3}{2})!(\nu-r)!l!(r-l)!}\left(\frac{2m_1c}{\pi i}\right)^{r-l} \\
&\times\ \frac{L_{m_2}^{\nu-r}(\psi)}{(c\tau+d)^{r-l}} L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma \\
&= \sum_{r=0}^{\nu}\sum_{l=0}^{r}\Bigg[\frac{(-1)^l}{(k_2+r-l-\frac{3}{2})!} m_1^{\nu-l}m_2^l\left(\frac{2m_2c}{\pi i}\right)^{\nu-r}\frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}} \\
&-\ \frac{(-1)^l}{(k_2+\nu-r-\frac{3}{2})!} m_1^{\nu-r}m_2^r\left(\frac{2m_1c}{\pi i}\right)^{r-l}\frac{L_{m_2}^{\nu-r}(\psi)}{(c\tau+d)^{r-l}}\Bigg] \\
&\times\ \frac{1}{l!(\nu-r)!(r-l)!(k_1+l-\frac{3}{2})!} L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma, \\
&= \sum_{l=0}^{\nu}\sum_{r=l}^{\nu}\Bigg[\frac{(-1)^l}{(k_2+r-l-\frac{3}{2})!} m_1^{\nu-l}m_2^l\left(\frac{2m_2c}{\pi i}\right)^{\nu-r}\frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}} \\
&-\ \frac{(-1)^l}{(k_2+\nu-r-\frac{3}{2})!} m_1^{\nu-r}m_2^r\left(\frac{2m_1c}{\pi i}\right)^{r-l}\frac{L_{m_2}^{\nu-r}(\psi)}{(c\tau+d)^{r-l}}\Bigg] \\
&\times\ \frac{1}{l!(\nu-r)!(r-l)!(k_1+l-\frac{3}{2})!} L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma, \\
&= \sum_{l=0}^{\nu}\frac{(-1)^l m_1^{\nu-l}m_2^l}{l!(k_1+l-\frac{3}{2})!} L_{m_1}^l(\phi)|_{k_1+2l,m_1}\gamma\ \sum_{r=l}^{\nu}\Bigg[\frac{1}{(k_2+r-l-\frac{3}{2})!} \\
&\times\ \left(\frac{2m_2c}{\pi i}\right)^{\nu-r}\frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}} - \frac{1}{(k_2+\nu-r-\frac{3}{2})!}\left(\frac{2m_2c}{\pi i}\right)^{r-l} \\
&\times\ \frac{L_{m_2}^{\nu-r}(\psi)}{(c\tau+d)^{r-l}}\Bigg]\frac{1}{(\nu-r)!(r-l)!}.
\end{aligned} \tag{4.10}$$

Now, consider the following expression

$$\sum_{r=l}^{\nu}\frac{1}{(k_2+r-l-\frac{3}{2})!(\nu-r)!(r-l)!}\left(\frac{2m_2c}{\pi i}\right)^{\nu-r}\frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}}.$$

Replace $r-l$ by p in the above expression to get

$$\sum_{r=l}^{\nu} \frac{1}{(k_2+r-l-\frac{3}{2})!(\nu-r)!(r-l)!}\left(\frac{2m_2c}{\pi i}\right)^{\nu-r}\frac{L_{m_2}^{r-l}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-r}}$$
$$=\sum_{p=0}^{\nu-l} \frac{1}{(k_2+p-\frac{3}{2})!p!(\nu-l-p)!}\left(\frac{2m_2c}{\pi i}\right)^{\nu-l-p}\frac{L_{m_2}^{p}(\psi|_{k_2,m_2}\gamma)}{(c\tau+d)^{\nu-l-p}}. \tag{4.11}$$

Similarly, consider the expression

$$\sum_{r=l}^{\nu} \frac{1}{(k_2+\nu-r-\frac{3}{2})!(\nu-r)!(r-l)!}\left(\frac{2m_2c}{\pi i}\right)^{r-l}\frac{L_{m_2}^{\nu-r}(\psi)}{(c\tau+d)^{r-l}}$$

and replace $\nu-r$ by p to obtain

$$\sum_{r=l}^{\nu} \frac{1}{(k_2+\nu-r-\frac{3}{2})!(\nu-r)!(r-l)!}\left(\frac{2m_2c}{\pi i}\right)^{r-l}\frac{L_{m_2}^{\nu-r}(\psi)}{(c\tau+d)^{r-l}}$$
$$=\sum_{p=0}^{\nu-l} \frac{1}{(k_2+p-\frac{3}{2})!p!(\nu-l-p)!}\left(\frac{2m_2c}{\pi i}\right)^{\nu-l-p}\frac{L_{m_2}^{p}(\psi)}{(c\tau+d)^{\nu-l-p}}. \tag{4.12}$$

Now using equations (4.11) and (4.12), the equation (4.10) reduces to

$$\begin{aligned}
0 &= \sum_{l=0}^{\nu}\frac{(-1)^l m_1^{\nu-l}m_2^l}{l!(k_1+l-\frac{3}{2})!}L_{m_1}^l(\phi)|_{k_1+2l,m_1}\sum_{p=0}^{\nu-l}\frac{1}{(k_2+p-\frac{3}{2})!p!(\nu-l-p)!}\\
&\times \left(\frac{2m_2c}{\pi i}\right)^{\nu-l-p}\frac{1}{(c\tau+d)^{\nu-l-p}}L_{m_2}^p(\psi|_{k_2,m_2}\gamma-\psi)\\
&= \sum_{l=0}^{\nu}\sum_{p=0}^{\nu-l}\frac{(-1)^l m_1^{\nu-l}m_2^l}{(k_1+l-\frac{3}{2})!(k_2+p-\frac{3}{2})!p!(\nu-l-p)!l!}\left(\frac{2m_2c}{\pi i}\right)^{\nu-l-p}\\
&\times \frac{1}{(c\tau+d)^{\nu-l-p}}L_{m_2}^p(\psi|_{k_2,m_2}\gamma-\psi)L_{m_1}^l(\phi)|_{k_1+2l,m_1}.
\end{aligned}$$

Again applying Lemma 4.3.2, we have

$$
\begin{aligned}
0 &= \sum_{l=0}^{\nu}\sum_{p=0}^{\nu-l} \frac{(-1)^l m_1^{\nu-l} m_2^l}{(k_1+l-\frac{3}{2})!(k_2+p-\frac{3}{2})!p!(\nu-l-p)!l!}\left(\frac{2m_2c}{\pi i}\right)^{\nu-l-p} \\
&\times \frac{1}{(c\tau+d)^{\nu-l-p}} L_{m_2}^p(\psi|_{k_2,m_2}\gamma-\psi)\sum_{r=0}^{l}(-1)^r\binom{l}{r}\frac{(k_1+l-\frac{3}{2})!}{(k_1+r-\frac{3}{2})!} \\
&\times \left(\frac{2m_1c}{\pi i}\right)^{l-r}\frac{L_{m_1}^r(\phi)}{(c\tau+d)^{l-r}} \\
&= \sum_{l=0}^{\nu}\sum_{r=0}^{l}\sum_{p=0}^{\nu-l}\frac{(-1)^{l+r}m_1^{\nu-r}m_2^{\nu-p}}{(k_1+r-\frac{3}{2})!(k_2+p-\frac{3}{2})!p!(\nu-l-p)!r!(l-r)!} \\
&\times \left(\frac{2c}{\pi i}\right)^{\nu-r-p} L_{m_2}^p(\psi|_{k_2,m_2}\gamma-\psi)\frac{L_{m_1}^r(\phi)}{(c\tau+d)^{\nu-r-p}} \\
&= \sum_{r=0}^{\nu}\sum_{l=r}^{\nu}\sum_{p=0}^{\nu-l}\frac{(-1)^{l+r}m_1^{\nu-r}m_2^{\nu-p}}{(k_1+r-\frac{3}{2})!(k_2+p-\frac{3}{2})!p!(\nu-l-p)!r!(l-r)!} \\
&\times \left(\frac{2c}{\pi i}\right)^{\nu-r-p} L_{m_2}^p(\psi|_{k_2,m_2}\gamma-\psi)\frac{L_{m_1}^r(\phi)}{(c\tau+d)^{\nu-r-p}} \\
&= \sum_{r=0}^{\nu} A_r(\gamma,\tau,z)L_{m_1}^r(\phi),
\end{aligned}
$$

where for $0 \leqslant r \leqslant \nu$, $A_r(\gamma,\tau,z)$ is given by

$$
\begin{aligned}
A_r(\gamma,\tau,z) &= \sum_{l=r}^{\nu}\sum_{p=0}^{\nu-l}\frac{(-1)^{l+r}m_1^{\nu-r}m_2^{\nu-p}}{(k_1+r-\frac{3}{2})!(k_2+p-\frac{3}{2})!p!(\nu-l-p)!r!(l-r)!} \\
&\times \left(\frac{2c}{\pi i}\right)^{\nu-r-p}\frac{1}{(c\tau+d)^{\nu-r-p}}L_{m_2}^p(\psi|_{k_2,m_2}\gamma-\psi).
\end{aligned}
$$

Thus we have

$$
\sum_{r=0}^{\nu} A_r(\gamma,\tau,z)L_{m_1}^r(\phi) = 0. \tag{4.13}
$$

As ϕ is a Jacobi form of weight k_1 and index m_1, it has a Fourier series expansion given by

$$\phi(\tau, z) = \sum_{\substack{N,R\in\mathbb{Z} \\ 4m_1N-R^2\geqslant 0}} c_\phi(N,R)q^N\zeta^R.$$

By applying the heat operator L_{m_1} repeatedly to the above Fourier series and using (4.13), we obtain

$$\sum_{\substack{N,R\in\mathbb{Z} \\ 4m_1N-R^2\geqslant 0}} \sum_{r=0}^{\nu} c_\phi(N,R)(4m_1N-R^2)^r A_r(\gamma,\tau,z)q^N\zeta^R = 0.$$

Hence, for every N and R with $4Nm_1 - R^2 \geqslant 0$, we get

$$\sum_{r=0}^{\nu} c_\phi(N,R)(4m_1N-R^2)^r A_r(\gamma,\tau,z) = 0.$$

Hence it follows that for each $r \in \{0,1,...,\nu\}$, $A_r(\gamma,\tau,z) = 0$, as the polynomial $\sum\limits_{r=0}^{\nu} A_r(\gamma,\tau,z)x^r$ can have only finitely many roots. In particular, for all $(\tau,z) \in \mathcal{H}\times\mathbb{C}$, we have

$$0 = A_\nu(\gamma,\tau,z) = \frac{m_2^\nu}{(k_1+\nu-\frac{3}{2})!(k_2-\frac{3}{2})!\nu!} m_2^\nu(\psi|_{k_2,m_2}\gamma - \psi)(\tau,z).$$

Therefore, in view of the above identity, we have

$$\psi|_{k_2,m_2}\gamma = \psi, \text{ for all } \gamma \in SL_2(\mathbb{Z}).$$

This proves the transformation property (1.5). Let us now prove the transformation property (1.6). Let $X = (\lambda,\mu) \in \mathbb{Z}^2$. Recall that

$$h(\tau,z) = \sum_{r=0}^{\nu}(-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r}m_2^r L_{m_1}^r(\phi)L_{m_2}^{\nu-r}(\psi).$$

Then, we have

$$\begin{aligned} h &= \sum_{r=0}^{\nu}(-1)^r\binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r}m_1^{\nu-r}m_2^rL_{m_1}^r(\phi)|_{k_1+2r,m_1}X \\ &\times \; L_{m_2}^{\nu-r}(\psi)|_{k_2+2(\nu-r),m_2}X, \\ &= \sum_{r=0}^{\nu}(-1)^r\binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r}m_1^{\nu-r}m_2^rL_{m_1}^r(\phi|_{k_1,m_1}X) \\ &\times \; L_{m_2}^{\nu-r}(\psi|_{k_2,m_2}X), \end{aligned}$$

where in the last line, we have used the commutativity of heat operator and lattice action. Hence, we obtain

$$h=\sum_{r=0}^{\nu}(-1)^r\binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r}m_1^{\nu-r}m_2^rL_{m_1}^r(\phi)L_{m_2}^{\nu-r}(\psi|_{k_2,m_2}X). \tag{4.14}$$

Subtracting (4.6) from (4.14), we obtain

$$\sum_{r=0}^{\nu}(-1)^r\binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r}m_1^{\nu-r}m_2^rL_{m_1}^r(\phi)L_{m_2}^{\nu-r}(\psi|_{k_2,m_2}X-\psi)=0$$

Rewrite the above equation as follows

$$\sum_{r=0}^{\nu}B_r(\lambda,\mu,\tau,z)L_{m_1}^r(\phi)=0,$$

where for $0\leqslant r\leqslant \nu$, $B_r(\lambda,\mu,\tau,z)$ is given by

$$B_r(\lambda,\mu,\tau,z)=(-1)^r\binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r}m_1^{\nu-r}m_2^rL_{m_2}^{\nu-r}(\psi|_{k_2,m_2}X-\psi).$$

Now proceeding as above (in the case of $SL_2(\mathbb{Z})$), we obtain the required transformation property of ψ with respect to the lattice $\mathbb{Z}^2$, i.e.,

$$\psi|_{k_2,m_2}X = \psi, \text{ for all } X = (\lambda, \mu) \in \mathbb{Z}^2.$$

Thus, ψ satisfies the transformation properties of a Jacobi form of weight k_2 and index m_2 and this completes the proof. $\square$

4.3.2 Proof of Theorem 4.2.2

Proof. By using Lemma 4.3.1, it is easy to see that the coefficients $c_f(n,r)$ of $f(\tau,z)$, satisfy the bound $c_f(n,r) = O((4n(m_1+m_2)-r^2)^{k_1-\frac{3}{2}+\nu+\epsilon})$ for any $\epsilon > 0$. Consider the generalized Poincaré series $\mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}(f)$ associated to the above base function f. To ensure its convergence the exponent in the bound of the coefficients of f which is $k_1 - \frac{3}{2} + \nu + \epsilon \leqslant \frac{k_1+k_2+2\nu}{2} - 6 - \epsilon$ reduces to $k_1 \leqslant k_2 - 9 - 4\epsilon$. By definition, we have

$$\begin{aligned}
\mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}(f) &= \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} \Bigg[\sum_{r=0}^{\nu}(-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r}m_2^r \\
&\times \quad (4Nm_2 - R^2)^{\nu-r} q^N \tau^R L_{m_1}^r(\phi)\Bigg]|_{k_1+k_2+2\nu,m_1+m_2}\gamma, \\
&= \quad \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} \sum_{r=0}^{\nu}(-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r}m_2^r \\
&\times \quad L_{m_2}^{\nu-r}(q^N\tau^R)|_{k_2+2(\nu-r),m_2}\gamma L_{m_1}^r(\phi)|_{k_1+2r,m_1}\gamma.
\end{aligned}$$

Using Lemma 4.3.2, we get

$$\begin{aligned}
\mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}(f) &= \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} \sum_{r=0}^{\nu}(-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r}m_2^r \\
\times L_{m_2}^{\nu-r}(q^N\tau^R)|_{k_2+2(\nu-r),m_2}\gamma &\sum_{l=0}^{r}\binom{r}{l}\left(\frac{2m_1c}{\pi i}\right)^{r-l}\frac{(k_1+r-\frac{3}{2})!}{(k_1+l-\frac{3}{2})!}\frac{L_{m_1}^l(\phi|_{k_1,m_1}\gamma)}{(c\tau+d)^{r-l}},
\end{aligned}$$

$$= \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}\sum_{l=0}^{\nu}(-1)^l L_{m_1}^l(\phi)\sum_{r=0}^{\nu-l}(-1)^{\nu-l-r}\binom{k_1+\nu-\frac{3}{2}}{r}\binom{k_2+\nu-\frac{3}{2}}{\nu-r}m_1^r m_2^{\nu-r}$$
$$\times \quad \binom{\nu-r}{l}\left(\frac{2m_1c}{\pi i}\right)^{\nu-r-l}\frac{(k_1+\nu-r-\frac{3}{2})!}{(k_1+l-\frac{3}{2})!}\frac{L_{m_2}^r(q^N\tau^R)|_{k_2+2(\nu-r),m_2}\gamma}{(c\tau+d)^{r-l}},$$

where in the last line we have replaced r by $\nu-r$. A simple calculation yields

$$\begin{aligned}
&\binom{k_1+\nu-\frac{3}{2}}{r}\binom{k_2+\nu-\frac{3}{2}}{\nu-r}\binom{\nu-r}{l}\frac{(k_1+\nu-r-\frac{3}{2})!}{(k_1+l-\frac{3}{2})!}\\
=\ &\frac{\Gamma(k_1+\nu-\frac{1}{2})\Gamma(k_2+\nu-\frac{1}{2})}{\Gamma(r+1)\Gamma(k_2+r-\frac{1}{2})\Gamma(l+1)\Gamma(\nu-l-r+1)\Gamma(k_1+l-\frac{1}{2})}\\
=\ &\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}\binom{\nu-l}{r}\frac{(k_2+\nu-l-\frac{3}{2})!}{(k_2+r-\frac{3}{2})!}.
\end{aligned}$$

Using the above equality, we see that $\mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}(f)$ equals

$$\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}\sum_{l=0}^{\nu}(-1)^l L_{m_1}^l(\phi)\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}m_1^{\nu-l}m_2^l$$
$$\times\sum_{r=0}^{\nu-l}(-1)^{\nu-l-r}\binom{\nu-l}{r}\left(\frac{2m_2c}{\pi i}\right)^{\nu-l-r}\frac{(k_2+\nu-l-\frac{3}{2})!}{(k_1+r-\frac{3}{2})!}\frac{L_{m_2}^r(q^N\tau^R)|_{k_2+2r,m_2}\gamma}{(c\tau+d)^{\nu-l-r}}.$$

Finally, using Lemma 4.3.3, we get that $\mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}(f)$ equals

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}\sum_{l=0}^{\nu}(-1)^l L_{m_1}^l(\phi)\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}m_1^{\nu-l}m_2^l L_{m_2}^{\nu-l}(q^N\zeta^R|_{k_2,m_2}\gamma),\\
=\ &\sum_{l=0}^{\nu}(-1)^l L_{m_1}^l(\phi)\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}m_1^{\nu-l}m_2^l\\
\times\ &\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}L_{m_2}^{\nu-l}(q^N\zeta^R|_{k_2,m_2}\gamma),
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\nu}(-1)^l\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}m_1^{\nu-l}m_2^{l}L_{m_1}^{l}(\phi)L_{m_2}^{\nu-l}(P_{k_2,m_2;N,R}),\\
&= [\phi, P_{k_2,m_2;N,R}]_\nu.
\end{aligned}
$$

Hence we have the theorem. □

4.3.3 Proof of Theorem 4.2.4

First, we state a lemma which we shall use to prove Theorem 4.2.4.

Lemma 4.3.4. *Let $\phi \in J^{cusp}_{k+2,\mathcal{M}}$. Then the sum*

$$
\sum_{\gamma\in\Gamma^J_{g,\infty}\backslash\Gamma^J_g}\int_{\Gamma^J_g\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \mid \phi(\tau,z)\overline{L_{k,\mathcal{M}}\left(e^{2\pi i(N\tau+Rz)}\mid_{k,\mathcal{M}}\gamma\right)}v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}} \mid dV_J
$$

converges.

For a proof, we refer to [20]. Now we prove Theorem 4.2.4. Let $L^*_{k,\mathcal{M}}(\phi)(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}^g,\\ 4n>\mathcal{M}^{-1}[r^t]>0}} a(n,r)e(n\tau+rz)$. Now consider the (N,R)-th Poincaré series of weight k and index $\mathcal{M}$. Then by Lemma 1.5.3, we have

$$
\langle L^*_{k,\mathcal{M}}(\phi), P_{k,\mathcal{M};N,R}\rangle = \lambda_{\kappa,\mathcal{M},D}\,a(N,R).
$$

Now using the definition of the adjoint map $\langle L^*_{k,\mathcal{M}}\phi, P_{k,\mathcal{M};N,R}\rangle = \langle\phi, L_{k,\mathcal{M}}(P_{k,\mathcal{M};N,R})\rangle$ we have

$$
a(N,R) = \frac{1}{\lambda_{\kappa,\mathcal{M},D}}\langle\phi, L_{k,\mathcal{M}}(P_{k,\mathcal{M};N,R})\rangle. \tag{4.15}
$$

We now compute $\langle\phi, L_{k,\mathcal{M}}(P_{k,\mathcal{M};N,R})\rangle$. By definition, $\langle\phi, L_{k,\mathcal{M}}(P_{k,\mathcal{M};N,R})\rangle$ equals

$$
= \int_{\Gamma^J_g\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{L_{k,\mathcal{M}}\left(P_{k,\mathcal{M};N,R}\right)}\,v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J
$$

$$= \int_{\Gamma_g^J \backslash \mathcal{H}\times\mathbb{C}^{g,1}} \sum_{\gamma\in\Gamma_{g,\infty}^J\backslash\Gamma_g^J} \phi(\tau,z)\overline{L_{k,\mathcal{M}}\left(e(N\tau+Rz)\right)|_{k+2,\mathcal{M}}\,\gamma}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J.$$

By Lemma 4.3.4, we can interchange the sum and integration in $\langle\phi, L_{k,\mathcal{M}}(P_{k,\mathcal{M};N,R})\rangle$. Hence we have,

$$\langle\phi, L_{k,\mathcal{M}}(P_{k,\mathcal{M};N,R})\rangle = \sum_{\gamma\in\Gamma_{g,\infty}^J\backslash\Gamma_g^J}\int_{\Gamma_g^J \backslash \mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{L_{k,\mathcal{M}}\left(e(N\tau+Rz)\right)|_{k+2,\mathcal{M}}\,\gamma}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J.$$

Using Rankin's unfolding argument, we see that $\langle\phi, L_{k,\mathcal{M}}(P_{k,\mathcal{M}}^{N,R})\rangle$ equals

$$\begin{aligned}
&\int_{\Gamma_{g,\infty}^J \backslash \mathcal{H}\times\mathbb{C}^{g}} \phi(\tau,z)\overline{L_{k,\mathcal{M}}\left(e(N\tau+Rz)\right)}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J \\
= &\int_{\Gamma_{\infty}^J \backslash \mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{(4N|\mathcal{M}| - \tilde{\mathcal{M}}[R^t])e(R\tau+Rz) - \frac{(k-\frac{g}{2})}{3}\left(1-24\sum_{j\geq 1}\sigma(j)e(j\tau)\right)} \\
&\times\overline{e(N\tau+Rz)}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J \\
= &\left(4N|\mathcal{M}| - \tilde{\mathcal{M}}[R^t] - \frac{(k-\frac{g}{2})}{3}\right)\int_{\Gamma_{g,\infty}^J \backslash \mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{e(N\tau+Rz)}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J \\
&+8\left(k-\frac{g}{2}\right|\mathcal{M}|\int_{\Gamma_{g,\infty}^J \backslash \mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{\left(\sum_{j\geq 1}\sigma(j)e(j\tau)\right)e(N\tau+Rz)}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J \\
= &(4N|\mathcal{M}| - \tilde{\mathcal{M}}[R^t] - \frac{(k-\frac{g}{2})}{3})I_1 + 8(k-\frac{g}{2})|\mathcal{M}|I_2,
\end{aligned}$$

where I_1 and I_2 are given by

$$I_1 = \int_{\Gamma_{g,\infty}^J \backslash \mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{e(N\tau+Rz)}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J,$$

and

$$I_2 = \int_{\Gamma^J_{g,\infty}\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{\left(\sum_{j\geq 1}\sigma(j)e(j\tau)\right) e(N\tau+Rz)}\, v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J.$$

Now, we calculate the integrals I_1 and I_2 separately. We have

$$\begin{aligned} I_1 &= \int_{\Gamma^J_{g,\infty}\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \phi(\tau,z)\overline{e(N\tau+Rz)}v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J \\ &= \int_{\Gamma^J_{g,\infty}\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \sum_{\substack{n,r\in\mathbb{Z}^g,\\ 4n>M^{-1}[rt]>0}} c_\phi(n,r)e(n\tau+rz)\overline{e(N\tau+Rz)}v^{k+2}e^{\frac{-4\pi\mathcal{M}[y]}{v}}dV_J. \end{aligned}$$

We put $\tau = u+iv$ and $z = x+iy$, where $x = (x_1, x_2, ..., x_g)$ and $y = (y_1, y_2, ..., y_g)$. A fundamental domain for the action of $\Gamma^J_{g,\infty}$ on $\mathcal{H}\times\mathbb{C}^g$ is given by $\{(\tau,z)\in\mathcal{H}\times\mathbb{C}^{g,1} : 0\leqslant u\leqslant 1, v>0, x_i\in[0,1], y\in\mathbb{R}^{g,1}\}$. Integrating over this region we obtain

$$I_1 = \frac{|\mathcal{M}|^{k+1-g}\Gamma(k-\frac{g}{2}+1)}{2^g\pi^{k-\frac{g}{2}+1}}\frac{c_\phi(N,R)}{(4N|\mathcal{M}|-\tilde{\mathcal{M}}[R^t])^{k-\frac{g}{2}+1}}.$$

Similarly, we can compute the integral I_2 and we obtain

$$I_2 = \frac{|\mathcal{M}|^{k+1-g}\Gamma(k-\frac{g}{2}+1)}{2^g\pi^{k-\frac{g}{2}+1}}\sum_{n\geqslant 1}\frac{c_\phi(n+N,R)\sigma(n)}{(4(n+N)|\mathcal{M}|-\tilde{\mathcal{M}}[R^t])^{k-\frac{g}{2}+1}}.$$

Finally, we have the Fourier coefficient $a(N,R)$ of the adjoint map of the heat operator

$$\begin{aligned} a(N,R) &= \frac{|\mathcal{M}|^{\frac{5-g}{2}}(k-\frac{g}{2})(k-\frac{g}{2}-1)(4N|\mathcal{M}|-\tilde{\mathcal{M}}[R])^{\mathcal{K}}}{\pi^2 2^{(g-1)(k-\frac{g}{2}-1)}} \\ &\times \Bigg[\frac{\left(4N|\mathcal{M}|-\tilde{\mathcal{M}}[R]-\frac{(k-\frac{g}{2})|\mathcal{M}|}{3}\right)}{(4N|\mathcal{M}|-\tilde{\mathcal{M}}[R])^{k-\frac{g}{2}+1}}c_\phi(N,R) \\ &\quad +8(k-\frac{g}{2})|\mathcal{M}|\sum_{n\geqslant 1}\frac{c_\phi(n+N,R)\sigma(n)}{(4(n+N)|\mathcal{M}|-\tilde{M}[R])^{k-\frac{g}{2}+1}}\Bigg]. \end{aligned}$$

4.4 Applications

Rankin [42] computed the Petersson scalar product $\langle f, gE_l\rangle$, where $f \in M_{k+l},\ g \in M_l$ and E_l is the Eisenstein series of weight l. To compute $\langle f, gE_l\rangle$, one can express gE_l in terms of Poincaré series and then use Lemma 1.2. The method is known as Rankin's method. Zagier [51] extended the result of Rankin and computed the Petersson scalar product $\langle f, [g, E_l]_\nu\rangle$, where $f \in M_{k+l+2\nu}$ and $g \in M_k$. We use Corollary 4.2.3 to give some applications in the case of Jacobi forms.

Example 4.4.1. *Let N and R be integers such that $4N - R^2 > 0$. Then from Theorem 4.2.2, we have*

$$\begin{aligned} E_{4,1}P_{14,1;N,R} &= \mathbb{P}_{18,2}(q^N\zeta^R E_{4,1}),\\ &= \mathbb{P}_{18,2}\left(\sum_{4n-r^2\geqslant 0}\frac{H(3,4n-r^2)}{\zeta(-5)}q^{n+N}\zeta^{r+R}\right),\\ &= \sum_{4n-r^2\geqslant 0}\frac{H(3,4n-r^2)}{\zeta(-5)}P_{18,2;n+N,r+R}. \end{aligned}$$

Thus we have

$$\langle E_{18,2}, E_{4,1}P_{14,1;N,R}\rangle = \frac{2^{\frac{61}{2}}15!}{\zeta(-5)\pi^{16}}\sum_{4n-r^2\geqslant 0}\frac{H(3,4n-r^2)e_{18,2}(17,8(n+N-(r+R)^2))}{(8(n+N)-(r+R)^2)^{16}},$$

where $H(n,r)$ are generalized class numbers defined by Cohen [10], and $e_{k,m}(n,r)$ are the Fourier coefficients of $E_{k,m}$ [13].

Example 4.4.2. *Let $k_1, k_2(\geq 11), m_1, m_2$ and ν be positive integers. Let $\psi \in J^{cusp}_{k_1,m_1}, \phi \in J_{k_1+k_2+2\nu,m_1+m_2}$, and E_{k_2,m_2} be the Jacobi-Eisenstein series of weight k_2 and index m_2.*

Assume that either ϕ or $[\psi, E_{k_2,m_2}]_\nu$ is a Jacobi cusp form. Then Corollary 4.2.3 implies

$$\begin{aligned}
[\psi, E_{k_2,m_2}]_\nu &= \mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}([\psi,1]_\nu), \\
&= (-1)^\nu \begin{pmatrix} k_2+\nu-\frac{3}{2} \\ \nu \end{pmatrix} m_2^\nu \mathbb{P}_{k_1+k_2+2\nu,m_1+m_2}(L_{m_1}^\nu(\psi)), \\
&= (-1)^\nu \begin{pmatrix} k_2+\nu-\frac{3}{2} \\ \nu \end{pmatrix} m_2^\nu \sum_{\substack{n,r\in\mathbb{Z} \\ 4n-r^2\geqslant 0}} c(n,r) P_{k_1+k_2+2\nu,m_1+m_2;n,r}.
\end{aligned}$$

Hence for $\phi \in J_{k_1+k_2+2\nu,m_1+m_2}$, we obtain

$$\langle \phi, [\psi, E_{k_2,m_2}]_\nu \rangle = \alpha_{k_1,k_2,\nu}^{m_1,m_2} \sum_{\substack{n,r\in\mathbb{Z} \\ 4n-r^2\geqslant 0}} \frac{c(n,r)d(n,r)}{(4n(m_1+m_2)-r^2)^{k_1+k_2+2\nu-\frac{3}{2}}},$$

where

$$\alpha_{k_1,k_2,\nu}^{m_1,m_2} = (-1)^\nu \begin{pmatrix} k_2+\nu-\frac{3}{2} \\ \nu \end{pmatrix} m_2^\nu \frac{(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{2\pi^{k_1+k_2+2\nu-\frac{3}{2}}}.$$

We give one more application of Theorem 4.2.4.

Example 4.4.3. *Let $\phi_{10,1} = \sum_{\substack{n,r\in\mathbb{Z}, \\ 4mn-r^2>0}} c_{\phi_{10,1}}(n,r) q^n \zeta^r \in J_{10,1}^{cusp}$ and $\phi_{12,1} = \sum_{\substack{n,r\in\mathbb{Z}, \\ 4mn-r^2>0}} c_{\phi_{12,1}}(n,r) q^n \zeta^r \in J_{12,1}^{cusp}$. Then we have the following identity:*

$$-\frac{1}{6}\frac{||\phi_{12,1}||^2}{||\phi_{10,1}||^2} C_{\phi_{10,1}}(N,R) = \frac{323(D_{N,R})^{\frac{17}{2}}}{4\pi}\left[\frac{(D_{N,R}-\frac{19}{6})}{(D_{N,R})^{\frac{21}{2}}} C_{\phi_{12,1}}(N,R) + 76 L_{\phi_{12,1}}(N,R;\frac{21}{2})\right],$$

where $D_{N,R} = 4N - R^2$ and $L_\phi(N,R;s) = \sum_{n\geqslant 1} \dfrac{C_\phi(n+N,R)}{(D_{n+N,R})^s}$.

Proof. We know that $J_{10,1}^{cusp}$ and $J_{12,1}^{cusp}$ are one dimensional and $\mathcal{L}_{10,1}(\phi_{10,1}) \in J_{12,1}^{cusp}$. Hence by comparing Fourier coefficients, we get

$$\mathcal{L}_{10,1}(\phi_{10,1}) = -\frac{1}{6}\phi_{12,1}.$$

Now let $\mathcal{L}^*_{10,1}(\phi_{12,1}) = \alpha\phi_{10,1}$, we have

$$\begin{aligned} \alpha||\phi_{10,1}||^2 &= \langle \alpha\phi_{10,1}, \phi_{10,1}\rangle \\ &= \langle \mathcal{L}^*_{10,1}(\phi_{12,1}), \phi_{10,1}\rangle \\ &= \langle \phi_{12,1}, \mathcal{L}_{10,1}(\phi_{10,1})\rangle \\ &= -\frac{1}{6}||\phi_{12,1}||^2. \end{aligned}$$

Now from Theorem 4.2.4, we get the desired identity. □

Example 4.4.4. *Observe that* $J_{4,1}$ *and* $J_{6,1}$ *are one dimensional spaces generated by* $E_{4,1}$ *and* $E_{6,1}$ *respectively. Comparing the constant term we get* $L_{4,1}(E_{4,1}) = -\frac{7}{6}E_{6,1}$. *Hence we get the following relation between generalized class numbers*

$$\begin{aligned} &\frac{1}{\zeta(-9)}H(5, 4n-r^2) \\ = &\frac{1}{\zeta(-5)}[(4n-r^2)H(3,4n-r^2) - \frac{7}{6}H(3,4n-r^2) - 28\sum_{\substack{n_1+n_2=n,\\ 4n_2-r^2\geqslant 0}}\sigma(n_1)H(3,4n_2-r^2)]. \end{aligned}$$

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