The Fluid-Membrane-Gravity Duality

By

Milan Patra

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Signature: S. Mukherji — 21.7.23
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Signature: Sayantani B — 21.7.2023
Guide / Convener - Dr. Sayantani Bhattacharyya — **Date:**

Signature: N Banerjee — 21.7.23
External examiner - Dr. Nabamita Banerjee — **Date:**

Signature: Yogesh — 21.7.23
Member 1 - Dr. Yogesh Kumar Srivastava — **Date:**

Signature: [signature] — 21/7/2023
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Sayantani B
Dr. Sayantani Bhattacharyya
(Guide)

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications

- **List of Publications arising from the thesis**

1. Sayantani Bhattacharyya, Parthajit Biswas, Anirban Dinda and **Milan Patra**, *" Fluid-gravity and membrane-gravity dualities: Comparison at subleading orders ",* JHEP 05 (2019) 054, [arXiv:1902.00854]

2. **Milan Patra**, *" Comparison between fluid-gravity and membrane-gravity dualities for Einstein-Maxwell system ",* Class.Quant.Grav. 38 (2021) 13, 135017, [arXiv: 1912.09402]

- **Other Publications (not included in the thesis)**

1. Sayantani Bhattacharyya, Parthajit Biswas and **Milan Patra**, *" A leading-order comparison between fluid-gravity and membrane-gravity dualities ",* JHEP 05 (2019) 022, [arXiv:1807.05058]

2. Sayantani Bhattacharyya , Prateksh Dhivakar , Anirban Dinda , Nilay Kundu, **Milan Patra** and Shuvayu Roy, *" An entropy current and the second law in higher derivative theories of gravity ",*JHEP 09 (2021) 169 ,[arXiv:2105.06455]

3. Sayantani Bhattacharyya, Pooja Jethwani, **Milan Patra** and Shuvayu Roy, *" Reparametrization Symmetry of Local Entropy Production on a Dynamical Horizon ",*[arXiv:2204.08447]

4. Chandranathan A, Sayantani Bhattacharyya , **Milan Patra** and Shuvayu Roy, *" Entropy Current and Fluid-Gravity Duality in Gauss-Bonnet theory ".*[arXiv:2208.07856]

5. Gilad Lifschytz and **Milan Patra**, *" Dressing bulk spinor fields in AdS_3 ",* JHEP 03 (2023) 245, [arXiv:2210.16845]

6. Sarthak Duary, Eliot Hijano and **Milan Patra** *" Towards an IR finite S-matrix in the flat limit of AdS/CFT ".*[arXiv:2211.13711]

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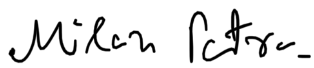

Milan Patra

DEDICATIONS

Dedicated to

My

Parents

And My

Maternal grandfather

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Contents

SUMMARY

The classical dynamics of black holes are governed by Einstein's equations. These are a set of nonlinear partial differential equations. These simple-looking equations appear to be too difficult to solve exactly, particularly when the geometry involves non-trivial dynamics. In such situations, to handle problems analytically one has to take recourse to perturbation theory. Two such perturbation techniques which can handle dynamical fluctuations around some static solutions even at non-linear level are 'derivative expansion' and 'large D expansion'. 'Derivative expansion' is used to generate 'black hole type' solutions to Einstein's equations with negative cosmological constant that are in one to one correspondence with the solutions of the relativistic generalization of Navier-Stokes equations of hydrodynamics. On the other hand 'large D expansion' generates similar 'black hole type' solutions to Einstein's equations with or without cosmological constant where the gravity solutions are dual to the dynamics of a co-dimension one non-gravitational membrane propagating without backreaction in the asymptotic geometry. In this thesis, we have compared these two perturbation schemes developed to handle both nonlinearity and dynamics in Einstein's equations in presence of negative cosmological constant. We have shown that in an appropriate regime of parameter space, the gravity solutions along with their corresponding horizon dynamics generated by these two perturbation techniques are equivalent up to the subleading order in expansion parameter for pure gravity systems and up to the first non-trivial order in expansion parameter for Einstein-Maxwell system [1, 2, 3].

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Chapter 1

Introduction

The classical dynamics of black holes are governed by Einstein's equations either in vacuum or in presence of different matter field stress tensors along with a separate set of equations controlling the dynamics of the matter fields. For example, the Einstein-Maxwell system of equations determines the coupled dynamics of charged black holes and electromagnetic fields around them. As we know these equations admit a simple but profound solution, namely black holes. These deceptively simple-looking equations are easy to state but they appear to be too difficult to solve exactly, particularly when the system involves non-trivial dynamics. In sufficiently dynamic situations, like the collision of two black holes and their subsequent merger, one may think to solve the problem numerically and in fact, much of our recent understanding of these process are due to numerics. But the numerics involve are sufficiently complicated for such complicated dynamical process. In that situation, one natural option might be to search for a parameter and then do perturbation theory. However, Einstein's equations lack any adjustable parameters to do perturbation theory. The main problem in solving the dynamics of 'Schwarzschild-type' black holes is that it has only one scale in the system, the horizon radius. One may think to simplify the problem in the point particle limit, where one considers particle moving in gravitational field of wavelength large compared to the size of the horizon size. But if one is interested in the dynamics of the horizon this assumption is not useful as one needs to consider finite horizon size. However if one considers black branes in asymptotically AdS (and also flat) space-time, the scale of variation in some directions is much larger than the variations along directions transverse to it. This is the famous 'derivative expansion' [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] (in AdS space) and the 'blackfold approach' [15, 16, 17, 18] (in flat space). In the limit of very large dimensions one can find additional length scale, the thickness of the black hole, after which the space-time ceases to be wrapped by black holes [19]. In this thesis, we will discuss two such perturbation techniques which are used to study dynamical fluctuation around some static solution even at non-linear level. One of these perturbation techniques as we have mentioned is 'derivative expansion' and another is 'expansion in inverse powers of dimensions'. In both cases, one can take theories of gravity and classically rewrite them as non-gravitational systems.

'Derivative expansion' is a perturbation technique that generates dynamical black-brane solutions to Einstein's equations in presence of negative cosmological constants that are dual to the solutions of relativistic generalization of Navier-Stokes equations of hydrodynamics. Here one studies a particular long wavelength limit of Einstein's equations with negative cosmological constant. In such a limit it has been found that Einstein's equations of general relativity in presence of negative cosmological constant, with appropriate boundary conditions and regularity restrictions, reduce to the equations of hydrodynamics, namely the relativistic generalization of Navier-Stokes equations. In other words, Einstein's equations in presence of negative cosmological constant in $(d+1)$ dimensions capture the relativistic generalization of Navier-Stokes equations in one lower dimension. It is called the famous fluid/gravity correspondence [13]. Here given an arbitrary fluid dynamical solution in the boundary of AdS, we can construct a time-dependent, inhomogeneous black hole-type solution with a regular event horizon, retaining full non-linearity whose properties mimic that of fluid flow. The fluid/gravity correspondence has many important applications and implications. It has confirmed the long-standing lore that space-time or black hole horizon dynamics is governed by hydrodynamics. This type of idea dates back to the old work of membrane paradigm [20, 21, 22], where it has been realized that black hole horizons behave like a fluid membrane, equipped with fluid-like properties, such as viscosity, conductivity and so on. However, this membrane paradigm is merely an analogy. On the other hand, the fluid-gravity correspondence is a real duality mapping the long-wavelength but arbitrary amplitude perturbations of black holes in asymptotically AdS background to the dynamics of conformal fluid, where one can realize that the membrane lives on the boundary of the space-time. This enables us to algorithmically construct dynamical black hole geometries with regular event horizons in the bulk, given solutions of the fluid equations for the fluid flow in the boundary field theory. This correspondence has provided useful insight not only for studying black holes in asymptotically AdS space-time but also helped in classifying various problems in fluid dynamics itself, e.g the appearance of new pseudo-vector contribution to charge current ignored by Landau & Lifshitz [23]. It has been used to precisely determine the higher-order transport coefficients of the conformal fluid which depends on the underlying microscopic structure and is hard to calculate on the field theory side. It has some experimental implications also since conformal fluid to an extent mimics the physics of Quark Gluon Plasma (QGP) discovered in heavy ion collisions.

Over the last few years, it has been observed that in the limit of large D (D is space-time dimension) , the equations of black hole dynamics simplifies a lot and there is a novel reformulation of black hole dynamics [24, 25, 26, 27, 28, 29]. As the space-time dimension grows large the gravitational lines of force get more and more diluted. In large dimensions, the interaction potential decays very fast at large distances and at short distances it increases more steeply. As a consequence of this, the gravitational field is localized near its source whereas at large distances it is suppressed non-perturbatively in inverse powers of space-time dimensions. Emparan, Suzuki, Tanabe and collaborators have noted that in large dimensions the Schwarzschild black holes have two widely separated length scales, the Schwarzschild radius and the thickness of the black hole's gravitational tail beyond the event horizon after which the gravitational force rapidly decays to zero [30, 31, 32, 33, 34, 35]. They have also computed the spectrum of quasinormal modes in large D and found that almost all but a finite number of modes are heavy with frequencies inversely proportional to the thickness of the region and the rest finite number of modes confined in the near horizon region are anomalously light. These two types of quasinormal modes are distinguished by frequencies parametrically separated in $1/D$. The separation of scale between heavy and light quasinormal modes suggests that it should be possible to 'integrate out' the heavy modes and get the full non-linear dynamics of black holes governed by only the effective non-linear theory of the light modes. In [24, 25, 26, 27, 28, 29] the effective dynamics of black hole horizons have been studied in a systematic expansion in $1/D$. Though very different in detail, the $1/D$ expansion is similar in spirit to the $1/N$ expansion, which t'Hooft invented to study $SU(3)$ Yang Mills theory in four dimensions generalizing to the study of $SU(N)$. It has been shown that black hole dynamics at large D reduces to the dynamics of a co-dimension one non-gravitational membrane propagating without backreaction in the asymptotic geometry. The dynamical degrees of freedom of this membrane are the shape of the membrane and a velocity field (and also charge density for Einstein-Maxwell system) that lives on the membrane world volume. One can correct the metric systematically order by order in $1/D$ provided the membrane shape and velocity field (and also charge density for Einstein-Maxwell system) obey an integrability constraint - membrane equation of motion. These equations together define a well-posed initial value problem for the shape of the membrane and also for the velocity field (and also charge density for Einstein-Maxwell system) and apply to arbitrarily non-linear and dynamical black hole motion.

Let us discuss very briefly its connection to the old membrane paradigm [20, 21, 22]. The equations of the membrane at large D arise from Einstein's constraint equations at the event horizon of the black hole. This can be regarded as the mathematical realization of the so-called membrane paradigm idea, where it has been realized that the black hole horizon dynamics resemble the equations of fluid on a membrane. But the membrane paradigm picture differs from our large D effective theory in crucial ways. Though like the large D effective theory the membrane paradigm consists of constraint equations on the horizon but unlike large D effective theory they are not imposed after integrating the dynamics near the horizon and hence is not an effective theory in the sense of large D effective theory. In fact the constraint equations in the 'membrane paradigm' apply to any null hypersurface in Einstein's theory in arbitrary dimension and do not require any separation of scales.

The analysis of black hole dynamics in large D has found applications in many directions [19]. For instance, the efficient analytical calculations of black hole quasinormal modes have been done using large D technique in perturbative expansion in $1/D$. It has also shed light on some basic problems about the instability of black holes and their evolution, cosmic censorship and so on. It has been noticed that the Gregory-Laflamme instability of black strings depends on D and it makes the large D technique a useful application to this instability problem [19]. In large D limit of AdS/CFT, one may gain further insight into explaining various phenomena in strongly interacting field theories. In this regard, it has found applications in condensed matter systems, nuclear physics, and hydrodynamics. When impact parameter and total angular momentum are small from the analysis of black hole collisions at large D one can get useful insight into black hole collision in $D = 4$. Since gravitational radiation is strongly suppressed in large D limit [37], no gravitational radiation is emitted in black hole collisions in large dimensions. As a result of this, the total mass and angular momentum of the black hole system will be conserved and this enormously simplifies the problem of getting a qualitative picture of the possible outcome of the merger.

The large D effective theory has several similarities with the fluid/gravity correspondence but with significant differences between them also. In both cases, one integrates the radial dependence orthogonal to the horizon and gets an effective theory for the fluctuations parallel to the horizon. These two methods of obtaining black hole solutions differ in how the expansion parameter is made small. The construction of large D effective theory works in an expansion in inverse powers of dimensions. Fluid/gravity works order by order in an expansion in derivatives in units of horizon radius and the coefficients of this expansion are determined exactly as functions of D. On the other side, in large D effective theory the membrane equations are constructed in perturbation in $1/D$ and so at any given order have terms of all orders in derivatives expansion, namely those that survive at large D limit. Also, the large D effective theory lives on a fluctuating surface rather than a fixed field theory background of fluid gravity.

To gain a clear understanding of the space of solutions of Einsteins equations generated by these two perturbation schemes, we have to chart out the interconnections between these two different perturbation techniques. In this thesis we have investigated the following questions.

- Is there any interconnection between these two perturbation schemes
- Whether it is possible to apply these two perturbation techniques simultaneously in any regime(s) of parameter space
- Are the solutions generated by these two perturbation techniques equivalent in any regime(s) of parameter space?

We have answered these questions in affirmative in chapter 2 and chapter 3 for pure gravity systems and in chapter 4 for Einstein-Maxwell system. Very briefly we have found the following answers.

- We can apply both perturbation schemes simultaneously in a regime of parameter space. Also, when D is large and derivatives are small in an appropriate sense, we can use $\left(\frac{1}{D}\right)$ and ∂_μ (with respect to some length scale) as two independent perturbation parameters, without any constraint on their ratio.
- Equivalently, if we expand the metric dual to hydrodynamics further in $1/D$, it matches with the metric dual to the membrane-dynamics generated using large D expansion, expanded in derivatives.

We have compared the gravity solutions generated by these two techniques and established the equivalence of the gravity solutions along with their dual systems generated by these two perturbation schemes up to subleading order in both perturbation parameters for pure gravity system and up to the first non-trivial order in expansion parameter on both sides for Einstein-Maxwell system [1, 2, 3]. The reason behind the matching is the following. Since we can use the same space-time geometry as the starting point for both techniques (namely the space-time of a Schwarzschild black brane) and given the starting point and hence the characterizing data both the technique generate the higher order corrections uniquely, it follows that in the regime where both the techniques are applicable, the solutions should be the same. However, this matching is not at all manifest and we could see it after a series of intricate gauge and variable transformations. The whole subtlety in the analysis is to find out the appropriate coordinate transformation. Very briefly we have essentially rewritten the metric dual to hydrodynamics in large number of dimension in terms of membrane degrees of freedom. In the course of the analysis we have worked out the field redefinition from the membrane to boundary fluid variables, i.e we have determined the membrane velocity and its shape function in terms of fluid variables. Then it is possible to rewrite the hydrodynamic metric (metric generated in derivative expansion) in terms of membrane degrees of freedom and hence the subsequent matching up to the order the solutions are known on both sides. We expect this equivalence to be exactly equivalent to all orders once we have worked out the field redefinition from the membrane to boundary hydrodynamic variable to all orders.

In this context let us mention the work of [36]. In [37, 38] it has been shown that the dual membrane in large D scheme hosts a stress tensor -'the membrane stress tensor' and the equations of motion of the dual membrane are determined from the conservation of the membrane stress tensor. In [36] the authors have determined an 'improved' membrane stress tensor which defines consistent probe membrane dynamics even at finite D. Though at leading order in large D limit, the improved stress tensor reproduces the earlier derived results, but in generic situations they do not exactly reproduce the finite D black hole physics (although they work surprisingly well in some equilibrium and non-equilibrium situations). The authors then studied the long wavelength dynamics of planar membrane in AdS space and computed the linearized gravitational fluctuations sourced by the membrane stress tensor. By using the AdS/CFT prescription they have calculated the boundary stress tensor induced by this linearized fluctuations. In this procedure, they have calculated the form of the boundary stress tensor in terms of membrane variables. Then performing a field redefinition to a local fluid velocity and temperature they rewritten the boundary stress tensor and showed that it takes the form of the stress tensor of the conformal fluid in derivative expansion [39]. They have compared this boundary stress tensor with the respective fluid-gravity stress tensor [39] and found that it matches exactly at zero and first order in derivative expansion with the fluid-gravity stress tensor even at finite D. But at second order in derivative expansion, they deviate from each other and the agreement only at large D. Now we have the field redefinition from the matching of the two metrics up to second order. Using this it might be possible to improve the stress tensor and understand this structure in more detail.

We conclude the introduction with a few interesting future directions and a brief overview of the chapters in the thesis.

Future directions

Let us now mention some interesting future directions.

We have matched the two metrics only within the membrane region. But it is possible to compute the gravitational radiation, sourced by the effective membrane stress tensor and extend outside the membrane region till infinity [37]. As we have mentioned previously, in [36] the authors have determined the boundary stress tensor from this radiation part and matched it with the dual fluid stress tensor of the hydrodynamic metric. Now since we know how to 'split' the hydrodynamic metric, we could also match the metric coefficients outside the membrane region that are exponentially falling off with D and therefore non-perturbative from the point of view of $\mathcal{O}\left(\frac{1}{D}\right)$ expansion.

In AdS space, there exist similar-looking constructions of horizon stress tensor in terms of boundary stress tensor by following a radial flow of constraint Einstein's equations on cut-off surfaces in the background AdS [40, 41]. In some way, they are a bit different from 'large-D' construction of the membrane stress tensor. For example, they do not resum the derivative expansion. Also, they remain like fluid equations all along the radial flow and reduce to non-relativistic fluid equations on the horizon. However, there must be some relations between this radial flow of Einstein's equations down to the horizon and the membrane stress tensor carried towards the AdS boundary via gravitational radiation [36]. It would be very interesting to explore these relations further.

We have compared the metric in the regime where both perturbation techniques are applicable and what we have shown is that the derivative and the $\left(\frac{1}{D}\right)$ expansion commute in this regime. However, we also know from [1] that there exists a regime where derivative expansion could not be applied but we could still apply 'large-D' expansion. This is an interesting regime to explore since here we would construct genuinely new dynamical black hole solutions that were not described previously by other perturbative techniques. It would be very interesting to isolate this regime in the general 'large-D' expansion technique.

We all know that surface tension is one very important physical property that we could associate with some membrane. These dualities, once explored further (which we leave to future work) might tell us that surface tension actually maps to some other well studied properties of fluid or gravity.

Another very interesting direction is the entropy analysis. Both large-D and derivative expansion are applicable to even higher derivative theories of gravity and coupled gauge theory. Concept of entropy in such gravitational systems is still not well understood, but our understanding about whole thermodynamics and entropy in particular is far more clear in both membrane and fluid systems. These explicit maps would probably lead to some progress about understanding these questions in such higher derivative gravity setups.

A brief overview of the chapters

We conclude the introduction with a brief overview of the chapters in the thesis.

Background

After a brief introduction to the fluid-gravity correspondence and the membrane-gravity correspondence in chapter 1, we have described the comparison between the gravity solutions generated by these two techniques up to the first subleading orders in the expansion parameters on both sides for pure gravity systems. We have described that in an appropriate regime of parameter space of the solutions, it is possible to apply both perturbation techniques simultaneously. We have explicitly matched the gravity solutions along with their dual systems in large numbers of dimensions.

Fluid-gravity and membrane-gravity dualities - Comparison at subleading orders

In chapter 3 (based on [2]) we have essentially extended the comparison of the two gravity solutions up to the second subleading orders. As the form of both the large D metric and the 'hydrodynamic metric' are very complicated and they look very different from each other the comparison at this order is far more non-trivial than at first subleading order [1]. An important outcome of this exercise is that at the intermediate steps often some pattern emerges and that would naturally lead to some all-order statement.

Comparison between fluid-gravity and membrane-gravity dualities for Einstein-Maxwell system

In chapter 4 (based on [3]) we have compared the two gravity solutions up to the first non-trivial orders. Unlike the pure gravity case, here even at first non-trivial orders the curvature of the background AdS space generates new term in the charge conservation equation and hence makes the checking more stringent than the pure gravity system.

Chapter 2

Background

2.1 The Membrane paradigm at large D

Following [24, 25, 26, 27, 28, 29] in this section we briefly discuss the effective dynamics of black hole horizons in large number of space-time dimensions. A few years ago, Emparan, Suzuki, Tanabe and collaborators have observed that classical dynamics of black holes in D dimension simplify a lot in large D limit. Motivated by this observation over the last few years there were several works [24, 25, 26, 27, 28, 29]. In those works, it has been demonstrated that black hole dynamics can be reformulated in terms of equations of motion of a co-dimension one non-gravitational membrane propagating without backreaction on any asymptotic geometry that solves Einstein's equations. Emparan, Suzuki, and Tanabe have introduced a parameter into Einstein's equations and studied dynamics of black holes in a systematic expansion in $1/D$. In order to gain more intuition about how it works let us first consider the Schwarzschild-Tangherlini black hole, written in Kerr-Schild coordinates. In Kerr-Schild coordinates, the black hole solution takes the following form

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2 + \left(\frac{r_0}{r}\right)^{D-3} (dr - dt)^2 \ . \tag{2.1}$$

This metric describes a black hole at rest, i.e moving with velocity $U = -dt$. Note that this metric manifestly reduces to background geometry (here flat space) at large r and also is smooth at the outer future event horizon. Also when D is large the deviation from the background is neat. Note that $(dr - dt)$ is null both in background and the full space-time geometry.

The metric for the black hole moving at arbitrary constant boost is obtained if we replace dt by a timelike one form U and then the metric reads

$$
\begin{aligned}
&g_{MN} = \eta_{MN} + \psi^{-D} O_M O_N \\
&O = n - U, \quad U = \text{constant}, \quad U \cdot U = -1, \quad U \cdot n = 0, \quad \psi^{-D} = \left(\frac{r_0}{r}\right)^{D-3} \\
&r^2(x) = P_{MN} x^M x^N, \quad P_{MN} = \eta_{MN} + U_M U_N, \quad n_M = \partial_M r
\end{aligned}
\tag{2.2}
$$

where all the '·' product are with respect to η_{MN}. Varying the fields; horizon radius r_0 and the one-form field U over the horizon much more slowly than the scale $\frac{D}{r_0}$ of the radial gradient we turn them into collective fields and call the coordinate collective coordinate ansatz. In this way, we can write the more general ansatz metric as

$$
\mathcal{W}^{(0)}_{AB} = \bar{\mathcal{W}}_{AB} + \psi^{-D} O_A O_B \tag{2.3}
$$

where $\bar{\mathcal{W}}_{AB}$ is a smooth metric that solves Einstein's equations and we refer to it as background metric. ψ is an unspecified function in the background geometry and $O = n - U$ with U as a one-form 'velocity' field in the background space such that

$$
\bar{\mathcal{W}}^{AB} U_A U_B = -1, \quad n_A = \frac{\partial_A \psi}{\sqrt{\bar{\mathcal{W}}^{AB} \partial_A \psi \partial_B \psi}}, \quad \bar{\mathcal{W}}^{AB} n_A U_B = 0 \ . \tag{2.4}
$$

We will deal with two derivative pure gravity theory in presence of cosmological constant. The action of this theory can be written as

$$
\mathcal{S} = \int \sqrt{\mathcal{W}} \left[R - \Lambda\right] \ . \tag{2.5}
$$

We will assume the following scaling of the cosmological constant Λ

$$
\Lambda = (D-1)(D-2)\lambda, \quad \lambda \sim \mathcal{O}(1) \ . \tag{2.6}
$$

The equation of motion is given by

$$
E_{AB} = R_{AB} - \left(\frac{R - \Lambda}{2}\right) \mathcal{W}_{AB} = 0 \ . \tag{2.7}
$$

Our goal is to find new 'black-hole type' solutions to these equations in large D limit. But let us first mention a complication in doing that. As we know Einstein's equations in D dimension is a collection of $\frac{D(D+1)}{2}$ equations that determine a metric tensor with $\frac{D(D+1)}{2}$ components, each of which is a function of D space-time coordinates. So, in large D limit, both the number of equations and the number of variables to be solved will blow up. To take a sensible large D limit, we will divide the space-time dimensions into two parts and assume that a large part of the space-time geometry is fixed by some symmetry and consider that the metric is dynamical only along some finite number of directions. In that case, we will write the metric in the following way.

$$ds^2 = \mathcal{W}_{AB} dX^A dX^B = \tilde{W}_{ab}\left(\{x^a\}\right) dx^a dx^b + f\left(\{x^a\}\right) d\Omega^2 \tag{2.8}$$

where $\tilde{W}_{ab}\left(\{x^a\}\right)$ with $\{a, b\} = \{0, 1, \cdots, p\}$ is a finite $(p+1)$ dimensional dynamical metric and $d\Omega^2$ is the line element on $(D - p - 1)$ dimensional symmetric space.

Now as we will see the metric generated in this large D technique will be dual to a probe membrane dynamics embedded in background space with a velocity field and shape function and hence the symmetry of the metric will be there in the membrane also. So, this dual membrane will be dynamical only along the x^a directions and simply wrap the symmetric space. The requirement that the solutions generated in this scheme have this particular symmetry is less restrictive than it sounds. Firstly, although in the derivation of the equations of this dual membrane, the symmetry has been assumed the final membrane equations are entirely covariant on the membrane world volume, and the isometry directions are not special. Secondly, a large class of interesting space-time (e.g the collision of black holes) indeed preserve this large isometry group.

In such cases, it could be shown that for any generic vector or one-form, the order of its divergence will be D times larger than the order of the vector or the one form and it will apply to any arbitrary tensor with any number of indices [24, 25, 26]. In the same way, it could be shown that if the background metric have that symmetry then we will has the following.

$$R_{ABCD}|_{\text{on } \bar{\mathcal{W}}_{AB}} = \mathcal{O}(1), \quad R_{AB}|_{\text{on } \bar{\mathcal{W}}_{AB}} = \mathcal{O}(D), \quad R|_{\text{on } \bar{\mathcal{W}}_{AB}} = \mathcal{O}\left(D^2\right) \tag{2.9}$$

So, Einstein's tensor evaluated on the background metric will be $\mathcal{O}\left(D^2\right)$. It will turn out that the leading ansatz (2.3) will solve Einstein's equations (at $\mathcal{O}\left(D^2\right)$) provided

$$\nabla^2 \psi^{-D} = 0, \quad \text{and} \quad \nabla \cdot U = \mathcal{O}(1), \quad \text{at } \psi = 1 \tag{2.10}$$

where, all contractions are with respect to the background metric.

Before proceeding further let us mention a few important properties of the ansatz metric (2.3).

- The static black hole solution of (2.2) is a special case of (2.3) up to corrections of $1/D$ and in that special case $\psi = 1$ is the equation for the event horizon.

- Note the inverse metric is

$$\mathcal{W}^{AB}_{(0)} = \bar{\mathcal{W}}^{AB} - \psi^{-D} O^A O^B$$

  , where all indices are raised using $\bar{\mathcal{W}}_{AB}$ Now it follows that

$$n_A n_B \mathcal{W}^{AB}_{(0)} = 1 - \frac{1}{\psi^D} \quad .$$

  So, $\psi = 1$ is a null co-dimension one submanifold of the ansatz metric of (2.3). At least when at late times the ansatz metric (2.3) settles down to a stationary black hole solution, this co-dimension one manifold may be identified with the event horizon of space-time. We will refer to this submanifold as 'membrane'.

- The deviation of this ansatz metric from the background space is proportional to $\psi^{-D} \approx e^{-D(\psi-1)}$. So, the ansatz space-time approaches the background space-time exponentially rapidly for $\psi - 1 \gg \frac{1}{D}$. Hence, the region of thickness $\mathcal{O}\left(\frac{1}{D}\right)$ around $\psi = 1$ is non-trivial and we call this region 'membrane region'.

- For $\psi - 1 \ll \frac{1}{D}$, as $D \to \infty$, the ansatz metric blows up. But this region lies entirely inside the event horizon and is causally disconnected from dynamics on and outside the membrane and hence is irrelevant for predicting solution outside.

In summary, our ansatz space-time reduces to background geometry for $\psi - 1 >> \frac{1}{D}$ and so solves Einstein's equations there. The region $\psi - 1 << \frac{1}{D}$ lies entirely inside the event horizon and so irrelevant for predicting solution outside. It remains to study the region $\psi - 1 \sim \frac{1}{D}$ around $\psi = 1$. So we need to solve Einstein's equations only within the 'membrane region'.

We use the ansatz metric of (2.3) as the starting point of our perturbation theory and solve Einstein's equations order by order in $1/D$. The leading ansatz is parametrized by the function ψ and the velocity one-form U. To solve Einstein's equation at the very leading order, these variables need to be satisfied some conditions only on $\psi = 1$ hypersurface (2.10). So, there is a ambiguity in defining the function ψ and the unit normalized velocity one-form U away from $\psi = 1$ hypersurface. To fix this ambiguity we will choose some conditions on ψ and U. These conditions are referred to as 'subsidiary conditions'. We will choose the subsidiary conditions on ψ as follows.

$$\nabla^2 \psi^{-D} = 0, \quad \text{everywhere} \quad . \tag{2.11}$$

where the covariant derivatives are with respect to the background metric.

$\psi = 1$ is the event horizon of the full space-time to all orders in $1/D$ expansion. With this initial condition we can always determine ψ in an expansion in $1/D$ solving (2.11).

Similarly, the condition on O is fixed by

$$\begin{aligned} &O \cdot O = 0, \quad O \cdot n = 1, \quad P^B_A \, (O \cdot \nabla) \, O^A = 0, \quad \text{everywhere} \quad . \\ &P^B_A = \delta^B_A - n_A O^B - O_A n^B + O_A O^B \end{aligned} \tag{2.12}$$

where all the contractions and covariant derivative are with respect to the background metric.

It is important to note that (2.11) and (2.12) are obeyed on $\psi = 1$.

Two different ansatz metrics whose shape function ψ and the velocity field U agree on $\psi = 1$ but differ at larger value of ψ, actually describe the same space-time on and outside $\psi = 1$ at leading order in $1/D$. In order to restrict attention only to inequivalent starting points of perturbation theory we need to define ψ and U everywhere in space-time in terms of the shape of the membrane and the values of the U on the membrane.

Note also that two different choices of these 'subsidiary conditions' produce the same space-time geometry. But they look different as they are expressed in terms of two different sets of variables. For example the 'subsidiary conditions' in [24, 25, 26] are different from the 'subsidiary conditions' in [27, 42, 43] and the metrics look different in these two different choices. These choices are made in such a way that the solutions look simpler.

Now we are in a stage to implement the perturbation theory and search for solution of the form

$$\begin{aligned} \mathcal{W}_{AB} &= \sum_{n=0}^{\infty} \frac{\mathcal{W}^{(n)}_{AB}}{D^n} \\ \mathcal{W}^{(0)}_{AB} &= \bar{\mathcal{W}}_{AB} + \psi^{-D} O_A O_B \end{aligned} \tag{2.13}$$

and attempt to find the corrections $\mathcal{W}^{(1)}_{AB}, \mathcal{W}^{(2)}_{AB}, \cdots$, solving Einstein's equations order by order in $1/D$. In order to find unambiguous solution we first fix coordinate redefinition freedom by choosing a gauge such that

$$O^A \mathcal{W}^{(n)}_{AB} = 0 \quad \text{for } n \geq 1 \quad . \tag{2.14}$$

It will turn out that we can find corrections to ansatz metrics consistently in $1/D$ only when the shape function ψ and the velocity field U satisfy some integrability constraint - a membrane equations of motion which follow from constraint equations of gravity. These membrane equations define a well-posed initial value problem for membrane dynamics. For every configuration that obeys these membrane equations, one can construct a metric that obeys Einstein's equations to appropriate order in $1/D$. Hence it follows that the solutions of these membrane equations are in one to one correspondence with dynamical black hole solutions that solve Einstein's equations order by order in $1/D$.

In this procedure, the solutions to Einstein's equations have been found in flat as well as AdS/dS background for pure gravity and Einstein-Maxwell system [24, 25, 26, 27, 28, 29].

2.2 The fluid-gravity correspondence

Here we shall very briefly describe the method of 'derivative expansion' [13]. In chapter 4 we have described this procedure in more detail while constructing the solutions for Einstein-Maxwell systems.

2.2.1 Perturbation parameter in 'derivative expansion'

'Derivative expansion' is used to study two of the best-studied non-linear partial differential equations in physics, namely, Einstein's equations of general relativity and Navier-Stokes equations of hydrodynamics. It is used to construct 'black hole type' solutions (i.e space-time with a singularity shielded by some horizon [44]) to Einstein's equations in presence of negative cosmological constant in dimension D.

Einstein's equations:

$$\mathcal{E}_{AB} \equiv R_{AB} + (D-1)\lambda^2 g_{AB} = 0 \tag{2.15}$$

λ is the inverse of AdS radius. We shall work in units where λ is set to one.
The solutions generated in this procedure are dual to the solutions of relativistic generalization of Navier-Stokes equations in $(D-1)$ dimensional Minkowski space (D is arbitrary). The different gravity solutions generated in this procedure are characterized by

1. Unit normalized velocity: $u^\mu(x)$

2. Local temperature: $T(x) = \left(\frac{D-1}{4\pi}\right) r_H(x)$

For the moment, r_H is just some arbitrary length scale. It will be related to the horizon scale of the dynamical black-brane geometry.
$\{x^\mu\}, \quad \mu = \{0, 1, \cdots, D-2\}$ are the coordinates on the flat space-time whose metric is simply given by the flat space metric, $\eta_{\mu\nu} = Diag\{-1, 1, 1, 1\cdots\}$.

In this procedure, the fluid velocity and the temperature are slowly varying functions with respect to the length scale set by $r_H(x)$. Mathematically, it can be stated as follows.

If we choose an arbitrary point x_0^μ and scale the coordinates in such a way that in the new coordinate $r_H(x_0) = 1$. Then the derivative expansion would be applicable if in this new scaled coordinate

$$
\begin{aligned}
&|\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_n} r_H|_{x_0} << |\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_{n-1}} r_H|_{x_0} << \cdots << |\bar\partial_{\alpha_1} r_H|_{x_0} << 1 \quad \forall\, n, \alpha_i, x_0\\
&|\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_n} u^\mu|_{x_0} << |\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_{n-1}} u^\mu|_{x_0} << \cdots << |\bar\partial_{\alpha_1} u^\mu|_{x_0} << |u^\mu| \quad \forall\, n, \alpha_i, x_0
\end{aligned}
\tag{2.16}
$$

Hence, the number of ∂_α derivatives in a given term determines how suppressed the term is[1]. Then in the original x^μ coordinates, each derivative ∂_μ corresponds to $r_H\bar\partial_\mu$. Therefore if we work in x^μ coordinate which is not defined around any given point like $\bar x^\mu$ then the perturbation parameter schematically is $\sim r_H^{-1}\partial_\mu$. [2].

In derivative expansion, the starting point of the perturbation is
(where $\{r, x^\mu\}, \quad \mu = \{0, 1, \cdots, D-2\}$. We will choose units such that, λ, appearing

[1]It may happen that at a point in space-time, some nth order term is smaller or comparable to some $(n+1)$th order term. In that situation, one needs to rearrange the expansion around such points.

[2]For a conformal fluid in finite dimension there is only one length scale which is set by the fluid temperature. But as D grows large $T(x)$ and $r_H \sim \frac{T(x)}{D}$ are two parametrically separated length scales. In (2.16) we have chosen r_H to be the relevant scale and set it to order $\mathcal{O}(1)$. Here the scaling of temperature with D is different from the D scaling of the temperature in [45].

in equation (2.15) is set to one)[3].

$$
\begin{aligned}
&ds^2 = -2u_\mu dx^\mu dr - r^2\ f\left(r/r_H\right)\ u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\
&\text{where} \quad f(z) = 1 - \frac{1}{z^{D-1}}, \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu
\end{aligned}
\tag{2.17}
$$

It is a boosted black-brane metric in AdS space.
If u_μ and r_H are constants then equation (2.17) is an exact solution to equation (2.15).
In 'derivative expansion' u^μ and r_H are allowed to be functions of the coordinates $\{x^\mu\}$. Then obviously (2.17) will not be a solution to equation (2.15). We need to add small corrections to (2.17). Treating derivatives on u^μ and r_H small we will be able to solve (2.15) order by order in derivatives. The r dependence of these small corrections could be determined exactly. It will turn out that equations (2.15) can be solved order by order if and only if u^μ and r_H satisfy the relativistic generalization of Navier-Stokes equations, which turn out to be the constraint equations of gravity.

2.3 Comparing fluid gravity and membrane gravity duality at leading order

Following [1] in this section we shall briefly describe the comparison between membrane-gravity and fluid-gravity duality in the first non-trivial order which will create the base for discussion of the next chapters.
As we have already discussed in the previous section, the metric constructed in 'large D expansion' [27] can be written in a 'split' form, where the metric is written as 'background' plus 'something else'. This means this metric always admits a particular point-wise map to the 'background' geometry (in our case pure AdS).
On the other hand, the metric constructed in derivative expansion, to begin with, does not have this 'split' and there is no guarantee that such a map would exist for this case also.

In this section, we will show that such a map also exists for the 'hydrodynamic metric'[4] also. We will see that such a map also exists for the 'hydrodynamic metric' even when the number of dimensions is finite. After that, we will match the two

[3] Note that we need to be careful about the scaling of λ while taking $D \to \infty$ limit. In this case, λ would be fixed to one in large D limit.

[4] In this section, we will refer to the dynamical black brane solution generated in 'derivative expansion' as the 'hydrodynamic metric'.

gravity solutions constructed in these two techniques up to the first non-trivial order in both perturbation parameters. As we will see in chapter 3, it would get more non-trivial at the next order.

In the analysis, we could show the equivalence of both the 'large D metric' and the 'hydrodynamic metric' in large dimensions. We have actually rewritten the 'hydrodynamic metric' in terms of membrane degrees of freedom in large dimensions. After implementing the correct coordinate transformations we could rewrite the labelling variables of the 'large D metric' (the membrane velocity and shape function) in terms of the labelling variables of the 'hydrodynamic metric' (the local fluid temperature and fluid velocity). This analysis in a sense gives hope of a possible duality between fluid and membrane dynamics in $D \to \infty$ limit (more ambitiously in finite dimensions) where gravity has no role to play (See [45], [36] for such a rewriting of fluid equations).

In section - (2.3.1) we have first discussed the overlap regime of these two perturbation techniques. Then in section - (2.3.2) we have discussed the map between the bulk of the 'black-brane' space-time and the pure AdS space, mentioned above and described an algorithm to construct the map, whenever it exists. In section - (2.3.3) we have compared the two metrics in the two perturbation techniques along with their dual systems (namely Navier-Stokes' equation and membrane equations) in large dimensions. We have also worked out the map between the two different sets of dual variables in these two perturbation schemes. In section - (3.7) we have concluded.

2.3.1 The overlap regime

Now, we will discuss whether we could apply both the 'large D expansion' and the 'derivative expansion' simultaneously in any regime of parameter space of the solutions. We have already discussed the perturbation parameter in the 'derivative expansion' in section 2.2.1. Now at first, we will discuss the perturbation parameter in 'large D expansion'. We shall see that these two perturbation parameters can be applied independently without any constraint on their ratio.

After that, we will compare the two metrics, generated using these two perturbation schemes, without any constraint on the ratio between these two perturbation parameters.

2.3.1.1 Perturbation parameter in $\left(\frac{1}{D}\right)$ expansion

'Large D expansion' technique is applicable in large number of space-time dimension (denoted as D), where gravity solutions are generated in a series expansion in powers of $\left(\frac{1}{D}\right)$. Here $\left(\frac{1}{D}\right)$ is the perturbation parameter (a dimensionless number) where,

$$\left(\frac{1}{D}\right) << 1$$

Note that 'derivative expansion' is applicable only in presence of negative cosmological constant in Einstein's equations. But the 'large D expansion' technique can be applied with or without the presence of cosmological constant provided λ, the AdS radius (see equation (2.15) in subsection - 2.2.1) has been kept fixed one takes D large. Note that the choice of λ defined previously for 'derivative expansion' is consistent with this large D scaling.

As we have discussed previously 2.1, the ansatz metric in 'large D expansion' can be written as follows.

$$dS^2 \equiv \mathcal{G}_{AB}\ dX^A dX^B = \bar{G}_{AB}\ dX^A dX^B + \psi^{-D}(O_A\ dX^A)^2 \tag{2.18}$$

where $\bar{G}_{AB}$, ψ and O_A are defined as follows.

1. $\bar{G}_{AB}$ is the 'background' metric. In our case, it is pure AdS metric.
   We have to choose coordinates in such a way that the metric $\bar{G}_{AB}$ is smooth and all components of the Riemann curvature tensors are of order $\mathcal{O}(1)$ or smaller in terms of the order counting in large D.

2. $\left(\psi^{-D}\right)$ is a harmonic function w.r.t $\bar{G}_{AB}$.

3. O_A is dual to a tangent vector to a null geodesic in the background metric such that $O_A n_B\ \bar{G}^{AB} = 1$. Where n_A is the unit normal to the constant ψ hypersurfaces.

It will turn out that the metric in equation (2.18) will solve Einstein's equations (2.15) at leading order (which is $\mathcal{O}(D^2)$) provided the divergence of $\mathcal{O}(1)$ vector field, $U^A \equiv n^A - O^A$ is also $\mathcal{O}(1)$. Mathematically it can be written as

$$\nabla\cdot U \equiv \left(\nabla\cdot n - \nabla\cdot O\right)_{\psi=1} = \mathcal{O}\left(1\right) \tag{2.19}$$

where $\nabla \equiv$ covariant derivative w.r.t. $\bar{G}_{AB}$

At first sight equation (2.19) does not seem to constraint the $\mathcal{O}(1)$ vector field U^A, since all the components of it along with their derivatives are $\mathcal{O}(1)$. But it is actually a constraint in the validity regime of large D technique, since in the validity regime of large D the divergence of a $\mathcal{O}(1)$ vector field is $\mathcal{O}(D)$.[5].
We can ensure such scaling in the dynamics by considering that the metric is dynamical only along some finite number of directions and there is some symmetry in the rest of the directions [27].
We will assume that such a scaling exists in all the geometry we will consider including the dual hydrodynamics. For example, we shall assume that the divergence of the fluid velocity u^μ, which we shall denote by $\Theta(\equiv \partial_\mu u^\mu)$ is of $\mathcal{O}(D)$, whereas u^μ itself is of $\mathcal{O}(1)$.

Now we shall mention some important properties of the leading ansatz metric in (2.23) and refer to [27] for a detailed discussion.
With the above mentioned conditions it can be shown that $\psi = 1$ hypersurface is a null hypersurface and we will identify it as the event horizon of the full space-time geometry
Note also that if we are finitely away from the $\psi = 1$ hypersurface, in large D limit the metric reduces to the background metric $\bar{G}_{AB}$.
Now we shall consider the region of thickness of the order of $\mathcal{O}\left(\frac{1}{D}\right)$ around $\psi = 1$ hypersurface. In this region[6], the $\left(\frac{1}{D}\right)$ expansion would lead to a nontrivial correction to the leading space-time geometry. To see how, let us do the following coordinate transformation.

$$X^A = X_0^A + \frac{\tilde{x}^A}{D} \quad \partial_A = D\ \tilde{\partial}_A$$

where $\{X_0^A\}$ is an arbitrary point on the $\psi = 1$ hypersurface. In these new coordi-

[5]This requirement is not restrictive as it seems. To explicitly see it let us consider a coordinate system $\{z, y^\mu\}$ in which the background metric takes the following form.

$$\begin{aligned} \bar{G}_{zz} &= \frac{1}{z^2}, \quad \bar{G}_{\mu\nu} = z^2\eta_{\mu\nu} \quad \mathrm{Det}[\bar{G}] = -z^{(D-2)} \\ \nabla\cdot V &= z^{-(D-2)}\partial_z\left[z^{(D-2)}V^z\right] + \partial_\mu V^\mu \\ &= \partial_z V^z + \partial_\mu V^\mu + (D-2)\left(\frac{V_z}{z}\right) \end{aligned} \tag{2.20}$$

Here obviously the first term is a $\mathcal{O}(1)$. Since in large D limit a large number of indices are summed over term, the second term could potentially be of order $\mathcal{O}(D)$. The third term is $\mathcal{O}(D)$ provided $\left(\frac{V_z}{z}\right)$ is not very small. The second and third terms can potentially cancel for a particular vector field . Equation (2.19) says that $U^A\partial_A$ is such a fine-tuned vector field.

[6]Following [27] , we shall refer to this region as 'membrane region'

nates

$$dS^2 = D^2 G_{AB}\ d\tilde{x}^A d\tilde{x}^B, \quad \text{where} \quad G_{AB} = \mathcal{G}_{AB}\left(X_0 + \frac{\tilde{x}}{D}\right) \tag{2.21}$$

Now, if $\tilde{x}^A$ is not as large as D, it is possible to expand ψ^{-D}, O_A and $\bar{G}_{AB}$ around X_0^A.

$$\begin{aligned} &\psi^{-D}(X^A) = e^{-\tilde{x}^A N_A} + \mathcal{O}\left(\frac{1}{D}\right), \quad \text{where} \quad N_A = [\partial_A \psi]_{X_0^A} \\ &O^A(X) = O^A|_{X_0^A} + + \mathcal{O}\left(\frac{1}{D}\right), \quad G_{AB}(X) = G_{AB}|_{X_0^A} + \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{2.22}$$

Note that from the second condition (see the discussion below equation (2.18)) it follows that

$$\text{Extrinsic curvature of } (\psi = 1) \text{ surface} = K|_{\psi=1} = D\sqrt{N_A N_B \bar{G}^{AB}} + \mathcal{O}(1)$$

Substituting equation (2.22) in equation (2.21) we find

$$\begin{aligned} G_{AB} = &\ O_A(X_0)\ n_B(X_0) + O_B(X_0)\ n_A(X_0) + P_{AB}(X_0) \\ &- \left(1 - e^{-\tilde{x}^A N_A}\right)\ O_A(X_0)\ O_B(X_0) + \mathcal{O}\left(\frac{1}{D}\right) \\ &\text{where} \quad P_{AB}(X^0) \equiv \text{projector perpendicular to } n_A(X_0) \text{ and } O_A(X_0) \\ &n_A = \frac{\partial_A \psi}{\sqrt{(\partial_A \psi)(\partial_B \psi)\bar{G}^{AB}}} \end{aligned} \tag{2.23}$$

Hence, at the leading order, the variation of the metric in 2.23 is non-trivial only along the directions N_A and variations along all the other directions are suppressed in large D. It is very similar to the starting metric in derivative expansion in equation (2.17) , where the metric has non-trivial variation only along the direction r. So, we can say that within the 'membrane region', the large D expansion almost reduces to the derivative expansion provided the metric in 2.23 solves Einstein's equations at the leading order. And the conditions mentioned below equation (2.18) along with equation (2.19) guarantee that this is true.
After finding the leading solution we can apply the above-mentioned procedure to find the subleading corrections which will handle the variations of N_A and O_A along the $\psi = \textit{constant}$ hypersurface provided the variations of them are suppressed in $1/D$ (in the unscaled X^A coordinates). It can be stated in the following way. We

should be able to choose a coordinate system along the $\psi = 1$ hypersurface such that

$$\left[\bar{G}^{AB}\left(\partial_A\, \psi^{-D}\right)\left(\partial_B\, \psi^{-D}\right)\right]^{-\frac{1}{2}} \partial_A\,|_{\text{horizon}} << 1 \quad . \tag{2.24}$$

We only need to impose this condition only on $\psi = 1$ and the conditions below equation (2.18) will guarantee that they are satisfied on all $\psi = constant$ hypersurfaces.

As already explained, the solutions generated in 'large D expansion' technique are expressed in terms ψ and the one-form $O_A\ dX^A$. These fields need to satisfy the conditions written below equation (2.18). Now to fix the fields completely we need to specify the boundary conditions along any fixed surface and we shall fix them on the $\psi = 1$ hypersurface. Different metric solutions are labelled by the auxiliary function ψ and the components of the one-form field O_A projected along the surface. Like 'derivative expansion', the metric correction can be solved order by order provided these defining data (The auxiliary field O_A and ψ) satisfy some constraint equation, which we call the 'membrane equation'.

2.3.1.2 Comparison between two perturbation schemes

In subsection-(2.3.1.1), we have explained that within the membrane region, large D expansion is *almost* reduces to 'derivative expansion' as discussed in subsection-(2.2.1). But they are not exactly the same. The leading ansatz for both 'large D expansion' and 'derivative expansion' looks quite different and we can not expect any 'overlap regime' between these two perturbation schemes if they compute perturbations around two completely different geometries. So, in finding an 'overlap regime', at first we need to understand where in the parameter-space and in what sense, equation (2.17) and (2.21) describe the same leading space-time.
Though the leading geometries of these two techniques look different , they both have similar geometric properties - for example the existence of a curvature singularity. In the black-brane metric (2.17) it is located at $r = 0$ and on the other hand the metric (2.21) is singular at $\psi = 0$. Also, in both cases the singularity is shielded by some event-horizon[7].
To explicitly see the similarities, let us now choose a coordinate system $X^A \equiv \{\rho, X^\mu\}$, such that the background metric- $\bar{G}_{AB}$ in equation (2.22) takes the follow-

[7]So far, the way both the techniques of 'large-D expansion' and 'derivative expansion' are developed, the existence of a horizon is a must.

ing form

$$\bar{G}_{AB}\ dX^A\ dX^B = \frac{d\rho^2}{\rho^2} + \rho^2\eta_{\mu\nu}dX^\mu dX^\nu, \tag{2.25}$$

Then in this coordinate, the following metric is an exact solution of equation (2.15)

$$ds^2 = \frac{d\rho^2}{\rho^2} + \rho^2\eta_{\mu\nu}dX^\mu dX^\nu + \left(\frac{\rho}{r_H}\right)^{-(D-1)} \left(\frac{d\rho}{\rho} - \rho\ dt\right)^2 \tag{2.26}$$

Note that this is the Schwarzschild black-brane metric, written in Kerr-Schild form. The metric [27] has the following properties.

- $\left(\frac{\rho}{r_H}\right)^{-(D-1)}$ is harmonic function with respect to the background up to correction of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

  $$\nabla^2\left(\frac{\rho}{r_H}\right)^{-(D-1)} = \mathcal{O}\left(\frac{1}{D}\right)^2$$

  Hence the function $\left(\frac{\rho}{r_H}\right)^{-(D-1)}$ could be identified with ψ^{-D} appearing in (2.18) up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

- It can easily be shown that the one form $\left(\frac{d\rho}{\rho} - \rho\ dt\right)$ is null. It also satisfies the geodesic equation. Further, one can easily check that the contraction of this one-form field with the unit normal to the constant ρ hypersurfaces is one. Hence, we can identify this with the null one form $O_A dX^A$

Hence we can recast the metric in (2.26), which is an exact solution of (2.15), in the form of our leading ansatz in large D expansion up to corrections that are subleading in $\left(\frac{1}{D}\right)$ expansion. The metric in equation (2.26) can also be expanded around a given point on the horizon $\rho = r_H$, in the same way that has been done (see equation (2.23)) in the previous subsection with the following identifications.

$$\begin{aligned} &N_A\ dX^A|_{\rho=1} = \frac{d\rho}{r_H}, \quad O_A\ dX^A|_{\rho=1} = \frac{d\rho}{r_H} - r_H\ dt \\ &n_A\ dX^A = \frac{N_A\ dX^A}{\sqrt{N_A N^A}} = \frac{d\rho}{r_H} \end{aligned} \tag{2.27}$$

So, the leading term in this expansion, after writing in terms of N_A and O_A would have exactly the same form as that of the metric written in equation (2.21). The main difference between the leading ansatz in equation (2.18) and in equation (2.26) is that in the later N_A and O_A satisfy (2.27) equation everywhere along the horizon, in the same $\{\rho, y^\mu\}$ coordinates. For the leading ansatz in (2.18) also a local $\{\rho, t\}$ coordinates can always be chosen by reversing the equations in (2.27). But for a generic ψ and O_A, we can not do this globally and this is the reason why the leading ansatz is not an exact solution of (2.15). However, the deviation from the exact solution will be proportional to the derivatives of N_A and O_A and therefore contribute in subleading order. Hence, locally around a point on the horizon, the leading ansatz for 'large D expansion' expansion looks like a Schwarzschild black-brane written in a Kerr-Schild form where the local ρ and t coordinates respectively are oriented along the direction of the normal N_A and the direction of the field O_A projected along the membrane $\psi = 1$.

Now we will discuss the leading ansatz for the metric in derivative expansion. As explained in [4], if we choose $r_H = constant$ and $u^\mu = \{1, 0, 0, \cdots\}$ then the leading ansatz in derivative expansion, equation (2.17) , reduces to Schwarzschild black-brane metric written in Eddington-Finkelstein coordinates. Also locally at any point $\{x_0^\mu\}$, we can always choose a coordinate system such that $u^\mu(x_0) = \{1, 0, 0, \cdots\}$. So, by appropriate choice of coordinates locally the metric described in (2.17) could always be made to look like a Schwarzschild black-brane metric, though in a different coordinate than described in equation (2.18). Hence, the starting point of these two different perturbation expansions are 'locally 'same and we can expect an overlap regime.

But the difference lies in the concept of 'locality' and also the space of defining data in these two perturbation techniques are different. In case of 'large-D' expansion, the labelling data of the gravity solutions is specified on the horizon whereas for the metric generated in 'derivative expansion' it is defined on the boundary of the AdS space.
The range of validity for 'large-D' expansion is given in equation (2.24). Now if we replace $\partial_A \psi^{-D}|_{\text{horizon}}$ by $(-DN_A)$ then the condition (2.24) reduces to the existence of coordinate system such that

$$\partial_A \,|_{\text{horizon}} << D \tag{2.28}$$

which looks similar to the validity regime for 'derivative expansion' , as we have mentioned in the subsection (2.2.1)

$$r_H^{-1} \partial_\mu << 1 \tag{2.29}$$

Now if somehow we could map each point on the boundary to a point on the horizon (which is viewed as a hypersurface embedded in the background space-time), then the same $\{x^\mu\}$ coordinates could be used as coordinates along the horizon. Then, whenever r_H is of order $\mathcal{O}(1)$ in terms of order counting in 'large-D', the inequality (2.29) would imply equation (2.28). Equivalently we could say, as $D \to \infty$, all solutions generated in 'derivative expansion' could be legitimately expanded further in $\left(\frac{1}{D}\right)$, though the reverse may not be true.

Now we know that for the case of exact Schwarzschild black-brane solutions ∂_A and ∂_μ are simply related (without any extra factor of D). It is just the the well-known coordinate transformation that one should use to go from Kerr-Schild form to Eddington-Finkelstein form of the black-brane metric. The required map from the horizon to boundary coordinates is also given by this coordinate transformation. After introducing the perturbations on both sides, relation between these two sets of coordinate systems are expected to get correction, but in a controlled and perturbative manner, and thus it will maintain the above argument for the existence of overlap.

In summary, a region of overlap exists between these two perturbative techniques. In this chapter, our goal is to match them in the regime of overlap. As we have discussed, the key step is to determine the map between ∂_A and ∂_μ. We shall elaborate that in the next section.

2.3.2 Transforming to 'large-D' gauge

From section (2.3.1) we have seen that whenever derivative expansion is applicable, we can always apply $1/D$ expansion but the reverse may not be true. So, a dynamical black-brane metric generated in 'derivative expansion' [5] when further expanded in $1/D$ should match with the metric generated in large D expansion (equation (8.1) of [27]) after appropriate coordinate transformations and field redefinition. In this section, we will construct the algorithm to find out the appropriate coordinate transformations.

Before proceeding further let us understand the question in a little more detail.

Both the large D expansion and 'derivative expansion' generate dynamical black-brane geometry in terms of labeling data defined on a co-dimension one hypersurface. In 'derivative expansion' the labelling data is defined on the conformal boundary of the AdS space and in the case of 'large D expansion' it is defined on a co-dimensional one fluctuating membrane embedded in the asymptotic geometry. Hence both procedure uses a particular map from the full space-time geometry to a co-dimension one hypersurface.

In derivative expansion given a unit normalized velocity field and a temperature field defined on a $(D-1)$ dimensional flat Minkowski space, which satisfies the relativistic generalization of Navier-Stokes' equations one can construct a D dimensional dynamical black-brane geometry solving Einstein's equations, which we refer to as bulk [4, 5]. In [46] the authors have also described how to reverse this construction. Starting from a dynamical black-brane metric they could read off the dual fluid variables.

Similarly, in large D expansion starting from a membrane velocity and shape function defined on a co-dimension one membrane one could systematically construct a dynamical black-brane geometry order by order in $1/D$ [27]. But with this technique, the correspondence has been shown only in one direction [27]. To complete the map we also need to know how to read off the 'membrane data' given an arbitrary dynamical black-brane geometry.

In this section, we will construct an algorithm to determine this 'membrane-bulk map', which is similar to the discussion of [46] in the context of writing the rotating black hole metric to the form of the hydrodynamic metric.

2.3.2.1 Bulk-Membrane map

The 'large-D expansion' technique [27], always generate the dynamical black-brane metric G_{AB} in a 'split' form. Here the 'split' is specified in terms of an auxiliary function ψ and an auxiliary one-form field $O^A\partial_A$. In terms of equation,

$$G_{AB} = \bar{G}_{AB} + G^{(\text{rest})}_{AB} \tag{2.30}$$

where $\bar{G}_{AB}$ is the background metric and $G^{(\text{rest})}_{AB}$ is such that there exists a null geodesic vector field $O^A\partial_A$ in the background, satisfying

$$O^A\ G_{AB} = O^A\ \bar{G}_{AB} \quad \Rightarrow \quad O^A\ G^{(\text{rest})}_{AB} = 0 \ \ . \tag{2.31}$$

The normalization of this null geodesic vector is determined in terms of the function ψ which is defined as follows.

1. $\left(\psi^{-D}\right)$ is a harmonic function with respect to the background metric $\bar{G}_{AB}$.

2. $\psi = 1$ hypersurface, viewed as an embedded surface in full space-time geometry, is the dynamical event horizon. The boundary condition on ψ is specified

in this way.

Once we have fixed ψ, the normalization of O^A is fixed through the following condition.

$$O^A n_A = 1.$$

where n_A is the unit normal on the constant ψ hypersurfaces (viewed as hypersurfaces embedded in the background space-time).

Equations (2.30) and (2.31) together specify a map between two entirely different geometries, with metric $\bar{G}_{AB}$ and G_{AB} respectively, both of which satisfy equation (2.15). Hence, for writing an arbitrary dynamical black-brane metric, which admits $\left(\frac{1}{D}\right)$ expansion, in the form described in (2.30), the first step would be to determine this map or the 'split' of the metric between 'background' and the 'rest', so that the equation (2.31) is obeyed.

Now in previous subsection, we have seen that this 'map' is crucially dependent on the vector field $O^A\partial_A$ and the function ψ. As both of them are defined using the 'background' geometry, we immediately face a problem, because given an arbitrary black-brane geometry, it is the 'background' that we are after.
For example, we could always determine the location of the event horizon for a given black-brane metric, but we would never know its embedding in the background, unless we know the 'split' and hence we would not be able to construct the ψ function, by solving the harmonicity condition on ψ^{-D}. If we do not know ψ we would not be able to normalize O^A, as required.

So we need to figure out some equivalent formulation of this 'split' just in terms of the full space-time metric. The following observation save us. It can be shown that if G_{AB} admits a split between $\bar{G}_{AB}$ and $G^{(\text{rest})}_{AB}$ satisfying $O^A G^{(\text{rest})}_{AB} = 0$, then the vector - $O^A\partial_A$, which is a null geodesic with respect to background metric $\bar{G}_{AB}$, is also a null geodesic with respect to the full space-time metric G_{AB}.

Proof:
We know that

$$(O\cdot\nabla)O^A = \kappa\, O^A$$

where ∇ denotes the covariant derivative with respect to the metric $\bar{G}_{AB}$ and κ is the proportionality factor. We want to show that

$$(O \cdot \bar{\nabla})O^A \propto O^A, \quad \text{where } \bar{\nabla} \text{ is covariant derivative w.r.t. } G_{AB}$$

Suppose $\bar{\Gamma}^A_{BC}$ denotes the Christoffel symbol corresponding to $\bar{\nabla}_A$ and Γ^A_{BC} denotes the Christoffel symbol corresponding to ∇_A. These two would be related as follows [27].

$$\bar{\Gamma}^A_{BC} = \Gamma^A_{BC} + \underbrace{\frac{1}{2}\left(\nabla_B \left[G^{(\text{rest})}\right]^A_C + \nabla_C \left[G^{(\text{rest})}\right]^A_B - \nabla^A \left[G^{(\text{rest})}\right]_{BC}\right)}_{\delta\Gamma^A_{BC}} \tag{2.32}$$

Here all raising and lowering of indices have been done using $\bar{G}_{AB}$. Note that

$$\begin{aligned} O^B O^C \;\delta\Gamma^A_{BC} =&\; O^B(O\cdot\nabla)\left[G^{(\text{rest})}\right]^A_B - \frac{1}{2}O^B O^C \nabla^A \left[G^{(\text{rest})}\right]_{BC} \\ =&\; -\left[G^{(\text{rest})}\right]^A_B \left[(O\cdot\nabla)O^B\right] + \frac{1}{2}\left(\nabla^A O^C\right)\left[G^{(\text{rest})}\right]_{BC} O^B \\ =&\; \kappa\left(O^C \left[G^{(\text{rest})}\right]^A_C\right) = 0 \end{aligned} \tag{2.33}$$

What we want to show simply follows from equation (2.33)

$$(O\cdot\bar{\nabla})O^A = (O\cdot\nabla)O^A = \kappa\, O^A \tag{2.34}$$

So by solving the null geodesic equation with respect to the full space-time metric G_{AB} we could determine O^A . But to fully determine it, we also need to know κ which is fixed by the normalization of O^A. As already mentioned, the normalization that has been used previously in the application of 'large-D' technique is not suitable for our purpose, since for it we need the knowledge of the 'background' beforehand. But luckily the form of the 'split' defined by the condition $\left[O^A G^{(\text{rest})}_{AB} = 0\right]$ is independent of the normalization of O^A.

So we will first determine another null geodesic field (we shall denote it by $\bar{O}_A$ to remind ourselves of the difference in normalization) which is affinely parametrized and whose inner-product with the normal to event horizon of the full space-time (which, up to normalization, could again be determined without the knowledge of the 'split') is one.

Then we will be at a stage to define the map between the 'background' and the full space-time metric.

Suppose $\{Y^A\}$ denote the coordinates in the background space-time (in our case it is the pure AdS, the metric is denoted by $\bar{G}_{AB}$) and $\{X^A\}$ are the coordinates of the full space-time geometry (the dynamical black-brane geometry, the metric is denoted by $\mathcal{G}_{AB}$). Let us now denote the invertible functions that give a one to one correspondence between these two spaces as $\{f^A\}$.

$$Y^A = f^A(\{X\}) \tag{2.35}$$

The equations that will determine f^A s are the following

$$\bar{O}^A \; \mathcal{G}_{AB}|_{\{X\}} = \bar{O}^A \left(\frac{\partial f^C}{\partial X^A}\right)\left(\frac{\partial f^{C'}}{\partial X^B}\right)\bar{G}_{CC'}|_{\{X\}} \tag{2.36}$$

[8] Here $\bar{O}^A$ are affinely parametrized null geodesics with respect to the full space-time metric i.e.,

$$\bar{O}\cdot\bar{\nabla}\bar{O}^A = 0 \tag{2.37}$$

Equation (2.37) would fix $\bar{O}_A$ completely if we specify the angles it would make with the tangents of the horizons, which is a set of $(D-1)$ numbers. Now what we are actually interested in is not $\bar{O}_A$ but O_A which is related to $\bar{O}_A$ with a normalization. Hence, at this stage we are free to choose the normalization of $\bar{O}_A$, since anyway, we have to re-normalize it again. It will fix one of the $(D-1)$ initial conditions. Rest we will keep arbitrary.
We shall assume

$$\begin{aligned} &\bar{O}^A N_A|_{\text{horizon}} = 1 \\ &\bar{O}^A l^{(i)}_A|_{\text{horizon}} = \text{ some arbitrary functions of horizon cordinates} \end{aligned} \tag{2.38}$$

where N_A is the null normal to the event horizon of the space-time (with some arbitrary normalization) and $l^A_{(i)}\partial_A$ s are the unit normalized space-like tangent vectors to the horizon.
It will turn out that the hydrodynamic metric could be split for a very specific choice of these spatial initial conditions and we will fix them order by order in derivative expansion by matching the the 'large-D' metric and hydrodynamic metric . Once $\bar{O}^A$ is fixed (in terms of these arbitrary angles), by solving equation (3.2) we could determine f^A s up to some integration constants .

If we apply the map (2.35) as a coordinate transformation on the 'background' then equation (3.2) further says that in the new $\{X^A\}$ coordinates the map would

[8] The subscript $\{X\}$ in equation (3.2) denotes that both LHS and RHS of equation (3.2) have to be evaluated in terms $\{X^A\}$ coordinates.

just be an 'identity' map and the full space-time metric $\mathcal{G}_{AB}$ would admit the split as given in equation (2.30) satisfying (2.31) [9].

Once we have figured out the splitting of the full space-time metric into 'background' and the 'rest', we know how to view the event horizon as a surface embedded in the 'background' geometry and therefore the auxiliary function ψ (by exploiting the harmonicity of ψ^{-D} with respect to the background metric) everywhere. Now we can normalize $\bar{O}^A$ as it has been done in [27]. Using these ψ and O^A (appropriately normalized) we will be able to recast any arbitrary metric, that admits large-D expansion, exactly in the form of [27].

2.3.3 Bulk-Membrane map in metric dual to Hydrodynamics

Now we shall implement the above described algorithm for the metric dual to hydrodynamics. For convenience, let us summarize the steps again.

- First determine the equation for the event horizon of the full space-time metric.
- Then Determine the null normal to the horizon.
- Solve equation (2.37) to determine $\bar{O}^A$ everywhere. We require the normal, derived in the previous step, to impose the boundary condition.
- Then choose any arbitrary coordinate system $\{Y^A\}$, where the 'background' has a smooth metric G_{AB}.
- After that solve the equation (3.2) to determine the mapping functions f^A 's.

It is not easy to implement all these steps for a generic dynamical metric. But in this case, the 'derivative expansion' and the fact that f^A 's are exactly known at zeroth order in derivative expansion (which is simply the coordinate transformation between Eddington-Finkelstein and Kerr-Schild form of a static black-brane spacetime) would help us.

Though the zeroth order transformation is already known, as a 'warm-up' exercise et us first re-derive it using the above mentioned algorithm. The condition of

[9]We would also like to emphasize that what we are describing here is not just a gauge or coordinate transformation. The 'split' mentioned in equation (2.30) is a genuine point-wise map between two entirely different geometries. Once we have figured out the 'map', we are free to transform the coordinates further; both G_{AB} and $\bar{G}_{AB}$ would change, but the 'map' will still be there.

'staticity' and translational symmetry of the metric will allow us to solve relevant equations exactly.

2.3.3.1 Zeroth order in 'derivative expansion':

At zeroth order in derivative expansion, the metric dual to hydrodynamics is given by

$$\begin{aligned} &ds^2 = -2u_\mu dx^\mu dr - r^2 f\left(r/r_H\right) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ &\text{where} \quad P_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu, \quad f(z) \equiv \left[1 - z^{-(D-1)}\right], \quad u_\mu u_\nu \eta^{\mu\nu} = -1 \end{aligned} \tag{2.39}$$

We could read off the components of the metric and its inverse.

$$\begin{aligned} &\mathcal{G}_{rr} = 0, \quad \mathcal{G}_{\mu r} = -u_\mu, \quad \mathcal{G}_{\mu\nu} = -r^2 f\left(r/r_H\right) u_\mu u_\nu + r^2 P_{\mu\nu} \\ &\mathcal{G}^{rr} = r^2 f\left(r/r_H\right), \quad \mathcal{G}^{\mu r} = u^\mu, \quad \mathcal{G}^{\mu\nu} = \frac{1}{r^2} P^{\mu\nu} \end{aligned} \tag{2.40}$$

At zero derivative order, both r_H and u^μ could be treated as constants. The event horizon and the null normal to the metric are given by

$$\text{Event Horizon} : \mathcal{S} = r - r_H = 0, \quad N_A \; dX^A = dX^A \partial_A \mathcal{S} = dr \tag{2.41}$$

Let us now figure out the 'map' that would lead to the desired 'split' between 'background' and 'rest'.

We have already determined the event horizon. Now, we have to solve for $\bar{O}^A$, satisfying the conditions

$$\bar{O}^B \bar{\nabla}_B \bar{O}^A = 0, \quad \bar{O}^A \bar{O}^B \mathcal{G}_{AB} = 0, \quad \bar{O}^A N_A|_{r=r_H} = \bar{O}^r|_{r=r_H} = 1$$

At zeroth order in derivative expansion, $\mathcal{G}_{AB}$ has translational symmetry in all the x^μ. The conditions on $\bar{O}^A$ does not break this symmetry. Hence $\bar{O}^A$ must have the form

$$\bar{O}^A \partial_A = h_1(r) \; \partial_r + h_2(r) \; u^\mu \partial_\mu \tag{2.42}$$

Now we shall process the condition that O^A is a null vector field.

$$\begin{aligned} &\bar{O}^A \bar{O}^B \mathcal{G}_{AB} = 0 \\ \Rightarrow \;& 2h_2(r) h_1(r) \mathcal{G}_{\mu r} u^\mu + h_2(r)^2 u^\mu u^\nu \mathcal{G}_{\mu\nu} = 0 \\ \Rightarrow \;& h_2(r) \left[2h_1(r) - r^2 f\left(r/r_H\right) h_2(r)\right] = 0 \\ \Rightarrow \;& h_2(r) = 0 \end{aligned} \tag{2.43}$$

So finally $\bar{O}^A\partial_A = h_1(r)\partial_r$[10].
Substituting this form of $\bar{O}^A$ in the geodesic equation we could see that $h_1(r)$ has to be a constant and then boundary condition simply says that $h_1(r) = 1$

$$\bar{O}^A\partial_A = \bar{O}^r\partial_r = \partial_r \tag{2.44}$$

Now let us choose a coordinate system $Y^A = \{\rho, y^\mu\}$ for the 'background' where the metric takes the following form

$$ds^2_{background} = \frac{d\rho^2}{\rho^2} + \rho^2\eta_{\mu\nu}\ dy^\mu\ dy^\nu \tag{2.45}$$

Again the symmetries of the background metric motivate us to take the following form for the mapping, which gives the one to one correspondence between the background coordinates $\{Y^A\} = \{\rho, y^\mu\}$ and the black-brane coordinates $\{X^A\} = \{r, x^\mu\}$

$$y^\mu = x^\mu + g(r)u^\mu, \quad \rho = h(r) \tag{2.46}$$

We shall now apply the map (2.46) as a coordinate transformation on the background. In the new coordinates (where the map is just an 'identity') the background metric will take the following form

$$\bar{\mathcal{G}}_{rr} = \left(\frac{h'}{h}\right)^2 - (g'h)^2\,,\ \ \bar{\mathcal{G}}_{\mu r} = g'h^2u_\mu,\ \ \bar{\mathcal{G}}_{\mu\nu} = h^2\ \eta_{\mu\nu} \tag{2.47}$$

Here we have suppressed the r dependence and derivative w.r.t r is denoted by prime ($'$). In this coordinates equation (3.2) takes the form

$$\left(\frac{h'}{h}\right)^2 - (g'h)^2 = 0,\ \ g'h^2 = -1 \tag{2.48}$$

These two equation could be solved very simply. The general solution is given by

$$h(r) = \pm(r + c_1), \quad g(r) = \frac{1}{r + c_1} + c_2 \tag{2.49}$$

---

[10]Actually, there is two solutions to (2.43). If we assume $h_2(r) \neq 0$ and finite everywhere, then

$$h_1(r) = \frac{r^2}{2} f\left(r/r_H\right) h_2(r)$$

This implies that $h_1(r)$ will vanish at the horizon $r = r_H$ (which is a zero of the function $f\left(r/r_H\right)$), contradicting the boundary condition on $\bar{O}^r$.

where c_1 and c_2 are two arbitrary constants.
We shall choose the plus sign in the function $h(r)$ to make sure that whenever r increases, ρ also increases.

Now we have to fix the integration constants. Note that once we know the map, we know the form of $\mathcal{G}^{(\text{rest})}_{AB}$, satisfying equation (2.31) by construction.

$$\begin{aligned}\mathcal{G}^{(\text{rest})}_{rr} &= \mathcal{G}^{(\text{rest})}_{r\mu} = 0\\ \mathcal{G}^{(\text{rest})}_{\mu\nu} &= \left[(r+c_1)^2 - r^2 f(r/r_H)\right] u_\mu u_\nu + \left[r^2 - (r+c_1)^2\right] P_{\mu\nu}\end{aligned} \tag{2.50}$$

We further want that as $D \to \infty$, the metric should reduce to its asymptotic form at any finite distance from the event horizon of the space-time or in other words, outside the 'membrane region' $\mathcal{G}^{(\text{rest})}_{\mu\nu}$ must vanish (a region with 'thickness' of the order of $\mathcal{O}\left(\frac{1}{D}\right)$ around the 'membrane', see section (2.3.1.1)). This condition will force us to set $c_1 = 0$. The other constant c_2 is not appearing in the final form of the metric, so, at this order this ambiguity will remain and it is simply a consequence of the translational symmetry in x^μ and y^μ directions. For simplicity, let us also choose $c_2 = 0$. So finally the final form of the map at zeroth order is

$$\rho = r, \qquad y^\mu = x^\mu + \frac{u^\mu}{r} \quad . \tag{2.51}$$

2.3.3.2 First order in derivative expansion

Now we extend the computation of the previous section up to the first order in derivative expansion. Here u^μ and r_H depends on x^μ but any term that has more than one derivatives of u^μ and r_H can be neglected. All calculations presented in this section generically have corrections at order $\mathcal{O}(\partial^2)$.

At first order in derivative expansion the metric dual to hydrodynamics is [5]

$$\begin{aligned}ds^2 = &\ -2u_\mu dx^\mu dr - r^2 f\left(r/r_H\right) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu\\ &+ r\left[-\left(u_\mu a_\nu + u_\nu a_\mu\right) + \frac{2\Theta}{D-2} u_\mu u_\nu + 2F\left(r/r_H\right)\sigma_{\mu\nu}\right] dx^\mu dx^\nu\end{aligned} \tag{2.52}$$

Where,

$$F(r) = r\int_r^\infty dx \frac{x^{D-2}-1}{x(x^{D-1}-1)}$$

And[11]

$$a_\mu = (u\cdot\partial)u_\mu\ , \qquad \Theta = \partial\cdot u\ , \qquad \sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2}\right) - \left(\frac{\Theta}{D-2}\right)P^{\mu\nu} \tag{2.53}$$

We will refer to this metric, described in equation (2.52), as 'hydrodynamic metric'. Here both r_H and u_μ are functions of x^μs. But the functional dependence is not completely arbitrary. It will turn out that the hydrodynamic metric will solve the Einstein's equations (up to corrections of order $\mathcal{O}(\partial^2)$) provided the derivatives of r_H and u_μ satisfies the following equations[12].

$$\frac{(u\cdot\partial)r_H}{r_H} + \frac{\Theta}{D-2} = 0, \quad P^{\mu\nu}\left(\frac{\partial_\mu r_H}{r_H}\right) + a^\nu = 0 \tag{2.54}$$

We read off the components of the metric and its inverse

$$\begin{aligned}
\mathcal{G}_{\mu r} =&\ - u_\mu, \quad \mathcal{G}_{rr} = 0\\
\mathcal{G}_{\mu\nu} =&\ - r^2 f\left(r/r_H\right) u_\mu u_\nu + r^2 P_{\mu\nu}\\
&+ r\left[- \left(u_\mu a_\nu + u_\nu a_\mu\right) + \left(\frac{2\Theta}{D-2}\right) u_\mu u_\nu + 2F\left(r/r_H\right)\sigma_{\mu\nu}\right]
\end{aligned} \tag{2.55}$$

$$\begin{aligned}
\mathcal{G}^{rr} &= r^2 f(r/r_H) - r\left(\frac{2\Theta}{D-2}\right), \qquad \mathcal{G}^{\mu r} = u^\mu - \frac{a^\mu}{r}\\
\mathcal{G}^{\mu\nu} &= \frac{P^{\mu\nu}}{r^2} - \frac{2F(r/r_H)}{r^3}\,\sigma^{\mu\nu}
\end{aligned} \tag{2.56}$$

The horizon is still given by the surface (no correction at first order in derivative, though the normal gets corrected since $\partial_\mu r_H$ is not negligible now.)

$$\text{Event Horizon} : \mathcal{S} = r - r_H = 0, \quad N_A\ dX^A = dX^A\partial_A\mathcal{S} = dr - dx^\mu\ \partial_\mu r_H \tag{2.57}$$

[11] Here '$\cdot$' denotes contraction with respect to $\eta_{\mu\nu}$

[12] These two equations are just the stress tensor conservation equation for a $(D-1)$ dimensional ideal conformal fluid.

We need the Christoffel symbols to compute the geodesic equation.

$$\begin{aligned}
&\Gamma^r_{rr} = 0, \qquad \Gamma^\mu_{rr} = 0\\
&\Gamma^r_{\alpha r} = \left[r f(r/r_H) + \frac{r^2}{2r_H} f'(r/r_H) - \frac{\Theta}{D-2} \right] u_\alpha\\
&\Gamma^\mu_{r\delta} = \frac{1}{2r^2} \left[2r P^\mu_\delta - \partial_\delta u^\mu - u_\delta a^\mu + \partial^\mu u_\delta + u^\mu a_\delta - 2F(r/r_H)\sigma^\mu_\delta + 2\,(r/r_H)\,F'(r/r_H)\sigma^\mu_\delta \right]
\end{aligned} \tag{2.58}$$

The most general correction that could be added to $\bar{O}^A$, at first order in derivative expansion, maintaining it as a null vector with respect to the first order corrected metric:

$$\bar{O}^A \partial_A = \partial_r + w_1(r)\ \Theta\ \partial_r + w_2(r)\ a^\mu \partial_\mu \tag{2.59}$$

We shall fix $w_1(r)$ and $w_2(r)$ using the geodesic equation.
The r component of the geodesic equation gives the following equation.

$$\begin{aligned}
&(\bar{O} \cdot \bar{\nabla})\bar{O}^r = 0\\
\Rightarrow &\bar{O}^r \bar{\nabla}_r \bar{O}^r + \bar{O}^\mu \bar{\nabla}_\mu O^r = 0\\
\Rightarrow &\bar{O}^r \partial_r \bar{O}^r + \Gamma^r_{rr} \bar{O}^r \bar{O}^r + 2\bar{O}^r \bar{O}^\alpha \Gamma^r_{\alpha r} = 0\\
\Rightarrow &(1 + w_1(r)\Theta) w_1'(r)\Theta + 2(1 + w_1(r)\Theta)(w_2(r) a^\alpha)\Gamma^r_{\alpha r} = 0\\
\Rightarrow &w_1'(r) = 0\\
\Rightarrow &w_1(r) = A_1, \qquad \text{where } A_1 \text{ is a constant}
\end{aligned}$$

From the μ component of the geodesic equation we find

$$\begin{aligned}
&(\bar{O} \cdot \bar{\nabla})\bar{O}^\mu = 0\\
\Rightarrow\ &\bar{O}^r \bar{\nabla}_r \bar{O}^\mu + \bar{O}^\lambda \bar{\nabla}_\lambda \bar{O}^\mu = 0\\
\Rightarrow\ &\bar{O}^r \partial_r \bar{O}^\mu + \bar{O}^r \bar{O}^r \Gamma^\mu_{rr} + 2\bar{O}^r \bar{O}^\delta \Gamma^\mu_{r\delta} = 0\\
\Rightarrow\ &\left[w_2'(r) + \frac{2w_2(r)}{r} \right] a^\mu = 0\\
\Rightarrow\ &w_2(r) = \left(\frac{A_2}{r^2} \right), \quad \text{where } A_2 \text{ is another integration constant}
\end{aligned}$$

At this stage

$$\bar{O}^A \partial_A = \partial_r + A_1 \Theta\ \partial_r + \left(\frac{A_2}{r^2} \right)\ a^\mu \partial_\mu \tag{2.60}$$

We can partially fix the integration constants using the boundary conditions.
At horizon

$$\begin{aligned}\bar{O}^A N_A|_{r=r_H} = 1 \quad &\Rightarrow \quad (1 + A_1\Theta) = 1 \Rightarrow \quad A_1 = 0\\ \bar{O}^\mu \partial_\mu r_H = \mathcal{O}\left(\partial^2\right) \quad &\Rightarrow \quad \text{No constraint on } A_2\end{aligned} \tag{2.61}$$

Hence it follows that .

$$\begin{aligned}&\bar{O}^A \partial_A = \partial_r + \left(\frac{A_2}{r^2}\right) \, a^\mu \partial_\mu + \text{terms 2nd order in derivative expansion}\\ \Rightarrow \;&\bar{O}_A \; dX^A = -u_\mu \; dx^\mu + A_2 \; a_\mu \; dx^\mu + \text{terms 2nd order in derivative expansion}\end{aligned} \tag{2.62}$$

Now, we need to solve for the 'mapping functions'. For that like in the previous section let us choose the same coordinates $\{Y^A\}$, so that the background metric takes the form of equation (2.45). It is expected that the mapping functions (2.51) will get corrected by first order terms in derivative expansion.

$$y^\mu = x^\mu + \frac{u^\mu(x)}{r} + f_1(r)\Theta \; u^\mu(x) + f_2(r) \; a^\mu(x), \quad \rho = r + f_3(r) \; \Theta \tag{2.63}$$

As we have done previously, we will apply the map (2.63) as a coordinate transformation on the background metric. In the new coordinates (where the map is just an 'identity') the background metric takes the following form

$$\begin{aligned}\bar{\mathcal{G}}_{rr} &= 2\left(f_1'(r) + \frac{f_3'(r)}{r^2} - \frac{2f_3(r)}{r^3}\right) \; \Theta\\ \bar{\mathcal{G}}_{\mu r} &= -\left[1 - \left(r^2 f_1'(r) - \frac{2f_3(r)}{r}\right)\Theta\right] u_\mu + r^2 f_2'(r) \; a_\mu\\ \bar{\mathcal{G}}_{\mu\nu} &= r^2\left(1 + \frac{2f_3(r)}{r} \; \Theta\right) \; \eta_{\mu\nu} + r\left(\partial_\nu u_\mu + \partial_\mu u_\nu\right)\end{aligned} \tag{2.64}$$

Substituting equation (2.64) in equation (3.2) we find

$$\begin{aligned}&\bar{\mathcal{G}}_{\mu r} + \left(\frac{A_2}{r^2}\right) a^\nu \bar{\mathcal{G}}_{\nu\mu} = -u_\mu + A_2 \; a_\mu + \mathcal{O}\left(\partial^2\right), \quad \bar{\mathcal{G}}_{rr} = 0\\ \Rightarrow \; &r^2 f_1'(r) - \frac{2f_3(r)}{r} = 0, \quad f_2'(r) = 0, \quad f_1'(r) + \frac{f_3'(r)}{r^2} - \frac{f_3(r)}{r^3} = 0\end{aligned} \tag{2.65}$$

The general solution for equation (2.65):

$$\begin{gathered} f_3(r) = C_3, \quad f_2(r) = C_2, \quad f_1(r) = C_1 - \frac{C_3}{r^2} \\ \text{where } C_1, C_2 \text{ and } C_3 \text{ are arbitray constants} \end{gathered} \tag{2.66}$$

In the new $X^A = \{r, x^\mu\}$ coordinates the metric of the background takes the following form

$$\begin{aligned} ds^2_{\text{background}} &= \bar{\mathcal{G}}_{AB} dX^A dX^B \\ &= -2u_\mu dx^\mu \; dr + r^2 \eta_{\mu\nu} dx^\mu \; dx^\nu \\ &\quad + r\left[2C_3\Theta \; \eta_{\mu\nu} + (\partial_\mu u_\nu + \partial_\nu u_\mu)\right] dx^\mu dx^\nu \\ \\ &= -2u_\mu dx^\mu \; dr + r^2 \eta_{\mu\nu} dx^\mu \; dx^\nu \\ &\quad + 2r\left[-C_3\Theta \; u_\mu u_\nu + \left(C_3 + \frac{1}{D-2}\right)\Theta \; P_{\mu\nu} - \left(\frac{a_\mu u_\nu + a_\nu u_\mu}{2}\right) + \sigma_{\mu\nu}\right] dx^\mu dx^\nu \end{aligned} \tag{2.67}$$

In the last step we have rewritten $\mathcal{G}_{\mu\nu}$ using the following identity

$$\partial_\mu u_\nu + \partial_\nu u_\mu = 2\sigma_{\mu\nu} + \left(\frac{2\Theta}{D-2}\right) P_{\mu\nu} - (a_\mu u_\nu + a_\nu u_\mu) \tag{2.68}$$

Once we know the background, we could determine $\bar{\mathcal{G}}^{\text{rest}}_{AB}$.

$$\begin{aligned} &\mathcal{G}^{(\text{rest})}_{rr} = 0, \quad \mathcal{G}^{(\text{rest})}_{\mu r} = 0 \\ &\mathcal{G}^{(\text{rest})}_{\mu\nu} = r^2 \left(\frac{r_H}{r}\right)^{D-1} u_\mu u_\nu - 2r \; \tilde{C}_3 \; \Theta \; \eta_{\mu\nu} + 2r\left[F(r/r_H) - 1\right]\sigma_{\mu\nu} \\ &\text{where} \quad \tilde{C}_3 \equiv C_3 + \frac{1}{D-2} \end{aligned} \tag{2.69}$$

2.3.4 Hydrodynamic metric in $\left(\frac{1}{D}\right)$ expansion

In the previous section, we split the 'hydrodynamic metric' into 'background' and 'rest' as required. In this section, we shall expand the metric further in $1/D$ and match it against the solution generated in large D technique described in [27].

The matching of the two metrics involves two steps. In the first step, we need to do an exact matching of the two metrics up to the orders we are interested in on both sides. In the second step, we need to show the mapping of the two sets of evolution data.

As we have already discussed, the 'hydrodynamic metric' and the 'large - D' metric are expressed in terms of data, defined on a co-dimension one hypersurface. In the case of 'hydrodynamic metric' it is defined on the conformal boundary of AdS space and in the case of the 'large D' technique, it is defined on a co-dimension one fluctuating membrane embedded in the asymptotic geometry. For the matching of the two gravity solutions, the evolution of data should also match. In other words, we should be able to reexpress the dynamical degrees of freedom of the membrane in terms of fluid variables and have to show that once the Navier-Stokes' equations are satisfied the 'membrane equations' are also satisfied.

Now at first, we match the two metrics up to the required order and then we show the equivalence of the evolution of the two sets of data.

2.3.4.1 Comparison between the two metrics

For matching of the hydrodynamic metric with the final metric described in [27], the first requirement is that $\bar{\mathcal{G}}^{\text{rest}}_{\mu\nu}$ must vanish as one goes finitely away from the horizon. This is possible if $\tilde{C}_3$ is zero and also the function $[F(r/r_H) - 1]$ at large r has a certain type of fall-off behavior. Now $\tilde{C}_3$ being an integration constant we can set it to zero. In appendix (A.1) we have analyzed the integral (2.53) and therefore the function $[F(r/r_H) - 1]$. We have shown that at large D this integral could be approximated as follows.

$$F(z) = F\left(1 + \frac{Z}{D}\right) = 1 - \left(\frac{1}{D}\right)^2 \sum_{m=1} \left(\frac{1+mZ}{m^2}\right) e^{-mZ} + \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{2.70}$$

Hence $[F(r/r_H) - 1]$ vanishes[13] up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.
After substituting equation (2.70) and the value for the integration constant $\tilde{C}_3$, the black-brane metric dual to hydrodynamics takes the following form

$$dS^2 = dS^2_{\text{background}} + r^2 \left(\frac{r_H}{r}\right)^{D-1} (u_\mu \, dx^\mu)^2 + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{2.71}$$

where $dS^2_{\text{background}}$ is given by equation (2.67)

As we have discussed before, the metric described in [27] is written in terms of one auxiliary function ψ and one auxiliary null one-form $O_A dX^A$. For convenience,

[13] Also, note that the vanishing has appropriate fall-off behaviour (exponential decay in the scaled Z variable) as required by large D corrections

let us quote the metric here again.

$$dS^2 = dS^2_{\text{background}} + \psi^{-D}\left(O_A\, dX^A\right)^2 + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{2.72}$$

Here ψ^{-D} is a harmonic function with respect to the background metric with $\psi = 1$ being the event horizon of the full space-time geometry and O_A is proportional to $\bar{O}_A$ determined in the previous section. The proportionality factor (we denote it by the scalar function $\Phi(X)$) is fixed by using the condition that the component of O_A along the unit normal to the $\psi =$ constant hypersurfaces is one everywhere. Mathematically, the above conditions could be expressed as

$$\bar{O}^A = \Phi(X)\; O^A, \quad \Phi(X) = \frac{\bar{O}^A\; \partial_A\psi}{\sqrt{(\partial_A\psi)(\partial^A\psi)}} \quad \text{where} \;\; \partial^A\psi \equiv \bar{\mathcal{G}}^{AB}\; \partial_B\psi \tag{2.73}$$

Rewriting (2.72) in terms of $\bar{O}_A$,

$$\begin{aligned} dS^2 =&\; dS^2_{\text{background}} + \left(\frac{\psi^{-D}}{\Phi^2}\right)\left(\bar{O}_A\; dX^A\right)^2 + \mathcal{O}\left(\frac{1}{D}\right)^2 \\ =&\; dS^2_{\text{background}} + \left(\frac{\psi^{-D}}{\Phi^2}\right)\left(u_\mu - A_2\; a_\mu\right)\left(u_\nu - A_2\; a_\nu\right)\; dx^\mu dx^\nu + \mathcal{O}\left(\frac{1}{D}\right)^2 \end{aligned} \tag{2.74}$$

For the exact matching of the metric in (2.74) with the metric in (2.71) we have to set A_2 to zero and identify $\left[\Phi^2 r^2 \left(\frac{r_H}{r}\right)^{D-1}\right]$ with the harmonic function ψ^{-D} up to corrections of order $\left(\frac{1}{D}\right)^2$. Hence, finally we have to verify is the following

$$\psi^{-D} - \Phi^2 r^2 \left(\frac{r_H}{r}\right)^{D-1} = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{2.75}$$

where ψ satisfies

$$\nabla^2\psi^{-D} = 0 \tag{2.76}$$

with the boundary condition that $\psi = 1$ hypersurface should reduce to the horizon, i.e., the hypersurface given by $r = r_H$, in an expansion in $\left(\frac{1}{D}\right)$.

Now we first determine ψ and then Φ. Note that both ψ and the norm of $\partial_A\psi$ are scalar functions and it would be easier to compute them in a coordinate system where the background metric has a simple form. Hence, we solve the equation in the $\{\rho, y^\mu\}$ coordinate system and then transform the final answer to the $\{r, x^\mu\}$ coordinates for final matching. For that at first, we need to know the position of the horizon in $\{Y^A\}$ coordinates since that will provide the required boundary condition for ψ. We know that in $\{X^A\} = \{r, x^\mu\}$ coordinates the horizon is at $r = r_H(x) + \mathcal{O}(\partial^2)$. Now $\{X^A\}$ and $\{Y^A\}$ coordinates are related as follows.

$$\begin{aligned}
\rho &= r - \frac{\Theta(x)}{D-2} + \mathcal{O}(\partial^2), \\
y^\mu &= x^\mu + \frac{u^\mu(x)}{r} + \left(\frac{\Theta(x)}{D-2}\right)\left(\frac{u^\mu(x)}{r^2}\right) + C_1\ \Theta(x)\ u^\mu(x) + C_2\ a^\mu(x) + \mathcal{O}(\partial^2)
\end{aligned} \tag{2.77}$$

The inverse transformation:

$$\begin{aligned}
r &= \rho + \frac{\Theta(y)}{D-2} + \mathcal{O}(\partial^2) \\
x^\mu &= y^\mu - \frac{u^\mu(x)}{\rho} - C_1\ \Theta(x)\ u^\mu(x) - C_2\ a^\mu(x) + \mathcal{O}(\partial^2) \\
&= y^\mu - \frac{u^\mu(y)}{\rho} + \frac{a^\mu(y)}{\rho^2} - C_1\ \Theta(y)\ u^\mu(y) - C_2\ a^\mu(y) + \mathcal{O}(\partial^2)
\end{aligned} \tag{2.78}$$

Therefore in terms of $\{Y^A\}$ coordinates the horizon is at

$$\begin{aligned}
\rho = &\ r_H\left(x^\mu\right) - \left(\frac{\Theta}{D-2}\right) + \mathcal{O}\left(\partial^2\right) \\
= &\ r_H(y^\mu) - \frac{(u\cdot\partial)\, r_H}{r_H} - \left(\frac{\Theta}{D-2}\right) + \mathcal{O}\left(\partial^2\right) = r_H(y^\mu) + \mathcal{O}\left(\partial^2\right)
\end{aligned} \tag{2.79}$$

Here, for terms that are of first order in derivative to begin with, this coordinate transformation will generate change of order $\mathcal{O}(\partial^2)$ and therefore negligible in our analysis. In the last line, we have used equation (2.54).
Once we know the position of the horizon, we will be able to solve for ψ. In $\{\rho, y^\mu\}$ coordinates the expressions for ψ and its norm are as follows (see appendix (A.2 for detail derivation).

$$\begin{aligned}
&\psi(\rho, y^\mu) = 1 + \left(1 - \frac{1}{D}\right)\left(\frac{\rho}{r_H(y)} - 1\right) + \mathcal{O}\left(\frac{1}{D}\right)^3 \\
\Rightarrow\quad & dY^A\ \partial_A\psi = \left(1 - \frac{1}{D}\right)\left(\frac{d\rho}{r_H(y)}\right) - \rho\left(1 - \frac{1}{D}\right)\left(\frac{\partial_\mu r_H(y)}{r_H^2(y)}\right)\ dy^\mu \\
\Rightarrow\quad & \partial^A\psi\ \partial_A\psi = \left(\frac{\rho}{r_H(y)}\right)^2\left(1 - \frac{1}{D}\right)^2 + \mathcal{O}(\partial)^2
\end{aligned} \tag{2.80}$$

Clearly this solution satisfies the boundary condition that $\psi = 1 \Rightarrow \rho = r_H(y) +$

$\mathcal{O}(\partial^2)$.

Now we have to transform these quantities in $\{X^A\}$ coordinates. We first transform the quantity $\left[\frac{\rho}{r_H(y)}\right]$.

$$\begin{aligned}\frac{\rho}{r_H(y)} &= \frac{r-\frac{\Theta}{D-2}}{r_H(x)+\frac{(u\cdot\partial)r_H}{r}}+\mathcal{O}(\partial^2)\\ &= \left(\frac{1}{r_H(x)}\right)\left(r-\frac{\Theta}{D-2}\right)\left(1-\frac{(u\cdot\partial)r_H}{r\ r_H}\right)+\mathcal{O}(\partial^2)\\ &= \left(\frac{1}{r_H(x)}\right)\left(r-\frac{\Theta}{D-2}-\frac{(u\cdot\partial)r_H}{r_H}\right)+\mathcal{O}(\partial^2)=\frac{r}{r_H(x)}+\mathcal{O}(\partial^2)\end{aligned}\tag{2.81}$$

From equation (2.81) it follows that

$$\begin{aligned}&\psi(r,x^\mu)=1+\left(1-\frac{1}{D}\right)\left(\frac{r}{r_H(x)}-1\right)+\mathcal{O}\left(\frac{1}{D^3},\ \partial^2\right)\\ \Rightarrow\quad & dX^A\ \partial_A\psi=\left(1-\frac{1}{D}\right)\left(\frac{dr}{r_H}\right)-r\left(1-\frac{1}{D}\right)\left(\frac{\partial_\mu r_H}{r_H^2}\right)\ dx^\mu+\mathcal{O}\left(\frac{1}{D^2},\ \partial^2\right)\\ \Rightarrow\quad & \partial^A\psi\ \partial_A\psi=\left(\frac{r}{r_H}\right)^2\left(1-\frac{1}{D}\right)^2+\mathcal{O}\left(\frac{1}{D^2},\ \partial^2\right)\end{aligned}\tag{2.82}$$

Substituting this solution in equation (2.73) we find $\Phi(X)=\frac{1}{r}$.

Now we have all the ingredients to verify equation (2.75). Let us introduce a new $\mathcal{O}(1)$ variable R such that

$$\frac{r}{r_H}=1+\frac{R}{D}$$

In terms of R we find

$$\begin{aligned}\psi^{-D}-\Phi^2r^2\left(\frac{r_H}{r}\right)^{D-1} &= \psi^{-D}-\left(\frac{r}{r_H}\right)^{-(D-1)}\\ &= \left[1+\left(1-\frac{1}{D}\right)\left(\frac{R}{D}\right)\right]^{-D}-\left(1+\frac{R}{D}\right)^{-(D-1)}\\ &= -\frac{1}{2}\left(\frac{R}{D}\right)^2e^{-R}+\mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned}\tag{2.83}$$

This is exactly what is required to have a match between the the 'large-D' metric and the 'hydrodynamic metric' up to the first subleading order on both sides.

2.3.4.2 Comparison between the evolution of two sets of data

As mentioned before, the ‘hydrodynamic metric’ is defined in terms of a unit normalized velocity and the temperature [14] of the relativistic conformal fluid living in a $(D-1)$ dimensional flat Minkowski space-time. In case of large - D expansion, the metric is written in terms of a $(D-1)$ dimensional time-like fluctuating membrane embedded in pure AdS space with a dynamical velocity field on it. In both the cases, these sets of data are controlled by separate equations. For ‘derivative expansion’ , the governing equation of data is given in (2.54). In ‘large-D’ technique, the governing equation is the following[27]

$$\hat{\nabla}\cdot U = 0, \qquad \left[\frac{\hat{\nabla}^2 U_\alpha}{\mathcal{K}} - \frac{\hat{\nabla}_\alpha \mathcal{K}}{\mathcal{K}} + U^\beta \mathcal{K}_{\beta\alpha} - U\cdot\hat{\nabla}U_\alpha\right]\mathcal{P}^\alpha_\gamma = 0 \tag{2.84}$$

This equation is written as an intrinsic equation on the membrane world-volume. Here all raising, lowering and contraction of the indices are done with respect to the induced metric on the membrane. U_α is the velocity of the membrane, expressed in terms of its intrinsic coordinates. $\mathcal{K}_{\beta\alpha}$ is the extrinsic curvature of the membrane, expressed as a symmetric tensor on the membrane world-volume. $\mathcal{K}$ is the trace of the extrinsic curvature. $\mathcal{P}^\alpha_\gamma$ is the projector perpendicular to U^α.
In this section, our aim is to show that equation (2.54) implies equation (2.84) up to the order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

Our first task would be to express the U^α and $\mathcal{K}_{\alpha\beta}$ in terms of velocity u^μ and temperature (or r_H) of the relativistic fluid. Note that though both u^μ and U^α are unit normalized velocity vector, they are defined on completely different spaces, one is defined in a flat Minkowski space and the other is defined in the curved (both intrinsic and extrinsic curvature, being nonzero) membrane world volume.
For convenience, we work in $\{Y^A\} = \{\rho, y^\mu\}$ coordinates where the background metric takes a simple form. At first we calculate the unit normal to the membrane and different components of its extrinsic curvature. First we will compute them in terms of background coordinates and then we will re-express it as an intrinsic symmetric tensor on the membrane.

[14] The temperature and the horizon radius are related by the following relation

$$r_H = \frac{4\pi\, T}{(D-1)}$$

In our choice of units

$$r_H \sim \mathcal{O}(1) \quad \Rightarrow \quad T \sim \mathcal{O}(D)$$

The unit normal to the membrane is given by

$$\begin{aligned} n_A \, dY^A|_{\text{membrane}} &\equiv dY^A \left[\frac{\partial_A \psi}{\sqrt{\partial^A \psi \; \partial_A \psi}} \right]_{\text{membrane}} \\ &= \frac{d\rho - dy^\mu \; \partial_\mu r_H(y)}{r_H(y)} \end{aligned} \tag{2.85}$$

The extrinsic curvature is defined as follows.

$$\begin{aligned} &K_{AB} = \Pi^C_A \; \nabla_C n_B = \Pi^C_A \left(\partial_C n_B - \Gamma^D_{CB} n_D \right) \\ &\text{where } \Pi^B_A = \delta^B_A - n_A \; n^B \quad \text{and } \nabla \text{ is the covariant derivative w.r.t background} \end{aligned} \tag{2.86}$$

Now let us choose $\{y^\mu\}$ as the intrinsic coordinate on the membrane world volume. In this choice of coordinates, the extrinsic curvature $\mathcal{K}_{\alpha\beta}$ will have the following structure.

$$\mathcal{K}_{\alpha\beta} = K_{\rho\rho} \left(\partial_\alpha r_H \right) \left(\partial_\beta r_H \right) + \left[K_{\rho\alpha} \left(\partial_\beta r_H \right) + K_{\rho\beta} \left(\partial_\alpha r_H \right) \right] + K_{\alpha\beta} \tag{2.87}$$

Note that the first term in the RHS of equation (2.87) does not contribute at first order in derivative expansion.
After using equation (2.86) and (2.87), at this order the final expression for $\mathcal{K}_{\mu\nu}$ turns out to be very simple (see appendix (A.3) for the details of the calculation).

$$\mathcal{K}_{\alpha\beta} = r_H^2 \; \eta_{\alpha\beta} + \mathcal{O}(\partial^2), \quad \mathcal{K} = (D-1) \tag{2.88}$$

The induced metric on the membrane is given by

$$g_{\alpha\beta} = r_H^2 \; \eta_{\alpha\beta} + \mathcal{O}(\partial^2) \tag{2.89}$$

Now we determine the membrane velocity U^α. The membrane velocity is defined as the projection of O^A on the membrane which, by construction, would be unit normalized with respect to the induced metric of the membrane. In $\{Y^A\}$ coordinates,

$O_A\ dY^A$ takes the following form

$$
\begin{aligned}
&O_A\ dX^A|_{\text{membrane}} = -\left[r\ u_\mu(x)\ dx^\mu\right]_{\text{membrane}} \\
= &\ -\left(r_H(y)+\frac{\Theta}{D-2}\right)\left[u_\mu(y)-\frac{a_\mu(y)}{r_H}\right]\left[\left(\frac{\partial x^\mu}{\partial\rho}\right)d\rho+\left(\frac{\partial x^\mu}{\partial y^\nu}\right)dy^\nu\right]_{\rho=r_H(y)} \\
= &\ -\left(r_H(y)+\frac{\Theta}{D-2}\right)\left[u_\mu(y)-\frac{a_\mu(y)}{r_H}\right]\left[\left(\frac{u^\mu(y)}{r_H^2(y)}-\frac{2a^\mu(y)}{r_H^3(y)}\right)d\rho+\left(\delta^\mu_\nu-\frac{\partial_\nu u^\mu}{r_H}\right)dy^\nu\right] \\
= &\ \left(\frac{1}{r_H(y)}+\frac{\Theta}{(D-2)r_H^2}\right)d\rho+\left[-r_H(y)\ u_\mu(y)-\left(\frac{\Theta}{D-2}\right)u_\mu+a_\mu(y)\right]\ dy^\mu \\
= &\ \left(\frac{1}{r_H(y)}+\frac{\Theta}{(D-2)r_H^2}\right)d\rho+\left[-r_H(y)\ u_\mu(y)-\left(\frac{\partial_\mu r_H}{r_H}\right)\right]\ dy^\mu
\end{aligned}
\tag{2.90}
$$

In the last line, we have used equation (2.54), which is the governing equation for the labelling data in the hydrodynamic side of the duality.

From equations (2.90) and (2.85) it follows that

$$
U_A\ dY^A \equiv -\,dY^A\left[O_A-n_A\right]_{\text{membrane}} = -\left(\frac{1}{r_H^2}\right)\left(\frac{\Theta}{D-2}\right)d\rho+r_H\ u_\mu\ dy^\mu \tag{2.91}
$$

Now U_α is just rewriting of U_A in terms of the intrinsic coordinates of the membrane. Following the same method as in equation (2.87) we find

$$
U_\alpha\ dy^\alpha \equiv \left[r_H\ u_\alpha+\mathcal{O}(\partial^2)\right]\ dy^\alpha \tag{2.92}
$$

Once we know $\mathcal{K}_{\alpha\beta}$, U^α and the induced metric on the membrane, we could compute each term in the equation (2.84).

$$
\begin{aligned}
&\hat\nabla\cdot U = \left(\frac{D-2}{r_H}\right)\left[\frac{\Theta}{D-2}+\frac{(u\cdot\partial)r_H}{r_H}\right]+\mathcal{O}\left(\partial^2\right)=\mathcal{O}\left(\partial^2\right) \\
&\hat\nabla^2 U_\alpha = \mathcal{O}\left(\partial^2\right) \\
&(U\cdot\hat\nabla)U_\beta = a_\beta+\frac{P^\alpha_\beta\partial_\alpha\ r_H}{r_H}+\mathcal{O}\left(\partial^2\right)=\mathcal{O}\left(\partial^2\right) \\
&U^\alpha\ \mathcal{K}_{\alpha\beta}\ \mathcal{P}^\beta_\gamma = \mathcal{O}(\partial^2) \\
&\hat\nabla_\alpha\mathcal{K} = \mathcal{O}(\partial^2)
\end{aligned}
\tag{2.93}
$$

As it is clear from the notation, in the LHS of each equation the relevant metric is the induced metric on the membrane on the other hand in RHS it is the flat

Minkowski metric $\eta_{\alpha\beta}$.
Substituting equations (2.93) in equation (2.84) we could show that membrane equation follows as a consequence of fluid equation.

In this context let us mention the work in [36]. In that paper, the authors have computed the boundary stress tensor dual to a slowly varying membrane embedded in AdS. They have calculated the dual fluid velocity in terms of the membrane velocity.One can easily check that the equation (2.92) is indeed the inverse of what the authors of [36] have found up to correction of order $\mathcal{O}(\partial^2)$.

2.4 Conclusion

In this chapter we have compared the gravity solutions generated by two different perturbation techniques, namely 'derivative expansion' and 'large D expansion'. We have seen that in large numbers of space-time dimensions there exists an overlap regime between these two techniques. In that regime we have shown that in large dimensions we can further expand the metric generated in 'derivative expansion' in $1/D$ and then it would match with the metric independently generated using 'large D technique'.
In chapter 3 we have extended this computation to the next order on both sides. In chapter 4 we have generalized this calculation to Einstein-Maxwell system in presence of negative cosmological constant.

In some sense, our computation serves as a consistency test for these two methods. But this analysis could teach us something more and that is about the dual systems of these two gravity solutions.
'Derivative expansion' generate dynamical black-brane metric in D space-time dimensions and the solutions are dual to the relativistic conformal hydrodynamics living in $(D-1)$ dimensional Minkowski space-time. The labelling variables of hydrodynamics are a unit normalized fluid velocity and temperature. They are the data that label different dynamical black-brane solutions in derivative expansion.
On the other hand, the gravity solutions generated in 'large D expansion' are dual to a co-dimension one dynamical non-gravitational membrane embedded in pure AdS and coupled with a velocity field. In this case also the labelling data of the metric is defined on a $(D-1)$ dimensional hypersurface and they consist of a unit normalized velocity field and a scalar function - the shape of the membrane. In terms of counting the number of variables this is very similar tot he hydrodynamic

metric, though the governing equations and the physical significance of the variables are totally different.

However, we have shown that these two sets of equations are approximately equivalent after an appropriate field redefinition. In this chapter, we have verified it at the very leading order and in chapter 3 we have extended this equivalence to the next order on both sides.

In fact, this equivalence is expected to valid to all orders[36]. In other words, in the overlap regime, these two systems of equations must be exactly equivalent to each other if we consider all orders on both sides [36] of the equations. But to see this equivalence we need to re-express one sets of variables in terms of the other sets of variables [36, 45, 47].

Chapter 3

Fluid-gravity and membrane-gravity dualities - Comparison at subleading orders

3.1 Introduction

This chapter is based on [2].
In this chapter, we shall compare two perturbation techniques, developed to handle both the nonlinearity and the dynamics in Einstein's equations in presence of negative cosmological constant, namely 'derivative expansion' [4, 13, 39] and 'large-D expansion' [24, 25, 26, 27, 42][1]. The initial set-up for such calculation has already been worked out in [1] and we have briefly described it in chapter 2 . Here we shall essentially extend it to the second subleading order. Very briefly, what we have done is to re express the metric dual to second order hydrodynamics [39] , derived using 'derivative expansion technique' in the form of the metric dual to membrane dynamics[42] derived using 'large-D expansion technique' upto corrections of order $\mathcal{O}\left(\frac{1}{\text{Dimension}}\right)^3$. As one might have expected, at this order the comparison and matching of the two gravity solutions in the regime of overlap, become far more non-trivial than what has been done in [1].

In the next subsection, we shall very briefly sketch the strategy we have used for comparison. In fact, we shall only give a sketch of the algorithm and shall refer to

[1] see [30, 34, 36, 43, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64] for work related to 'large-D expansion', see [7, 9, 10, 11, 46, 65, 66, 67, 68] for work related to 'derivative expansion'

[1] for any proof or other logical details.

3.1.1 Strategy

The 'large-D expansion' is a technique to generate a perturbative gravity solution expanded around space-time dimension $D \to \infty$ in inverse power of D. The metric $\mathcal{W}_{AB}$, constructed using this method, always has a 'split' form between background $\bar{\mathcal{W}}_{AB}$ and rest $\mathcal{W}^{(rest)}_{AB}$. $\bar{\mathcal{W}}_{AB}$ is the metric of the asymptotic geometry, which is also an exact solution of Einstein's equations. For our case, it is just the pure AdS.
The classifying data for different $\mathcal{W}_{AB}$ is encoded by the shape of a co-dimension one dynamical hypersurface embedded in pure ADS, coupled with a velocity field. We shall denote the equation that governs the dynamics of this membrane and the velocity field as 'membrane equation'. For every solution of this 'membrane equation', the 'large-D expansion' technique generates one unique dynamical metric that solves Einstein's equations in the presence of negative cosmological constant.[2]

The technique of 'derivative expansion' generates gravity solutions in D dimension that are dual to $(D-1)$ dimensional dynamical fluids, i.e., the characterizing data of the solution is given by a $(D-1)$ dimensional fluid velocity and temperature field. The velocity and the temperature are assumed to be slowly varying functions of the $(D-1)$ dimensional space. Therefore, the derivatives of these fields are the small parameters that control the perturbation here. The dynamics of the fluids are governed by a relativistic (and also higher order) generalization of Navier-Stokes equations, which we shall refer to as 'fluid equation'. The duality states that for every solution to the fluid equation, there exists a solution to Einstein's equations in the presence of negative cosmological constant, constructed in derivative expansion. For convenience, we shall refer to this metric as 'hydrodynamic metric'. This technique works in any number of space-time dimension. Also, note that the metric here is not in a 'split form' as we have in the case of 'large-D' expansion.

In [69] it has been argued that there exists an overlap in the allowed parameter regimes where these two perturbation techniques are applicable and also the starting points for both of these techniques (i.e., the solution at zeroth order) could be chosen to be the same space-time - namely the black-brane. Now given the zeroth order solution, both 'large-D' and 'derivative expansion' technique generates the higher

[2] The presence of cosmological constant is not a must for the applicability of the 'large-D expansion' technique, but it has to be present for the other technique, namely 'derivative expansion' to work. Since our goal is to compare the solutions generated by these two perturbative techniques, we have to deal with Einstein's equations in the presence of cosmological constant for both the cases.

order solutions uniquely in terms of the characterizing data. Hence it follows that in the overlap regime the two metrics generated by these two techniques must be the same or at least coordinate equivalent to each other.
Our goal is to show this equivalence in this overlap regime in the space of the perturbation parameters. We shall do it in the following three key steps.

3.1.1.1 Part-1:

As mentioned above, the metric generated in the 'large-D' expansion technique would always be expressed as a sum of two metrics - the background $\bar{\mathcal{W}}_{AB}$ and $\mathcal{W}^{(rest)}_{AB}$. The split is such that the contraction of a certain null geodesic vector $O^A\partial_A$ with $\mathcal{W}^{(rest)}_{AB}$ vanishes to all orders. However, the hydrodynamic metric, to begin with, does not have this 'split' form.

Our first step is to split the hydrodynamic metric into 'background ' and 'rest' such that the background is a pure AdS (though would have a complicated look if we stick to the coordinate system used in [39]) and the 'rest' part of the metric is such that its contraction with a certain null geodesic vector always vanishes.
The procedure is as follows.

1. We determine the position of the horizon (in an expansion in terms of the derivatives of the fluid data) in the hydrodynamic metric following the method described in [44].

2. Next we determine a null geodesic field (affinely parametrized) $\bar{O}^A\partial_A$, that passes through the horizon.
   Because of the specific gauge of the hydrodynamic metric, we could guess a simple form for $\bar{O}^A\partial_A$ that would work to all orders in derivative expansion. We gave some heuristic argument in support of this all order statement.

3. Next we pick up a coordinate system denoted as $\{Y^A\} \equiv \{\rho, y^\mu\}$ such that the 'background' of the hydrodynamic metric takes the following form

$$ds^2_{\text{background}} = \bar{G}_{AB}\ dY^A dY^B = \frac{d\rho^2}{\rho^2} + \rho^2\eta_{\mu\nu}\ dy^\mu\ dy^\nu \tag{3.1}$$

   The $\{Y^A\}$ coordinates are related to the $\{X^A\}$ coordinates (the coordinates used in [39] to express the hydrodynamic metric) by some (yet unknown) mapping functions $f^A(X)$.

$$Y^A = f^A(X)$$

4. Now we demand that the following set of equations.

$$\bar{O}^A \, \mathcal{G}_{AB}|_{\{X\}} = \bar{O}^A \left(\frac{\partial f^C}{\partial X^A}\right)\left(\frac{\partial f^{C'}}{\partial X^B}\right) \bar{G}_{CC'}|_{\{X\}} \tag{3.2}$$

   Where, $\mathcal{G}_{AB}$ is the full hydrodynamic metric in $\{X^A\}$ coordinates. Here the subscript $\{X\}$ denotes that both LHS and RHS of the above equation has been expressed in terms of $\{X\}$ variables.

5. Solving equation (3.2) we determine the mapping functions f^A s.
   However, it turns out that equation (3.2) cannot fix f^A s uniquely. To fix this ambiguity we demanded some extra 'conformal type' symmetry (see section (3.4) for the details) on the background metric .
   As with the case of null geodesic $\bar{O}^A\partial_A$, here also we try to guess some 'all order 'expressions for the mapping functions.

6. Once we know the mapping functions, it is not difficult to see the split of the hydrodynamic metric.

7. Finally we take the large - D limit of the hydrodynamic metric written in a 'split' form. Our goal is to match this metric with the large-D metric as determined in [42] after expressing the later in terms of fluid - data.

Note all but the last step in this part has been done exactly in D. We have also tried to make some 'all order statements' in terms of derivative expansion, whenever possible.

3.1.1.2 Part-2:

Next comes the relation between the data of the 'large - D' expansion technique and that of the derivative expansion. The metric generated in large - D expansion is expressed in terms of a very specific function ψ and the geodesic form field O_A, which is not affinely parametrized. It turns out that this O_A is related to the dual form field $\bar{O}_A$, determined in the previous part (which was affinely parametrized by construction), by an overall normalization. The normalization crucially depends on the shape of the constant ψ hypersurfaces.
So in the second part, we first determine ψ and then the normalization of $O^A\partial_A$ in terms of the 'fluid data'. The steps are as follows.

1. According [42] the function ψ is such that ψ^{-D} is a harmonic function in the embedding space of the background and also $\psi = 1$ is the hypersurface given

by the equation of the horizon.

$$\begin{aligned} &\nabla^2\psi^{-D} = 0, \quad \text{where } \nabla \equiv \text{covariant derivative w.r.t background} \\ &\text{Equation of the bulk horizon}: \ \psi = 1 \end{aligned} \tag{3.3}$$

Since we already know the explicit form of the background geometry, the above condition could be solved exactly in D using derivative expansion.

2. The null geodesic O^A is normalized such that

$$\begin{aligned} &O^A n_A = 1 \ \text{ everywhere in the background} \\ &\text{where } n_A \text{ is the unit normal to constant } \psi \text{ hypersurfaces} \end{aligned} \tag{3.4}$$

Now we already know the expressions for $\bar{O}^A$, which is proportional to O^A, the required geodesic. Let us denote the proportionality constant as Φ.

$$\bar{O}^A = \Phi\ O^A, \ \Rightarrow \ \Phi = n_A \bar{O}^A \tag{3.5}$$

Clearly, once we know both ψ and $\bar{O}^A$, it is easy to determine Φ and therefore O^A in terms of fluid data, all are exact in D.

3. We substitute these expressions of ψ and $O^A\partial_A$ in the 'large-D' metric as derived in [42] and convert it to metric in terms of fluid data.

After following the above steps in this part, we find a metric which we expect to match with the metric found in the previous part up to appropriate orders in derivative and $\left(\frac{1}{D}\right)$ expansion.

3.1.1.3 Part-3:

As mentioned before, in case of 'large-D' expansion, the characterizing data of the metric consist of the shape of $\psi = 1$ membrane viewed as a hypersurface embedded in the background pure AdS and a coupled $D-1$ dimensional velocity field (we shall refer to this data set as 'membrane data'). In 'derivative expansion' the data are the velocity and temperature of a relativistic fluid living on a $(D-1)$ dimensional Minkowski space (referred to as 'fluid data').

But for both cases, we are not allowed to choose these data completely arbitrarily; they are constrained by some equations. For 'large-D' expansion this is the membrane equation that governs the coupled dynamics of the membrane shape and the velocity. For 'derivative expansion' it is simply the relativistic generalization of the Navier Stokes equation.

After completing the previous two parts, we would be able to identify the membrane data in terms of the fluid data. However, as we shall see in the later sections, this identification will be done locally point by point, both in time and space. If we want the relations between these two sets of data to be valid always and everywhere, then their dynamics must be compatible. In other words, if we rewrite the membrane equation in terms of the fluid data it should reduce to 'relativistic Navier Stokes equation' in the appropriate limit of large dimension.

The fluid equation (the governing equation for fluid data) could be expressed as conservation of a specific stress tensor $T_{\mu\nu}$ living on a flat $(D-1)$ dimensional space-time .

$$\partial_\mu T^\mu_\nu = 0, \tag{3.6}$$

In [37] the authors have expressed the membrane equation also in terms of a stress tensor $\hat{T}_{ab}$ living on $\psi = 1$ hypersurface and conserved with respect to the induced metric of membrane upto correction of $\mathcal{O}\left(\frac{1}{D}\right)^2$. The membrane equation (the governing equation for large-D data) could be expressed as

$$\begin{aligned} &\tilde{\nabla}_a \hat{T}^a_b = 0, \\ &\tilde{\nabla}_a = \text{covariant derivative w.r.t the induced metric of the membrane} \end{aligned} \tag{3.7}$$

It turns out that the existence of $\hat{T}_{ab}$ makes the comparison quite easy. We took the following steps.

1. To begin with $\hat{T}_{ab}$ is a function of the membrane data encoded in the extrinsic curvature of the $\psi = 1$ hypersurface (the horizon in the bulk geometry) and the velocity field of the membrane, read off from the horizon generators. Fluid stress tensor $T_{\mu\nu}$ is a function of fluid velocity and the temperature.

2. But we already know the precise form of horizon generator and the $\psi = 1$ hypersurface in terms of fluid data. Therefore we could easily compute the extrinsic curvature of the surface as well as the induced metric on it in terms of the coordinates of the flat Minkowski space-time.

3. Inserting these relations in the membrane equation (3.7), we first convert the equation in the form $\partial_\mu W^{\mu\nu} = 0$ for some tensor $W^{\mu\nu}$

4. Finally we match $W^{\mu\nu}$ with $T^{\mu\nu}$ up to the appropriate order in large-D and derivative expansion.

Unfortunately, the expression for $\hat{T}_{ab}$ is not known at order $\mathcal{O}\left(\frac{1}{D}\right)^2$ though we know the form of membrane equation at that order[42]. So at that order, we had to deal with the full membrane equation and showed the equivalence with the help of Mathematica.

This chapter is organized as follows.
In section-(3.2) and section-(3.3) we simply quote the hydrodynamic and the 'large-D' metric along with the corresponding constraint equations from [39] and [42] respectively. Next in sections-(3.4), (3.5) and (3.6) we implement the strategy we described in subsection-(3.1.1). Finally in section-(3.7) we summarize our work and discussed the future directions.
At this statge we should emphasize that though, in principle, the strategy used in this chapter is very similar to [69], it differs a lot in details. We believe that now we have a more streamlined and simplified procedure to implement the strategy, mentioned in the previous section. However, to establish a clear connection with [69] we have also worked out every details by following [69] exactly and we have presented this method of work in appendix-(B.1).
Various computational details are collected in the appendices -(C.1), (B.3) and (B.4). In appendix-(B.5) we summarized the notations used in this chapter.

3.2 Hydrodynamic metric and its large D limit

The hydrodynamic metric in arbitrary dimension has been derived in [39], correctly up to second order in derivative expansion. In this section, we shall simply quote the final result for the metric, position of the horizon and the dual stress tensor from [39].

3.2.1 Hydrodynamic metric up to 2nd order in derivative expansion

The metric dual to relativistic hydrodynamics in any dimension could be expressed in terms of the basic variables of the dual fluid, living on $(D-1)$ dimensional flat space. In this case, it is the relativistic velocity, given by the unit normalized four-vector u^μ and the temperature scale, set by $r_H(x)$ (local fluid temperature is given by the following formula $T(x) = \left(\frac{D-1}{4\pi}\right) r_H(x))$. At nth order in derivative expansion the metric has terms with n number of derivatives, acting on $u^\mu(x)$ and $r_H(x)$. The authors in [39] have determined the metric corrections for $n = 0, 1$ and 2. Independent fluid data at first and second order in derivative expansion are listed in Table-4.1 and Table-3.2.

$$dS^2 = dS_0^2 + dS_1^2 + dS_2^2 \tag{3.8}$$

where,

Zeroth order Piece:

$$\begin{aligned} dS_0^2 =& -2u_\mu \, dx^\mu \, dr - r^2 f(\mathbf{r}) \, u_\mu u_\nu \, dx^\mu \, dx^\nu + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu \\ \mathcal{P}_{\mu\nu} &= \eta_{\mu\nu} + u_\mu u_\nu, \qquad \mathbf{r} = r/r_H, \qquad f(z) = 1 - z^{-(D-1)} \end{aligned} \tag{3.9}$$

First order Piece:

$$\begin{aligned} & dS_1^2 = -r \left(u_\mu A_\nu + u_\nu A_\mu\right) dx^\mu \, dx^\nu + 2r \; F(\mathbf{r}) \; \sigma_{\mu\nu} \; dx^\mu \; dx^\nu \\ & \text{where,} \\ & A_\mu = (u \cdot \partial) u_\mu - \left(\frac{\partial \cdot u}{D-2}\right) u_\mu, \qquad \sigma_{\mu\nu} = \mathcal{P}_\mu^\alpha \mathcal{P}_\nu^\beta \left[\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \eta_{\alpha\beta}\left(\frac{\partial \cdot u}{D-2}\right)\right] \\ & \mathbf{r} = r/r_H, \qquad F(y) = y \int_y^\infty \frac{dx}{x} \left[\frac{x^{D-2}-1}{x^{D-1}-1}\right] \end{aligned} \tag{3.10}$$

Second order Piece:

$$\begin{aligned} & dS_2^2 = \Big[X_1 \; u_\mu u_\nu + X_2 \mathcal{P}_{\mu\nu} + (Y_\mu u_\nu + Y_\nu u_\mu) + Z_{\mu\nu}\Big] dx^\mu \; dx^\nu \\ & \text{where,} \\ & X_1 = -\left[2\left(\frac{\partial \cdot A}{D-3}\right) - A^2 + \omega^2\left(\frac{1}{2\mathbf{r}^{D-1}} + \frac{2}{D-3}\right) + \frac{\sigma^2}{D-2}\left\{\frac{K_2(\mathbf{r})}{\mathbf{r}^{D-3}} - 2\left(\frac{D-2}{D-3}\right)\right\}\right] \\ & X_2 = \Big[2\,[F(\mathbf{r})]^2 - K_1(\mathbf{r})\Big]\left(\frac{\sigma^2}{D-2}\right) + \frac{\omega^2}{D-2} \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
Y_\mu &= \left[\frac{2}{\mathbf{r}^{D-3}}\right]\left[L(\mathbf{r}) + \frac{\mathbf{r}^{D-3}}{2(D-3)}\right](\mathcal{D}_\lambda \sigma^\lambda_\mu) - \left[\frac{1}{D-3}\right](\mathcal{D}_\lambda {\omega^\lambda}_\mu) \\
Z_{\mu\nu} &= \left(2\,[F(\mathbf{r})]^2 - H_1(\mathbf{r})\right)\sigma^\lambda_\mu \sigma_{\lambda\nu} - \mathcal{P}_{\mu\nu}\left(2\,[F(\mathbf{r})]^2 - H_1(\mathbf{r})\right)\left(\frac{\sigma^2}{D-2}\right) \\
&\quad + \left[H_2(\mathbf{r}) - H_1(\mathbf{r})\right](u\cdot\mathcal{D})\sigma_{\mu\nu} + H_2(\mathbf{r})\left({\omega_\mu}^\lambda \sigma_{\lambda\nu} + {\omega_\nu}^\lambda \sigma_{\lambda\mu}\right) + \left[{\omega_\mu}^\lambda \omega_{\nu\lambda} - \mathcal{P}_{\mu\nu}\left(\frac{\omega^2}{D-2}\right)\right]
\end{aligned}
\tag{3.12}
$$

where,

$$
\begin{aligned}
&\mathbf{r} = r/r_H \\
&\omega_{\mu\nu} = \mathcal{P}^\alpha_\mu \mathcal{P}^\beta_\nu \left(\frac{\partial_\alpha u_\beta - \partial_\beta u_\alpha}{2}\right), \quad \sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}, \quad \omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}, \\
&(u\cdot\mathcal{D})\sigma_{\mu\nu} = \mathcal{P}^\alpha_\mu \mathcal{P}^\beta_\nu\ (u\cdot\partial)\sigma_{\alpha\beta} + \left(\frac{\Theta}{D-2}\right)\sigma_{\mu\nu} \\
&\mathcal{D}^\lambda \sigma_{\mu\lambda} = \mathcal{P}^\alpha_\mu\ \partial^\lambda \sigma_{\alpha\lambda} - (D-2)A^\lambda\ \sigma_{\mu\lambda} \\
&\mathcal{D}^\lambda \omega_{\mu\lambda} = \mathcal{P}^\alpha_\mu\ \partial^\lambda\ \omega_{\alpha\lambda} - (D-4)A^\lambda \omega_{\mu\lambda}
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
H_1(y) &= 2y^2 \int_y^\infty \frac{dx}{x}\left[\frac{x^{D-3}-1}{x^{D-1}-1}\right] \\
H_2(y) &= F(y)^2 - 2\ y^2 \int_y^\infty \frac{dx}{x(x^{D-1}-1)} \int_1^x \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \\
K_1(y) &= 2y^2 \int_y^\infty \frac{dx}{x^2} \int_x^\infty \frac{dz}{z^2}\left[z\ F'(z) - F(z)\right]^2 \\
K_2(y) &= \int_y^\infty \left(\frac{dx}{x^2}\right)\left[1 - 2(D-2)\ x^{D-2} - \left(1 - \frac{1}{x}\right)\left(xF'(x) - F(x)\right)\right. \\
&\qquad \left. + \left(2(D-2)x^{D-1} - (D-3)\right)\int_x^\infty \frac{dz}{z^2}\left(zF'(z) - F(z)\right)^2\right] \\
L(y) &= \int_y^\infty dx\ x^{D-2} \int_x^\infty \frac{dz}{z^3}\left[\frac{z-1}{z^{D-1}-1}\right]
\end{aligned}
\tag{3.14}
$$

This is a dynamical black-brane metric with a singularity ar $r = 0$ and the location of the horizon is given by

$$
\begin{aligned}
H(x) = r_H(x) + \frac{1}{r_H(x)}\Bigg[& h_1\ \sigma^{\mu\nu}\sigma_{\mu\nu} + h_2\ \omega^{\mu\nu}\omega_{\mu\nu} \\
&+ h_3(D-3)\left\{\left(\frac{\Theta}{D-2}\right)^2 - a^2 + 2\ (u\cdot\partial)\left(\frac{\Theta}{D-2}\right)\right\}\Bigg]
\end{aligned}
\tag{3.15}
$$

Where,

$$h_1 = \frac{4}{(D-1)^2(D-2)} - \frac{K_{2H}}{(D-1)(D-2)}, \quad h_2 = -\frac{1}{2(D-1)} \quad \text{and,} \quad h_3 = -\frac{1}{(D-1)(D-3)}$$

$$\text{with,} \quad K_{2H} = \int_1^\infty \left(\frac{dx}{x^2}\right)\left[1 - 2(D-2)\; x^{D-2} - \left(1-\frac{1}{x}\right)\Big(xF'(x) - F(x)\Big) + \Big(2(D-2)x^{D-1} - (D-3)\Big)\int_x^\infty \frac{dz}{z^2}\Big(zF'(z) - F(z)\Big)^2\right] \tag{3.16}$$

The fluid dual to the metric, described above, is characterized by the following stress tensor, living on $(D-1)$ dimensional flat Minkowski space

$$T_{\mu\nu} = p\;[\;\eta_{\mu\nu} + (D-1)u_\mu u_\nu] - 2\;\eta\;\sigma_{\mu\nu} \tag{3.17}$$

Where,

$$p = \frac{r_H^{D-1}}{16\pi G_{\text{AdS}}} \quad \text{and,} \quad \eta = \frac{r_H^{D-2}}{16\pi G_{\text{AdS}}} \tag{3.18}$$

The hydrodynamic metric would solve the D dimensional Einstein's equations in presence of negative cosmological constant provided the stress tensor described in equation (3.17) is conserved.

Table 3.1: Data at 1st order in derivative

	Independent Data
Scalar	$\frac{\Theta}{D-2} = \left(\frac{\partial\cdot u}{D-2}\right)$
Vector	$a_\mu = (u\cdot\partial)u_\mu$
Tensor	$\sigma_{\mu\nu} = \mathcal{P}_\mu^\alpha \mathcal{P}_\nu^\beta \left[\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \eta_{\alpha\beta}\left(\frac{\Theta}{D-2}\right)\right]$

Where, $\mathcal{P}_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$

3.3 Large-D metric and Membrane equation

Just like the previous section, here we shall simply quote the form of the large-D metric from [42], correctly upto order $\mathcal{O}\left(\frac{1}{D}\right)^2$. Schematically, the solution generated

Table 3.2: Data at 2nd order in derivative

	Independent Data
Scalars	$\mathfrak{s}_1 \equiv \left(\frac{\Theta}{D-2}\right)^2, \quad \mathfrak{s}_2 \equiv a^2, \ \mathfrak{s}_3 \equiv \omega^{\mu\nu}\omega_{\mu\nu}, \ \mathfrak{s}_4 = \sigma^{\mu\nu}\sigma_{\mu\nu}, \ \mathfrak{s}_5 = (u\cdot\partial)\left(\frac{\Theta}{D-2}\right)$
Vectors	$\mathfrak{v}^{(1)}_\mu \equiv \left(\frac{\Theta}{D-2}\right)a_\mu, \ \mathfrak{v}^{(2)}_\mu \equiv a^\nu\omega_{\nu\mu}, \ \mathfrak{v}^{(3)}_\mu \equiv a^\nu\sigma_{\nu\mu},$ $\mathfrak{v}^{(4)}_\mu \equiv \mathcal{P}_{\mu\nu}\partial^\nu\left(\frac{\Theta}{D-2}\right), \ \mathfrak{v}^{(5)}_\mu \equiv \mathcal{P}_{\mu\nu}\left(\frac{\partial_\lambda\sigma^{\nu\lambda}}{D-2}\right)$
Tensors	$\mathfrak{t}^{(1)}_{\mu\nu} \equiv \sigma_\mu{}^\alpha\sigma_{\alpha\nu}, \ \mathfrak{t}^{(2)}_{\mu\nu} \equiv \omega_\mu{}^\alpha\omega_{\alpha\nu}, \ \mathfrak{t}^{(3)}_{\mu\nu} \equiv \omega_\mu{}^\alpha\sigma_{\alpha\nu} - \sigma_\mu{}^\alpha\omega_{\alpha\nu}$ $\mathfrak{t}^{(4)}_{\mu\nu} \equiv \mathcal{P}^\alpha_\mu\mathcal{P}^\beta_\nu(u\cdot\partial)\sigma_{\alpha\beta}, \ \mathfrak{t}^{(5)}_{\mu\nu} \equiv a_\mu a_\nu$

by Large-D technique takes the following form

$$\mathcal{W}_{AB} = \mathcal{W}^{(0)}_{AB} + \left(\frac{1}{D}\right)\mathcal{W}^{(1)}_{AB} + \left(\frac{1}{D}\right)^2\mathcal{W}^{(2)}_{AB} + \cdots \tag{3.19}$$

Where, the starting ansatz $\mathcal{W}^{(0)}_{AB}$ is given by

$$\mathcal{W}^{(0)}_{AB} = \bar{\mathcal{W}}_{AB} + \psi^{-D}O_AO_B \tag{3.20}$$

Here $\bar{\mathcal{W}}_{AB}$ is the background metric which could be any smooth solution of Einstein's equations. The function $\psi(X^A)$ and the one-form field $O_A \equiv O_A dX^A$ are defined in section- (3.1.1). Rest of the metric corrections could be expressed in terms of O_A, ψ and their derivatives.
For convenience, one velocity field has been defined on the constant ψ slices as follows.

$$\begin{aligned} &U_A = n_A - O_A \\ &\text{where,} \\ &n_A \equiv \text{unit normal to constnt } \psi \text{ hypersurfaces embedded in background.} \end{aligned} \tag{3.21}$$

And the derivatives of O_A has been replaced by derivatives of U_A and n_A or the extrinsic curvature of the constant ψ surfaces.

It turns out that $\mathcal{W}^{(1)}_{AB}$ - 1st order metric correction simply vanishes.

$\mathcal{W}^{(2)}_{AB}$- 2nd order metric correction is non-zero. It can be decomposed as follows.

$$\mathcal{W}^{(2)}_{AB} = \left[O_A O_B \left(\sum_{n=1}^{2} f_n(R)\ S_n\right) + v(R)\ \left(\ V_A O_B + V_B O_A\right) + t(R)\ T_{AB}\right] \tag{3.22}$$

where,

$$\begin{aligned} T_{AB} = P^C_A P^D_B \Bigg[\bar{R}_{FCDE} O^E O^F + \frac{K}{D}\Bigg(K_{CD} - \frac{\nabla_C U_D + \nabla_D U_C}{2}\Bigg) \\ - P^{EF}(K_{EC} - \nabla_E U_C)(K_{FD} - \nabla_F U_D)\Bigg] \end{aligned} \tag{3.23}$$

$$\begin{aligned} V_A = P^B_A \Bigg[\frac{K}{D}\left(n^D U^E O^F \bar{R}_{FBDE}\right) + \frac{K^2}{2D^2}\left(\frac{\nabla_B K}{K} + (U\cdot\nabla)U_B - 2\ U^D K_{DB}\right) \\ - P^{FD}\left(\frac{\nabla_F K}{D} - \frac{K}{D}(U^E K_{EF})\right)(K_{DB} - \nabla_D U_B)\Bigg] \end{aligned} \tag{3.24}$$

$$\begin{aligned} S_1 &= U^E U^F n^D n^C \bar{R}_{CEFD} + \left(\frac{U\cdot\nabla K}{K}\right)^2 + \frac{\hat{\nabla}_A K}{K}\left[4\ U^B K^A_B - 2\left[(U\cdot\nabla)U^A\right] - \frac{\hat{\nabla}^A K}{K}\right] \\ &- (\hat{\nabla}_A U_B)(\hat{\nabla}^A U^B) - (U\cdot K\cdot U)^2 - \left[(U\cdot\hat{\nabla})U_A\right]\left[(U\cdot\hat{\nabla})U^A\right] + 2\left[(U\cdot\nabla)U^A\right](U^B K_{BA}) \\ &\quad - 3\ (U\cdot K\cdot K\cdot U) - \frac{K}{D}\left(\frac{U\cdot\nabla K}{K} - U\cdot K\cdot U\right) \\ S_2 &= \frac{K^2}{D^2}\Bigg[-\frac{K}{D}\left(\frac{U\cdot\nabla K}{K} - U\cdot K\cdot U\right) - 2\ \lambda - (U\cdot K\cdot K\cdot U) + 2\left(\frac{\nabla_A K}{K}\right)U^B K^A_B - \left(\frac{U\cdot\nabla K}{K}\right)^2 \\ &+ 2\left(\frac{U\cdot\nabla K}{K}\right)(U\cdot K\cdot U) - \left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right) - (U\cdot K\cdot U)^2 + n^B n^D U^E U^F \bar{R}_{FBDE}\Bigg] \end{aligned} \tag{3.25}$$

$\bar{R}_{ABCD}$ is the Riemann tensor of the background metric $\bar{\mathcal{W}}_{AB}$.
∇ denotes the covariant derivative with respect to $\bar{\mathcal{W}}_{AB}$. $\hat{\nabla}$ is defined as follows: for any general tensor with n indices $W_{A_1 A_2 \cdots A_n}$

$$\hat{\nabla}_A W_{A_1 A_2 \cdots A_n} = \Pi^C_A\ \Pi^{C_1}_{A_1}\Pi^{C_2}_{A_2}\cdots\Pi^{C_n}_{A_n}\left(\nabla_C W_{C_1 C_2 \cdots C_n}\right), \quad \text{with} \quad \Pi_{AB} = \bar{\mathcal{W}}_{AB} - n_A n_B \tag{3.26}$$

and,

$$
\begin{aligned}
t(R) &= -\,2\left(\frac{D}{K}\right)^2 \int_R^\infty \frac{y\ dy}{e^y - 1}\\
v(R) &= 2\left(\frac{D}{K}\right)^3 \left[\int_R^\infty e^{-x}dx \int_0^x \frac{y\ e^y}{e^y-1}dy \ - \ e^{-R}\int_0^\infty e^{-x}dx\int_0^x \frac{y\ e^y}{e^y-1}dy\right]\\
f_1(R) &= -2\left(\frac{D}{K}\right)^2\int_R^\infty x\ e^{-x}dx + 2\ e^{-R}\left(\frac{D}{K}\right)^2\int_0^\infty x\ e^{-x}dx\\
f_2(R) &= \left(\frac{D}{K}\right)\left[\int_R^\infty e^{-x}dx\int_0^x \frac{v(y)}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx\int_0^x\frac{v(y)}{1-e^{-y}}dy\right]\\
&\quad -\left(\frac{D}{K}\right)^4\left[\int_R^\infty e^{-x}dx\int_0^x\frac{y^2\ e^{-y}}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx\int_0^x\frac{y^2\ e^{-y}}{1-e^{-y}}dy\right]\\
&\text{Where,}\quad R\equiv D(\psi-1)
\end{aligned}
\tag{3.27}
$$

The above expressions for $\mathcal{W}_{AB}$ would solve Einstein's equations in presence of negative cosmological constant[3] provided the following constraint equation is satisfied,

[3]Note that each component of the metric corrections described above vanishes exponentially in D when $R \sim \mathcal{O}(D)$. Now this 'large -D metric', by construction, solves Einstein's equations (in presence of negative cosmological constant) upto correction of order $\mathcal{O}\left(\frac{1}{D}\right)^3$; and therefore, whenever the metric corrections become of the order of $\mathcal{O}(e^{-D})$, they are no longer trustable. In other words, the above metric solves Einstein's equations as long as $R = D(\psi - 1) << D$.
It follows that while comparing with hydrodynamic metric we would expect a perfect match only within this region of validity of the large-D metric. Also a 'match' requires a similar exponential fall off in D for the hydrodynamic metric if one goes away distance of order $\mathcal{O}(D)$ from the horizon - the $\psi = 1$ hypersurface.

$$
\begin{aligned}
&P^A_C\left[\frac{\hat\nabla^2 U_A}{K} - \frac{\hat\nabla_A K}{K} + U^B K_{BA} - U\cdot\hat\nabla U_A\right] + P^A_C\left[-\frac{U^B K_{BD}K^D_A}{K} + \frac{\hat\nabla^2\hat\nabla^2 U_A}{K^3} - \frac{(\hat\nabla_A K)(U\cdot\hat\nabla K)}{K^3}\right.\\
&-\frac{(\hat\nabla_B K)(\hat\nabla^B U_A)}{K^2} - \frac{2K^{DE}\hat\nabla_D\hat\nabla_E U_A}{K^2} - \frac{\hat\nabla_A\hat\nabla^2 K}{K^3} + \frac{\hat\nabla_A(K_{BD}K^{BD}K)}{K^3} + 3\frac{(U\cdot K\cdot U)(U\cdot\hat\nabla U_A)}{K}\\
&-3\frac{(U\cdot K\cdot U)(U^B K_{BA})}{K} - 6\frac{(U\cdot\hat\nabla K)(U\cdot\hat\nabla U_A)}{K^2} + 6\frac{(U\cdot\hat\nabla K)(U^B K_{BA})}{K^2} + 3\frac{U\cdot\hat\nabla U_A}{D-3}\\
&\left.-3\frac{U^B K_{BA}}{D-3} - \frac{(D-1)\lambda}{K^2}\left(\frac{\hat\nabla_A K}{K} - 2U^D K_{DA} + 2(U\cdot\hat\nabla)U_A\right)\right] = \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned}
$$

$$
\text{and,}\quad \hat\nabla\cdot U - \frac{1}{2K}\nabla_{(A}U_{B)}\nabla_{(C}U_{D)}P^{AC}P^{BD} = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{3.28}
$$

where, $U_A = n_A - O_A$, $P_{AB} = \bar{\mathcal{W}}_{AB} - n_A n_B + U_A U_B$ and $\nabla_{(A}U_{B)} = \nabla_A U_B + \nabla_B U_A$

If we truncate the membrane equation at first subleading order, it takes the following form

$$
P^A_C\left[\frac{\hat\nabla^2 U_A}{K} - \frac{\hat\nabla_A K}{K} + U^D K_{DA} - (U\cdot\hat\nabla)U_A\right] = \mathcal{O}\left(\frac{1}{D}\right),\quad \hat\nabla\cdot U = \mathcal{O}\left(\frac{1}{D}\right) \tag{3.29}
$$

In [37] this part of the equation has been expressed as a conservation of some stress tensor, defined on the $\psi = 1$ hypersurface. The form this stress tensor is as follows.

$$
\begin{aligned}
T^{(m)}_{AB} = &\left(\frac{K}{2}\right)U_A U_B + \left(\frac{1}{2}\right)K_{AB} - \frac{1}{2}\left(\hat\nabla_A U_B + \hat\nabla_B U_A\right) - \frac{1}{K}\left(U_A\hat\nabla^2 U_B + U_B\hat\nabla^2 U_A\right)\\
&+\frac{1}{2}\left(U_A\frac{\hat\nabla_B K}{K} + U_B\frac{\hat\nabla_A K}{K}\right) - \frac{1}{2}\left(U\cdot K\cdot U + \frac{K}{D}\right)\Pi_{AB}
\end{aligned} \tag{3.30}
$$

As explained in section-(3.1.1), we shall use this form of the membrane equation to show equivalence between the two sets of the constraint equations.

3.4 Implementing part-1: The split of the hydrodynamic metric

In this section, we shall see how to split the hydrodynamic metric as a sum of the background and the rest. The hydrodynamic metric that we shall work with is correct up to second order in derivative expansion and therefore in this section, we shall neglect all terms of third order or higher. As we have mentioned before, all these steps are already executed in [69] accurately up to first order in derivative expansion. Here we shall use the results derived in [69] whenever possible. Also, we shall try to generalize the results and the derivation, as much as possible, to higher orders on both sides of the perturbation. It turns out that often some general pattern emerges which would naturally lead to some 'all-order' statements at the intermediate steps.

3.4.1 The null geodesic $\bar{O}^A\partial_A$

As summarized in the introduction, the 'split ' of the metric would be done in terms of a geodesic field $O^A\partial_A$ which is null with respect to the full space-time and also with respect to the background. In this subsection, our task is to fix this O_A field.

Before getting into any details of this second order calculation let us describe few general features of the hydrodynamic metric $\mathcal{G}_{AB}$, which would allow us to determine a null vector field that would satisfy the geodesic equation to all order in derivative expansion.

According to the derivation of [39], the coordinates are fixed in a way such that $\mathcal{G}_{rr} = 0$ and $\mathcal{G}_{r\mu} = -u_\mu$ to all order in derivative expansion. In this gauge Γ^r_{rr} and Γ^μ_{rr} vanish identically to all order. It follows that in this metric, any vector of the form $\left(k^A\partial_A \equiv \zeta(x^\mu)\partial_r\right)$ would be an affinely parametrized null geodesic to all order in derivative expansion as long as the function ζ depends only on x^μ .

$$\begin{aligned}(k^A\bar{\nabla}_A)k^r &= k^r\partial_r k^r + k^r\Gamma^r_{rr}k^r = 0\\(k^A\bar{\nabla}_A)k^\mu &= k^r\Gamma^\mu_{rr}k^r = 0\end{aligned} \tag{3.31}$$

Now at zeroth order in derivative expansion we know that $\bar{O}^A\partial_A$ is simply ∂_r. In fact this turns out to be true even at first order in derivative expansion[69]. It is very tempting to conjecture that to all order in derivative expansion

$$\bar{O}^A\partial_A = \partial_r \tag{3.32}$$

We could simply set the function $\zeta(x^\mu)$ to be one, since anyway we have to normalize $\bar{O}^A$ further to get the O^A vector field (see the previous section) that appears in the large-D metric

We could construct some inductive proof for this statement. Suppose at some nth order in derivative expansion $\bar{O}^A\partial_A = \partial_r$. At $(n+1)$ th order, after setting the norm to zero and normalization by fixing the coefficient of ∂_r to be one, the form of $\bar{O}^A$ would be

$$\bar{O}^A\partial_A = \partial_r + V^\mu(r)\partial_\mu$$

where V^μ is some vector structure, perpendicular to u^μ and containing $(n+1)$ derivatives. Now since $V^\mu(r)$ already contains $(n+1)$ derivatives, in the geodesic equation at $(n+1)$ th order, it is the zeroth order metric that will multiply this term and we could solve for the r dependence of $V^\mu(r)$ without any details of the higher order metric correction. $V^\mu(r)$ turns out to be

$$V^\mu(r) = \frac{\tilde{V}^\mu}{r^2}, \quad \text{where } \tilde{V}^\mu \text{ is independent of } r$$

Upon lowering the index we find $\bar{O}_A\ dX^A = \left[-u_\mu + \tilde{V}_\mu + \mathcal{O}\left(\partial^{n+2}\right)\right] dx^\mu$. Substituting this $\bar{O}_A$ in the expression of large-D metric and using the facts that $\psi^{-D} = \left(\frac{r_H}{r}\right)^{D-1} + \mathcal{O}\left(\frac{1}{D}\right)$ and $\bar{O}_A$ is proportional to O_A, we could see that the leading term in $\left(\frac{1}{D}\right)$ expansion (i.e., the terms $\psi^{-D}O_AO_B$) itself will generate a term of the form $\sim \left(\frac{r_H}{r}\right)^{D-1}(u_\mu\tilde{V}_\nu + u_\nu\tilde{V}_\mu)$. Using AdS-CFT correspondence one could deduce that such a term in the metric will generate a term of the form $(u_\mu\tilde{V}_\nu + u_\nu\tilde{V}_\mu)$ in the dual fluid stress tensor, thus making it out of Landau frame. But since u_μ of the hydrodynamic metric is defined to be the fluid velocity in Landau frame (see [39],[4]), such a term in O^A must vanish once we equate the resultant large-D metric with the hydrodynamic metric.

Hence equation (3.32) gives an all order expression for $\bar{O}^A\partial_A$

3.4.2 The mapping functions and the 'split' of the hydrodynamic metric

Next we come to the computation of the mapping functions f^A s that relate the $\{Y^A\} = \{\rho, y^\mu\}$ coordinates (where the background pure AdS has simple metric given by equation (3.1)) with the $\{X^A\} = \{r, x^\mu\}$, the coordinates in which the hydrodynamic metric $\mathcal{G}_{AB}$ is expressed in section (3.2).

As before we shall start with some general observation and try to get some all order statements about the mapping functions. We shall use equation (3.32) for the expression of $\bar{O}^A$. Now the mapping functions f^A s are determined by solving equation (3.2). We could view the RHS of equation (3.2) as pure AdS expressed in $\{X^A\}$ coordinates and contracted with $\bar{O}^A$. Let us rewrite equation (3.2) in this language.
Suppose $\bar{\mathcal{G}}_{AB}$ denotes the pure AdS metric in $\{X^A\}$ coordinates, i.e.,

$$\bar{\mathcal{G}}_{AB} = \left(\frac{\partial f^C}{\partial X^A}\right)\left(\frac{\partial f^{C'}}{\partial X^B}\right)\bar{G}_{CC'}|_{\{X\}} \tag{3.33}$$
where $\bar{G}_{CC'}$ is given in equation (3.1)

After using the fact that $\bar{O}^A\partial_A = \partial_r$, equation (3.2) simply implies

$$\bar{O}^A\left(\mathcal{G}_{AB} - \bar{\mathcal{G}}_{AB}\right) \equiv \bar{O}^A\mathcal{G}^{\text{rest}}_{AB} = 0 \;\Rightarrow\; \mathcal{G}^{\text{rest}}_{rB} = 0 \tag{3.34}$$

Now we know that the hydrodynamic metric as presented in [39] is in a gauge where, to all orders in derivative expansion,

$$\mathcal{G}_{rr} = 0, \quad \mathcal{G}_{r\mu} = -u_\mu$$

Clearly equation (3.34) could be satisfied provided $\bar{\mathcal{G}}_{AB}$ is also in the same gauge. In other words, f^As should be such that it transforms the pure AdS metric in a gauge where the $(r\mu)$ component is equal to minus of u_μ as read off from the hydrodynamic metric and the (rr) component simply vanishes.
Note that in any general metric the above condition does not fix the gauge completely; we are left with a residual coordinate transformation symmetry within the x^μ coordinates. For example, consider the following set of mapping functions.

$$\rho = r + \chi(x), \quad y^\mu = x^\mu + \frac{u^\mu}{r + \chi(x)} + \xi^\mu(x) \tag{3.35}$$

The above transformation will take the pure AdS metric to the required gauge (i.e., $\bar{\mathcal{G}}_{rr} = 0$ and $\bar{\mathcal{G}}_{r\mu} = -u_\mu$) to all order in derivative expansion, as along as the function χ is independent of the r coordinate and $\xi^\mu(x)$ is an arbitrary four-vector, independent of r, satisfying,

$$u_\mu\left(\frac{\partial \xi^\mu}{\partial x^\nu}\right) = 0 \tag{3.36}$$

If we demand an exact match between the large-D and hydrodynamic metric, the mapping functions must be in terms of the fluid data. In other words $\chi(x)$ and $\xi^\mu(x)$ must be functions of u^μ, r_H and their derivatives. On top of that $\xi^\mu(x)$, once expressed in terms of independent fluid data, is further constrained to satisfy equation (3.36) as an identity. Now we would like to show that only solution to (3.36) is

$$\xi^\mu = c\ u^\mu, \quad c = \text{constant} \tag{3.37}$$

Suppose

$$\xi^\mu = C(x)\ u^\mu + \xi^\mu_\perp(x) \ \text{ such that } \ u_\mu \xi^\mu_\perp = 0$$

Substituting this expression of ξ^μ in (3.36) we find

$$\begin{aligned} &u_\alpha \partial_\nu \left[C(x) u^\alpha + \xi^\alpha_\perp(x)\right] = 0 \\ \Rightarrow \quad &- \partial_\nu C(x) - \xi^\alpha_\perp(x) \partial_\nu u_\alpha = 0 \end{aligned} \tag{3.38}$$

$\xi^\alpha_\perp(x)\partial_\nu u_\alpha$ is a gradient function and therefore must satisfy the following integrability condition.

$$\begin{aligned} &\partial_\mu(\xi^\alpha_\perp \partial_\nu u_\alpha) - \partial_\nu(\xi^\alpha_\perp \partial_\mu u_\alpha) = 0 \\ \Rightarrow \quad &(\partial_\mu \xi^\alpha_\perp)(\partial_\nu u_\alpha) - (\partial_\nu \xi^\alpha_\perp)(\partial_\mu u_\alpha) = 0 \end{aligned} \tag{3.39}$$

Dotted with u_μ,

$$\begin{aligned} &[(u\cdot\partial)\xi^\alpha_\perp](\partial_\nu u_\alpha) - a_\alpha(\partial_\nu \xi^\alpha_\perp) = 0 \\ \Rightarrow \quad &[\partial_\beta \xi^\alpha_\perp]\left[-a_\alpha \mathcal{P}^\beta_\nu + \sigma_{\nu\alpha}u^\beta + \omega_{\nu\alpha}u^\beta + \left(\frac{\Theta}{D-2}\right)\mathcal{P}_{\nu\alpha}u^\beta\right] = 0 \end{aligned} \tag{3.40}$$

In the above equation the four terms multiplying $\partial_\beta \xi^\alpha_\perp$ are independent fluid data and therefore their linear combination can never vanish identically. The only way to satisfy equation (3.40) is to set $\partial_\beta \xi^\alpha_\perp$ to zero.

$$\partial_\beta \xi^\alpha_\perp = 0, \quad \Rightarrow \quad \xi^\alpha_\perp = \text{constant} \tag{3.41}$$

Now in the hydrodynamic metric there is no special special vector apart from u^μ, which is not a constant. Therefore, if we want a term by term matching of the 'large-D' metric (written in $\{X^A\}$ coordinates) with the hydrodynamic metric, $\xi^\mu_\perp$ itself must vanish. Substituting in equation (3.36) we find

$$\partial_\nu C(x) = 0 \quad \Rightarrow \quad C(x) = c = \text{constant} \tag{3.42}$$

From the above discussion, it also follows that exact matching of the two metrics (upto the required order) would be possible only for a very specific choice of $\chi(x)$ and the constant c in equation (3.37). Any other choice, apart from this specific one, would result in a hydrodynamic metric which would not be exactly same, but coordinate-equivalent to the metric presented in section-(3.2). The corresponding coordinate transformation would simply be a x^μ dependent shift in the r and x^μ coordinates.

Note that the constant c could not have any derivative correction and the computation of [69], which is correct up to first order in derivative expansion, has already told us that c has to be set to zero for an exact match between these two metrics. It turns out that if we choose the function $\chi(x) = \left(\frac{\partial_\mu u^\mu}{D-2}\right)$, it does cast the pure AdS metric to the required gauge and everything works out as we wanted i.e., upto second order in derivative expansion, both the metrics match term by term without any further coordinate redefinition. So the final form of the mapping functions[4] .

$$\rho = r - \left(\frac{\Theta}{D-2}\right) + \mathcal{O}\left(\partial^3\right), \quad y^\mu = x^\mu + \frac{u^\mu}{\rho}, \quad \text{where} \quad \Theta \equiv \partial \cdot u \tag{3.43}$$

After imposing this coordinate transformation the final form of the background metric is as follows,

$$\begin{aligned}
\bar{\mathcal{G}}_{rr} =&\ 0, \quad \bar{\mathcal{G}}_{r\mu} = -u_\mu \\
\bar{\mathcal{G}}_{\mu\nu} =&\ r^2(\mathcal{P}_{\mu\nu} - u_\mu u_\nu) + 2r\left(\frac{\Theta}{D-2}\right)u_\mu u_\nu + r\left[2\,\sigma_{\mu\nu} - a_\mu u_\nu - a_\nu u_\mu\right] + \left[\mathfrak{t}^{(1)}_{\mu\nu} - \mathfrak{t}^{(2)}_{\mu\nu} + \mathfrak{t}^{(3)}_{\mu\nu}\right] \\
&+ u_\mu\left[\mathfrak{v}^{(2)}_\nu - \mathfrak{v}^{(3)}_\nu + \mathfrak{v}^{(4)}_\nu\right] + u_\nu\left[\mathfrak{v}^{(2)}_\mu - \mathfrak{v}^{(3)}_\mu + \mathfrak{v}^{(4)}_\mu\right] - u_\mu u_\nu\left[\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5\right]
\end{aligned} \tag{3.44}$$

Once we know the background in $\{r, x^\mu\}$ coordinates, we could determine $\mathcal{G}^{\text{rest}}_{AB}$ by simply subtracting the background from the full hydrodynamic metric $\mathcal{G}_{AB}$. $\mathcal{G}^{\text{rest}}_{rr}$ and $\mathcal{G}^{\text{rest}}_{r\mu}$ are zero by construction. The structure of $\mathcal{G}^{\text{rest}}_{\mu\nu}$ is a bit complicated. We first decompose it into the scalar vector and the tensor sectors.

$$\boxed{\mathcal{G}^{\text{rest}}_{\mu\nu} = \mathcal{G}^{(1)}_s u_\mu u_\nu + \mathcal{G}^{(2)}_s \mathcal{P}_{\mu\nu} + \left(\mathcal{G}^{(v)}_\mu u_\nu + \mathcal{G}^{(v)}_\nu u_\mu\right) + \mathcal{G}^{(t)}_{\mu\nu}} \tag{3.45}$$

[4]At this stage it is very tempting to conjecture equation (3.43) to be an all order statement for the coordinate transformation since generically we should have an order $\mathcal{O}\left(\partial^2\right)$ term in the ρ redefinition, but it does not appear.

Where, $\mathcal{G}_s^{(1)}$, $\mathcal{G}_s^{(2)}$, $\mathcal{G}_\mu^{(v)}$ and $\mathcal{G}_{\mu\nu}^{(t)}$ have the following forms

$$\begin{aligned}
\mathcal{G}_s^{(1)} &= r^2[1-f(\mathbf{r})] - \left(\frac{1}{2\ \mathbf{r}^{D-1}}\right)\mathfrak{s}_3 - \frac{1}{D-2}\left(\frac{K_2(\mathbf{r})}{\mathbf{r}^{D-3}}\right)\mathfrak{s}_4 \\
\mathcal{G}_s^{(2)} &= \frac{1}{D-2}\left[2\ [F(\mathbf{r})]^2 - K_1(\mathbf{r}) - 1\right]\mathfrak{s}_4 \\
\mathcal{G}_\mu^{(v)} &= \frac{2(D-2)}{\mathbf{r}^{D-3}}L(\mathbf{r})\left(\mathfrak{v}_\mu^{(5)} - \mathfrak{v}_\mu^{(3)}\right) \\
\mathcal{G}_{\mu\nu}^{(t)} &= -2\ r[1-F(\mathbf{r})]\sigma_{\mu\nu} + \left[2\ [F(\mathbf{r})]^2 - H_1(\mathbf{r}) - 1\right]\left[\mathfrak{t}_{\mu\nu}^{(1)} - \mathcal{P}_{\mu\nu}\frac{\mathfrak{s}_4}{D-2}\right] + [H_2(\mathbf{r})-1]\ \mathfrak{t}_{\mu\nu}^{(3)} \\
&\quad + [H_2(\mathbf{r}) - H_1(\mathbf{r})]\left[\mathfrak{t}_{\mu\nu}^{(4)} + \left(\frac{\Theta}{D-2}\right)\sigma_{\mu\nu}\right]
\end{aligned} \tag{3.46}$$

See Table-(3.2) for the definition of $\mathfrak{s}_i$, $\mathfrak{v}_\mu^{(i)}$ and $\mathfrak{t}_{\mu\nu}^{(i)}$

3.5 Implementing part-2: Large-D metric in terms of fluid data

In the previous section, we have recast the hydrodynamic metric, $\mathcal{G}_{AB}$ as a sum of 'background' (which is just pure AdS but looks complicated in the coordinate system where the full hydrodynamic metric has the simple form) and the 'rest'.

$$\mathcal{G}_{AB} = \bar{\mathcal{G}}_{AB} + \mathcal{G}_{AB}^{\text{rest}}$$

The large-D metric $\mathcal{W}_{AB}$ (see section (3.3)) has exactly this form. We shall simply identify $\bar{\mathcal{W}}_{AB}$ with $\bar{\mathcal{G}}_{AB}$. Next to show that the hydrodynamic is exactly the same as the large-D metric in the appropriate regime, we need to match $\mathcal{W}_{AB}^{\text{rest}}$ expanded in terms of boundary derivative up to second order in derivative expansion with $\mathcal{G}_{AB}^{\text{rest}}$, expanded in inverse power of dimension up to order $\mathcal{O}\left(\frac{1}{D}\right)^2$. In this section our goal is to rewrite $\mathcal{W}_{AB}^{\text{rest}}$ in terms of fluid data.
As we have explained before, $\mathcal{W}_{AB}^{\text{rest}}$ is expressed in terms of a harmonic function ψ and a null geodesic one-form field O_A (normalized so that the component of O_A along the normal to the constant ψ hypersurfaces is always one). We have already determined the null geodesic field up to the normalization. Our next task is to determine ψ in terms of fluid data.

3.5.1 Determining ψ

The function ψ is a harmonic function in the background AdS.

1. ψ satisfies the following differential equation everywhere on the background.

$$\nabla^2\psi^{-D} = 0 \text{ where } \nabla \text{ denotes covariant derivative with respect to the background.} \tag{3.47}$$

2. $\psi = 1$ hypersurface corresponds to the horizon, viewed as a surface embedded in the background. More precisely $\psi = 1 \Rightarrow r - H(x) = 0$ where $H(x)$ is the location of the horizon in the hydrodynamic metric as quoted in equation (3.15).

In this subsection, we shall determine ψ solving the above two conditions. We shall do it in two steps.
We shall first solve equation (3.47) in $\{\rho, y^\mu\}$ coordinates, because the expression of Laplacian is far simpler in this coordinate system (the background pure AdS metric is just diagonal here) as compared to the $\{X^A\} = \{r, x^\mu\}$ system (the one that has been used to describe the hydrodynamic metric in section (3.2)). We shall assume that in $\{Y^A\}$ coordinates, $\psi = 1$ hypersurface is given by

$$\psi = 1 \Rightarrow r = H(x) \Rightarrow \rho = \rho_H(y) \tag{3.48}$$

Note that the above condition will provide only one boundary condition for the differential equation on ψ and this is not sufficient to determine a function uniquely. We need one more condition. The other boundary condition is implicitly given by writing the harmonic function as ψ^{-D}. It implies that at a point which is order $\mathcal{O}(1)$ distance away (along any arbitrary direction) from the $\psi = 1$ hypersurface, this harmonic function falls off exponentially with D. Now clearly increasing ρ keeping all other y^μ coordinates constant is one way to go away from the $\psi = 1$ hypersurface and therefore the harmonic function ψ^{-D} must vanish as ρ goes to ∞. This will provide the required boundary condition.

Still, for a generic $\rho_H(y)$, it is difficult to solve the equation explicitly even in $\{Y^A\}$ coordinate system where the pure AdS has a simple form. However, in this case, we have two perturbation parameters and we know the solution at leading order in terms of both of them.

$$\psi^{-D} = \left(\frac{\rho}{\rho_H}\right)^{-(D-1)} + \mathcal{O}(\partial) = \left(\frac{r}{r_H}\right)^{-(D-1)} + \mathcal{O}(\partial)$$

This is what will help us to solve the equation. We shall use derivative expansion and determine ψ up to second order. As usual, at every order in the derivative expansion we shall encounter a universal and also simple second order ordinary differential equation in ρ with some source. For an explicit solution, we need two integration constants. One of them is fixed by the condition that $\psi = 1$ is the horizon. As we have explained above, the other boundary condition we fix by demanding that ψ^{-D} vanishes as $\rho \to \infty$. At the moment, we do not require $\left(\frac{1}{D}\right)$ expansion to solve for ψ.

In $\{\rho, y^\mu\}$ coordinates, the form of ψ turns out to be the following

$$\psi = \left(\frac{\rho}{\rho_H}\right)^{1-\frac{1}{D}} - \frac{(D-1)}{2\ D(D+1)\rho_H^2}\left(\frac{\rho}{\rho_H}\right)^{1-\frac{1}{D}}\left[1-\left(\frac{\rho}{\rho_H}\right)^{-2}\right][(D-2)\ \mathfrak{t}_1 +\ \mathfrak{t}_2] + \mathcal{O}(\partial)^3 \tag{3.49}$$

where,

$$\mathfrak{t}_1 = \left(\frac{\partial^\mu \rho_H}{\rho_H}\right)\left(\frac{\partial_\mu \rho_H}{\rho_H}\right), \quad \mathfrak{t}_2 = \left(\frac{\partial^\mu \partial_\mu \rho_H}{\rho_H}\right) \tag{3.50}$$

After transforming to $\{r, x^\mu\}$ (see appendix-B.3 for the details of the derivation)

$$\begin{aligned}\psi(r, x^\mu) = &\left(\frac{r}{H}\right)^{1-\frac{1}{D}} + \left(\frac{r}{r_H}\right)^{1-\frac{1}{D}}\left(\frac{1}{r^2} - \frac{1}{r_H^2}\right)\frac{D-1}{D(D+1)}\left[\mathfrak{s}_1 - \mathfrak{s}_2 + \frac{1}{2}(\mathfrak{s}_3 - \mathfrak{s}_4) + 2\ \mathfrak{s}_5\right] \\ &- \left(\frac{r}{r_H}\right)^{1-\frac{1}{D}}\ \mathfrak{s}_4\left[\frac{2}{D(D-2)\ r_H}\left(\frac{1}{r} - \frac{1}{r_H}\right)\right] + \mathcal{O}(\partial)^3\end{aligned} \tag{3.51}$$

3.5.2 Fixing the normalization of $\bar{O}^A$

As we have explained in section-(3.1.1), the null geodesic field $\bar{O}^A\partial_A$ is related to the geodesic field $\bar{O}^A\partial_A$ (determined in section - (3.4)) upto an overall normalization. The proportionality factor Φ is given by the component of $\bar{O}_A$ in the direction of n_A-the unit normal to the constant ψ hypersurfaces (see equation (3.5)). More explicitly

$$\Phi \equiv \bar{O}^A n_A = n_r = \left(\frac{\partial_r \psi}{\mathcal{N}}\right), \quad \text{where} \ \ \mathcal{N} = \sqrt{(\partial_A\psi)\ \bar{\mathcal{G}}^{AB}\ (\partial_B\psi)} \tag{3.52}$$

However, in $\{X^A\}$ coordinates it is difficult to compute $\mathcal{N}$ and therefore n_A since the background metric $\bar{\mathcal{G}}_{AB}$ and its inverse $\bar{\mathcal{G}}^{AB}$ are complicated. Fortunately we also know ψ in $\{Y^A\}$ coordinates where the background has a simple diagonal form. It is easier to compute n_A first in $\{Y^A\}$ coordinates and then convert to $\{X^A\}$ coordinates. Note in $\{X^A\}$ coordinates we only need the r component of n_A.

In $\{Y^A\}$ coordinates :

$$n_A dY^A = \left[\frac{1}{\rho} - \frac{1}{2\,\rho^3}\left(\frac{\partial_\mu \rho_H}{\rho_H}\right)^2\right] d\rho - \frac{1}{\rho_H(y)}\left(\frac{\partial \rho_H}{\partial y^\mu}\right) dy^\mu \tag{3.53}$$

In $\{X^A\}$ coordinates :

$$\begin{aligned} n_r &= \left(\frac{\partial \rho}{\partial r}\right) n_\rho + \left(\frac{\partial y^\mu}{\partial r}\right) n_\mu \\ &= \left[\frac{1}{\rho} - \frac{1}{2\,\rho^3}\left(\frac{\partial_\mu \rho_H}{\rho_H}\right)^2\right] + \left[\frac{1}{r^2} u^\mu(x) + \frac{2}{r^3}\left(\frac{\Theta}{D-2}\right) u^\mu\right]\left[\frac{\partial_\mu \rho_H(y)}{\rho_H(y)} - \frac{1}{\rho}\left(\frac{\partial_\nu \rho_H}{\rho_H}\right)(\partial_\mu u^\nu)\right] \end{aligned} \tag{3.54}$$

After some simplifications the above expression becomes

$$n_r = \frac{1}{r} - \frac{1}{2\,r^3}\left[\mathfrak{s}_1 - \mathfrak{s}_2 - 4\left(\frac{r}{r_H}\right)\frac{\mathfrak{s}_4}{(D-1)(D-2)} + 2\mathfrak{s}_5\right] = \Phi \tag{3.55}$$

Substituting the normalization we get the following expression for O_A

$$O_A\, dX^A = \; -r\left[1 + \frac{1}{2r^2}\left(\mathfrak{s}_1 - \mathfrak{s}_2 - 4\left(\frac{r}{r_H}\right)\frac{\mathfrak{s}_4}{(D-1)(D-2)} + 2\mathfrak{s}_5\right)\right] u_\mu dx^\mu \tag{3.56}$$

3.5.2.1 Large-D metric in terms of fluid data

In section-(3.3) we have described the large-D metric upto corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^3$. It is written in terms of the extrinsic curvatures of the ($\psi = 1$) hypersurface and the derivatives of the membrane velocity field $U_A \equiv n_A - O_A$. Since ψ and O_A are already determined in terms of the fluid data, it is easy to express all the structures that appear in the large-D metric in terms of the fluid data. We are listing it in tables - (4.2), (3.4) and (3.5).

Using these tables we can convert the scalar, vector and the tensor structures as

Table 3.3: Scalar large-D Data in terms of fluid Data

Large-D Data	Corresponding Fluid Data
$\mathcal{S}_1 \equiv \left(\frac{U\cdot\nabla K}{K}\right)$	$= \ 0$
$\mathcal{S}_2 \equiv U\cdot K\cdot U$	$= \ -1 + \left(\frac{1}{2r^2}\right)(\mathfrak{s}_1 - \mathfrak{s}_2 + 2\ \mathfrak{s}_5)$
$\mathcal{S}_3 \equiv U\cdot K\cdot K\cdot U$	$= \ -1 + \left(\frac{1}{r^2}\right)(\mathfrak{s}_1 - \mathfrak{s}_2 + 2\ \mathfrak{s}_5)$
$\mathcal{S}_4 \equiv \Pi^{AB}\left(\frac{\nabla_A K}{K}\right)\left(\frac{\nabla_B K}{K}\right)$	$= \ 0$
$\mathcal{S}_5 \equiv \Pi^{AB}\Pi^{CD}(\nabla_A U_C)(\nabla_B U_D)$	$= \ \left(\frac{1}{r^2}\right)(\mathfrak{s}_4 + \mathfrak{s}_3)$
$\mathcal{S}_6 \equiv \Pi^{AB}[(U\cdot\nabla)U_A][(U\cdot\nabla)U_B]$	$= \ 0$
$\mathcal{S}_7 \equiv U\cdot K\cdot\left(\frac{\nabla K}{K}\right)$	$= \ 0$
$\mathcal{S}_8 \equiv \Pi^{AB}\left(\frac{\nabla_A K}{K}\right)[(U\cdot\nabla)U_B]$	$= \ 0$
$\mathcal{S}_9 \equiv [(U\cdot\nabla)U^A][U^B K_{BA}]$	$= \ 0$
$\mathcal{S}_{10} \equiv \Pi^{AB}(\nabla_A U_B)$	$= \ \frac{2}{r\ r_H}\left(\frac{\mathfrak{s}_4}{D-1}\right)$
$\mathcal{S}_{11} \equiv \Pi^{AD}\Pi^{BC}(\nabla_A U_B)(\nabla_C U_D)$	$= \ \frac{1}{r^2}(\mathfrak{s}_4 - \mathfrak{s}_3)$

Table 3.4: Tensor large-D Data in terms of fluid Data

Large-D Data	Corresponding Fluid Data
$\mathcal{T}^{(1)}_{AB}\ dX^A dX^B \equiv P^C_A P^D_B P^{EF}(K_{EC} - \nabla_E U_C) \times (K_{FD} - \nabla_F U_D)\ dX^A dX^B$	$= \Big\{ r^2\ \mathcal{P}_{\mu\nu} + \left[\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5 - \left(\frac{4}{(D-1)(D-2)}\right)\left(\frac{r}{r_H}\right)\mathfrak{s}_4\right]\mathcal{P}_{\mu\nu} -2\left(\frac{r}{r_H}\right)\left(\frac{D-2}{D-1}\right)\left[u_\mu\left(\mathfrak{v}^{(5)}_\nu - \mathfrak{v}^{(3)}_\nu\right) + u_\nu\left(\mathfrak{v}^{(5)}_\mu - \mathfrak{v}^{(3)}_\mu\right)\right] - 2\ \mathfrak{t}^{(3)}_{\mu\nu} + \left[u_\mu\left(\mathfrak{v}^{(4)}_\nu + \mathfrak{v}^{(2)}_\nu - \mathfrak{v}^{(3)}_\nu\right) + u_\nu\left(\mathfrak{v}^{(4)}_\mu + \mathfrak{v}^{(2)}_\mu - \mathfrak{v}^{(3)}_\mu\right)\right]\Big\} dx^\mu dx^\nu$
$\mathcal{T}^{(2)}_{AB}\ dX^A dX^B \equiv P^C_A P^D_B\left[K_{CD} - \frac{\nabla_C U_D + \nabla_D U_C}{2}\right] \times dX^A dX^B$	$= \Big\{ r^2\mathcal{P}_{\mu\nu} + r\sigma_{\mu\nu} + \mathcal{P}_{\mu\nu}\left[\frac{\mathfrak{s}_1 - \mathfrak{s}_2}{2} + \mathfrak{s}_5 - \left(\frac{2}{(D-1)(D-2)}\right)\left(\frac{r}{r_H}\right)\mathfrak{s}_4\right] + u_\mu\left[\mathfrak{v}^{(2)}_\nu - \mathfrak{v}^{(3)}_\nu + \mathfrak{v}^{(4)}_\nu + 2\left(\frac{D-2}{D-1}\right)\frac{r}{r_H}\left(\mathfrak{v}^{(3)}_\nu - \mathfrak{v}^{(5)}_\nu\right)\right] + u_\nu\left[\mathfrak{v}^{(2)}_\mu - \mathfrak{v}^{(3)}_\mu + \mathfrak{v}^{(4)}_\mu + 2\left(\frac{D-2}{D-1}\right)\frac{r}{r_H}\left(\mathfrak{v}^{(3)}_\mu - \mathfrak{v}^{(5)}_\mu\right)\right]\Big\} dx^\mu dx^\nu$

described in equations (3.23) and (3.25) in terms of fluid data.

$$
\begin{aligned}
S_1 &= \frac{1}{D} - \frac{1}{r^2}\left(2 - \frac{1}{D}\right)(\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5) - \frac{1}{r^2}\left(1 - \frac{1}{D}\right)\mathfrak{s}_3 - \frac{1}{r^2}\left(1 + \frac{1}{D}\right)\mathfrak{s}_4 \\
S_2 &= \frac{1}{D}\left(1 - \frac{1}{D}\right)^2 + \frac{2}{r^2}\left(\frac{(D-2)(D-1)}{D^3}\right)(\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5) + \frac{1}{r^2}\left(\frac{(D-3)(D-1)}{D^3}\right)(\mathfrak{s}_3 - \mathfrak{s}_4)
\end{aligned}
\tag{3.57}
$$

Table 3.5: Vector large-D Data in terms of fluid Data

Large-D Data	Corresponding Fluid Data
$\mathcal{V}^{(1)}_A dX^A \equiv P^B_A \left(\frac{K^2}{D^2}\right) [(U\cdot\nabla)U_B]\ dX^A$	$= \frac{(D-2)(D-1)}{D^2}\left(\frac{2}{r_H}\right)[\mathfrak{v}^{(5)}_\mu - \mathfrak{v}^{(3)}_\mu]dx^\mu$
$\mathcal{V}^{(2)}_A dX^A \equiv P^B_A \left(\frac{K^2}{D^2}\right)\left(U^C K_{CB}\right) dX^A$	$= \left(\frac{D-1}{D}\right)^2\left(\frac{1}{r}\right)\left[\mathfrak{v}^{(4)}_\mu + \mathfrak{v}^{(2)}_\mu - \mathfrak{v}^{(3)}_\mu\right] dx^\mu$
$\mathcal{V}^{(3)}_A dX^A \equiv P^B_A P^F_D \left(\frac{\nabla_F K}{D} - \frac{K}{D}U^E K_{EF}\right)\times \left(K_{DB} - \nabla_D U_B\right) dX^A$	$= -\left(\frac{D-1}{D}\right)^2\left(\frac{1}{r}\right)\left[\mathfrak{v}^{(4)}_\mu + \mathfrak{v}^{(2)}_\mu - \mathfrak{v}^{(3)}_\mu\right] dx^\mu$
$\mathcal{V}^{(4)}_A dX^A \equiv P^B_A \left(\frac{K^2}{D^2}\right)\left(\frac{\nabla_B K}{K}\right) dX^A$	$= 0$
$\mathcal{V}^{(5)}_A dX^A \equiv P^A_C \left(\frac{\hat\nabla^2 U_A}{K}\right) dX^C$	$= \frac{1}{r}\left[2\left(\frac{D-2}{D-1}\right)\mathfrak{v}^{(5)}_\mu - \mathfrak{v}^{(3)}_\mu - \left(\frac{D-3}{D-1}\right)\left(\mathfrak{v}^{(2)}_\mu + \mathfrak{v}^{(4)}_\mu\right)\right] dx^\mu$
$\mathcal{V}^{(6)}_A dX^A \equiv \frac{1}{K} P^A_C \left(U^B K_{BD} K^D_A\right) dX^C$	$= \frac{1}{r}\left(\frac{2}{D-1}\right)\left(\mathfrak{v}^{(4)}_\mu - \mathfrak{v}^{(3)}_\mu + \mathfrak{v}^{(2)}_\mu\right) dx^\mu$
$\mathcal{V}^{(7)}_A dX^A \equiv \frac{1}{K}\Pi^{BA}\Pi^D_C\left(\frac{\nabla_B K}{K}\right)(\nabla_A U_D) dX^C$	$= 0$
$\mathcal{V}^{(8)}_A dX^A \equiv P^A_C \left(\frac{\hat\nabla^2\hat\nabla^2 U_A}{K^3}\right) dX^C$	$= \left(\frac{1}{D-1}\right)^2 \frac{1}{r^5}\left[2\left(\frac{D-2}{D-1}\right)\mathfrak{v}^{(5)}_\mu - \mathfrak{v}^{(3)}_\mu - \left(\frac{D-3}{D-1}\right)\left(\mathfrak{v}^{(2)}_\mu + \mathfrak{v}^{(4)}_\mu\right)\right] dx^\mu$
$\mathcal{V}^{(9)}_A dX^A \equiv \frac{1}{K^2} P^A_C K^{DE}\left(\hat\nabla_D\hat\nabla_E U_A\right) dX^C$	$= \left(\frac{1}{D-1}\right)\frac{1}{r}\left[2\left(\frac{D-2}{D-1}\right)\mathfrak{v}^{(5)}_\mu - \mathfrak{v}^{(3)}_\mu - \left(\frac{D-3}{D-1}\right)\left(\mathfrak{v}^{(2)}_\mu + \mathfrak{v}^{(4)}_\mu\right)\right] dx^\mu$
$\mathcal{V}^{(10)}_A dX^A \equiv \frac{1}{K^3} P^A_C \nabla_A (K_{BD}K^{BD}K) dX^C$	$= 0$
$\mathcal{V}^{(11)}_A dX^A \equiv P^A_C \left(\frac{\hat\nabla_A\hat\nabla^2 K}{K^3}\right) dX^C$	$= 0$

$$
\begin{aligned}
V_r &= 0\\
V_\mu &= -\frac{1}{r_H}\left(\frac{(D-2)(D-1)}{D^2}\right)\left(\mathfrak{v}^{(3)}_\mu - \mathfrak{v}^{(5)}_\mu\right)
\end{aligned} \tag{3.58}
$$

$$
\begin{aligned}
T_{rr} =&~ 0, \quad T_{r\mu} = 0 \\
T_{\mu\nu} =&~ -\left(\frac{r^2}{D}\right)\mathcal{P}_{\mu\nu} + r\sigma_{\mu\nu}\left(1-\frac{1}{D}\right) + 2~\mathfrak{t}^{(3)}_{\mu\nu} \\
&+ \mathcal{P}_{\mu\nu}\left[-\frac{2}{D}(\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5) - \frac{1}{D}(\mathfrak{s}_3 - \mathfrak{s}_4) + \frac{r}{r_H}\left(1+\frac{1}{D}\right)\frac{2}{(D-2)(D-1)}\mathfrak{s}_4\right] \\
&+ u_\mu\left[-\frac{1}{D}\left(\mathfrak{v}^{(2)}_\nu - \mathfrak{v}^{(3)}_\nu + \mathfrak{v}^{(4)}_\nu\right) - \frac{r}{r_H}\left(\frac{2(D-2)}{D(D-1)}\right)\left(\mathfrak{v}^{(3)}_\nu - \mathfrak{v}^{(5)}_\nu\right)\right] \\
&+ u_\nu\left[-\frac{1}{D}\left(\mathfrak{v}^{(2)}_\mu - \mathfrak{v}^{(3)}_\mu + \mathfrak{v}^{(4)}_\mu\right) - \frac{r}{r_H}\left(\frac{2(D-2)}{D(D-1)}\right)\left(\mathfrak{v}^{(3)}_\mu - \mathfrak{v}^{(5)}_\mu\right)\right]
\end{aligned}
\tag{3.59}
$$

Next we have to expand the functions (i.e., $f_1(R),~ f_2(R),~ v(R)$ and $t(R)$) appearing in the large-D metric in terms of fluid data. Note that the arguments of these functions are $R \equiv D\,(\psi - 1)$. Since ψ admits an expansion in terms of derivatives so does these functions.
Let us define a new variable $\tilde{R} \equiv D\left(\frac{r}{r_H} - 1\right)$, which is of zeroth order derivative expansion. Now we express R in terms of $\tilde{R}$.

$$
R = D\left[\left(1+\frac{\tilde{R}}{D}\right)^{1-\frac{1}{D}} - 1\right] + \delta\tilde{R} \tag{3.60}
$$

where,

$$
\begin{aligned}
\delta\tilde{R} =& -\left(1+\frac{\tilde{R}}{D}\right)^{-1-\frac{1}{D}}\frac{1}{r_H^2}\left(2~\tilde{R} + \frac{\tilde{R}^2}{D}\right)\frac{D-1}{D(D+1)}\left[\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5 + \frac{1}{2}(\mathfrak{s}_3 - \mathfrak{s}_4)\right] \\
&- D\left(1-\frac{1}{D}\right)\left(1+\frac{\tilde{R}}{D}\right)^{1-\frac{1}{D}}\frac{1}{r_H^2}\left[h_1~\mathfrak{s}_4 + h_2~\mathfrak{s}_3 + (D-3)h_3(\mathfrak{s}_1 - \mathfrak{s}_2 + 2~\mathfrak{s}_5)\right] \\
&+ \left(1+\frac{\tilde{R}}{D}\right)^{-\frac{1}{D}}\frac{2~\tilde{R}}{D(D-2)}\frac{\mathfrak{s}_4}{r_H^2}
\end{aligned}
\tag{3.61}
$$

Now the functions appearing in the large-D metric could easily be expanded in derivative expansion upto the required order.

$$
\begin{aligned}
f_1(R) &= f_1(\tilde{\mathbf{R}}) + \delta\tilde{R}\left(\frac{\partial f_1(\tilde{\mathbf{R}})}{\partial R}\right), \quad f_2(R) = f_2(\tilde{\mathbf{R}}) + \delta\tilde{R}\left(\frac{\partial f_2(\tilde{\mathbf{R}})}{\partial R}\right)\\
v(R) &= v(\tilde{\mathbf{R}}) + \delta\tilde{R}\left(\frac{\partial v(\tilde{\mathbf{R}})}{\partial R}\right), \quad t(R) = t(\tilde{\mathbf{R}}) + \delta\tilde{R}\left(\frac{\partial t(\tilde{\mathbf{R}})}{\partial R}\right)\\
\text{where,}\quad \tilde{\mathbf{R}} &= D\left[\left(1+\frac{\tilde{R}}{D}\right)^{1-\frac{1}{D}} - 1\right]
\end{aligned}
\tag{3.62}
$$

In equation (3.62) we did not explicitly evaluate the functions in terms of $\tilde{R}$ and we do not need to. Let us explain why.
We know the large-D metric only upto corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^3$. Also note that the functions $f_1(R)$, $f_2(R)$, $v(R)$ and $t(R)$ appear in the second order correction to the metric. In other words, whenever they occur, they always come with an explicit factor of $\left(\frac{1}{D}\right)^2$. Therefore it follows that in equation (3.62), any term of the order $\mathcal{O}\left(\frac{1}{D}\right)$ or higher is of no relevance. Expanding equation (3.60) further in $\left(\frac{1}{D}\right)$ we find[5]

$$
R = \tilde{\mathbf{R}} + \frac{1}{2\,r_H^2}\left(1-\frac{\tilde{\mathbf{R}}}{D}\right)\left[\mathfrak{s}_3 - \mathfrak{s}_4 + 2\left(\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5\right)\right] + \mathcal{O}\left(\frac{1}{D}\right) \tag{3.63}
$$

Here we have used the large-D expansion of the coefficients h_i appearing in equation (3.61)

$$
h_1 = \frac{1}{2D} + \mathcal{O}\left(\frac{1}{D}\right)^2, \quad h_2 = -\frac{1}{2D} + \mathcal{O}\left(\frac{1}{D}\right)^2 \quad \text{and,} \quad (D-3)h_3 = -\frac{1}{D} + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{3.64}
$$

Now examining the scalar, vector and the tensor structures in equations (3.57), (3.58) and (3.59), we see that the terms are either of first or second order in terms of derivative expansion or of order $\mathcal{O}\left(\frac{1}{D}\right)$ in terms of large-D expansion. In either case the $\mathcal{O}\left(\partial^2\right)$ terms in equation (3.63), which are actually the leading terms of $\delta\tilde{R}$ in terms of $\left(\frac{1}{D}\right)$ expansion, are negligible.

[5]When both R and $\tilde{R}$ are of order $\mathcal{O}(1)$ in terms of $\left(\frac{1}{D}\right)$ expansion, in the functions we could simply replace R by $\tilde{R}$. For regions, where R is of order D, we have to use the full relation as given as equation (3.60). We could still neglect δR but R has to be replaced by $\tilde{\mathbf{R}}$ and not by $\tilde{R}$. However, as we have mentioned in a previous footnote, in these regions, the metric correction will fall exponentially with D and therefore are not accurately captured by a power series expansion in $\left(\frac{1}{D}\right)$.

So, in $f_1(R)$, $f_2(R)$, $v(R)$ and $t(R)$ finally we could simply replace R by $\tilde{\mathbf{R}}$.

Now we have all the ingredients to express the large-D metric particularly the 'rest' part - $\mathcal{W}^{\text{rest}}_{AB}$ in terms of the fluid data. We substitute the data set presented in tables (4.2), (3.4) and (3.5) in the metric described in section-(3.3). By construction, $\mathcal{W}^{\text{rest}}_{rr}$ and $\mathcal{W}^{\text{rest}}_{r\mu}$ will vanish and only non-trivial components are $\mathcal{W}^{\text{rest}}_{\mu\nu}$. For convenience of comparison, we shall decompose the resultant expression for $\mathcal{W}^{\text{rest}}_{\mu\nu}$ again in scalar, vector and the tensor sectors as we have done for $\mathcal{G}^{\text{rest}}_{AB}$.

$$\mathcal{W}^{\text{rest}}_{\mu\nu} = \mathcal{W}^{(1)}_S u_\mu u_\nu + \mathcal{W}^{(2)}_S P_{\mu\nu} + \left(\mathcal{W}^{(V)}_\mu u_\nu + \mathcal{W}^{(V)}_\nu u_\mu\right) + \mathcal{W}^{(T)}_{\mu\nu} \tag{3.65}$$

where,

$$\begin{aligned}
\mathcal{W}^{(1)}_S =\;& r^2\left(\frac{r_H}{r}\right)^{D-1} - 2\left(\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5\right)\left[\frac{f_1(\tilde{R})}{D^2} + \frac{1}{D+1}\left(\frac{r_H}{r}\right)^{D-3}\left\{1-\left(\frac{r_H}{r}\right)^2\right\}\right] \\
&+ \frac{\mathfrak{s}_3}{2}\left[-\left(\frac{r_H}{r}\right)^{D-3} - 2\,\frac{f_1(\tilde{R})}{D^2} + \left(\frac{r_H}{r}\right)^{D-3}\left(\frac{D-1}{D+1}\right)\left\{1-\left(\frac{r_H}{r}\right)^2\right\}\right] \\
&+ \mathfrak{s}_4\left(\frac{r_H}{r}\right)^{D-3}\left[\frac{4}{(D-2)(D-1)}\left(1-\frac{r_H}{r}\right) - \left(\frac{r_H}{r}\right)^{-(D-3)}\left(\frac{f_1(\tilde{R})}{D^2}\right)\right. \\
&\qquad\left. - \frac{2}{D-2}\left(1-\frac{r_H}{r}\right) - \frac{1}{2}\left(\frac{D-1}{D+1}\right)\left\{1-\left(\frac{r_H}{r}\right)^2\right\} - \frac{K_{2H}}{D-2}\right] \\
\mathcal{W}^{(2)}_S =\;& \mathcal{O}\left(\frac{1}{D}\right)^3 \\
\mathcal{W}^{(V)}_\mu =\;& \frac{1}{D^2}\left(\frac{r}{r_H}\right)v(\tilde{R})\left(\mathfrak{v}^{(3)}_\mu - \mathfrak{v}^{(5)}_\mu\right) + \mathcal{O}\left(\frac{1}{D}\right)^3 \\
\mathcal{W}^{(T)}_{\mu\nu} =\;& \frac{r\,t(\tilde{R})}{D^2}\sigma_{\mu\nu} + \left(\frac{2\,t(\tilde{R})}{D^2}\right)t^{(3)}_{\mu\nu} + \mathcal{O}\left(\frac{1}{D}\right)^3
\end{aligned} \tag{3.66}$$

where, $\tilde{R} \equiv D\left(\frac{r}{r_H} - 1\right)$ and

$$\begin{aligned}
t(R) &= -\,2\left(\frac{D}{K}\right)^2\int_R^\infty \frac{y\,dy}{e^y-1} \\
v(R) &= 2\left(\frac{D}{K}\right)^3\left[\int_R^\infty e^{-x}dx\int_0^x \frac{y\,e^y}{e^y-1}dy \;-\; e^{-R}\int_0^\infty e^{-x}dx\int_0^x \frac{y\,e^y}{e^y-1}dy\right] \\
f_1(R) &= -2\left(\frac{D}{K}\right)^2\int_R^\infty x\,e^{-x}dx + 2\,e^{-R}\left(\frac{D}{K}\right)^2\int_0^\infty x\,e^{-x}dx
\end{aligned} \tag{3.67}$$

$$f_2(R) = \left(\frac{D}{K}\right)\left[\int_R^\infty e^{-x}dx\int_0^x \frac{v(y)}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx\int_0^x \frac{v(y)}{1-e^{-y}}dy\right]$$
$$-\left(\frac{D}{K}\right)^4\left[\int_R^\infty e^{-x}dx\int_0^x \frac{y^2\,e^{-y}}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx\int_0^x \frac{y^2\,e^{-y}}{1-e^{-y}}dy\right] \tag{3.68}$$

3.5.3 Comparison between $\mathcal{G}^{\text{rest}}_{\mu\nu}$ and $\mathcal{W}^{\text{rest}}_{\mu\nu}$

We expect each component of $\mathcal{G}^{\text{rest}}_{\mu\nu}$ to be equal to $\mathcal{W}^{\text{rest}}_{\mu\nu}$ upto corrections of order $\mathcal{O}\left(\partial^3,\ (1/D)^3\right)$. This would be true provided the coefficients (functions of r only) of independent scalar vector and tensor types of fluid data, appearing in both the metrics agree upto corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^3$. Below we are simply listing the equations that must be true for the equality of the two metrics to be valid. In the next subsection we shall explicitly verify them by doing the integrations in the limit of large D.

Table 3.6: Matching of $\mathcal{G}^{\text{rest}}_{\mu\nu}$ and $\mathcal{W}^{\text{rest}}_{\mu\nu}$

Coefficient of different structures	The resultant equation
Coefficient of $\sigma_{\mu\nu}$	$F(\mathbf{r}) = 1 + \frac{t(\tilde{R})}{2\ D^2} + \mathcal{O}\left(\frac{1}{D}\right)^3$
Coefficient of $t^{(1)}_{\mu\nu} - \mathcal{P}_{\mu\nu}\left(\frac{\mathfrak{s}_4}{D-2}\right)$	$H_1(\mathbf{r}) = 2\ [F(\mathbf{r})]^2 - 1 = 1 + 2\ \frac{t(\tilde{R})}{D^2} + \mathcal{O}\left(\frac{1}{D}\right)^3$
Coefficient of $t^{(3)}_{\mu\nu}$	$H_2(\mathbf{r}) = 1 + 2\ \frac{t(\tilde{R})}{D^2} + \mathcal{O}\left(\frac{1}{D}\right)^3$
Coefficient of $t^{(4)}_{\mu\nu} + \left(\frac{\Theta}{D-2}\right)\sigma_{\mu\nu}$	$H_2(\mathbf{r}) = H_1(\mathbf{r}) + \mathcal{O}\left(\frac{1}{D}\right)^3$
Coefficient of $\mathcal{P}_{\mu\nu}$	$K_1(\mathbf{r}) = 2\ [F(\mathbf{r})]^2 - 1 = 1 + 2\ \frac{t(\tilde{R})}{D^2} + \mathcal{O}\left(\frac{1}{D}\right)^3$
Coefficient of $\mathfrak{v}^{(3)}_\mu - \mathfrak{v}^{(5)}_\mu$	$L(\mathbf{r}) = -\frac{1}{2D^3}\left(\frac{r}{r_H}\right)\mathbf{r}^{D-3}v(\tilde{R})$ $= -\left(\frac{e^{\tilde{R}}}{2D^3}\right)v(\tilde{R}) + \mathcal{O}\left(\frac{1}{D}\right)^3$
Coefficient of $\mathfrak{s}_1 - \mathfrak{s}_2 + 2\ \mathfrak{s}_5 + \frac{\mathfrak{s}_3}{2}$	$f_1(\tilde{R}) = -2\tilde{R}\ e^{-\tilde{R}}$
Coefficient of $\mathfrak{s}_4$	$K_2(\mathbf{r}) = K_{2H} + \tilde{R} - \frac{1}{D}\left(4\ \tilde{R} + \frac{3}{2}\tilde{R}^2\right) + \mathcal{O}\left(\frac{1}{D}\right)^3$

$$\text{Where}\quad \mathbf{r} \equiv \frac{r}{r_H},\quad \tilde{R} \equiv D\,(\mathbf{r}-1)$$

3.5.3.1 $(1/D)$ expansion of the functions appearing in Hydrodynamic metric

In this subsection we shall verify the relations appearing in table (3.6) upto order $\mathcal{O}\left(\frac{1}{D}\right)^3$. For this, we need to evaluate the different integrals appearing in the hydrodynamic metric and expand it upto the required order in inverse power of dimension. For convenience we are quoting the integrals here again.

$$
\begin{aligned}
H_1(y) &= 2y^2 \int_y^\infty \frac{dx}{x}\left[\frac{x^{D-3}-1}{x^{D-1}-1}\right] \\
H_2(y) &= F(y)^2 - 2\, y^2 \int_y^\infty \frac{dx}{x(x^{D-1}-1)} \int_1^x \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \\
K_1(y) &= 2y^2 \int_y^\infty \frac{dx}{x^2}\int_x^\infty \frac{dz}{z^2}\left[z\, F'(z) - F(z)\right]^2
\end{aligned}
\tag{3.69}
$$

$$
\begin{aligned}
K_2(y) = \int_y^\infty \left(\frac{dx}{x^2}\right)\Bigg[&1 - 2(D-2)\, x^{D-2} - \left(1-\frac{1}{x}\right)\left(xF'(x) - F(x)\right) \\
&+ \left(2(D-2)x^{D-1} - (D-3)\right)\int_x^\infty \frac{dz}{z^2}\left(zF'(z) - F(z)\right)^2\Bigg] \\
L(y) = \int_y^\infty dx\; x^{D-2} &\int_x^\infty \frac{dz}{z^3}\left[\frac{z-1}{z^{D-1}-1}\right]
\end{aligned}
$$

Note that the expansion in inverse power of D would crucially depend on how we choose to scale the variable y or the coordinate $\mathbf{r}$ with D. This is what we expect and we want a detailed match in the regime where $(\mathbf{r}-1) \sim \mathcal{O}\left(\frac{1}{D}\right)$.
Below we shall first report the results of the integration in this regime, i.e., in equation (3.69) we shall substitute $y = 1 + \frac{Y}{D}$ with $Y \sim \mathcal{O}(1)$ and then evaluate the integral in an expansion in inverse powers of D (see appendix (C.1) for the details of the computation).

Large D expansion of different functions in the 'membrane region':

$$
\begin{aligned}
F(y) = F\left(1+\frac{Y}{D}\right) &= 1-\left(\frac{1}{D}\right)^2\sum_{m=1}^{\infty}\left(\frac{1+mY}{m^2}\right)e^{-mY}+\mathcal{O}\left(\frac{1}{D^3}\right)\\
H_1(y) = H_1\left(1+\frac{Y}{D}\right) &= 1-\left(\frac{2}{D}\right)^2\sum_{m=1}^{\infty}\left(\frac{1+mY}{m^2}\right)e^{-mY}+\mathcal{O}\left(\frac{1}{D}\right)^3\\
K_1(y) = K_1\left(1+\frac{Y}{D}\right) &= 1-\left(\frac{1}{D}\right)^3\sum_{m=1}^{\infty}\left(\frac{4}{m^3}\right)(2+mY)e^{-mY}+\mathcal{O}\left(\frac{1}{D}\right)^4\\
K_2(y) = K_2\left(1+\frac{Y}{D}\right) &= -\left(\frac{D}{2}\right)+(3+Y)-\left(\frac{1}{2D}\right)\left[Y(8+3Y)\right]+\mathcal{O}\left(\frac{1}{D}\right)^2\\
L(y) = L\left(1+\frac{Y}{D}\right) &= \mathcal{O}\left(\frac{1}{D}\right)^3\\
H_2(y) = H_2\left(1+\frac{Y}{D}\right)&\\
&= 1-\frac{1}{D^2}\Bigg(\frac{\pi^2}{3}\left(e^Y-1\right)-4\,Y\,\text{Log}\left[1-e^{-Y}\right]+\left(e^Y-1\right)\left(\,\text{Log}\left[1-e^{-Y}\right]\,\right)^2\\
&\quad+2\,\left(e^Y-1\right)\text{Log}\left[1-e^{-Y}\right]\,\text{Log}\left[\frac{1}{1-e^Y}\right]+2\left(e^Y+1\right)\text{PolyLog}[2,\,e^{-Y}]\\
&\quad-2\left(e^Y-1\right)\text{PolyLog}\left[2,\,\frac{e^Y}{e^Y-1}\right]\Bigg)+\mathcal{O}\left(\frac{1}{D}\right)^3
\end{aligned}
\tag{3.70}
$$

Once we use this expansion in the equations we derived in the previous subsection, they are just trivially satisfied thus proving the equivalence of the two metrics within the membrane region.

Next, we have performed these integrations outside the membrane-region. In this region $(\mathbf{r}-1)\sim\mathcal{O}(1)$, so here we have substitute $y=1+\zeta$ with $\zeta\sim\mathcal{O}(1)$. It turns out that in this regime of y , all the above functions evaluate to one up to corrections exponentially falling in D, and therefore non-perturbative from the point of view of $\left(\frac{1}{D}\right)$ expansion (see appendix (C.1) for the details of the computation). Substituting this fact in the hydrodynamic metric, we see that outside the membrane region $\mathcal{G}^{\text{rest}}_{AB}$ vanishes exponentially fast in D, exactly as we have in case of large-D metric.

3.6 Implementing part-3: Equivalence of the constraint equations

In the previous subsection, we have seen that the hydrodynamic metric is exactly the same as the large-D metric once we have correctly identified the membrane data of the large-D expansion with the fluid data. However, as we have mentioned before, the matching of the two metrics is not enough to show that the two gravity solutions are identical since the time-evolution of both the large-D data and the fluid data are constrained by two sets different looking equations. In this subsection, our goal is to show these two sets of equations are also equivalent. More precisely what we would like to show is that whenever the fluid data would satisfy the appropriate relativistic Navier-Stokes equation, the corresponding 'membrane data' would satisfy the membrane equation.

The evolution equation of a fluid dual to D dimensional gravity in presence of cosmological constant, could be expressed as a conservation of a stress tensor $T^{\mu\nu}_{\text{fluid}}$, living on the $(D-1)$ dimensional flat space-time. Up to first order in derivative expansion, $T^{\mu\nu}_{\text{fluid}}$ has the following structure, once expressed in terms of the fluid velocity u^μ and the temperature scale r_H.

$$\begin{aligned} T^{\mu\nu}_{\text{fluid}} &= r_H^{D-1}\left[(D-1)\,u^\mu u^\nu + \eta^{\mu\nu} - \left(\frac{2}{r_H}\right)\sigma^{\mu\nu}\right] + \mathcal{O}\left(\partial^2\right) \\ \text{Fluid equation :}\quad & \partial_\mu T^{\mu\nu}_{\text{fluid}} = 0 \end{aligned} \tag{3.71}$$

The membrane equation could also be expressed as conservation of some membrane stress tensor, living on the $D-1$ dimensional membrane embedded in the empty AdS.

Suppose $\{z^a,\quad a = 0, 1, 2, \cdots, D-2\}$ denotes the $(D-1)$ induced coordinates on the membrane. In terms of the membrane data (i.e., the membrane velocity U^a and the membrane-shape encoded in the extrinsic curvature tensor K_{ab}) the stress tensor

$\hat{T}^{ab}$ and conservation equation would have the following structure

$$\begin{aligned}\hat{T}^{ab} = & \left(\frac{\mathcal{K}}{2}\right) U^a U^b + \left(\frac{1}{2}\right) \mathcal{K}^{ab} - \frac{1}{2}\left(\tilde{\nabla}^a U^b + \tilde{\nabla}^b U^a\right) - \frac{1}{\mathcal{K}}\left(U^a \tilde{\nabla}^2 U^b + U^b \tilde{\nabla}^2 U^a\right) \\ & + \frac{1}{2}\left(U^a \frac{\hat{\nabla}^b \mathcal{K}}{\mathcal{K}} + U^b \frac{\hat{\nabla}^a \mathcal{K}}{\mathcal{K}}\right) - \frac{1}{2}\left(U \cdot \mathcal{K} \cdot U + \frac{\mathcal{K}}{D}\right) g^{ab}_{(\text{ind})} + \mathcal{O}\left(\frac{1}{D}\right) \\ \text{where} \;\; & \mathcal{K} \equiv g^{ab}_{(\text{ind})} K_{ab}, \;\; \tilde{\nabla} \equiv \text{Covariant derivative w.r.t. } g^{(\text{ind})}_{ab} \\ \text{Membrane equation}: \;\; & \tilde{\nabla}_a \hat{T}^{ab} = 0 \end{aligned} \tag{3.72}$$

Now, we shall process the membrane equation and after rewriting U^a and K_{ab} in terms of the fluid data and their derivatives, we shall try to express the membrane equation as fluid equation plus terms identically vanishing upto the required order. As before, for computational convenience we shall work in the $\{Y^A\} = \{\rho, y^\mu\}$ coordinates. In these coordinates, the location of the membrane is given by $[\rho - \rho_H(y) = 0]$. Let us choose $\{z^a\}$ to be $\{y^\mu\}$ themselves so that the form of the induced metric is simple. Also with this choice of coordinates along the membrane, there is no distinction between $\{a, b\}$ and $\{\mu, \nu\}$ indices and now onwards we shall use only the $\{\mu, \nu\}$ ones.

Note that since we are neglecting all terms of third or higher order in derivative expansion, and since both fluid and the membrane equation already have one overall derivative, we need to know the membrane stress tensor $\hat{T}^{\mu\nu}$ only upto first order in derivative expansion. In [69] the membrane velocity U^μ and the membrane shape have already worked out upto first order in derivative expansion. We shall simply take their results and compute the other relevant quantities.

$$\begin{aligned} & g^{\text{ind}}_{\mu\nu} = r_H^2 \; \eta_{\mu\nu} + \mathcal{O}(\partial)^2 \\ & U_\mu \; dy^\mu = r_H(y) \; u_\mu(y) dy^\mu + \mathcal{O}(\partial)^2 \\ & \mathcal{K}_{\mu\nu} = r_H^2(y) \; \eta_{\mu\nu} + \mathcal{O}(\partial)^2, \quad \mathcal{K} = (D-1) + \mathcal{O}(\partial)^2, \\ & \frac{1}{2}\left(\tilde{\nabla}_\mu U_\nu + \tilde{\nabla}_\nu U_\mu\right) = r_H \; \sigma_{\mu\nu} + \mathcal{O}(\partial)^2, \quad \hat{\nabla}^2 U_\mu = \mathcal{O}(\partial)^2 \end{aligned} \tag{3.73}$$

Substituting equation (3.73) in equation (3.72) we find

$$\hat{T}^{\mu\nu} = \frac{1}{2}\left(\frac{1}{r_H^2}\right)\left[(D-1)\, u^\mu u^\nu + \eta^{\mu\nu} - \left(\frac{2}{r_H}\right)\sigma^{\mu\nu}\right] + \mathcal{O}\left(\frac{1}{D}, \partial^2\right) \tag{3.74}$$

The Christoffel symbols are given by

$$\Gamma^{\delta}_{\beta\alpha} = \left[\delta^{\delta}_{\alpha}\left(\frac{\partial_\beta r_H}{r_H}\right) + \delta^{\delta}_{\beta}\left(\frac{\partial_\alpha r_H}{r_H}\right) - \eta_{\beta\alpha}\left(\frac{\partial^\delta r_H}{r_H}\right)\right] + \mathcal{O}(\partial)^3 \tag{3.75}$$

Rewriting the membrane equation in terms ∂_μ and the Christoffel symbols we find

$$\begin{aligned}\tilde{\nabla}_\mu \hat{T}^{\mu\nu} &= \partial_\mu T^{\mu\nu}_{\text{membrane}} + \Gamma^{\mu}_{\mu\alpha}\hat{T}^{\alpha\nu} + \Gamma^{\nu}_{\mu\alpha}T^{\mu\alpha}_{\text{membrane}}\\ &= \partial_\mu \hat{T}^{\mu\nu} + (D+1)\left(\frac{\partial_\alpha r_H}{r_H}\right)\hat{T}^{\alpha\nu} - \left(\frac{\partial^\nu r_H}{r_H}\right)\left(\eta_{\alpha\beta}\hat{T}^{\alpha\beta}\right) + \mathcal{O}(\partial)^3\\ &= \left(\frac{1}{r_H^{D+1}}\right)\partial_\mu\left[r_H^{D+1}\;\hat{T}^{\mu\nu}\right] - \left(\frac{\partial^\nu r_H}{r_H}\right)\left(\eta_{\alpha\beta}\hat{T}^{\alpha\beta}\right) + \mathcal{O}\left(1,\partial^3\right)\end{aligned} \tag{3.76}$$

Now substituting equation (3.74) in equation (3.76) we find (upto corrections of order $\mathcal{O}(\partial^3)$ in derivative expansion and $\mathcal{O}(1)$ in large D expansion)

$$\begin{aligned}0 =\;& \tilde{\nabla}_\mu \hat{T}^{\mu\nu}\\ =\;& \left(\frac{1}{r_H^{D+1}}\right)\partial_\mu\left[r_H^{D+1}\;\hat{T}^{\mu\nu}\right] - \left(\frac{\partial^\nu r_H}{r_H}\right)\left(\eta_{\alpha\beta}\hat{T}^{\alpha\beta}\right)\\ =\;& \frac{1}{2}\left(\frac{1}{r_H^{D+1}}\right)\partial_\mu\left(r_H^{D-1}\left[(D-1)\,u^\mu u^\nu + \eta^{\mu\nu} - \left(\frac{2}{r_H}\right)\sigma^{\mu\nu}\right]\right)\\ =\;& \frac{1}{2}\left(\frac{1}{r_H^{D+1}}\right)\partial_\mu T^{\mu\nu}_{\text{fluid}}\end{aligned} \tag{3.77}$$

Equation (3.77) clearly proves that upto the order $\mathcal{O}\left(1/D^2,\partial^3\right)$, the two sets of constraint equations are equivalent once the data of the solution are appropriately identified with each other.

Unfortunately, we still do not have the expression for the membrane stress tensor to second subleading order, though we do know the final membrane equation at this order (see equation (3.28)). Using the tables (4.2) and (3.5) we could easily rewrite each term of the membrane equation in terms of independent fluid data. We have checked (with the help of Mathematica version-11) that the membrane equation vanishes up to second subleading order provided the fluid equation is satisfied up to order $\mathcal{O}\left(\partial^2\right)$.

3.7 Discussion and future directions

In this chapter, we have compared two dynamical 'black-hole' type solutions of Einstein's equations in the presence of negative cosmological constant. These two solutions were already known and were determined using two different perturbation techniques - one is the 'derivative expansion' and the other is an expansion in inverse powers of dimensions. We have shown that in the regime of overlap of the two perturbation parameters, the metric of these two apparently different spaces are exactly the same, to the order the solutions are known on both sides.
Very briefly our procedure is as follows.
We have taken the metric generated in the derivative expansion (known up to second order) in an arbitrary number of space-time dimension-D and expanded it in $\left(\frac{1}{D}\right)$ up to order $\left(\frac{1}{D}\right)^2$ assuming D to be very large.
Next, we have taken the metric generated in $\left(\frac{1}{D}\right)$ expansion (this is also known up to second order) and expanded it in terms of boundary derivatives up to second order. The final result is that these two metrics just agree with each other.

The key reasons for this exact match are the following.
Firstly both the perturbation techniques use the same space-time (namely the space-time of a Schwarzschild black-brane) as the starting point and secondly given the starting point and therefore the characterizing data, both of the techniques generate the higher order corrections uniquely. Hence in the regime where both perturbations are applicable, there must exist just one solution with a given starting point. We could determine this solution by first applying derivative expansion and then expanding the answer further in $\mathcal{O}\left(\frac{1}{D}\right)$ or vice versa.

As stated above, the matching of the two metrics seems quite simple in principle. But in practice, it is quite complicated because the two metrics look very different from each other. In particular, unlike the hydrodynamic metric, the 'large-D' metric is always generated in a 'split' form - as a sum of 'background' which is just a pure AdS metric and the 'rest' which is nontrivial only within the 'membrane region' (a region of thickness of order $\mathcal{O}\left(\frac{1}{D}\right)$ around the horizon). The matching of these two metrics would imply that from hydrodynamic metric if we subtract off its decaying part (i.e, the part that falls off like r^{-D} in the large D limit), the remaining would be a metric for a regular space-time and would also satisfy the Einstein's equations in presence of negative cosmological constant. It does not follow just from the 'derivative expansion' technique. In this note, we have shown that this 'non-decaying part of the hydrodynamic metric is actually a coordinate transformation of pure AdS and

this we have done without using any 'large-D' expansion.
This is one of the main results of this note.

This work could be extended to several directions.
We have matched the two metrics only within the membrane region. But it is possible to compute the gravitational radiation, sourced by the effective membrane stress tensor and extended outside the membrane region till infinity [37]. In [36] the authors have determined the boundary stress tensor from this radiation part and matched with the dual fluid stress tensor of the hydrodynamic metric. Now since we know how to 'split' the hydrodynamic metric, we could also match the metric coefficients outside the membrane region that are exponentially falling off with D and therefore non-perturbative from the point of view of $\mathcal{O}\left(\frac{1}{D}\right)$ expansion.

The membrane equation essentially is a rewriting of the fluid equation in the limit of a large number of space-time dimensions. At a given order in large D expansion, membrane equation resums the fluid equation to all orders in derivative expansion and the resummed equation could be compactly expressed in terms of membrane velocity, which is related to fluid velocity by field redefinition. The field redefinition contains terms of all orders in derivative expansion. In this sense, it is some non-local field redefinition. But if we truncate the derivative expansion at some given order, it would look local [36].

In AdS space, there exist similar-looking constructions of horizon stress tensor in terms of boundary stress tensor by following a radial flow of constraint Einstein's equations on cut-off surfaces in the background AdS [40, 41]. In some way, they are a bit different from 'large-D' construction of the membrane stress tensor. For example, they do not resum the derivative expansion. Also, they remain like fluid equations all along the radial flow and reduce to non-relativistic fluid equations on the horizon. However, there must be some relations between this radial flow of Einstein's equations down to the horizon and the membrane stress tensor carried towards the AdS boundary via gravitational radiation [36]. It would be very interesting to explore these relations further.

We have compared the metric in the regime where both perturbation techniques are applicable and what we have shown is that the derivative and the $\left(\frac{1}{D}\right)$ expansion commute in this regime. However, we also know [69] that there exists a regime where derivative expansion could not be applied but we could still apply 'large-D' expansion. This is an interesting regime to explore since here we would construct genuinely new dynamical black hole solutions that were not described previously by other perturbative techniques. It would be very interesting to isolate this regime in the general ‘large-D’ expansion technique.

Chapter 4

Comparison between fluid-gravity and membrane-gravity dualities for Einstein-Maxwell system

This chapter is based on [3].

4.1 Introduction :

As we have already discussed in chapter 2, the effective dynamics of these black hole horizons can be described by a co-dimension one massive membrane with dynamical degrees of freedom as charge density, shape and a divergence-free velocity field moving in the background space. This duality has been studied for both asymptotically flat and AdS/dS background and also for Einstein-Maxwell systems in [24, 25, 26, 27, 28, 29].

On the other hand 'derivative expansion', which is a perturbation technique in boundary derivative expansion [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], generates dynamical black-brane solutions of Einstein's equations in the bulk with negative cosmological constant. These dynamical black brane solutions are dual to an arbitrary fluid dynamical solutions in the boundary. In other words, for every solution to the relativistic generalization of Navier-Stokes' equation in the boundary, one can construct an asymptotically AdS black hole type solution equivalent to a boundary fluid flow. These solutions thus generated are characterized by a local temperature field, a unit normalized velocity field and charge density living on the boundary.

Like chapter 3, in this chapter the questions that we would like to address in this chapter are the following.

- Is there any interconnection between these two perturbation techniques

- Can we apply these two techniques simultaneously in any regime of parameter space

- Are the solutions generated by these two techniques equivalent in any regime of parameter space?

We will try to understand these questions in the context of Einstein-maxwell system. These questions have been answered in affirmative[1, 2] for pure gravity systems. In fact in [1, 2] authors have argued on a very general ground that for these two perturbation techniques such an overlap regime should exist and the two solutions should match. The argument runs as follows.

Since we can use the same space-time geometry as the starting point for both the techniques and since given a starting point both the techniques generate the solutions uniquely, it follows that in the regime where both the techniques are applicable, the solutions should be same. But this is not at all manifest and it involves a series of intricate gauge and variable transformations. Given the general argument stated above, these explicit comparisons and matching between two differently looking and complicated metrices, serve as a consistency check for both sides of calculation.

In this chapter, we we have extended this elaborate checking to Einstein-Maxwell system. The solutions (dynamical charged black hole metric and a $U(1)$ gauge field) are known both in $\frac{1}{D}$ expansion and derivative expansion up to first subleading order. Also unlike the uncharged case worked out before, here even in the first subleading order in $\frac{1}{D}$ the curvature of the background AdS space generates new term in the membrane charge conservation equation, thus making this check even more stringent for a far more complicated set of gravity calculations as compared to that of pure gravity system.

Just in [1, 2] here also the heart of the chapter lies in these set of subtle gauge transformation needed to see the equivalence of the two metrices and gauge fields. In fact these maps or relations between these two coordinate systems, one used for large D expansion and the other for derivative expansion, are the main results of this chapter.

Though the algorithm used here is very generic, the details of the map of course depend on the details of the metrices. This map establishes a duality between three different systems, namely the dynamical charged black hole, a co-dimension one dynamical membrane and dynamical fluid living in one lower dimension. Therefore it allows us to translate between the physical quantities defined for these three different systems.

4.1.1 Strategy

In this subsection we will discuss briefly the procedure we have used to show the equivalence of the two gravity solutions and refer to [1, 2] for any logical discussion and proof. As we know the metric W_{AB} generated in large-D technique is written in a split form, a background metric $\bar{W}_{AB}$ and $W^{(rest)}_{AB}$. The metric $\bar{W}_{AB}$ is the metric of the asymptotic geometry (in our case it is pure AdS metric) and $W^{(rest)}_{AB}$ is written in a way such that contraction of a certain null geodesic $O^A\partial_A$ (not affinely parametrized) with it is zero to all order. On the contrary, the hydrodynamic metric is not written in such split form. So it is obvious that to compare the solutions generated by these two techniques the first step should be to split the hydrodynamic metric into background and 'rest'. We will do it by the following steps

At first we determine an affinely parametrized null geodesic field $\bar{O}^A\partial_A$ (with respect to full hydrodynamic metric), which passes through the event horizon of the full space-time. Then we pick up a coordinate system $Y^A \equiv \{\rho, y^\mu\}$, where the background of the hydrodynamic metric can be written in the following form

$$ds^2_{background} = \bar{G}_{AB}dY^A dY^B = \frac{d\rho^2}{\rho^2} + \rho^2\eta_{\mu\nu}dy^\mu dy^\nu \tag{4.1}$$

And determine the mapping function $f^A(X)$ that relates Y^A to the $X^A \equiv \{r, x^\mu\}$ coordinates (in which the hydrodynamic metric is written) by the following equation

$$\bar{O}^A\mathcal{G}_{AB}|_{\{X\}} = \bar{O}^A\frac{\partial f^C}{\partial X^A}\frac{\partial f^{C'}}{\partial X^B}\bar{G}_{CC'}|_{\{X\}} \tag{4.2}$$

where $\mathcal{G}_{AB}$ is the full metric written in X^A coordinates and the subscripts $\{X\}$ refer to the fact that all the terms in the left and right are calculated in X^A coordinates.

However it cannot fix f^A completely and we require some conformal type symmetry on the background metric to fix it.

After determining the mapping function we can find out the background metric in $\{X^A\}$ coordinates and hence can split the hydrodynamic metric into background plus 'rest'. Then we take the large-D limit of the hydrodynamic metric.

The next step is to write the large-D data in terms of fluid data. The metric and gauge field generated in large-D technique are written in terms of a smooth function ψ (such that ψ^{-D} is a harmonic function w..r.t the background), a charge field $\widetilde{Q}$ and a non-affinely parametrized null geodesic O_A. It will turn out that O_A is related to $\bar{O}_A$ (determined from the hydrodynamic metric) by an overall normalization constant.

At first we determine ψ and then O_A and $\widetilde{Q}$ in terms of fluid data. After that we substitute these expressions in the large-D metric and gauge field and write those in terms of fluid data. Then it is easy to check that the metric and gauge field in large-D side matches with those in fluid side up to appropriate orders in both the perturbation parameters.

But the hydrodynamic data and large-D data cannot be chosen arbitrarily. They have to satisfy some constraint equations, named as fluid equations and membrane equations for the metric and gauge field to be a solution of Einstein's equations. So to show the equivalence of the gravity solutions generated by these two techniques the final step would be to show the equivalence of these constraint equations.

The organisation of this chapter is as follows.
In section §4.2 we determine the hydrodynamic metric and gauge field up to first order in derivative in arbitrary dimensions. In section §4.3 we have noted the gravity solutions for Einstein-Maxwell systems in large-D technique. In section §4.4 we have rewritten the large-D data in terms of fluid data and compared the two gravity solutions. And finally in section §5 we have concluded.

4.2 Hydrodynamics from charged black-branes in arbitrary dimensions :

In this section we shall review the work on fluid-gravity correspondence for charged black-branes [7, 8, 70] by determining the metric and gauge field dual to charged fluid configuration up to first order in boundary derivative expansion for all $D \geq 3$. The results of this section were previously recorded in [70] in a bit different language. As our Lagrangian and notations are slightly different from the authors of [70], we will redo everything with our Lagrangian and notations.

We start with the D dimensional action[1]

$$
\begin{aligned}
\mathbf{S} &= \frac{1}{16\pi G_D}\int d^Dx\,\sqrt{-\mathrm{g}}\left[R - 2\Lambda - \frac{F_{MN}F^{MN}}{4}\right] \\
&\text{with negative cosmological constant } \Lambda = \frac{(D-1)\,(D-2)}{2}\lambda
\end{aligned} \tag{4.3}
$$

By varying the above action we shall get the D dimensional Einstein-Maxwell equations with negative cosmological constant

$$
\begin{aligned}
&R_{AB} - \frac{1}{2}R\; g_{AB} - \frac{(D-1)(D-2)}{2}g_{AB} + \frac{1}{2}\;F_{AC}F^{C}_{\;B} + \frac{1}{8}\;g_{AB}F_{CD}F^{CD} = 0 \\
&\nabla_B F^{AB} = 0 \Rightarrow \frac{1}{\sqrt{-g}}\partial_B\left(\sqrt{-g}\;F^{AB}\right) = 0
\end{aligned} \tag{4.4}
$$

where g_{AB} is the D dimensional metric tensor and $F_{AB} = \partial_A A_B - \partial_B A_A$.

We know that these equations (4.4) admit an AdS-Reisner-Nordstrom 'boosted black-brane solutions', which we write in ingoing Eddington-Finkelstein coordinates as

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu dr - r^2 V\left(r, m, Q\right) u_\mu u_\nu dx^\mu dx^\nu + r^2 \mathcal{P}_{\mu\nu}dx^\mu dx^\nu \\
A &= \frac{\sqrt{3}\;Q}{2r^{D-3}}u_\mu dx^\mu
\end{aligned} \tag{4.5}
$$

with

$$
\begin{aligned}
&V(r, m, Q) = 1 - \frac{m}{r^{D-1}} + \frac{1}{4}\,\frac{3(D-3)}{2(D-2)}\frac{Q^2}{r^{2(D-2)}}\;, \quad \mathcal{P}^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu \\
&u^v = \frac{1}{\sqrt{1-\beta^2}}\;,\; u^i = \frac{\beta^i}{\sqrt{1-\beta^2}}\;,\; \beta^2 = \beta_i\beta^i
\end{aligned} \tag{4.6}
$$

As in the metric described above we shall use coordinates $X^A \equiv \underbrace{\{r, v, x^i\}}_{D}$ for our bulk spaces, on the other hand coordinates $x^\mu \equiv \underbrace{\{v, x^i\}}_{D-1}$ parametrize our boundary and r is the radial coordinate.

Now we allow the temperature, velocity and charge field in the black-brane metric (4.5) to vary slowly in the boundary coordinates and determine the metric and gauge

[1] We shall use the Latin indices $\{M, N, \cdots\}$ to denote the bulk indices while the Greek indices $\{\mu, \nu, \cdots\}$ refer to the boundary indices. And the $\{\mu, \nu, \cdots\}$ indices are raised and lowered by the Minkowski metric $\eta^{\mu\nu}$. And for our case $\lambda = -1$.

field in boundary derivative expansion. To start we shall take the ansatz as

$$\begin{aligned} g_{MN} &= g^{(0)}_{MN} + g^{(1)}_{MN} + g^{(2)}_{MN} + \cdots \\ A_N &= A^{(0)}_N + A^{(1)}_N + A^{(2)}_N + \cdots \end{aligned} \tag{4.7}$$

where the leading order ansatz $g^{(0)}_{MN}$ and $A^{(0)}_N$ are given by

$$\begin{aligned} g^{(0)} &= -2u_\mu dx^\mu dr - r^2 V(r,m,Q) u_\mu u_\nu dx^\mu dx^\nu + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu \\ A^{(0)} &= \frac{\sqrt{3}\,Q}{2r^{D-3}} u_\mu dx^\mu \end{aligned} \tag{4.8}$$

and $g^{(k)}_{MN}$, $A^{(k)}_N$, which are corrections to the bulk metric and gauge field, are determined by solving Einstein-Maxwell equations order by order in derivative expansion. To solve these equations by our perturbation technique it is useful to work in a particular gauge. Following[7] we work in the following gauge

$$g_{rr} = 0\ ,\ g_{r\mu} = -u_\mu\ ,\ \mathrm{Tr}\left[\left(g^{(0)}\right)^{-1} g^{(k)}\right] = 0\ ,\ A_r = 0 \tag{4.9}$$

Now to define velocity field uniquely we will work in Landau frame defined by $u_\mu T^{\mu\nu}_{(k)} = 0$, where $T^{\mu\nu}_{(k)}$ is the kth order stress tensor with $k \geq 1$ and in the proper frame of a fluid element the longitudinal component of the stress tensor to the fluid velocity give the local energy density in the fluid.

In this section our goal is to find out the metric and gauge field up to first order in derivative expansion. To implement our perturbation technique we set our velocity field u^μ to be $\{1, 0, 0, \cdots\}$ by a boundary Lorentz transformation at a boundary point x^μ and solve these equations about this special point. Since our perturbation procedure is ultra-local we can easily write the result thus obtained in covariant form with respect to the boundary metric. Our velocity, temperature and charge field expanded in taylor series about this special point x^μ in terms of boundary derivatives as

$$\begin{aligned} \beta_i &= x^\mu \partial_\mu \beta^{(0)}_i + \cdots \\ m &= m^{(0)} + x^\mu \partial_\mu m^{(0)} + \cdots \\ Q &= Q^{(0)} + x^\mu \partial_\mu Q^{(0)} + \cdots \end{aligned} \tag{4.10}$$

Using the expressions written above if we expand the 0^{th} order ansatz up to first order in derivative we have

$$
\begin{aligned}
ds^2_{(0)} =& 2\ dv\ dr - r^2 V^{(0)}(r)\ dv^2 + r^2 dx^i\ dx_i \\
& - 2r^2 \left(1 - V^{(0)}(r)\right) x^\mu \partial_\mu \beta_i^{(0)}\ dx^i\ dv - 2x^\mu \partial_\mu \beta_i^{(0)}\ dx^i dr \\
& - \left(-\frac{x^\mu \partial_\mu m^{(0)}}{r^{D-3}} + \frac{1}{4}\frac{3(D-3)}{2(D-2)}\frac{2Q^{(0)} x^\mu \partial_\mu Q^{(0)}}{r^{2(D-3)}} \right) dv^2 \\
A =& -\frac{\sqrt{3}}{2r^{D-3}} \left[\left(Q^{(0)} + x^\mu \partial_\mu Q^{(0)}\right) dv - Q^{(0)} x^\mu \partial_\mu \beta_i^{(0)} dx^i \right] \\
\text{with} \quad & V^{(0)}(r) = 1 - \frac{m^{(0)}}{r^{D-1}} + \frac{1}{4}\ \frac{3(D-3)}{2(D-2)} \frac{{Q^{(0)}}^2}{r^{2(D-2)}}
\end{aligned}
\tag{4.11}
$$

Obviously the metric and gauge field of equation (4.11) will not solve the Einstein-Maxwell's equations up to the order we are interested. We need to add corrections containing first order derivatives to solve these equations. While solving these equations we find that the bulk Einstein-Maxwell equations decompose into constraint equations and dynamical equations.

Since the background metric have the $SO(D-2)$ symmetry, the Einstein-Maxwell equations will split into scalar, vector, and traceless symmetric two tensor sector and we can solve each sector separately.

4.2.1 Scalars at first order

The scalar components of the first order corrections added to the metric and gauge field are parametrized by $h_1(r), k_1(r)$ and $w_1(r)$ and we can write these as

$$
\begin{aligned}
g^{(1)}_{vv}(r) &= \frac{k_1(r)}{r^2} \\
g^{(1)}_{vr}(r) &= -\frac{(D-2)}{2} h_1(r) \\
\sum_i g^{(1)}_{ii}(r) &= (D-2)\ r^2\ h_1(r) \\
A^{(1)}_v(r) &= -\frac{\sqrt{3}\ w_1(r)}{2\ r^{D-3}}
\end{aligned}
\tag{4.12}
$$

Here one should note that the metric corrections $g^{(1)}_{ii}$ and $g^{(1)}_{vr}$ are related to each other by the gauge choice $\mathrm{Tr}\left[\left(g^{(0)}\right)^{-1} g^{(1)}\right] = 0$. At first we determine the constraint equations. These equations are determined by taking dot product of Einstein-Maxwell equations with the vector dual to the one form dr. We have these equations as follows

$$\frac{\partial_v m^{(0)}}{m^{(0)}} + \frac{(D-1)}{(D-2)}\partial_i \beta_i^{(0)} = 0 \tag{4.13}$$

which is identical to the conservation of the stress tensor in the boundary

$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \tag{4.14}$$

The other constraint equation is given by

$$\frac{\partial_v Q^{(0)}}{Q^{(0)}} + \partial_i \beta_i^{(0)} = 0 \tag{4.15}$$

which is equivalent to the conservation of boundary current density.

$$\partial_\mu J^\mu_{(0)} = 0 \tag{4.16}$$

In the scalar sector we have 6 differential equations, the rr, vv, rv component of the Einstein tensor along with the trace over the boundary spatial part and the r and v components of the Maxwell equations. Among these 6 equations we have to use only 3 equations to determine the three unknown parameters h_1, k_1 and w_1. The solutions thus obtained should satisfy the rest equations. Solving these equations and demanding the appropriate normalizability at infinity we have the following solutions

$$\begin{aligned}
g^{(1)}_{vv}(r) &= 2r\left(\frac{\partial_i \beta_i^{(0)}}{D-2}\right)\\
g^{(1)}_{vr}(r) &= 0\\
\sum_i g^{(1)}_{ii}(r) &= 0\\
A^{(1)}_v(r) &= 0
\end{aligned} \tag{4.17}$$

4.2.2 Vectors at first order

The vector components of first order metric and gauge field are parametrized by $g_i^{(1)}(r)$ and $j_i^{(1)}(r)$ as

$$\begin{aligned} g_{vi}^{(1)}(r) &= r^2\left(1-V(r)\right) j_i^{(1)}(r) \\ A_i^{(1)}(r) &= -\frac{\sqrt{3}Q^{(0)}}{2r^{D-3}} j_i^{(1)}(r) + g_i^{(1)}(r) \end{aligned} \tag{4.18}$$

Here also at first we have determined the constraint equation which is given by

$$\frac{\partial_i m^{(0)}}{m^{(0)}} + (D-1)\partial_v \beta_i^{(0)} = 0 \tag{4.19}$$

This follows from the conservation of boundary stress tensor.

The dynamical equations in the vector sector are the ri, vi components of the Einstein tensor and i th component of the Maxwell equation. Solving these equations with appropriate boundary conditions (regularity at the future event horizon and appropriate fall off at boundary) we have the following corrected first order metric and gauge field in the vector sector

$$\begin{aligned} g_{vi}^{(1)}(r) &= r\ \partial_v \beta_i^{(0)} + \frac{3(D-3)\ r^2\ Q^{(0)}\left(\partial_i Q^{(0)} + (D-2)\ Q^{(0)}\ \partial_v \beta_i^{(0)}\right)}{R^{2(D-1)}} R\ F_1(\rho, M) \\ A_i^{(1)}(r) &= -2\sqrt{3}\frac{r^D}{R^{2(D-1)}}\left(\partial_i Q^{(0)} + (D-2)\ Q^{(0)}\ \partial_v \beta_i^{(0)}\right) F_1^{(1,0)}(\rho, M) \end{aligned} \tag{4.20}$$

where F_1 is given by

$$\begin{aligned} F_1(\rho, M) &= \frac{1}{4(D-2)}\left(1 + \frac{1}{4}\frac{3(D-3)}{2(D-2)}\frac{Q_1^2}{\rho^{2(D-2)}} - \frac{M}{\rho^{D-1}}\right) F_3(\rho, M) \\ \text{where}\quad F_3(\rho, M) &= \int_\rho^\infty dp \frac{1}{\left(1 + \frac{1}{4}\frac{3(D-3)}{2(D-2)}\frac{Q_1^2}{p^{2(D-2)}} - \frac{M}{p^{D-1}}\right)^2}\left(\frac{1}{p^{2(D-1)}} - \frac{c_1}{p^{2D-3}}\right) \end{aligned} \tag{4.21}$$

with

$$c_1 = \frac{D-2}{D-1}\left(1 + \frac{2}{M(D-3)}\right) \tag{4.22}$$

and also $F_1^{(1,0)}(\rho, M)$ is defined as

$$F_1^{(1,0)}(\rho, M) = \frac{d}{d\rho} F_1(\rho, M) \tag{4.23}$$

Where we have used the following rescaled variables

$$\rho = \frac{r}{R} \quad , M = \frac{m}{R^{D-1}} \quad , Q_1 = \frac{Q}{R^{D-2}}, \quad \text{and } Q_1^2 = 4\,\frac{2(D-2)}{3(D-3)}(M-1) \tag{4.24}$$

And then the Hawking temperature is given by

$$T = \frac{(D-1)R}{4\pi}\left(1 - \left(\frac{D-3}{D-1}\right)(M-1)\right) \tag{4.25}$$

where R is the radius of the outer horizon. In terms of these rescaled variables the outer horizon is given by $\rho_+ \equiv 1$.

4.2.3 Tensors at first order

The tensor component of the metric at first order can be parametrized by $\pi_{ij}^{(1)}(r)$ as

$$g_{ij}^{(1)} = r^2 \pi_{ij}^{(1)}(r) \tag{4.26}$$

This unknown parameter $\pi_{ij}^{(1)}$ can be determined by solving the dynamical equation obtained from the ij component of Einstein equation. Then demanding regularity at the future event horizon and appropriate fall off at boundary we have following corrected first order metric in the tensor sector .

$$\pi_{ij}^{(1)}(r) = \frac{2}{R}\sigma_{ij}\; F_2(\rho, M) \tag{4.27}$$

where,

$$F_2(\rho, M) = \int_\rho^\infty \frac{\rho^D\left(\rho^D - \rho^2\right)}{\rho^2\left(\rho^{2D} - M\rho^{D+1} + (M-1)\rho^4\right)} d\rho \tag{4.28}$$

4.2.4 The global metric and gauge field at first order

We have done our computation about a special point $x^\mu = 0$ on the boundary. However, our perturbation procedure is ultralocal and we could set any arbitrary velocity u^μ to be $\{1, 0, \cdots\}$ by a boundary coordinate transformation. So we could do our computation about any arbitrary point on the boundary. Hence the results recorded in the previous subsections contain enough information to write down the metric and gauge field in covariant form w.r.t the boundary metric. we have the following covariant form of the metric and gauge field

$$
\begin{aligned}
ds^2 =& g_{AB}dx^A dx^B \\
=& -2u_\mu dx^\mu dr - r^2 V(r,m,Q) u_\mu u_\nu dx^\mu dx^\nu + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu \\
& -2u_\mu dx^\mu \; r \left[u^\lambda \partial_\lambda u_\nu - \frac{\partial_\lambda u^\lambda}{D-2} u_\nu\right] dx^\nu + \frac{2r^2}{R} F_2(\rho, M) \sigma_{\mu\nu} dx^\mu dx^\nu \\
& -2u_\mu dx^\mu \left[\frac{3\,(D-3)\;Q\;r^2}{R^{2D-3}} \mathcal{P}^\lambda_\nu \left(\mathcal{D}_\lambda Q\right) F_1(\rho, M)\right] dx^\nu + \cdots
\end{aligned}
\tag{4.29}
$$

$$
A = \left[\frac{\sqrt{3}Q}{2\;r^{D-3}} u_\mu - \frac{2\sqrt{3} r^D}{R^{2(D-1)}} \mathcal{P}^\lambda_\mu \left(\mathcal{D}_\lambda Q\right) F_1^{(1,0)}(\rho, M)\right] dx^\mu + \cdots
$$

where

$$
\mathcal{P}^\lambda_\mu \left(\mathcal{D}_\lambda Q\right) = \mathcal{P}^\lambda_\mu \left(\partial_\lambda Q\right) + (D-2)\left(u^\lambda \partial_\lambda u_\mu\right) Q \tag{4.30}
$$

and

$$
\sigma_{\mu\nu} = \mathcal{P}^\alpha_\mu \mathcal{P}^\beta_\nu \left[\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \eta_{\alpha\beta}\left(\frac{\Theta}{D-2}\right)\right] \tag{4.31}
$$

And the constraint equations can be written in covariant form as

$$
\begin{aligned}
&\frac{(u\cdot\partial)\,Q}{Q} + \partial\cdot u = 0 \\
&\frac{(u\cdot\partial)\,m}{m} + (D-1)\frac{\partial\cdot u}{D-2} = 0 \\
&\frac{\mathcal{P}^\alpha_\mu\;\partial_\alpha m}{m} + (D-1) u^\lambda \partial_\lambda u_\mu = 0
\end{aligned}
\tag{4.32}
$$

4.2.5 The boundary stress tensor and the charge current

By using AdS/CFT correspondence we can determine the boundary stress tensor from the bulk space-time metric. Here we have to add suitable counter terms to the action to regularise the divergence arising from integrating the full space-time volume. The expression for the boundary stress tensor dual to the metric presented in the previous subsection can be obtained by the prescription of [71].

The stress tensor for the metric up to first order in derivative expansion is given by

$$T^{\mu\nu} = p\,(\eta^{\mu\nu} + (D-1)u^\mu u^\nu) - 2\eta\ \sigma^{\mu\nu}$$
$$\text{where} \quad p = \frac{M\ R^{D-1}}{16\ \pi\ G_D} \quad \text{and} \quad \eta = \frac{R^{D-2}}{16\ \pi\ G_D} \tag{4.33}$$

One can explicitly find out the following expression for charge current for the gauge field

$$J_\mu = n\ u_\mu - \mathfrak{D}\ \mathcal{P}^\nu_\mu \mathcal{D}_\nu n \tag{4.34}$$

where the charge density n and diffusion constant $\mathfrak{D}$ are given by

$$n = \frac{\sqrt{3}Q}{16\ \pi\ G_D} \quad \text{and} \quad \mathfrak{D} = \frac{(D-3)M + 2}{R\ M\ (D-1)(D-3)} \tag{4.35}$$

4.2.6 Hydrodynamic metric and gauge field up to first order in derivative

The metric dual to hydrodynamics in arbitrary dimension D is written in terms of fluid variable u, a charge field and a temperature field T living on the $(D-1)$ dimensional boundary of the space-time. The independent data in first order is written in table (4.1). In this subsection we shall write the hydrodynamic metric, gauge field and the constraint equations recorded in previous subsections in the following way.

Table 4.1: Data at 1st order in derivative

	Independent Data
Scalar	$\frac{\Theta}{D-2} \equiv \frac{\partial \cdot u}{D-2}$
Vector	$a_\mu \equiv (u\cdot\partial)u_\mu\ ,\ \mathcal{P}^\lambda_\mu \partial_\lambda Q_C$
Tensor	$\sigma_{\mu\nu} = \mathcal{P}^\alpha_\mu \mathcal{P}^\beta_\nu \left[\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \eta_{\alpha\beta}\left(\frac{\Theta}{D-2}\right)\right]$

The constraint equations can be written as

$$
\begin{aligned}
&\frac{(u\cdot\partial)\,Q_C}{Q_C} = 0\\
&\frac{(u\cdot\partial)\,r_H}{r_H} + \frac{\Theta}{D-2} = 0\\
&\frac{\mathcal{P}_\mu^\alpha\ \partial_\alpha r_H}{r_H} + a_\mu + f(Q_C)\mathcal{P}_\mu^\alpha\ \partial_\alpha Q_C = 0
\end{aligned}
\tag{4.36}
$$

where

$$
f(Q_C) = \frac{6\frac{(D-3)}{D-1}Q_C}{8(D-2)+3(D-3)Q_C^2} \tag{4.37}
$$

We write metric and gauge field in two part as

$$
dS^2 = dS_0^2 + dS_1^2 \tag{4.38}
$$

where

0^{th} Order Piece:

$$
\begin{aligned}
dS_0^2 =& -2u_\mu\ dx^\mu\ dr - r^2V(r)\ u_\mu u_\nu\ dx^\mu\ dx^\nu + r^2\mathcal{P}_{\mu\nu}dx^\mu dx^\nu\\
\mathcal{P}_{\mu\nu} =&\ \eta_{\mu\nu} + u_\mu u_\nu
\end{aligned}
\tag{4.39}
$$

1^{st} Order Piece:

$$
dS_1^2 = -r\,(u_\mu B_\nu + u_\nu B_\mu)\,dx^\mu\ dx^\nu + \frac{2r^2}{r_H}\ F_2(\rho, M)\ \sigma_{\mu\nu}\ dx^\mu\ dx^\nu
$$

where

$$
B_\mu = a_\mu - \left(\frac{\Theta}{D-2}\right)u_\mu + \frac{3\ (D-3)\ r\ Q_C}{r_H^2}\mathcal{P}_\mu^\lambda\,(\partial_\lambda Q_C)\left[1-(D-2)Q_C\ f(Q_C)\right]F_1(\rho, M) \tag{4.40}
$$

The gauge field up to first order in derivative expansion can be written as

$$
A = A_0 + A_1 \tag{4.41}
$$

where

0^{th} Order Piece:

$$
A_0 = \frac{\sqrt{3}\ r\ Q_C}{2}\left(\frac{r_H}{r}\right)^{D-2}u_\mu dx^\mu \tag{4.42}
$$

1^{st} **Order Piece:**

$$A_1 = -2\sqrt{3}\left(\frac{r}{r_H}\right)^D \left[1-(D-2)Q_C\ f(Q_C)\right] F_1^{(1,0)}(\rho, M)\ \mathcal{P}^\lambda_\mu\left(\partial_\lambda Q_C\right) dx^\mu \quad (4.43)$$

where,

$$V(r) = 1-\left(1+\frac{1}{4}\,\frac{3(D-3)}{2(D-2)}Q_C^2\right)\left(\frac{r_H}{r}\right)^{D-1}+\frac{1}{4}\,\frac{3(D-3)}{2(D-2)}Q_C^2\left(\frac{r_H}{r}\right)^{2(D-2)} \quad (4.44)$$

Here $r = r_H$ is the position of the outer event horizon of the space-time and Q_C is just Q_1, defined in (4.24) and also r_H is just R defined in previous section.

4.3 The large D metric, gauge field and membrane equations:

In this section we simply quote the results of large D metric and gauge field as in paper [29].
The metric in the large D expansion is written in a split form as $\bar{W}_{AB}$ plus $W^{(rest)}_{AB}$.

$$W_{AB} = \bar{W}_{AB} + W^{(rest)}_{AB}$$

$\bar{W}_{AB}$ is the metric of the nonsingular asymptotic geometry. In our case it would be the metric of just pure AdS. In general, $\bar{W}_{AB}$ is any smooth solution (where all components of the Riemann tensor are of order $\mathcal{O}(1)$ in terms of large D counting) to Einstein's equations with cosmological constant and vanishing electromagnetic field. $W^{(rest)}_{AB}$ captures the effect of black hole and singularity. Using the large D technique one can determine $W^{(rest)}_{AB}$ and in this case also the gauge field A_M order by order in an expansion in inverse powers of D. In other words the metric and the gauge field will have the following form

$$\begin{aligned} W^{(rest)}_{AB} &= \mathcal{W}^{(0)}_{AB} + \left(\frac{1}{D}\right)\mathcal{W}^{(1)}_{AB} + \cdots \\ A_M &= A^{(0)}_M + \left(\frac{1}{D}\right)A^{(1)}_M + \cdots \end{aligned} \quad (4.45)$$

The final solution is expressed in terms of a null geodesic field $O^A\partial_A$ and two scalar fields ψ and $\tilde{Q}$. ψ is a smooth function with $\psi = 1$ as event horizon of the full space-time and harmonic w.r.t the background $\bar{W}_{AB}$. $\widetilde{Q}$ is also a smooth function satisfying

$$n \cdot \partial \widetilde{Q} = 0 \ , \ \text{and} \ \widetilde{Q}|_{\psi=1} = \frac{1}{\sqrt{2}} \left(U^M A_M\right)|_{\psi=1} \tag{4.46}$$

where n_A is the unit normal to the constant ψ slices and U is the membrane velocity defined by $U = n - O$.
The gauge is fixed by demanding that to all orders

$$O^A W^{(rest)}_{AB} = 0 \ \ \text{and} \ \ O^M A_M = 0$$

Now we shall present the final solutions up to the first subleading order in $\mathcal{O}\left(\frac{1}{D}\right)$ i.e., the explicit expressions for $\mathcal{W}^{(0)}_{AB}$, $A^{(0)}_M$ and $\mathcal{W}^{(1)}_{AB}$, $A^{(1)}_M$

$$\begin{aligned} \mathcal{W}^{(0)}_{AB} &= f O_A O_B \\ A^{(0)}_M &= \sqrt{2}\ \widetilde{f} O_M \\ \mathcal{W}^{(1)}_{AB} &= \mathcal{Z}^{(s1)} O_A O_B + \left(\mathcal{Z}^{(v)}_A O_B + \mathcal{Z}^{(v)}_B O_A\right) + \mathcal{Z}^{(T)}_{AB} \\ \text{and } A^{(1)}_M &= \mathcal{A}^{(s)} O_M + \mathcal{A}^{(v)}_M \end{aligned} \tag{4.47}$$

where

$$f = \left(1 + \widetilde{Q}^2\right) \psi^{-D} \ , \ \ \widetilde{f} = \widetilde{Q} \psi^{-D} \tag{4.48}$$

and

$$\begin{aligned} \mathcal{Z}^{(s1)} &= \sum_{i=1}^{N_S} S^{(i)}_1(\zeta) \mathcal{S}^{(i)} \ , \ \mathcal{Z}^{(s2)} = \sum_{i=1}^{N_S} S^{(i)}_2(\zeta) \mathcal{S}^{(i)} \ , \ \mathcal{A}^{(s)} = \sum_{i=1}^{N_S} a^{(i)}_s(\zeta) \mathcal{S}^{(i)} \\ \mathcal{Z}^{(v)}_A &= \sum_{i=1}^{N_V} \mathcal{V}^{(i)}(\zeta) V^{(i)}_A \ , \ \mathcal{A}^{(v)}_A = \sum_{i=1}^{N_V} a^{(i)}_v(\zeta) V^{(i)}_A \ , \ \mathcal{Z}^{(T)}_{AB} = \sum_{i=1}^{N_T} \mathcal{T}^{(i)}(\zeta) t^{(i)}_{AB} \end{aligned} \tag{4.49}$$

Here $O_A \ dX^A$ is a null one-form with respect to both the background metric $\bar{W}_{AB}$ and full metric W_{AB}. This is dual to the geodesic vector field $O^A \partial_A$ mentioned before. The function ψ and $\tilde{Q}$ are already defined (see the discussion around equation (4.46)).

Now both the functions ψ and $\widetilde{Q}$ admit $\frac{1}{D}$ expansion in the 'membrane region' - the region where the gravitational field is non trivial in the limit of large D. It follows that up to first order in $\frac{1}{D}$, the functions f and $\widetilde{f}$ can be written as

$$\begin{aligned} f(W) &= \left(1+\widetilde{Q}^2\right)\ e^{-W} - \widetilde{Q}^2\ e^{-2W} + \left(\frac{1}{D}\right) \\ \widetilde{f}(W) &= \widetilde{Q}\ e^{-W} + \left(\frac{1}{D}\right) \end{aligned} \tag{4.50}$$

where the parameter W is a $\mathcal{O}(1)$ variable, defined by $W = D\,(\psi - 1)$.

And then the solution up to first order in metric and gauge field can be written as

$$\mathcal{V}^{(i)}(y) = -2\int_y^\infty dx\ e^{-x}\ \widetilde{Q}\ a_v^{(i)}(x) - e^{-y}\mathcal{K}_{vector} - 2\left(\frac{D}{K}\right)\int_y^\infty dx\ e^{-x}\int_0^x dt\frac{e^t\ \mathfrak{v}^{(i)}_{\text{metric}}(t)}{1-f(t)}$$

$$\text{where } \mathcal{K}_{vector} = -2\int_0^\infty dx\ e^{-x}\ \widetilde{Q}\ a_v^{(i)}(x) - 2\left(\frac{D}{K}\right)\int_0^\infty dx\ e^{-x}\int_0^x dt\frac{e^t\ \mathfrak{v}^{(i)}_{\text{metric}}(t)}{1-f(t)} \tag{4.51}$$

and

$$\begin{aligned} a_v^{(i)}(t) &= -e^t(f-\widetilde{f}^2)\int_t^\infty dx\frac{e^{-3x}}{(1-f)(f-\widetilde{f}^2)^2}\int_0^x dy\ M^{(i)}(y) \\ &\quad + \left(2\int_0^\infty dz\ \widetilde{f}\ a_v^{(i)}(z)\right)e^t(f-\widetilde{f}^2)\int_t^\infty dx\frac{e^{-3x}}{(1-f)(f-\widetilde{f}^2)^2}\int_0^x dy\ e^y\ \widetilde{f}\ (f-\widetilde{f}^2) \end{aligned} \tag{4.52}$$

where

$$\begin{aligned} M^{(i)}(x) = \int_0^x dx\ e^{2x}\left(f-\widetilde{f}^2\right)&\left(\widetilde{f}\ e^{-x}\left(-\frac{2D}{K}\right)\int_0^\infty dz\ e^{-z}\int_0^z dt\frac{e^t\ \mathfrak{v}^{(i)}_{\text{metric}}(t)}{1-f(t)} - \frac{1}{N}\mathfrak{v}^{(i)}_{\text{gauge}}(x)\right. \\ &\left. + 2\ \widetilde{f}\left(\frac{D}{K}\right)e^{-x}\int_0^x dt\ \frac{e^t\ \mathfrak{v}^{(i)}_{\text{metric}}(t)}{1-f(t)}\right) \end{aligned} \tag{4.53}$$

and

$$\int_0^\infty dt\ \widetilde{f}\ a_v^{(i)} = \frac{A}{B}$$
$$\text{where } B = 1 - 2\int_0^\infty dz\ \widetilde{Q}\ (f - \widetilde{f}^2) \int_z^\infty dx \frac{e^{-3x}}{(1-f)(f-\widetilde{f}^2)^2} \int_0^x dy\ e^y\ \widetilde{f}\ (f - \widetilde{f}^2)$$
$$\text{And } A = -\int_0^\infty dz\ \widetilde{Q}\ (f - \widetilde{f}^2) \int_z^\infty dx \frac{e^{-3x}}{(1-f)(f-\widetilde{f}^2)^2} \int_0^x dy\ M^{(i)}(y) \tag{4.54}$$

and

$$a_s^{(i)}(y) = -e^{-y}\left(\frac{1}{N}\right)\int_0^\infty\ d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^\zeta\ s^{(i)}_{\text{gauge}}(\zeta) + \left(\frac{1}{N}\right)\int_y^\infty d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^\zeta\ s^{(i)}_{\text{gauge}}(\zeta) \tag{4.55}$$

and

$$S_1^{(i)}(y) = -4\int_y^\infty d\rho\ \widetilde{f}\ a_s^{(i)}(\rho) - e^{-y}\ A_{scalar} + \left(\frac{2}{N}\right)\int_y^\infty d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^\zeta\ s^{(i)}_{\text{metric}}(\zeta)$$
$$\text{where } A_{scalar} = -4\int_0^\infty d\rho\ \widetilde{f}\ a_s^{(i)}(\rho) + \left(\frac{2}{N}\right)\int_0^\infty d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^\zeta\ s^{(i)}_{\text{metric}}(\zeta) \tag{4.56}$$

and correction to the tensor sector

$$\mathcal{T}(W) = 2\frac{D}{K}\log\left[1 - \widetilde{Q}^2 e^{-W}\right] \tag{4.57}$$

where

$$\begin{gathered}\mathfrak{v}^{(1)}_{\text{metric}} = \frac{\widetilde{f}^2 - f}{2}\ ,\ \mathfrak{v}^{(2)}_{\text{metric}} = \mathfrak{v}^{(4)}_{\text{metric}} = \frac{f}{2}\ ,\ \mathfrak{v}^{(\ 3)}_{\text{metric}} = -\frac{\widetilde{f}^2}{2} \\ \mathfrak{v}^{(1)}_{\text{gauge}} = \widetilde{f}\ ,\ \mathfrak{v}^{(2)}_{\text{gauge}} = -\widetilde{f}\ ,\ \mathfrak{v}^{(\ 4)}_{\text{gauge}} = -\widetilde{f}\end{gathered} \tag{4.58}$$

and

$$
\begin{aligned}
&s^{(1)}_{\text{metric}} = \frac{N}{2}\left(\dot{\mathcal{V}}^{(2)} + \dot{\mathcal{V}}^{(3)} + \dot{\mathcal{V}}^{(4)}\right) - \widetilde{f}^2 \ , \ s^{(2)}_{\text{metric}} = \frac{N}{2}\left(-\dot{\mathcal{V}}^{(2)} + \dot{\mathcal{V}}^{(3)}\right) + \widetilde{f}^2 \\
&s^{(3)}_{\text{metric}} = \frac{f - \widetilde{f}^2}{2} \ , \ s^{(4)}_{\text{metric}} = \frac{N}{2}\dot{\mathcal{V}}^{(1)} \ , \ s^{(5)}_{\text{metric}} = \left(\widetilde{Q}\widetilde{f} - \widetilde{f}^2\right) \ , \ s^{(6)}_{\text{metric}} = 0 \\
&s^{(7)}_{\text{metric}} = -\widetilde{f}^2 \ , \ s^{(8)}_{\text{metric}} = \frac{N}{2}\left(-\dot{\mathcal{V}}^{(2)} + \dot{\mathcal{V}}^{(3)}\right) \\
&s^{(1)}_{\text{gauge}} = N\left(\widetilde{f}\mathcal{V}^{(2)} + f\dot{a}^{(2)}_v + \widetilde{f}\mathcal{V}^{(4)} + f\dot{a}^{(4)}_v\right) \ , \ s^{(2)}_{\text{gauge}} = N\left(-\widetilde{f}\mathcal{V}^{(2)} - f\dot{a}^{(2)}_v + \widetilde{f}\mathcal{V}^{(3)} + f\dot{a}^{(3)}_v\right) \\
&s^{(4)}_{\text{gauge}} = N\left(\widetilde{f}\mathcal{V}^{(1)} + f\dot{a}^{(1)}_v\right) \ , \ s^{(6)}_{\text{gauge}} = \widetilde{f} \ , \ s^{(3)}_{\text{gauge}} = s^{(5)}_{\text{gauge}} = s^{(7)}_{\text{gauge}} = 0 \\
&s^{(8)}_{\text{gauge}} = N\left(-\widetilde{f}\mathcal{V}^{(2)} - f\dot{a}^{(2)}_v + \widetilde{f}\mathcal{V}^{(3)} + f\dot{a}^{(3)}_v\right) + \widetilde{f}
\end{aligned}
\tag{4.59}
$$

Table 4.2: Membrane Data

Scalar	Vector	Tensor
$\mathcal{S}^{(1)} \equiv \frac{(U\cdot\nabla)K}{K}$	$V^{(1)}_A \equiv P^C_A\left(\frac{\nabla_C K}{K}\right)$	$t_{AB} \equiv P^C_A P^{C'}_B\left[\left(\frac{\nabla_C O_{C'} + \nabla_{C'} O_C}{2}\right) - \frac{P_{CC'}}{D}\left(\nabla\cdot O\right)\right]$
$\mathcal{S}^{(2)} \equiv U\cdot K\cdot U$	$V^{(2)}_A \equiv P^C_A\left(U\cdot\nabla\right)O_C$	
$\mathcal{S}^{(3)} \equiv \hat{\nabla}\cdot U$	$V^{(3)}_A \equiv P^C_A\left(U\cdot\nabla\right)U_C$	
$\mathcal{S}^{(4)} \equiv \frac{\hat{\nabla}^2 K}{K^2}$	$V^{(4)}_A \equiv P^C_A\left(\frac{\hat{\nabla}^2 U_C}{K}\right)$	
$\mathcal{S}^{(5)} \equiv U\cdot\left(\frac{\nabla\widetilde{Q}}{\widetilde{Q}}\right)$		
$\mathcal{S}^{(6)} \equiv \frac{1}{K}\nabla\cdot\left(\frac{\nabla\widetilde{Q}}{\widetilde{Q}}\right)$		
$\mathcal{S}^{(7)} \equiv \frac{K}{D}$		
$\mathcal{S}^{(8)} \equiv \frac{R_{uu}}{K}$		

4.3.1 The dual system

The large-D gravity solutions, described in the previous subsection, are dual to a co-dimension one, massive and charged, membrane embedded in the asymptotic geometry (AdS for our purpose). The membrane is characterized by a velocity field

U, named as 'membrane velocity', a charge field $\widetilde{Q}$ and a shape function ψ (the same scalar fields that appear in the bulk metric and gauge fields). The function ψ is a harmonic function with respect to the background geometry. Just like in fluid gravity correspondence, here also the velocity field U and the shape function ψ cannot be chosen arbitrarily. They have to satisfy some constraint equations, which we shall refer to as membrane equations. For every solution to these membrane equations, we have one solution to Einstein's equations. The membrane equations are given by

$$\begin{aligned}
&P^A_C\left[\frac{\widehat{\nabla}^2 U_A}{K} - \left(1+\widetilde{Q}^2\right)\frac{\widehat{\nabla}_A K}{K} + U^B K_{BA} - \left(1+\widetilde{Q}^2\right)\left(U\cdot\widehat{\nabla} U_A\right)\right] = \mathcal{O}\left(\frac{1}{D}\right)\\
&\hat{\nabla}\cdot U = \mathcal{O}\left(\frac{1}{D}\right)\\
&\frac{\widehat{\nabla}^2\widetilde{Q}}{K} - U\cdot\widehat{\nabla}\widetilde{Q} - \widetilde{Q}\left[\frac{U\cdot\widehat{\nabla}K}{K} - U\cdot K\cdot U - \frac{R_{uu}}{K}\right] = \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{4.60}$$

where

$$P_{AB} = \bar{W}_{AB} - n_A n_B + U_A U_B\ ,\ \ R_{uu} = U^A \bar{R}_{AB} U^B\ ,\ \text{and}\ \bar{R}_{AB} = (D-1)\,\lambda \bar{W}_{AB} \tag{4.61}$$

4.4 Comparing fluid-gravity and membrane-gravity dualities :

In this section we compare the two perturbation techniques, 'derivative expansion' and 'large-D expansion' which are used to generate dynamical black-brane solutions to Einstein equations.

'Derivative expansion' is used to solve Einstein's equations with negative cosmological constant, whereas large-D expansion technique is used to solve Einstein equations with or without cosmological constant. 'Derivative expansion' generate gravity solutions that are dual to the relativistic Navier-Stokes' equations of fluid dynamics. On the other hand large-D expansion techniques generate solutions that are dual to a co-dimension one dynamical membrane embedded in some background space. Like in the previous papers [1, 2], here also our goal is to compare these two gravity solutions along with their dual systems for the charged case. We will show that in appropriate regime of parameter space there exists an overlap regime between these two different looking gravity solutions generated by two different perturbation techniques, which we can see after a coordinate transformation.

4.4.1 The split of the hydrodynamic metric

As we have mentioned earlier, the metric generated in large-D expansion technique are written in a split form, background plus 'rest'. Here we have a null geodesic, which when contracted with the 'rest' part vanishes to all order in $\frac{1}{D}$. This is not the case for hydrodynamic metric.
So to compare these two solutions, the first step would be to split the hydrodynamic metric into background plus 'rest'.

We shall do it in the following way.

First we find out an affinely parametrized null geodesic $\bar{O}^A\partial_A$ w.r.t the full space-time metric which passes thorough the event horizon of the space-time. This null geodesic vector is normalized in a way such that $O^A n_A = 1$ (O_A is related to $\bar{O}_A$ by an overall normalization constant), everywhere in the background, where n_A is the unit normal to the constant ψ hypersurfaces. Now the hydrodynamic metric is written in a gauge where $\mathcal{G}_{rr} = 0$, and $\mathcal{G}_{r\mu} = -u_\mu$ to all order in derivative expansion. In this gauge $k^A\partial_A = \zeta(x)\partial_r$ is an affinely parametrized null geodesic to all order in derivative expansion. However we have to normalize this null geodesic and hence we can set $\zeta(x)$ to be one. Ultimately we shall find that $\bar{O}^A\partial_A = \partial_r$ is the null geodesic which split the hydrodynamic metric as we want. After that we shall choose a coordinate system $Y^A \equiv (\rho, y^\mu)$ where the background of hydrodynamic metric take the following form

$$dS^2_{\text{background}} = \bar{G}_{AB}dY^A dY^B = \frac{d\rho^2}{\rho^2} + \rho^2\eta_{\mu\nu} \tag{4.62}$$

The Y^A coordinates are related to $X^A \equiv (r, x^\mu)$ coordinate by the mapping f

$$Y^A = f^A(X) \tag{4.63}$$

We can determine this mapping function by the following equation

$$\bar{O}^A\mathcal{G}_{AB}|_{\{X\}} = \bar{O}^A\left(\frac{\partial f^C}{\partial X^A}\right)\left(\frac{\partial f^{C'}}{\partial X^B}\right)\bar{G}_{CC'}|_{\{X\}} \tag{4.64}$$

If we use the fact that $\bar{O}^A\partial_A = \partial_r$, then (4.64) can be written as

$$\mathcal{G}^{(rest)}_{rB} = 0 \tag{4.65}$$

where $\mathcal{G}^{(rest)}_{AB} = \left(\mathcal{G}_{AB} - \bar{\mathcal{G}}_{AB}\right)$, all written in $\{X^A\}$ coordinates. As previously noted the hydrodynamic metric is written in a particular gauge $\mathcal{G}_{rr} = 0,$ and $\mathcal{G}_{r\mu} = -u_\mu$ to all order in derivative expansion. We shall find that coordinate transformation of the form

$$\rho = r + \chi(x) \quad \text{and} \quad y^\mu = x^\mu + \frac{u^\mu}{r + \chi(x)} + \zeta^\mu(x) \tag{4.66}$$

where $u_\mu\partial_\nu\zeta^\mu = 0$, keep the hydrodynamic metric in this required gauge. Further it will turn out that for the exact matching of the two metric and gauge field we should have $\zeta^\mu = 0$ and $\chi(x) = -\frac{\Theta}{D-2}$. So finally we have[2]

$$\rho = r - \frac{\Theta}{D-2} \quad \text{and} \quad y^\mu = x^\mu + \frac{u^\mu}{r - \frac{\Theta}{D-2}} \tag{4.67}$$

If we apply these coordinate transformations, the background metric in $\{X^A\}$ coordinate can be written as[3]

$$\begin{aligned}
&\bar{\mathcal{G}}_{rr} = 0\\
&\bar{\mathcal{G}}_{\mu r} = -u_\mu\\
&\bar{\mathcal{G}}_{\mu\nu} = r^2\left(\mathcal{P}_{\mu\nu} - u_\mu u_\nu\right) - r\left(u_\mu a_\nu + u_\nu a_\mu\right) + 2r\sigma_{\mu\nu} + 2r\frac{\Theta}{D-2}u_\mu u_\nu + \mathcal{O}\left(\partial^2\right)
\end{aligned} \tag{4.68}$$

Once we know the background metric in $\{X^A\}$ coordinates, by subtracting it from the full metric we can determine $\mathcal{G}^{(rest)}_{AB}$. Now by our construction $\mathcal{G}^{(rest)}_{rr}$ and $\mathcal{G}^{(rest)}_{r\mu}$ are identically zero to all order in derivative expansion and the $\mathcal{G}^{(rest)}_{\mu\nu}$ component

[2]For a detail discussion see [2].

[3]The inverse of the background metric and the christoffel symbols w.r.t background metric are give in Appendix C.2

can be written as

$$
\mathcal{G}^{(rest)}_{\mu\nu} = \mathcal{G}^{(S1)} u_\mu u_\nu + \mathcal{G}^{(S2)} \mathcal{P}_{\mu\nu} + \left(\mathcal{G}^{(V)}_\mu \ u_\nu + \mathcal{G}^{(V)}_\nu \ u_\mu\right) + \mathcal{G}^{(T)}_{\mu\nu}
$$

where,

$$
\begin{aligned}
\mathcal{G}^{(S1)} &= r^2 \Bigg(1 - V(r)\Bigg) \\
\mathcal{G}^{(S2)} &= \mathcal{O}\left(\partial^2\right) \\
\mathcal{G}^{(V)}_\mu &= -\frac{3(D-3)\ r^2\ Q_C}{r_H^2}\Bigg[1-(D-2)Q_C f(Q_C)\Bigg] F_1(\rho, M)\ \mathcal{P}^\lambda_\mu \left(\partial_\lambda Q_C\right) \\
\mathcal{G}^{(T)}_{\mu\nu} &= 2r\left(\frac{r}{r_H} F_2\left(\rho, M\right) - 1\right)\sigma_{\mu\nu}
\end{aligned}
\tag{4.69}
$$

4.4.2 Membrane data in terms of fluid data

In derivative expansion the solutions are characterized by a velocity filed u, called fluid velocity, a temperature field T and a charge field Q, whereas in large-D expansion characterising data are the shape function ψ, the charge $\tilde{Q}$ and the membrane velocity U. The number of data does match on both sides, as it should be. But these variables are not the same and we need to rewrite one in terms of the other, to perform a comparison. In this subsection we rewrite the characterising data of the membrane in terms of the fluid variables.

4.4.2.1 Determining ψ

As we have described before, ψ is a scalar function, harmonic with respect to the background geometry. The hypersurface $\psi = 1$ is identified with the dynamical horizon of the black brane solution. So we have to solve the differential equation $\nabla^2 \psi^{-D} = 0$ in this background geometry order by order in both the perturbation parameters.

After solving we find the following expression for the function ψ (see appendix-B of [1] for the details of the calculation)

$$
\psi\left(r, x^\mu\right) = 1 + \left(1 - \frac{1}{D}\right)\left(\frac{r}{r_H} - 1\right) + \mathcal{O}\left(\partial^2, \frac{1}{D^3}\right)
\tag{4.70}
$$

4.4.2.2 Determining U^A

Once we have the ψ field everywhere, we could compute the unit normal to the constant ψ surfaces.

$$
\begin{aligned}
n_r &= \frac{1}{r} + \mathcal{O}\left(\partial^2\right) & n_\mu &= -\frac{\Theta}{D-2}u_\mu + a_\mu + f(Q_C)\mathcal{P}^\alpha_\mu \partial_\alpha Q_C + \mathcal{O}\left(\partial^2\right) \\
n^r &= r - \frac{\Theta}{D-2} + \mathcal{O}\left(\partial^2\right) & n^\mu &= \frac{u^\mu}{r} + \frac{f(Q_C)}{r^2}\,\mathcal{P}^{\mu\alpha}\,\partial_\alpha Q_C + \mathcal{O}\left(\partial^2\right)
\end{aligned}
\tag{4.71}
$$

Now U^A is defined as follows

$$U_A = n_A - O_A \tag{4.72}$$

After properly normalizing our null geodesic field by $O^A = \frac{\bar{O}^A}{\left(n_A \cdot \bar{O}^A\right)}$, we have

$$
\begin{aligned}
O_r &= 0 & O_\mu &= -r u_\mu + \mathcal{O}\left(\partial^2\right) \\
O^r &= r + \mathcal{O}\left(\partial^2\right) & O^\mu &= 0
\end{aligned}
\tag{4.73}
$$

Then we can find out the membrane velocity U_A as

$$
\begin{aligned}
U_r &= \frac{1}{r} & U_\mu &= r u_\mu - \frac{\Theta}{D-2}u_\mu + a_\mu + f(Q_C)\mathcal{P}^\alpha_\mu \partial_\alpha Q_C + \mathcal{O}\left(\partial^2\right) \\
U^r &= -\frac{\Theta}{D-2} + \mathcal{O}\left(\partial^2\right) & U^\mu &= \frac{u^\mu}{r} + \frac{f(Q_C)}{r^2}\,\mathcal{P}^{\mu\alpha}\,\partial_\alpha Q_C + \mathcal{O}\left(\partial^2\right)
\end{aligned}
\tag{4.74}
$$

4.4.2.3 Determining $\widetilde{Q}$

Next our goal is to write the smooth function $\widetilde{Q}$ present in the large D metric and gauge field in terms of fluid data. This function satisfies the subsidiary condition $(n \cdot \nabla)\,\widetilde{Q} = 0$. The boundary condition which fix it completely is given by

$$
\begin{aligned}
\widetilde{Q}|_{\psi=1} &= \frac{1}{\sqrt{2}}\left(U^M A_M\right)|_{\psi=1} \\
&= -\frac{1}{\sqrt{2}}\,\frac{\sqrt{3}Q_C}{2}\left(\frac{r_H}{r}\right)^{D-2}|_{\psi=1} \\
&= -\frac{\sqrt{3}Q_C}{2\sqrt{2}}
\end{aligned}
\tag{4.75}
$$

We want to solve $\widetilde{Q}$ such that $(n \cdot \partial)\,\widetilde{Q} = 0$ and for that we will take the following expansion in $\widetilde{Q}$.

$$\begin{split}\widetilde{Q} &= \widetilde{Q}_0 + \widetilde{Q}_1\ \ (r - r_H) + \cdots \\ \text{where} \quad \widetilde{Q}_0 &= -\frac{\sqrt{3} Q_C}{2\sqrt{2}}\end{split} \tag{4.76}$$

Collecting coefficients of $(r - r_H)^0$ after applying $(n \cdot \partial)\,\widetilde{Q} = 0$ and using the fact that $(n \cdot \partial)\,\widetilde{Q}_0 = -\frac{\sqrt{3}Q_C}{2\sqrt{2}\ r}\left(\frac{(u\cdot\partial)Q_C}{Q_C}\right) = 0$, we have $\widetilde{Q}_1 = 0$. So finally we have

$$\widetilde{Q} = -\frac{\sqrt{3}\ Q_C}{2\sqrt{2}} + \mathcal{O}\left(\partial^2\right) \tag{4.77}$$

4.4.2.4 Relevant derivatives of the basic data

Large-D metric is determined in terms of the basic functions ψ, $\widetilde{Q}$, U^A and their derivatives with respect to the induced coordinates on the membrane. In this subsection we shall convert these 'membrane derivatives' of the basic 'membrane data' in terms of the fluid data.

One of the key structure that arises repeatedly in large-D construction is the extrinsic curvature of the membrane, viewed as a hypersurface embedded in the background. The expressions for extrinsic curvature can be re-expressed in terms of fluid variables as

$$\begin{split}K_{rr} &= -\frac{1}{r^2} + \mathcal{O}\left(\partial^2\right) \\ K_{r\mu} &= -u_\mu + \frac{1}{r}\left(\frac{\Theta}{D-2}u_\mu - a_\mu\right) + \mathcal{O}\left(\partial^2\right) \\ K_{\mu\nu} &= r^2\left(\mathcal{P}_{\mu\nu} - u_\mu u_\nu\right) + 2\ r\left(\frac{\Theta}{D-2}u_\mu u_\nu - \frac{u_\mu a_\nu + u_\nu a_\mu}{2} + \sigma_{\mu\nu}\right) + \mathcal{O}\left(\partial^2\right) \\ K &= (D-1) + \mathcal{O}\left(\partial^2\right)\end{split} \tag{4.78}$$

where K_{AB} is defined as

$$\begin{split}K_{AB} &= \Pi_A^C\ \nabla_C\ n_B \\ \text{with} \quad \Pi_{AB} &= \bar{\mathcal{G}}_{AB} - n_A\ n_B\end{split} \tag{4.79}$$

The rest of the data that are relevant for our purpose are presented in the tables

-(4.3), (4.4) and (4.5). Here $\hat{\nabla}$ is defined for a general n index tensor $X_{A_1A_2\cdots A_n}$ as

$$\hat{\nabla}_A X_{A_1A_2\cdots A_n} = \Pi^C_A \Pi^{C_1}_{A_1}\Pi^{C_2}_{A_2}\cdots\Pi^{C_n}_{A_n}\nabla_C X_{C_1C_2\cdots C_n} \tag{4.80}$$

Table 4.3: Scalar large-D Data in terms of fluid Data

Large-D Data	Corresponding Fluid Data
$\mathcal{S}^{(1)} \equiv \frac{(U\cdot\nabla)K}{K}$	$=\ 0$
$\mathcal{S}^{(2)} \equiv U\cdot K\cdot U$	$=\ -1$
$\mathcal{S}^{(3)} \equiv \hat{\nabla}\cdot U = \Pi^{AB}(\nabla_A U_B)$	$=\ 0$
$\mathcal{S}^{(4)} \equiv \frac{\hat{\nabla}^2 K}{K^2}$	$=\ 0$
$\mathcal{S}^{(5)} \equiv U\cdot\left(\frac{\nabla\widetilde{Q}}{\widetilde{Q}}\right)$	$=\ 0$
$\mathcal{S}^{(6)} \equiv \frac{1}{K}\nabla\cdot\left(\frac{\nabla\widetilde{Q}}{\widetilde{Q}}\right)$	$=\ 0$
$\mathcal{S}^{(7)} \equiv \frac{K}{D}$	$=\ 1-\frac{1}{D}$
$\mathcal{S}^{(8)} \equiv \frac{R_{uu}}{K}$	$=\ -\lambda$

Table 4.4: Vector large-D Data in terms of fluid Data

Large-D Data	Corresponding Fluid Data
$V^{(1)}_A\ dX^A \equiv P^C_A\left(\frac{\nabla_C K}{K}\right)$	$=\ 0$
$V^{(2)}_A\ dX^A \equiv P^C_A\,(U\cdot\nabla)\,O_C$	$=\ f\,(Q_C)\,\mathcal{P}^\lambda_\mu\partial_\lambda Q_C\ dx^\mu$
$V^{(3)}_A\ dX^A \equiv P^C_A\,(U\cdot\nabla)\,U_C$	$=\ -f\,(Q_C)\,\mathcal{P}^\lambda_\mu\partial_\lambda Q_C\ dx^\mu$
$V^{(4)}_A\ dX^A \equiv P^C_A\left(\frac{\hat{\nabla}^2 U_C}{K}\right)$	$=\ -f\,(Q_C)\,\mathcal{P}^\lambda_\mu\partial_\lambda Q_C\ dx^\mu$

Table 4.5: Tensor large-D Data in terms of fluid Data

Large-D Data		Corresponding Fluid Data
$t_{AB} \equiv P_A^C P_B^{C'} \left[\left(\frac{\nabla_C O_{C'} + \nabla_{C'} O_C}{2}\right) - \frac{P_{CC'}}{D}\left(\nabla \cdot O\right)\right]$	$=$	$-r\sigma_{\mu\nu} + \frac{1}{D}\Big(r^2 \mathcal{P}_{\mu\nu} + 2r\sigma_{\mu\nu} + r f\left(Q_C\right)\left(u_\mu \mathcal{P}_\nu^\lambda \partial_\lambda Q_C + u_\nu \mathcal{P}_\mu^\lambda \partial_\lambda Q_C\right)\Big)$

4.4.3 Comparing the metrics and gauge fields

In this subsection we shall take the large D limit of the fluid metric and gauge field and match with the metric and gauge field in large D side after expressing them in terms of fluid data.

4.4.3.1 Comparing the gauge fields

At first we decompose the fluid gauge field into scalar and vector components. As from the gauge condition A_r component of the gauge field is zero to all order in derivative, we write only the components in the boundary directions.

$$A_\mu^{(\text{fluid})} = \mathcal{B}^{(S)}\ u_\mu + \mathcal{B}_\mu^{(V)} \tag{4.81}$$

where

$$\begin{aligned} \mathcal{B}^{(S)} &= \frac{\sqrt{3}\ r\ Q_C}{2}\left(\frac{r_H}{r}\right)^{D-2} \\ \mathcal{B}_\mu^{(V)} &= -2\sqrt{3}\left(\frac{r}{r_H}\right)^D \Big[1 - (D-2)Q_C\ f(Q_C)\Big] F_1^{(1,0)}(\rho, M)\ \mathcal{P}_\mu^\lambda\left(\partial_\lambda Q_C\right) \end{aligned} \tag{4.82}$$

The gauge field in large-D side is given as

$$\begin{aligned} A_M &= \sqrt{2}\left[\widetilde{f}\ O_M + \frac{1}{D}A_M^{(1)} + \mathcal{O}\left(\frac{1}{D}\right)^2\right] \\ &\text{where} \quad \mathcal{A}_M^{(1)} = \mathcal{A}^{(s)} O_M + \mathcal{A}_M^{(v)} \end{aligned} \tag{4.83}$$

Also in this case the radial component of the gauge field vanishes and only the components in boundary directions are non-zero. We decompose this into scalar and vector components as follows

$$A^{(\mathrm{D})}_\mu = \mathcal{Y}^{(S)}\ u_\mu + \mathcal{Y}^{(V)}_\mu \tag{4.84}$$

where

$$\begin{aligned}
\mathcal{Y}^{(S)} &= -\sqrt{2}\ r\left(\widetilde{f} + \frac{1}{D}\mathcal{A}^{(s)}\right) + \mathcal{O}\left(\frac{1}{D}\right)^2 \\
\mathcal{Y}^{(V)}_\mu &= \frac{1}{D}\sqrt{2}\ \mathcal{A}^{(v)}_\mu + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned} \tag{4.85}$$

$\mathcal{A}^{(s)}$ is defined as follows

$$\begin{aligned}
\mathcal{A}^{(s)} &= \sum_{i=1}^{N_s} a_s^{(i)}(Y)\ \mathcal{S}^{(i)} \\
&= \left(a_s^{(2)}(Y)\ \mathcal{S}^{(2)} + a_s^{(7)}(Y)\ \mathcal{S}^{(7)} + K\ a_s^{(R_{uu})}(Y)\ \mathcal{S}^{(8)}\right) \\
&= \left(-a_s^{(2)}(Y) + a_s^{(7)}(Y) + K\ a_s^{(R_{uu})}(Y)\right) \\
&= a_s^{(\text{total})}(Y)
\end{aligned} \tag{4.86}$$

where $Y = D(\psi - 1)$.[4]
where

$$a_s^{(\text{total})}(Y) = -e^{-Y}\left(\frac{1}{N}\right)\int_0^\infty d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^{\zeta}\ s^{(\text{total})}_{\text{gauge}}(\zeta) + \left(\frac{1}{N}\right)\int_Y^\infty d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^{\zeta}\ s^{(\text{total})}_{\text{gauge}}(\zeta) \tag{4.87}$$

And $s^{(\text{total})}_{\text{gauge}}$ is defined as

$$\begin{aligned}
s^{(\text{total})}_{\text{gauge}} &= -s^{(2)}_{\text{gauge}} + s^{(7)}_{\text{gauge}} + s^{(R_{uu})}_{\text{gauge}} \\
&= \widetilde{f} \\
&= \widetilde{Q}\ e^{-Y} + \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{4.88}$$

[4]Here one should note that the integrations in the large-D side are parametrized by $W = D(\psi - 1)$. On the other hand we can define another parameter $\widetilde{R} = D\left(\frac{r}{r_H} - 1\right)$ to expand other functions in inverse power of dimensions. But it is easy to check that $W = \widetilde{R} + \mathcal{O}\left(\frac{1}{D}\right)$. And hence up to the order we are interested the two parameters are just equal and we simply denote both of them by the $\mathcal{O}(1)$ parameter Y without any further confusion.

Substituting (4.88) into (4.87) we have

$$a_s^{(\text{total})}(Y) = Y\ \widetilde{Q}\ e^{-Y} \tag{4.89}$$

Putting this in (4.86) we have

$$\mathcal{A}^{(s)} = Y\ \widetilde{Q}\ e^{-Y} \tag{4.90}$$

The vector component calculated up to leading order in large-D as

$$\begin{aligned}\mathcal{A}^{(v)}_\mu &= \sum_{i=1}^{N_s} a_v^{(i)}(Y)\ V_\mu^{(i)} \\ &= f\left(Q_C\right)\left(a_v^{(2)}(Y) - a_v^{(3)}(Y) - a_v^{(4)}(Y)\right)\mathcal{P}_\mu^\lambda\left(\partial_\lambda Q_C\right) \\ &= \mathcal{O}\left(\frac{1}{D}\right)\end{aligned} \tag{4.91}$$

So finally we have

$$\begin{aligned}\mathcal{Y}^{(S)} &= -\sqrt{2}\ r\left(\widetilde{f} + \frac{1}{D}Y\ \widetilde{Q}\ e^{-Y}\right) + \mathcal{O}\left(\frac{1}{D}\right)^2 \\ \mathcal{Y}^{(V)}_\mu &= \mathcal{O}\left(\frac{1}{D}\right)^2\end{aligned} \tag{4.92}$$

Now taking the derivative of (4.21) and then expanding in $\frac{1}{D}$ and also using the result of C.1.1 we have

$$F_1^{(1,0)}\left(1 + \frac{Y}{D}, M\right) = \left(\frac{1}{D}\right)^2 \tag{4.93}$$

Now if we expand both large-D and fluid gauge field up to $\mathcal{O}\left(\frac{1}{D}\right)$ by substituting $r = r_H\left(1 + \frac{Y}{D}\right)$, we find

$$\begin{aligned}\mathcal{Y}^{(S)} - \mathcal{B}^{(S)} &= \mathcal{O}\left(\frac{1}{D}\right)^2 \\ \mathcal{Y}^{(V)}_\mu - \mathcal{B}^{(V)}_\mu &= \mathcal{O}\left(\frac{1}{D}\right)^2\end{aligned} \tag{4.94}$$

So within the membrane region, the two gauge fields are equivalent to one another.

4.4.3.2 Comparing the metric

As we have discussed earlier we can determine $\mathcal{G}^{(rest)}_{\mu\nu}$, by simply subtracting the background piece from the full hydrodynamic metric once we know the background metric in X^A coordinates. After decomposing $\mathcal{G}^{(rest)}_{\mu\nu}$ into scalar, vector and tensor components, we can write it as follows

$$\mathcal{G}^{(rest)}_{\mu\nu} = \mathcal{G}^{(S1)} u_\mu u_\nu + \mathcal{G}^{(S2)} \mathcal{P}_{\mu\nu} + \left(\mathcal{G}^{(V)}_\mu \ u_\nu + \mathcal{G}^{(V)}_\nu \ u_\mu\right) + \mathcal{G}^{(T)}_{\mu\nu} \tag{4.95}$$

where

$$\begin{aligned}
\mathcal{G}^{(S1)} &= r^2 \Big(1 - V(r)\Big) \\
\mathcal{G}^{(S2)} &= \mathcal{O}\left(\partial^2\right) \\
\mathcal{G}^{(V)}_\mu &= -\frac{3(D-3)\ r^2\ Q_C}{r_H^2}\Big[1-(D-2)Q_C f(Q_C)\Big] F_1(\rho, M)\ \mathcal{P}^\lambda_\mu \left(\partial_\lambda Q_C\right) \\
\mathcal{G}^{(T)}_{\mu\nu} &= 2r\left(\frac{r}{r_H} F_2\left(\rho, M\right) - 1\right)\sigma_{\mu\nu}
\end{aligned} \tag{4.96}$$

Now we will write the large D metric in terms of fluid data. The large D metric up to first order in $\frac{1}{D}$ can be written as

$$\begin{aligned}
W_{AB} &= \bar{W}_{AB} + f O_A O_B + \frac{1}{D}\mathcal{W}^{(1)}_{AB} + \mathcal{O}\left(\frac{1}{D}\right)^2 \\
\text{where} \quad \mathcal{W}^{(1)}_{AB} &= \mathcal{Z}^{(s1)} O_A O_B + \Big(\mathcal{Z}^{(v)}_A O_B + \mathcal{Z}^{(v)}_B O_A\Big) + \mathcal{Z}^{(T)}_{AB}
\end{aligned} \tag{4.97}$$

Subtracting the background part from this full large-D metric, we will get, $W^{(rest)}_{AB}$. Now from the construction of this metric $W^{(rest)}_{rr}$ and $W^{(rest)}_{r\mu}$ vanishes, and we only have $W^{(rest)}_{\mu\nu}$, which we decompose into scalar, vector and tensor components as follows

$$W^{(rest)}_{\mu\nu} = \mathcal{W}^{(S1)} u_\mu u_\nu + \mathcal{W}^{(S2)} \mathcal{P}_{\mu\nu} + \left(\mathcal{W}^{(V)}_\mu \ u_\nu + \mathcal{W}^{(V)}_\nu \ u_\mu\right) + \mathcal{W}^{(T)}_{\mu\nu} \tag{4.98}$$

where

$$\begin{aligned}
\mathcal{W}^{(S1)} &= r^2\left(f+\frac{1}{D}\mathcal{Z}^{(s1)}\right)+\mathcal{O}\left(\frac{1}{D}\right)^2\\
\mathcal{W}^{(S2)} &= \mathcal{O}\left(\frac{1}{D}\right)^2\\
\mathcal{W}^{(V)}_{\mu} &= -\frac{1}{D}\ r\ \mathcal{Z}^{(v)}_{\mu}+\mathcal{O}\left(\frac{1}{D}\right)^2\\
\mathcal{W}^{(T)}_{\mu\nu} &= \frac{1}{D}\mathcal{Z}^{(T)}_{\mu\nu}+\mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned}\tag{4.99}$$

So at first we determine $\mathcal{Z}^{(s1)}$ in terms of fluid variables. We write the leading order expressions in large D expansion by substituting $r = r_H\left(1+\frac{Y}{D}\right)$.

$$\begin{aligned}
\mathcal{Z}^{(s1)} &= \sum_{i=1}^{N_s} S_1^{(i)}(Y)\ \mathcal{S}^{(i)}\\
&= \left(S_1^{(2)}(Y)\ \mathcal{S}^{(2)}+S_1^{(7)}(Y)\ \mathcal{S}^{(7)}+K\ S_1^{(R_{uu})}(Y)\ \mathcal{S}^{(8)}\right)\\
&= \left(-S_1^{(2)}(Y)+S_1^{(7)}(Y)+K\ S_1^{(8)}(Y)\right)\\
&= S_{1s}^{(\text{total})}(Y)
\end{aligned}\tag{4.100}$$

where

$$\begin{aligned}
&S_{1s}^{(\text{total})}(y) = -4\int_y^\infty d\rho\ \widetilde{f}\ a_s^{(\text{total})}(\rho) - e^{-y}\ A^{(\text{total})}_{\text{scalar}} + \left(\frac{2}{N}\right)\int_y^\infty d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^{\zeta}\ s^{(\text{total})}_{\text{metric}}(\zeta)\\
&\text{where } A^{(\text{total})}_{\text{scalar}} = -4\int_0^\infty d\rho\ \widetilde{f}\ a_s^{(\text{total})}(\rho) + \left(\frac{2}{N}\right)\int_0^\infty d\rho\ e^{-\rho}\int_0^\rho d\zeta\ e^{\zeta}\ s^{(\text{total})}_{\text{metric}}(\zeta)
\end{aligned}\tag{4.101}$$

and $s^{(\text{total})}_{\text{metric}}$ is defined as

$$\begin{aligned}
s^{(\text{total})}_{\text{metric}} &= -s^{(2)}_{\text{metric}}+s^{(7)}_{\text{metric}}+s^{(R_{uu})}_{\text{metric}}\\
&= -2\widetilde{f}^2\\
&= -2\widetilde{Q}^2\ e^{-2\ Y}+\mathcal{O}\left(\frac{1}{D}\right)
\end{aligned}\tag{4.102}$$

Substituting (4.102) into (4.101), we have

$$\begin{aligned} A^{\text{(total)}}_{\text{scalar}} &= -3\widetilde{Q}^2 \\ S^{\text{(total)}}_{1s}(Y) &= \widetilde{Q}^2 e^{-2Y}\left(1 - 2Y - e^Y\right) \end{aligned} \tag{4.103}$$

And hence we have

$$\mathcal{Z}^{(s1)} = \widetilde{Q}^2 e^{-2Y}\left(1 - 2Y - e^Y\right) + \mathcal{O}\left(\frac{1}{D}\right) \tag{4.104}$$

Now we will calculate $\mathcal{Z}^{(V)}_\mu$ component

$$\begin{aligned} \mathcal{Z}^{(v)}_\mu &= f\left(Q_C\right)\left(\mathcal{V}^{(2)} - \mathcal{V}^{(3)} - \mathcal{V}^{(4)}\right)\mathcal{P}^\lambda_\mu\left(\partial_\lambda Q_C\right) \\ &= \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{4.105}$$

And the tensor component $\mathcal{Z}^{(T)}_{\mu\nu}$ is given by

$$\mathcal{Z}^{(T)}_{\mu\nu} = -2r\left(\frac{D}{K}\right)\log\left(1 - \widetilde{Q}^2 e^{-Y}\right)\sigma_{\mu\nu} + \mathcal{O}\left(\frac{1}{D}\right) \tag{4.106}$$

So finally we have

$$\begin{aligned} \mathcal{W}^{(S1)} &= r^2\left[f + \frac{1}{D}\left(\widetilde{Q}^2 e^{-2Y}\left(1 - 2Y - e^Y\right)\right)\right] + \mathcal{O}\left(\frac{1}{D}\right)^2 \\ \mathcal{W}^{(S2)} &= \mathcal{O}\left(\frac{1}{D}\right)^2 \\ \mathcal{W}^{(V)}_\mu &= \mathcal{O}\left(\frac{1}{D}\right)^2 \\ \mathcal{W}^{(T)}_{\mu\nu} &= -r\left(\frac{2}{K}\right)\log\left(1 - \widetilde{Q}^2 e^{-Y}\right)\sigma_{\mu\nu} + \mathcal{O}\left(\frac{1}{D}\right)^2 \end{aligned} \tag{4.107}$$

From C.1.1, we have

$$F_1\left(1 + \frac{Y}{D}, M\right) = \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{4.108}$$

hence

$$\mathcal{G}^{(V)}_\mu = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{4.109}$$

and from C.1.2, we have

$$\mathcal{G}^{(T)}_{\mu\nu} = -r\left(\frac{2}{D}\right)\log\left(1-\widetilde{Q}^2 e^{-Y}\right)\sigma_{\mu\nu} + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{4.110}$$

So finally we have

$$\begin{aligned}
\mathcal{G}^{(S1)} &= r^2\Big(1-V(r)\Big)\\
\mathcal{G}^{(S2)} &= \mathcal{O}\left(\partial^2\right)\\
\mathcal{G}^{(V)}_{\mu} &= \mathcal{O}\left(\frac{1}{D}\right)^2\\
\mathcal{G}^{(T)}_{\mu\nu} &= -r\left(\frac{2}{D}\right)\log\left(1-\widetilde{Q}^2 e^{-Y}\right)\sigma_{\mu\nu} + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned} \tag{4.111}$$

So, now if we subtract the fluid metric from the large-D metric both expanded up to $\mathcal{O}\left(\frac{1}{D}\right)^2$, we shall find that in the membrane region the two metric matches.

4.4.4 Comparing the evolution of two sets of data

In the previous subsection we have seen that the metric and gauge fields of both the perturbation techniques are equivalent in their overlap regime. However, our intention is to show that the solutions generated by these two perturbation techniques are equivalent. Hence we also have to show the equivalence of the differential equations that govern the time evolution of the defining data of both the systems. But the defining data of hydrodynamic metric and large-D metric are constrained by two different looking sets of differential equations. For hydrodynamic metric these equations are given by the equations in (4.36), on the other hand the constraint equations for the large-D case are given by equations in (4.60). To show the equivalence of these two gravity solutions we have to show that once 'membrane equations' are satisfied, the constraint equations in hydrodynamics are also satisfied. One of these equations is given by[5]

$$P^A_C\left[\frac{\widehat{\nabla}^2 U_A}{K} - \left(1+\widetilde{Q}^2\right)\frac{\widehat{\nabla}_A K}{K} + U^B K_{BA} - \left(1+\widetilde{Q}^2\right)\left(U\cdot\widehat{\nabla}U_A\right)\right] = \mathcal{O}\left(\frac{1}{D}\right) \tag{4.112}$$

We have calculated the different components of these equations in terms of fluid data as follows

$$\begin{aligned} P^A_C\frac{\widehat{\nabla}^2 U_A}{K} &= -f\left(Q_C\right)\mathcal{P}^\lambda_\mu\partial_\lambda Q_C \\ P^A_C\frac{\widehat{\nabla}_A K}{K} &= 0 \\ P^A_C U^B K_{BA} &= \mathcal{O}\left(\partial^2\right) \\ P^A_C\left(U\cdot\widehat{\nabla}U_A\right) &= -f\left(Q_C\right)\mathcal{P}^\lambda_\mu\partial_\lambda Q_C \end{aligned} \tag{4.113}$$

Substituting these in the L.H.S of equation (4.112) evaluate to

$$\begin{aligned} &= -f\left(Q_C\right)\mathcal{P}^\lambda_\mu\partial_\lambda Q_C - \left(1+\widetilde{Q}^2\right)\left(-f\left(Q_C\right)\mathcal{P}^\lambda_\mu\partial_\lambda Q_C\right) \\ &= \widetilde{Q}^2\ f\left(Q_C\right)\mathcal{P}^\lambda_\mu\partial_\lambda Q_C \\ &= \frac{3}{8}Q_C^2\ f\left(Q_C\right)\mathcal{P}^\lambda_\mu\partial_\lambda Q_C \\ &= \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{4.114}$$

As $f\left(Q_C\right) = \mathcal{O}\left(\frac{1}{D}\right)$, so our membrane velocity and shape function satisfies (4.112).

The second constraint equation is

$$\hat{\nabla}\cdot U = \mathcal{O}\left(\frac{1}{D}\right) \tag{4.115}$$

now $\hat{\nabla}\cdot U = \mathcal{O}\left(\partial^2\right)$. So up to the order we are interested our membrane velocity satisfies this equation.

[5] It is well known that fluid equations can be written as conservation equations of a stress tensor and charge current living on $(D-1)$ dimensional flat space. Also from paper [37, 38] we know that we can define a stress tensor and charge current on the membrane order by order in inverse power of dimensions such that the membrane equations is simply the conservation equations of this stress tensor and charge current. So it would be interesting and easier to show that the conservation of membrane stress tensor and charge current follows from the conservation equations of fluid stress tensor and charge current. But unfortunately at this time we do not have the expressions for the stress tensor and charge current for a charged membrane propagating in AdS space. Hence we are forced to check the equivalence of the two different looking sets of constraint equations, namely membrane equations and fluid equations.

And the third equation is given by

$$\frac{\widehat{\nabla}^2\widetilde{Q}}{K} - U \cdot \widehat{\nabla}\widetilde{Q} - \widetilde{Q}\left[\frac{U \cdot \widehat{\nabla}K}{K} - U \cdot K \cdot U - \frac{R_{uu}}{K}\right] = \mathcal{O}\left(\frac{1}{D}\right) \tag{4.116}$$

The large-D structures appeared in this equation can be calculated in terms of fluid data as

$$\begin{aligned}
&\frac{\widehat{\nabla}^2\widetilde{Q}}{K} = 0 \\
&U \cdot \widehat{\nabla}\widetilde{Q} = 0 \\
&\frac{U \cdot \widehat{\nabla}K}{K} = 0 \\
&U \cdot K \cdot U = -1 \\
&\frac{R_{uu}}{K} = -\lambda
\end{aligned} \tag{4.117}$$

Substituting these the L.H.S of (4.116) we can check that (4.116) is also satisfied. So up to the order we are interested our membrane velocity satisfies this equation. So we have shown that our membrane equations follow as a consequence of fluid constraint equations.

4.5 Conclusions:

In this chapter, we have compared two different perturbation techniques - namely derivative expansion and expansion in inverse powers of dimension, in the regime where both techniques are applicable.
We have considered the case, when these techniques are used to generate asymptotically AdS, dynamical black hole type solutions of Einstein-Maxwell systems. We have shown that in the appropriate regime of the parameter space, the two solutions are equivalent to one another up to the first non-trivial order in both the perturbation parameters. It turns out that after a series of gauge transformations and field redefinitions, the metrics and the gauge fields generated by these two different techniques are exactly same up to the order the solutions are known on both sides. This work could be extended to many directions. Below we are listing few of them.

We have discussed in the introduction the cause of the equivalence. But it is also important to chart out the subtle details of the redefinitions and transformation it involves. If we know how and when the two perturbation techniques generate the same solution, it will help us to find out when they are really different where one is generating new set of dynamical black hole solutions that could not be generated from the other. So once we identified and studied the overlapping regime of the parameter space, it would be interesting to look at the non-overlapping regimes.

Because of the very generic nature of the physical intuition that asserts this equivalence, we believe that it exists not only for pure Einstein systems or Einstein-Maxwell systems but also for Einstein-dilaton systems, higher-derivative gravity theory [46, 55, 72, 73] or any other systems where we can apply both the perturbation techniques. However, our explicit calculations are very much system-specific, which somehow obscures this genericity. It would be interesting to set these calculations more physically or abstractly, without using too much details of a given theory.

In some sense, this chapter is also describing a duality between the dynamics of a $(D-1)$ dimensional charged and massive membrane, embedded in D dimensional AdS space and that of charged fluid, living on the boundary of the AdS space. The basic variables of charged fluids are temperature (T), velocity (u^μ) and charge density (Q); whereas any charged dynamical membrane would be characterized by the embedding function (ψ), the charge density field $(\tilde{Q})$ and the velocity field (U^A)[6]. The statement of the duality could be as follows.

If it is possible to have an all order completion for both the membrane equations and the relativistic fluid equations, then they are actually the same equations, just written in terms of two different sets of variables.

In this chapter, we have worked out this variable redefinition (see equations (4.70),(4.73) and (4.74)), needed to show the equivalence, up to the order the equations are known on both sides.

As it is clear from all previous discussions, the key reason for this equivalence is the fact that both the systems are dual to the same gravity solution in the overlap regime of the two perturbation parameters. Still it might be possible to formulate the duality, removing the gravity altogether, because once we know the appropriate variable redefinition, that is enough to show the equivalence of the membrane and fluid equations. However, we should emphasize that at this stage it is a duality between a very specific membrane and a very specific fluid, the ones that could have gravity duals. It would be really interesting to see if we can extend such duality to more generic fluids and membranes. It will provide new ways to analyze the

[6]Just like the velocity field in fluids, the membrane velocity U^A captures the charge or mass redistribution within the membrane.

unsolved problems in both sets of equations.

Chapter 5

Conclusion and future directions

In this thesis, we have compared two different gravity solutions to Einstein's equations in presence of negative cosmological constant produced by two different perturbation techniques, namely 'derivative expansion' and 'large D expansion' [1, 2, 3].

The gravity solution generated in 'derivative expansion' in D space-time dimension is dual to the dynamics of the relativistic conformal fluid living on the $(D-1)$ dimensional flat space-time. The different dynamical black-brane geometry generated in 'derivative expansion' is characterized by a unit normalized four velocity $u^\mu(x)$, temperature scale set by the horizon radius $r_H(x)$ (the local fluid temperature is related to $r_H(x)$ by $T(x) = \frac{(D-1)}{4\pi} r_H(x)$) and charge density (for Einstein-Maxwell system) living on $(D-1)$ dimensional flat space-time. On the other hand, the different gravity solutions generated in 'large D expansion' are dual to the dynamics of a co-dimension one probe membrane embedded in the asymptotic geometry (in our case pure AdS) coupled with a velocity field and charge density (for Einstein-Maxwell system). In this case, the different 'black hole type' solutions are labelled by the shape of the membrane, a unit normalized velocity field and charge density (for Einstein-Maxwell system) all of them live on a $(D-1)$ dimensional hypersurface. In terms of counting the number of variables the solutions generated in these two different techniques are very similar though the governing data and the space of defining data are very different.

We have found that one could apply both the perturbation techniques simultaneously in a regime of parameter space and argued that there should be an overlap regime between these two techniques and in that regime, the two solutions generated by the two different techniques should match. In large number of space-time dimensions whenever derivative expansion is applicable we could always apply 'large D

expansion' though the reverse may not be true. In a nutshell, we have re-expressed the metric generated in 'derivative expansion' in the form of the metric generated in the large D expansion up to the second subleading order for the pure gravity system and up to the first subleading order for the Einstein-Maxwell system in both the perturbation parameter. But the matching of the two gravity solutions is not at all manifest and we could match the two gravity solutions generated in these two different perturbation techniques after a series of coordinate transformations and field redefinitions.

On physical ground, the reason behind the matching of the two gravity solutions is as follows. Since we can use the same space-time geometry (namely the space-time geometry of Schwarzschild black brane) as the starting point for both the perturbation procedure and given the starting point and the labelling data both the perturbation schemes generate the higher order corrections uniquely, it follows that the solutions should be the same.

Though the matching of the two gravity solutions is expected on physical ground, the detailed comparison between these two different gravity solutions is important for the following many reasons. First of all, it would help to explicitly work out the map between dynamical black-brane geometry generated in 'derivative expansion' technique and the pure AdS space (which exists even when the space-time dimension D is finite). In that process, we could find out the field redefinition between the fluid and membrane degrees of freedom and it implies some interesting re-summation of one series into another. For example, even the leading term in the Navier-Stokes equations of hydrodynamics contains arbitrarily higher order corrections in $1/D$ of the membrane equations. It essentially says that they are essentially the same equations but written in different variables. It can also give us some insight into the finite D completion of the membrane stress tensor [36].

The dynamical black-brane geometry generated in 'large D expansion' is written in a particular 'split' form, where the full space-time metric is written as the background AdS metric and something else to all orders in $1/D$. So, the space-time generated in this technique always admits a particular map from the full space-time space to the background AdS space. But the metric generated in 'derivative expansion' to begin with does not have such a 'split' form. In this work, we have described that this particular map also exists for the space-time generated in 'derivative expansion'. And the map also exists even at finite number of dimensions irrespective of large D limit. After determining the 'split' of the hydrodynamic metric we could show the equivalence of the two gravity solutions in the appropriate regime of parameter space.

Let us now mention one important point for the comparison in the Einstein-Maxwell system. Naively from the large D effective theory it seems that as the gravitational field is localized only within the near horizon region, it does not care about the asymptotic geometry. But for the Einstein-Maxwell system even at the very leading order in membrane equations the curvature of background AdS appears. For that reason, the comparison, in this case, is more rigorous.

In a sense, we could view the equivalence of the two gravity solutions generated in these two perturbation techniques as describing a duality between the dynamics of a $(D-1)$ dimensional membrane propagating in pure AdS space and that of a fluid, which lives on the boundary of the AdS space-time. We could get a field redefinition between fluid and membrane variables (which we believe will exist to all orders). We hope that such a writing could lead to a new duality between fluid and membrane in large dimensions (more ambitiously in finite dimensions also) where gravity has no role to play. In this work, we have explicitly shown that up to subleading order in pure gravity and up to leading order in the Einstein-Maxwell system the membrane equations correctly reproduce the Navier-Stokes equations of hydrodynamics. Even the leading term in the membrane equations contains arbitrarily higher derivative corrections to Navier Stokes equations and keeps only those terms that survive in large D limit. All these things imply a particular resummation of one series into another series. We end the conclusion with a few future directions.

We have compared the solutions in the overlap regime of parameter space where both perturbation techniques can be applied. However, in chapter 2, we have described that there is a non-overlap regime, where we could apply 'large D' but 'derivative expansion' could not be applied. In this regime, we can construct some genuinely new black hole solutions that have not been described by other.

The very generic nature of the arguments and algorithm used in the comparison of the two gravity solutions implies that the equivalence not only exists for pure gravity or the Einstein-Maxwell system but also will exist for higher derivative theory and some other possible system. But our explicit calculation is very much dependent on the detail of the metric. Hence, it would be interesting to set up the calculation more abstractly, without using too much details of the information about the specific systems.

We have established the equivalence of the two metrics only within the membrane region. In [36] the authors have computed the boundary stress tensor from the linearized gravitational fluctuations sourced by the membrane stress tensor and matched it with the dual hydrodynamic stress tensor. Now as we know the 'split' of the fluid-gravity metric, we can match the metric coefficients outsides the membrane region also, which are non-perturbative in large D expansion.

In AdS space, there exist similar-looking constructions of the horizon stress tensor in terms of boundary stress tensor by following a radial flow of constraint Einstein's equations on cut-off surfaces in the background AdS [40, 41]. In some way, they are a bit different from 'large-D' construction of the membrane stress tensor. For example, they do not resum the derivative expansion. Also, they remain like fluid equations all along the radial flow and reduce to non-relativistic fluid equations on the horizon. However, there must be some relations between this radial flow of Einstein's equations down to the horizon and the membrane stress tensor carried towards the AdS boundary via gravitational radiation [36]. It would be very interesting to explore these relations further.

Appendix A

Appendices for chapter 2

A.1 Analysis of $F(r/r_H)$

In this section, we shall evaluate the integral (2.53) in large D limit. For convenience we are quoting the equation here.

$$F(y) = y \int_y^\infty dx \frac{x^{D-2} - 1}{x(x^{D-1} - 1)} \tag{A.1}$$

We would like to evaluate this integral systematically for large D. Let us first evaluate the integral for $y \geq 2$. In this case, since D is very large, $x^D >> 1$ throughout the range of integration. So we shall expand the integrand in the following way.

$$\begin{aligned}
\frac{x^{D-2} - 1}{x(x^{D-1} - 1)} &= \left(\frac{1}{x^2}\right)\left(1 - x^{-(D-2)}\right)\left(1 - x^{-(D-1)}\right)^{-1} \\
&= \left(\frac{1}{x^2}\right)\left(1 - x^{-(D-2)}\right)\left(1 + \sum_{m=1} x^{-m(D-1)}\right) \\
&= \left(\frac{1}{x^2}\right)\left(1 + \sum_{m=1}\left[x^{-m(D-1)} - x^{-m(D-1)+1}\right]\right)
\end{aligned} \tag{A.2}$$

Integrating (A.2) we find

$$y \int_{y\geq 2}^\infty dx \frac{x^{D-2} - 1}{x(x^{D-1} - 1)} = 1 + \sum_{m=1}\left[\left(\frac{1}{(D-1)m + 1}\right) y^{-(D-1)m} - \left(\frac{1}{(D-1)m}\right) y^{-(D-1)m+1}\right] \tag{A.3}$$

Clearly, the sums in the RHS of (A.3) are convergent for $y \geq 2$. Let us denote the RHS as $k(y)$.

However, the expansion in (A.2) is not valid inside the 'membrane region', i.e., when $y - 1 \sim \mathcal{O}\left(\frac{1}{D}\right)$ and naively $k(y)$ is not the answer for the integral.

But consider the function $\tilde{k}(y) = F(y) - k(y)$. This function vanishes for all $y \geq 2$ and also by construction, it is a smooth function at $y = 2$ (none of the derivatives diverge). Hence $\tilde{k}(y)$ must vanish for every y. So we conclude, for every allowed y (i.e., $y \geq 1$)

$$F(y) = 1 + \sum_{m=1} \left[\left(\frac{1}{(D-1)m+1} \right) y^{-(D-1)m} - \left(\frac{1}{(D-1)m} \right) y^{-(D-1)m+1} \right] \quad \text{(A.4)}$$

Note that $F(y)$ reduces to 1 as $y \to \infty$ as required in section (2.3.3.2).

Now we would like to expand $F(y)$ in a series in $\left(\frac{1}{D}\right)$, where y is in the membrane regime.

$$y = 1 + \frac{Y}{D}, \quad Y \sim \mathcal{O}(1)$$

In this regime $F(y)$ takes the following form

$$F(y) = F\left(1 + \frac{Y}{D}\right) = 1 - \left(\frac{1}{D}\right)^2 \sum_{m=1} \left(\frac{1 + mY}{m^2}\right) e^{-mY} + \mathcal{O}\left(\frac{1}{D^3}\right) \quad \text{(A.5)}$$

In this appendix, we consider only the first subleading correction in $\left(\frac{1}{D}\right)$ expansion. Therefore $F(y)$ could be set to 1 for our purpose.

A.2 Derivation of ψ in $\{Y^A\} = \{\rho, y^\mu\}$ coordinates

In this section, we shall give the derivation of ψ as mentioned in eq (2.76). We want to solve ψ such that $\nabla^2 \psi^{-D} = 0$. Where ∇ is the covariant derivative with respect to the background metric

$$ds^2_{background} = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu} \, dy^\mu \, dy^\nu \quad \text{(A.6)}$$

we can expand ψ as follows

$$\psi = 1 + \left(A_{10} + \epsilon \, B_{10} + \frac{A_{11} + \epsilon \, B_{11}}{D}\right)(\rho - r_H) + (A_{20} + \epsilon \, B_{20})(\rho - r_H)^2 + \mathcal{O}\left(\frac{1}{D^3}\right) \quad \text{(A.7)}$$

Here ϵ denotes that B_{ij}'s are $\mathcal{O}(\partial)$ terms.

$$\begin{aligned}
&\nabla^2\left(\psi^{-D}\right) = 0\\
\Rightarrow\ &\psi\left(\nabla^2\psi\right) - (D+1)(\nabla^A\psi)(\nabla_A\psi) = 0\\
\Rightarrow\ &\psi\ \rho^2\left[\partial_\rho\partial_\rho\psi - \Gamma^\rho_{\rho\rho}(\partial_\rho\psi) - \Gamma^\mu_{\rho\rho}(\partial_\mu\psi)\right] + \frac{\psi}{\rho^2}\ \eta^{\mu\nu}\left[-\Gamma^\rho_{\mu\nu}(\partial_\rho\psi) - \Gamma^\alpha_{\mu\nu}\partial_\alpha\psi\right]\\
&\qquad - (D+1)\ \rho^2\ (\partial_\rho\psi)^2 + \mathcal{O}(\partial)^2 = 0
\end{aligned} \tag{A.8}$$

The required Christoffel symbols are

$$\Gamma^\rho_{\rho\rho} = -\frac{1}{\rho}; \qquad \Gamma^\mu_{\rho\rho} = 0; \qquad \Gamma^\rho_{\mu\nu} = -\rho^3\eta_{\mu\nu}; \qquad \Gamma^\alpha_{\mu\nu} = 0; \tag{A.9}$$

Using the above Christoffel symbol we get

$$\psi\left[\rho^2\ \partial^2_\rho\psi + D\rho\ \partial_\rho\psi\right] - (D+1)\ \rho^2\ (\partial_\rho\psi)^2 = 0 \tag{A.10}$$

Now,

$$\begin{aligned}
\partial_\rho\psi &= \left(A_{10} + \epsilon\ B_{10} + \frac{A_{11} + \epsilon\ B_{11}}{D}\right) + 2\ (A_{20} + \epsilon\ B_{20})(\rho - r_H)\\
\partial^2_\rho\psi &= 2\ (A_{20} + \epsilon\ B_{20})
\end{aligned} \tag{A.11}$$

Solving, (A.10) order by order in derivative expansion we get the following solution

$$\psi(\rho, y^\mu) = 1 + \left(1 - \frac{1}{D}\right)\left(\frac{\rho}{r_H(y^\mu)} - 1\right) + \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{A.12}$$

A.3 Computing different terms in membrane equation

In this section we shall give the details of calculations of different terms that appear in the membrane equation. The different components of the projector defined in (2.86) are given by

$$\Pi^\rho_\rho = 0; \qquad \Pi^\rho_\mu = \partial_\mu r_H; \qquad \Pi^\mu_\rho = \frac{1}{r_H^4}(\partial^\mu r_H); \qquad \Pi^\mu_\nu = \delta^\mu_\nu \tag{A.13}$$

The different components of the Christoffel symbol of the background metric in $Y^A = \{\rho, y^\mu\}$ co-ordinates are given by

$$\Gamma^\rho_{\rho\rho} = -\frac{1}{\rho}; \qquad \Gamma^\rho_{\mu\rho} = 0; \qquad \Gamma^\rho_{\mu\nu} = -\rho^3 \eta_{\mu\nu}; \qquad \Gamma^\nu_{\mu\rho} = \frac{1}{\rho}\delta^\nu_\mu; \qquad \Gamma^\alpha_{\mu\nu} = 0; \qquad \Gamma^\mu_{\rho\rho} = 0; \tag{A.14}$$

From (2.87) it is clear that we need only $K_{\rho\alpha}$ and $K_{\alpha\beta}$ component of extrinsic curvature

$$\begin{aligned} K_{\rho\mu} &= \Pi^C_\rho \Big(\partial_C n_\mu - \Gamma^D_{C\mu} n_D \Big) \\ &= \Pi^\nu_\rho \Big(\partial_\nu n_\mu - \Gamma^\rho_{\nu\mu} n_\rho \Big) \\ &= \frac{\partial_\mu r_H}{r_H^2} \\ K_{\mu\nu} &= \Pi^C_\mu \Big(\partial_C n_\nu - \Gamma^D_{C\nu} n_D \Big) \\ &= \Pi^\rho_\mu \Big(\partial_\rho n_\nu - \Gamma^\rho_{\rho\nu} n_\rho \Big) + \Pi^\alpha_\mu \Big(\partial_\alpha n_\nu - \Gamma^\rho_{\alpha\nu} n_\rho \Big) \\ &= -\delta^\alpha_\mu \; \Gamma^\rho_{\alpha\nu} n_\rho \\ &= \rho^2 \; \eta_{\mu\nu} \end{aligned} \tag{A.15}$$

Now, as mentioned in (2.87) in terms of the intrinsic coordinates on the membrane the extrinsic curvature will have the structure

$$\begin{aligned} \mathcal{K}_{\alpha\beta} &= K_{\rho\rho} \left(\partial_\alpha r_H\right) \left(\partial_\beta r_H\right) + \left[K_{\rho\alpha} \left(\partial_\beta r_H\right) + K_{\rho\beta} \left(\partial_\alpha r_H\right)\right] + K_{\alpha\beta} \\ &= r_H^2 \; \eta_{\alpha\beta} + \mathcal{O}(\partial)^2 \end{aligned} \tag{A.16}$$

The trace of the extrinsic curvature

$$\mathcal{K} = (D-1) + \mathcal{O}(\partial^2) \tag{A.17}$$

For the calculation of the extrinsic curvature we need background metric, where for the rest of the calculation we require induced metric on the horizon. The induced metric on the horizon is given by

$$g_{\alpha\beta} = r_H^2 \; \eta_{\alpha\beta} + \mathcal{O}(\partial^2) \tag{A.18}$$

The Christoffel symbol of the induced metric

$$\Gamma^{\delta}_{\beta\alpha} = \left(\delta^{\delta}_{\beta} \frac{\partial_{\alpha} r_H}{r_H} + \delta^{\delta}_{\alpha} \frac{\partial_{\beta} r_H}{r_H} - \eta_{\alpha\beta} \frac{\partial^{\delta} r_H}{r_H} \right) \tag{A.19}$$

Now we shall calculate all the terms mentioned in (2.93). First, we shall calculate

$$\begin{aligned}
\hat{\nabla} \cdot U &= g^{\alpha\beta}\ \hat{\nabla}_{\alpha} U_{\beta} \\
&= \frac{\eta^{\alpha\beta}}{r_H^2} \left[\partial_{\alpha} U_{\beta} - \Gamma^{\delta}_{\alpha\beta} U_{\delta} \right] + \mathcal{O}(\partial)^2 \\
&= \frac{\eta^{\alpha\beta}}{r_H^2} \left[\partial_{\alpha} \left(r_H\ u_{\beta} \right) - \left(r_H\ u_{\delta} \right) \left(\delta^{\delta}_{\beta} \frac{\partial_{\alpha} r_H}{r_H} + \delta^{\delta}_{\alpha} \frac{\partial_{\beta} r_H}{r_H} - \eta_{\alpha\beta} \frac{\partial^{\delta} r_H}{r_H} \right) \right] + \mathcal{O}(\partial)^2 \\
&= (D-2) \left(\frac{(u \cdot \partial) r_H}{r_H^2} \right) + \frac{\partial \cdot u}{r_H} + \mathcal{O}(\partial)^2
\end{aligned} \tag{A.20}$$

Now we shall calculate $\hat{\nabla}^2 U_{\mu}$ and $\left(U \cdot \hat{\nabla} \right) U_{\alpha}$

$$\begin{aligned}
\hat{\nabla}^2 U_{\mu} &= g^{\alpha\beta} \hat{\nabla}_{\alpha} \hat{\nabla}_{\beta} U_{\mu} \\
&= g^{\alpha\beta} \left[\partial_{\alpha} (\hat{\nabla}_{\beta} U_{\mu}) - \Gamma^{\delta}_{\alpha\beta} (\hat{\nabla}_{\delta} U_{\mu}) - \Gamma^{\delta}_{\alpha\mu} (\hat{\nabla}_{\beta} U_{\delta}) \right] \\
&= \mathcal{O}(\partial)^2
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
\left(U \cdot \hat{\nabla} \right) U_{\alpha} &= U^{\beta} (\partial_{\beta} U_{\alpha}) - U^{\beta}\ \Gamma^{\delta}_{\beta\alpha} U_{\delta} \\
&= \frac{u^{\beta}}{r_H} \left(r_H (\partial_{\beta} u_{\alpha}) + u_{\alpha} (\partial_{\beta} r_H) \right) - \frac{u^{\beta}}{r_H} (r_H\ u_{\delta}) \left(\delta^{\delta}_{\beta} \frac{\partial_{\alpha} r_H}{r_H} + \delta^{\delta}_{\alpha} \frac{\partial_{\beta} r_H}{r_H} - \eta_{\alpha\beta} \frac{\partial^{\delta} r_H}{r_H} \right) + \mathcal{O}(\partial^2) \\
&= (u \cdot \partial) u_{\alpha} + u_{\alpha} \left(\frac{(u \cdot \partial) r_H}{r_H} \right) + \frac{\partial_{\alpha} r_H}{r_H} + \mathcal{O}(\partial^2)
\end{aligned} \tag{A.22}$$

Now,

$$\begin{aligned}
U^{\alpha}\ \mathcal{K}_{\alpha\beta}\ \mathcal{P}^{\beta}_{\gamma} &= (\delta^{\beta}_{\gamma} + U^{\beta}\ U_{\gamma})(U^{\alpha}\ r_H^2\ \eta_{\alpha\beta}) + \mathcal{O}(\partial^2) \\
&= (\delta^{\beta}_{\gamma} + U^{\beta}\ U_{\gamma}) U_{\beta} + \mathcal{O}(\partial^2) \\
&= \mathcal{O}(\partial^2)
\end{aligned} \tag{A.23}$$

Appendix B

Appendices for chapter 3

B.1 Comparison up to $\mathcal{O}\left(\partial^2, \frac{1}{D}\right)$ following [1] exactly:

As we have already mentioned in chapter 3, the computation of this note is quite different in its approach from that of [69] and in fact a bit conjectural in few steps. We could see these simple general patterns, conjectured to be true to all orders (for example see equations (3.32) and (3.43)) only after we have done some detailed calculations, keeping all allowed arbitrariness, to begin with at every stage and fixing them one by one exactly the way it has been done in [69]. In this appendix, we shall record this way of doing the calculation. Though it looks clumsy, the clear advantage in this brute-force method is that it is bound to give the correct result at the end.

Assuming derivative expansion, the most general form of $\bar{O}^A\partial_A$ at second order in derivative expansion which is null with respect to the full hydrodynamic metric $\mathcal{G}_{AB}$ is the following

$$\bar{O}^A\partial_A = \partial_r + \left(\sum_{i=1}^{5} B_i(r)\ \mathfrak{s}_{(i)}\right)\partial_r + \sum_{i=1}^{5} B_{5+i}(r)\ \mathfrak{v}^{\mu}_{(i)}\partial_\mu \tag{B.1}$$

Now imposing the condition that $\bar{O}_A$ is affinely parametrized null geodesic with respect to $\mathcal{G}_{AB}$ i.e., $(\bar{O}^A\nabla_A)\bar{O}^B = 0$, the form of $\bar{O}^A$ becomes

$$\bar{O}^A\partial_A = \partial_r + \left(\sum_{i=1}^{5} b_i\ \mathfrak{s}_{(i)}\right)\partial_r + \sum_{i=1}^{5}\left(\frac{b_{5+i}}{r^2}\right)\ \mathfrak{v}^{\mu}_{(i)}\partial_\mu \tag{B.2}$$

Where b_i's are arbitrary constants.

As before, we shall start with the most general possible form of f^A 's upto second order in derivative expansion, substituing the answer for zeoth and first order correction from [1].

$$
\begin{aligned}
\rho &= r - \frac{\Theta}{D-2} + \sum_{i=1}^{5} c_i(r)\ \mathfrak{s}_i \\
y^\mu &= x^\mu + \frac{u^\mu(x)}{r} + \left(\frac{\Theta}{D-2}\right)\left(\frac{u^\mu}{r^2}\right) + K^\mu_{(\text{old})} + \left(\sum_{i=1}^{5} \tilde{c}_i(r)\ \mathfrak{s}_i\right) u^\mu + \left(\sum_{i=1}^{5} \tilde{v}_i(r)\ \mathfrak{v}_i^\mu\right)
\end{aligned}
\tag{B.3}
$$

Where, $K^\mu_{(\text{old})}$ is defined as

$$
K^\mu_{(\text{old})} = k_1(r_H)\left(\frac{\Theta}{D-2}\right) u^\mu + k_2(r_H)\ a^\mu \tag{B.4}
$$

We shall determine f^A 's by using

$$
\mathcal{E}_B \equiv \bar{O}^A\left(\mathcal{G}_{AB} - \bar{\mathcal{G}}_{AB}\right) = 0 \tag{B.5}
$$

Here, $\mathcal{G}_{AB}$ is full hydrodynamic metric in $\{X\}$ and $\bar{\mathcal{G}}_{AB}$ is background metric in $\{X\}$. Different components of $\bar{\mathcal{G}}_{AB}$ are as follows

$$
\begin{aligned}
\bar{\mathcal{G}}_{rr} &= \frac{2}{r^2}\left[\sum_{i=1}^{5} \mathfrak{s}_i\left(c_i'(r) + r^2\tilde{c}_i'(r) - \frac{2c_i}{r}\right) + \left(\frac{3}{r^2}\right)\mathfrak{s}_1\right] \\
\bar{\mathcal{G}}_{r\mu} &= -u_\mu + \left(\frac{3}{r^2}\right)\mathfrak{s}_1\ u_\mu + u_\mu \sum_{i=1}^{5}\left(r^2\tilde{c}_i' - \frac{2c_i}{r}\right)\mathfrak{s}_i + \sum_{i=1}^{5}\left(r^2\tilde{v}_i'\mathfrak{v}^i_\mu\right) - u_\beta\partial_\mu K^\beta_{(\text{old})} \\
\bar{\mathcal{G}}_{\mu\nu} &= r^2(\mathcal{P}_{\mu\nu} - u_\mu u_\nu) + 2r\left(\frac{\Theta}{D-2}\right)u_\mu u_\nu + r\left[2\ \sigma_{\mu\nu} - a_\mu u_\nu - a_\nu u_\mu\right] \\
&\quad + (\mathcal{P}_{\mu\nu} - u_\mu u_\nu)\left[\sum_{i=1}^{5} 2r\ c_i(r)\mathfrak{s}_i\right] - u_\mu u_\nu\left[\mathfrak{s}_1 - \mathfrak{s}_2 + 2\mathfrak{s}_5\right] + u_\mu\left[\mathfrak{v}^{(2)}_\nu - \mathfrak{v}^{(3)}_\nu + \mathfrak{v}^{(4)}_\nu\right] \\
&\quad + u_\nu\left[\mathfrak{v}^{(2)}_\mu - \mathfrak{v}^{(3)}_\mu + \mathfrak{v}^{(4)}_\mu\right] + \left[\mathfrak{t}^{(1)}_{\mu\nu} - \mathfrak{t}^{(2)}_{\mu\nu} + \mathfrak{t}^{(3)}_{\mu\nu}\right] + r^2\left[\partial_\mu K^{(\text{old})}_\nu + \partial_\nu K^{(\text{old})}_\mu\right]
\end{aligned}
\tag{B.6}
$$

Solving, $\mathcal{E}_r = 0$ and $u^\mu \mathcal{E}_\mu = 0$ we get the following solution

$$\begin{aligned}
&c_1(r) = r\left(r_H \frac{\partial k_1}{\partial r_H}\right) + p_1, \quad \tilde{c}_1(r) = \frac{1}{r^3} - \frac{p_1}{r^2} - \frac{r_H}{r}\left(\frac{\partial k_1}{\partial r_H}\right) + \tilde{p}_1,\\
&c_2(r) = -k_2 r + p_2, \quad \tilde{c}_2(r) = \frac{k_2}{r} - \frac{p_2}{r^2} + \tilde{p}_2,\\
&c_3(r) = p_3, \quad \tilde{c}_3(r) = -\frac{p_3}{r^2} + \tilde{p}_3,
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
&c_4(r) = p_4, \quad \tilde{c}_4(r) = -\frac{p_4}{r^2} + \tilde{p}_4,\\
&c_5(r) = -k_1 r + p_5, \quad \tilde{c}_5(r) = \frac{k_1}{r} - \frac{p_5}{r^2} + \tilde{p}_5.
\end{aligned}$$

Solving, $\mathcal{P}^{\mu\nu}\mathcal{E}_\nu = 0$

$$\begin{aligned}
&\tilde{v}_1(r) = \frac{1}{r}\left(k_2 - r_H \frac{\partial k_1}{\partial r_H}\right) + q_1, \quad \tilde{v}_2(r) = -\frac{k_2}{r} + q_2, \quad \tilde{v}_3(r) = \frac{k_2}{r} + q_3\\
&\tilde{v}_4(r) = \frac{k_1}{r} + q_4, \quad \tilde{v}_5(r) = q_5
\end{aligned} \tag{B.8}$$

Substituting the solutions for c_i, $\tilde{c}_i$ and $\tilde{v}_i$ it is easy to figure out the form of background metric $\bar{\mathcal{G}}_{AB}$ in $\{X\}$ coordinates. Now, to get the $\mathcal{G}^{\text{rest}}_{AB}$, we have to subtract off $\bar{\mathcal{G}}_{AB}$ from the hydrodynamic metric presented in section-(3.2). The $\mathcal{G}^{\text{rest}}_{rr}$ and $\mathcal{G}^{\text{rest}}_{r\mu}$ simply vanish.

$$\mathcal{G}^{(\text{rest})}_{rr} = 0, \quad \mathcal{G}^{(\text{rest})}_{r\mu} = 0 \tag{B.9}$$

The structure of $\mathcal{G}^{\text{rest}}_{\mu\nu}$ is a bit complicated. We first decompose it into the scalar vector and the tensor sectors.

$$\mathcal{G}^{(\text{rest})}_{\mu\nu} = \mathcal{G}^{(1)}_S u_\mu u_\nu + \mathcal{G}^{(2)}_S P_{\mu\nu} + \left(\mathcal{G}^{(V)}_\mu u_\nu + \mathcal{G}^{(V)}_\nu u_\mu\right) + \mathcal{G}^{(T)}_{\mu\nu} \tag{B.10}$$

Where, $\mathcal{G}_S^{(1)}$, $\mathcal{G}_S^{(2)}$, $\mathcal{G}_\mu^{(V)}$ and $\mathcal{G}_{\mu\nu}^{(T)}$ have the following forms

$$\begin{aligned}
\mathcal{G}_S^{(1)} &= r^2\left(\frac{r_H}{r}\right)^{D-1} + r^2\left(\frac{r_H}{r}\right)^{D-1}\left[-\frac{1}{r_H^2}\left(\frac{1}{2}-\frac{\tilde{R}}{D}\right)\mathfrak{s}_{(3)} + \frac{1}{r_H^2}\left(\frac{1}{2}-\frac{\tilde{R}+2}{D}\right)\mathfrak{s}_{(4)}\right]\\
&\quad + 2\ r\left[\left(r\ r_H\ \frac{\partial k_1}{\partial r_H}+p_1\right)\ \mathfrak{s}_{(1)} - (k_2\ r - p_2)\mathfrak{s}_{(2)} + p_3\ \mathfrak{s}_{(3)} + p_4\ \mathfrak{s}_{(4)} - (k_1\ r - p_5)\mathfrak{s}_{(5)}\right]\\
&\quad - 2\ r^2\left[r_H\left(\frac{\partial k_1}{\partial r_H}\right)\mathfrak{s}_{(1)} - k_1\ \mathfrak{s}_{(5)} - k_2\ \mathfrak{s}_{(2)}\right]\\
\mathcal{G}_S^{(2)} &= -2\ r\left[\left(r\ r_H\ \frac{\partial k_1}{\partial r_H} + k_1\ r + k_2\ r + p_1\right)\ \mathfrak{s}_{(1)} - (k_2\ r - p_2)\mathfrak{s}_{(2)} + p_3\ \mathfrak{s}_{(3)}\right.\\
&\qquad\left. + p_4\ \mathfrak{s}_{(4)} - (k_1\ r - k_2\ r - p_5)\mathfrak{s}_{(5)}\right]\\
\mathcal{G}_\mu^{(V)} &= r^2\left[(k_1-k_2) + r_H\ \frac{\partial}{\partial r_H}(k_1-k_2)\right]\mathfrak{v}_\mu^{(1)} - (k_1-k_2)\ r^2\ \mathfrak{v}_\mu^{(4)} - 2\ k_2\ r^2\left(\mathfrak{v}_\mu^{(3)} - \mathfrak{v}_\mu^{(2)}\right)\\
\mathcal{G}_{\mu\nu}^{(T)} &= -2\ k_2\ r^2\ \mathfrak{t}_{\mu\nu}^{(1)} - 2\ k_1\ r^2\left(\frac{\Theta}{D-2}\right)\sigma_{\mu\nu}
\end{aligned} \tag{B.11}$$

Now, in the large-D side the metric is quite simple if we neglect terms of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

$$\begin{aligned}
G_{AB} &= \bar{G}_{AB} + G_{AB}^{\text{rest}}\\
\text{where}\ \ G_{AB}^{\text{rest}} &= \left(\frac{\psi^{-D}}{\Phi^2}\right)\bar{O}_A\bar{O}_B
\end{aligned} \tag{B.12}$$

Where,

$$\begin{aligned}
\psi^{-D} &= \left(\frac{r}{H}\right)^{-D+1} - \left(\frac{r}{r_H}\right)^{-D+1}\left(\frac{1}{r^2}-\frac{1}{r_H^2}\right)\frac{D-1}{(D+1)}\left[\mathfrak{s}_1 - \mathfrak{s}_2 + \frac{1}{2}\left(\mathfrak{s}_3 - \mathfrak{s}_4\right) + 2\ \mathfrak{s}_5\right]\\
&\quad + \left(\frac{r}{r_H}\right)^{-D+1}\mathfrak{s}_4\left[\frac{2}{(D-2)\ r_H}\left(\frac{1}{r}-\frac{1}{r_H}\right)\right] + \mathcal{O}(\partial)^3\\
\Phi^2 &= \frac{1}{r^2} - \frac{1}{r^4}\left[\mathfrak{s}_1 - \mathfrak{s}_2 - 4\left(\frac{r}{r_H}\right)\frac{\mathfrak{s}_4}{(D-1)(D-2)} + 2\mathfrak{s}_5\right] + \mathcal{O}(\partial)^3\\
\bar{O}_A\ dX^A &= -u_\mu\ dx^\mu - \left(\sum_{i=1}^{5} b_i\ \mathfrak{s}_{(i)}\ u_\mu - \sum_{i=1}^{5} b_{5+i}\ \mathfrak{v}_\mu^{(i)}\right) dx^\mu
\end{aligned} \tag{B.13}$$

After using equation (B.13) we could rewrite G_{AB}^{rest} in terms of scalar, vector and tensor type fluid data.

$$
\begin{aligned}
G^{\text{rest}}_{rr} &= 0, \quad G^{\text{rest}}_{r\mu} = 0 \\
G^{\text{rest}}_{\mu\nu} &= G^{(1)}_S u_\mu u_\nu + G^{(2)}_S \mathcal{P}_{\mu\nu} + \left(G^{(V)}_\mu u_\nu + G^{(V)}_\nu u_\mu\right) + G^{(T)}_{\mu\nu}
\end{aligned}
\tag{B.14}
$$

where

$$
\begin{aligned}
G^{(1)}_S &= r^2\left(\frac{r_H}{r}\right)^{D-1} + r^2\left(\frac{r_H}{r}\right)^{D-1}\left[\frac{1}{r_H^2}\left(\frac{\tilde{R}}{D} - \frac{1}{2}\right)\mathfrak{s}_3 + \frac{1}{r_H^2}\left(\frac{1}{2} - \frac{\tilde{R}+2}{D}\right)\mathfrak{s}_4 + 2\sum_{i=1}^{5} b_i\ \mathfrak{s}_{(i)}\right] \\
G^{(2)}_S &= 0 \\
G^{(V)}_\mu &= -\, r^2\left(\frac{r_H}{r}\right)^{D-1}\left(\sum_{i=1}^{5} b_{5+i}\ \mathfrak{v}^{(i)}_\mu\right) \\
G^{(T)}_{\mu\nu} &= 0
\end{aligned}
\tag{B.15}
$$

Now we demand that each component of G^{rest}_{AB} should be exactly equal to the same component of $\mathcal{G}^{\text{rest}}_{AB}$. We shall start from the tensor sector. $G^{(T)}_{\mu\nu}$ vanishes and therefore we shall set $\mathcal{G}^{(T)}_{\mu\nu}$ to zero implying

$$
k_1 = 0; \quad k_2 = 0 \tag{B.16}
$$

Now once we set k_1 and k_2 to zero, $\mathcal{G}^{(V)}_\mu$ vanishes. Therefore $G^{(V)}_\mu$ also must vanish and that determines the constants b_{5+i}s.

$$
b_{5+i} = 0, \quad i = \{1, \cdots, 5\} \tag{B.17}
$$

Next we come to the comparison of $\mathcal{G}^{(2)}_S$ and $G^{(2)}_S$. After setting k_1 and k_2 to zero, $\mathcal{G}^{(2)}_s$ turns out to be

$$
\mathcal{G}^{(2)}_s = -2r\sum_{i=1}^{5} p_i\ \mathfrak{s}_{(i)} \tag{B.18}
$$

whereas from equation (B.15) we see $G^{(2)}_S$ vanish. These two would be equal if we set all the constants p_i s to zero.

$$
p_i = 0, \quad i = \{1, \cdots, 5\} \tag{B.19}
$$

Substituting equation (B.16) and (B.18) in equation (B.11), and equating $\mathcal{G}^{(1)}_s$ with

$G_S^{(1)}$ we find

$$b_i = 0, \quad i = \{1, \cdots, 5\} \tag{B.20}$$

The final form of O^A becomes

$$O^A \partial_A = \partial_r + \mathcal{O}\left(\partial^3\right) \tag{B.21}$$

The final form of the mapping functions are

$$\begin{aligned} \rho &= r - \frac{\Theta}{D-2} + \mathcal{O}(\partial)^3 \\ y^\mu &= x^\mu + \frac{u^\mu(x)}{\left(r - \frac{\Theta(x)}{D-2}\right)} + \left(\sum_{i=1}^{5} \tilde{p}_i \, \mathfrak{s}_i\right) u^\mu + \left(\sum_{i=1}^{5} \tilde{q}_i \, \mathfrak{v}_i^\mu\right) + \mathcal{O}(\partial)^3 \end{aligned} \tag{B.22}$$

Where, $\tilde{p}_i$ and $\tilde{q}_i$ are some arbitrary constants.
They clearly match with (3.32) and (3.43) upto the required order[1]

B.2 Large-D limit of the functions appearing in hydrodynamic metric

In this appendix, we shall evaluate the integrations appearing in equation (3.69). We are interested in some expressions, which could be further expanded in inverse powers of dimension. But unfortunately we have not been able to do these integrations exactly in arbitrary D and therefore we had to use several tricks to get to the answers, required for our comparison.

Integration would be done separately for two cases. One is for those ranges of y so that the metric remains within the 'membrane region'. Here we have to be careful so that we could fix each factor up to the corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^3$. This is the regime where we expect a detailed match between the large D and the hydrodynamic metric.
Next, we perform these integrations outside the membrane regime. Here it is enough

[1]The last two set of terms in the expression of y^μ do not get fixed by the matching at order $\mathcal{O}(\partial)^2$. They are equivalent to the terms K^μ_{old}, which were $\mathcal{O}(\partial)$ terms in the expression of y^μ and did not get fixed from matching at $\mathcal{O}(\partial)$. But K^μ_{old} do get fixed from matching at order $\mathcal{O}(\partial)^2$ and which turns out to be zero. We think that these undetermined terms in y^μ will get fixed from the matching at the next order (i.e., $\mathcal{O}(\partial)^3$). We can very easily see that the expression of y^μ is consistent with (3.32) $u^\mu(\frac{\partial \xi^\mu}{\partial x^\nu}) = \mathcal{O}(\partial)^3$.

for us to show the overall falloff behavior of these integrations as a function of D.

B.2.1 Within the membrane region

B.2.1.1 $F(y)$:

$$F(y) = 1 + \sum_{m=1}^{\infty} \left[\frac{y^{-m(D-1)}}{m(D-1)+1} - \frac{y^{-m(D-1)+1}}{m(D-1)} \right] \tag{B.23}$$

Expanding in $\frac{1}{D}$, after putting $y = 1 + \frac{Y}{D}$,

$$\begin{aligned} F(y) = F\left(1 + \frac{Y}{D}\right) &= 1 - \left(\frac{1}{D}\right)^2 \sum_{m=1}^{\infty} \left(\frac{1+mY}{m^2}\right) e^{-mY} + \mathcal{O}\left(\frac{1}{D^3}\right) \\ &= 1 + \frac{1}{D^2}\left(Y \text{ Log}\left[1 - e^{-Y}\right] - \text{PolyLog}\left[2,\ e^{-Y}\right]\right) + \mathcal{O}\left(\frac{1}{D}\right)^3 \end{aligned} \tag{B.24}$$

B.2.1.2 $H_1(y)$:

Expanding the integrand:

$$\begin{aligned} Integrand = &\ \frac{1}{x}\left[\frac{x^{D-3}-1}{x^{D-1}-1}\right] \\ = &\ \frac{1}{x^3}\left[1 - x^{-(D-3)}\right]\left[\sum_{m=0}^{\infty} x^{-m(D-1)}\right] \\ = &\ \frac{1}{x^3}\left[1 + \sum_{m=1}^{\infty}\left(x^{-m(D-1)}\right) - \sum_{m=1}^{\infty}\left(x^{-m(D-1)+2}\right)\right] \end{aligned} \tag{B.25}$$

After integration

$$\begin{aligned} H_1(y) = &\ 2y^2 \int_y^{\infty} \frac{dx}{x}\left[\frac{x^{D-3}-1}{x^{D-1}-1}\right] \\ = &\ 2y^2 \int_y^{\infty} \frac{dx}{x^3}\left[1 + \sum_{m=1}^{\infty}\left(x^{-m(D-1)}\right) - \sum_{m=1}^{\infty}\left(x^{-m(D-1)+2}\right)\right] \\ = &\ 1 + 2\sum_{m=1}^{\infty}\left[\frac{y^{-m(D-1)}}{m(D-1)+2}\right] - 2\sum_{m=1}^{\infty}\left[\frac{y^{-m(D-1)+2}}{m(D-1)}\right] \end{aligned} \tag{B.26}$$

Large-D expansion after substituting $y = 1 + \frac{Y}{D}$

$$H_1\left(1+\frac{Y}{D}\right) = 1 - \left(\frac{2}{D}\right)^2 \sum_{m=1}^{\infty}\left(\frac{1+mY}{m^2}\right) e^{-mY} + \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{B.27}$$

B.2.1.3 $K_1(y)$:

$$F(y) = 1 + \sum_{m=1}^{\infty}\left[\frac{y^{-m(D-1)}}{m(D-1)+1} - \frac{y^{-m(D-1)+1}}{m(D-1)}\right] \tag{B.28}$$

$$yF'(y) - F(y) = -1 + (y-1)\sum_{m=1}^{\infty} y^{-m(D-1)} = -1 + \frac{y-1}{y^{D-1}-1} \tag{B.29}$$

$$\left[\frac{yF'(y)-F(y)}{y}\right]^2 = \frac{1}{y^2} + \sum_{m=1}^{\infty}\left[m\left(\frac{1-y}{y}\right)^2 y^{-(m+1)(d-1)} - 2\left(\frac{y-1}{y^2}\right) y^{-m(D-1)}\right] \tag{B.30}$$

$$\begin{aligned}
&\int_y^{\infty}\left(\frac{dz}{z^2}\right)\left(z\ F'(z) - F(z)\right)^2 \\
&= \frac{1}{y} + \sum_{m=1}^{\infty}\left[\frac{1}{y}\left(\frac{2}{m(D-1)+1}\right) - \frac{2}{m(D-1}\right] y^{-m(D-1)} \\
&\quad + \sum_{m=1}^{\infty}\left[-\frac{2\ m}{(D-1)(m+1)} + \frac{1}{y}\left(\frac{m}{m(D-1)+D}\right) + y\left(\frac{m}{(D-1)m+D-2}\right)\right] y^{-(D-1)(m+1)}
\end{aligned} \tag{B.31}$$

$$
\begin{aligned}
&K_1(x)\\
&= 2x^2 \int_x^\infty \frac{dy}{y^2} \int_y^\infty \left(\frac{dz}{z^2}\right)\left(z\ F'(z) - F(z)\right)^2\\
&= 1 + 4\sum_{m=1}^{\infty}\left[\frac{1}{[(D-1)m+1][(D-1)m+2]} - \frac{x}{[(D-1)m+1][m(D-1)]}\right] x^{-m(D-1)}\\
&\quad + 2\sum_{m=1}^{\infty} m\left[\frac{1}{[(D-1)m+D][D(m+1)-(m-1)]} - \frac{2\ x}{[(D-1)(m+1)][D(m+1)-m]}\right.\\
&\qquad \left. + \frac{x^2}{[(D-1)(m+1)][D(m+1)-(m+2)]}\right] x^{-(D-1)(m+1)}
\end{aligned}
\tag{B.32}
$$

Substituting $x = 1 + \frac{X}{D}$ and taking the large-D limit

$$
K_1\left(1+\frac{X}{D}\right) = 1 - \left(\frac{1}{D}\right)^3 \sum_{m=1} \left(\frac{4}{m^3}\right)(2 + m\ X)e^{-m\ X} + \mathcal{O}\left(\frac{1}{D}\right)^4 \tag{B.33}
$$

B.2.1.4 $K_2(y)$:

$$
\begin{aligned}
K_2(y) = \int_y^\infty \left(\frac{dx}{x^2}\right) \Bigg[1 - 2(D-2)\ x^{D-2} - \left(1 - \frac{1}{x}\right)\left(xF'(x) - F(x)\right)\\
+ \left(2(D-2)x^{D-1} - (D-3)\right)\int_x^\infty \frac{dz}{z^2}\left(zF'(z) - F(z)\right)^2 \Bigg]
\end{aligned}
\tag{B.34}
$$

Naively the integration of $K_2(y)$ seems to be diverging. But upon appropriate expansion the singular terms cancel among themselves. After substituting (B.29) and (B.31) in (B.34) we can integrate the whole expression.

$$
\begin{aligned}
&K_2(y)\\
&= \frac{2}{y} - \frac{1}{y^2}\left(\frac{D-2}{2}\right)\\
&+ \sum_{m=1}^{\infty}\Bigg[\left(\frac{D(m-1)-3m+2}{[(d-1)m][D(m+1)-(m+2)]}\right) + \frac{1}{y}\left(\frac{2\left[-(D-3)m^2+2(D-2)m+D-3\right]}{[(D-1)m(m+1)][(D-1)m+1]}\right)\\
&\qquad + \frac{1}{y^2}\left(\frac{D^2\left(m^2-3m-2\right)+D\left(-4m^2+10m+5\right)+3(m-3)m}{[(D-1)m+1][(D-1)m+2][D(m+1)-m]}\right)\Bigg]y^{-m(D-1)}\\
&+ \sum_{m=1}^{\infty}\Bigg[\frac{1}{y}\left(\frac{2(D-3)m}{[(D-1)(m+1)][D(m+1)-m]}\right) - \left(\frac{(D-3)m}{[(D-1)(m+1)][D(m+1)-(m+2)]}\right)\\
&\qquad - \frac{1}{y^2}\left(\frac{(D-3)m}{[D(m+1)-m][D(m+1)-(m-1)]}\right)\Bigg]y^{-(D-1)(m+1)}\\
&+ \sum_{m=1}^{\infty}\left[\frac{1}{y}\left(\frac{4(d-2)}{[d(m-1)-m+3][(d-1)m+1]}\right) - \frac{4(d-2)}{[(d-1)m][d(m-1)-m+2]}\right]y^{-D(m-1)+(m-2)}
\end{aligned}
\tag{B.35}
$$

Now, after putting $x = 1 + \frac{X}{D}$ we get

$$
K_2\left(1+\frac{X}{D}\right) = -\left(\frac{D}{2}\right) + (3+X) - \left(\frac{1}{2D}\right)\left[X\ (\ 8+3X\ )\right] + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{B.36}
$$

B.2.1.5 $L(y)$:

$$
\begin{aligned}
L(y) =& \int_y^{\infty} dx\ x^{D-2}\int_x^{\infty}\frac{dz}{z^3}\left[\frac{z-1}{z^{D-1}-1}\right]\\
=& \int_y^{\infty} dx\ x^{D-2}\int_x^{\infty} dz\sum_{m=1}^{\infty}\left[z^{-m(D-1)-2}-z^{-m(D-1)-3}\right]\\
=& \int_y^{\infty} dx\ x^{D-2}\sum_{m=1}^{\infty}\left[\frac{x^{-m(D-1)-1}}{m(D-1)+1}-\frac{x^{-m(D-1)-2}}{m(D-1)+2}\right]\\
=& \int_y^{\infty} dx\ \sum_{m=0}^{\infty}\left[\frac{x^{-m(D-1)-2}}{(m+1)(D-1)+1}-\frac{x^{-m(D-1)-3}}{(m+1)(D-1)+2}\right]\\
=& \sum_{m=0}^{\infty}\left[\frac{y^{-m(D-1)-1}}{[(m+1)(D-1)+1]\,[m(D-1)+1]}-\frac{y^{-m(D-1)-2}}{[(m+1)(D-1)+2]\,[m(D-1)+2]}\right]
\end{aligned}
\tag{B.37}
$$

Substituting $y = 1 + \frac{Y}{D}$ we find

$$
L\left(1+\frac{Y}{D}\right) = \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{B.38}
$$

B.2.1.6 $H_2(y)$:

$$H_2(y) = F(y)^2 - 2\, y^2 \int_y^\infty \frac{dx}{x(x^{D-1}-1)} \int_1^x \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \tag{B.39}$$

We shall first process the integral

$$\begin{aligned}
&\int_y^\infty \frac{dx}{x(x^{D-1}-1)} \int_1^x \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \\
=\ & \left(\int_1^\infty \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right]\right)\int_y^\infty \frac{dx}{x(x^{D-1}-1)} - \int_y^\infty \frac{dx}{x(x^{D-1}-1)} \int_x^\infty \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \\
=\ & -\mathcal{Q}(D)\frac{\mathrm{Log}[1-y^{-(D-1)}]}{D-1} - \int_y^\infty \frac{dx}{x(x^{D-1}-1)} \int_x^\infty \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right]
\end{aligned} \tag{B.40}$$

In the third step we have used the following identities.

$$\begin{aligned}
&\int_1^\infty \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] = -\left(\frac{1}{D-1}\right)\left[\mathrm{EulerGamma} + \mathrm{PolyGamma}\left(0, \frac{2}{D-1}\right)\right] \equiv \mathcal{Q}(D) \\
&\int \frac{dx}{x(x^{D-1}-1)} = \frac{\mathrm{Log}[1-x^{-(D-1)}]}{D-1}
\end{aligned} \tag{B.41}$$

Now the second term could be processed in an expansion.

$$\begin{aligned}
\int_y^\infty \frac{dx}{x(x^{D-1}-1)} \int_x^\infty \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] &\equiv \int_y^\infty dx \left[\frac{1}{x(x^{D-1}-1)}\right] S(x) \\
\text{Where,}\quad S(x) &\equiv \int_x^\infty \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right]
\end{aligned} \tag{B.42}$$

Now, first we will do the indefinite integral then will take the proper limit

$$\begin{aligned}
\int dx \left[\frac{S(x)}{x(x^{D-1}-1)}\right] &= S(x)\int \frac{dx}{x(x^{D-1}-1)} - \int dx\left(\frac{dS}{dx}\int \frac{dx}{x(x^{D-1}-1)}\right) \\
&= S(x)\int \frac{dx}{x(x^{D-1}-1)} + \int dx\left(\frac{x^{D-3}-1}{x(x^{D-1}-1)}\int \frac{dx}{x(x^{D-1}-1)}\right) \\
&= S(x)\int \frac{dx}{x(x^{D-1}-1)} + \int dx\,\left(x^{D-3}-1\right)\left(\frac{dG}{dx}\right) G(x)
\end{aligned} \tag{B.43}$$

Where we have defined

$$G(x) = \int \frac{dx}{x(x^{D-1}-1)}$$

So, we are getting

$$\begin{aligned}
&\int dx \left[\frac{S(x)}{x(x^{D-1}-1)}\right] \\
&= S(x)\int \frac{dx}{x(x^{D-1}-1)} + \frac{1}{2}\int dx \left(x^{D-3}-1\right)\frac{d}{dx}[G(x)]^2 \\
&= S(x)\int \frac{dx}{x(x^{D-1}-1)} + \frac{1}{2}\int dx \, \frac{d}{dx}\Big[\left(x^{D-3}-1\right)[G(x)]^2\Big] - \frac{1}{2}\int dx \; [G(x)]^2(D-3)x^{D-4}
\end{aligned} \tag{B.44}$$

Now,

$$G(x) \equiv \int \frac{dx}{x(x^{D-1}-1)} = \frac{\text{Log}[1-x^{-(D-1)}]}{D-1} \tag{B.45}$$

$$\begin{aligned}
&\int dx \left[\frac{S(x)}{x(x^{D-1}-1)}\right] \\
&= S(x)\int \frac{dx}{x(x^{D-1}-1)} + \frac{1}{2\,(D-1)^2}\left(x^{D-3}-1\right)\left(\text{Log}\left[1-x^{-(D-1)}\right]\right)^2 \\
&\quad - \frac{1}{2}\frac{D-3}{(D-1)^2}\int dx \; x^{D-4}\left(\text{Log}\left[1-x^{-(D-1)}\right]\right)^2 \\
&= S(x)\left(\frac{\text{Log}[1-x^{-(D-1)}]}{D-1}\right) + \frac{1}{2\,(D-1)^2}\left(x^{D-3}-1\right)\left(\text{Log}\left[1-x^{-(D-1)}\right]\right)^2 \\
&\quad - \frac{1}{2}\frac{D-3}{(D-1)^2}\int dx \; x^{D-4}\left(\text{Log}\left[1-x^{-(D-1)}\right]\right)^2
\end{aligned} \tag{B.46}$$

Restoring the limit, we get

$$\begin{aligned}
&\int_y^\infty \frac{dx}{x(x^{D-1}-1)}\int_x^\infty \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \\
&= \underbrace{\left[S(x)\left(\frac{\text{Log}[1-x^{-(D-1)}]}{D-1}\right)\right]_y^\infty}_{\text{Term1}} \underbrace{-\frac{1}{2\,(D-1)^2}\left(y^{D-3}-1\right)\left(\text{Log}\left[1-y^{-(D-1)}\right]\right)^2}_{\text{Term2}} \\
&\quad \underbrace{-\frac{1}{2}\frac{D-3}{(D-1)^2}\int_y^\infty dx \; x^{D-4}\left(\text{Log}\left[1-x^{-(D-1)}\right]\right)^2}_{\text{Term3}}
\end{aligned} \tag{B.47}$$

First we will calculate 'Term3'. We can expand the integrand in 'Term3' as follows

$$x^{D-4}\left(\text{Log}\left[1-x^{-(D-1)}\right]\right)^2=\sum_{m=2}^{\infty}2\ x^{D-4}x^{-m(D-1)}\frac{\text{HarmonicNumber}[m-1]}{m} \tag{B.48}$$

Now, we can integrate term by term

$$\begin{aligned}&-\frac{1}{2}\frac{D-3}{(D-1)^2}\int_y^\infty dx\ 2\ x^{D-4}x^{-m(D-1)}\frac{\text{HarmonicNumber}[m-1]}{m}\\ &=-\frac{D-3}{(D-1)^2}\left(\frac{y^{-m(D-1)+D-3}}{D(m-1)-m+3}\right)\frac{\text{HarmonicNumber}[m-1]}{m}\end{aligned} \tag{B.49}$$

Putting, $y=1+\frac{Y}{D}$

$$\begin{aligned}&-\frac{D-3}{(D-1)^2}\left(\frac{y^{-m(D-1)+D-3}}{D(m-1)-m+3}\right)\frac{\text{HarmonicNumber}[m-1]}{m}\\ &=\frac{1}{D^2}\left(\frac{e^{Y(1-m)}}{1-m}\right)\frac{\text{HarmonicNumber}[m-1]}{m}+\mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{B.50}$$

Now, if we take the summation we get,

$$\begin{aligned}\text{Term3}&=\sum_{m=2}^{\infty}\frac{1}{D^2}\left(\frac{e^{Y(1-m)}}{1-m}\right)\frac{\text{HarmonicNumber}[m-1]}{m}+\mathcal{O}\left(\frac{1}{D}\right)^3\\ &=-\frac{1}{6\ D^2}\Bigg[6\ e^Y\text{PolyLog}\left[2,e^{-Y}\right]\\ &+\left(e^Y-1\right)\left(\pi^2+6\ \text{Log}[1-e^{-Y}]\ \text{Log}\left[\frac{1}{1-e^Y}\right]-6\ \text{PolyLog}\left[2,\frac{e^Y}{e^Y-1}\right]\right)\Bigg]+\mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{B.51}$$

Now we will calculate 'Term2'

$$\text{Term2}=-\frac{1}{2\ (D-1)^2}\left(y^{D-3}-1\right)\left(\text{Log}\left[1-y^{-(D-1)}\right]\right)^2 \tag{B.52}$$

Putting, $y=1+\frac{Y}{D}$

$$\text{Term2}=-\frac{1}{2\ D^2}\left(e^Y-1\right)\left(\ \text{Log}\left[1-e^{-Y}\right]\ \right)^2 \tag{B.53}$$

Now we will calculate 'Term1'

$$\begin{aligned} S(x)\left[\frac{\text{Log}[1-x^{-(D-1)}]}{D-1}\right] &= \left[\frac{\text{Log}[1-x^{-(D-1)}]}{D-1}\right]\int_x^\infty \frac{dz}{z}\left(\frac{z^{D-3}-1}{z^{D-1}-1}\right) \\ &= \left[\frac{\text{Log}[1-x^{-(D-1)}]}{D-1}\right]\int_x^\infty dz\sum_{m=0}^{\infty}\left[z^{-m(D-1)-3}-z^{-(m+1)(D-1)-1}\right] \\ &= \left[\frac{\text{Log}[1-x^{-(D-1)}]}{D-1}\right]\sum_{m=0}^{\infty}\left[\frac{x^{-m(D-1)-2}}{(D-1)m+2}-\frac{x^{-(D-1)(m+1)}}{(D-1)(m+1)}\right] \end{aligned} \tag{B.54}$$

$$\begin{aligned} \text{Term1} &= \left[S(x)\left(\frac{\text{Log}[1-x^{-(D-1)}]}{D-1}\right)\right]_y^\infty \\ &= -\left[\frac{\text{Log}[1-y^{-(D-1)}]}{D-1}\right]\sum_{m=0}^{\infty}\left[\frac{y^{-m(D-1)-2}}{(D-1)m+2}-\frac{y^{-(D-1)(m+1)}}{(D-1)(m+1)}\right] \\ &= -\left[\frac{\text{Log}[1-y^{-(D-1)}]}{D-1}\right]\left[\frac{y^{-2}}{2}-\frac{y^{-(D-1)}}{D-1}\right] \\ &\quad -\left[\frac{\text{Log}[1-y^{-(D-1)}]}{D-1}\right]\sum_{m=1}^{\infty}\left[\frac{y^{-m(D-1)-2}}{(D-1)m+2}-\frac{y^{-(D-1)(m+1)}}{(D-1)(m+1)}\right] \\ &= -\left[\frac{\text{Log}[1-y^{-(D-1)}]}{D-1}\right]\left[\frac{y^{-2}}{2}-\frac{y^{-(D-1)}}{D-1}\right]-\left[\frac{\text{Log}[1-y^{-(D-1)}]}{D-1}\right]\left[\frac{y^{-(D+1)}}{D-1}\right]\times \\ &\quad \left(y^2+\text{LerchPhi}\left[y^{1-D},\ 1,\ \frac{D+1}{D-1}\right]+y^{D+1}\text{Log}[1-y^{-(D-1)}]\right) \end{aligned} \tag{B.55}$$

Now putting $y = 1 + \frac{Y}{D}$

$$\text{Term1} = -\frac{1}{2\ D}\text{Log}\left[1-e^{-Y}\right]+\frac{Y(Y+2)+2(e^Y-1)(2\ Y-1)\text{Log}[1-e^{-Y}]}{4\ D^2(e^Y-1)}+\mathcal{O}\left(\frac{1}{D}\right)^3 \tag{B.56}$$

Now, the integration (B.40) becomes

$$\begin{aligned} &\int_y^\infty \frac{dx}{x(x^{D-1}-1)}\int_1^x \frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \\ &= -\mathcal{Q}(D)\frac{\text{Log}[1-y^{-(D-1)}]}{D-1}-[\text{Term1}+\text{Term2}+\text{Term3}] \end{aligned} \tag{B.57}$$

Putting, $y = 1 + \frac{Y}{D}$ we get

$$-\mathcal{Q}(D)\frac{\text{Log}[1-y^{-(D-1)}]}{D-1} = -\frac{\text{Log}[1-e^{-Y}]}{2\ D}+\frac{Y(Y+2)-2\ (e^Y-1)\text{Log}[1-e^{-Y}]}{4\ D^2\ (e^Y-1)}+\mathcal{O}\left(\frac{1}{D}\right)^3 \tag{B.58}$$

Now, the expression for $H_2(y)$

$$H_2(y) = F(y)^2 - 2y^2\left[-\mathcal{Q}(D)\frac{\text{Log}[1-y^{-(D-1)}]}{D-1} - [\text{Term1}+\text{Term2}+\text{Term3}]\right] \tag{B.59}$$

Putting (B.24), (B.58), (B.56), (B.53) and (B.51) we get the final expression

$$\begin{aligned}H_2\left(1+\frac{Y}{D}\right) = 1 - \frac{1}{D^2}\Bigg(&\frac{\pi^2}{3}\left(e^Y-1\right) - 4\ Y\ \text{Log}\left[1-e^{-Y}\right] + \left(e^Y-1\right)\left(\ \text{Log}\left[1-e^{-Y}\right]\ \right)^2 \\ &+ 2\ \left(e^Y-1\right)\text{Log}\left[1-e^{-Y}\right]\ \text{Log}\left[\frac{1}{1-e^Y}\right] + 2\left(e^Y+1\right)\text{PolyLog}[2,\ e^{-Y}] \\ &- 2\left(e^Y-1\right)\text{PolyLog}\left[2,\ \frac{e^Y}{e^Y-1}\right]\Bigg) + \mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{B.60}$$

B.2.2 Outside membrane region

Here we shall show that fluid metric vanishes to any orders in $\frac{1}{D}$ expansion outside the membrane region. To show this we will use the following equation

$$(1+\zeta)^{-(\alpha\ D-\beta)} = e^{-(\alpha\ D-\beta)\ \text{Log}[1+\zeta]} \tag{B.61}$$

Now, if ζ is some $\mathcal{O}(1)$ number then the right hand side is non perturbative in $\frac{1}{D}$ expansion. Now to show how the fluid metric behaves outside horizon we need to calculate $F(y), K_1(y), H_1(y)$ and $H_2(y)$ as the terms containing $L(y)$ and $K_2(y)$ are already multiplied by $y^{-(D-3)}$, hence non perturbative in $\frac{1}{D}$ expansion.

B.2.2.1 $F(y)$:

$$F(y) = y\int_y^\infty \frac{dx}{x}\left[\frac{x^{D-2}-1}{x^{D-1}-1}\right] \tag{B.62}$$

For, $y = 1+\zeta$, where ζ is some $\mathcal{O}(1)$ number, we can write the above integration (B.23) as

$$
\begin{aligned}
&F(1+\zeta)\\
&= 1+\sum_{m=1}^{\infty}\left[\left(\frac{1}{(D-1)m+1}\right)(1+\zeta)^{-(D-1)m} - \left(\frac{1}{(D-1)m}\right)(1+\zeta)^{-(D-1)m+1}\right]\\
&= 1+\sum_{m=1}^{\infty}\left[\left(\frac{1}{(D-1)m+1}\right)e^{-(m\ D-m)\ \mathrm{Log}[1+\zeta]} - \left(\frac{1}{(D-1)m}\right)e^{-(m\ D-m-1)\ \mathrm{Log}[1+\zeta]}\right]\\
&= 1+\text{terms non-perturbative in } \frac{1}{D}
\end{aligned}
\tag{B.63}
$$

In the last line we have used (B.61).

B.2.2.2 $K_1(y)$:

$$
K_1(y) = \ 2y^2\int_y^{\infty}\frac{dx}{x^2}\int_x^{\infty}\left(\frac{dz}{z^2}\right)\left(z\ F'(z)-F(z)\right)^2 \tag{B.64}
$$

From (B.32), for $y = 1+\zeta$, where ζ is some $\mathcal{O}(1)$ number

$$
\begin{aligned}
&K_1(1+\zeta)\\
&= 1+4\sum_{m=1}^{\infty}\left[\frac{1}{[(D-1)m+1][(D-1)m+2]} - \frac{(1+\zeta)}{[(D-1)m+1][m(D-1)]}\right](1+\zeta)^{-m(D-1)}\\
&\quad +2\sum_{m=1}^{\infty}m\Bigg[\frac{1}{[(D-1)m+D][D(m+1)-(m-1)]} - \frac{2\ (1+\zeta)}{[(D-1)(m+1)][D(m+1)-m]}\\
&\qquad + \frac{(1+\zeta)^2}{[(D-1)(m+1)][D(m+1)-(m+2)]}\Bigg](1+\zeta)^{-(D-1)(m+1)}
\end{aligned}
\tag{B.65}
$$

Now, using (B.61), we get

$$
K_1(1+\zeta) = 1+\text{terms non-perturbative in } \frac{1}{D} \tag{B.66}
$$

B.2.2.3 $H_1(y)$:

$$
H_1(y) = 2y^2\int_y^{\infty}\frac{dx}{x}\left[\frac{x^{D-3}-1}{x^{D-1}-1}\right] \tag{B.67}
$$

From (B.27), for $y = 1 + \zeta$, where ζ is some $\mathcal{O}(1)$ number

$$H_1(1+\zeta) = 1 + 2\sum_{m=1}^{\infty}\left[\frac{(1+\zeta)^{-m(D-1)}}{m(D-1)+2}\right] - 2\sum_{m=1}^{\infty}\left[\frac{(1+\zeta)^{-m(D-1)+2}}{m(D-1)}\right] \tag{B.68}$$

Now, using (B.61) we can very easily see that

$$H_1(1+\zeta) = 1 + \text{terms non-perturbative in } \frac{1}{D} \tag{B.69}$$

B.2.2.4 $H_2(y)$:

$$H_2(y) = F(y)^2 - 2\ y^2\int_y^{\infty}\frac{dx}{x(x^{D-1}-1)}\int_1^x\frac{dz}{z}\left[\frac{z^{D-3}-1}{z^{D-1}-1}\right] \tag{B.70}$$

From(B.59), for $y = 1 + \zeta$, where ζ is some $\mathcal{O}(1)$ number

$$H_2(y) = F(y)^2 - 2y^2\ \mathcal{Q}(D)\left[\sum_{m=1}^{\infty}\frac{y^{-m(D-1)}}{m(D-1)}\right] + 2\ y^2\,[\text{Term1} + \text{Term2} + \text{Term3}] \tag{B.71}$$

where,

$$\begin{aligned}
\mathcal{Q}(D) &= -\left(\frac{1}{D-1}\right)\left[\text{EulerGamma} + \text{PolyGamma}\left(0, \frac{2}{D-1}\right)\right]\\
\text{Term1} &= \left[\sum_{m=1}^{\infty}\frac{y^{-m(D-1)}}{m(D-1)}\right]\left[\frac{y^{-2}}{2} - \frac{y^{-(D-1)}}{D-1}\right]\\
&\quad + \left[\sum_{m=1}^{\infty}\frac{y^{-m(D-1)}}{m(D-1)}\right]\left(\frac{y^{-(D-1)}}{D-1} + y^{-(D+1)}\sum_{n=0}^{\infty}\frac{y^{-n(D-1)}}{n(D-1)+(D+1)} - \sum_{n=1}^{\infty}\frac{y^{-n(D-1)}}{n(D-1)}\right)
\end{aligned} \tag{B.72}$$

$$\begin{aligned}
\text{Term2} &= -\frac{1}{2\ (D-1)^2}\left(y^{D-3}-1\right)\left(\sum_{m,n=1}^{\infty}\frac{y^{-(m+n)(D-1)}}{m\ n}\right)\\
\text{Term3} &= -\frac{D-3}{(D-1)^2}\sum_{m=2}^{\infty}\left(\frac{y^{-m(D-1)+D-3}}{D(m-1)-m+3}\right)\frac{\text{HarmonicNumber}[m-1]}{m}
\end{aligned} \tag{B.73}$$

Now using (B.61) we can easily see that

$$H_2(1+\zeta) = 1 + \text{terms non-perturbative in } \frac{1}{D} \tag{B.74}$$

Using the above results in (3.45) we see that $\mathcal{G}^{\rm rest}_{\mu\nu}$ vanishes for all order in $\frac{1}{D}$ expansion outside the membrane region.

B.3 Relation between Horizon $\rho_H(y)$ in $Y^A(\equiv \{\rho, y^\mu\})$ coordinates and $H(x)$ in $X^A(\equiv \{r, x^\mu\})$ coordinates:

In this appendix we shall determine the relation between the position of the horizon($\rho_H(y^\mu)$) in Y^A - coordinates with the position of the horizon ($H(x^\mu)$) in X^A - coordinates. ($X^A \equiv \{r, x^\mu\}$) and ($Y^A \equiv \{\rho, y^\mu\}$) are related through the following coordinate transformation

$$\rho = r - \left(\frac{\Theta(x)}{D-2}\right), \quad y^\mu = x^\mu + \frac{u^\mu(x)}{r - \frac{\Theta(x)}{D-2}} \tag{B.75}$$

The inverse of the above coordinate transformation is

$$\begin{aligned} r &= \rho + \left(\frac{\Theta(y)}{D-2}\right) - \frac{1}{\rho}(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial)^3 \\ x^\mu &= y^\mu - \frac{u^\mu(y)}{\rho} + \frac{a^\mu(y)}{\rho^2} + \mathcal{O}(\partial)^2 \end{aligned} \tag{B.76}$$

Now, the equation of the horizon is

$$\begin{aligned} & r = H(x) \\ \Rightarrow\ & \rho = H\left(y^\mu - \frac{u^\mu(y)}{\rho} + \frac{a^\mu(y)}{\rho^2}\right) - \left(\frac{\Theta(y)}{D-2}\right) + \frac{1}{\rho}(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial)^3 \\ & = H(y) - \frac{(u\cdot\partial)H(y)}{\rho} + \frac{(a\cdot\partial)H(y)}{\rho^2} + \frac{u^\mu u^\nu \partial_\mu\partial_\nu H(y)}{2\rho^2} - \left(\frac{\Theta(y)}{D-2}\right) \\ & \quad + \frac{1}{\rho}(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial)^3 \end{aligned} \tag{B.77}$$

Using (3.15) we can write the equation of the horizon as

$$\begin{aligned} \rho = H(y) &- \left(\frac{\Theta(y)}{D-2}\right) - \frac{(u\cdot\partial)r_H(y)}{\rho} + \frac{(a\cdot\partial)r_H(y)}{\rho^2} + \frac{u^\mu u^\nu\partial_\mu\partial_\nu r_H(y)}{2\rho^2} \\ &+ \frac{1}{\rho}(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial)^3 \end{aligned} \tag{B.78}$$

Up to $\mathcal{O}(\partial)^2$ the equation of the horizon is

$$\begin{aligned}\rho &= H(y) - \left(\frac{\Theta(y)}{D-2}\right) - \frac{(u\cdot\partial)r_H(y)}{r_H(y)} + \mathcal{O}(\partial)^2 \\ &= r_H(y) + \mathcal{O}(\partial)^2\end{aligned} \tag{B.79}$$

Where, In the last line we have used (B.88) and (3.15).
Using, (B.79) in (B.78) we get,

$$\begin{aligned}\rho = H(y) &- \left(\frac{\Theta(y)}{D-2}\right) - \frac{(u\cdot\partial)r_H(y)}{r_H(y)} + \frac{(a\cdot\partial)r_H(y)}{r_H^2(y)} + \frac{u^\mu u^\nu \partial_\mu\partial_\nu r_H(y)}{2\ r_H^2(y)} \\ &+ \frac{1}{r_H(y)}(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial)^3\end{aligned} \tag{B.80}$$

After some simplifications the above expression becomes

$$\rho = H(y) + \frac{1}{2\ r_H}(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \frac{1}{2\ r_H}\left(\frac{\Theta}{D-2}\right)^2 - \frac{a^2}{2\ r_H} - \frac{2}{r_H}\frac{\sigma^2}{(D-1)(D-2)} + \mathcal{O}(\partial)^3 \tag{B.81}$$

In $(Y^A \equiv \{\rho, y^\mu\})$ coordinate the equation of the horizon is

$$\rho = \rho_H(y) \tag{B.82}$$

From (B.81) and (B.82) we get

$$\rho_H(y) = H(y) + \frac{1}{2\ r_H}(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \frac{1}{2\ r_H}\left(\frac{\Theta}{D-2}\right)^2 - \frac{a^2}{2\ r_H} - \frac{2}{r_H}\frac{\sigma^2}{(D-1)(D-2)} + \mathcal{O}(\partial)^3 \tag{B.83}$$

We can express $H(y)$ in terms of $H(x)$

$$\begin{aligned}H(y) = H(x) &- \frac{r_H(x)}{r}\left(\frac{\Theta(x)}{D-2}\right) - \frac{r_H}{r^2}\left(\frac{\Theta(x)}{D-2}\right)^2 + \frac{2}{r}\left(\frac{\sigma^2}{(D-1)(D-2)}\right) \\ &+ \frac{r_H}{2\ r^2}\left[\left(\frac{\Theta}{D-2}\right)^2 + a^2 - (u\cdot\partial)\frac{\Theta}{D-2}\right] + \mathcal{O}(\partial)^3\end{aligned} \tag{B.84}$$

Substituting (B.84) in (B.83) we get the final expression

$$\begin{aligned}\rho_H(y) = H(x) - \frac{r_H(x)}{r}\left(\frac{\Theta(x)}{D-2}\right) - \frac{2}{r_H}\left(1-\frac{r_H}{r}\right)\frac{\mathfrak{s}_4}{(D-1)(D-2)} \\ + \frac{1}{2\, r_H}\left(1-\frac{r_H^2}{r^2}\right)[\mathfrak{s}_1 - \mathfrak{s}_2 + \mathfrak{s}_5] + \mathcal{O}(\partial)^3\end{aligned} \tag{B.85}$$

B.4 Identities

In this appendix we shall give the derivation of the identities we have used in subsection 3.5.

Identity 1:

$$\begin{aligned}\left(\frac{\partial_\mu r_H}{r_H}\right)^2 &= \left[-a_\mu + u_\mu\left(\frac{\Theta}{D-2}\right)\right]\left[-a^\mu + u^\mu\left(\frac{\Theta}{D-2}\right)\right] + \mathcal{O}(\partial)^3 \\ &= a^2 - \left(\frac{\Theta}{D-2}\right)^2 + \mathcal{O}(\partial)^3 \\ &= \mathfrak{s}_2 - \mathfrak{s}_1 + \mathcal{O}(\partial)^3\end{aligned} \tag{B.86}$$

Identity 2:

$$\begin{aligned}\left(\frac{\partial^\mu\partial_\mu r_H}{r_H}\right) &= \frac{1}{r_H}\partial^\mu\left[-r_H\ a_\mu + r_H\ u_\mu\left(\frac{\Theta}{D-2}\right)\right] \\ &= -(\partial\cdot a) - \frac{(a\cdot\partial)r_H}{r_H} + \frac{\Theta^2}{D-2} + \left(\frac{\Theta}{D-2}\right)\left(\frac{(u\cdot\partial)r_H}{r_H}\right) + (u\cdot\partial)\left(\frac{\Theta}{D-2}\right) \\ &= -\left(\sigma^2 - \omega^2 + \frac{\Theta^2}{D-2} + (u\cdot\partial)\Theta\right) + a^2 + \frac{\Theta^2}{D-2} - \left(\frac{\Theta}{D-2}\right)^2 \\ &\qquad + (u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial)^3 \\ &= \omega^2 - \sigma^2 - (D-3)(u\cdot\partial)\left(\frac{\Theta}{D-2}\right) + a^2 - \left(\frac{\Theta}{D-2}\right)^2 + \mathcal{O}(\partial)^3 \\ &= \mathfrak{s}_3 - \mathfrak{s}_4 - (D-3)\mathfrak{s}_5 + \mathfrak{s}_2 - \mathfrak{s}_1 + \mathcal{O}(\partial)^3\end{aligned} \tag{B.87}$$

Identity 3:

In this identity we shall just quote the fluid equation upto second subleading order. The derivation is quite straight forward. We have to calculate divergence of fluid stress tensor (3.17) and have to project it along u_μ direction and perpendicular to u^μ direction.

$$\begin{aligned} & u_\nu \partial_\mu (T^{\mu\nu}) = \mathcal{O}\left(\partial\right)^3 \\ \Rightarrow \quad & \frac{(u\cdot\partial) r_{\rm H}}{r_H} = -\frac{\Theta}{D-2} + \left(\frac{2}{r_{\rm H}}\right)\frac{\sigma^2}{(D-1)(D-2)} + \mathcal{O}\left(\partial\right)^3 \end{aligned} \tag{B.88}$$

$$\begin{aligned} & \mathcal{P}^\alpha_\nu \partial_\mu (T^{\mu\nu}) = \mathcal{O}\left(\partial\right)^3 \\ \Rightarrow \quad & \mathcal{P}^\alpha_\nu \left(\frac{\partial^\nu r_H}{r_H}\right) = \mathcal{P}^\alpha_\nu \left[-a^\nu - \frac{2}{r_H}\left(\frac{D-2}{D-1}\right)(a_\mu \sigma^{\mu\nu}) + \frac{2}{r_H}\left(\frac{1}{D-1}\right)\partial_\mu \sigma^{\mu\nu}\right] + \mathcal{O}\left(\partial\right)^3 \end{aligned} \tag{B.89}$$

B.5 Notations

Table B.1: Notations

Fluid velocity	u_μ
Membrane velocity	U_A
Boundary metric	$\eta_{\mu\nu}$
Background metric in $(Y^A = \{\rho, y^\mu\})$	$\bar{G}_{AB}$
Background metric in $(X^A = \{r, x^\mu\})$	$\bar{\mathcal{G}}_{AB}$
Full metric in $(X^A = \{r, x^\mu\})$	$\mathcal{G}_{AB}$
Background metric in arbitrary coordinates	$\bar{\mathcal{W}}_{AB}$
Full metric in arbitrary coordinates	$\mathcal{W}_{AB}$
Projector perpendicular to n_A	$\Pi_{AB} = \bar{\mathcal{W}}_{AB} - n_A\, n_B$
Projector perpendicular to both n_A and U_A	$P_{AB} = \bar{\mathcal{W}}_{AB} - n_A\, n_B + U_A U_B$
Projector perpendicular to u_μ	$\mathcal{P}_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$
Covariant derivative w.r.t background	∇_A
Covariant derivative w.r.t $\mathcal{G}_{AB}$	$\bar{\nabla}_A$
Covariant derivative w.r.t. induced metric on the membrane	$\tilde{\nabla}_\mu$
Covariant derivative projected along the membrane	$\hat{\nabla}_A$ See equation (3.26) for detail definition

Appendix C

Appendices for chapter 4

C.1 The Large-D limit of the integrations appearing in hydrodynamic metric

In this section we shall try to do the integrations appearing in the fluid metric and we shall do those integrals in $\frac{1}{D}$ expansion.

C.1.1 Analysis of the integral in the function F_1

The integral appearing in F_1 is given by

$$\begin{aligned} F_3(\rho, M) &= \int_\rho^\infty dp \frac{1}{\left(1 + \frac{1}{4}\frac{3(D-3)}{2(D-2)}\frac{Q_1^2}{p^{2(D-2)}} - \frac{M}{p^{D-1}}\right)^2}\left(\frac{1}{p^{2(D-1)}} - \frac{c_1}{p^{2D-3}}\right) \\ &= \int_\rho^\infty dp \frac{1}{\left(1 + \frac{(M-1)}{p^{2(D-2)}} - \frac{M}{p^{D-1}}\right)^2}\left(\frac{1}{p^{2(D-1)}} - \frac{c_1}{p^{2D-3}}\right) \end{aligned} \tag{C.1}$$

We would like to evaluate this integral for large D. Now following the appendix A of [1] let's first evaluate this integral for $\rho \geq 2$. For large D we can expand this integrand in the following way

$$\begin{aligned} &\frac{1}{\left(1 + \frac{(M-1)}{p^{2(D-2)}} - \frac{M}{p^{D-1}}\right)^2}\left(\frac{1}{p^{2(D-1)}} - \frac{c_1}{p^{2D-3}}\right) \\ &= \left(1 + \frac{(M-1)}{p^{2(D-2)}} - \frac{M}{p^{D-1}}\right)^{-2}\left(\frac{1}{p^{2(D-1)}} - \frac{c_1}{p^{2D-3}}\right) \\ &= \sum_{k=0} (-1)^k (k+1)\left(\frac{(M-1)}{p^{2(D-2)}} - \frac{M}{p^{D-1}}\right)^k \left(\frac{1}{p^{2(D-1)}} - \frac{c_1}{p^{2D-3}}\right) \end{aligned} \tag{C.2}$$

Now one can do the integration (we have done it using *Mathematica*, the answer is a bit long) and let's call it $k(\rho)$. However, this expansion is not possible in the membrane region (when $\rho - 1 \sim \mathcal{O}\left(\frac{1}{D}\right)$) and naively $k(\rho)$ is not the answer to the integral. But now consider the function $\widetilde{k}(\rho) = F_3(\rho) - k(\rho)$. This $\widetilde{k}(\rho)$ vanishes for all $\rho \geq 2$ and also by construction it is a smooth function at $\rho = 2$. Hence $\widetilde{k}(\rho)$ must vanish for every ρ and $k(\rho)$ is the answer. Now we would expand it with $\rho = 1 + \frac{Y}{D}$ and finally we will get

$$F_3(1 + \frac{Y}{D}, M) = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.3}$$

C.1.2 Analysis of the integral in the function F_2

The integral appearing in F_2 is given by

$$F_2(\rho, M) = \int_\rho^\infty \frac{p^D\left(p^D - p^2\right)}{p^2\left(p^{2D} - Mp^{D+1} + (M-1)p^4\right)} dp \tag{C.4}$$

This integration is difficult to do other than $M = 1$ and hence at first we expand the integrand around $M = 1$ by putting $M = 1 + \delta M$. Then the integral becomes

$$\begin{aligned}\int_\rho^\infty \frac{p^D\left(p^D - p^2\right)}{p^2\left(p^{2D} - Mp^{D+1} + (M-1)p^4\right)} dp &= \int_\rho^\infty dp \frac{p^{D-2} - 1}{p\left(p^{D-1} - 1\right)} + \int_\rho^\infty dp \frac{\left(p^D - p^3\right)\left(p^D - p^2\right)}{p^{D+1}\left(p^D - p\right)^2} \delta M \\ &+ \int_\rho^\infty dp \frac{\left(p^D - p^3\right)^2\left(p^D - p^2\right)}{p^{2D}\left(p^D - p\right)^3} \delta M^2 + \cdots \\ &= \widetilde{F}_0(\rho) + \widetilde{F}_1(\rho)\delta M + \widetilde{F}_2(\rho)\delta M^2 + \cdots \quad \text{(say)}\end{aligned} \tag{C.5}$$

Now we will integrate each of the functions $\widetilde{F}$ and then expand in $\frac{1}{D}$ by setting $\rho = 1 + \frac{Y}{D}$. Then we will get for all $\widetilde{F}_k(\rho)$ with $k \geq 1$

$$\widetilde{F}_k\left(1 + \frac{Y}{D}\right) = e^{-kY}\left(\frac{1}{kD}\right) + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.6}$$

Now if we add all these $\widetilde{F}$, we have

$$F_2\left(1 + \frac{Y}{D}, M\right) - \widetilde{F}_0\left(1 + \frac{Y}{D}\right) = -\frac{1}{D}\log[1 - \widetilde{Q}^2 e^{-Y}] + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.7}$$

where we have used the fact that $\widetilde{Q}^2 = (M-1) + \mathcal{O}\left(\frac{1}{D}\right)$.

Now following the appendix A of [1] we have

$$\hat{F}\left(\rho\right) = \rho\ \widetilde{F}_0(\rho) = 1 + \sum_{m=1}\left[\left(\frac{1}{(D-1)m+1}\right)\rho^{-m(D-1)} - \left(\frac{1}{(D-1)m}\right)\rho^{-m(D-1)+1}\right] \tag{C.8}$$

Now if we expand it in $\frac{1}{D}$ by setting $\rho = 1 + \frac{Y}{D}$ we will get

$$\hat{F}\left(1 + \frac{Y}{D}\right) = 1 + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.9}$$

Hence finally expanding in $\frac{1}{D}$ we have (remember $\rho = \frac{r}{r_H}$)

$$\frac{r}{r_H}F_2\left(\rho, M\right) \Rightarrow 1 - \frac{1}{D}\log[1 - \widetilde{Q}^2 e^{-Y}] + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.10}$$

C.2 The inverse of the background metric and christoffel symbols w.r.t background metric

We can find out the inverse of the background metric order by order in derivative expansion. These expressions are given by

$$\begin{aligned}
\bar{\mathcal{G}}^{rr} &= r^2 - 2r\frac{\Theta}{D-2} + \mathcal{O}\left(\partial^2\right)\\
\bar{\mathcal{G}}^{\mu r} &= u^\mu - \frac{a^\mu}{r} + \mathcal{O}\left(\partial^2\right)\\
\bar{\mathcal{G}}^{\mu\nu} &= \frac{1}{r^2}\mathcal{P}^{\mu\nu} - \frac{2}{r^3}\sigma^{\mu\nu} + \mathcal{O}\left(\partial^2\right)
\end{aligned} \tag{C.11}$$

Having these expressions for the background metric we have the christoffel symbols for the background metric as

$$
\begin{aligned}
&\bar{\Gamma}^r_{rr} = 0 \\
&\bar{\Gamma}^\alpha_{rr} = 0 \\
&\bar{\Gamma}^r_{\mu r} = r\ u_\mu - \frac{\Theta}{D-2} u_\mu \\
&\bar{\Gamma}^\alpha_{\mu r} = \frac{1}{r}\mathcal{P}^\alpha_\mu - \frac{1}{2\ r^2}\left(u_\mu a^\alpha - a_\mu u^\alpha\right) + \frac{1}{2\ r^2}\left(\partial^\alpha u_\mu - \partial_\mu u^\alpha\right) - \frac{1}{r^2}\sigma^\alpha_\mu \\
&\bar{\Gamma}^r_{\alpha\mu} = -r^3\left(\mathcal{P}_{\alpha\mu} - u_\alpha u_\mu\right) + r^2\left(u_\alpha a_\mu + u_\mu a_\alpha\right) - 2\ r^2\sigma_{\alpha\mu} + r^2\frac{\Theta}{D-2}\mathcal{P}_{\alpha\mu} - 3r^2\frac{\Theta}{D-2}u_\alpha u_\mu \\
&\bar{\Gamma}^\beta_{\alpha\mu} = u^\beta\left(-r\left(\mathcal{P}_{\alpha\mu} - u_\alpha u_\mu\right) + \left(u_\alpha a_\mu + u_\mu a_\alpha\right) - 2\sigma_{\alpha\mu} - \frac{\Theta}{D-2}u_\alpha u_\mu - \frac{\Theta}{D-2}\mathcal{P}_{\alpha\mu}\right) + a^\beta\left(\mathcal{P}_{\alpha\mu} - u_\alpha u_\mu\right)
\end{aligned}
\tag{C.12}
$$

C.3 Notation

In this section we write down the various symbols we have used in 4.

$\bar{G}_{AB}$	:	Background metric in $Y^A \equiv \{\rho, y^\mu\}$ coordinates
$\bar{\mathcal{G}}_{AB}$	:	Background metric in $X^A \equiv \{r, x^\mu\}$ coordinates
$\mathcal{G}_{AB}$	:	Full metric in $X^A \equiv \{r, x^\mu\}$ coordinates
$\bar{W}_{AB}$	:	Background metric in arbitrary coordinates
W_{AB}	:	Full metric in arbitrary coordinates
$\eta_{\mu\nu}$	:	The boundary metric
u_μ	:	Fluid velocity
U_A	:	Membrane velocity
Q	:	Charge field in hydrodynamic metric
$\widetilde{Q}$	:	Charge field in large-D metric
Π_{AB}	:	Projector perpendicular to n_A
P_{AB}	:	Projector perpendicular to both n_A and U_A
$\mathcal{P}_{\mu\nu}$	:	Projector perpendicular to u_μ
∇_A	:	Covarient derivative w.r.t background metric
$\hat{\nabla}$	:	Covarient derivative projected along the membrane

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