NON-COMMUTATIVE NEVEU DECOMPOSITION AND ASSOCIATED ERGODIC THEOREMS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

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Dedications

To my parents and sister.

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ABSTRACT

Given a von Neumann algebra M with faithful, normal tracial state τ and an automorphism defined on it, Grabarnik and Katz characterised the existence of an invariant state with the existence of a weakly wandering operator. This characterisation is done by proving a non-commutative analogue of Neveu decomposition. In the first part of this thesis, we will consider a covariant system (M, G, α), where G is an amenable group and arrive at a similar result.

The second part of this thesis is devoted to pointwise ergodic theorems in the non-commutative L^1 spaces. First, we will consider a dynamical system (M, G, α), where M is a von Neumann algebra with faithful normal tracial state τ and assume that the action α is preserved by a faithful, normal state ρ on M . When G is a group of polynomial growth with a compact, symmetric generating set V, we show that the ergodic averages associated with the predual action corresponding to the Folner sequence $\{V^n\}$ converge bilateral almost uniformly. Furthermore, these results are extended to multi-parameter cases and for finitely generated free group actions.

Finally we combine the Neveu decomposition and the pointwise ergodic theorems as discussed above to show a stochastic ergodic theorem for a covariant system consisting of a finite von Neumann algebra.

Contents

Summary

This thesis deals with non-commutative Neveu decomposition, ergodic theorems and stochastic ergodic theorems for various group and semigroup actions. This study sits at the juncture of ergodic theory and operator algebra. Since the development of the theory of operator algebras, there has been a connection between ergodic theory and von Neumann algebras. This connection is still an active area of research. More often, the ergodic theory on measure spaces provides interesting motivations to study ergodic theory on various non-commutative spaces, such as von Neumann algebras and several other Banach spaces. Among many others, the existence of finite invariant measures is a well-studied area of research in classical ergodic theory (see [14],[35], [15] and the references therein).

The relation between invariant measures and weekly wandering functions in classical measure spaces is well known. The notion of weakly wandering sets in a measure space was introduced in [13]. In the same paper, the authors established various equivalent conditions regarding the existence of finite invariant measures. In 1965, Neveu characterized the existence of invariant measures in terms of weakly wandering functions for positive contractions on L^1-spaces (see [23, theorem 3.4.6]). In particular, Neveu decomposed the underlying measure space uniquely in two disjoint sets, one of them being the support of an invariant measure and the other one being the support of a weakly wandering function. This decomposition is known as Neveu decomposition in the literature. Further, in 1995, Grabarnik and Katz generalized it for an automorphism acting on a finite von Neumann algebra in [8].

We consider a finite von Neumann algebra M and a covariant system (M, G, α), where G is a locally compact, second countable, Hausdorff, amenable group (henceforth abbreviated as *s.c.l.c*), and α is an action of G on M via automorphisms. Then we decompose the identity projection in M into two mutually orthogonal projections e_1 and e_2 with the properties that e_1 is the support of a faithful, normal invariant state and e_2 is the support of a weakly wandering operator in M.

Historically, the study of non-commutative ergodic theorems appeared first in the pio-

neering work of Lance in [26]. In this article, the author proved an individual ergodic theorem on a von Neumann algebra. After that, in his seminal papers [46], and [47], Yeadon extended the results to the non-commutative L^1-spaces. Although many profound results of classical ergodic theory have already been generalized for the actions of more general amenable groups on non-L^1-spaces the very basic pointwise ergodic theorem for amenable group actions on L^1-spaces is only proved in 2001 by Lindenstrauss [27].

We prove a pointwise ergodic theorem in the predual of a von Neumann algebra with f.n.s trace τ. To prove this, we assume that the G-action is preserved by a faithful normal state and use an inequality, proved very recently in [18, Proposition 4.8]. It is worth mentioning that we do not assume that the trace τ is preserved by the action α and as a result, we work with the predual action of α on $L^1(M, \tau)$, which may not be an extension of α. For this reason, the main techniques used in our context are nonidentical with [18] and this makes our results novel and strong.

Then we asked the same question beyond the group of polynomial growth, and we obtained similar results for the actions of finitely generated free groups and actions of $\mathbb{Z}_+^d$ or $\mathbb{R}_+^d$ ($d \geq 1$).

In the end, we return to the setting of finite von Neumann algebra M and a covariant system (M, G, α), where G is a *s.c.l.c* amenable group, and alpha is an action of G on M via automorphisms. Then we use the Neveu decomposition and pointwise convergence as obtained before to derive a stochastic ergodic theorem.

Chapter 1

Background and preliminaries

We compile the preliminary information needed for this thesis in this chapter. A few essential properties of linear functionals defined on a von Neumann algebra are collected in the first section. In the second section, we review key concepts and fundamental characteristics of non-commutative L_p spaces that are important for the following chapters. The third section presents appropriate group actions on various ordered Banach spaces, which form the core of this thesis.

1.1 von Neumann algebra and linear functionals

Let M denote a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. The norm on M induced from $\mathbf{B}(\mathcal{H})$ will be denoted by $\|\cdot\|$. Other than the norm topology, M possesses several locally convex topologies. For the definition of these topologies and various other facts regarding von Neumann algebras, we refer to [36, chapter 1]. We will write the space of self-adjoint elements of M as M_s, and by M_+, we will denote the positive cone of the von Neumann algebra M.

Let M_* be the set of all w-continuous linear functionals on M. It is a norm closed subspace of M^*, the Banach space dual of M. The dual space norm on M^* and its restriction to M_* will be denoted by $\|\cdot\|_1$ in the sequel. Then M is isomorphic to $(M_*)^*$ via a canonical sesquilinear form defined on $M \times M_*$. For the proof of this fact, we again refer to [36, Lemma 1.9]. Under this identification, M_* is called the predual of M.

Let $\varphi \in M^*$. Then φ is called self-adjoint if $\varphi(x) = \overline{\varphi(x^*)}$ for all $x \in M$ and it is called positive if it takes positive values at all positive elements of M, and it will be further denoted by $\varphi \geq 0$. It is well-known that a positive linear functional φ on M is bounded. φ is called a state if it is positive and normalised, i.e, $\varphi(1) = 1$. Henceforth, the set of all w-continuous positive linear functionals will be denoted by M_{*+} and self-adjoint elements of M_* will be denoted by M_{*s}. The elements of M_{*+} is also known as normal linear functionals in the literature. For $\varphi \in M^*$, φ is normal iff whenever $x_i \uparrow x$ then $\varphi(x_i) \uparrow \varphi(x)$, where x_i's and x are in M_+.

The support of a self-adjoint element x in M, subsequently denoted by $s(x)$, is the smallest projection in M such that $s(x)x = x$ (equivalently, $xs(x) = x$). Furthermore, for a $\varphi \in M_{*+}$, the set $\{e \in M : e$ is a projection and $\varphi(e) = 0\}$ is increasingly directed. Now, if the projection $p \in M$ be the least upper bound of the family then we can easily infer that $\varphi(p) = 0$. Then, the projection $1 - p$ is called the support of φ and will be denoted by $s(\varphi)$ in the sequel. Note that φ is faithful iff $s(\varphi) = 1$.

To prevent repetition and for notational discomfort, we now fix some abbreviations. The family of all (resp. non-zero) projections in a von Neumann algebra M will be denoted by $\mathcal{P}(M)$ (resp. $\mathcal{P}_0(M)$). If two projections $p, q \in \mathcal{P}(M)$ are orthogonal, we will write $p \perp q$. We also abbreviate the faithful normal linear functional on M as f.n. linear functional.

A positive linear functional φ on a von Neumann algebra M is said to be singular if there exists a positive normal linear functional ψ on M such that $\varphi \geq \psi$, then $\psi = 0$. The characterisation of a singular linear functional and the unique decomposition of a positive linear functional into a singular and normal part will be of utmost importance to us. We recall the following two theorems from [37] for that purpose. We also refer to [39] and [40] for the same.

Theorem 1.1.1. *A positive linear functional φ on M is singular if and only if for any $e \in$*

$\mathcal{P}_0(M)$, there exists a $f \in \mathcal{P}_0(M)$ with $f \leq e$ such that $\varphi(f) = 0$.

Theorem 1.1.2. *Let φ be a positive linear functional on M. Then there exists a unique normal linear functional φ_n and a singular linear functional φ_s on M such that*

$$\varphi = \varphi_n + \varphi_s.$$

A functional $\omega : M_+ \to [0, \infty]$ is called a weight if it satisfies additivity and positive homogeneity, that is, $\omega(\lambda a + b) = \lambda\omega(a) + \omega(b)$ for all $x, y \in M_+$ and $\lambda \geq 0$. An weight ω is called normal if whenever $x_i \uparrow x$ then $\omega(x_i) \uparrow \omega(x)$, where x_i's and x are in M_+. ω is called faithful if $\omega(x^*x) = 0$ for some $x \in M$, then $x = 0$. ω is called semifinite if the set $\mathcal{M}_\varphi := span\{a^*b : a, b \in M, \varphi(a^*a) < \infty, \varphi(b^*b) < \infty\}$ is w^*-dense in M. We will abbreviate faithful, normal, semifinite weight as f.n.s. weight.

Let τ be a f.n.s. weight on M. τ is said to have tracial property if $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$.

For more information about weights on a von Neumann algebra, their GNS representation and unbounded operators we refer to [36].

1.2 τ-measurable operators

Non-commutative integration theory was initiated by Segal in [33], then it was studied in great depth by many authors. Throughout this section we will assume that M is a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and equipped with a f.n.s. trace τ.

A closed densly defined operator $X : \mathcal{D}(X) \subseteq \mathcal{H} \to \mathcal{H}$ is called affiliated to M if $u'X = Xu'$ for all unitary u' in M', where M' is the commutant of M in $\mathbf{B}(\mathcal{H})$. If X is affiliated to M, we will denote it by $X\eta M$. Now an operator $X\eta M$ is called τ-measurable if for every $\epsilon > 0$, there exists $e \in \mathcal{P}(M)$ such that $e\mathcal{H} \subseteq \mathcal{D}(X)$ and $\tau(1 - e) < \epsilon$. The set of all τ-measurable operators on M is denoted by $L^0(M, \tau)$. The set of positive operators in $L^0(M, \tau)$ will be denoted by $L^0(M, \tau)_+$.

Let $X, Y \in L^0(M, \tau)$. Then it is clear that $X + Y$ and XY are densely defined and closable. Moreover, $L^0(M, \tau)$ is a $*$-algebra with respect to adjoint operation $*$, the strong sum $\overline{X + Y}$, and the strong product $\overline{XY}$, where the overline denotes closure of an operator. In the sequel, we will adhere to the convention that for $X, Y \in L^0(M, \tau)$, $X + Y$ and XY will stand in for the the strong sum and product, respectively, unless otherwise mentioned.

One can also define a topology on $L^0(M, \tau)$ in the following way.
For all $\epsilon, \delta > 0$, let us consider the following set,

$$\mathcal{N}(\epsilon, \delta) := \{X \in L^0(M, \tau) : \exists e \in \mathcal{P}(M) \text{ such that } \|Xe\| \leq \epsilon \text{ and } \tau(1 - e) \leq \delta\}.$$

Note that the collection of sets $\{X + \mathcal{N}(\epsilon, \delta) : X \in L^0(M, \tau), \epsilon > 0, \delta > 0\}$ forms a neighborhood basis on $L^0(M, \tau)$.

Definition 1.2.1. *The topology generated by the neighborhood basis* $\{X + \mathcal{N}(\epsilon, \delta) : X \in L^0(M, \tau), \epsilon > 0, \delta > 0\}$ *is called the measure topology on* $L^0(M, \tau)$.

We observe that $L^0(M, \tau)$ transforms into a complete, metrizable, Hausdorff space. Furthermore, under this topology, M is dense in $L^0(M, \tau)$ [cf. [16], Theorem 4.12]. As a result, the convergence in measure of a sequence in $L^0(M, \tau)$ can also be defined as follows.

Definition 1.2.2. *A sequence* $\{X_n\}_{n \in \mathbb{N}}$ *in* $L^0(M, \tau)$ *is said to converge in measure (or converge stochastically) to* $X \in L^0(M, \tau)$ *if for all* $\epsilon, \delta > 0$ *there exists a sequence of projections* $\{e_n\}_{n \in \mathbb{N}}$ *in* M *and* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$

$$\tau(1 - e_n) < \delta \text{ and } \|(X_n - X)e_n\| < \epsilon.$$

An equivalent definition for the convergence of a sequence in measure is given by the following theorem from [3]. From this point onward, we define convergence in measure according to the criteria in the following theorem.

Theorem 1.2.3 (Theorem 2.2, [3]). *A sequence of operators $\{X_n\}_{n\in\mathbb{N}} \subseteq L^0(M,\tau)$ converges in measure to $X \in L^0(M,\tau)$ iff for all $\epsilon > 0$ and $\delta > 0$ there exists $n_0 \in \mathbb{N}$ and a sequence of projections $\{e_n\}_{n\in\mathbb{N}}$ in M such that for all $n \geq n_0$,*

$$\tau(1-e_n) < \delta \text{ and } \|e_n(X_n - X)e_n\| < \epsilon.$$

Let $X \in L^0(M,\tau)$. We define the distribution function of X as

$$\lambda_s(X) = \tau(\chi_{(s,\infty)}(|X|)), \quad s \geq 0.$$

Here, $\chi_{(s,\infty)}(|X|)$ is the spectral projection of $|X|$ corresponding to the characteristic function $\chi_{(s,\infty)}$. Now we define the generalised singular number of X as

$$\mu_t(X) := \inf\{s \geq 0 : \lambda_s(X) \leq t\}, \quad t \geq 0.$$

The semifinite trace τ primarily defined on M_+ can be extended to $L^0(M,\tau)_+$ as an additive, positive homogeneous (i.e, $\tau(\alpha A + B) = \alpha\tau(A) + \tau(B)$ for $A, B \in L^0(M,\tau)_+$ and $\alpha \geq 0$) functional by the following formula.
For $X \in L^0(M,\tau)_+$,

$$\tau(X) := \int_0^\infty \lambda d\tau(e_\lambda),$$

where $X = \int_0^\infty \lambda d(e_\lambda)$ is the spectral decomposition.

The following proposition summarises some basic properties of τ defined above. For proof and other relevant facts we refer to [16].

Proposition 1.2.4. *(i) For all $X \in L^0(M,\tau)_+$,*

$$\tau(X) = \int_0^\infty \mu_t(X)dt.$$

Furthermore, if $f : [0,\infty) \to [0,\infty)$ be any continuous non-decreasing function with $f(0) \geq 0$, then

$$\tau(f(X)) = \int_0^\infty f(\mu_t(X))dt.$$

(ii) $\tau(X) \leq \tau(Y)$ *for all* $X, Y \in L^0(M,\tau)_+$ *and* $X \leq Y$.

(iii) $\tau(X^*X) = \tau(XX^*)$ *for all* $X \in L^0(M,\tau)$.

1.2.1 Non-commutative tracial L^p spaces

The study of non-commutative tracial L^p spaces were initiated in the works of Dixmier, Kunze, Segal and Stinespring in [6],[25],[33] and [34]. Later on it was developed through the approach of τ-measurable operators by Nelson and Yeadon in [28] and [45].

Definition 1.2.5. *For* $0 < p \leq \infty$*, the non-commutative tracial* L^p*-space on* (M,τ) *is defined by*

$$L^p(M,\tau) := \begin{cases} \{X \in L^0(M,\tau) : \|X\|_p := \tau(|X|^p)^{1/p} < \infty\} & \textit{for } p \neq \infty, \\ (M, \|\cdot\|) & \textit{for } p = \infty. \end{cases}$$

With the definition above, $L^p(M,\tau)$ satisfies many properties similar to L^p spaces defined on a measure space. In the following we mention few of those which are necessary for our purpose.

Theorem 1.2.6. *(i) (Minkowski's inequality) For all* $1 \leq p \leq \infty$ *and* $X, Y \in L^0(M,\tau)$*,*

$$\|X+Y\|_p \leq \|X\|_p + \|Y\|_p .$$

(ii) (Holder's inequality) For $p, q, r \in (0,\infty]$ *such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$*, and* $X, Y \in L^0(M,\tau)$ *we have*

$$\|XY\|_r \leq \|X\|_p \|Y\|_q .$$

Theorem 1.2.7. *(i) For all* $1 \leq p \leq \infty$*,* $L^p(M,\tau)$ *is a Banach space with respect to the norm* $\|\cdot\|_p$*. Especially,* $L^p(M,\tau)$ *becomes a Hilbert space when* $p = 2$ *with the inner product defined as* $\langle X, Y\rangle = \tau(XY^*)$*. Further more,* $M \cap L^1(M,\tau)$ *is a dense subspace of* $L^p(M,\tau)$ *for all* $1 \leq p < \infty$*.*

(ii) *($L^p - L^q$ duality) Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. Then the map $(X, Y) \ni L^p(M,\tau) \times L^q(M,\tau) \to \tau(XY) \in \mathbb{C}$ defines the duality between the spaces $L^p(M,\tau)$ and $L^q(M,\tau)$.*

Corollary 1.2.8. *The map $\Psi : L^1(M,\tau) \to M_*$ defined by $\Psi(X)(a) = \tau(Xa)$ for $X \in L^1(M,\tau), a \in M$ is a surjective linear isometry. Furthermore, X is positive if and only if $\Psi(X)$ is positive.*

The following discussion is also going to play an important role in the sequel. Let $e \in \mathcal{P}(M)$ and assume that $M_e := \{x_e := ex_{\restriction e\mathcal{H}} : x \in M\}$ denotes the reduced von Neumann algebra. Define the reduced trace on M_e as

$$\tau_e(x_e) = \tau(exe), \text{ for all } x \in M.$$

Remark 1.2.9. (i) *Since τ is a faithful, normal, semifinite trace, τ_e also has similar properties.*

(ii) *Let $1 \leq p \leq \infty$ and $X \in L^p(M_e, \tau_e)$. Define $\tilde{X}$ on $\mathcal{H}$ by*

$$\tilde{X}\xi = Xe\xi, \textit{ for all } \xi \in \mathcal{D}(\tilde{X}) := \mathcal{D}(X) \oplus (1-e)\mathcal{H}.$$

Then the mapping $L^p(M_e, \tau_e) \ni X \mapsto \tilde{X} \in eL^p(M,\tau)e$ defines an isomorphism as Banach space for $1 \leq p \leq \infty$. From now onwards we identify $L^p(M_e, \tau_e)$ with $eL^p(M,\tau)e$.

(iii) *Since $L^p(M,\tau)$ is a left-right M ideal for every $0 < p \leq \infty$, we have $eL^p(M,\tau)e \subseteq L^p(M,\tau)$.*

Now we define bilateral almost uniform convergence of a sequence of measurable operators. This type of convergence is one of the key components of this thesis. Due to Egorov's theorem, this definition of convergence is actually refers to the almost everywhere convergence in classical measure space. In chapters 4 and 5, we shall demonstrate this form of convergence of various sequences.

Definition 1.2.10. *A sequence of operators* $\{X_n\}_{n\in\mathbb{N}} \subseteq L^0(M,\tau)$ *converges bilaterally almost uniformly (b.a.u) to* $X \in L^0(M,\tau)$ *if for every* $\epsilon > 0$ *there exists* $e \in \mathcal{P}(M)$ *with* $\tau(1-e) < \epsilon$ *and*

$$\lim_{n\to\infty} \|e(X_n - X)e\| = 0.$$

Remark 1.2.11. *We immediately notice that the bilateral almost uniform convergence of a sequence implies the convergence of the sequence in measure.*

We put down the following proposition for our future reference. The proof is simple, and hence we omit it.

Proposition 1.2.12. *Suppose* $\{X_n\}_{n\in\mathbb{N}}$ *and* $\{Y_n\}_{n\in\mathbb{N}}$ *are two sequences in* $L^0(M,\tau)$ *such that* $\{X_n\}_{n\in\mathbb{N}}$ *converges in measure (resp. b.a.u) to* X *and* $\{Y_n\}_{n\in\mathbb{N}}$ *converges in measure (resp. b.a.u) to* Y. *Then, for all* $c \in \mathbb{C}$, $\{cX_n + Y_n\}_{n\in\mathbb{N}}$ *converges in measure (resp. b.a.u) to* $cX + Y$.

1.3 Amenable action

Throughout this thesis, unless otherwise mentioned, G will be referred to as a second countable, locally compact, Hausdorff (henceforth abbreviated as *s.c.l.c*) group with a fixed right invariant Haar measure m (i.e, $m(Eg) = m(E)$ for all $g \in G$ and Borel subset E of G). G is said to be amenable if there exists a sequence of compact subsets of non-zero measure $\{K_n\}_{n\in\mathbb{N}}$ of G, satisfying the condition:

$$\lim_{n\to\infty} \frac{m(K_n \Delta K_n g)}{m(K_n)} = 0 \text{ for all } g \in G, \tag{1.3.1}$$

where Δ denotes the symmetric difference of two sets.

Such a sequence of subsets $\{K_n\}_{n\in\mathbb{N}}$ of G satisfying eq. 1.3.1 is called *Følner sequence*.

Recall that a real Banach space E paired with a closed convex subset K satisfying $K \cap -K = \{0\}$ and $\lambda K \subseteq K$ for $\lambda \geq 0$ induces a partial order on E given by $x \leq y$ if and only if $y - x \in K$ where $x, y \in E$. With such a partial order, E will be referred to as an ordered Banach space. In our context, it is easy to verify that $M, M^*, M_*, L^p(M,\tau)$ for $1 \leq p \leq \infty$ become ordered Banach spaces with respect to the natural order.

Let us now define an action of G on ordered Banach spaces.

Definition 1.3.1. *Let E be an ordered Banach space. A map Λ defined by*

$$G \ni g \xrightarrow{\Lambda} \Lambda_g \in \mathbf{B}(E)$$

is called an action if $\Lambda_g \circ \Lambda_h = \Lambda_{gh}$ for all $g, h \in G$. In this article, unless otherwise stated, we only consider the actions $\Lambda = \{\Lambda_g\}_{g\in G}$ which satisfy the following conditions.

(C) *For all $x \in E$, the map $g \to \Lambda_g(x)$ from G to E is continuous. Here we take w^*-topology when $E = M$ and norm topology otherwise.*

(UB) $\sup_{g\in G} \|\Lambda_g\| < \infty$.

(P) *For all $g \in G$ and $x \in E$ with $x \geq 0$, $\Lambda_g(x) \geq 0$.*

We refer the triple (E, G, Λ) as a non-commutative dynamical system.

Remark 1.3.2. *We note that when $E = M$ and G is a group as above, then in our definition for the non-commutative dynamical system (M, G, Λ), we did not assume $\Lambda_g(1) = 1$, for all $g \in G$. However, in this thesis, for most of the results we assume $\Lambda_g(1) \leq 1$, for all $g \in G$, which further implies $\Lambda_g(1) = 1$, for all $g \in G$. Thus, when $\Lambda_g(1) = 1$ for all $g \in G$, Λ_g becomes a Jordan $*$-isomorphism of M (see [22]). Then, we point out that any Jordan $*$-isomorphism of M is sum of a $*$-isomorphism and a $*$-anti-isomorphism of M. Thus, in this case it is enough to assume for all $g \in G$, Λ_g are $*$-isomorphism of M.*

Let Aut(M) be the set of all $*$-algebra isomorphisms on M. In view of the Remark 1.3.2, when $E = M$ and G is a group as above, we rephrase the definition of non-commutative dynamical system (M, G, Λ).

Definition 1.3.3. *Let G be a s.c.l.c amenable group and M be a von Neumann algebra. A map Λ defined by*

$$G \ni g \xrightarrow{\Lambda} \Lambda_g \in Aut(M)$$

is called an action if $\Lambda_g \circ \Lambda_h = \Lambda_{gh}$ for all $g, h \in G$ and

(i) for all $x \in M$, the map $g \to \Lambda_g(x)$ from G to M is continuous with respect to w^*-topology,

(ii) $\Lambda_{1_G}(1) = 1$ and, for all $g \in G$ and $x \in M_+$, $\Lambda_g(x) \geq 0$.

The non-commutative dynamical system (M, G, Λ) will be referred to as a covariant system.

Now let (E, G, Λ) be a non-commutative dynamical system and $\{K_n\}_{n\in\mathbb{N}}$ be a *Følner sequence* in G. For all $x \in E$, we consider the following average

$$A_n(x) := \frac{1}{m(K_n)} \int_{K_n} \Lambda_g(x) dm(g). \tag{1.3.2}$$

We note that for $x \in E$, the map $G \ni g \to \Lambda_g(x) \in E$ is continuous in norm of E. Further, when $E = M$, then it is w^*-continuous. Therefore, in both cases the integration in eq. 1.3.2 is well defined. In addition by [38, Proposition 1.2, pp-238], for all $n \in \mathbb{N}$, $\varphi \in M_*$ and $x \in M$ the following holds

$$\varphi(A_n(x)) = \frac{1}{m(K_n)} \int_{K_n} \varphi(\Lambda_g(x)) dm(g).$$

Now for all $g \in G$, consider

$$\Lambda_g^* : M^* \to M^* \text{ by } \Lambda_g^*(\varphi)(x) = \varphi(\Lambda_g(x)) \text{ for all } \varphi \in M^*, x \in M. \tag{1.3.3}$$

For all $n \in \mathbb{N}$, we also consider the following average defined by

$$A_n^* : M_* \to M_*;\ \varphi \mapsto A_n^*(\varphi)(\cdot) := \varphi(A_n(\cdot)) = \frac{1}{m(K_n)} \int_{K_n} \Lambda_g^*(\varphi)(\cdot) dm(g). \quad (1.3.4)$$

We note that for all $n \in \mathbb{N}$, $\|A_n\| \leq \sup_{g \in G} \|\Lambda_g\|$ and $\|A_n^*\| \leq \sup_{g \in G} \|\Lambda_g\|$. These will be called averaging operators for future references. We make no difference between these two averaging operators A_n and A_n^* for the convenience of notation, unless and otherwise, it is not clear from the context.

The following proposition is standard, but for completeness, we provide a sleek proof.

Proposition 1.3.4. *Let (E, Λ, G) be a non-commutative dynamical system and $\{K_n\}_{n \in \mathbb{N}}$ be a Følner sequence. Then for $x \in E$ and $h \in G$, we have*

$$\|A_n(\Lambda_h x) - A_n(x)\| \leq \frac{m(K_n h \Delta K_n)}{m(K_n)} \|\Lambda_g(x)\| \leq \frac{m(K_n h \Delta K_n)}{m(K_n)} \sup_{g \in G} \|\Lambda_g\| \|x\| . \quad (1.3.5)$$

Therefore, $\|A_n(\Lambda_h x) - A_n(x)\|$ converges to 0 as n tends to ∞.

Proof. First part of proof follows from the following observation. Let $x \in E$, $h \in G$ and $n \in \mathbb{N}$, then we have

$$\begin{aligned}
&\|A_n(\Lambda_h x) - A_n(x)\| \\
=& \left\| \frac{1}{m(K_n)} \Big(\int_{K_n} \Lambda_{gh}(x) dm(g) - \int_{K_n} \Lambda_g(x) dm(g) \Big) \right\| \\
=& \left\| \frac{1}{m(K_n)} \Big(\int_{K_n h} \Lambda_g(x) dm(g) - \int_{K_n} \Lambda_g(x) dm(g) \Big) \right\| \\
=& \left\| \frac{1}{m(K_n)} \Big(\int_{K_n h \setminus K_n} \Lambda_g(x) dm(g) - \int_{K_n \setminus K_n h} \Lambda_g(x) dm(g) \Big) \right\| \\
\leq& \frac{1}{m(K_n)} \Big(\int_{K_n h \setminus K_n} \|\Lambda_g(x)\| dm(g) + \int_{K_n \setminus K_n h} \|\Lambda_g(x)\| dm(g) \Big) \\
\leq& \frac{1}{m(K_n)} \int_{K_n h \Delta K_n} \|\Lambda_g(x)\| dm(g) \\
\leq& \frac{m(K_n h \Delta K_n)}{m(K_n)} \sup_{g \in G} \|\Lambda_g\| \|x\| .
\end{aligned}$$

Hence, the convergence of $\|A_n(\Lambda_h x) - A_n(x)\|$ follows from Følner condition 1.3.1 and $\sup_{g \in G} \|\Lambda_g\| < \infty$. □

Chapter 2

Neveu Decomposition

This chapter is dedicated to the study of existence of invariant states associated to a covariant system. We study various necessary and sufficient conditions for the same.

2.1 Introduction

Consider a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. Transformations on Ω leaving a measure invariant is important in various branch of mathematics, notably in dynamics and probability theory. It is therefore of some interest to find a finite invariant measure on Ω given a fixed non singular transformation S. This problem has been studied extensively in the literature and many necessary and sufficient conditions have already been determined. In [13], the authors characterised the existence of finite S-invariant measure on Ω which is equivalent to μ with the non existence of weakly wandering set of positive measure. It is worth mentioning here that a measurable set B is called weakly wandering if there exist a sequence of non-negetive integers $\{n_k\}$ such that the image sets $\{S^{n_k}(B)\}$ are mutually disjoint. The result is then extended to arbitrary group of non-singular transformations by Hajian and Ito in [12].

On the other hand, given a positive contraction T on $L^1(\Omega, \mu)$, one might ask a similar question. In this case, the existence of a finite measure ν on Ω which is absolutely continuous with respect to μ, is equivalent to the existence of a positive $f \in L^1(\Omega, \mu)$ such that $Tf = f$. In [20], Ito first studied conditions for existence of finite invariant measures associated to positive contractions on $L^1(\Omega,\mu)$. Later on, Neveu proved the following theorem.

Theorem 2.1.1. *[29, Theorem 3] Let T be a positive contraction on $L^1(\Omega,\mu)$. Then there exists a measurable subset C of Ω (uniquely determined upto null sets), which is characterized by each of the following properties.*

(i) For every T-invariant $g \in L^1(\Omega,\mu)$ we have $\{g \neq 0\} \subset C$. Conversely, there exists a T-invariant $g_0 \in L^1(\Omega,\mu)_+$ such that $\{g_0 > 0\} = C$.

(ii) Let $f \in L^1(\Omega,\mu)$ be such that $f > 0$ a.e. and $0 \leq n_0 < n_1 < n_2 < \ldots$ be an infinite sequence of integers. Then $C \subseteq \{\sum_0^\infty T^{n_k} f = \infty\}$. Conversely, there exists an infinite sequence of integers $0 \leq \tilde{n_0} < \tilde{n_1} < \tilde{n_2} < \ldots$ such that $\{\sum_0^\infty T^{\tilde{n_k}} f = \infty\} = C$.

Afterwards, Krengel strengthened Neveu's results in [23]. He characterised the set C in the previous theorem using the support of a m-weakly wandering function in $L^\infty(\Omega,\mu)_+$. An element $h \in L^\infty(\Omega,\mu)_+$ is called weakly wandering if there exists an infinite sequence $0 = n_0 < n_1 < n_2 < \ldots$ of integers satisfying $\|\sum_0^\infty T^{*n_k} h\| < \infty$. If a stronger condition $\sup_{j\geq 0} \left\|\sum_0^\infty T^{*(n_k - j)} h\right\| < \infty$ holds, then it is called m-weakly wandering. Krengel proved the following result.

Theorem 2.1.2. *[23, Theorem 3.4.6] Let T be a positive contraction on $L^1(\Omega,\mu)$. Then Ω has a decomposition in two disjoint sets C and D, uniquely determined upto null sets by the following properties.*

(i) There exists a $g \in L^1(\Omega,\mu)_+$ with $Tg = g$ and $\{g > 0\} = C$, and

(ii) there exists a $h \in L^\infty(\Omega,\mu)_+$ such that h is m-weakly wandering and $\{h > 0\} = D$.

The sets C and D in the previous theorem are called positive and null part respectively for T and the theorem is known as Neveu Decomposition in the literature. In this chapter, we are mainly concerned about a non-commutative analogue of Theorem 2.1.2 for a given covariant system (M, G, α).

Throughout this chapter, we fix a covariant system (M, G, α) and a Følner sequence $\{K_n\}_{n\in\mathbb{N}}$ in G. Further, we always consider the ergodic averages with respect to the Følner sequence $\{K_n\}_{n\in\mathbb{N}}$ and it will be denoted again by A_n, i.e,

$$A_n(x) = \frac{1}{m(K_n)} \int_{K_n} \alpha_g(x)dm(g),\ n \in \mathbb{N}, x \in M.$$

Observe that for all $n \in \mathbb{N}$, A_n defines a contraction on M and by eq. 1.3.4, it induces contractions on M^* and M_* respectively. In this chapter, although we denote these averaging operators by same notation A_n, it is translucent from the context. It is worth mentioning here that for a covariant system (M, G, α), a state φ on M is said to be G-invariant if $\varphi \circ \alpha_g = \varphi$ for all $g \in G$.

The existence of G-invariant states and weakly wandering operators associated to (M, G, α) will now be addressed, and we will subsequently derive the corresponding Neveu Decomposition.

2.2 Invariant states

In the following proposition, we obtain a sufficient condition for existence of G-invariant normal states for a covariant system (M, G, α).

Proposition 2.2.1. *Let φ be a f.n state on M and (M, G, α) be a covariant system. Suppose there exists an $e \in \mathcal{P}_0(M)$ such that*

$$p \in \mathcal{P}_0(M) \text{ and } p \leq e \Rightarrow \inf_{n\in\mathbb{N}} A_n(\varphi)(p) > 0. \tag{2.2.1}$$

Then, there exists a G-invariant normal state ν_φ in M such that $s(\nu_\varphi) \geq e$.

Proof. First, we note that $\|A_n(\varphi)\|_1 \leq \|\varphi\|_1$ for all $n \in \mathbb{N}$. Hence, by Banach- Alaoglu theorem, there exists a subsequence $\{A_{n_k}(\varphi)\}_{k\in\mathbb{N}}$ and a $\bar{\varphi} \in M^*$ with $\bar{\varphi} \geq 0$ and $\|\varphi\|_1 \leq 1$

such that

$$\bar{\varphi}(x) = \lim_{k\to\infty} A_{n_k}(\varphi)(x) \text{ for all } x \in M.$$

Now observe that, for all $h \in G$, $x \in M$ and $k \in \mathbb{N}$

$$\begin{aligned}|\bar{\varphi}(\alpha_h(x)) - \bar{\varphi}(x)| &\leq |\bar{\varphi}(\alpha_h(x)) - A_{n_k}(\varphi)(\alpha_h(x))| + \\ &\quad |A_{n_k}(\varphi)(\alpha_h(x)) - A_{n_k}(\varphi)(x)| + |A_{n_k}(\varphi)(x) - \bar{\varphi}(x)| \\ &\leq |\bar{\varphi}(\alpha_h(x)) - A_{n_k}(\varphi)(\alpha_h(x))| + \\ &\quad \frac{m(K_{n_k} h \Delta K_{n_k})}{m(K_{n_k})} + |A_{n_k}(\varphi)(x) - \bar{\varphi}(x)|\,, \quad \text{(by Proposition 1.3.4).}\end{aligned}$$

Then, Følner condition (eq. 1.3.1) and definition of $\bar{\varphi}$ immediately imply that right hand side of the above equation converges to 0. Hence, conclude that $\alpha_h^*(\bar{\varphi}) = \bar{\varphi}$ for all $h \in G$. Further note that as φ is a state, so is $\bar{\varphi}$. Now by Theorem 1.1.2, $\bar{\varphi}$ can be decomposed as

$$\bar{\varphi} = \bar{\varphi}_n + \bar{\varphi}_s,$$

where $\bar{\varphi}_n$ and $\bar{\varphi}_s$ are normal and singular parts of $\bar{\varphi}$ respectively. We claim that normal component $\bar{\varphi}_n$ is non-zero. Indeed, if $\bar{\varphi}_n(x) = 0$ for all $x \in M$, then clearly $\bar{\varphi} = \bar{\varphi}_s$, i.e, $\bar{\varphi}$ is singular.

As $e \in \mathcal{P}_0(M)$, so by Theorem 1.1.1 it follows that, there is a non-zero sub-projection p of e in M such that $\bar{\varphi}(p) = 0$. Then, we have

$$\lim_{k\to\infty} A_{n_k}(\varphi)(p) = 0 \Rightarrow \inf_{k\in\mathbb{N}} A_{n_k}(\varphi)(p) = 0 \Rightarrow \inf_{n\in\mathbb{N}} A_n(\varphi)(p) = 0,$$

which is a contradiction to the hypothesis. So, $\bar{\varphi}_n$ is non-zero. As it is already observed that $\alpha_g^*(\bar{\varphi}) = \bar{\varphi}$ for all $g \in G$, so we have

$$\alpha_g^*(\bar{\varphi}_n) + \alpha_g^*(\bar{\varphi}_s) = \bar{\varphi}_n + \bar{\varphi}_s.$$

Note that, for each $g \in G$, $\alpha_g^*(\bar{\varphi}_n)$ is a normal linear functional defined on M. Secondly, since for all $g \in G$, α_g is an automorphism of M, so by Theorem 1.1.1 it is straightforward that $\alpha_g^*(\bar{\varphi}_s)$ is also a singular linear functional defined on M. Hence, by uniqueness part of the Theorem 1.1.2, it follows that

$$\alpha_g^*(\bar{\varphi}_n) = \bar{\varphi}_n \text{ and } \alpha_g^*(\bar{\varphi}_s) = \bar{\varphi}_s, \text{ for all } g \in G.$$

Our last claim is $s(\bar{\varphi}_n) \geq e$. Indeed, if $p \in \mathcal{P}_0(M)$ with $p \leq e$, then again by Theorem 1.1.1, there is a $p' \in \mathcal{P}_0(M)$ with $p' \leq p$ such that $\bar{\varphi}_s(p') = 0$. Therefore, we have

$$\begin{aligned}
\bar{\varphi}_n(p) \geq \bar{\varphi}_n(p') &= \bar{\varphi}_n(p') + \bar{\varphi}_s(p') \\
&= \bar{\varphi}(p') = \lim_{k\to\infty} A_{n_k}(\varphi)(p') \\
&\geq \inf_{k\in\mathbb{N}} A_{n_k}(\varphi)(p') \\
&\geq \inf_{n\in\mathbb{N}} A_n(\varphi)(p') > 0, \quad \text{since } 0 \neq p' \leq e.
\end{aligned}$$

So, $s(\bar{\varphi}_n) \geq e$. Consider $\nu_\varphi = \frac{\bar{\varphi}_n}{\bar{\varphi}_n(1)}$, which is the required normal state. □

Given a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ and a positive contraction T on $L^1(\Omega, \mu)$, the existence of a finite invariant measure ν, which is absolutely continuous with respect to μ, is equivalent to the existence of a function $f \in L^1(\Omega, \mu)_+$ such that $T(f) = f$. Hence Proposition 2.2.1 is a non-commutative analogue of Krengel's theorem [23, see Theorem 3.4.2].

Let N be a von Neumann algebra equipped with a f.n state φ. Recall that N is represented in standard form on the GNS Hilbert space $L^2(N, \varphi)$ with the cyclic and separating vector Ω_φ. The following known provides a norm dense subset of N_{*+} associated with a f.n state φ on N. Even though the proof is well known in the literature, we write the proof here for the sake of completeness.

Theorem 2.2.2. *Let N be a von Neumann algebra with a f.n state φ. Then $\{\psi \in N_{*+} : \psi \leq k\varphi$ for some $k > 0\}$ is a norm dense subset of N_{*+}.*

Proof. Let $\psi \in N_{*+}$. Then there exists a sequence $\{\xi_n\} \subset L^2(N, \varphi)$ with $\sum_1^\infty \|\xi_n\|^2 < \infty$ such that

$$\psi(x) = \sum_1^\infty \langle x\xi_n, \xi_n \rangle \,,\; x \in N.$$

Denote for $m \in \mathbb{N}$, $\psi_m(x) = \sum_1^m \langle x\xi_n, \xi_n\rangle$, $x \in N$. Let $\epsilon > 0$ and choose $m \in \mathbb{N}$ such that $\|\psi - \psi_m\|_1 \leq \epsilon/2$. Since Ω_φ is separating for N, we can approximate each ξ_n by $x_n'\Omega_\varphi$, where $x_n' \in N'$ such that if we define $\tilde{\psi}_m(\cdot) := \sum_1^m \langle \cdot x_n'\Omega_\varphi, x_n'\Omega_\varphi\rangle$, then $\tilde{\psi}_m \in N_{*+}$ and $\left\|\psi - \tilde{\psi}_m\right\|_1 \leq \epsilon$. Further note that for $x \in M_+$,

$$\begin{aligned}\tilde{\psi}_m(x) := \sum_1^m \langle x x_n'\Omega_\varphi, x_n'\Omega_\varphi\rangle &= \sum_1^m \left\langle (x_n')^* x_n' x^{1/2}\Omega_\varphi, x^{1/2}\Omega_\varphi\right\rangle \\ &\leq (\sum_1^m \|x_n'\|^2) \,\langle x\Omega_\varphi, \Omega_\varphi\rangle \,.\end{aligned}$$

This completes the proof. □

As an immediate application of the theorem above, we write the following proposition, which will be essential for subsequent results.

Proposition 2.2.3. *Let (M, G, α) be a covariant system and φ be a f.n state on M. If there exists a $\rho \in M_{*+}$ such that $\alpha_g^*(\rho) = \rho$ for all $g \in G$, then for any $x \in M_+$ with $\rho(x) \neq 0$, we have* $\inf_{g\in G} \alpha_g^*(\varphi)(x) > 0$.

Proof. Fix a $x \in M_+$ with $\|x\| \leq 1$ and $\rho(x) \neq 0$. Write $\epsilon = \rho(x)$. Then by Theorem 2.2.2, we conclude that there exist $k > 0$ and $\nu \in M_{*+}$ with $\nu \leq k\varphi$, such that $\|\rho - \nu\|_1 < \frac{\epsilon}{2}$. Hence, for all $g \in G$, we have

$$\begin{aligned}\alpha_g^*(\varphi)(x) &\geq \frac{1}{k}\alpha_g^*(\nu)(x) \\ &\geq \frac{1}{k}(\alpha_g^*(\rho)(x) - \|\rho - \nu\|_1) \\ &\geq \frac{1}{k}(\rho(x) - \frac{\epsilon}{2}) = \frac{\epsilon}{2k}.\end{aligned}$$

Therefore, it is clear that $\inf_{g\in G} \alpha_g^*(\varphi)(x) \geq \frac{\epsilon}{2k} > 0$. Thus, for all $x \in M_+$ with $\rho(x) \neq 0$, we have $\inf_{g\in G} \alpha_g^*(\varphi)(x) > 0$. □

The following corollary is an immediate consequence of Proposition 2.2.1 and Proposition 2.2.3.

Corollary 2.2.4. *Let $e \in \mathcal{P}_0(M)$, then the following are equivalent.*

(i) $p \in \mathcal{P}_0(M)$ *and* $p \leq e \Rightarrow \inf_{n\in\mathbb{N}} A_n(\varphi)(p) > 0$.

(ii) $p \in \mathcal{P}_0(M)$ *and* $p \leq e \Rightarrow \inf_{g\in G} \alpha_g^*(\varphi)(p) > 0$.

Remark 2.2.5. *Let (M, G, α) be a covariant system and φ be a f.n. state defined on M. If there exists a $e \in \mathcal{P}_0(M)$ satisfying eq. 2.2.1 in Proposition 2.2.1, we will get a G-invariant normal state ν_φ on M. However, we note that such a non zero projection e may not exist and as a result we may not get any G-invariant normal state. For example, consider $l^2(\mathbb{Z})$ and let $\{e_n : n \in \mathbb{Z}\}$ be the standard orthonormal basis of $l^2(\mathbb{Z})$. Further, let $U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be the unitary defined by $U(e_n) = e_{n+1}$ for all $n \in \mathbb{Z}$. Now consider the automorphism α on $\mathbf{B}(l^2(\mathbb{Z}))$ defined by $\alpha(\cdot) = U(\cdot)U^*$. Then it is easy to check that for the covariant system $(\mathbf{B}(l^2(\mathbb{Z})), \mathbb{Z}, \alpha)$ such a projection does not exist.*

Theorem 2.2.6. *Let (M, G, α) be a covariant system and φ be a f.n. state on M. Further, let $e \in \mathcal{P}_0(M)$, then the following statements are equivalent.*

(i) There exists a G-invariant normal state ρ on M with $s(\rho) = e$ such that, if ν is any G-invariant normal state on M, then $s(\nu) \leq e$.

(ii) e is the maximal projection satisfying the following condition:

$$p \in \mathcal{P}_0(M) \text{ and } p \leq e \;\Rightarrow\; \inf_{n\in\mathbb{N}} A_n(\varphi)(p) > 0. \tag{2.2.2}$$

Proof. $(i) \Rightarrow (ii)$: Let $p \leq e$ be a non-zero projection in M. We note that $\rho(p) \neq 0$. Then, by Proposition 2.2.3 we get $\inf_{g\in G} \alpha_g^*(\varphi)(p) > 0$, which implies

$$\inf_{n\in\mathbb{N}} A_n(\varphi)(p) > 0.$$

Now suppose e is not the maximal projection, but it satisfies the condition in eq. 2.2.2. Therefore, there exists a $e_1 \in \mathcal{P}_0(M)$ with $e_1 \nleq e$ and for all $p \in \mathcal{P}_0(M)$ with $p \leq e_1$ we have $\inf_{g\in G} \alpha_g^*(\varphi)(p) > 0$. Then, by Proposition 2.2.1 there exists a G-invariant normal state ν_φ on M with $s(\nu_\varphi) \geq e_1$, which contradicts the assumption.

$(ii) \Rightarrow (i)$: It is given that for all $p \in \mathcal{P}(M)$ with $0 \neq p \leq e$ we have $\inf_{n\in\mathbb{N}} A_n^*(\varphi)(p) > 0$. Hence, by applying Proposition 2.2.1 we find a G-invariant normal state ρ on M with $s(\rho) \geq e$. We claim that $s(\rho) = e$.

Indeed, first observe that $\rho(q) \neq 0$ for all $q \in \mathcal{P}(M)$ with $0 \neq q \leq s(\rho)$. Hence, by Proposition 2.2.3 we conclude that $\inf_{g\in G} \alpha_g^*(\varphi)(q) > 0$. Further, it implies $\inf_{n\in\mathbb{N}} A_n^*(\varphi)(q) > 0$. Therefore, $s(\rho)$ satisfies the condition in eq. 2.2.2. Since e is the maximal projection satisfying eq. 2.2.2, so it follows that $s(\rho) \leq e$. Hence, $e = s(\rho)$.

For the later part let ν be a G-invariant normal state on M. Then we aim to show that $s(\nu) \leq e$. Indeed, as ν is a G-invariant normal state on M, then again by Proposition 2.2.3, we find that $s(\nu)$ satisfies the condition in eq. 2.2.2. Since e is the maximal projection satisfying the same condition, so, we conclude that $s(\nu) \leq e$. □

2.3 Weakly wandering operators

In this section, we introduce weakly wandering operators associated with a covariant system (M, G, α). We begin with the definition.

Definition 2.3.1. *Let (M, G, α) be a covariant system and x be a positive operator in M. Then x is said to be a weakly wandering operator if*

$$\lim_{n\to\infty} \|A_n x\| = 0.$$

In the next set of results we discuss the existence of weakly wandering operators and its relation with the existence of G-invariant states. Further, from now onwards we assume that M is a finite von Neumann algebra with a f.n tracial state τ.

Lemma 2.3.2. *Let (M, G, α) be a covariant system and τ be a f.n tracial state on M such that there exists a non-zero projection q in M satisfying*

$$\inf_{n\in\mathbb{N}} \tau(A_n(q)) = 0. \tag{2.3.1}$$

Then there exists a non-zero weakly wandering projection $q_2 \in M$ with $q_2 \leq q$.

Proof. Let $\lambda = \tau(q) > 0$. Since $\inf_{n\in\mathbb{N}} \tau(A_n(q)) = 0$, so we have $\inf_{g\in G} \tau(\alpha_g(q)) = 0$. Hence, there exists a $g_1 \in G$ such that

$$\tau(\alpha_{g_1}(q)) < \frac{\lambda}{2^2}.$$

We claim that there exists a sequence $\{g_j\}_{j\in\mathbb{N}} \subseteq G$ such that for all $k \in \mathbb{N} \setminus \{1\}$

$$\begin{aligned} &(1) \quad g_k g_j^{-1} \neq 1 \text{ for all } j \in \{1, \ldots, (k-1)\} \text{ and} \\ &(2) \quad \tau\Big(\bigvee_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q)\Big) < \frac{\lambda}{2^{k+1}}. \end{aligned} \tag{2.3.2}$$

We prove the above claim by induction. Indeed, we note that $\inf_{g\in G} \tau(\alpha_{gg_1^{-1}}(q)) = 0$. Hence there exists $g_2 \in G$ such that

$$\tau(\alpha_{g_2 g_1^{-1}}(q)) < \frac{\lambda}{2^3}.$$

Also note that $g_2 g_1^{-1} \neq 1$. Otherwise we have, $\lambda = \tau(q) < \frac{\lambda}{2^3}$, which is a contradiction. Now we assume that there exists a set $\{g_1, \ldots, g_k\} \subseteq G$ satisfying eq. 2.3.2. Then we find a $g_{k+1} \in G$ such that

$$\begin{aligned} &(1) \quad g_{k+1} g_j^{-1} \neq 1 \text{ for all } j \in \{1, \ldots, k\} \text{ and} \\ &(2) \quad \tau\Big(\bigvee_{j=1}^{k} \alpha_{g_{k+1} g_j^{-1}}(q)\Big) < \frac{\lambda}{2^{k+2}}. \end{aligned}$$

For all $n \in \mathbb{N}$, observe that

$$\begin{aligned}\tau(A_n(\sum_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q) + q)) &= \sum_{j=1}^{k-1} \tau(A_n(\alpha_{g_k g_j^{-1}}(q))) + \tau(A_n(q)) \\ &= \sum_{j=1}^{k-1} \tau(A_n(\alpha_{g_k g_j^{-1}}(q)) - A_n(q)) + k\tau(A_n(q)) \\ &\leq \sum_{j=1}^{k-1} \frac{m(K_n(g_k g_j^{-1})\Delta K_n)}{m(K_n)} + k\tau(A_n(q)), \text{ by Proposition 1.3.4.}\end{aligned}$$

Note that by Følner condition $\sum_{j=1}^{k-1} \frac{m(K_n(g_k g_j^{-1})\Delta K_n)}{m(K_n)}$ converges to 0 and we have $\inf_{n\in\mathbb{N}} \tau(A_n(q)) = 0$. Therefore, it follows that

$$\inf_n \tau(A_n(\sum_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q) + q)) = 0.$$

Further, it implies $\inf_{g\in G} \tau(\alpha_g(\sum_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q) + q)) = 0$. Hence, there exists $g_N \in G$ such that

$$\tau(\alpha_{g_N}(\sum_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q) + q)) < \frac{\lambda}{2^{k+2}}. \tag{2.3.3}$$

Let $g_{k+1} := g_N g_k$. Then we get

$$\begin{aligned}\tau\Big(\bigvee_{j=1}^{k} \alpha_{g_{k+1} g_j^{-1}}(q)\Big) \leq \tau\Big(\sum_{j=1}^{k} \alpha_{g_{k+1} g_j^{-1}}(q)\Big) &= \tau\Big(\alpha_{g_{k+1} g_k^{-1}}\Big(\sum_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q) + q\Big)\Big) \\ &= \tau(\alpha_{g_N}(\sum_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q) + q)) \\ &< \frac{\lambda}{2^{k+2}}.\end{aligned}$$

It is also clear that $g_{k+1} g_j^{-1} \neq 1$ for all $j \in \{1, \ldots, k\}$. Indeed, if for some $j \in \{1, \ldots, k\}$ $g_{k+1} = g_j$, then $g_N = g_j g_k^{-1}$. Hence, eq. 2.3.3 will imply $\tau(q) < \frac{\lambda}{2^{k+2}}$, which is a contradiction, since $\tau(q) = \lambda$.

Let,

$$q_1 := \bigvee_{k=2}^{\infty} \bigvee_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q) \text{ and } q_2 := q \wedge (1 - q_1). \tag{2.3.4}$$

We claim that, $q_2 \neq 0$. Indeed,

$$\begin{aligned}
\tau(1 - q_2) = \tau((1 - q) \vee q_1) &= \tau\Big((1 - q) \bigvee \Big(\bigvee_{k=2}^{\infty} \bigvee_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q)\Big)\Big) \\
&\leq \tau(1 - q) + \tau\Big(\bigvee_{k=2}^{\infty} \bigvee_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q)\Big) \\
&\leq \tau(1) - \tau(q) + \sum_{k=2}^{\infty} \tau\Big(\bigvee_{j=1}^{k-1} \alpha_{g_k g_j^{-1}}(q)\Big) \\
&\leq 1 - \lambda + \sum_{k=2}^{\infty} \frac{\lambda}{2^{k+1}}, \text{ (by eq. 2.3.2)} \\
&= 1 - \frac{3\lambda}{4} < 1.
\end{aligned}$$

Therefore, we have $q_2 \neq 0$. Now we show that the projections $\{\alpha_{g_k^{-1}}(q)\}_{k \in \mathbb{N}}$ are mutually orthogonal. We note that for all $k \in \mathbb{N} \setminus \{1\}$ and $j \in \{1, \ldots, k-1\}$ we have $\alpha_{g_k g_j^{-1}}(q) \leq q_1$ and

$$\alpha_{g_k g_j^{-1}}(q) \perp (1 - q_1).$$

So, by definition of q_2 (eq. 2.3.4), for all $k \in \mathbb{N} \setminus \{1\}$ and $j \in \{1, \ldots, k-1\}$, it follows that

$$\begin{aligned}
&\alpha_{g_k g_j^{-1}}(q_2) = \alpha_{g_k g_j^{-1}}(q) \wedge \alpha_{g_k g_j^{-1}}(1 - q_1) \leq \alpha_{g_k g_j^{-1}}(q) \leq q_1 \\
\Rightarrow &\alpha_{g_k g_j^{-1}}(q_2) \perp q_2, \quad \text{(since } q_2 \leq 1 - q_1) \\
\Rightarrow &\alpha_{g_k^{-1}}(\alpha_{g_k g_j^{-1}}(q_2)) \perp \alpha_{g_k^{-1}}(q_2) \\
\Rightarrow &\alpha_{g_k^{-1}}(q_2) \perp \alpha_{g_j^{-1}}(q_2).
\end{aligned}$$

Hence, $\sum_{j=1}^{\infty} \alpha_{g_j^{-1}}(q_2)$ is a projection in M. Therefore,

$$\Big\| \sum_{j=1}^{\infty} \alpha_{g_j^{-1}}(q_2) \Big\| = 1. \tag{2.3.5}$$

We now show that q_2 is a weakly wandering operator. Indeed, for $n, k \in \mathbb{N}$, we have

$$\begin{aligned}
\|A_n(q_2)\| &= \frac{1}{k}\|k \cdot A_n(q_2)\| \\
&\leq \frac{1}{k}\sum_{j=1}^{k}\left\|A_n(q_2) - A_n(\alpha_{g_j^{-1}}(q_2))\right\| + \frac{1}{k}\left\|\sum_{j=1}^{k} A_n(\alpha_{g_j^{-1}}(q_2))\right\| \\
&\leq \frac{1}{k}\sum_{j=1}^{k}\frac{m(K_n \Delta K_n g_j^{-1})}{m(K_n)} + \frac{1}{k}\left\|A_n(\sum_{j=1}^{k}\alpha_{g_j^{-1}}(q_2))\right\| \quad \text{(by Proposition 1.3.4)} \\
&\leq \frac{1}{k}\sum_{j=1}^{k}\frac{m(K_n \Delta K_n g_j^{-1})}{m(K_n)} + \frac{1}{k}\left\|\sum_{j=1}^{k}\alpha_{g_j^{-1}}(q_2)\right\| \\
&= \frac{1}{k}\sum_{j=1}^{k}\frac{m(K_n \Delta K_n g_j^{-1})}{m(K_n)} + \frac{1}{k} \quad \text{(by eq. 2.3.5)}.
\end{aligned}$$

Now let $\epsilon > 0$. We choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\epsilon}{2}$. For each $j \in \{1, \ldots, k\}$, by Følner condition (see eq. 1.3.1) there exists a $N_j \in \mathbb{N}$ such that

$$\frac{m(K_n \Delta K_n g_j^{-1})}{m(K_n)} < \frac{\epsilon}{2}, \text{ for all } n \geq N_j.$$

Choose $N := \max\{N_1, \ldots, N_k, k\} \in \mathbb{N}$ and for all $n \geq N$, note that

$$\|A_n(q_2)\| \leq \epsilon.$$

This completes the proof. □

The following lemma establishes that support of a weakly wandering operator and support of a G-invariant state are orthogonal.

Lemma 2.3.3. *Let (M, G, α) be a covariant system and ν be a G-invariant normal state on M. Suppose $x_0 \in M_+$ is any weakly wandering operator, then $s(\nu) \perp s(x_0)$.*

Proof. Since $x_0 \in M_+$ is a weakly wandering operator, we have $\lim_{n\to\infty}\|A_n(x_0)\| = 0$. Since ν is G-invariant, for all $n \in \mathbb{N}$, we have

$$\nu(x_0) = \frac{1}{m(K_n)}\int_{K_n}\nu(\alpha_g(x_0))dm(g) = \nu(A_n(x_0)) \leq \|A_n(x_0)\| \xrightarrow{n\to\infty} 0. \qquad (2.3.6)$$

So, $\nu(x_0) = 0$, which implies $\nu(s(x_0)) = 0$. Therefore, $s(\nu) \perp s(x_0)$. □

2.4 Neveu decomposition

In this section, we derive Neveu decomposition for a covariant system (M, G, α) possessing a f.n. tracial state τ. Following is the main theorem of this chapter.

Theorem 2.4.1. *Let (M, G, α) be a covariant system and τ be a f.n tracial state on M. Suppose $e \in \mathcal{P}_0(M)$, then the following statements are equivalent.*

(i) There exists a G-invariant normal state ρ on M with $s(\rho) = e$ such that, if ν is any G-invariant normal state on M, then $s(\nu) \leq e$.

(ii) There is a weakly wandering operator $x_0 \in M_+$ with support $s(x_0) = 1 - e$ such that, if $x \in M_+$ is any weakly wandering operator, then $s(x) \leq 1 - e$.

Proof. $(i) \Rightarrow (ii)$: Suppose $e = 1$, i.e, $s(\rho) = 1$. So, ρ is a f.n G-invariant state. Therefore, if $x \in M_+$ is any weakly wandering operator, then by similar argument as in eq. 2.3.6, find that $\rho(x) = 0$. Hence, $x = 0$. Therefore, if $e = 1$, then 0 is the only weakly wandering operator.

Now suppose $e \neq 1$ and let $p \in \mathcal{P}_0(M)$ with $p \leq 1 - e$. Then we claim that there exists a $q \in \mathcal{P}_0(M)$ with $q \leq p$ such that $\inf_{n\in\mathbb{N}} \tau(A_n(q)) = 0$. On the contrary, suppose for every $q \in \mathcal{P}_0(M)$ with $q \leq p$, we have

$$\inf_{n\in\mathbb{N}} \tau(A_n(q)) > 0.$$

Then by Proposition 2.2.1, there exists a G-invariant normal state ν_τ on M such that $s(\nu_\tau) \geq p$. Hence, $s(\nu_\tau) \nleq e$, which is a contradiction to the assumption. Thus, $\inf_{n\in\mathbb{N}} \tau(A_n(q)) = 0$ for a non zero $q \leq p$ in M. Therefore, by Lemma 2.3.2, there is a weakly wandering projection $q_2 \in M$ with $0 \neq q_2 \leq q$. Define $q_{2,1} := q_2$. Let, $\{q_{2,j}\}_{j\in\Lambda}$ be the maximal family of mutually orthogonal weakly wandering projections in M such that $q_{2,j} \leq 1 - e$ for all $j \in \Lambda$. Since, M is σ-finite, we may take $\Lambda = \mathbb{N}$.

We claim that $\bar{q} := \sum_{j=1}^{\infty} q_{2,j} = 1 - e$. Note that, $\bar{q} \leq 1 - e$. Now if $\bar{q} \neq 1 - e$, then by the same construction, we get a weakly wandering non-zero subprojection of $1 - e - \bar{q}$, which contradicts the maximality of $\{q_{2,j}\}_{j \in \mathbb{N}}$.

Now define, $x_0 := \sum_{j=1}^{\infty} \frac{1}{2^j} q_{2,j} \in M_+$. Since $\bar{q} = 1 - e$, we have $s(x_0) = 1 - e$. We claim that x_0 is a weakly wandering operator. Indeed, for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\|A_n(x_0)\| &\leq \sum_{j=1}^{m} \frac{1}{2^j} \|A_n(q_{2,j})\| + \frac{1}{2^m} \left\| A_n(\sum_{j=1}^{\infty} \frac{1}{2^j} q_{2,m+j}) \right\| \\
&\leq \sum_{j=1}^{m} \|A_n(q_{2,j})\| + \frac{1}{2^m} \left\| \sum_{j=1}^{\infty} \frac{1}{2^j} q_{2,m+j} \right\| \\
&\leq \sum_{j=1}^{m} \|A_n(q_{2,j})\| + \frac{1}{2^m} \left\| \sum_{j=1}^{\infty} q_{2,m+j} \right\| \\
&\leq \sum_{j=1}^{m} \|A_n(q_{2,j})\| + \frac{1}{2^m}.
\end{aligned}$$

Let $\epsilon > 0$. We choose $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \frac{\epsilon}{2}$. Since for all $j \in \{1, \dots, m\}$, $q_{2,j}$ is weakly wandering, so there exists $N_j \in \mathbb{N}$ such that $\|A_n q_{2,j}\| < \frac{\epsilon}{2^j}$, for all $n \geq N_j$. We choose, $N := \max\{N_1, \dots, N_m, m\} \in \mathbb{N}$. Hence, $\|A_n(x_0)\| < \epsilon$ for all $n \geq N$. Therefore, $x_0 \in M_+$ is a weakly wandering operator with $s(x_0) = 1 - e$.

$(ii) \Rightarrow (i)$: We like to show that for all $p \in \mathcal{P}(M)$ with $0 \neq p \leq 1 - s(x_0) = e$, we have

$$\inf_{n \in \mathbb{N}} \tau(A_n(p)) > 0.$$

Indeed, if there exists a $p \in \mathcal{P}_0(M)$ with $p \leq 1 - s(x_0)$ such that $\inf_{n \in \mathbb{N}} \tau(A_n(p)) = 0$, then by Lemma 2.3.2, we find a non-zero weakly wandering projection p_1 such that $p_1 \leq p$. Hence, we have $p_1 \leq p \leq 1 - s(x_0)$, which violates the maximality condition on $s(x_0)$.

Thus, by Proposition 2.2.1 there exists a G-invariant normal state $\rho \in M_*$ with $s(\rho) \geq 1 - s(x_0)$. Then by Lemma 2.3.3, we have $s(\rho) \perp s(x_0)$. Therefore, we deduce that

$s(\rho) = 1 - s(x_0)$.

For the remaining part, if μ is any G-invariant normal state, then again by Lemma 2.3.3, we have $s(\mu) \perp s(x_0)$. Hence, $s(\mu) \leq 1 - s(x_0) = e$. □

Remark 2.4.2. *Suppose there is no $e \in \mathcal{P}_0(M)$ such that for every $p \in \mathcal{P}_0(M)$ with $p \leq e$ implies $\inf_{n\in\mathbb{N}} A_n(\tau)(p) > 0$, equivalently, for every $e \in \mathcal{P}_0(M)$, there exists a $p \in \mathcal{P}_0(M)$ with $p \leq e$ such that $\inf_{n\in\mathbb{N}} A_n(\tau)(p) = 0$, then by the proof of Theorem 2.4.1 ($(i) \Rightarrow (ii)$), we deduce that there exists a weakly wandering operator $x_0 \in M$ with $s(x_0) = 1$. Conversely, if $\inf_{n\in\mathbb{N}} A_n(\tau)(p) > 0$ for every $p \in \mathcal{P}_0(M)$, then there exists a G-invariant normal state ρ such that $s(\rho) = 1$, i.e, ρ is faithful.*

Let e be the maximal projection satisfying eq. 2.2.2. Note that e could be $0, 1$ or a proper projection. Let $e_1 = e$ and $e_2 = 1 - e$. For the next theorem, we refer 0-linear functional as 0-state. Thus, summarizing Theorem 2.4.1 and Remark 2.4.2, we obtain the following decomposition.

Theorem 2.4.3 (Neveu Decomposition)**.** *Let (M, G, α) be a covariant system and τ be a f.n tracial state on M. Then there exist two projections $e_1, e_2 \in \mathcal{P}(M)$ such that $e_1 + e_2 = 1$ and*

(i) there exists a G-invariant normal state ρ on M with support $s(\rho) = e_1$ and

(ii) there exists a weakly wandering operator $x_0 \in M$ with support $s(x_0) = e_2$.

Further, $s(\rho)$ and $s(x_0)$ are unique.

Chapter 3

Pointwise Ergodic Theorem

3.1 Introduction

The study of pointwise ergodic theorem or almost everywhere convergence of ergodic averages has long history. The non-commutative analogue of this type of convergence in the von Neumann algebra setting is termed as bilateral almost uniform convergence. The groundbreaking work of Lance [26] in 1976 yielded the first results in this direction. In this article, the author established a pointwise ergodic theorem for a state-preserving automorphism on a von Neumann algebra. Moreover, the author proved an individual ergodic theorem on a von Neumann algebra M. Later on, several generalizations were obtained in the works of Yeadon, Kummerer, Conze, Dang-Ngoc and many others [see, [46], [47], [24], [5] and the references therein]. In particular, in his seminal articles [46] and [47], Yeadon extended the results of [26] to the non-commutative L^1-spaces. Moreover, in [21], Junge and Xu extended Yeadon's weak (1,1) maximal ergodic inequality and used it to prove an individual ergodic theorem in non-commutative L^p-spaces for $1 < p < \infty$. Although many profound results of classical ergodic theory have already been generalized for the actions of more general amenable groups on non-L^1-spaces but the very basic pointwise ergodic theorem for amenable group actions on L^1-spaces is only proved in 2001 by Lindenstrauss [27]. Very recently, in 2021, Hong, Liao, and Wang extended Junge-Xu's work (see [21]). They obtained maximal type inequalities and various individual ergodic theorems on non-commutative L^p-spaces for the actions of locally compact groups with polynomial growth (see [18]). Notably in [21], Junge and Xu further extended the pointwise ergodic and maximal theorem to tobtained similar results for the action of $\mathbb{R}_+^d$ on L^p spaces for $1 < p < \infty$.

Study of ergodic theorems for actions of groups which are not amenable were first initiated in classical setting by Arnold and Krylov in [1]. After that, it was carried out in great detail by many authors. We refer to [9], [11], [30], [31] and references therein. The free group action in the setting of von Neumann algebra M was first considered by Walker in [44]. He proved that if the homomorphism ϕ is invariant under a faithful, normal state ρ, then for all $x \in M$ the sequence $\{S_n(x)\}$ will converge almost uniformly to an element $\hat{x}$ in M. In fact, Walker dealt with more general maps on M. In particular, he considered a sequence $\{\sigma_n\}$ of completely positive maps on M which preserves a faithful, normal state ρ and satisfies $\sigma_1\sigma_n = w\sigma_{n+1} + (1-w)\sigma_{n-1}$ for some $w \in [1/2, 1]$ with $\sigma_0(x) = x$. He then proved the almost uniform convergence of $\{\frac{1}{n+1}\sum_0^n \sigma_k\}$ under some natural spectral condition on σ_1. In [3], Walker's result was extended to $L^1(M,\tau)$, where τ is a faithful, normal, semifinite trace such that $\tau(\sigma_1(x)) \leq \tau(x)$ for all $x \in M \cap L^1(M,\tau)$ with $0 \leq x \leq 1$.

Later on, inspired by the seminal work of Junge and Xu ([21]), in [19], Hu obtained a maximal ergodic theorem and associated individual ergodic theorem in the case of $L^p(M,\tau)$ corresponding to $\{\sigma_n\}$. In particular, he considered the sequence $\{\sigma_n\}$ as described above on (M,φ), where φ is a faithful, normal state, together with $\varphi \circ \sigma_n \leq \varphi$ and $\sigma_1 \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \sigma_1$. Under these assumptions, Hu proved maximal and individual ergodic theorems associated with $\{\frac{1}{n+1}\sum_0^n \sigma_k\}$ in Haagerup non-commutative L^p spaces for $1 < p < \infty$. Furthermore, if M is assumed to be semifinite with faithful, normal, semifinite trace τ (that is, it is assumed that $\tau(\sigma_1(x)) \leq \tau(x)$ for all $x \in M \cap L^1(M,\tau)$ with $0 \leq x \leq 1$), then Hu recovered the result in [3] as mentioned above.

In this chapter, we first prove a maximal ergodic inequality and individual ergodic theorems on non-commutative L^1-spaces for the actions of locally compact groups with polynomial growth with a symmetric compact generating set V. To be exact, we consider an action α of G on M (as defined in 1.3.1). Consequently, we obtain that the sequence of averages defined by

$$A_n(x) = \frac{1}{m(V^n)} \int_{V^n} \alpha_g(x) dm(g),\ x \in M,$$

and study maximal and individual ergodic theorems associated to this sequence.

After that we extend these results to the multiparameter case, that is, the action of $\mathbb{Z}_+^d$ or $\mathbb{R}_+^d$ for an integer $d \geq 1$. If $G = \mathbb{Z}_+^d$, we first consider d many w-continuous commuting operators $T_1, \ldots, T_d$ on M and then the associated action of G is naturally defined as $T_{(i_1,\cdots,i_d)}(\cdot) = T_1^{i_1} T_2^{i_2} \cdots T_d^{i_d}(\cdot)$ for $(i_1, \cdots, i_d) \in \mathbb{Z}_+^d$. If $G = \mathbb{R}_+^d$, the we consider a continuous action $T = \{T_t\}_{t \in \mathbb{R}_+^d}$ of G on M. Then we study the following averages.

$$M_a(\cdot) := \begin{cases} \frac{1}{a^d} \sum_{0 \leq i_1 < a} \cdots \sum_{0 \leq i_d < a} T_1^{i_1} \cdots T_d^{i_d}(\cdot) & \text{when } G = \mathbb{Z}_+^d,\ a \in \mathbb{N}, \\ \frac{1}{a^d} \int_{Q_a} T_t(\cdot) dt & \text{when } G = \mathbb{R}_+^d,\ a \in \mathbb{R}_+. \end{cases}$$

On similar grounds, we study ergodic convergence of spherical averages associated to a sequence of w-continuous maps $\sigma := \{\sigma_n\}_{n=0}^{\infty}$ on a von Neumann algebra M. Such research was inspired by the study of the convergence of the spherical averages' associated to free group actions. To be precise, let $\mathcal{F}_r$ denote the free group generated by r elements and its inverses. Let ϕ be a homomorphism from $\mathcal{F}_r$ to $Aut(M)$ and consider the spherical averages

$$\sigma_n(x) := \frac{1}{|W_n|} \sum_{a \in W_n} \phi(a)(x),\ n \in \mathbb{N},$$

where W_n is the set of elements of $\mathcal{F}_r$ of length n. Then it is interesting to determine the convergence of the sequence $S_n(x) = \frac{1}{n+1} \sum_0^n \sigma_k(x)$. In fact, we consider a generalized noncommutative dynamical system (M, σ), where $\sigma = \{\sigma_n\}_{n=0}^{\infty}$ is a sequence of w-continuous maps on M satisfying

(A_1). σ_1 is completely positive, $\sigma_1(1) \leq 1$ and σ_n is positive.

(A_2). $\sigma_1 \circ \sigma_n = w\sigma_{n+1} + (1-w)\sigma_{n-1}$ for all $n \in \mathbb{N}$ and $\sigma_0(x) = x$, where $1/2 < w < 1$,

and consider the sequence of averages defined by

$$S_n(x) = \frac{1}{n+1} \sum_{0}^{n} \sigma_k(x), \; n \in \mathbb{N}.$$

Let us now discuss the predual map in our context which will be essential for this thesis. Let M be a von Neumann algebra with f.n.s trace τ and $\alpha : L^1(M,\tau) \to L^1(M,\tau)$ be a bounded linear map. Since $M = L^1(M,\tau)^*$, the dual map of α, denoted by α^*, defined on M is determined by the equation

$$\tau(\alpha^*(x)Y) = \tau(x\alpha(Y)) \text{ for all } x \in M \text{ and } Y \in L^1(M,\tau). \tag{3.1.1}$$

On the other hand if β is a bounded linear map on M, the predual transformation of β, denoted by $\hat{\beta}$, defined on $L^1(M,\tau)$, and it is determined by the equation

$$\tau(\beta(x)Y) = \tau(x\hat{\beta}(Y)) \text{ for all } x \in M \text{ and } Y \in L^1(M,\tau). \tag{3.1.2}$$

Further, we have $(\hat{\beta})^* = \beta$. This identification will be used in the sequel.

3.2 Action of group of polynomial growth

First we study the non-commutative dynamical system $(M, \mathbb{Z}_+, \alpha)$. Then we note that it is determined by a w-continuous positive linear map $T : M \to M$. Corresponding to this T, we consider the dual operator $T^* : M^* \to M^*$. In this context, we always consider ergodic averages with respect to the Følner sequence $K_n = \{0, 1, 2, \cdots, n-1\}$, $n \in \mathbb{N}$. Write the ergodic averages corresponding to the operator T^* as

$$S_n(\mu) := \frac{1}{n} \sum_{k=0}^{n-1} (T^*)^k(\mu), \; \mu \in M^*.$$

We note that since T is assumed to be w-continuous, so for all $n \in \mathbb{N}$, we have $(T^*)^n(\mu)$, $S_n(\mu) \in M_*$ whenever $\mu \in M_*$.

The following lemma will be used in the proof of successive theorems. Maybe it is well known in the literature, but for lack of reference, we provide a sleek proof.

Lemma 3.2.1. *Let $M \subseteq \mathbf{B}(\mathcal{H})$ be a von Neumann algebra and $c \in M$ such that $0 \leq c \leq 1$. Suppose $x \in M$ such that $0 \leq x \leq s(c)$, then*

$$x \in \overline{\{y \in M : 0 \leq y \leq nc \textit{ for some } n \in \mathbb{N}\}}^{w\text{-topology}}.$$

Proof. Let $0 \leq c \leq 1$ and $e = s(c)$. For every $n \in \mathbb{N}$ consider the projection $e_n = \chi_{(\frac{1}{n},1]}(c)$ and observe that

$$e_n \uparrow e \text{ in SOT and } ce_n \geq \frac{1}{n} e_n.$$

Let $x \in M$ be such that $0 \leq x \leq e$. Hence, we have $0 \leq e_n x e_n \leq \|x\| e_n \leq e_n \leq nce_n \leq nc$ for all $n \in \mathbb{N}$. It is also straightforward to check that $e_n x e_n$ converges to x in SOT. Consequently, $e_n x e_n$ converges to x in w-topology. □

In the next theorem, we consider a positive, sub-unital and state preserving map on M. Then prove a maximal inequality type theorem for the induced map T^* on M_*. We follow similar idea as in [46], but in a different context.

Theorem 3.2.2. *Let M be a von Neumann algebra with a f.n state ρ on M. Assume that $T : M \to M$ is a sub-unital, ρ-preserving and w-continuous positive linear map. Let $\mu \in M_{*s}$ and $\epsilon > 0$. Then for any $N \in \mathbb{N}$ there exists $e \in \mathcal{P}(M)$ such that $\rho(1-e) < \|\mu\| / \epsilon$ and*

$$|S_n(\mu)(x)| \leq \epsilon\rho(x) \textit{ for all } x \in (eMe)_+ \textit{ and } n \in \{1, \ldots, N\}.$$

Proof. Let $N \in \mathbb{N}$ and consider the w-compact subset $\mathcal{L}$ of the von Neumann algebra $R := \bigoplus_{n=1}^{2N} M$ defined by

$$\mathcal{L} := \Big\{(\underline{\mathrm{x}}, \underline{\mathrm{y}}) := (x_1, \ldots, x_N, y_1, \ldots, y_N) \in R : x_i, y_i \geq 0 \text{ and } \sum_{i=1}^{N}(x_i + y_i) \leq 1\Big\}.$$

Now consider the w-continuous linear functional κ on R defined by

$$\kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}})) := \sum_{n=1}^{N} n\Big[S_n(\mu')(x_n) - \rho(x_n) - S_n(\mu')(y_n) - \rho(y_n)\Big], \text{ for all } (\underline{\mathrm{x}}, \underline{\mathrm{y}}) \in R,$$

where, $\mu' := \mu/\epsilon$. Since $\mathcal{L}$ is w-compact, the supremum value of the function κ on the set $\mathcal{L}$ will be attained. Let $(\underline{\mathrm{x}}, \underline{\mathrm{y}}) \in \mathcal{L}$ be such that $\kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}})) \geq \kappa((\underline{\mathrm{a}}, \underline{\mathrm{b}}))$ for all $(\underline{\mathrm{a}}, \underline{\mathrm{b}}) \in \mathcal{L}$. We define $c := 1 - \sum_{n=1}^{N}(x_i + y_i)$. Note that $c \geq 0$. Let c' be an element in M such that $0 \leq c' \leq c$. For $m \in \{1, \ldots, N\}$, take $\underline{x}' = (x_1, \ldots, x_m + c', \ldots, x_N)$. We consider the point $(\underline{\mathrm{x}}', \underline{\mathrm{y}}) \in \mathcal{L}$, i.e,

$$(\underline{\mathrm{x}}', \underline{\mathrm{y}}) = (x_1, \ldots, x_m + c', \ldots, x_N, y_1, \ldots, y_N).$$

Then we have,

$$\kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}})) \geq \kappa((\underline{\mathrm{x}}', \underline{\mathrm{y}})).$$

Therefore, we deduce the following

$$\begin{aligned}
& \kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}})) - \kappa((\underline{\mathrm{x}}', \underline{\mathrm{y}})) \geq 0 \\
\Rightarrow \quad & \sum_{n=1}^{N} n\Big[S_n(\mu')(x_n) - \rho(x_n) - S_n(\mu')(y_n) - \rho(y_n)\Big] \\
& - \sum_{n=1}^{N} n\Big[S_n(\mu')(x_n) - \rho(x_n) - S_n(\mu')(y_n) - \rho(y_n)\Big] - m\Big[S_m(\mu')(c') - \rho(c')\Big] \geq 0 \\
\Rightarrow \quad & - m\Big[S_m(\mu')(c') - \rho(c')\Big] \geq 0 \\
\Rightarrow \quad & S_m(\mu')(c') \leq \rho(c'). \qquad (3.2.1)
\end{aligned}$$

We again consider the following point in $\mathcal{L}$

$$(\underline{\mathrm{x}}, \underline{\mathrm{y}}') := (x_1, \ldots, x_N, y_1, \ldots, y_m + c', \ldots, y_N).$$

Then we have

$$\kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}})) \geq \kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}}')).$$

Likewise as eq. 3.2.1, we deduce

$$-S_m(\mu')(c') \leq \rho(c'). \tag{3.2.2}$$

Hence by eq. 3.2.1 and eq. 3.2.2, we have

$$|S_m(\mu)(c')| \leq \epsilon\rho(c') \text{ for all } c' \in M \text{ such that } 0 \leq c' \leq c. \tag{3.2.3}$$

Now if $x \in M$ with $0 \leq x \leq 1$ and $x \leq nc$ for some $n \in \mathbb{N}$, consider $c' := \frac{x}{n}$. As, $c' \leq c$, hence by eq. 3.2.3 we have

$$|S_m(\mu)(x)| \leq \epsilon\rho(x).$$

Let $e = s(c)$ and $x \in eMe$ with $0 \leq x \leq e$. Since $|S_m(\mu)(\cdot)|$, ρ are w-continuous, so by Lemma 3.2.1, we have

$$|S_m(\mu)(x)| \leq \epsilon\rho(x).$$

And consequently we achieve

$$|S_n(\mu)(x)| \leq \epsilon\rho(x) \text{ for all } x \in (eMe)_+ \text{ and } n \in \{1, \ldots, N\}.$$

We claim that $\rho(1-e) < \|\mu\| / \epsilon$. Consider the point $(Tx_2, \ldots, Tx_N, 0, Ty_2, \ldots, Ty_N, 0) \in R$. Since $T(1) \leq 1$, we note that $(Tx_2, \ldots, Tx_N, 0, Ty_2, \ldots, Ty_N, 0) \in \mathcal{L}$. Hence, we have

$$\kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}})) \geq \kappa((Tx_2, \ldots, Tx_N, 0, Ty_2, \ldots, Ty_N, 0)).$$

From this we deduce the following

$$\kappa((\underline{\mathrm{x}}, \underline{\mathrm{y}})) \geq \kappa((Tx_2, \ldots, Tx_N, 0, Ty_2, \ldots, Ty_N, 0))$$

$$
\begin{aligned}
\Rightarrow \quad & \mu'(x_1) - \rho(x_1) + \cdots + \sum_{k=0}^{N-1} \mu'(T^k(x_N)) - N\rho(x_N) \\
& -\mu'(y_1) - \rho(y_1) - \cdots - \sum_{k=0}^{N-1} \mu'(T^k(y_N)) - N\rho(y_N) \\
& \geq \\
& \mu'(Tx_2) - \rho(Tx_2) + \cdots + \sum_{k=0}^{N-2} \mu'(T^{k+1}x_N) - (N-1)\rho(Tx_N) \\
& -\mu'(Ty_2) - \rho(Ty_2) - \cdots - \sum_{k=0}^{N-2} \mu'(T^{k+1}y_N) - (N-1)\rho(Ty_N) \\
\Rightarrow \quad & \mu'(x_1) + \sum_{k=0}^{1} \mu'(T^k(x_2)) + \cdots + \sum_{k=0}^{N-1} \mu'(T^k(x_N)) - \sum_{k=1}^{N} k\rho(x_k) \\
& -\mu'(Tx_2) - \sum_{k=0}^{1} \mu'(T^{k+1}x_k) - \cdots - \sum_{k=0}^{N-2} \mu'(T^{k+1}x_N) + \sum_{k=1}^{N} (k-1)\rho(x_k) \\
& \geq \\
& \mu'(y_1) + \sum_{k=0}^{1} \mu'(T^k(y_2)) + \cdots + \sum_{k=0}^{N-1} \mu'(T^k(y_N)) + \sum_{k=1}^{k} k\rho(y_k) \\
& -\mu'(Ty_2) - \sum_{k=0}^{1} \mu'(T^{k+1}y_3) - \cdots - \sum_{k=0}^{N-2} \mu'(T^{k+1}y_N) - \sum_{k=1}^{N} (k-1)\rho(y_k) \\
\Rightarrow \quad & \mu'(\sum_{k=1}^{N} x_k) - \rho(\sum_{k=1}^{N} x_k) \geq \mu'(\sum_{k=1}^{N} y_k) + \rho(\sum_{k=1}^{N} y_k) \\
\Rightarrow \quad & \mu'(\sum_{k=1}^{N} x_k - \sum_{k=1}^{N} y_k) \geq \rho(\sum_{k=1}^{N} x_k + \sum_{k=1}^{N} y_k) \\
\Rightarrow \quad & \|\mu\|/\epsilon \geq \rho(\sum_{i=1}^{N} (x_i + y_i)) \\
\Rightarrow \quad & \rho(c) \geq 1 - \|\mu\|/\epsilon, \quad \text{(since } c = 1 - \sum_{i=1}^{N} (x_i + y_i)) \\
\Rightarrow \quad & \rho(e) \geq 1 - \|\mu\|/\epsilon, \quad \text{(since } e = s(c)).
\end{aligned}
$$

This completes the proof. □

Now we need to recall the GNS representation and some basic facts about the Tomita-

Takesaki modular theory associated with a f.n state ρ on a von Neumann algebra M. Let $L^2(M,\rho)$ be the GNS Hilbert space and identify M as a von Neumann subalgebra of $\mathbf{B}(L^2(M,\rho))$. Further, let Ω_ρ be the cyclic, separating vector in $L^2(M,\rho)$ and $\langle\cdot,\cdot\rangle_\rho$ denotes the inner product in $L^2(M,\rho)$.

Simply write J and Δ for the modular conjugation operator J_ρ and modular operator Δ_ρ respectively. Let $M(\sigma)$ be the collection of analytic vectors of M corresponding to the modular automorphism group $\{\sigma_t\}_{t\in\mathbb{R}}$. Similarly, one can define $M'(\sigma)$ to be the collection of analytic vectors of M' corresponding to $\{\sigma_t\}_{t\in\mathbb{R}}$. Recall that $M(\sigma)$ (resp. $M'(\sigma)$) is a σ_t-invariant subalgebra of M (resp. M') which is strong* dense in M (resp. M') (see [16] and [36]). Further, recall that for all $z'\in M'(\sigma)$ there exists a $z\in M$ such that $z'\Omega_\rho = z\Omega_\rho$, indeed, $z = J\sigma_{i/2}(z'^*)J$ (see [7]).

Remark 3.2.3. *In this remark we compare Theorem 3.2.2 and a result obtained in [48, Theorem 2.1].*

(i) We show that in the context of a semifinite von Neumann algebras, Theorem 3.2.2 can be used to get a similar result obtained in [48, Theorem 2.1] and vice versa. Let M be a semifinite von Neumann algebra with a f.n.s trace τ and identify M_* with $L^1(M,\tau)$.

Consider T and ρ as in Theorem 3.2.2. Then note that ρ can be identified with $d_\rho \in L^1(M,\tau)_+$, satisfying $\rho(x) = \tau(d_\rho x)$ for all $x\in M$. Further, we have $\tau(T^*(a)y) = \tau(aT(y))$, for all $y\in M$ and $a\in L^1(M,\tau)$. Now let $\mu\in M_{*+}$, then there exists a unique $a\in L^1(M,\tau)_+$ such that $\mu(y)=\tau(ay)$ for all $y\in M$ and write $\mu=\tau_a$. Let $\epsilon>0$, then by Theorem 3.2.2, we deduce that for any $N\in\mathbb{N}$ there exist $e\in\mathcal{P}(M)$ such that $\rho(1-e)<\frac{\|\tau_a\|}{\epsilon}$ and for all $x\in(eMe)_+$ and $n\in\{1,\dots,N\}$, we have

$$\begin{aligned} &|S_n(\tau_a)(x)| \leq \epsilon\rho(x) \\ \Rightarrow\ &\tau(eS_n(a)ex) \leq \epsilon\tau(ed_\rho ex), \text{ as } \rho(\cdot)=\tau(d_\rho(\cdot)). \end{aligned}$$

Thus we obtain, $eS_n(a)e \leq ed_\rho e$ *for* $1 \leq n \leq N$*, which is similar to one of the conclusions obtained in [48, Theorem 2.1].*

Conversely, let τ' *be the trace on* M' *defined by* $\tau'(\cdot) = \tau(J_\tau(\cdot)J_\tau))$*, then the Hilsum spatial* L^1*-space* $L^1(M;\tau')$ *is same as* $L^1(M,\tau)$*. Let* $\frac{d\rho}{d\tau'}$ *be the Connes spatial derivative. Then it follows that* $d_\rho = \frac{d\rho}{d\tau'}$*. For details on Hilsum spatial* L^1*-space and Connes spatial derivatives, we refer to [4], [17] [41]. Thus, it is routine to deduce Theorem 3.2.2 from [48, Theorem 2.1] in the tracial context.*

(ii) *Suppose* T *is an automorphism of* M *and* ρ *is a f.n state on* M *such that* $\rho \circ T = \rho$*. Assume that* $M \subseteq \mathbf{B}(L^2(M,\rho))$ *and* ψ *be a f.n state on* M'*. Then* T^{-1} *satisfies all the conditions of [48, Theorem 2.1] and* T^{-1} *extends to a positive contraction on Hilsum spatial* L^1*-space* $L^1(M;\psi)$ *by*

$$T^{-1} : d^{1/2}Md^{1/2} \to d^{1/2}Md^{1/2};\ \ T^{-1}(d^{1/2}xd^{1/2}) = d^{1/2}T^{-1}(x)d^{1/2},\ \ x \in M,$$

where $d = \frac{d\rho}{d\psi}$*. Moreover, since* M *is* σ*-finite, for each* $x \in M_+$*,* $d^{1/2}xd^{1/2}$ *corresponds to* $\rho_x \in M_{*+}$*, where* ρ_x *is determined by the equation (see [42, Section 2.3])*

$$\rho_x(y) = \langle J_\rho x^* J_\rho y \Omega_\rho, \Omega_\rho \rangle_\rho,\ \ y \in M.$$

Further, we have $\left\| d^{1/2}xd^{1/2} \right\|_1 = \|\rho_x\|_1$*. However, for* $x \in M(\rho)_+$ *and* $y \in M$*, we have*

$$\begin{aligned} T^*(\rho_x)(y) = \rho_x(T(y)) &= \langle J_\rho x^* J_\rho T(y)\Omega_\rho, \Omega_\rho \rangle_\rho \\ &= \langle T(y)\Omega_\rho, J_\rho x J_\rho \Omega_\rho \rangle_\rho \\ &= \left\langle T(y)\Omega_\rho, \sigma_{i/2}(x^*)\Omega_\rho \right\rangle_\rho \end{aligned}$$

$$
\begin{aligned}
&= \rho(\sigma_{-i/2}(x)T(y)) \\
&= \rho(\sigma_{-i/2}(T^{-1}(x))y), \ \ \textit{since } \rho \circ T = \rho \\
&= \left\langle y\Omega_\rho, \sigma_{i/2}(T^{-1}(x)^*)\Omega_\rho \right\rangle_\rho \\
&= \left\langle y\Omega_\rho, J_\rho T^{-1}(x) J_\rho \Omega_\rho \right\rangle_\rho \\
&= \rho_{T^{-1}(x)}(y).
\end{aligned}
$$

Hence, we note that the extension T^{-1} to the Hilsum spatial L^1-space $L^1(M;\psi)$ is same as T^* on M_*. Thus, in this context, it is also straightforward to deduce Theorem 3.2.2 from [48, Theorem 2.1] and vice versa.

But when T is not an automorphism on M in Theorem 3.2.2, and M is not a semifinite von Neumann algebra, we do not find an immediate way to compare Theorem 3.2.2 with the results obtained in [48, Theorem 2.1]. As the article [48] works with the natural extension of T on the Hilsum spatial L^1-space, but we work with the predual map of T. Further, our approach is fairly standard and elementary, even though the core idea of both the proofs are borrowed from [46, Theorem 1].

Before we move on to our next results, we need few important lemmas. Let M' denotes the commutant of the von Neumann algebra M in $\mathbf{B}(\mathcal{H})$. The following Radon-Nikodym type lemma is well-known in the literature; we still give a proof for the sake of completeness and for the interest of reference.

Lemma 3.2.4. *Let $M \subseteq \mathbf{B}(\mathcal{H})$ be a von Neumann algebra and ω be a positive linear functional on M. Suppose there exists a $C > 0$ and $\xi \in \mathcal{H}$ such that $\omega(x) \leq C \langle x\xi, \xi \rangle_{\mathcal{H}}$ for all $x \in M_+$, then there exists a $x' \in M'_+$ such that $x' \leq C$ and $\omega(x) = \langle x'x\xi, \xi \rangle_{\mathcal{H}}$ for all $x \in M$. Moreover, if ξ is a cyclic vector for M in $\mathcal{H}$, then x' is unique.*

Proof. Consider the following sesquilinear form on $\mathcal{H}_0 := \overline{M\xi}^{\|\cdot\|_{\mathcal{H}}}$:

$$B(x\xi, y\xi) := \omega(y^*x),\ x, y \in M.$$

Note that it is positive. Further, use the condition $\omega(\cdot) \leq C\langle(\cdot)\xi, \xi\rangle$ and Cauchy-Schwarz inequality to show that it is indeed a bounded sesquilinear form on $\mathcal{H}_0$. Hence there exists a unique positive, bounded operator $a \in \mathbf{B}(\mathcal{H}_0)$ such that $0 \leq a \leq C$ and

$$\omega(y^*x) = \langle ax\xi, y\xi\rangle_{\mathcal{H}} \text{ for all } x, y \in M.$$

Also for all $x, y, z \in M$, we have

$$\langle azx\xi, y\xi\rangle_{\mathcal{H}} = \omega(y^*zx) = \omega((z^*y)^*x) = \langle ax\xi, z^*y\xi\rangle_{\mathcal{H}} = \langle zax\xi, y\xi\rangle_{\mathcal{H}}.$$

Hence,

$$a(z_{\restriction \mathcal{H}_0}) = (z_{\restriction \mathcal{H}_0})a.$$

Thus if we define, p to be the projection onto $\mathcal{H}_0 \subseteq \mathcal{H}$, then $p \in M'$ and $pap \in M'_+$. Assign $x' = pap$. Clearly, $x' \leq C$. Then for any $x \in M$

$$\omega(x) = \langle ax\xi, \xi\rangle_{\mathcal{H}} = \langle papx\xi, \xi\rangle_{\mathcal{H}} = \langle x'x\xi, \xi\rangle_{\mathcal{H}}.$$

Now let $\xi \in \mathcal{H}$ be a cyclic vector, and there exist $x', y' \in M'_+$ such that $\omega(x) = \langle x'x\xi, \xi\rangle_{\mathcal{H}} = \langle y'x\xi, \xi\rangle_{\mathcal{H}}$ for all $x \in M$. Then $x'_{\restriction \mathcal{H}_0} = y'_{\restriction \mathcal{H}_0}$, which implies $x' = y'$. □

Lemma 3.2.5. *Let (M, G, α) be a non-commutative dynamical system and ρ be a G-invariant f.n state on M. Then there exists an action α' on M' such that*

$$\langle \alpha'_g(y')x\Omega_\rho, \Omega_\rho\rangle_\rho = \langle y'\alpha_{g^{-1}}(x)\Omega_\rho, \Omega_\rho\rangle_\rho \textit{ for all } x \in M, y' \in M', g \in G.$$

Proof. For $y' \in M'_+$ and $g \in G$, consider the linear functional $\nu^g_{y'} : M \to \mathbb{C}$ defined by

$$\nu^g_{y'}(x) = \langle y'\alpha_{g^{-1}}(x)\Omega_\rho, \Omega_\rho\rangle_\rho,\ x \in M.$$

For $x \in M_+$, note that

$$\begin{aligned}\nu_{y'}^{g}(x) &= \langle y'\alpha_{g^{-1}}(x)\Omega_\rho, \Omega_\rho\rangle_\rho \\ &\leq \|y'\|_\infty \langle \alpha_{g^{-1}}(x)\Omega_\rho, \Omega_\rho\rangle_\rho \\ &= \|y'\|_\infty \rho(\alpha_{g^{-1}}(x)) \\ &= \|y'\|_\infty \rho(x) \\ &= \|y'\|_\infty \langle x\Omega_\rho, \Omega_\rho\rangle_\rho.\end{aligned}$$

Hence, by Lemma 3.2.4 there exists a unique $z' \in M'$ such that $\nu_{y'}^{g}(x) = \langle z'x\Omega_\rho, \Omega_\rho\rangle_\rho$. Write $z' = \alpha'_g(y')$ and then we have $\langle y'\alpha_{g^{-1}}(x)\Omega_\rho, \Omega_\rho\rangle_\rho = \langle \alpha'_g(y')x\Omega_\rho, \Omega_\rho\rangle_\rho$. It is straightforward to check that $\alpha' := (\alpha'_g)$ is an action of G on the von Neumann algebra M' satisfying the required conditions. $\square$

Let (M, G, α) be a non-commutative dynamical system and $\{K_n\}_{n\in\mathbb{N}}$ be a Følner sequence in G. Note that it follows from eq. 1.3.3 that $\beta := \{\beta_g\}_{g\in G} := \{\alpha^*_{g^{-1}}\}_{g\in G}$ defines an action of the group G on M_*. We consider the following averages on M_*;

$$B_n(\mu) := \frac{1}{m(K_n)}\int_{K_n} \beta_g(\mu)dm(g) := \frac{1}{m(K_n)}\int_{K_n} \alpha^*_{g^{-1}}(\mu)dm(g), \quad \mu \in M_*, n \in \mathbb{N}.$$

And of course when $x \in M$, we write

$$B_n(x) = \frac{1}{m(K_n)}\int_{K_n} \alpha_{g^{-1}}(x)dm(g), n \in \mathbb{N}.$$

Further, if G is unimodular and $\{K_n\}_{n\in\mathbb{N}}$ are symmetric, then for all $n \in \mathbb{N}$, we note that

$$\begin{aligned}A_n(\cdot) &= \frac{1}{m(K_n)}\int_{K_n} \alpha_g(\cdot)dm(g) \\ &= \frac{1}{m(K_n)}\int_{K_n} \alpha_{g^{-1}}(\cdot)dm(g) = B_n(\cdot), \text{ as } K_n \text{ is symmetric and G is unimodular.}\end{aligned}$$

We now prove the following abstract mean ergodic theorem on M_*, which will be used to prove our main theorem.

Theorem 3.2.6. *Let (M, G, α) be a non-commutative dynamical system and $\{K_n\}_{n\in\mathbb{N}}$ be a Følner sequence in G. Also assume that there exists a f.n state ρ satisfying $\rho(\alpha_g(x)^2) \leq \rho(x^2)$ for all $x \in M_s$ and $g \in G$. Then for all $\mu \in M_*$, there exists a $\bar{\mu} \in M_*$ such that*

$$\bar{\mu} = \|\cdot\|_1 - \lim_{n\to\infty} B_n(\mu).$$

Further, if G is a unimodular group and $\{K_n\}$ are symmetric, then $\bar{\mu}$ is G-invariant.

Proof. Let $L^2(M_s, \rho)$ be the closure of M_s with respect to the norm induced from the inner product $\langle\cdot,\cdot\rangle_\rho$. Then we can define the following contractions on the Hilbert space $L^2(M_s, \rho)$.

$$u_g(x\Omega_\rho) = \alpha_g(x)\Omega_\rho, x \in M_s, g \in G.$$

Now consider $T_n := \frac{1}{m(K_n)} \int_{K_n} u^*_{g^{-1}}\, dm(g)$. Then by von Neumann mean ergodic theorem, it follows that for all $\xi \in L^2(M_s, \rho), T_n(\xi)$ converges to $P\xi$ strongly, where P is the orthogonal projection of $L^2(M_s, \rho)$ onto the subspace $\{\xi \in L^2(M_s, \rho) : u_g\xi = \xi$ for all $g \in G\}$.
Now let $x \in M$. Further write x as $x_1 + ix_2$, where $x_1, x_2 \in M_s$ and then by the previous argument, it follows that $T_n(x\Omega) := T_n(x_1\Omega_\rho) + iT_n(x_2\Omega_\rho)$ converges in $L^2(M, \rho)$.
Let $y_1, y_2 \in M'(\sigma)$ and define $\psi_{y_1,y_2}(x) = \langle xy_1\Omega_\rho, y_2\Omega_\rho\rangle$ for all $x \in M$. Then by the discussion just before Remark 3.2.3, there exists a $z \in M$ such that $y_1^*y_2\Omega_\rho = z\Omega_\rho$, where $z = J\sigma_{i/2}(y_2^*y_1)J$. Consequently, for $x \in M$

$$\begin{aligned}
B_n(\psi_{y_1,y_2})(x) &= \frac{1}{m(K_n)} \int_{K_n} \langle \alpha_{g^{-1}}(x)y_1\Omega_\rho, y_2\Omega_\rho\rangle_\rho \, dm(g)\\
&= \frac{1}{m(K_n)} \int_{K_n} \langle \alpha_{g^{-1}}(x)\Omega_\rho, y_1^*y_2\Omega_\rho\rangle_\rho \, dm(g)\\
&= \frac{1}{m(K_n)} \int_{K_n} \langle \alpha_{g^{-1}}(x)\Omega_\rho, z\Omega_\rho\rangle_\rho \, dm(g)\\
&= \frac{1}{m(K_n)} \int_{K_n} \left\langle x\Omega_\rho, u^*_{g^{-1}}(z\Omega_\rho)\right\rangle_\rho dm(g)\\
&= \langle x\Omega_\rho, T_n(z\Omega_\rho)\rangle_\rho \, dm(g).
\end{aligned}$$

Hence, for all $x \in M$, $B_n(\psi_{y_1,y_2})(x) \to \langle x\Omega_\rho, \eta\rangle_\rho$, where $\eta = \lim_{n\to\infty} T_n(z\Omega_\rho)$. Consider $\overline{\psi}_{y_1,y_2} \in M_*$, defined by

$$\overline{\psi}_{y_1,y_2}(x) = \langle x\Omega_\rho, \eta\rangle_\rho\,, \quad x \in M.$$

Then by standard argument it follows that $\overline{\psi}_{y_1,y_2} \circ \alpha_g = \overline{\psi}_{y_1,y_2}$ for all $g \in G$ and

$$\overline{\psi}_{y_1,y_2} = \|\cdot\|_1 - \lim_{n\to\infty} B_n(\psi_{y_1,y_2}).$$

Hence the result follows since the set $\{\psi_{y_1,y_2} : y_1, y_2 \in M'(\sigma)\}$ is total in M_*. When G is unimodular group and $\{K_n\}_{n\in\mathbb{N}}$ are symmetric, then G-invariance of μ follows by standard argument. □

Definition 3.2.7. *Let (M, G, α) be a non-commutative dynamical system and ρ be a f.n. state on M. (M, G, α, ρ) is called kernel if*

1. *ρ is G-invariant and*
2. *α is sub-unital, i.e, $\alpha_g(1) \leq 1$ for all $g \in G$.*

Remark 3.2.8. *Let (M, G, α, ρ) be a kernel, then observe the following.*

1. *By Kadison's inequality [22], for all $x \in M_s$ we have $\alpha_g(x)^2 \leq \alpha_g(x^2)$, and further as ρ is G-invariant, it follows that*

$$\rho(\alpha_g(x)^2) \leq \rho(x^2) \text{ for all } x \in M_s \text{ and } g \in G.$$

2. *As α_g is a positive map on M, so by Russo-Dye theorem [32, Corollary 2.9] $\|\alpha_g\| = \|\alpha_g(1)\| \leq 1$. Thus, we have $\sup_{g\in G} \|\alpha_g\| \leq 1$.*

Let (M, G, α, ρ) be a kernel. Then by Remark 3.2.8 and Theorem 3.2.6, it follows that $\lim_{n\to\infty} B_n(\mu)$ exists in $\|\cdot\|_1$, for all $\mu \in M_*$. We denote the limit by $\bar{\mu}$. The following is one of the important results to obtain ergodic type theorem.

Lemma 3.2.9. *Let (M, G, α, ρ) be a kernel. Further assume that G is unimodular and $\{K_n\}_{n\in\mathbb{N}}$ are symmetric. Consider the following set*

$$\mathcal{W}_1 := \{\nu - B_k(\nu) + \bar{\nu} : \quad k \in \mathbb{N},\ \nu \in M_{*+} \text{ with } \nu \leq \lambda\rho \text{ for some } \lambda > 0\}.$$

(i) Write $\mathcal{W} = \mathcal{W}_1 - \mathcal{W}_1$, then $\mathcal{W}$ is dense in M_{*s} and

(ii) for all $\nu \in \mathcal{W}$, we have

$$\lim_{n\to\infty} \sup_{x\in M_+, x\neq 0} |(B_n(\nu) - \bar{\nu})(x)| / \rho(x) = 0. \tag{3.2.4}$$

Proof. *(i):* Let $\mu \in M_{*+}$ and $\epsilon > 0$. From Theorem 2.2.2, find a $\nu \in M_{*+}$ with $\nu \leq \lambda\rho$ for some $\lambda > 0$, such that $\|\mu - \nu\| < \epsilon/2$. Further, by Theorem 3.2.6 we know that $B_n(\nu)$ is convergent, and write $\bar{\nu} = \lim_{n\to\infty} B_n(\nu)$. So there exists a $n_0 \in \mathbb{N}$ such that $\|\bar{\nu} - B_{n_0}(\nu)\| \leq \epsilon/2$. Therefore by triangle inequality, we have $\|\mu - (\nu - B_{n_0}(\nu) - \bar{\nu})\| \leq \|\mu - \nu\| + \|B_{n_0}(\nu) - \bar{\nu}\| < \epsilon$.
Now for $\mu \in M_{*s}$, we write $\mu = \mu_+ - \mu_-$, where μ_+, μ_- are normal positive linear functional. Thus, it follows that $\mathcal{W}$ is dense in M_{*s}.

(ii): Fix $k \in \mathbb{N}$ and consider $\nu_k := \nu - B_k(\nu) + \bar{\nu}$ and it is enough to prove eq. 3.2.4 for ν_k. First we claim that $\bar{\nu}_k = \bar{\nu}$. Since $\nu \leq \lambda\rho$, so by Lemma 3.2.4 there exists a unique $y_1' \in M_+'$ with $y_1' \leq \lambda$ such that

$$\nu(x) = \langle y_1' x\Omega_\rho, \Omega_\rho\rangle_\rho \text{ for all } x \in M.$$

Let $y' \in M'$, write $B_n'(y') := \frac{1}{m(K_n)} \int_{K_n} \alpha_g'(y') dm(g)$ and by Lemma 3.2.5, we have

$$\langle B_n'(y') x\Omega_\rho, \Omega_\rho\rangle_\rho = \langle y' B_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho, \quad x \in M, y' \in M'.$$

Now for all $n \in \mathbb{N}$ and $x \in M_+$, we note that

$$|(B_n(\nu_k) - \bar{\nu})(x)|$$

$$
\begin{aligned}
&= |(\nu_k - \bar{\nu})(B_n(x))| \\
&= |(\nu - B_k(\nu))(B_n(x))| \quad (\text{since } \nu_k := \nu - B_k(\nu) + \bar{\nu}) \\
&= |\nu(B_n(x)) - \nu(B_k(B_n(x)))| \ (\text{as } B_k(\nu)(\cdot) = \nu(B_k(\cdot))) \\
&= \left| \langle y_1' B_n(x)\Omega_\rho, \Omega_\rho \rangle_\rho - \langle y_1' B_k(B_n(x))\Omega_\rho, \Omega_\rho \rangle_\rho \right| \ (\text{as } \nu(\cdot) = \langle y_1'(\cdot)\Omega_\rho, \Omega_\rho \rangle_\rho) \\
&= \left| \langle y_1' B_n(x)\Omega_\rho, \Omega_\rho \rangle_\rho - \langle B_k'(y_1')(B_n(x))\Omega_\rho, \Omega_\rho \rangle_\rho \right| \ (\text{by Lemma 3.2.5}) \\
&= \left| \langle (y_1' - B_k'(y_1'))B_n(x)\Omega_\rho, \Omega_\rho \rangle_\rho \right| \\
&= \left| \langle B_n'(y_1' - B_k'(y_1'))x\Omega_\rho, \Omega_\rho \rangle_\rho \right| \ (\text{by Lemma 3.2.5}) \\
&\leq \|B_n'(y_1' - B_k'(y_1')\| \, \rho(x).
\end{aligned}
$$

Further, for all $n \in \mathbb{N}$, note that

$$
\begin{aligned}
&B_n'(y_1' - B_k'(y_1')) \\
&\quad = \frac{1}{m(K_n)} \int_{K_n} \Big[\alpha_g'(y_1') - \alpha_g'(B_k'(y_1'))\Big] dm(g) \\
&\quad = \frac{1}{m(K_n)} \int_{K_n} \Big[\alpha_g'(y_1') - \alpha_g'\Big(\frac{1}{m(K_k)} \int_{K_k} \alpha_h'(y_1') dm(h)\Big)\Big] dm(g) \\
&\quad = \frac{1}{m(K_n)} \frac{1}{m(K_k)} \int_{K_n} \int_{K_k} \Big(\alpha_g'(y_1') - \alpha_{gh}'(y_1')\Big) dm(h) dm(g) \\
&\quad = \frac{1}{m(K_n)} \frac{1}{m(K_k)} \int_{K_k} \int_{K_n} \Big(\alpha_g'(y_1') - \alpha_{gh}'(y_1')\Big) dm(g) dm(h) \\
&\quad = \frac{1}{m(K_n)} \frac{1}{m(K_k)} \int_{K_k} \Big(\int_{K_n} \alpha_g'(y_1') dm(g) - \int_{K_n} \alpha_{gh}'(y_1') dm(g)\Big) dm(h) \\
&\quad = \frac{1}{m(K_n)} \frac{1}{m(K_k)} \int_{K_k} \Big(\int_{K_n} \alpha_g'(y_1') dm(g) - \int_{K_n h} \alpha_g'(y_1') dm(g)\Big) dm(h).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\|B_n'(y_1' - B_k'(y_1'))\| \\
&\quad = \left\| \frac{1}{m(K_n)} \frac{1}{m(K_k)} \int_{K_k} \Big(\int_{K_n h} \alpha_g'(y_1') dm(g) - \int_{K_n} \alpha_g'(y_1') dm(g)\Big) dm(h) \right\| \\
&\quad \leq \|y_1'\| \frac{1}{m(K_k)} \int_{K_k} \frac{m(K_n h \Delta K_n)}{m(K_n)} dm(h).
\end{aligned}
$$

Now for fixed k, consider the function $K_k \ni h \mapsto \frac{m(K_n \Delta K_n h)}{m(K_n)}$. It is a real valued measurable function defined on the compact set K_k and bounded by 2. Thus, applying DCT and Følner condition we get $\lim_{n\to\infty} \|B'_n(y'_1 - B'_k(y'_1))\| = 0$. Thus we have

$$\lim_{n\to\infty} \|B_n(\nu_k) - \bar{\nu}\| = 0.$$

Hence, $\bar{\nu}_k = \bar{\nu}$ and we finally obtain

$$\lim_{n\to\infty} \sup_{x\in M_+, x\neq 0} |(B_n(\nu_k) - \bar{\nu}_k)(x)| / \rho(x) = \lim_{n\to\infty} \sup_{x\in M_+, x\neq 0} |(B_n(\nu_k) - \bar{\nu})(x)| / \rho(x)$$
$$= 0.$$

This completes the proof. □

Let us now recall the definition of a group with polynomial growth.

Definition 3.2.10. *A locally compact group G is said to be of polynomial growth if there exists a compact generating subset V of G satisfying the following condition.*

There exists $k > 0$ and $r \in \mathbb{N}$ such that $m(V^n) \leq kn^r$ for all $n \in \mathbb{N}$.

Remark 3.2.11. *It is known from [43] that if G is a group as in Definition 3.2.10 then G is amenable and for the compact generating set V, the sequence $\{V^n\}_{n\in\mathbb{N}}$ satisfies the Følner condition in eq. 1.3.1.*

Now we like to prove a result similar to Theorem 3.2.2 for an action of a group with polynomial growth. From here onwards, we assume that it is compactly generated, locally compact and has polynomial growth and the compact generating set V is symmetric. Moreover, the ergodic averages will be considered with respect to the Følner sequence $\{V^n\}_{n\in\mathbb{N}}$ in the group case.

It is known from [10] that a locally compact group with polynomial growth is unimodular, hence for $\mu \in M_*$ and an action α of the group G on M, we have $A_n(\mu) = B_n(\mu)$, as

V^n is symmetric.

We next recall the following proposition from [18] which will be helpful in obtaining our next results.

Proposition 3.2.12. *[18, Proposition 4.8] Let G be a locally compact group with polynomial growth and let V be a compact generating set. Let α be a strongly continuous action of G on an ordered Banach space E such that $\alpha_g(x) \geq 0$ for all $g \in G$ and $x \in E_+$. Define an operator $\mathcal{T}$ on E by*

$$\mathcal{T}(x) = \frac{1}{m(V)} \int_V \alpha_g(x) dm(g), \text{ for all } x \in E_+.$$

Then, there exists a constant c only depending on G such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g(x) dm(g) \leq \frac{c}{2n^2} \sum_{k=1}^{2n^2} \mathcal{T}^k(x), \text{ for all } x \in E_+. \tag{3.2.5}$$

Theorem 3.2.2 proves a version of maximal ergodic inequality for a w-continuous linear map defined on M. Our next few results, though consequence of Theorem 3.2.2 and Proposition 3.2.12, but they establish a version of maximal ergodic inequality for group actions on M.

Theorem 3.2.13. *Let (M, G, α, ρ) be a kernel. Further assume that G is a group of polynomial growth with V being a compact symmetric generating set. Now let $\mu \in M_{*s}$ and $\epsilon > 0$, then for any $N \in \mathbb{N}$ there exists a projection $e \in M$ such that $\rho(1-e) < c \|\mu\| / \epsilon$ and*

$$|A_n(\mu)(x)| \leq \epsilon \rho(x) \text{ for all } x \in (eMe)_+ \text{ and } n \in \{1, \dots, N\},$$

where c is the constant obtained using Proposition 3.2.12.

Proof. First we prove the result for $\mu \in M_{*+}$. Consider the w-continuous positive linear operator T, defined by

$$T(x) = \frac{1}{m(V)} \int_V \alpha_g(x) dm(g), \; x \in M.$$

Since ρ is G-invariant and α is sub-unital, so T is ρ-preserving and sub-unital. Then the result for $\mu \in M_{*+}$ follows from Theorem 3.2.2 and Proposition 3.2.12.

Now suppose $\mu \in M_{*s}$, then there exist $\mu_1, \mu_2 \in M_{*+}$ such that $\mu = \mu_1 - \mu_2$ and $\|\mu\| = \|\mu_1\| + \|\mu_2\|$. Consider $\nu = \mu_1 + \mu_2$ and note that $\nu \in M_{*+}$. Then by applying the preceding case, for any $N \in \mathbb{N}$ and $\epsilon > 0$, find a projection e such that $\rho(1 - e) < \frac{c\|\nu\|}{\epsilon}$ and

$$|A_n(\nu)(x)| \leq \epsilon\rho(x) \text{ for all } x \in (eMe)_+ \text{ and } n \in \{1, \ldots, N\}.$$

Further, note that $\rho(1 - e) < \frac{c\|\nu\|}{\epsilon} \leq \frac{c(\|\mu_1\|+\|\mu_2\|)}{\epsilon} = \frac{c\|\mu\|}{\epsilon}$. Also for all $x \in (eMe)_+$ and $n \in \{1, \ldots, N\}$, we have

$$\begin{aligned} |A_n(\mu)(x)| &= |A_n(\mu_1 - \mu_2)(x)| \\ &\leq |A_n(\mu_1)(x)| + |A_n(\mu_2)(x)| = A_n(\mu_1)(x) + A_n(\mu_2)(x) \\ &= A_n(\mu_1 + \mu_2)(x) = A_n(\nu)(x) \leq \epsilon\rho(x). \end{aligned}$$

This completes the proof. □

Remark 3.2.14. *Let (M, G, α) be a covariant system and G be a group as in Theorem 3.2.13. Suppose ρ is a G-invariant f.n state on M. Consider the following average;*

$$T(a) = \frac{1}{m(V)} \int_V \alpha_g(a) dm(g), \quad a \in M.$$

As V is symmetric and G is unimodular, we have

$$T(a) = \frac{1}{m(V)} \int_V \alpha_{g^{-1}}(a) dm(g), \quad a \in M.$$

Thus, by similar discussion as in point (ii) of Remark 3.2.3, T^* becomes natural extension of T on the Hilsum L^1-space. Then one can use a result of [48] (instead of using Theorem 3.2.2) to obtain Therem 3.2.13 as discussed in point (ii) of Remark 3.2.3.

For the next set of results we assume that M is a finite von Neumann algebra with a f.n tracial state τ. In due course, we prove the main results in this section which deal with the $b.a.u$ convergence of the ergodic averages in $L^1(M,\tau)$. We start with the following lemma.

Lemma 3.2.15. *Let $Y \in L^1(M,\tau)$ be a self-adjoint element. Let $p \in \mathcal{P}(M)$ and $\delta > 0$ be such that $|\tau(Ypxp)| \leq \delta\tau(pxp)$ for all $x \in M_+$, then $pYp \in M$ and $\|pYp\| \leq \delta$.*

Proof. We first observe that $pYp \in L^1(M,\tau)$. Hence pYp is τ-measurable. Therefore, for all $n \in \mathbb{N}$ there exists a projection $e_n \in M$ such that $\tau(1-e_n) < \frac{1}{n}$ and $e_n\mathcal{H} \subseteq \mathcal{D}(pYp)$. Now for all $n \in \mathbb{N}$ and $z \in M$, observe that

$$\begin{aligned} |\tau(Ype_nzz^*e_np)| \leq \delta\tau(pe_nzz^*e_np) &\Rightarrow |\tau(e_npYpe_nzz^*)| \leq \delta\tau(e_nzz^*e_n) \\ &\Rightarrow |\langle e_npYpe_nz, z\rangle_\tau| \leq \delta\langle e_nz, e_nz\rangle_\tau \\ &\Rightarrow |\langle pYpe_nz, e_nz\rangle_\tau| \leq \delta\langle e_nz, e_nz\rangle_\tau. \end{aligned}$$

Thus, as the set $\{e_nz : n \in \mathbb{N}, z \in M\}$ is $\|\cdot\|_\tau$-dense in $L^2(M,\tau)$, so pYp is bounded and $\|pYp\|_\infty \leq \delta$. □

Theorem 3.2.16. *Let (M,G,α,ρ) be a kernel and τ be a f.n tracial state on M. Then, for any $\mu \in M_{*s}$ there exists a G-invariant $\overline{\mu} \in M_{*s}$, such that for all $\epsilon > 0$, there exists a projection $e \in M$ with $\tau(1-e) < \epsilon$ and*

$$\lim_{n\to\infty} \sup_{x \in eM_+e,\ x\neq 0} \left| \frac{(A_n(\mu) - \bar{\mu})(x)}{\tau(x)} \right| = 0.$$

Proof. First we note that, since $\rho \in M_{*+}$, there exists a unique $X \in L^1(M,\tau)_+$ such that $\rho(x) = \tau(Xx)$ for all $x \in M$. Then for any $s > 0$ consider the projection $q_s := \chi_{(1/s,s)}(X) \in M$. Observe that $(1-q_s) \xrightarrow{s\to\infty} 0$ in SOT and hence there exists a $s_0 > 0$ such that $\tau(1-q_{s_0}) < \epsilon/2$. Further, it implies $Xq_{s_0} \leq s_0q_{s_0}$. Thus, for all $0 \neq x \in (q_{s_0}Mq_{s_0})_+$

we have

$$\begin{aligned}\frac{\rho(x)}{\tau(x)} = \frac{\tau(Xx)}{\tau(x)} &= \frac{\tau(Xq_{s_0}x)}{\tau(x)} \quad \text{(since } q_{s_0}x = x\text{)} \\ &\leq \frac{\tau(s_0 x)}{\tau(x)} = s_0.\end{aligned} \tag{3.2.6}$$

Now we use Lemma 3.2.9 recursively to obtain a sequence $\{\nu_{n_1}, \nu_{n_2}, \cdots\} \subseteq \mathcal{W}$ satisfying

$$\begin{aligned}&(1)\ n_1 < n_2 < \cdots, \\ &(2)\ \left\|\mu - \nu_{n_j}\right\| < \frac{1}{4^j c} \text{ for all } j \in \mathbb{N}, \text{ and,} \\ &(3)\ \sup_{x \in M_+, x \neq 0} \frac{\left|(A_n(\nu_{n_j}) - \bar{\nu}_{n_j})(x)\right|}{\rho(x)} < \frac{1}{2^j} \text{ for all } n \geq n_j.\end{aligned} \tag{3.2.7}$$

Further, note that $\mu - \nu_{n_j}, \bar{\mu} - \bar{\nu}_{n_j} \in M_{*s}$ for all $j \in \mathbb{N}$. For every $j \in \mathbb{N}$, take a $N_j \in \mathbb{N}$ and use the Theorem 3.2.13 to get sequences of projections $\{e_1, e_2, \ldots\}$ and $\{f_1, f_2, \ldots\}$ in M such that

$$\begin{aligned}&(1)\ \rho(1 - e_j) < \frac{1}{2^j} \text{ and } \rho(1 - f_j) < \frac{1}{2^j}, \text{ for all } j \in \mathbb{N}, \\ &(2)\ \sup_{x \in e_j M_+ e_j, x \neq 0} \frac{\left|A_n(\mu - \nu_{n_j})(x)\right|}{\rho(x)} < \frac{1}{2^{j-1}} \text{ for all } n \in \{1, \ldots, N_j\}, \\ &(3)\ \sup_{x \in f_j M_+ f_j, x \neq 0} \left|\frac{A_n(\bar{\mu} - \bar{\nu}_{n_j})(x)}{\rho(x)}\right| < \frac{1}{2^{j-1}} \text{ for all } n \in \{1, \ldots, N_j\}.\end{aligned} \tag{3.2.8}$$

Now it immediately follows that both $\rho(1 - e_j)$ and $\rho(1 - f_j)$ converges to 0 as j tends to infinity. Therefore, both $\tau(1 - e_j)$ and $\tau(1 - f_j)$ converges to 0 as j tends to infinity. Hence choose a subsequence $(j_k)_{k \in \mathbb{N}}$ such that

$$\tau(1 - e_{j_k}) < \frac{\epsilon}{2^{k+2}} \text{ and } \tau(1 - f_{j_k}) < \frac{\epsilon}{2^{k+2}}. \tag{3.2.9}$$

Now consider $e := \wedge_{k \geq 1}(e_{j_k} \wedge f_{j_k}) \wedge q_{s_0}$ and observe that

$$\tau(1 - e) \leq \sum_{k \geq 1}(\tau(1 - e_{j_k}) + \tau(1 - f_{j_k})) + \tau(1 - q_s) < \epsilon.$$

Therefore, for all $0 \neq x \in eM_+e$, and $n, n_k \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{|(A_n(\mu) - \bar{\mu})(x)|}{\tau(x)} \\
\leq& \Big(\frac{|(A_n(\mu - \nu_{n_k})(x)|}{\rho(x)} + \frac{|(\bar{\mu} - \bar{\nu}_{n_k})(x)|)}{\rho(x)} + \frac{|(A_n(\nu_{n_k}) - \bar{\nu}_{n_k})(x)|}{\rho(x)} \Big) \frac{\rho(x)}{\tau(x)} \\
\leq& s_0 \Big(\big(\frac{|A_n(\mu - \nu_{n_k})(x)|}{\rho(x)} + \frac{|A_n(\bar{\mu} - \bar{\nu}_{n_k})(x)|}{\rho(x)} \big) + \frac{|(A_n(\nu_{n_k}) - \bar{\nu}_{n_k})(x)|}{\rho(x)} \Big) \\
&(\text{ by eq. 3.2.6, } \frac{\rho(x)}{\tau(x)} \leq s_0, \text{ as } e \leq q_{s_0}) \\
\leq& s_0 \Big(\big(\frac{|A_n(\mu - \nu_{n_k})(x)|}{\rho(x)} + \frac{|A_n(\bar{\mu} - \bar{\nu}_{n_k})(x)|}{\rho(x)} \big) + \sup_{x \in M_+, x \neq 0} \frac{|(A_n(\nu_{n_k}) - \bar{\nu}_{n_k})(x)|}{\rho(x)} \Big).
\end{aligned}$$

Hence, by taking supremum over the set $eM_+e \setminus \{0\}$ on both sides of the above inequality and applying eq. 3.2.7 and eq. 3.2.8 we obtain

$$\lim_{n\to\infty} \sup_{x \in eM_+e, x\neq 0} \frac{|(A_n(\mu) - \bar{\mu})(x)|}{\tau(x)} = 0.$$

This completes the proof. □

Theorem 3.2.17. *Let (M, G, α, ρ) be a kernel and τ be a f.n tracial state on M. Further assume that G is a group of polynomial growth with a symmetric compact generating set V. Then for all $Y \in L^1(M, \tau)$, there exists $\overline{Y} \in L^1(M, \tau)$ such that $A_n(Y)$ converges to $\overline{Y}$ bilaterally almost uniformly.*

Proof. Let $Y \in L^1(M, \tau)_s$. Then there exists a unique $\mu \in M_{*s}$ such that $\mu(x) = \tau(Yx)$ for all $x \in M$. Apply Theorem 3.2.6 and define $\bar{\mu} \in M_{*s}$ as

$$\bar{\mu} := \|\cdot\|_1 - \lim_{n\to\infty} A_n(\mu).$$

Hence, there exists a unique $\bar{Y} \in L^1(M, \tau)_s$ such that $\bar{\mu}(x) = \tau(\bar{Y}x)$ for all $x \in M$. Note that $\bar{\mu}$ is G-invariant. Let $\epsilon, \delta > 0$. Then by Theorem 3.2.16 there exists a projection $e \in M$ with $\tau(1 - e) < \epsilon$ and there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in (eMe)_+ \setminus \{0\}} \left| \frac{(A_n(\mu) - \bar{\mu})(x)}{\tau(x)} \right| < \delta, \text{ for all } n \geq N.$$

Therefore, for all $n \geq N$ we have

$$|(A_n(\mu) - \bar{\mu})(x)| < \delta\tau(x) \text{ for all } x \in (eMe)_+ \setminus \{0\}$$
$$\Rightarrow \left|\tau((A_n(Y) - \overline{Y})eye)\right| < \delta\tau(eye) \text{ for all } y \in M_+ \setminus \{0\}$$
$$\Rightarrow \left|\tau(e(A_n(Y) - \overline{Y})ey)\right| < \delta\tau(eye) \text{ for all } y \in M_+ \setminus \{0\}.$$

Hence by Lemma 3.2.15, we have $\left\|e(A_n(Y) - \overline{Y})e\right\| \leq \delta$ for all $n \geq N$. Now let $Y \in L^1(M,\tau)$ and write $Y = Y_1 + iY_2$ where $Y_1, Y_2 \in L^1(M,\tau)_s$, then the rest follows from Proposition 1.2.12. □

For our next result we consider the action α of G on $L^1(M,\tau)$. We recall the discussion at the beginning of section 3.2 and observe that α induces an action of the group G, namely the dual action, on the von Neumann algebra M, and it is defined by $g \mapsto \alpha^*_{g^{-1}}$ for all $g \in G$.

Theorem 3.2.18. *Let τ be a f.n tracial state on M and $(L^1(M,\tau), G, \alpha)$ be a non-commutative dynamical system satisfying the following:*

1. *It has polynomial growth and V being its compact symmetric generating set, and*
2. *there exists a f.n state ρ on M such that (M, G, α^*, ρ) is a kernel.*

Then, for all $Y \in L^1(M,\tau)$ there exists $\overline{Y} \in L^1(M,\tau)$ such that $A_n(Y)$ converges to $\overline{Y}$ in $b.a.u.$

Proof. First note that $\hat{\alpha^*_g} = \alpha_g$ for all $g \in G$. Then the result follows by applying Theorem 3.2.17 on the action β, where $\beta = (\alpha^*_{g^{-1}})$. □

Remark 3.2.19. *Note that if $\sup_{g\in G} \|\alpha_g\| \leq 1$, then $\left\|\alpha^*_g(1)\right\| \leq 1$. Therefore, $\alpha^*_g(1) \leq 1$. Thus, the condition $\sup_{g\in G} \|\alpha_g\| \leq 1$ together with ρ is G-invariant can be put in Theo-*

rem 3.2.18 instead of assuming (M, G, α^*, ρ) is a kernel. However, former conditions are stronger than assuming (M, G, α^*, ρ) is a kernel.

Corollary 3.2.20. *Let τ be a f.n tracial state on M and $(L^1(M,\tau), G, \alpha)$ be a non-commutative dynamical system. Assume the following.*

1. *G has polynomial growth and V be a compact symmetric generating set,*
2. *$\alpha_g(1) = 1$ and $\alpha_g^*(1) \leq 1$ for all $g \in G$.*

Then, for all $Y \in L^1(M,\tau)$ there exists $\overline{Y} \in L^1(M,\tau)$ such that $A_n(Y)$ converges b.a.u. to $\overline{Y}$.

Proof. Since τ is finite, observe that $1 \in L^1(M,\tau)$ and it follows from eq. 3.1.1 that

$$\tau(\alpha_g^*(x)) = \tau(x) \text{ for all } x \in M \text{ and } g \in G.$$

Hence, (M, G, α^*, τ) becomes a kernel and thus the result follows from Theorem 3.2.18. □

Remark 3.2.21. *In [46], the author considered a positive linear transformation α on $L^1(M,\tau)$ (here M is a von Neumann algebra with f.n.s trace τ) satisfying $0 \leq \alpha(X) \leq 1$ and a sub traciality condition (i.e, $\tau(\alpha(X)) \leq \tau(X)$) for all $X \in L^1(M,\tau) \cap M$ and $0 \leq X \leq 1$. Under these conditions, for all $X \in L^1(M,\tau)$ the author proved b.a.u convergence of the averages $A_n(X) := \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(X)$.*

Later on, in [18], the authors considered a strongly continuous action α of G (of polynomial growth and having symmetric, compact generating set) of τ-preserving automorphisms on M, where τ is a f.n.s trace on M. Then the author extended this action to $L^1(M,\tau)$ and generalized the results of [46] by proving the almost uniform convergence of the averages of the form $A_n(X) := \frac{1}{m(V^n)}\int_{V^n} \alpha_g(X)dm(g)$, for $X \in L^1(M,\tau)$.

Note that, in Theorem 3.2.17, we do not necessarily assume that the trace τ is preserved by the action α on M, but we assume that it is preserved by a f.n state. As a result, we work with the predual action on $L^1(M,\tau)$ and proved similar theorems. Note that the predual action may not be an extension of α. Further, we do not compulsorily assume that the predual action is an extension of a τ-preserving automorphic action on M.

Theorem 3.2.18 and Corollary 3.2.20 mainly focus on the action of G (G is a group with polynomial growth and having symmetric, compact generating set) on $L^1(M,\tau)$. In Theorem 3.2.18 and Corollary 3.2.20, we assume some natural conditions to make the dual action a kernel. In both cases, we showed b.a.u convergence on $L^1(M,\tau)$. The techniques that are used in proving both these results are independent of the techniques that are used in [18] and it reflects the fact that τ-preserving condition is not necessary in Theorem 3.2.18.

3.3 Action of semigroup

In this section, we will consider G to be a semigroup and m be a σ-finite measure on G which is both left and right invariant (i.e. $m(uB) = m(B)$ and $m(Bu) = m(B)$ for all $u \in G$). In the sequel, $\mathbb{K}$ will always denote either $\mathbb{Z}_+$ or $\mathbb{R}_+$. We consider a collection $\{I_l\}_{l\in\mathbb{K}}$ of measurable subsets of G having the following properties.

(P1) $0 < m(I_l) < \infty$ for all $l \in \mathbb{K}$.

(P2) $\lim_{l\to\infty} \frac{m(I_l \Delta I_l u)}{m(I_l)} = 0$ and $\lim_{l\to\infty} \frac{m(I_l \Delta u I_l)}{m(I_l)} = 0$ for all $u \in G$.

Definition 3.3.1. *Let E be an ordered Banach space. A map Λ defined by*

$$G \ni g \xrightarrow{\Lambda} \Lambda_g \in \mathcal{B}(E)$$

is called an action if $\Lambda_g \circ \Lambda_h = \Lambda_{gh}$ for all $g, h \in G$. It is called anti-action if $\Lambda_g \circ \Lambda_h = \Lambda_{hg}$ for all $g, h \in G$. In this article, we consider both actions and anti-actions $\Lambda = \{\Lambda_g\}_{g\in G}$ which satisfy the following conditions.

(C) *For all* $x \in E$*, the map* $g \to \Lambda_g(x)$ *from* G *to* E *is continuous. Here we take* w^**-topology when* $E = M$ *and norm topology otherwise.*

(UB) $\sup_{g \in G} \|\Lambda_g\| < \infty$.

(P) *For all* $g \in G$ *and* $x \in E$ *with* $x \geq 0$*,* $\Lambda_g(x) \geq 0$*.*

We refer the triple (E, G, Λ) *as a non-commutative dynamical system.*

Lemma 3.3.2. *Let* (M, G, T) *be a non-commutative dynamical system and* ρ *be a* G*-invariant f.n state on* M*. Then there exists a non-commutative dynamical system* (M', G, T') *such that*

$$\langle T'_g(y')x\Omega_\rho, \Omega_\rho\rangle_\rho = \langle y'T_g(x)\Omega_\rho, \Omega_\rho\rangle_\rho \text{ for all } x \in M, y' \in M', g \in G.$$

Proof. For $y' \in M'_+$ and $g \in G$, consider the linear functional $\nu^g_{y'} : M \to \mathbb{C}$ defined by

$$\nu^g_{y'}(x) = \langle y'T_g(x)\Omega_\rho, \Omega_\rho\rangle_\rho.$$

For $x \in M_+$, note that

$$\begin{aligned}
\nu^g_{y'}(x) &= \langle y'T_g(x)\Omega_\rho, \Omega_\rho\rangle_\rho \\
&\leq \|y'\|_\infty \langle T_g(x)\Omega_\rho, \Omega_\rho\rangle_\rho \\
&= \|y'\|_\infty \rho(T_g(x)) \\
&= \|y'\|_\infty \rho(x) \\
&= \|y'\|_\infty \langle x\Omega_\rho, \Omega_\rho\rangle_\rho.
\end{aligned}$$

Hence, by Lemma 3.2.4 there exists a unique $z' \in M'$ such that $\nu^g_{y'}(x) = \langle z'x\Omega_\rho, \Omega_\rho\rangle_\rho$. Write $z' = T'_g(y')$ and then we have $\langle y'T_g(x)\Omega_\rho, \Omega_\rho\rangle_\rho = \langle T'_g(y')x\Omega_\rho, \Omega_\rho\rangle_\rho$. □

The following is the mean ergodic type theorem for the action of semigroup. This will be useful for the subsequent results.

Theorem 3.3.3. *Let (M, G, T) be a non-commutative dynamical system. Suppose there exists a f.n state ρ satisfying $\rho(T_g(x)^2) \leq \rho(x^2)$ for all $x \in M_s$ and $g \in G$. Then for all $\mu \in M_*$, there exists a $\bar{\mu} \in M_*$ such that*

$$\bar{\mu} = \|\cdot\|_1 - \lim_{l \to \infty} B_l(\mu),$$

where for $l \in \mathbb{K}$, $B_l(\mu) := \frac{1}{m(I_l)} \int_{I_l} \beta_g(\mu) dm(g)$ and $\beta_g(\mu) = \mu \circ T_g$ for all $g \in G$ and $\mu \in M_*$.

Proof. Let $L^2(M_s, \rho)$ be the closure of M_s with respect to the norm induced from the inner product $\langle \cdot, \cdot \rangle_\rho$. Then we can define the following contractions on the Hilbert space $L^2(M_s, \rho)$.

$$u_g(x\Omega_\rho) = T_g(x)\Omega_\rho, x \in M_s, g \in G.$$

For $l \in \mathbb{K}$, consider $T_l := \frac{1}{m(I_l)} \int_{I_l} u_g^* \, dm(g)$. Then by von Neumann mean ergodic theorem, it follows that for all $\xi \in L^2(M_s, \rho)$, $T_l(\xi)$ converges to $P\xi$ strongly, where P is the orthogonal projection of $L^2(M_s, \rho)$ onto the subspace $\{\xi \in L^2(M_s, \rho) : u_g^* \xi = \xi \text{ for all } g \in G\}$. Now let $x \in M$. Further write x as $x_1 + ix_2$, where $x_1, x_2 \in M_s$ and then by the previous argument, it follows that $T_l(x\Omega) := T_l(x_1\Omega_\rho) + iT_l(x_2\Omega_\rho)$ converges in $L^2(M, \rho)$.
Let $y_1, y_2 \in M'(\sigma)$ and define $\psi_{y_1,y_2}(x) = \langle xy_1\Omega_\rho, y_2\Omega_\rho \rangle$ for all $x \in M$. Then there exists a $z \in M$ such that $y_1^* y_2 \Omega_\rho = z\Omega_\rho$, where $z = J\sigma_{i/2}(y_2^* y_1)J$. Consequently for $x \in M$,

$$\begin{aligned} B_l(\psi_{y_1,y_2})(x) &= \frac{1}{m(I_l)} \int_{I_l} \langle T_g(x)y_1\Omega_\rho, y_2\Omega_\rho \rangle_\rho \, dm(g) \\ &= \frac{1}{m(I_l)} \int_{I_l} \langle T_g(x)\Omega_\rho, y_1^* y_2 \Omega_\rho \rangle_\rho \, dm(g) \\ &= \frac{1}{m(I_l)} \int_{I_l} \langle T_g(x)\Omega_\rho, z\Omega_\rho \rangle_\rho \, dm(g) \\ &= \frac{1}{m(I_l)} \int_{I_l} \left\langle x\Omega_\rho, u_g^*(z\Omega_\rho) \right\rangle_\rho dm(g) \\ &= \langle x\Omega_\rho, T_l(z\Omega_\rho) \rangle_\rho . \end{aligned}$$

Hence, for all $x \in M$, $B_l(\psi_{y_1,y_2})(x) \to \langle x\Omega_\rho, \eta\rangle_\rho$, where $\eta = \lim_{l\to\infty} T_l(z\Omega_\rho)$. Consider $\overline{\psi}_{y_1,y_2} \in M_*$, defined by

$$\overline{\psi}_{y_1,y_2}(x) = \langle x\Omega_\rho, \eta\rangle_\rho, \;\; x \in M.$$

Then by standard argument it follows that $\overline{\psi}_{y_1,y_2} \circ T_g = \overline{\psi}_{y_1,y_2}$ for all $g \in G$ and

$$\overline{\psi}_{y_1,y_2} = \|\cdot\|_1 - \lim_{l\to\infty} B_l(\psi_{y_1,y_2}).$$

Hence the result follows since the set $\{\psi_{y_1,y_2} : y_1, y_2 \in M'(\sigma)\}$ is total in M_*. □

In this section we consider state preserving actions of $\mathbb{Z}_+^d$ and $\mathbb{R}_+^d$ on a von Neumann algebra M. Then prove a version of maximal inequality for the induced action on the predual von Neumann algebra M_*.

Definition 3.3.4. *Let (M, G, T) be a non-commutative dynamical system and ρ be a f.n. state on M. Then (M, G, T, ρ) is called kernel if*

1. *ρ is G-invariant and*
2. *T is sub-unital, i.e, $T_g(1) \leq 1$ for all $g \in G$.*

3.3.1 Action of $\mathbb{Z}_+^d$

In this subsection, we consider $(M, \mathbb{Z}_+^d, T)$. Now note that there exists positive d-commuting maps $T_1, T_2, \cdots, T_d$ on M such that $T_{(i_1,\cdots,i_d)}(\cdot) = T_1^{i_1}T_2^{i_2}\cdots T_d^{i_d}(\cdot)$ for $(i_1, \cdots, i_d) \in \mathbb{Z}_+^d$. Then we prove a maximal inequality for the induced action on M_*. An analogue of the following result is proved in [23, Theorem 3.4, pp-213] for the case of classical L^1 spaces. In our setup we require the exact same result in a general ordered Banach space E. Although the proof is similar, we write it for the sake of completeness. We recall the following known result which is the main ingredient in this context.

Lemma 3.3.5. *[23] Let* $\xi(x) = 1 - \sqrt{1-x}$ *for* $0 \le x \le 1$. *For* $n \in \mathbb{N}$, *write* $[\xi(x)]^n = \sum_{p=0}^{\infty} \alpha_p^{(n)} x^p$. *Then*

(i) $\alpha_p^{(n)} = \begin{cases} 0 & \text{for } p < n, \\ \frac{n}{2p} 2^{n+1-2p} \binom{2p-n-1}{p-1} & \text{for } p \ge n. \end{cases}$

(ii) If $\varphi(n)$ *denotes the greatest integer less than or equal to* $\sqrt{n}+1$, *then there exists* $c > 0$ *such that*

$$\frac{1}{\varphi(n)} \sum_{0 \le j < \varphi(n)} \alpha_{v+j}^{(j)} \alpha_{w+j}^{(j)} \ge \frac{c}{n^2} \text{ holds for all } 0 \le v, w < n.$$

First we fix the following notation. For for an integer $d > 1$ notice that there is an unique $m \in \mathbb{N}$ such that $2^{m-1} < d \le 2^m$. For $n \in \mathbb{N}$ and $d > 1$ we fix $n_d := \varphi^m(n)$, where $\varphi^m := \varphi \circ \cdots \circ \varphi$ (m times).

Theorem 3.3.6. *Let* E *be an ordered Banach space and* $d > 1$ *be an integer. Then there exists* $\chi_d > 0$ *and a family* $\{a(u) : u = (u_1, \ldots, u_d) \in \mathbb{Z}_+^d\}$ *of strictly positive numbers summing to* 1 *such that the following holds: If* $T_1, \ldots, T_d$ *are commuting positive contractions of* E, *then the operator*

$$U = \sum_{u \in Z_+^d} a(u) T_1^{u_1} \cdots T_d^{u_d}$$

satisfies

$$\frac{1}{n^d} \sum_{0 \le i_1 < n} \cdots \sum_{0 \le i_d < n} T_1^{i_1} \cdots T_d^{i_d} f \le \frac{\chi_d}{n_d} \sum_{j=0}^{n_d - 1} U^j f$$

for all $n \in \mathbb{N}$ *and* $f \in E_+$.

Proof. It is enough to consider $d = 2^m$ for some positive integer m. Because if $d < 2^m$, one can put $T_j = I$ for all $j \in \{d+1, \ldots, 2^m\}$.

Let us denote by $\overline{\xi}(x) = \frac{\xi(x)}{x}$ for $0 < x \le 1$. Then, by (i) of Lemma 3.3.5 we have

$$\overline{\xi}(x) = \sum_{\lambda=0}^{\infty} \alpha_{\lambda+1}^{(1)} x^{\lambda}.$$

Now if T is any contraction, then $\overline{\xi}(T) = \sum_{\lambda=0}^{\infty} \alpha_{\lambda+1}^{(1)} T^{\lambda}$ is again a contraction. Consequently, since $\alpha_{\lambda}^{(j)} = 0$ for all $\lambda < j$ we have

$$(\overline{\xi}(T))^j = \sum_{\lambda=0}^{\infty} \alpha_{j+\lambda}^{(j)} T^{\lambda}$$

for every $j \in \mathbb{N}$. When $d = 2$ we first consider the contraction

$$U = \overline{\xi}(T_1) \circ \overline{\xi}(T_2).$$

Note that since T_1 and T_2 commutes we have

$$\begin{aligned}
\frac{1}{\varphi(n)} \sum_{j<\varphi(n)} U^j &\geq \frac{1}{\varphi(n)} \sum_{0\leq i_1,i_2<n} \sum_{j<\varphi(n)} \alpha_{j+i_1}^{(j)} \alpha_{j+i_1}^{(j)} T_1^{i_1} T_2^{i_2} \\
&\geq \frac{c}{n^2} \sum_{0\leq i_1,i_2<n} T_1^{i_1} T_2^{i_2} \quad \text{(by Lemma 3.3.5 } (ii)).
\end{aligned}$$

Therefore, the result holds for $d = 2$ with $\chi_d = 1/c$. Next, assume that $d = 4$. In this case consider the contraction

$$U = \overline{\xi}(U_1) \circ \overline{\xi}(U_2),$$

where $U_1 = \overline{\xi}(T_1) \circ \overline{\xi}(T_2)$ and $U_2 = \overline{\xi}(T_3) \circ \overline{\xi}(T_4)$. Hence, we obtain

$$\begin{aligned}
\frac{1}{\varphi(\varphi(n))} \sum_{0\leq j<n_4} U^j &\geq \frac{1}{\varphi(\varphi(n))} \sum_{0\leq j<\varphi(n)} U^j \\
&\geq \frac{1}{\varphi(\varphi(n))} \sum_{0\leq i_1,i_2<\varphi(n)} \sum_{j<\varphi(\varphi(n))} \alpha_{j+i_1}^{(j)} \alpha_{j+i_1}^{(j)} U_1^{i_1} U_2^{i_2} \\
&\geq \frac{c}{\varphi(n)^2} \sum_{0\leq i_1,i_2<\varphi(n)} U_1^{i_1} U_2^{i_2} \quad \text{(by Lemma 3.3.5 } (ii)) \\
&\geq c\Big(\frac{1}{\varphi(n)} \sum_{0\leq i_1<\varphi(n)} U_1^{i_1}\Big) \circ \Big(\frac{1}{\varphi(n)} \sum_{0\leq i_2<\varphi(n)} U_2^{i_2}\Big) \\
&\geq \frac{c^3}{n^4} \sum_{0\leq i_1,i_2,i_3,i_4<n} T_1^{i_1} T_2^{i_2} T_3^{i_3} T_4^{i_4}.
\end{aligned}$$

Therefore, the result holds for $d = 3, 4$ with $\chi_3 = \frac{1}{c^3}$ and $\chi_4 = \frac{1}{c^3}$ respectively.

Now if for $d > 1$ the contractions $\{T_1, \dots, T_{2d}\}$ are given, then following the method described above, first construct a contraction U_1 using $\{T_1, \dots, T_d\}$, and then construct another contraction U_2 using $\{T_{d+1}, \dots, T_{2d}\}$. Finally consider $U := \overline{\xi}(U_1) \circ \overline{\xi}(U_2)$.

Moreover, observe that it follows from the construction that

$$U = \sum_{u \in Z_+^d} a(u) T_1^{u_1} \cdots T_d^{u_d},$$

where $a(u) > 0$ for all $u \in \mathbb{Z}_+^d$ and $\sum_{u \in Z_+^d} a(u) = 1$. □

Definition 3.3.7. *For $n \in \mathbb{Z}_+$,*

$$A_n(f) := \frac{1}{n^d} \sum_{0 \le i_1 < n} \cdots \sum_{0 \le i_d < n} T_1^{i_1} \cdots T_d^{i_d}(f), \;\; f \in E.$$

Theorem 3.3.8. *Let $(M, \mathbb{Z}_d^+, T, \rho)$ be a kernel. Also assume $\mu \in M_{*s}$ and $\epsilon > 0$, then for any $N \in \mathbb{N}$ there exists a projection $e \in M$ such that $\rho(1-e) < \frac{\chi_d \|\mu\|}{\epsilon}$ and*

$$|A_n(\mu)(x)| \le \epsilon \rho(x) \text{ for all } x \in (eMe)_+ \text{ and } n \in \{1, \dots, N\}.$$

Proof. Let $\epsilon > 0$ and $N \in \mathbb{N}$. First consider $\mu \in M_{*+}$. Then there exists $e \in \mathcal{P}_0(M)$ such that $\rho(1-e) < \frac{\chi_d \|\mu\|}{\epsilon}$ and

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} U^j(\mu)(x) \right| \le \frac{\epsilon}{\chi_d} \rho(x) \text{ for all } x \in (eMe)_+ \text{ and } n \in \{1, \dots, N_d\}.$$

Now consider $n \in \{1, \dots, N\}$. Note that $n_d \le N_d$. Therefore,

$$A_n(\mu)(x) \le \frac{\chi_d}{n_d} \sum_{j=0}^{n_d - 1} U^j(\mu)(x) \le \epsilon \rho(x) \text{ for all } x \in (eMe)_+.$$

This completes the proof □

3.3.2 Action of $\mathbb{R}_+^d$

In this subsection, we consider $(M, \mathbb{R}_+^d, T)$ and deduce a maximal inequality for the induced action on the predual M_*. We follow the same technique as it is obtained in [2] for the case of classical L^1- spaces to establish our case for the interest to ameliorate the exposition of this article.

Definition 3.3.9. *Let* $a \in \mathbb{R}_+$ *and define the set* $Q_a := \{(t_1, \ldots, t_d) \in \mathbb{R}_+^d : t_1 < a, \ldots, t_d < a\}$. *Then consider the following averages.*

$$M_a(f) := \frac{1}{a^d} \int_{Q_a} T_t(f) dt.$$

Remark 3.3.10. *In particular, when* $d > 1$ *be an integer and* $n \in \mathbb{Z}_+$*, note that*

$$M_n(f) := \frac{1}{n^d} \sum_{(j_1,\ldots,j_d) \in \mathbb{Z}_+^d \cap Q_n} \int_{Q_{j_1,\ldots,j_d}} T_t(f) dt,$$

where for any $(j_1, \ldots, j_d) \in \mathbb{Z}_+^d$, $Q_{j_1,\ldots,j_d} := \{(t_1, \ldots, t_d) \in \mathbb{R}_+^d : j_1 \leq t_1 < j_1 + 1, \ldots, j_d \leq t_d < j_d + 1\}$. *It also follows that*

$$M_1(f) = \int_{Q_{0,\ldots,0}} T_t(f) dt = \int_{[0,1)^d} T_t(f) dt.$$

Further, denoting $S_1 = T_{(1,0,\ldots,0,0)}, \; S_2 = T_{(0,1,0,\ldots,0)}, \; \cdots, S_d = T_{(0,0,\ldots,0,1)}$ *we observe that*

$$M_n(f) := \frac{1}{n^d} \sum_{(j_1,\ldots,j_d) \in \mathbb{Z}_+^d \cap Q_n} S_1^{j_1} \cdots S_d^{j_d}(M_1(f)).$$

Theorem 3.3.11. *Let* E *be an ordered Banach space and* $d > 1$ *be an integer. Then for all* $a > 0$ *there exists* $n \in \mathbb{N}$ *such that*

$$M_a(f) \leq \frac{1}{n^d} \sum_{(j_1,\ldots,j_d) \in \mathbb{Z}_+^d \cap Q_{n+1}} S_1^{j_1} \cdots S_d^{j_d} M_1(f) = \frac{1}{n^d} \sum_{0 \leq j_1 < n+1} \cdots \sum_{0 \leq j_d < n+1} S_1^{j_1} \cdots S_d^{j_d} M_1(f)$$

for all $f \in E_+$.

Proof. Let $a > 0$. Choose $n = [a]$. Then

$$M_a(f) \leq \frac{1}{n^d} \int_{Q_{n+1}} T_t(f) dt$$

$$= \frac{1}{n^d} \sum_{(j_1,\ldots,j_d)\in\mathbb{Z}_+^d\cap Q_{n+1}} \int_{Q_{j_1,\ldots,j_d}} T_t(f)dt$$
$$= \frac{1}{n^d} \sum_{(j_1,\ldots,j_d)\in\mathbb{Z}_+^d\cap Q_{n+1}} S_1^{j_1}\cdots S_d^{j_d}(M_1(f))$$
$$= \frac{1}{n^d} \sum_{0\le j_1<n+1} \cdots \sum_{0\le j_d<n+1} S_1^{j_1}\cdots S_d^{j_d} M_1(f).$$

□

Proposition 3.3.12. *Let E be an ordered Banach space and $d > 1$ be an integer. Then there exists $\chi_d > 0$ and a family $\{a(u) : u = (u_1, \ldots, u_d) \in \mathbb{Z}_+^d\}$ of strictly positive numbers summing to 1 such that the following holds:*

$$U = \sum_{u\in Z_+^d} a(u) S_1^{u_1}\cdots S_d^{u_d}$$

and for all $a > 0$ and $f \in E_+$

$$M_a(f) \le \frac{\chi_d}{([a]+1)_d} \sum_{j=0}^{([a]+1)_d-1} U^j M_1(f).$$

Proof. It is clear from Theorem 3.3.6 and Theorem 3.3.11. □

Theorem 3.3.13. *Let $(M, \mathbb{R}_+^d, T, \rho)$ be a kernel. Also assume $\mu \in M_{*s}$ and $\epsilon > 0$. Then for any $N \in \mathbb{N}$ there exists a projection $e \in M$ such that $\rho(1-e) < \frac{\chi_d\|\mu\|}{\epsilon}$ and*

$$|M_a(\mu)(x)| \le \epsilon\rho(x) \text{ for all } x \in (eMe)_+ \text{ and } 1 \le a \le N.$$

Proof. Let $\epsilon > 0$ and $N \in \mathbb{N}$. Enough to consider $\mu \in M_{*+}$. Then there exists $e \in \mathcal{P}_0(M)$ such that $\rho(1-e) < \frac{\chi_d\|\mu\|}{\epsilon}$ and

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} U^j(M_1(\mu))(x)\right| \le \frac{\epsilon}{\chi_d}\rho(x) \text{ for all } x \in (eMe)_+ \text{ and } n \in \{1, \ldots, (N+1)_d\}.$$

Now consider $1 \le a \le N$. Note that $[a]_d \le N_d$. Therefore,

$$M_a(\mu)(x) \le \frac{\chi_d}{([a]+1)_d} \sum_{j=0}^{([a]+1)_d-1} U^j(M_1(\mu))(x) \le \epsilon\rho(x) \text{ for all } x \in (eMe)_+.$$

This completes the proof. □

We assume that G is either $\mathbb{Z}_+^d$ or $\mathbb{R}_+^d$. Let (G, T, M) be a non-commutative dynamical system and we summarize the notation as follows.

$$M_a(\cdot) := \begin{cases} \frac{1}{a^d}\sum_{0\leq i_1<a}\cdots\sum_{0\leq i_d<a} T_1^{i_1}\cdots T_d^{i_d}(\cdot) & \text{when } G = \mathbb{Z}_+^d,\ a\in\mathbb{N}, \\ \frac{1}{a^d}\int_{Q_a} T_t(\cdot)dt & \text{when } G = \mathbb{R}_+^d,\ a\in\mathbb{R}_+. \end{cases}$$

We also combine the obtained maximal inequality as follows;

Theorem 3.3.14. *Let G be either $\mathbb{Z}_+^d$ or $\mathbb{R}_+^d$ and (M, G, T, ρ) be a kernel. Let $\mu \in M_{*s}$ and $\epsilon > 0$, then for any $N \in \mathbb{N}$ there exists a projection $e \in M$ such that $\rho(1-e) < \frac{\chi_d\|\mu\|}{\epsilon}$ and*

$$|M_a(\mu)(x)| \leq \epsilon\rho(x) \text{ for all } x \in (eMe)_+ \text{ and } 1 \leq a \leq N.$$

In this section, we assume that M is finite von Neumann algebra with a faithful normal tracial state τ. We also assume G is either $\mathbb{Z}_+^d$ or $\mathbb{R}_+^d$ and (G, T, M) be a non-commutative dynamical system. Then we prove a non-commutative version of pointwise ergodic theorem for the induced action on the pre-dual M_*.

Let (M, G, T, ρ) be a kernel. Then by Remark 3.2.8 and Theorem 3.2.6, it follows that for all $\mu \in M_*$, $\lim_{n\to\infty} B_n(\mu)$ exists in $\|\cdot\|_1$ in M_*. We denote the limit by $\bar{\mu}$.

Lemma 3.3.15. *Let (M, G, T, ρ) be a kernel. Consider the following set*

$$\mathcal{W}_1 := \{\nu - B_k(\nu) + \bar{\nu} : \quad k \in \mathbb{K},\ \nu \in M_{*+} \text{ with } \nu \leq \lambda\rho \text{ for some } \lambda > 0\}.$$

(i) Write $\mathcal{W} = \mathcal{W}_1 - \mathcal{W}_1$, then $\mathcal{W}$ is dense in M_{*s} and

(ii) for all $\nu \in \mathcal{W}$, we have

$$\lim_{l\to\infty} \sup_{x\in M_+, x\neq 0} |(B_l(\nu) - \bar{\nu})(x)| / \rho(x) = 0. \tag{3.3.1}$$

Proof. *(i):* Let $\mu \in M_{*+}$ and $\epsilon > 0$. From Theorem 2.2.2, find a $\nu \in M_{*+}$ with $\nu \leq \lambda\rho$ for some $\lambda > 0$, such that $\|\mu - \nu\| < \epsilon/2$. Further, by Theorem 3.2.6 we know that

$B_l(\nu)$ is convergent, and write $\bar{\nu} = \lim_{l\to\infty} B_l(\nu)$. So there exists a $l_0 \in \mathbb{K}$ such that $\|\bar{\nu} - B_{l_0}(\nu)\| \leq \epsilon/2$. Therefore by triangle inequality, we have $\|\mu - (\nu - B_{l_0}(\nu) + \bar{\nu})\| \leq \|\mu - \nu\| + \|B_{l_0}(\nu) - \bar{\nu}\| < \epsilon$.

Now for $\mu \in M_{*s}$, we write $\mu = \mu_+ - \mu_-$, where μ_+, μ_- are normal positive linear functional. Thus, it follows that $\mathcal{W}$ is dense in M_{*s}.

(ii): Fix $k \in \mathbb{K}$ and consider $\nu_k := \nu - B_k(\nu) + \bar{\nu}$ and it is enough to prove eq. 3.3.1 for ν_k. First we claim that $\bar{\nu}_k = \bar{\nu}$.

Since $\nu \leq \lambda\rho$, so by Lemma 3.2.4 there exists a unique $y_1' \in M_+'$ with $y_1' \leq \lambda$ such that

$$\nu(x) = \langle y_1' x\Omega_\rho, \Omega_\rho\rangle_\rho \text{ for all } x \in M.$$

Let $y' \in M'$, write $B_l'(y') := \frac{1}{m(I_l)} \int_{I_l} T_g'(y')dm(g)$ and by Lemma 3.2.5, we have

$$\langle B_l'(y')x\Omega_\rho, \Omega_\rho\rangle_\rho = \langle y' B_l(x)\Omega_\rho, \Omega_\rho\rangle_\rho, \quad x \in M, y' \in M'.$$

Now for all $l \in \mathbb{K}$ and $x \in M_+$, we note that

$$\begin{aligned}
|(B_l(\nu_k) - \bar{\nu})(x)| &= |(\nu_k - \bar{\nu})(B_l(x))| \\
&= |(\nu - B_k(\nu))(B_l(x))| \quad (\text{since } \nu_k := \nu - B_k(\nu) + \bar{\nu}) \\
&= |\nu(B_l(x)) - \nu(B_k(B_l(x)))| \quad (\text{as } B_k(\nu)(\cdot) = \nu(B_k(\cdot))) \\
&= \left|\langle y_1' B_l(x)\Omega_\rho, \Omega_\rho\rangle_\rho - \langle y_1' B_k(B_l(x))\Omega_\rho, \Omega_\rho\rangle_\rho\right| \quad (\text{as } \nu(\cdot) = \langle y_1'(\cdot)\Omega_\rho, \Omega_\rho\rangle_\rho) \\
&= \left|\langle y_1' B_l(x)\Omega_\rho, \Omega_\rho\rangle_\rho - \langle B_k'(y_1')(B_l(x))\Omega_\rho, \Omega_\rho\rangle_\rho\right|, \text{ (by Lemma 3.2.5)} \\
&= \left|\langle (y_1' - B_k'(y_1'))B_l(x)\Omega_\rho, \Omega_\rho\rangle_\rho\right| \\
&= \left|\langle B_l'(y_1' - B_k'(y_1'))x\Omega_\rho, \Omega_\rho\rangle_\rho\right|, \text{ (by Lemma 3.2.5)} \\
&\leq \|B_l'(y_1' - B_k'(y_1')\| \, \rho(x).
\end{aligned}$$

Further, for all $n \in \mathbb{K}$, note that

$$B_l'(y_1' - B_k'(y_1')) = \frac{1}{m(I_l)} \int_{I_l} \left[T_g'(y_1') - T_g'(B_k'(y_1'))\right] dm(g)$$

$$
\begin{aligned}
&= \frac{1}{m(I_l)} \int_{I_l} \Big[T'_g(y'_1) - T'_g\Big(\frac{1}{m(I_k)} \int_{I_k} T'_h(y'_1) dm(h)\Big)\Big] dm(g) \\
&= \frac{1}{m(I_l)} \frac{1}{m(I_k)} \int_{I_l} \int_{I_k} \Big(T'_g(y'_1) - T'_{hg}(y'_1)\Big) dm(h) dm(g) \\
&= \frac{1}{m(I_k)} \frac{1}{m(I_l)} \int_{I_k} \int_{I_l} \Big(T'_g(y'_1) - T'_{hg}(y'_1)\Big) dm(g) dm(h).
\end{aligned}
$$

Hence, we have

$$
\|B'_l(y'_1 - B'_k(y'_1))\| \leq \frac{1}{m(I_k)} \int_{I_k} \frac{m(I_l \Delta h I_l)}{m(I_l)} dm(h).
$$

Now, for all $l \in \mathbb{K}$, consider the function $I_k \ni h \mapsto \frac{m(I_l \Delta h I_l)}{m(I_l)}$. It is a real valued measurable function defined on the compact set I_k and bounded by 2. Thus, applying DCT we get

$$
\lim_{l\to\infty} \|B'_l(y'_1 - B'_k(y'_1))\| \leq \frac{1}{m(I_k)} \int_{I_k} \lim_{l\to\infty} \frac{m(I_l \Delta h I_l)}{m(I_l)} dm(h) = 0. \tag{3.3.2}
$$

Hence, we also obtain

$$
\lim_{l\to\infty} \|B_l(\nu_k) - \overline{\nu}\| = 0.
$$

Therefore, $\bar{\nu}_k = \bar{\nu}$ and we have

$$
\lim_{l\to\infty} \sup_{x\in M_+, x\neq 0} |(B_l(\nu_k) - \bar{\nu}_k)(x)| / \rho(x) = \lim_{l\to\infty} \sup_{x\in M_+, x\neq 0} |(B_l(\nu_k) - \bar{\nu})(x)| / \rho(x) = 0.
$$

This completes the proof. □

For the next set of results we assume that M is a finite von Neumann algebra with a f.n tracial state τ. In due course, we prove the main results in this section which deal with the $b.a.u$ convergence of the ergodic averages in $L^1(M, \tau)$. We start with the following theorem.

Theorem 3.3.16. *Let M be a finite von Neumann algebra with f.n trace τ and (M, G, T, ρ) be a kernel. Then for any $\mu \in M_{*s}$ there exists an invariant $\overline{\mu} \in M_{*s}$, such that for all $\epsilon > 0$, there exists a projection $e \in M$ with $\tau(1 - e) < \epsilon$ and*

$$
\lim_{a\to\infty} \sup_{x\in eM_+e,\ x\neq 0} \left| \frac{(M_a(\mu) - \bar{\mu})(x)}{\tau(x)} \right| = 0.
$$

Proof. First we note that, since $\rho \in M_{*+}$, there exists a unique $X \in L^1(M,\tau)_+$ such that $\rho(x) = \tau(Xx)$ for all $x \in M$. Then for any $s > 0$ consider the projection $q_s := \chi_{(1/s,s)}(X) \in M$. Observe that $(1-q_s) \xrightarrow{s\to\infty} 0$ in SOT and hence there exists a $s_0 > 0$ such that $\tau(1-q_{s_0}) < \epsilon/2$. Further, it implies $Xq_{s_0} \leq s_0 q_{s_0}$. Thus, for all $0 \neq x \in (q_{s_0} M q_{s_0})_+$ we have

$$\begin{aligned}\frac{\rho(x)}{\tau(x)} = \frac{\tau(Xx)}{\tau(x)} &= \frac{\tau(Xq_{s_0}x)}{\tau(x)} \quad \text{(since } q_{s_0}x = x\text{)} \\ &\leq \frac{\tau(s_0 x)}{\tau(x)} = s_0.\end{aligned} \tag{3.3.3}$$

Now we use Lemma 3.3.15 recursively to obtain a sequence $\{\nu_{a_1}, \nu_{a_2}, \cdots\} \subseteq \mathcal{W}$ satisfying

$$\begin{aligned}&(1)\ a_1 < a_2 < \cdots,\\ &(2)\ \left\|\mu - \nu_{a_j}\right\| < \frac{1}{4^j c} \text{ for all } j \in \mathbb{N}, \text{ and,}\\ &(3)\ \sup_{x\in M_+, x\neq 0} \frac{\left|(M_a(\nu_{a_j}) - \bar{\nu}_{a_j})(x)\right|}{\rho(x)} < \frac{1}{2^j} \text{ for all } a \geq a_j.\end{aligned} \tag{3.3.4}$$

Further, note that $\mu - \nu_{a_j}, \bar{\mu} - \bar{\nu}_{a_j} \in M_{*s}$ for all $j \in \mathbb{N}$. For every $j \in \mathbb{N}$, take a $N_j \in \mathbb{N}$ and use the Theorem 3.3.13 to get sequences of projections $\{e_1, e_2, \ldots\}$ and $\{f_1, f_2, \ldots\}$ in M such that

$$\begin{aligned}&(1)\ \rho(1-e_j) < \frac{1}{2^j} \text{ and } \rho(1-f_j) < \frac{1}{2^j}, \text{ for all } j \in \mathbb{N},\\ &(2)\ \sup_{x\in e_jM_+e_j, x\neq 0} \frac{\left|M_a(\mu - \nu_{a_j})(x)\right|}{\rho(x)} < \frac{1}{2^{j-1}} \text{ for all } 1 \leq a \leq N_j,\\ &(3)\ \sup_{x\in f_jM_+f_j, x\neq 0} \left|\frac{M_a(\bar{\mu} - \bar{\nu}_{a_j})(x)}{\rho(x)}\right| < \frac{1}{2^{j-1}} \text{ for all } 1 \leq a \leq N_j.\end{aligned} \tag{3.3.5}$$

Now it immediately follows that both $\rho(1-e_j)$ and $\rho(1-f_j)$ converges to 0 as j tends to infinity. Therefore, both $\tau(1-e_j)$ and $\tau(1-f_j)$ converges to 0 as j tends to infinity. Hence choose a subsequence $(j_k)_{k\in\mathbb{N}}$ such that

$$\tau(1-e_{j_k}) < \frac{\epsilon}{2^{k+2}} \text{ and } \tau(1-f_{j_k}) < \frac{\epsilon}{2^{k+2}}.$$

Now consider $e := \wedge_{k\geq 1}(e_{j_k} \wedge f_{j_k}) \wedge q_{s_0}$ and observe that

$$\tau(1-e) \leq \sum_{k\geq 1}(\tau(1-e_{j_k}) + \tau(1-f_{j_k})) + \tau(1-q_s) < \epsilon.$$

Therefore, for all $0 \neq x \in eM_+e$, and $a, a_k \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\frac{|(M_a(\mu)-\bar{\mu})(x)|}{\tau(x)} \\
\leq&\Big(\frac{|(M_a(\mu-\nu_{a_k})(x)|}{\rho(x)} + \frac{|(\bar{\mu}-\bar{\nu}_{a_k})(x)|)}{\rho(x)} + \frac{|(M_a(\nu_{a_k})-\bar{\nu}_{a_k})(x)|}{\rho(x)}\Big)\frac{\rho(x)}{\tau(x)} \\
\leq& s_0\Big(\big(\frac{|M_a(\mu-\nu_{a_k})(x)|}{\rho(x)} + \frac{|M_a(\bar{\mu}-\bar{\nu}_{a_k})(x)|}{\rho(x)}\big) + \frac{|(M_a(\nu_{a_k})-\bar{\nu}_{a_k})(x)|}{\rho(x)}\Big) \\
&(\text{ by eq. 3.3.3, } \frac{\rho(x)}{\tau(x)} \leq s_0, \text{ as } e \leq q_{s_0}) \\
\leq& s_0\Big(\big(\frac{|M_a(\mu-\nu_{a_k})(x)|}{\rho(x)} + \frac{|M_a(\bar{\mu}-\bar{\nu}_{a_k})(x)|}{\rho(x)}\big) + \sup_{x\in M_+, x\neq 0}\frac{|(M_a(\nu_{a_k})-\bar{\nu}_{a_k})(x)|}{\rho(x)}\Big).
\end{aligned}$$

Hence, by taking supremum over the set $eM_+e\setminus\{0\}$ on both sides of the above inequality and applying eq. 3.3.4 and eq. 3.3.5 we obtain

$$\lim_{a\to\infty}\sup_{x\in eM_+e, x\neq 0}\frac{|(M_a(\mu)-\bar{\mu})(x)|}{\tau(x)} = 0.$$

This completes the proof. □

Theorem 3.3.17. *Let (M, G, T, ρ) be a kernel and τ be a f.n tracial state on M. Then for all $Y \in L^1(M, \tau)$, there exists $\overline{Y} \in L^1(M, \tau)$ such that $M_a(Y)$ converges to $\overline{Y}$ bilaterally almost uniformly.*

Proof. Let $Y \in L^1(M, \tau)_s$. Then there exists a unique $\mu \in M_{*s}$ such that $\mu(x) = \tau(Yx)$ for all $x \in M$. by Theorem 3.3.3, we note that $\|\cdot\|_1 - \lim_{a\to\infty} M_a(\mu)$ exists and denote it by $\bar{\mu} \in M_{*s}$, i.e,

$$\bar{\mu} := \|\cdot\|_1 - \lim_{a\to\infty} M_a(\mu).$$

As, $\bar{\mu} \in M_{*s}$, so there exists a unique $\bar{Y} \in L^1(M,\tau)_s$ such that $\bar{\mu}(x) = \tau(\bar{Y}x)$ for all $x \in M$. Note that $\bar{\mu}$ is G-invariant. Let $\epsilon, \delta > 0$. Then by Theorem 3.3.16 there exists a projection $e \in M$ with $\tau(1-e) < \epsilon$ and there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in (eMe)_+ \setminus \{0\}} \left| \frac{(M_a(\mu) - \bar{\mu})(x)}{\tau(x)} \right| < \delta, \text{ for all } a \geq N.$$

Therefore, for all $a \geq N$ we have

$$\begin{aligned} &|(M_a(\mu) - \bar{\mu})(x)| < \delta\tau(x) \text{ for all } x \in (eMe)_+ \setminus \{0\} \\ \Rightarrow &\left|\tau((M_a(Y) - \overline{Y})eye)\right| < \delta\tau(eye) \text{ for all } y \in M_+ \setminus \{0\} \\ \Rightarrow &\left|\tau(e(M_a(Y) - \overline{Y})ey)\right| < \delta\tau(eye) \text{ for all } y \in M_+ \setminus \{0\}. \end{aligned}$$

Hence by Lemma 3.2.15, we have $\left\|e(M_a(Y) - \overline{Y})e\right\| \leq \delta$ for all $a \geq N$. □

Theorem 3.3.18. *Let τ be a f.n tracial state on M and $(L^1(M,\tau), G, T)$ be a non-commutative dynamical system. Also assume that there exists a f.n state ρ on M such that (M, G, T^*, ρ) is a kernel. Then, for all $Y \in L^1(M,\tau)$ there exists $\overline{Y} \in L^1(M,\tau)$ such that $M_a(Y)$ converges to $\overline{Y}$ in $b.a.u.$*

Proof. First note that $\hat{T_g^*} = T_g$ for all $g \in G$. Consider the anti-action $\beta = (\beta_g)_{g \in G} := (T_g^*)_{g \in G}$ on M. Then the result follows by applying Theorem 3.3.17 on the action β. □

Remark 3.3.19. *Note that if $\sup_{g \in G} \|T_g\| \leq 1$, then $\left\|T_g^*(1)\right\| \leq 1$. Therefore, $T_g^*(1) \leq 1$. Thus, the condition $\sup_{g \in G} \|T_g\| \leq 1$ together with ρ is G-invariant can be put in Theorem 3.3.18 instead of assuming (M, G, T^*, ρ) is a kernel. However, former conditions are stronger than assuming (M, G, T^*, ρ) is a kernel.*

Corollary 3.3.20. *Let τ be a f.n tracial state on M and $(L^1(M,\tau), G, T)$ be a non-commutative dynamical system. Assume that $T_g(1) = 1$ and $T_g^*(1) \leq 1$ for all $g \in G$. Then, for all $Y \in L^1(M,\tau)$ there exists $\overline{Y} \in L^1(M,\tau)$ such that $M_a(Y)$ converges $b.a.u.$ to $\overline{Y}$.*

Proof. Since τ is finite, observe that $1 \in L^1(M, \tau)$ and it follows that

$$\tau(T_g^*(x)) = \tau(x) \text{ for all } x \in M \text{ and } g \in G.$$

Hence, (M, G, T^*, τ) becomes a kernel and thus the result follows from Theorem 3.3.18.

□

3.4 Action of finitely generated free group

In this section we study convergence of ergodic averages associated a sequence of w-continuous maps $\sigma := \{\sigma_n\}_{n=0}^{\infty}$ on a von Neumann algebra M. We obtain mean convergence and an auxiliary maximal ergodic inequality of the spherical averages associated to σ. In the end, we assume that M is finite von Neumann algebra and prove b.a.u convergence of the spherical averages associated to the maps on the predual of M. We also remark that same can be obtained for the free group action. We begin with following definition.

Definition 3.4.1. *Let $\sigma := \{\sigma_n\}_{n=0}^{\infty}$ be a sequence of w-continuous maps on a von Neumann algebra M. Then (M, σ) is called generalized noncommutative dynamical system if it satisfies the following*

A_1. *σ_1 is completely positive, $\sigma_1(1) \leq 1$ and σ_n is positive.*

A_2. *$\sigma_1 \circ \sigma_n = w\sigma_{n+1} + (1-w)\sigma_{n-1}$ for all $n \in \mathbb{N}$ and $\sigma_0(x) = x$, where $1/2 < w < 1$.*

We remark that such examples naturally arises to study ergodic converges of spherical averages of actions of free group action. Let G be free group with generators $\{a_1, \ldots, a_r, a_r^{-1}, \ldots, a_r^{-1}\}$. Let $\phi : G \to Aut(M)$ denote a group homomorphism. Define,

$$\sigma_n := \frac{1}{|W_n|} \sum_{a \in W_n} \phi(a), \; n \in \mathbb{N},$$

where, W_n denotes the set of words of length n. It is well known that $\sigma_m \circ \sigma_n = \sigma_m \circ \sigma_n$ for all $m, n \in \mathbb{N}$. Furthermore,

$$\sigma_1 \circ \sigma_n = w\sigma_{n+1} + (1-w)\sigma_{n-1},\ n \in \mathbb{N},$$

and $\sigma_0(x) = x$ for all $x \in M$, where $1/2 < w < 1$ is a fixed number.

Given a noncommutative dynamical system (M, σ), we consider the following average

$$S_n(x) = \frac{1}{n+1}\sum_0^n \sigma_k(x),\ n \in \mathbb{N}.$$

Although, it is referred as spherical average in the context of free group action, we freely use the same terminology in our context as well.

In this subsection we prove mean convergence of spherical averages associated to a generalized kernel.

Definition 3.4.1. Consider a sequence of w-continuous maps $\sigma := \{\sigma_n\}_{n=0}^{\infty}$ and a f.n state ρ on M. Then (M, σ, ρ) is call called generalized kernel if it satisfy the following

1. σ_1 is completely positive, $\sigma_1(1) \leq 1$ and σ_n is positive.
2. $\sigma_1 \circ \sigma_n = w\sigma_{n+1} + (1-w)\sigma_{n-1}$ for all $n \in \mathbb{N}$ and $\sigma_0(x) = x$, where $1/2 < w < 1$.
3. $\rho \circ \sigma_1 = \sigma_1$ and $\rho(y\sigma_1(x)) = \rho(\sigma_1(y)x)$for all $x, y \in M$.

For a generalized kernel (M, σ, ρ), there exists a sequence of operators $\{u_n\}$ on $L^2(M, \rho)$ such that

$$u_n(x\Omega) = \sigma_n(x)\Omega,\ \ x \in M.$$

Then immediately observe that

1. u_n is self adjoint and $u_m \circ u_n = u_n \circ u_m$ for all $m, n \in \mathbb{N}$.

2. There exists $1/2 < w < 1$ such that $u_1 \circ u_n = w u_{n+1} + (1-w) u_{n-1},\ n \in \mathbb{N}$.

Remark 3.4.2. *Consider the following set;*

$$D_w := \{z \in \mathbb{C} : \left|\sqrt{z + 4w - 4w^2} + \sqrt{z - 4w + 4w^2}\right| \leq 2\sqrt{w} \text{ and } \left|\sqrt{z + 4w - 4w^2} - \sqrt{z - 4w + 4w^2}\right| \leq 2\sqrt{w}\}.$$

Then by [44], we note that $\sigma(u_1) \subseteq D_w$.

Then we recall the following mean ergodic type theorem in this context, the proof follows from Remark 3.4.2 and [44].

Theorem 3.4.3. *Let* (M, σ, ρ) *be a kernel, then the ergodic averages* $\frac{1}{n+1} \sum_0^n u_n$ *converges strongly on* $L^2(M, \rho)$ *to a projection onto the space* $\{\eta \in L^2(M, \rho) : u_1 \eta = \eta\}$.

Remark 3.4.4. *We note that the above situation is automatic for free group action. Indeed, let* ρ *be a f.n state on* M *such that* $\rho \circ \phi(a)(x) = \rho(x)$ *for all* $x \in M$ *and for all* $a \in G$*, Then consider the sequence of following spherical averages*

$$\sigma_n = \frac{1}{|W_n|} \sum_{a \in W_n} \phi_a.$$

Then observe that $(M, \sigma := \{\sigma_n\}, \rho)$ *becomes a kernel.*

Theorem 3.4.5. *Let* (M, σ, ρ) *be a kernel. For all* $\mu \in M_*$*, there exists a* $\bar{\mu} \in M_*$ *such that*

$$\bar{\mu} = \|\cdot\|_1 - \lim_{n \to \infty} S_n(\mu),$$

where, for all $n \in \mathbb{N}$, $\sigma_n(\mu) = \mu \circ \sigma_n$, $\mu \in M_*$.

Proof. For $n \in \mathbb{N}$, let u_n be the associated self adjoint operator defined by $u_n(x\Omega_\rho) = \sigma_n(x)\Omega_\rho$ for $x \in M$. Consider $T_n = \frac{1}{n+1} \sum_0^n u_k$. Let $y_1, y_2 \in M'(\sigma)$ and define $\psi_{y_1,y_2}(x) =$

$\langle xy_1\Omega_\rho, y_2\Omega_\rho\rangle$ for all $x \in M$. Then there exists a $z \in M$ such that $y_1^* y_2\Omega_\rho = z\Omega_\rho$, where $z = J\sigma_{i/2}(y_2^* y_1)J$. Consequently, for $x \in M$

$$\begin{aligned}
S_n(\psi_{y_1,y_2})(x) &= \frac{1}{n+1}\sum_0^n \langle \sigma_k(x)y_1\Omega, y_2\Omega\rangle \\
&= \frac{1}{n+1}\sum_0^n \langle u_k(x\Omega), z\Omega\rangle \\
&= \frac{1}{n+1}\sum_0^n \langle x\Omega, u_k(z\Omega)\rangle \\
&= \left\langle x\Omega, \frac{1}{n+1}\sum_0^n u_k(z\Omega)\right\rangle \\
&= \langle x\Omega, T_n(z\Omega)\rangle .
\end{aligned}$$

Therefore, by Theorem 3.4.3 we obtain that for all $x \in M$, $S_n(\psi_{y_1,y_2})(x) \to \langle x\Omega, \eta\rangle$, where $\eta = \lim_{n\to\infty} T_n(z\Omega)$. Observe that $u_1\eta = \eta$. By induction argument $u_n\eta = \eta$ for all $n \in \mathbb{N}$.

Now let $x \in M$. Further write x as x_1+ix_2, where $x_1, x_2 \in M_s$ and then by the previous argument, it follows that $T_n(x\Omega) := T_n(x_1\Omega_\rho) + iT_n(x_2\Omega_\rho)$ converges in $L^2(M, \rho)$.

Now consider $\overline{\psi}_{y_1,y_2} \in M_*$, defined by

$$\overline{\psi}_{y_1,y_2}(x) = \langle x\Omega_\rho, \eta\rangle_\rho, \;\; x \in M.$$

Then by a standard argument it follows that $\overline{\psi}_{y_1,y_2} \circ \sigma_n = \overline{\psi}_{y_1,y_2}$ for all $n \in \mathbb{N}$ and

$$\overline{\psi}_{y_1,y_2} = \|\cdot\|_1 - \lim_{n\to\infty} S_n(\psi_{y_1,y_2}).$$

Hence the result follows since the set $\{\psi_{y_1,y_2} : y_1, y_2 \in M'(\sigma)\}$ is total in M_*. □

In this subsection, we obtain an auxiliary maximal ergodic inequality for a generalized noncommutative dynamical system (M, σ).

Theorem 3.4.6. *Let E be an ordered Banach space. Consider a sequence of positive maps $\{\kappa_n\}_{n=0}^\infty$ on E satisfying*

$$\kappa_1 \circ \kappa_n = w\kappa_{n+1} + (1-w)\kappa_{n-1} \; and$$

$$\kappa_0(x) = x \textit{ for all } x \in E.$$

Then there exists a $C_w > 0$ *such that for all* $x \in E_+$

$$\frac{1}{n+1}\sum_{l=0}^{n} \kappa_l(x) \leq C_w \frac{1}{3n+1}\sum_{l=0}^{3n} \kappa_1^l(x).$$

Proof. Observe that a simple calculation implies that the composition power κ_1^n is a convex combination of κ_l, $0 \leq l \leq n$. That is $\kappa_1^n = \sum_{l=0}^{n} a_n(l)\kappa_l$, where for all $n \in \mathbb{N}$, $\sum_{l=0}^{n} a_n(l) = 1$ and $0 \leq a_n(l) \leq 1$.

Now a calculation similar to the proof of Lemma 1 of [31] estimates the coefficients $a_n(l)$ and the results follows. □

Now consider a (M, σ) as described in the beginning of this section. Then we can consider a sequence of positive maps, on M_*, again denoting by $\{\sigma_n\}$ by the abuse of notation, which are defined by $\sigma_n(\mu) = \mu \circ \sigma_n$, $\mu \in M_*$. Then for all $n \in \mathbb{N}$ we consider the averaging operators on M_* defined by

$$S_n(\mu) = \frac{1}{n+1}\sum_{k=0}^{n} \sigma_k(\mu).$$

Theorem 3.4.7. *Let* (M, σ) *be a generalized noncommutative dynamical system and* ρ *be a f.n. state on* M *such that* $\rho \circ \sigma_1 = \rho$. *Further assume that* $\mu \in M_{*s}$ *and* $\epsilon > 0$. *Then for any* $N \in \mathbb{N}$ *there exists a projection* $e \in M$ *such that* $\rho(1-e) < \frac{C_w\|\mu\|}{\epsilon}$ *and*

$$|S_n(\mu)(x)| \leq \epsilon\rho(x) \textit{ for all } x \in (eMe)_+ \textit{ and } 1 \leq n \leq N.$$

Proof. Proof follows immediately from Theorem 3.2.2 and Theorem 3.4.6. □

In this subsection, we assume M to be finite von Neumann algebra and then we prove b.a.u convergence of spherical averages associated to a kernel.

Lemma 3.4.8. *Let* (M, σ) *be a generalized noncommutative dynamical system and* ρ *be a f.n. state on* M *such that* $\rho \circ \sigma_1 = \rho$. *Then there exists a sequence of maps* $\{\sigma'_n\}_{n\in\mathbb{N}}$ *on* M' *such that*

1. $\langle \sigma'_n(y')x\Omega_\rho, \Omega_\rho\rangle_\rho = \langle y'\sigma_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho$ *for all* $x \in M, y' \in M', n \in \mathbb{N}$.

2. $\sigma'_m \circ \sigma'_n = \sigma'_n \circ \sigma'_m$ *for all* $m, n \in \mathbb{N}$.

3. $\sigma'_1 \circ \sigma'_n = w\sigma'_{n+1} + (1-w)\sigma'_{n-1},\ n \in \mathbb{N}$.

Consequently, (M', σ') *becomes a noncommutative dynamical system.*

Proof. For (1) let $y' \in M'_+$ and $n \in \mathbb{N}$, consider the linear functional $\nu^n_{y'} : M \to \mathbb{C}$ defined by

$$\nu^n_{y'}(x) = \langle y'\sigma_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho.$$

For $x \in M_+$, note that

$$\begin{aligned}
\nu^n_{y'}(x) &= \langle y'\sigma_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho \\
&\leq \|y'\| \, \langle \sigma_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho \\
&= \|y'\| \, \rho(\sigma_n(x)) \\
&= \|y'\| \, \rho(x) \\
&= \|y'\| \, \langle x\Omega_\rho, \Omega_\rho\rangle_\rho.
\end{aligned}$$

Hence, by Lemma 3.2.4 there exists a unique $z' \in M'$ such that $\nu^n_{y'}(x) = \langle z'x\Omega_\rho, \Omega_\rho\rangle_\rho$. Write $z' = \sigma'_n(y')$ and then we have $\langle y'\sigma_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho = \langle \sigma'_n(y')x\Omega_\rho, \Omega_\rho\rangle_\rho$. Note that (2) and (3) follows immediately from (1). □

Proposition 3.4.9. *With the above notation, for all* $k \in \mathbb{Z}_+$ *we have,*

$$\lim_{n\to\infty} \|S'_n(y' - \sigma'_k(y'))\| = 0.$$

Proof. First, we note that for all $n, k \in \mathbb{N}$ and $y' \in M'$, we have

$S'_n(y' - \sigma'_{k+1}(y'))$

$$= (1 - \frac{1}{w})S'_n(y' - \sigma'_{k-1}(y')) - \frac{1}{w}\sigma'_1(S'_n(\sigma'_k(y') - y')) - \frac{1}{w}S'_n(\sigma'_1(y') - y'). \quad (3.4.1)$$

Indeed, For all $n, k \in \mathbb{N}$ we note that

$$\begin{aligned} S'_n(y' - \sigma'_{k+1}(y')) &= S'_n\Big[y' - \frac{1}{w}\Big(\sigma'_1 \circ \sigma'_k(y') - (1-w)\sigma'_{k-1}(y')\Big)\Big] \\ &= S'_n\Big(y' - \sigma'_{k-1}(y')\Big) - \frac{1}{w}S'_n\Big(\sigma'_1 \circ \sigma'_k(y') - \sigma'_{k-1}(y')\Big). \end{aligned}$$

Also we have,

$$\begin{aligned} S'_n\Big(\sigma'_1 \circ \sigma'_k(y') - \sigma'_{k-1}(y')\Big) &= S'_n\Big(\sigma'_1 \circ \sigma'_k(y') - \sigma'_1(y') + \sigma'_1(y') - y' + y' - \sigma'_{k-1}(y')\Big) \\ &= \sigma'_1(S'_n(\sigma'_k(y') - y')) + S'_n(\sigma'_1(y') - y') + S'_n(y' - \sigma'_{k-1}(y')). \end{aligned}$$

Therefore, combining the above two equalities we obtain Eq. 3.4.1.

For $k = 0$, the result follows immediately. For $k = 1$, we obtain

$$\begin{aligned} \|S'_n(y' - \sigma'_1(y'))\| &= \left\| \frac{1}{n+1}\sum_{l=0}^{n} \sigma'_l(y' - \sigma'_1(y')) \right\| \\ &= \left\| \frac{w}{n+1}(y' - \sigma'_{n+1}(y')) + \frac{1-w}{n+1}\sigma'_n(y') \right\| \\ &\leq \frac{1+w}{n+1}\|y'\|. \end{aligned}$$

Therefore, the result is true for $k = 1$. Also when $k = 2$, then we have

$$\begin{aligned} \|S'_n(y' - \sigma'_2(y'))\| &= \left\| \frac{1}{n+1}\sum_{l=0}^{n} \sigma'_l(y' - \sigma'_2(y')) \right\| \\ &= \frac{1}{w}\|\sigma'_1(S'_n(y' - \sigma'_1(y'))) + S'_n(y' - \sigma'_1(y'))\| \text{ (by eq. 3.4.1)} \\ &\leq \frac{2}{w}\|S'_n(y' - \sigma'_1(y'))\| \\ &\leq \frac{2(1+w)}{w(n+1)}\|y'\| \text{ (from the case } k = 1). \end{aligned}$$

Hence the result is true for $k = 2$. Now let us assume that the result is true for $\{3, 4, \ldots, k\}$ where $k \in \mathbb{N}$. Now, by Eq. 3.4.1 it follows that

$$\left\|S'_n(y' - \sigma'_{k+1}(y'))\right\|$$

$$\leq (1-\frac{1}{w})\left\|S_n'(y'-\sigma_{k-1}'(y'))\right\| + \frac{1}{w}\left\|\sigma_1'(S_n'(\sigma_k'(y')-y'))\right\| + \frac{1}{w}\left\|S_n'(\sigma_1'(y')-y')\right\|$$
$$\leq (1-\frac{1}{w})\left\|S_n'(y'-\sigma_{k-1}'(y'))\right\| + \frac{1}{w}\left\|S_n'(\sigma_k'(y')-y')\right\| + \frac{1}{w}\left\|S_n'(\sigma_1'(y')-y')\right\|.$$

Hence by induction hypothesis, the result follows for $k+1$. Therefore, the results holds for any $k \in \mathbb{N}$. □

Lemma 3.4.10. *Let (M, σ, ρ) be a generalized kernel. Consider the following set*

$$\mathcal{W}_1 := \{\nu - S_k(\nu) + \bar{\nu} : \quad k \in \mathbb{N},\ \nu \in M_{*+} \text{ with } \nu \leq \lambda\rho \text{ for some } \lambda > 0\}.$$

(i) Write $\mathcal{W} = \mathcal{W}_1 - \mathcal{W}_1$, then $\mathcal{W}$ is dense in M_{*s} and

(ii) for all $\nu \in \mathcal{W}$, we have

$$\lim_{n\to\infty} \sup_{x \in M_+, x \neq 0} |(S_n(\nu) - \bar{\nu})(x)| / \rho(x) = 0. \tag{3.4.2}$$

Proof. (i): Let $\mu \in M_{*+}$ and $\epsilon > 0$. From Theorem 2.2.2, find a $\nu \in M_{*+}$ with $\nu \leq \lambda\rho$ for some $\lambda > 0$, such that $\|\mu - \nu\| < \epsilon/2$. Further, by Theorem 3.4.5 we know that $S_n(\nu)$ is convergent, and write $\bar{\nu} = \lim_{n\to\infty} S_n(\nu)$. So there exists a $n_0 \in \mathbb{N}$ such that $\|\bar{\nu} - S_{n_0}(\nu)\| \leq \epsilon/2$. Therefore by triangle inequality, we have $\|\mu - (\nu - S_{n_0}(\nu) + \bar{\nu})\| \leq \|\mu - \nu\| + \|S_{n_0}(\nu) - \bar{\nu}\| < \epsilon$.

Now for $\mu \in M_{*s}$, we write $\mu = \mu_+ - \mu_-$, where μ_+, μ_- are normal positive linear functional. Thus, it follows that $\mathcal{W}$ is dense in M_{*s}.

(ii): Fix $k \in \mathbb{N}$ and consider $\nu_k := \nu - S_k(\nu) + \bar{\nu}$ and it is enough to prove eq. 3.4.2 for ν_k. First we claim that $\bar{\nu}_k = \bar{\nu}$.

Since $\nu \leq \lambda\rho$, so by Lemma 3.2.4 there exists a unique $y_1' \in M_+'$ with $y_1' \leq \lambda$ such that

$$\nu(x) = \langle y_1' x \Omega_\rho, \Omega_\rho \rangle_\rho \text{ for all } x \in M.$$

Let $y' \in M'$, write $S_n'(y') := \frac{1}{n+1}\sum_0^n \sigma_n'(y')$ and by Lemma 3.4.8, we have

$$\langle S_n'(y')x\Omega_\rho, \Omega_\rho\rangle_\rho = \langle y' S_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho, \quad x \in M, y' \in M'.$$

Now for all $n \in \mathbb{N}$ and $x \in M_+$, we note that

$$\begin{aligned}
|(S_n(\nu_k) - \bar{\nu})(x)| &= |(\nu_k - \bar{\nu})(S_n(x))| \\
&= |(\nu - S_k(\nu))(S_n(x))| \quad (\text{since } \nu_k := \nu - S_k(\nu) + \bar{\nu}) \\
&= |\nu(S_n(x)) - \nu(S_k(S_n(x)))| \quad (\text{as } S_k(\nu)(\cdot) = \nu(S_k(\cdot))) \\
&= \left| \langle y_1' S_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho - \langle y_1' S_k(S_n(x))\Omega_\rho, \Omega_\rho\rangle_\rho \right| \quad (\text{as } \nu(\cdot) = \langle y_1'(\cdot)\Omega_\rho, \Omega_\rho\rangle_\rho) \\
&= \left| \langle y_1' S_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho - \langle S_k'(y_1')(S_n(x))\Omega_\rho, \Omega_\rho\rangle_\rho \right|, \ (\text{by Lemma } 3.4.8) \\
&= \left| \langle (y_1' - S_k'(y_1')) S_n(x)\Omega_\rho, \Omega_\rho\rangle_\rho \right| \\
&= \left| \langle S_n'(y_1' - S_k'(y_1')) x\Omega_\rho, \Omega_\rho\rangle_\rho \right|, \ (\text{by Lemma } 3.4.8) \\
&\leq \| S_n'(y_1' - S_k'(y_1'))\| \, \rho(x).
\end{aligned}$$

Further for $n, k \in \mathbb{N}$ we have

$$\begin{aligned}
S_n'(y_1' - S_k'(y_1')) &= S_n'\Big[\frac{1}{k+1}\sum_{l=0}^{k}\Big(y_1' - \sigma_l'(y_1')\Big)\Big] \\
&= \frac{1}{k+1}\sum_{l=0}^{k}\Big[S_n'\Big(y_1' - \sigma_l'(y_1')\Big)\Big].
\end{aligned}$$

Now by Proposition 3.4.9 we obtain

$$\lim_{n\to\infty} \|S_n(\nu_k) - \bar{\nu}\| = 0.$$

Therefore, $\bar{\nu}_k = \bar{\nu}$ and we have

$$\lim_{n\to\infty} \sup_{x \in M_+, x \neq 0} |(S_n(\nu_k) - \bar{\nu}_k)(x)| / \rho(x) = \lim_{n\to\infty} \sup_{x \in M_+, x \neq 0} |(S_n(\nu_k) - \bar{\nu})(x)| / \rho(x) = 0.$$

This completes the proof. □

Now we assume that M is a finite von Neumann algebra with a f.n tracial state τ and prove the main results in this section which deal with the $b.a.u$ convergence of the ergodic averages in $L^1(M,\tau)$ for a kernel associated to $\sigma = \{\sigma_n\}$.

Theorem 3.4.11. *Let M be a finite von Neumann algebra with f.n trace τ and let (M,σ,ρ) be a generalized kernel. Then we have the following;*

1. *for any $\mu \in M_{*s}$ there exists an invariant $\overline{\mu} \in M_{*s}$, such that for all $\epsilon > 0$, there exists a projection $e \in M$ with $\tau(1-e) < \epsilon$ and*
$$\lim_{n\to\infty} \sup_{x \in eM_+e,\ x\neq 0} \left| \frac{(S_n(\mu) - \bar{\mu})(x)}{\tau(x)} \right| = 0, \text{ and}$$
2. *for all $Y \in L^1(M,\tau)$, there exists $\overline{Y} \in L^1(M,\tau)$ such that $S_n(Y)$ converges to $\overline{Y}$ bilaterally almost uniformly.*

Proof. The proofs of (1) and (2) follow similarly as Theorem 3.2.16 and Theorem 3.2.17. □

Remark 3.4.12. *We like to highlight that simillar results as obtained in Theorem 3.2.18 and Corollary 3.2.20 for a generalized noncommuative dynamical system $(L^1(M,\tau),\sigma)$ associated to a sequence of maps $\sigma = \{\sigma_n\}$ on $L^1(M,\tau)$ can also be obtained under similar assumption.*

Chapter 4

Stochastic Ergodic Theorem

In this section we combine the results obtained in §2 and §3 to prove a stochastic ergodic theorem.

4.1 Introduction

Consider a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ and a positive contraction T on $L^1(\Omega, \mu)$. Recall the Neveu decomposition (Theorem 2.1.2) in this case. In 1966, Krengel proved the following Ergodic theorem.

Theorem 4.1.1. *[23, Theorem 3.4.9] If T is a positive contraction on $L^1(\Omega, \mu)$ then, for all $f \in L^1(\Omega, \mu)$ the averages $\{A_n(f) := \frac{1}{n}\sum_0^{n-1} T^k f\}$ converge in measure. The limit is T-invariant and vanishes on D. Moreover, for $f \in L^1(\Omega, \mu)_+$ the limit is equal almost everywhere to the pointwise limit* $\liminf A_n(f)$.

In this chapter, we will prove a non-commutative analogue of this theorem. Throughout this chapter, we assume that $M \subseteq \mathbf{B}(\mathcal{H})$ is a von Neumann algebra with a f.n tracial state τ and we consider $L^1(M, \tau)$. We further assume that G is a group of polynomial growth with a compact, symmetric generating set V. Then consider the covariant system (M, G, α) and prove stochastic ergodic theorem for this covariant system. We also assume throughout this section that the ergodic averages will be considered with respect to the Følner sequence $\{V^n\}_{n\in\mathbb{N}}$.

4.2 Stochastic ergodic theorem

We begin with the following useful proposition.

Proposition 4.2.1. *Let (M, G, α) be a covariant system and let e_1, e_2 be the projections as in Theorem 2.4.3. Then for $i = 1, 2$, we have $\alpha_g(e_i) = e_i$ for all $g \in G$.*

Proof. Let ρ be the G-invariant normal state with support e_1 as in Theorem 2.4.3. Therefore, $\rho(e_1) = 1$ and so is $\rho(\alpha_g(e_1)) = 1$ for all $g \in G$. Hence, $\alpha_g(e_1) \geq e_1$ for all $g \in G$. Since G is a group, so we further obtain $\alpha_g(e_1) = e_1$ for all $g \in G$. Consequently, we also have $\alpha_g(e_2) = e_2$ for all $g \in G$. □

Remark 4.2.2. *Let (M, G, α) be a covariant system and τ be a f.n tracial state on M. We observe the following.*

(i) There exists two mutually orthogonal projections, $e_1, e_2 \in M$ such that $e_1 + e_2 = 1$ and satisfying the conditions in Theorem 2.4.3.

(ii) It is also evident from Proposition 4.2.1 that for $i = 1, 2$, the restriction of α to the reduced von Neumann algebra M_{e_i} ($= e_i M e_i$) defines an action by automorphisms and we denote this induced action by same notation α. Thus, (M_{e_i}, G, α) for $i = 1, 2$ becomes a covariant system.

(iii) For $i = 1, 2$, it also follows from Proposition 4.2.1 and eq. 3.1.2 that for all $g \in G$, the predual transformation $\hat{\alpha_g}$ defined on $L^1(M, \tau)$ satisfies

$$\hat{\alpha_g}(e_i X e_i) = e_i \hat{\alpha_g}(X) e_i \text{ for all } X \in L^1(M, \tau) \text{ and for all } g \in G.$$

As a consequence, the ergodic averages satisfy

$$e_i A_n(X) e_i = A_n(e_i X e_i) \text{ for all } X \in L^1(M, \tau), n \in \mathbb{N} \text{ and } i = 1, 2.$$

We write $\tau_{e_i} = \frac{1}{\tau(e_i)}\tau|_{e_iMe_i}$ for $i = 1, 2$. Now we have the following theorem.

Theorem 4.2.3. *Let (M, G, α) be a covariant system and τ be a f.n tracial state on M. Consider the projections $e_1, e_2 \in M$ as mentioned in Remark 4.2.2. Then we have the following results.*

(i) For all $B \in L^1(M_{e_1}, \tau_{e_1})$, there exists $\bar{B} \in L^1(M_{e_1}, \tau_{e_1})$ such that $A_n(B)$ converges b.a.u to $\bar{B}$. Moreover, $A_n(B)$ converges in measure to $\bar{B}$.

(ii) For all $B \in L^1(M_{e_2}, \tau_{e_2})$, $A_n(B)$ converges to 0 in measure.

Proof. (i): We note that ρ is a f.n state on M_{e_1} such that $\rho(\alpha_g(x)) = \rho(x)$ for all $g \in G$ and $x \in M_{e_1}$. Let $B \in L^1(M_{e_1}, \tau_{e_1})$, then it follows from Theorem 3.2.17 that there exists a $\bar{B} \in L^1(M_{e_1}, \tau_{e_1})$ such that $A_n(B)$ converges to $\bar{B}$ in b.a.u. Furthermore, the convergence in measure follows from Remark 1.2.11.

(ii): From Corollary 2.4.3, it follows that there exists a weakly wandering operator $x_0 \in M_+$ such that $s(x_0) = e_2$. Hence $e_2x_0e_2 = x_0$, which implies $x_0 \in M_{e_2}$.

Now let B be a non-zero element of $L^1(M_{e_2}, \tau_{e_2})_+$. Let us choose $0 < \epsilon \leq 1$ and $\delta > 0$. Since $e_2 = \chi_{(0,\infty)}(x_0)$, observe that there exists $m \in \mathbb{N}$ such that the projection $p := \chi_{(\frac{1}{m},\infty)}(x_0) \in M_{e_2}$ satisfies $\tau(e_2 - p) < \frac{\delta}{2}$. Now we define the projections

$$r_n := \chi_{[\epsilon,\infty)}(pA_n(B)p), n \in \mathbb{N},$$

and claim that $\tau(r_n) < \delta/2$ for all $n \in \mathbb{N}$. Indeed, since $\frac{1}{m}p \leq x_0$ we have, $A_n(p) \leq mA_n(x_0)$ for all $n \in \mathbb{N}$, which implies $\|A_n(p)\| \leq m\|A_n(x_0)\|$. Now since x_0 is a weakly wandering operator, there exists $N_0 \in \mathbb{N}$ such that

$$\|A_n(p)\| \leq \frac{\epsilon\delta}{2\tau(B)} \text{ for all } n \geq N_0.$$

Therefore, for all $n \in \mathbb{N}$ we have,

$$\tau(pA_n(B)p) = \tau(A_n(B)p) = \tau(BA_n(p)) \leq \tau(B)\|A_n(p)\|.$$

Note that $\epsilon r_n \leq pA_n(B)p$ for all $n \in \mathbb{N}$. Therefore, we have $\tau(r_n) \leq \frac{\delta}{2}$ for all $n \geq N_0$. Define the projections $q_n := p - r_n$, $n \in N$ and observe that for all $n \geq N_0$,

$$\tau(e_2 - q_n) = \tau(e_2 - p + r_n) = \tau(e_2 - p) + \tau(r_n) \leq \delta/2 + \delta/2 = \delta, \ n \geq N_0.$$

We also note that, for all $n \in \mathbb{N}$

$$\begin{aligned} q_n A_n(B) q_n = q_n p A_n(B) p q_n &\leq \chi_{[0,\epsilon]}(pA_n(B)p)(pA_n(B)p)\chi_{[0,\epsilon]}(pA_n(B)p) \\ &\leq \chi_{[0,\epsilon]}(pA_n(B)p). \end{aligned}$$

Hence, for all $n \in \mathbb{N}$ we have

$$\|q_n A_n(B) q_n\| \leq \epsilon.$$

The result for arbitrary $B \in L^1(M_{e_2}, \tau_{e_2})$ then follows from Proposition 1.2.12. □

Remark 4.2.4. *Let $X \in L^1(M, \tau)$. Then as a consequence of Theorem 4.2.3, we get the following.*

(i) There exists $Y \in L^1(M, \tau)$ such that for all $\epsilon, \delta > 0$ there exists $N_0 \in \mathbb{N}$ and a projection $p \in M_{e_1}$ such that

$$\tau(e_1 - p) < \delta/2, \text{ and, } \|p(e_1 A_n(X) e_1 - e_1 Y e_1) p\| < \epsilon \text{ for all } n \geq N_0.$$

(ii) For all $\epsilon, \delta > 0$, there exists a sequence of projections $\{q_n\}_{n \in \mathbb{N}}$ in M_{e_2} and $N_1 \in \mathbb{N}$ such that

$$\tau(e_2 - q_n) < \delta/2, \text{ and, } \|q_n e_2 A_n(X) e_2 q_n\| < \epsilon \text{ for all } n \geq N_1.$$

Consider the following projection

$$r_n := p + q_n, \ n \in \mathbb{N}.$$

Note the for all $n \in \mathbb{N}$, r_n is a projection in M and

$$\tau(1 - r_n) = \tau(e_1 - p) + \tau(e_2 - q_n) < \delta.$$

Lemma 4.2.5. *Let $X \in L^1(M,\tau)_+$. Then there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $\|r_n e_1 A_n(X) e_2 r_n\| \leq \sqrt{\epsilon(\epsilon + \|pe_1 Y e_1 p\|)}$ and $\|r_n e_2 A_n(X) e_1 r_n\| \leq \sqrt{\epsilon(\epsilon + \|pe_1 Y e_1 p\|)}$.*

Proof. Observe that for all $n \in \mathbb{N}$, $A_n(X) \in L^1(M,\tau)_+$ and for all $n \geq N_0$, $pe_1 Y e_1 p$ and $A_n(X)e_1 p$ are bounded operators. Then we claim that for all $n \geq N_0$, $pe_1 A_n(X)^{1/2}$ is also a bounded operator. Indeed, let $n \geq N_0$ and $\xi \in \mathcal{D}(A_n(X)e_1 p)$. Then,

$$\begin{aligned}
\langle A_n(X)^{1/2} e_1 p\xi, A_n(X)^{1/2} e_1 p\xi\rangle &= \langle A_n(X) e_1 p\xi, e_1 p\xi\rangle \\
&= \langle pe_1 A_n(X) e_1 p\xi, \xi\rangle \\
&\leq \|pe_1 A_n(X) e_1 p\| \, \|\xi\| \\
&= \|pe_1 (A_n(X) - Y) e_1 p + pe_1 Y e_1 p\| \, \|\xi\| \\
&\leq (\|pe_1 (A_n(X) - Y) e_1 p\| + \|pe_1 Y e_1 p\|) \, \|\xi\| \\
&\leq (\epsilon + \|pe_1 Y e_1 p\|) \, \|\xi\| \, .
\end{aligned}$$

Since, for all $n \in \mathbb{N}$, $\overline{\mathcal{D}(A_n(X)e_1 p)} = \mathcal{H}$, we get $\left\|A_n(X)^{1/2} e_1 p\right\| \leq \sqrt{\epsilon + \|pe_1 Y e_1 p\|}$ for all $n \geq N_0$. Also we note that

$$\left\|pe_1 A_n(X)^{1/2}\right\| = \left\|(A_n(X)^{1/2} e_1 p)^*\right\| \leq \sqrt{\epsilon + \|pe_1 Y e_1 p\|} \text{ for all } n \geq N_0. \tag{4.2.1}$$

Again observe that for all $n \geq N_1$, $A_n(X)e_2 q_n$ is a bounded operator. We also claim that for all $n \geq N_1$, $A_n(X)^{1/2} e_2 q_n$ is a bounded operator. Indeed, let $n \geq N_1$ and $\xi \in \mathcal{D}(A_n(X)e_2 q_n)$. Then,

$$\begin{aligned}
\langle A_n(X)^{1/2} e_2 q_n \xi, A_n(X)^{1/2} e_2 q_n \xi\rangle &= \langle A_n(X) e_2 q_n \xi, e_2 q_n \xi\rangle \\
&= \langle q_n e_2 A_n(X) e_2 q_n \xi, \xi\rangle \\
&\leq \|q_n e_2 A_n(X) e_2 q_n\| \, \|\xi\| \\
&\leq \epsilon \, \|\xi\| \text{ for all } n \geq N_1.
\end{aligned}$$

Since, $\overline{\mathcal{D}(A_n(X)e_2q_n)} = \mathcal{H}$ for all $n \in \mathbb{N}$, we get, $\left\|A_n(X)^{1/2}e_2q_n\right\| \leq \sqrt{\epsilon}$ for all $n \geq N_1$. Now define $N_2 := \max\{N_0, N_1\}$ and note that for all $n \geq N_2$

$$\begin{aligned}\|r_ne_1A_n(X)e_2r_n\| &= \|pe_1A_n(X)e_2q_n)\| \\ &= \left\|pe_1A_n(X)^{1/2}A_n(X)^{1/2}e_2q_n\right\| \\ &\leq \left\|pe_1A_n(X)^{1/2})\right\| \left\|A_n(X)^{1/2}e_2q_n)\right\| \\ &\leq \sqrt{\epsilon(\epsilon + \|pe_1Ye_1p\|)}.\end{aligned}$$

Now since $r_ne_2A_n(X)e_1r_n = (r_ne_1A_n(X)e_2r_n)^*$ holds for all $n \in \mathbb{N}$, we have

$$\|r_ne_2A_n(X)e_1r_n\| \leq \sqrt{\epsilon(\epsilon + \|pe_1Ye_1p\|)}.$$

□

Theorem 4.2.6. *Let (M, G, α) be a covariant system, τ be a f.n tracial state on M and assume that G has polynomial growth. Let $X \in L^1(M, \tau)$, then there exists $Z \in L^1(M, \tau)$ such that $A_n(X)$ converges to Z in measure.*

Proof. Enough to consider $X \in L^1(M, \tau)_+$. Then for all $n \in \mathbb{N}$, $A_n(X) \in L^1(M, \tau)_+$ and

$$A_n(X) = e_1A_n(X)e_1 + e_1A_n(X)e_2 + e_2A_n(X)e_1 + e_2A_n(X)e_2.$$

Observe that, it follows from Proposition 4.2.1 that for all $n \in \mathbb{N}$, $e_1A_n(X)e_1 = A_n(e_1Xe_1) \in L^1(M_{e_1}, \tau_{e_1})_+$ and $e_2A_n(X)e_2 = A_n(e_2Xe_2) \in L^1(M_{e_2}, \tau_{e_2})_+$.

Let $\epsilon, \delta > 0$. Consider the element $Y \in L^1(M, \tau)$ and projections r_n in M as in Remark 4.2.4. Let $Z := e_1Ye_1$ and note that for all $n \in \mathbb{N}$

$$\begin{aligned}r_n(A_n(X) - Z)r_n = r_n\Big(e_1A_n(X)e_1 - e_1Ye_1\Big)r_n + r_ne_1A_n(X)e_2r_n + r_ne_2A_n(X)e_1r_n \\ + r_ne_2A_n(X)e_2r_n.\end{aligned}$$

We also note that for all $n \in \mathbb{N}$, $r_n\Big(e_1A_n(X)e_1 - e_1Ye_1\Big)r_n = p\Big(e_1A_n(X)e_1 - e_1Ye_1\Big)p$ and $r_ne_2A_n(X)e_2r_n = q_ne_2A_n(X)e_2q_n$.

Hence the result follows from Remark 4.2.4 and Lemma 4.2.5. □

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