ENUMERATIVE GEOMETRY OF CURVES IN A MOVING FAMILY OF SURFACES

By

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A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

December, 2021

Homi Bhaba National Institute

Recommendations of the Viva Voce Committee

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Date: December 13, 2021

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List of publications arising from the thesis

Journal

1. "Elliptic Gromov-Witten Invariants of Del-Pezzo Surfaces", Chitrabhanu Chaudhuri and Nilkantha Das, *J. Gökova Geom. Topol. GGT* , 2019, 13, 1–14.

2. "Counting planar curves in $\mathbb{P}^3$ with degenerate singularities", Nilkantha Das and Ritwik Mukherjee, *Bull. Sci. Math.*, 2021, 173, Paper No. 103065, 64 pp.

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Dedicated to my family

ACKNOWLEDGEMENTS

I would like to thank my advisor, Prof. Ritwik Mukherjee, for his encouragement and great support throughout the process of writing my thesis. I have been fortunate to have an advisor who gave me the freedom to explore on my own and at the same time the guidance to recover when my steps faltered. I am deeply indebted to him for his spirit of tireless group discussions for hours, and his involvement with the subject always inspires me. I am also grateful for an opportunity to work under him since the third year of my doctoral program. I would also like to thank my former advisor Dr. Amit Tripathi who gave me a deep insight into the subject of algebraic geometry till the second year of my doctoral program. Having different perspectives on math, I benefited tremendously from both of them. I am indebted to my co-author Dr. Chitrabhanu Chaudhuri for all his help, especially in the final year of my doctoral program. I have learned a lot from him. A special thanks to him for a careful reading of the thesis and several suggestions. I thank my doctoral committee members for their support and encouragement. Having vibrant research surrounding, I benefited a lot from NISER. I thank Dr. Senthil Kumar, Dr. Jaban Meher, Dr. Sanjay Parui, Dr. Sutanu Roy for several academic help and discussions. I am also very grateful to HBNI and DAE, Govt. of India for financing me to carry on this thesis work.

Thanks are also due to friends and colleagues for making my doctoral journey a memorable and enjoyable one. In particular, I thank Abhrajyoti Sen, Anantadulal Paul, Atibur Rahaman, Bappa Bisai, Buddhadev Hajra, Diptesh Saha, Gorekh Prasad Sena, Milan Pramanik, Mrityunjoy Charan, Puspendu Pradhan, Rajeev Ranjan Mohanta, Tattwamasi Amrutam for many useful discussions and help. A special thanks to Anantadulal Paul for several fruitful group discussions. I am also thankful to my friends Ajoy, Debajyoti, Kalachand, Pinaki, Swarup for their help and support.

I am deeply indebted to my parents for their generous help and support throughout my whole life. A special thanks to Sweta, my better half and the love of my life, for caring and

motivating me in my doctoral studies. Her keen interest and passionate encouragement were always a great help.

Contents

Summary

This thesis is based on the variants of the following two classical enumerative questions of $\mathbb{CP}^2$.

Question 0.0.1. *How many degree* d *and* δ *nodal curves are there in* $\mathbb{CP}^2$ *that pass through* $\frac{d(d+3)}{2} - \delta$ *points of* $\mathbb{CP}^2$ *in general position?*

Question 0.0.2. *How many degree* d *curves of arithmetic genus* g *are there in* $\mathbb{CP}^2$ *that pass through* $3d+g-1$ *points of* $\mathbb{CP}^2$ *in general position?*

Thus the thesis can be divided into two parts.

In the first part of the thesis, we study enumerative geometry of planar curves in $\mathbb{CP}^3$ with one or more singularities such that the sum of the codimensions of the singularities is at most 4. A curve in $\mathbb{CP}^3$ is said to be *planar* if its image lies inside some $\mathbb{CP}^2$ in $\mathbb{CP}^3$. Enumerative geometry of nodal planar curves is a fiber bundle version of Question 0.0.1. We consider enumerative geometry of curves with more degenerate singularities in a moving family of $\mathbb{CP}^2$. The set of all $\mathbb{CP}^2$ in $\mathbb{CP}^3$ is the Grassmannian $\mathbb{G}(3,4)$, the space of 3 planes in $\mathbb{C}^4$. The parameter space of planar curves is a fiber bundle over $\mathbb{G}(3,4)$. The singularities as well as the other constraints can be expressed as a section, transverse to the zero section, of some appropriate vector bundle. The degree of the Euler class of this bundle produces an 'expected' answer to the enumerative question that we have started with. To get the actual answer, we first need to find the degenerate loci of the Euler class, and then, subtract the degenerate contributions from the total degree. We use smooth deformation theory to debug the degenerate locus of the Euler class.

In the second part of the thesis, we study enumerative geometry of curves of a fixed degree, having arithmetic genus 0 and 1, respectively, of special type of surfaces which

are known as *del Pezzo* surface. A smooth projective algebraic surface is said to be a del Pezzo surface if the anti-canonical divisor is ample. A classification theorem of del Pezzo surfaces shows that a del Pezzo surface is either isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, or isomorphic to blow-up of $\mathbb{CP}^2$ up to 8 general points. We study the intersection theory of $\overline{M}_{0,n}(X, \beta)$, the moduli space of stable maps of a del Pezzo surface X, and define some numerical invariants which are known as *Gromov-Witten invariants*. We compute rational $(g = 0)$, and elliptic $(g = 1)$ Gromov-Witten invariants of X. Gromov-Witten invariants do not necessarily have enumerative significance always. However, Ravi Vakil showed that Gromov-Witten invariants of del Pezzo surfaces are enumerative. This establishes a relation between the study of the Gromov-Witten invariants and enumerative geometry of del Pezzo surfaces.

Notations and conventions

$\mathbb{Z}$	The ring of integers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{CP}^n$ or $\mathbb{P}^n$	The complex projective n-space
X	Complex projective variety
$H^*(X, R)$	The cohomology ring of X with coefficients in R graded by complex degree
$H^i(X, R)$	i-th cohomology group of X
$H_i(X, R)$	i-th homology group of X
$A_*(X)$	The Chow group of X
$A^*(X)$	The Chow ring of X
$A_i(X)$	i-dimensional cycle modulo rational equivalence
$A^i(X)$	i-codimensional cycle modulo rational equivalence
K_X	Canonical divisor of X
$\mathcal{O}_X(1)$	Twisting sheaf of Serre on the projective space X
$h^i(X, \mathcal{F})$	Dimension of $H^i(X, \mathcal{F})$

Throughout the thesis, we will assume the following convensions.

All the varieties are assumed over $\mathbb{C}$ unless otherwise specified. All the manifolds are complex manifolds, and their dimensions are complex dimensions. The cohomology ring of a complex manifold is graded by complex degree. Rings are assumed to be commutative ring with unity 1.

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Chapter 1

Introduction

1.1 History and motivation

Intersection theory is at the heart of the subject algebraic geometry that studies the geometry of the zero loci of a bunch of polynomial equations over some fixed field. From the very beginnings of the subject, the fact that the number of solutions to a system of polynomial equations is, in many circumstances, constant as we vary the coefficients of those polynomials has fascinated algebraic geometers. At the outset of the 19th century, it was to extend this "preservation of number" that algebraic geometers made two important choices: to work over the field of complex numbers rather than the real numbers, and to work in projective space rather than affine space.

Despite a lack of proper and systematic treatments, 19th-century enumerative geometry rose to impressive heights: for example, the works of Schubert [39], and Zeuthen [44]. Schubert calculated the number of twisted cubics tangent to 12 quadrics — and got the correct answer (5,819,539,783,680).

At the outset of the 20th-century, Hilbert made finding rigorous foundations for Schubert calculus one of his celebrated problems. It took a long period in the search of a proper definition of multiplicity, including the work of van der Waerden, Zariski, Samuel, Weil, and Serre in the subject of commutative algebra. This progress culminated, towards the end of the century, in the work of Fulton and MacPherson, and then in Fulton's landmark book Intersection Theory [16]. In the last decade of the century, a new insight has been brought into the subject from physics. From then, enumerative geometry has made significant progress including the extension of intersection theory from schemes to stacks, virtual fundamental cycles, Gromov–Witten theory, and quantum intersection rings.

1.2 Classical enumerative geometric questions

A general question in enumerative geometry would be about the number of certain types of sub-objects in some ambient space that satisfy certain constraints to ensure that the number of sub-objects is at least finite. In this thesis, we deal with only curve counting problems. Enumerative geometry of curves in $\mathbb{CP}^2$ is very deep and classical.

Question 1.2.1. *How many degree d and δ nodal curves are there in $\mathbb{CP}^2$ that pass through $\dfrac{d(d+3)}{2} - \delta$ points of $\mathbb{CP}^2$ in general position?*

Question 1.2.2. *How many degree d curves of arithmetic genus g are there in $\mathbb{CP}^2$ that pass through $3d + g - 1$ points of $\mathbb{CP}^2$ in general position?*

We discuss each of the questions as we proceed further. Intuitively, a curve in $\mathbb{CP}^2$ has a node at a point p if it locally looks like a pair of intersecting lines, and p is the point of intersection. A set of n-points in some variety X is said to be in *general position* with respect to some statement if the statement is true for a dense open subset of X^n.

1.3 General approach

A general approach to deal with enumerative problems about curves is the following.

First, find a suitable space that parametrizes the curves (of certain types) mentioned in the enumerative problem. These spaces are known as *parameter spaces* or *moduli spaces* depending upon the context. It is natural to desire that the resulting space belongs to that category in which the ambient space belongs. For example, if the ambient space is an algebraic variety, we seek the parameter (or moduli resp.) to be an algebraic variety. If the ambient space is an object in the category of smooth (or complex) manifold, we want the parameter space to be an object in that category. The word moduli or parameter space is employed to distinguish between the problems which focus on purely intrinsic data and those which involve, to a greater or lesser degree, extrinsic data. We reserve the terminology parameter space for the problems of the latter type and moduli for the former one. In that sense, for any $g \geq 0$, $\mathcal{M}_g$, the space of isomorphism classes of smooth genus g curves is a moduli space, whereas the space of degree d curves in $\mathbb{CP}^n$ having arithmetic genus g is a parameter space. A variant of the former one mentioned above will be studied extensively in Chapter 3, and the latter in Chapter 2. For more details about the construction of various parameter (or moduli resp.) space of curves, we refer to [20].

Second, find an appropriate intersection ring of the parameter (or moduli resp.) space that parametrizes the space of curves. One motivation of the intersection product is the Theorem of Bézout. It states that if a degree d curve and a degree e curve in $\mathbb{CP}^2$ do not have a common component, they intersect in de many points. This statement consists of two important facts about defining intersection product. First of all, the final number de does not depend on the choice of the curves (only their degrees are enough), and secondly, the number de suggests some sort of product. This leads to the definition of the Chow ring, which we will denote by $A^*(X)$ for any smooth projective variety X. However, for an algebraic variety X, the group $A^*(X)$ need not has a ring structure in general. More restrictions on X are needed to define product between two elements. For the discussion about the Chow ring with more details, we refer to [14] and [16]. If we work in the algebraic category, the Chow ring will serve as the intersection ring, and for the category of manifold (smooth or complex), cohomology ring will do the job. However, for an algebraic variety, we have the notion of cohomology as well. In general, cohomology ring and the Chow ring of an algebraic variety are not always the same. It has been seen that the Chow ring is a better fit for the intersection ring in algebraic cases. We will refer to the rings as intersection rings associated with an enumerative problem, and elements of it are referred to as cycles.

Third, express the constraints in terms of cycles of the parameter space. There are ways to express the cycles involving the constraints associated with the enumerative problem. The intersection ring is contravariant in nature. One way is that first consider the cycle in the ambient space X (for example, in Question 1.2.1, the class of point in $\mathbb{CP}^2$), and pull it back to the intersection ring of the parameter (or moduli resp.) space, provided there is some natural morphism from the parameter space to X. Another way is to express the constraints as the sections of some appropriate vector bundle over the parameter space and consider its Euler class which yields a cycle. We will see both the approach while dealing with Question 1.2.1.

Fourth, take the intersection product of the cycles coming from the constraints and evaluate the degree of the intersection product (for the definition of degree of a cycle, see [16]). Note that the total amount of constraints will be such, for which one expects the answer to the enumerative problem is finite. Due to this restriction, the intersection product will always have a component of top codimension (as the intersection ring is graded by codimension), and degree will give some number in this case (at least it is not zero every time).

Fifth, verify that whether the obtained degree gives the correct answer to the enumerative problem we started. This step is one of the most crucial steps. It may happen that some extra pieces of stuff, which we do not want to count, contribute to the degree calculation. In that case, simply the computation of degree is not enough. We need to subtract these extra contributions from the degree to obtain the correct answer to the enumerative problem. For example, let $f(z) = z^2(z-1)$ be a degree 3 polynomial in one variable. We want to count the number of non-zero solutions to $f(z) = 0$. But after homogenizing, we deduce f as a section of the bundle $\mathcal{O}(3) \longrightarrow \mathbb{P}^1$, where $\mathcal{O}(1)$ is the dual of the tautological line bundle over $\mathbb{P}^1$. It can immediately be checked that

$$\deg e(\mathcal{O}(3)) = 3,$$

where $e(\mathcal{O}(3))$ is the first Chern class of the line bundle $\mathcal{O}(3)$. The Euler class computation also include 0 as a solution to $f(z) = 0$. But $z = 0$ is a solution here with multiplicity 2. The required number is

$$3 - 2 \cdot 1 = 1.$$

1.4 Curves with singularities

We now proceed with the setup of Question 1.2.1. A more general question is to enumerate the characteristic number of curves that have more degenerate singularities. To make this precise, let us make the following definition.

Definition 1.4.1. Let $f : \mathbb{P}^2 \longrightarrow \mathcal{O}(d)$ be a holomorphic section. A point $q \in f^{-1}(0)$ is said to have a singularity of type A_k or D_k if there exists an analytic coordinate system $(x, y) : (U, q) \longrightarrow (\mathbb{C}^2, 0)$ such that $f^{-1}(0) \cap U$ is given by

$$A_{k\geq 1} : y^2 + x^{k+1} = 0 \qquad \text{and} \qquad D_{k\geq 4} : y^2 x + x^{k-1} = 0.$$

In more common terminology, q is a *simple node* (or just node) if its singularity type is A_1; a *cusp* if its type is A_2; a *tacnode* if its type is A_3 and an *ordinary triple point* if its type is D_4.

Remark 1.4.2. We frequently use the phrase "a singularity of codimension k". Intuitively, this refers to the number of conditions having that singularity imposes on the space of curves. More precisely, it is the expected codimension of the equisingular strata. Hence, a singularity of type A_k or D_k is a singularity of codimension k.

Let us denote the number of degree d curves in $\mathbb{CP}^2$, that have δ distinct (ordered) nodes, that pass through $\frac{d(d+3)}{2}-\delta$ generic points by $N_d(A_1^\delta)$. Then the answer to Question 1.2.1 is $N_d(A_1^\delta)$. More generally, one can ask what is $N_d(A_1^\delta\mathfrak{X})$, the number of degree d curves in $\mathbb{P}^2$, that have δ distinct (ordered) nodes and one singularity of type $\mathfrak{X}$, that pass through $\frac{d(d+3)}{2}-\delta-c_{\mathfrak{X}}$ generic points, where $c_{\mathfrak{X}}$ is the codimension of the singularity $\mathfrak{X}$?

The question of computing $N_d(A_1^\delta)$ and $N_d(A_1^\delta\mathfrak{X})$ has been studied for a very long time starting with Zeuthen [44] more than a hundred years ago. It has been studied extensively in the last thirty years from various perspectives by numerous mathematicians including amongst others, Z. Ran ([37], [38]), I. Vainsencher ([41]), L. Caporaso and J. Harris ([9]), M. Kazarian ([21]), S. Kleiman and R. Piene ([25]), D. Kerner ([22] and [23]), F. Block ([7]), Y. Tzeng and J. Li ([40], [32]), M. Kool, V. Shende and R. Thomas ([29]), S. Fomin and G. Mikhalkin ([15]), and S. Basu and R. Mukherjee ([1], [2] and [3]).

We mainly follow the idea of Basu-Mukherjee [1] to enumerate curves with singularities. We will give a outline of the procedure by calculating the number $N_d(A_1)$. The space of degree d curves in $\mathbb{CP}^2$ is the space $\mathbb{P}(H^0(\mathbb{CP}^2,\mathcal{O}(d)))$. Then the parameter space will be the set

$$\{([f],p)\in\mathbb{P}(H^0(\mathbb{CP}^2,\mathcal{O}(d)))\times\mathbb{CP}^2|f \text{ has a node at } p\}.$$

That is, we keep track of the nodal point as well. The cohomology ring will serve as the role of intersection ring in this case. Now express the condition that f has a node at p to a section of some appropriate bundle. Morse lemma gives us that f has a node at p if and only if $f(p)=0$, and the directional derivative $\nabla f|_p=0$, but the Hessian is non-degenerate. All these vanishing conditions as well as the other constraints can be expressed as a section of an appropriate vector bundle. We will now use the following proposition from differential topology to obtain the degree of the intersection product:

Proposition 1.4.3. *Let M be a compact complex manifold and V be a holomorphic vector bundle over M such that the rank of V is same as $\dim_{\mathbb{C}} M$. If $s : M \longrightarrow V$ is a section* *that is transverse to zero, then the cardinality of the set* $s^{-1}(0)$ *is the Euler class of* V *evaluated on the fundamental class of* M*, i.e.,*

$$|s^{-1}(0)| = \int_M e(V).$$

Proof. See [8]. □

We say the section $s : M \longrightarrow V$ is transverse to the zero section if the submanifolds M and $s(M)$ intersect transversely inside the total space V, where M is identified with the image of the zero section.

With the help of the above result, we compute the degree of the intersection product described in the fourth step of the general approach. Now come to the crucial step, verify whether the above degree is the same as $N_d(A_1)$. Fortunately, it can be justified that they are the same indeed (see [2]). But often the Euler class carries a degenerate contribution. To detect these contributions, we perturb the section smoothly, and count the number of zeros (counted with multiplicity) in a neighborhood of the degenerate locus. Subtracting the degenerate contribution from the number obtained from Euler class calculation, the answer to the enumerative question can be derived. We encounter with this situation several times in Chapter 2. Observe that a smooth perturbation of the section is considered as opposed to the holomorphic perturbation. In other words, we use smooth deformation theory of the space of curves instead of holomorphic (or regular) ones. This is a *topological* method as opposed to an *algebro-geometric* one. This method is an extension of the method that originates in the paper by A. Zinger [46] and which is further pursued by S. Basu and R. Mukherjee in [1] , [2] and [3].

In Chapter 2, we consider a natural generalization of curve counting questions in the complex plane introduced above. We define a **planar curve** in $\mathbb{CP}^3$ to be a curve in $\mathbb{CP}^3$, whose image lies inside some $\mathbb{CP}^2$. Note that here $\mathbb{CP}^2$ inside $\mathbb{CP}^3$ is not fixed. One can ask the counting problems in this setup. Note that intersection with a line and passing through a point are the constraints here. Let us define

$$N_d^{\mathrm{Planar},\mathbb{P}^3}(A_1^{\delta}\mathfrak{X}; r, s)$$

to be the number of planar degree d curves in $\mathbb{P}^3$, intersecting r lines and passing through s points, and having δ distinct nodes and one singularity of type $\mathfrak{X}$, where $r + 2s = \frac{d(d+3)}{2} + 3 - (\delta + c_{\mathfrak{X}})$ and $c_{\mathfrak{X}}$ is the codimension of the singularity $\mathfrak{X}$. This situation can be thought of as a fiber bundle analogue of the above situation, and in fact, the space of curves here will be a fiber bundle over $\mathbb{G}(3,4)$, the Grassmannian of 3-planes in $\mathbb{C}^4$, where the fiber over each point is the space of curves in the corresponding plane. This generalization was first considered by Kleiman and Piene [26], where they study the enumerative geometry of nodal curves in a moving family of surfaces. In 2018, a formula for $N_d^{\mathrm{Planar},\mathbb{P}^3}(A_1^{\delta}; r, s)$ is obtained by T. Laarakker [30] for all δ. Very recently, a stable map version of the planar curves is considered by Mukherjee-Paul-Singh [34]. In this thesis, we consider the situation where the singularities are worse than nodes, and we compute $N_d^{\mathrm{Planar},\mathbb{P}^3}(A_1^{\delta}\mathfrak{X}; r, s)$, where $\mathfrak{X}$ is a singularity, other than a node, of codimension $c_{\mathfrak{X}}$, and $\delta + c_{\mathfrak{X}} \leq 4$. This is a joint work with Ritwik Mukherjee [12]. Chapter 2 is entirely devoted to the study of planar curves. A complete list of closed formulas in terms of the degree of curves is given in Appendix A.

1.5 Gromov-Witten theory

We now move towards the discussion of the setup of Question 1.2.2. In this case, the space of curves is a moduli space. We can discuss the moduli space for general projective variety X. Let $\beta \in A_1(X)$ (in the case of Question 1.2.2, $X = \mathbb{CP}^2$ and $\beta = dH$, where H is the hyperplane class in $\mathbb{CP}^2$). Let $M_g(X, \beta)$ be the set of isomorphism classes of maps $(f : C \longrightarrow X)$ where C is a smooth genus g projective curve such that $f_*([C]) = \beta$. Two such map $(f : C \longrightarrow X)$ and $(f' : C' \longrightarrow X)$ are said to be isomorphic if there is an isomorphism of schemes $\phi : C \longrightarrow C'$. One major problem is that the moduli space $M_g(X, \beta)$ is not compact which is necessary in order to do intersection theory. There are several compactifications of $M_g(X, \beta)$ obtained by slightly generalizing the moduli problem. In this thesis, we consider one such compactification proposed by Kontsevich which is known as the so-called *moduli space of stable maps*. He considered $(f : C \longrightarrow X)$, where C is allowed to be singular projective marked curve (at most nodal) of arithmetic genus g, and then defined some stabilizing conditions in terms of the number of marked points (more precisely, he allowed all such curve whose automorphism group is finite). This moduli space is denoted by $\overline{M}_{g,n}(X, \beta)$, where n is the number of marked points on the curve. The moduli space $\overline{M}_{g,n}(X, \beta)$ is compact but in general it is not smooth, irreducible, reduced, connected, and of constant dimension. So it does not always carry a natural fundamental class; however, it always carries a so-called *virtual fundamental class* which is shown by Li-Tian [31] and Behrend-Fantechi [5]. In Chapter 3, we study this moduli space in detail and define some intersection numbers which are known as *Gromov-Witten invariants*. In Chapter 4, we will compute Gromov-Witten invariants (in the case of $g = 0$ and 1) of special type of surfaces which are known as *del Pezzo* surface. Note that $\mathbb{CP}^2$ is a del Pezzo surface. The computations of elliptic Gromov-Witten invariants are done jointly with Chitrabhanu Chaudhuri [10]. Though Gromov-Witten invariants do not have enumerative significance always (strictly speaking, it can be negative and rational as well), Vakil has shown that any genus g Gromov-Witten invariants of del Pezzo surfaces are enumerative (cf. [42]). In particular, this answers Question 1.2.2.

Chapter 2

Planar curves in $\mathbb{CP}^3$ with degenerate singularities

This chapter is devoted to the study of the fiber bundle analogue of Question 1.2.1. That is, we study enumerative geometry of planar curves in $\mathbb{P}^3$.

Definition 2.0.1. A curve in $\mathbb{P}^3$ is said to be *planar* if its image lies inside some $\mathbb{P}^2$ in $\mathbb{P}^3$.

In other words, a planar curve in $\mathbb{P}^3$ is a one-dimensional complete intersection given by two polynomials and one of which is linear among them. Recall that $N_d^{\mathrm{Planar},\mathbb{P}^3}(A_1^\delta \mathfrak{X}; r, s)$ is defined to be the number of planar degree d curves in $\mathbb{P}^3$, intersecting r lines and passing through s points, and having δ distinct nodes and one singularity of type $\mathfrak{X}$, where $r+2s = \frac{d(d+3)}{2} + 3 - (\delta + c_{\mathfrak{X}})$ and $c_{\mathfrak{X}}$ is the codimension of the singularity $\mathfrak{X}$. The main result in this chapter is the following.

Theorem 2.0.2. *Let $\mathfrak{X}$ be a singularity of codimension $c_{\mathfrak{X}}$ and δ a non negative integer. We obtain an explicit formula for $N_d^{\mathrm{Planar},\mathbb{P}^3}(A_1^\delta \mathfrak{X}, r, s)$, when $\delta + c_{\mathfrak{X}} \leq 4$, provided $d \geq d_{\mathrm{min}}$, where $d_{\mathrm{min}} := c_{\mathfrak{X}} + 2\delta$.*

All the recursive formulas are explicitly described in Section 2.4. Also, a list of closed formulas (in terms of degree) is tabulated in Appendix A for convenience.

2.1 Parameter space of planar curves

Let us now describe the parameter space and develop some notations which we use throughout the chapter. Our basic objects are planar degree d curves in $\mathbb{P}^3$, that is, degree d curves in $\mathbb{P}^3$ whose image lies inside a $\mathbb{P}^2$. Let us denote the dual of $\mathbb{P}^3$ by $\widehat{\mathbb{P}}^3$; this is the space of $\mathbb{P}^2$ inside $\mathbb{P}^3$. An element of $\widehat{\mathbb{P}}^3$ can be thought of as a nonzero linear functional $\eta : \mathbb{C}^4 \longrightarrow \mathbb{C}$ up to scaling (i.e., it is the projectivization of the dual of $\mathbb{C}^4$). Given such an η, we define the projectivization of its zero set as $\mathbb{P}^2_\eta$. In other words,

$$\mathbb{P}^2_\eta := \mathbb{P}(\eta^{-1}(0)).$$

Note that this $\mathbb{P}^2_\eta$ is a subset of $\mathbb{P}^3$. Under this identification $\widehat{\mathbb{P}}^3 \cong \mathbb{G}(3,4)$.

Next, given a positive integer δ, let us define

$$\mathcal{S}_\delta := \{([\eta], q_1, \ldots, q_\delta) \in \widehat{\mathbb{P}}^3 \times (\mathbb{P}^3)^\delta : \eta(q_1) = 0, \ldots, \eta(q_\delta) = 0\}.$$

Clearly $\mathcal{S}_\delta$ is a fiber bundle over $\widehat{\mathbb{P}}^3$ with fiber $(\mathbb{P}^2)^\delta$. This is a plane in $\mathbb{P}^3$ and a collection of δ points that lie on that plane. We will often abbreviate $\mathcal{S}_1$ as $\mathcal{S}$. Let us consider the section of the following line bundle induced by the evaluation map, that is,

$$\mathrm{ev} : \widehat{\mathbb{P}}^3 \times \mathbb{P}^3 \longrightarrow \gamma^*_{\widehat{\mathbb{P}}^3} \otimes \gamma^*_{\mathbb{P}^3}, \qquad \text{given by} \qquad \{\mathrm{ev}([\eta],[q])\}(\eta \otimes q) := \eta(q),$$

where $\gamma^*_{\widehat{\mathbb{P}}^3}$ and $\gamma^*_{\mathbb{P}^3}$ are dual of the tautological line bundles over $\widehat{\mathbb{P}}^3$ and $\mathbb{P}^3$ respectively (or equivalently $\mathcal{O}_{\widehat{\mathbb{P}}^3}(1)$ and $\mathcal{O}_{\mathbb{P}^3}(1)$ respectively). Note that

$$\mathcal{S} = \mathrm{ev}^{-1}(0). \tag{2.1}$$

Next, let us denote $\mathcal{D} \longrightarrow \widehat{\mathbb{P}}^3$ to be the fiber bundle over $\widehat{\mathbb{P}}^3$, such that the fiber over each $[\eta] \in \widehat{\mathbb{P}}^3$ is the space of degree d curves in $\mathbb{P}^2_\eta$. Let us denote $\gamma_{3,4} \longrightarrow \mathbb{G}(3,4)$ to be the tautological three plane bundle over the Grassmannian. Hence,

$$\mathcal{D} \approx \mathbb{P}(\mathrm{Sym}^d \gamma_{3,4}^*) \longrightarrow \widehat{\mathbb{P}}^3$$

is a fiber bundle over $\widehat{\mathbb{P}}^3$, whose fibers are isomorphic to $\mathbb{P}^{\frac{d(d+3)}{2}}$. An element of $\mathcal{D}$ will be denoted by $([f],[\eta])$; this means that f is a homogeneous degree d-polynomial defined on $\mathbb{P}^2_\eta$. This is the space of all planar degree d curves in $\mathbb{P}^3$.

Next, given a positive integer δ, let us define

$$\mathcal{S}_{\mathcal{D}_\delta} := \{([f],[\eta],q_1,\ldots,q_\delta) \in \mathcal{D} \times (\mathbb{P}^3)^\delta : ([\eta],q_1,\ldots,q_\delta) \in \mathcal{S}_\delta\}.$$

Note that $\mathcal{S}_{\mathcal{D}_\delta}$ can be considered as pull back bundle of $\mathcal{D}$ via the fiber bundle map $\pi : \mathcal{S}_\delta \to \widehat{\mathbb{P}}^3$, that is, the following diagram

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{D}_\delta} & \longrightarrow & \mathcal{D} \\ {\scriptstyle \pi_{\mathcal{D}}^*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{D}}} \\ \mathcal{S}_\delta & \longrightarrow & \widehat{\mathbb{P}}^3 \end{array}$$

is Cartesian. We will abbreviate $\mathcal{S}_{\mathcal{D}_1}$ as $\mathcal{S}_{\mathcal{D}}$. Next, let $X_1, X_2, \ldots, X_\delta$ be subsets of $\mathcal{S}_{\mathcal{D}}$. We define

$$X_1 \circ X_2 \circ \ldots \circ X_\delta := \{([f],[\eta],q_1,\ldots,q_\delta) \in \mathcal{S}_{\mathcal{D}_\delta} : ([f],[\eta],q_i) \in X_i \;\; \forall\, i = 1 \text{ to } \delta \quad \text{and} \quad q_i \neq q_j \qquad \text{if } i \neq j\}.$$

We will make the following abbreviation

$$X_1^{\delta_1} \circ X_2^{\delta_2} \ldots X_m^{\delta_m} := \underbrace{X_1 \circ \ldots \circ X_1}_{\delta_1 \text{ times}} \circ \underbrace{X_2 \circ \ldots \circ X_2}_{\delta_2 \text{ times}} \circ \ldots \circ \underbrace{X_m \circ \ldots \circ X_m}_{\delta_m \text{ times}}.$$

When $\delta_i = 1$, we will omit writing the superscript. For example,

$$X_1 \circ X_2^3 \circ X_3 = X_1^1 \circ X_2^3 \circ X_3^1 = X_1 \circ X_2 \circ X_2 \circ X_2 \circ X_3.$$

Next, let $\mathfrak{X}$ be a singularity of a given type. We will also denote $\mathfrak{X}$ to be the space of curves and a marked point such that the curve has a singularity of type $\mathfrak{X}$ at the marked point. More precisely,

$$\mathfrak{X} := \{([f], [\eta], q) \in \mathcal{S}_{\mathcal{D}} : \ f \ \text{ has a singularity of type } \mathfrak{X} \text{ at } q\}.$$

For example,

$$A_2 := \{([f], [\eta], q) \in \mathcal{S}_{\mathcal{D}} : \ f \ \text{ has a singularity of type } A_2 \text{ at } q\}.$$

For example, $A_1^2 \circ A_2$ is the space of curves with three ordered points, where the curve has a simple node at the first two points with a cusp at the last point, and all the three points are distinct. Similarly, $A_1^2 \circ \overline{A}_2$ is the space of curves with three distinct ordered points, where the curve has a simple node at the first two points and a singularity at least as degenerate as a cusp at the last point; the curve could have a tacnode at the last marked point (here $\overline{X}$ indicates the closure of X).

Next, consider $\pi : W \longrightarrow \mathcal{S}$ to be the relative tangent bundle of $\mathcal{S} \longrightarrow \widehat{\mathbb{P}}^3$, where the fiber over each point $([\eta], q)$ is the tangent space of $\mathbb{P}^2_\eta$ at the point q, that is,

$$\pi^{-1}([\eta], q) := T\mathbb{P}^2_\eta|_q. \tag{2.2}$$

Let $W_{\mathcal{D}} \longrightarrow \mathcal{S}_{\mathcal{D}}$ denote the pullback of W to $\mathcal{S}_{\mathcal{D}}$ and let $\mathbb{P}W_{\mathcal{D}} \longrightarrow \mathcal{S}_{\mathcal{D}}$ denote the projectivization of $W_{\mathcal{D}}$. We can now define the space of curves having a singularity of a certain type together with a direction, i.e., if $\mathfrak{X}$ is a singularity of a given type, then define

$$\widehat{\mathfrak{X}} := \{([f], [\eta], l_q) \in \mathbb{P}W_{\mathcal{D}} : f \text{ has a singularity of type } \mathfrak{X} \text{ at } q\}.$$

We can also define the space of curves with a singularity and a specific direction along which certain directional derivatives vanish, that is,

$$\mathcal{P}A_k := \{([f],[\eta],l_q) \in \mathbb{P}W_{\mathcal{D}} : ([f],[\eta],q) \in A_k,\;\; \nabla^2 f|_q(v,\cdot) = 0\,\forall v \in l_q\} \quad \text{if} \quad k \geq 2.$$

For example, $\mathcal{P}A_2$ is the space of curves with a marked point and a marked direction, such that the curve has a cusp at the marked point and the marked direction belongs to the kernel of the Hessian. Note that the projection map $\pi : \mathcal{P}A_k \longrightarrow A_k$ is one to one. Next, let us define

$$\begin{aligned}
\mathcal{P}A_1 &:= \{([f],[\eta],l_q) \in \mathbb{P}W_{\mathcal{D}} : ([f],[\eta],q) \in A_1,\;\; \nabla^2 f|_q(v,v) = 0\,\forall v \in l_q\}, \quad \text{and} \\
\mathcal{P}D_4 &:= \{([f],[\eta],l_q) \in \mathbb{P}W_{\mathcal{D}} : ([f],[\eta],q) \in D_4,\;\; \nabla^3 f|_q(v,v,v) = 0\,\forall v \in l_q\}.
\end{aligned}$$

In other words, $\mathcal{P}A_1$ is the space of curves with a marked point and a marked direction, such that the curve has a node at the marked point and the second derivative along the marked direction vanishes. Note that there are two such distinguished directions. Hence, the projection map $\pi : \mathcal{P}A_1 \longrightarrow A_1$ is two to one. Similarly, the projection map $\pi : \mathcal{P}D_4 \longrightarrow D_4$ is three to one.

Next, let $\mathcal{S}_{\mathcal{D}_\delta} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}}$ denote the fibered product of $\mathcal{S}_{\mathcal{D}_\delta}$ and $\mathbb{P}W_{\mathcal{D}}$ over $\mathcal{D}$ via the natural forgetful map. It can be considered as a fiber bundle over $\widehat{\mathbb{P}}^3$ whose fiber over each point $[\eta] \in \widehat{\mathbb{P}}^3$ is

$$\mathbb{P}(H^0(\mathcal{O}(d),\mathbb{P}^2_\eta)) \times (\mathbb{P}^2_\eta)^\delta \times \mathbb{P}(T\mathbb{P}^2_\eta).$$

Let $\pi : \mathcal{S}_{\mathcal{D}_\delta} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} \longrightarrow \mathcal{S}_{\mathcal{D}_{\delta+1}}$ denote the projection map. If S is a subset of $\mathcal{S}_{\mathcal{D}_{\delta+1}}$, then we define

$$\widehat{S} := \{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \mathcal{S}_{\mathcal{D}_\delta} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} : ([f],[\eta],q_1,\ldots,q_{\delta+1}) \in S\} = \pi^{-1}(S). \tag{2.3}$$

Finally, if $S_1, \dots S_n$ are subsets of $\mathcal{S}_{\mathcal{D}}$ and T is a subset of $\mathbb{P}W_{\mathcal{D}}$, then we define

$$\begin{aligned} S_1 \circ S_2 \circ \dots \circ S_\delta \circ T := \{([f], [\eta], q_1, \dots, q_\delta, l_{q_{\delta+1}}) \in \mathcal{S}_{\mathcal{D}_\delta} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} : ([f], l_{q_{\delta+1}}) \in T, \\ ([f], q_1) \in S_1, \dots, ([f], q_\delta) \in S_\delta, \text{ and} \\ q_1, \dots, q_\delta, q_{\delta+1} \text{ are all distinct}\}. \end{aligned}$$

As an example, $A_1^2 \circ \mathcal{P}A_2$ is the space of curves with three distinct ordered points, where the curve has a simple node at the first two points and a cusp at the last point and a distinguished direction at the last marked point, such that the Hessian vanishes along that direction.

2.2 Intersection ring structure of projective fiber bundle

We now recapitulate some basic facts about the cohomology ring of the various spaces we will encounter. We need to consider the splitting principle and the Leray-Hirch Theorem that can be found in [8]. We recall that via the annihilation map, $\widehat{\mathbb{P}}^3$ is isomorphic to $\mathbb{G}(3,4)$. Via this isomorphism, we can think of a (which is actually a generator of $H^*(\widehat{\mathbb{P}}^3)$) as a generator of $H^*(\mathbb{G}(3,4))$. We note that

$$c(\gamma_{3,4}^*) = 1 + a + a^2 + a^3.$$

Next, using the splitting principle, we conclude that

$$c(\text{Sym}^d \gamma_{3,4}^*) = 1 + s_1 a + s_2 a^2 + s_3 a^3, \qquad \text{where} \tag{2.4}$$

$$s_1 := \frac{d(d+1)(d+2)}{6}, \qquad s_2 := \frac{d(d+1)(d+2)(d+3)(d^2+2)}{72} \qquad \text{and}$$

$$s_3 := \frac{d(d+1)(d+2)(d+3)(d^2+2)(d^3+3d^2+2d+12)}{1296}. \tag{2.5}$$

Notice that $\mathcal{D} = \mathbb{P}(\mathrm{Sym}^d \gamma_{3,4}^*)$, is a $\mathbb{P}^{n-1}$ bundle, where

$$n := 1 + \frac{d(d+3)}{2}. \tag{2.6}$$

Hence, we conclude (by the Leray-Hirsch Theorem) that the cohomology ring structure of $\mathcal{D}$ is given by

$$H^*(\mathcal{D}) \approx \frac{\mathbb{Z}[a, \lambda]}{\langle a^4, \ \ \lambda^n + s_1 a \lambda^{n-1} + s_2 a^2 \lambda^{n-2} + s_3 a^3 \lambda^{n-3} \rangle}, \tag{2.7}$$

where $\gamma_{\mathcal{D}} \longrightarrow \mathbb{P}(\mathrm{Sym}^d \gamma_{3,4}^*)$ denotes the tautological line bundle and $\lambda := c_1(\gamma_{\mathcal{D}}^*)$.

2.3 Intersection numbers

Let $\gamma_W \longrightarrow \mathbb{P}W$ denote the tautological line bundle over the projective bundle $\mathbb{P}W \longrightarrow \mathcal{S}$. We denote $\lambda_W := c_1(\gamma_W^*)$ and H to be the standard generator of $H^*(\mathbb{P}^3)$ (i.e., the class of a hyperplane in $\mathbb{P}^3$). Let us denote the cycle given by the subspace of planar degree d-curves in $\mathbb{P}^3$ that intersects a generic line, pass through a generic point in $\mathbb{P}^3$ by $\mathcal{H}_L$ and $\mathcal{H}_p$, respectively.

Since we will primarily be dealing with planar degree d-curves in $\mathbb{P}^3$, we will usually use the prefix N as opposed to the more elaborate notation $N_d^{\mathrm{Planar}, \mathbb{P}^3}$. If there is a chance for confusion, we will use the latter notation. We will occasionally be dealing with curves in $\mathbb{P}^2$. In such a case we will use the notation $N_d^{\mathbb{P}^2}$; we will never use N in such a case.

We now define a few intersection numbers related to the enumeration of planar curves satisfying various constraints. Our goal in this chapter is to compute these intersection numbers with some appropriate restrictions. Let us define

$$N(A_1^\delta \mathfrak{X}, r, s, n_1, n_2, n_3) := \langle a^{n_1} \lambda^{n_2} \pi_{\delta+1}^* H^{n_3} \mathcal{H}_L^r \mathcal{H}_p^s, \ \ [\overline{A_1^\delta \circ \mathfrak{X}}] \rangle,$$

$$N(A_1^\delta \mathcal{P} \mathfrak{X}, r, s, n_1, n_2, n_3, \theta) := \langle a^{n_1} \lambda^{n_2} \pi_{\delta+1}^* H^{n_3} \lambda_W^\theta \mathcal{H}_L^r \mathcal{H}_p^s, \ \ [\overline{A_1^\delta \circ \mathcal{P} \mathfrak{X}}] \rangle, \text{ and}$$

$$N(A_1^\delta\widehat{\mathfrak{X}}, r, s, n_1, n_2, n_3, \theta) := \langle a^{n_1}\lambda^{n_2}\pi_{\delta+1}^* H^{n_3}\lambda_W^\theta \mathcal{H}_L^r\mathcal{H}_p^s,\ \ [\overline{A_1^\delta \circ \widehat{\mathfrak{X}}}]\rangle. \tag{2.8}$$

Here π_i denotes the projection onto the i^{th} point. Note we are identifying the pairing between cohomology and homology with the intersection of cycles in homology (via Poincaré duality). More precisely, if α is a cohomology class and μ is a homology class, then

$$\langle \alpha, \mu\rangle = \mu \cdot \alpha, \tag{2.9}$$

where $\cdot$ denotes intersection in homology. Note that in the right hand side of eq. (2.9) we are making a slight abuse of notation; α denotes the homology class Poincaré dual to the cohomology class α (which we are denoting by the same letter). This is a standard abuse of notation (the cohomology class and its Poincaré dual are usually represented by the same symbol). Hence, the right hand side of the first equation of (2.8) can also be written as

$$[\overline{A_1^\delta \circ \mathfrak{X}}] \cdot a^{n_1} \cdot \lambda^{n_2} \cdot \pi_{\delta+1}^* H^{n_3} \cdot \mathcal{H}_L^r \cdot \mathcal{H}_p^s.$$

Similarly with the other two quantities defined in eq. (2.8).

Remark 2.3.1. We note that the right hand sides of eq. (2.8) denote the degree of intersection of certain cycles. In Section 2.6, we show that when d satisfies the appropriate bound, the spaces $A_1^\delta\mathfrak{X}$, $A_1^\delta\widehat{\mathfrak{X}}$ and $A_1^\delta\mathcal{P}\mathfrak{X}$ are smooth complex manifolds of the expected dimension. Hence, the intersection of cycles are transverse, and as a result, the corresponding intersection numbers are enumerative.

Next, we note that if $\theta \geq 2$, then

$$\begin{aligned} N(A_1^\delta\mathcal{P}\mathfrak{X}, r, s, n_1, n_2, n_3, \theta) &= -3N(A_1^\delta\mathcal{P}\mathfrak{X}, r, s, n_1, n_2, n_3+1, \theta-1) \\ &\quad + N(A_1^\delta\mathcal{P}\mathfrak{X}, r, s, n_1+1, n_2, n_3, \theta-1) \end{aligned}$$

$$
\begin{aligned}
&- N(A_1^{\delta}\mathcal{P}\mathfrak{X}, r, s, n_1+2, n_2, n_3, \theta-2)\\
&+ 2N(A_1^{\delta}\mathcal{P}\mathfrak{X}, r, s, n_1+1, n_2, n_3+1, \theta-2)\\
&- 3N(A_1^{\delta}\mathcal{P}\mathfrak{X}, r, s, n_1, n_2, n_3+2, \theta-2). \qquad (2.10)
\end{aligned}
$$

This is because

$$
\lambda_W^2 = -c_1(W)\lambda_W - c_2(W) \Rightarrow \lambda_W^2 = -(3H-a)\lambda_W - (a^2 - 2aH + 3H^2). \qquad (2.11)
$$

The Chern classes $c_1(W)$ and $c_2(W)$ are given by eq. (2.21). Next, we note that

$$
N(A_1^{\delta}\mathfrak{X}, r, s, n_1, n_2, n_3) = \frac{1}{\deg(\pi)} N(A_1^{\delta}\mathcal{P}\mathfrak{X}, r, s, n_1, n_2, n_3, 0), \qquad (2.12)
$$

where $\deg(\pi)$ is the degree of the projection map $\pi : \mathcal{P}\mathfrak{X} \longrightarrow \mathfrak{X}$. We remind the reader that the degree is one when $\mathfrak{X} = A_{k\geq 2}$, it is two when $\mathfrak{X} = A_1$ and it is three when $\mathfrak{X} = D_4$. We also note that

$$
\begin{aligned}
N(A_1^{\delta}\widehat{\mathfrak{X}}, n_1, n_2, n_3, \theta) =& 0 \quad \text{if } \theta = 0,\\
N(A_1^{\delta}\widehat{\mathfrak{X}}, n_1, n_2, n_3, \theta) =& N(A_1^{\delta}\mathfrak{X}, n_1, n_2, n_3) \quad \text{if } \theta = 1, \text{ and}\\
N(A_1^{\delta}\widehat{\mathfrak{X}}, n_1, n_2, n_3, \theta) =& - 3N(A_1^{\delta}\widehat{\mathfrak{X}}, r, s, n_1, n_2, n_3+1, \theta-1)\\
&+ N(A_1^{\delta}\widehat{\mathfrak{X}}, r, s, n_1+1, n_2, n_3, \theta-1)\\
&- N(A_1^{\delta}\widehat{\mathfrak{X}}, r, s, n_1+2, n_2, n_3, \theta-2)\\
&+ 2N(A_1^{\delta}\widehat{\mathfrak{X}}, r, s, n_1+1, n_2, n_3+1, \theta-2)\\
&- 3N(A_1^{\delta}\widehat{\mathfrak{X}}, r, s, n_1, n_2, n_3+2, \theta-2) \quad \text{if } \theta > 1. \qquad (2.13)
\end{aligned}
$$

The last equality of eq. (2.13) follows from eq. (2.11). Finally, let us define

$$
N(r, s, n_1, n_2) := \langle a^{n_1}\lambda^{n_2}\mathcal{H}_L^r\mathcal{H}_p^s,\ [\mathcal{D}]\rangle. \qquad (2.14)
$$

We now claim that

$$\mathcal{H}_L = \lambda + da \qquad \text{and} \qquad \mathcal{H}_p = \lambda a. \tag{2.15}$$

We note that the cohomology ring of $\mathcal{D}$ is generated by λ and a. Since $\mathcal{H}_L$ is a codimension one cycle, we can conclude that $\mathcal{H}_L$ is a linear combination of λ and a. Let $\Sigma \subseteq \mathcal{D}$ be a general pencil $\{([f]_k, [\eta])\}$ of planar degree d curves in a general plane $\mathbb{P}^2_\eta \subseteq \mathbb{P}^3$, and Ω be the space of planar curves that corresponds to a general pencil of plane sections $\{\mathbb{P}^2_\tau \cap Q\}$ for a fixed hypersurface Q of degree d in $\mathbb{P}^3$. We also denote their (co)homology classes in $H^*(X, \mathbb{Z})$ by σ and ω, respectively. Then we get the following intersection table:

	a	λ	$\mathcal{H}_L$
σ	0	1	1
ω	1	0	d

To obtain the intersection numbers of the first row, observe that Σ can be identified with a general pencil of degree d plane curves in the fixed plane $\mathbb{P}^2_\eta$. Since the plane $\mathbb{P}^2_\eta$ is already fixed, $a\sigma = 0$. Also, $\mathcal{H}_L\sigma$ will be the same as the number of plane degree d plane curves that pass through $\dfrac{d(d+3)}{2}$ generic points, and this number is one. Similarly, we can conclude that $\lambda\sigma = 1$. To obtain the intersection numbers of the second row, observe that Ω is considered by fixing a hypersurface of degree d in $\mathbb{P}^3$ and the family of planes contain a common line. Since a hypersurface of degree d intersects a generic line in $\mathbb{P}^3$ at d many points, and there is unique plane in $\mathbb{P}^3$ that contains a line and a point outside the line. We deduce that $\mathcal{H}_L\omega = d$. We also conclude that $a\omega = 1$ as there is a unique plane section that belongs to Ω for each plane $\mathbb{P}^2_\tau$. To show $\lambda\omega = 0$, it is enough to show that the tautological bundle $\gamma_{\mathcal{D}} \longrightarrow \mathcal{D}$ restricted to Ω is trivial. Let us assume that Q is given by a degree d homogeneous polynomial F. Then we see that restrictions of F to the planes $\mathbb{P}^2_\tau$ produce an everywhere non-zero section of $\gamma_{\mathcal{D}}$ over Ω. Thus $\gamma_{\mathcal{D}}$ is trivial over Ω.

Let us now assume $\mathcal{H}_L = n_1\lambda + n_2 a$, where n_1 and n_2 are to be determined. Intersecting $\mathcal{H}_L$ with σ, we conclude that $n_1 = 1$ and intersecting $\mathcal{H}_L$ with ω, we conclude that $n_2 = d$

(where we use the multiplication table). Hence, we conclude that

$$\mathcal{H}_L = \lambda + da.$$

Next, let us compute the expression for the class $\mathcal{H}_p$. We will follow the discussion from [45, Pages 18 and 19]. First note that if a planar curve $([f], [\eta])$ passes though a point $p \in \mathbb{P}^3$, then p lies on the plane $\mathbb{P}^2_\eta$, and after that p lies on the curve $[f]$. Note that the space of planes that pass through p is again a hyperplane in $\widehat{\mathbb{P}}^3$. Let us denote the hyperplane by Λ_p. Let us define a section

$$\phi_p : \mathbb{P}(\text{Sym}^d \gamma^*_{3,4})|_{\Lambda_p} \longrightarrow \gamma^*_{\mathcal{D}}, \text{ given by}$$
$$\{\phi_p([f], [\eta])\}(f) := f(p).$$

Observe that the class $\mathcal{H}_p$ is the same as the cycle given by the subspace $\phi_p^{-1}(0)$ which is again same as λ restricted to $\mathbb{P}(\text{Sym}^d \gamma^*_{3,4})|_{\Lambda_p}$. Hence we conclude that $\mathcal{H}_p = \lambda a$. This completes the justification of eq. (2.15).

Now, using the ring structure of $H^*(\mathcal{D})$ (as given by eq. (2.7)), we can compute $N(r, s, n_1, n_2)$ by extracting the coefficient of $a^3\lambda^{n-1}$ from the expression

$$P := (\lambda + da)^r (\lambda a)^s a^{n_1} \lambda^{n_2},$$

by thinking it as an element of $H^*(\mathcal{D})$. Hence, $N(r, s, n_1, n_2)$ can be computed for any r, s, n_1 and n_2.

Remark 2.3.2. The coefficient of $a^3\lambda^{n-1}$ is computed in the following way: first the polynomial P is expanded. The terms involving λ, $\lambda^2, \ldots, \lambda^{n-1}$ are kept as it is. Next, any expression involving λ^n is converted to an expression involving λ^{n-1}, using the ring structure, namely

$$\lambda^n = -s_1 a\lambda^{n-1} - s_2 a^2 \lambda^{n-2} - s_3 a^3 \lambda^{n-3}.$$

Similarly, any expression involving λ^{n+1} is converted to an expression involving λ^{n-1}, by applying the above identity twice. And so on. The final expression will comprise of terms of the type $a^k\lambda^j$ where j is strictly less than n. The coefficient of $a^3\lambda^{n-1}$ in this final expression is what we need to extract.

2.4 Recursive formulas

We are now ready to state the recursive formulas.

Theorem 2.4.1. *Consider the ring*

$$\mathcal{R} = \frac{\mathbb{Z}[a, H, \lambda]}{\langle a^4, \quad H^4, \quad \lambda^n + s_1 a\lambda^{n-1} + s_2 a^2\lambda^{n-2} + s_3 a^3\lambda^{n-3}\rangle},$$

where s_1, s_2, s_3 and n are as defined in eq. (2.5) and eq. (2.6). Let

$$e := (\lambda + H)(\lambda + da)^r(\lambda a)^s a^{n_1}\lambda^{n_2}H^{n_3}(\lambda + dH)\Big((\lambda + dH)^2 - (3H - a)(\lambda + dH) + a^2 - 2aH + 3H^2\Big).$$

Then $N(A_1, r, s, n_1, n_2, n_3)$ is the coefficient of $\lambda^{n-1}a^3H^3$ in the polynomial e, seen as an element of the ring $\mathcal{R}$.

Remark 2.4.2. We remind the reader to look at Remark 2.3.2 to understand the precise meaning of the phrase "coefficient of $\lambda^{n-1}a^3H^3$ in the polynomial e".

Next, we will give a formula for $N(A_1^{\delta+1}, r, s, n_1, n_2, n_3)$, when $1 \leq \delta \leq 3$. First let us make several definitions.

$$\begin{aligned}
\mathrm{Eul}(\delta, r, s, n_1, n_2, 0) &:= (d - 2d^2 + d^3)N(A_1^\delta, r, s, n_1 + 1, n_2, 0) \\
&\quad + (3 - 6d + 3d^2)N(A_1^\delta, r, s, n_1, n_2 + 1, 0) \\
\mathrm{Eul}(\delta, r, s, n_1, n_2, 1) &:= (d^2 - d)N(A_1^\delta, r, s, n_1 + 2, n_2, 0)
\end{aligned}$$

$$
\begin{aligned}
&+ (3d^2-4d+1)N(A_1^{\delta}, r, s, n_1+1, n_2+1, 0)\\
&+ (3d-3)N(A_1^{\delta}, r, s, n_1, n_2+2, 0),\\
\mathrm{Eul}(\delta, r, s, n_1, n_2, 2) &:= dN(A_1^{\delta}, r, s, n_1+3, n_2, 0)\\
&+ (2d-1)N(A_1^{\delta}, r, s, n_1+2, n_2+1, 0)\\
&+ (3d-2)N(A_1^{\delta}, r, s, n_1+1, n_2+2, 0)\\
&+ N(A_1^{\delta}, r, s, n_1, n_2+3, 0)\\
\mathrm{Eul}(\delta, r, s, n_1, n_2, 3) &:= N(A_1^{\delta}, r, s, n_1+3, n_2+1, 0)\\
&+ N(A_1^{\delta}, r, s, n_1+2, n_2+2, 0)\\
&+ N(A_1^{\delta}, r, s, n_1+1, n_2+3, 0)\\
\mathrm{Eul}(\delta, r, s, n_1, n_2, n_3) &= 0 \qquad \text{if } \ n_3 > 3. \qquad (2.16)
\end{aligned}
$$

We also define

$$
\begin{aligned}
\mathrm{B}(\delta, r, s, n_1, n_2, n_3) &:= \binom{\delta}{1}B_1 + \binom{\delta}{2}B_2 + \binom{\delta}{3}B_3, \qquad \text{where}\\
B_1 &:= \Big(N(A_1^{\delta}, r, s, n_1, n_2+1, n_3)\\
&\quad + dN(A_1^{\delta}, r, s, n_1, n_2, n_3+1)\\
&\quad + 3N(A_1^{\delta-1}\mathcal{P}A_2, r, s, n_1, n_2, n_3, 0)\Big),\\
B_2 &:= 4\Big(N(A_1^{\delta-2}\mathcal{P}A_3, r, s, n_1, n_2, n_3, 0)\Big),\\
B_3 &:= \frac{18}{3}\Big(N(A_1^{\delta-3}\mathcal{P}D_4, r, s, n_1, n_2, n_3, 0)\Big). \qquad (2.17)
\end{aligned}
$$

We also make use of the convention that if an exponent of A_1 of any term is negative, then the term is defined to be zero.

We are now ready to state the formula for $N(A_1^{\delta+1}, r, s, n_1, n_2, n_3)$.

Theorem 2.4.3. *Let* $\mathrm{Eul}(\delta, r, s, n_1, n_2, n_3)$ *and* $\mathrm{B}(\delta, r, s, n_1, n_2, n_3)$ *be defined as in eq.* (2.16) *and eq.* (2.17) *respectively. If* $1 \leq \delta \leq 3$*, then*

$$N(A_1^{\delta+1}, r, s, n_1, n_2, n_3) = \mathrm{Eul}(\delta, r, s, n_1, n_2, n_3) - \mathrm{B}(\delta, r, s, n_1, n_2, n_3),$$

provided $d \geq 2\delta + 1$*.*

We now state the remaining formulas.

Theorem 2.4.4. *If* $0 \leq \delta \leq 2$*, then*

$$\begin{aligned} N(A_1^{\delta}\mathcal{P}A_1, r, s, n_1, n_2, n_3, 0) &= 2N(A_1^{\delta+1}, r, s, n_1, n_2, n_3), \\ N(A_1^{\delta}\mathcal{P}A_1, r, s, n_1, n_2, n_3, 1) &= N(A_1^{\delta+1}, r, s, n_1, n_2+1, n_3) \\ &+ (d-6)N(A_1^{\delta+1}, r, s, n_1, n_2, n_3+1) \\ &+ 2N(A_1^{\delta+1}, r, s, n_1+1, n_2, n_3) \\ &- 2\binom{\delta}{2}N(A_1^{\delta-2}\mathcal{P}D_4, r, s, n_1, n_2, n_3, 0), \end{aligned}$$

provided $d \geq 2\delta + 2$*.*

Remark 2.4.5. To compute $N(A_1^{\delta}\mathcal{P}A_1, r, s, n_1, n_2, n_3, \theta)$ when $\theta \geq 2$, we use eq. (2.10).

Theorem 2.4.6. *Let* $0 \leq \delta \leq 2$ *and* θ *a non negative integer with the following property: if* δ *is either* 0 *or* 1*, then* θ *can be anything, but if* $\delta = 2$*, then* $\theta = 0$*. Then,*

$$\begin{aligned} N(A_1^{\delta}\mathcal{P}A_2, r, s, n_1, n_2, n_3, \theta) &= N(A_1^{\delta}\mathcal{P}A_1, r, s, n_1+1, n_2, n_3, \theta) \\ &+ N(A_1^{\delta}\mathcal{P}A_1, r, s, n_1, n_2+1, n_3, \theta) \\ &+ (d-3)N(A_1^{\delta}\mathcal{P}A_1, r, s, n_1, n_2, n_3+1, \theta) \\ &- 2\binom{\delta}{1}N(A_1^{\delta-1}\mathcal{P}A_3, r, s, n_1, n_2, n_3, \theta) \\ &- 3\binom{\delta}{1}N(A_1^{\delta-1}\widehat{D}_4, r, s, n_1, n_2, n_3, \theta) \\ &- 4\binom{\delta}{2}N(A_1^{\delta-2}\mathcal{P}D_4, r, s, n_1, n_2, n_3, \theta), \end{aligned}$$

provided $d \geq 2\delta + 2$.

Remark 2.4.7. If $\delta = 2$ and $\theta > 0$, then the formula given by Theorem 2.4.6 is not valid; there is a further correction term (the interested reader can refer to [3] to see what the extra correction term is). However, to compute $N(A_1^2 A_2, r, s, n_1, n_2, n_3)$ we only need to know what is $N(A_1^2 \mathcal{P} A_2, r, s, n_1, n_2, n_3, 0)$ and hence for the purposes of this paper, this Theorem is sufficient. We would require $N(A_1^2 \mathcal{P} A_2, r, s, n_1, n_2, n_3, \theta)$ for $\theta > 0$ if we were computing any of the codimension five (or higher) numbers; in this paper we are computing numbers till codimension four.

Theorem 2.4.8. *If* $0 \leq \delta \leq 1$, *then*

$$\begin{aligned} N(A_1^\delta \mathcal{P} A_3, r, s, n_1, n_2, n_3, \theta) = {} & N(A_1^\delta \mathcal{P} A_2, r, s, n_1, n_2 + 1, n_3, \theta) \\ & + 3N(A_1^\delta \mathcal{P} A_2, r, s, n_1, n_2, n_3, \theta + 1) \\ & + dN(A_1^\delta \mathcal{P} A_2, r, s, n_1, n_2, n_3 + 1, \theta) \\ & - 2\binom{\delta}{1} N(A_1^{\delta-1} \mathcal{P} A_4, r, s, n_1, n_2, n_3, \theta), \end{aligned}$$

provided $d \geq 2\delta + 3$.

Theorem 2.4.9. *If* $d \geq 4$, *then*

$$\begin{aligned} N(\mathcal{P} A_4, r, s, n_1, n_2, n_3, \theta) = {} & 2N(\mathcal{P} A_3, r, s, n_1, n_2 + 1, n_3, \theta) \\ & + 2N(\mathcal{P} A_3, r, s, n_1, n_2, n_3, \theta + 1) \\ & + 2N(\mathcal{P} A_3, r, s, n_1 + 1, n_2, n_3, \theta) \\ & + (2d - 6)N(\mathcal{P} A_3, r, s, n_1, n_2, n_3 + 1, \theta). \end{aligned}$$

Theorem 2.4.10. *If* $d \geq 3$, *then*

$$\begin{aligned} N(\mathcal{P} D_4, r, s, n_1, n_2, n_3, \theta) = {} & N(\mathcal{P} A_3, r, s, n_1, n_2 + 1, n_3, \theta) \\ & - 2N(\mathcal{P} A_3, r, s, n_1, n_2, n_3, \theta + 1) \end{aligned}$$

$$+\,2N(\mathcal{P}A_3, r, s, n_1+1, n_2, n_3, \theta)$$
$$+\,(d-6)N(\mathcal{P}A_3, r, s, n_1, n_2, n_3+1, \theta).$$

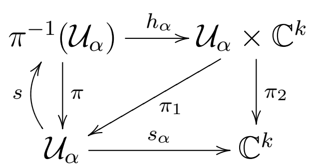
2.5 Definition of Vertical Derivative

We now recapitulate the concept of the vertical derivative which will be used extensively in proving the formulas stated in Section 2.4.

Definition 2.5.1. Let $\pi : V \longrightarrow M$ be a holomorphic vector bundle of rank k and $s : M \longrightarrow V$ a holomorphic section. Suppose $h_\alpha : V|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{U}_\alpha \times \mathbb{C}^k$ is a trivialization of V and

$$\pi_1, \pi_2 : \mathcal{U}_\alpha \times \mathbb{C}^k \longrightarrow \mathcal{U}_\alpha, \mathbb{C}^k$$

the projection maps. Let $s_\alpha := \pi_2 \circ h_\alpha \circ s$.



(2.18)

For $q \in \mathcal{U}_\alpha$, we define the **vertical derivative** of s at q to be the $\mathbb{C}$-linear map

$$\nabla s|_q : T_qM \longrightarrow V_q, \qquad \nabla s|_q \;:= (\pi_2 \circ h_\alpha)\,|_{V_q}^{-1} \circ ds_\alpha|_q. \tag{2.19}$$

Lemma 2.5.2. *The linear map $\nabla s|_q$ is well defined (i.e. independent of the trivialization) if $s(q) = 0$.*

Proof. Follows immediately from the product rule for derivatives. □

Remark 2.5.3. The vertical derivative $\nabla s|_q$ is also called the **linearization** of s at the point q. We will occasionally use this phrase in our paper. We note that the section s is transverse to zero if and only if for all points q such that $s(q) = 0$, the linear map $\nabla s|_q$ is surjective (that is, the linearization of s at the point q is surjective).

2.6 Proof of the recursive formulas

We are now ready to prove the formulas stated in Section 2.4. We will use a topological method to compute the degenerate contribution to the Euler class which is mentioned in the introduction.

When there is no cause for confusion, we abbreviate $N(A_1^\delta A_1, r, s, n_1, n_2, n_3)$ and $N(A_1^\delta \mathcal{P}\mathfrak{X}, r, s, n_1, n_2, n_3, \theta)$ as $N(A_1^\delta A_1)$ and $N(A_1^\delta \mathcal{P}\mathfrak{X})$ (for the sake of notational simplicity).

2.6.1 Proof of Theorem 2.4.1 and Theorem 2.4.3: computation of $N(A_1^\delta A_1)$ when $0 \leq \delta \leq 3$.

We will justify our formula for $N(A_1^{\delta+1}, r, s, n_1, n_2, n_3)$, when $0 \leq \delta \leq 3$. Recall that we have defined

$$A_1^\delta \circ \mathcal{S}_{\mathcal{D}} := \{([f], [\eta]; q_1, \ldots, q_\delta, q_{\delta+1}) \in \mathcal{D} \times (\mathbb{P}^3)^{\delta+1} : \eta(q_i) = 0, \ \forall i = 1 \ \text{ to } \ \delta+1,$$
$$f \text{ has a singularity of type } A_1 \text{ at } q_1, \ldots, q_\delta,$$
$$q_1, \ldots, q_{\delta+1} \text{ all distinct}\}.$$

Let μ be a generic cycle given by

$$\mu := \mathcal{H}_L^r \cdot \mathcal{H}_p^s \cdot a^{n_1} \lambda^{n_2} (\pi_{\delta+1}^* H)^{n_3}.$$

Here π_i denotes the projection onto the i^{th} point. We will often omit writing down $\pi_{\delta+1}^*$, if there is no cause for confusion. We now consider sections of the following two bundles that are induced by the evaluation map and the vertical derivative at the last point, namely:

$$\Psi_{A_0} : A_1^\delta \circ \mathcal{S}_{\mathcal{D}} \longrightarrow \mathcal{L}_{A_0} := \gamma_{\mathcal{D}}^* \otimes \pi_{\delta+1}^* \gamma_{\mathbb{P}^3}^{*d}, \text{ and}$$
$$\Psi_{A_1} : \psi_{A_0}^{-1}(0) \longrightarrow \mathcal{V}_{A_1} := \gamma_D^* \otimes \pi_{\delta+1}^* W^* \otimes \pi_{\delta+1}^* \gamma_{\mathbb{P}^3}^{*d},$$

defined by

$$\{\Psi_{A_0}([f],[\eta],q_1,\ldots,q_{\delta+1})\}(f) := f(q_{\delta+1}), \text{ and}$$
$$\{\Psi_{A_1}([f],[\eta],q_1,\ldots,q_{\delta+1})\}(f) := \nabla f|_{q_{\delta+1}}.$$

We will show shortly that Ψ_{A_0} and Ψ_{A_1} are transverse to zero, provided $d \geq 2\delta+1$.

Next, let us define

$$\mathcal{B} := \overline{A_1^\delta \circ \mathcal{S}}_{\mathcal{D}} - A_1^\delta \circ \mathcal{S}_{\mathcal{D}}.$$

Hence

$$\langle e(\mathcal{L}_{A_0}) e(\mathcal{V}_{A_1}), \ [\overline{A_1^\delta \circ \mathcal{S}_{\mathcal{D}}}] \cap \widetilde{\mu} \rangle = N(A_1^\delta A_1, r, s, n_1, n_2, n_3) + \mathcal{C}_{\mathcal{B}\cap\mu}, \tag{2.20}$$

where e denote the Euler class and $\mathcal{C}_{\mathcal{B}\cap\mu}$ denotes the contribution of the section to the Euler class from the points of $\mathcal{B} \cap \mu$.

Let us first explain how to compute the left hand side of eq. (2.20) (i.e., the Euler class). From eq. (2.1) and eq. (2.15), we note that

$$\mathcal{H}_L = \lambda + da, \ \mathcal{H}_p = \lambda a, \text{ and } [\pi_{\delta+1}^* \mathcal{S}_{\mathcal{D}}] = a + \pi_{\delta+1}^* H.$$

Next, we need to compute the Chern classes of W. We note that over $\mathcal{S}$, we have the following short exact sequence of vector bundles:

$$0 \longrightarrow W \longrightarrow T\mathbb{P}^3 \longrightarrow \gamma_{\widehat{\mathbb{P}}^3}^* \otimes \gamma_{\mathbb{P}^3}^* \longrightarrow 0.$$

Here the first map is the inclusion map and the second map is $\nabla\eta|_q$. Hence,

$$c(W)c(\gamma_{\widehat{\mathbb{P}}^3}^* \otimes \gamma_{\mathbb{P}^3}^*) = c(T\mathbb{P}^3).$$

From this relation we conclude that

$$c_1(W) = 3H - a, \text{ and } c_2(W) = a^2 - 2aH + 3H^2. \tag{2.21}$$

Next, using the splitting principle, we conclude that

$$e(\gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^3}^{*d})e(\gamma_{\mathcal{D}}^* \otimes W^* \otimes \gamma_{\mathbb{P}^3}^{*d}) = (\lambda + dH)((\lambda + dH)^2 - c_1(W)(\lambda + dH) + c_2(W)). \tag{2.22}$$

Note that we have made an abuse of notation by omitting to write down $\pi_{\delta+1}^*$; henceforth we will make this abuse of notation. Now, suppose $\delta = 0$. Then, using the ring structure of $\mathcal{D}$ (as given by eq. (2.7)) and by extracting the coefficient of $\lambda^{n-1}a^3H^3$ from

$$(a + H)(\lambda + dH)((\lambda + dH)^2 - c_1(W)(\lambda + dH) + c_2(W))(\lambda + da)^r(\lambda a)^s a^{n_1}\lambda^{n_2}H^{n_3},$$

we obtain the Euler class. When $\delta = 0$, using eq. (2.21), we get the formula of Theorem 2.4.1. When $\delta > 0$, we get $\mathrm{Eul}(\delta, r, s, n_1, n_2, n_3)$ as defined in eq. (2.16).

Let us now explain how to compute $\mathcal{C}_{\mathcal{B}\cap\mu}$, the degenerate contribution to the Euler class. When $\delta = 0$, the boundary $\mathcal{B}$ is empty, and hence we get the result of Theorem 2.4.1. Let us now consider the case when $\delta \geq 1$. Given k distinct integers $i_1, i_2, \ldots, i_k \in [1, \delta + 1]$, let us define

$$\Delta_{i_1,\ldots,i_k} := \{([f], [\eta]; q_1, \ldots, q_\delta, q_{\delta+1}) \in \mathcal{S}_{\mathcal{D}_{\delta+1}} : \quad q_{i_1} = q_{i_2} = \ldots = q_{i_k}\}, \text{ and}$$
$$\mathcal{B}(q_{i_1}, \ldots, q_{i_k}) := \mathcal{B} \cap \Delta_{i_1,\ldots,i_k}.$$

Let us now consider $\mathcal{B}(q_i, q_{\delta+1})$. We claim that

$$\mathcal{B}(q_i, q_{\delta+1}) \approx \overline{A_1^{\delta-1} \circ A_1} \,\forall\;\; i = 1 \text{ to } \delta, \tag{2.23}$$

where $\mathcal{B}(q_i, q_{\delta+1})$ is identified as a subset of $\mathcal{S}_{\mathcal{D}_\delta}$ in the obvious way (namely via the inclusion of $\mathcal{S}_{\mathcal{D}_\delta}$ inside $\mathcal{S}_{\mathcal{D}_{\delta+1}}$ where the $(\delta+1)^{\text{th}}$ point is equal to the i^{th} point). Next, we claim that the contribution from $\mathcal{B}(q_i, q_{\delta+1}) \cap \mu$ is given by

$$\langle e(\mathcal{L}_{A_0}),\; [\overline{A_1^{\delta-1} \circ A_1}] \cap [\mu]\rangle + 3N(A_1^{\delta-1}\mathcal{P}A_2, r, s, n_1, n_2, n_3, 0). \tag{2.24}$$

We will explain the reason for both the claims shortly. The expression given by eq. (2.24) is precisely equal to B_1 as defined in eq. (2.17). Hence, the sum total of the contribution from $\mathcal{B}(q_i, q_{\delta+1})$ for $i = 1$ to δ is $\binom{\delta}{1}\mathrm{B}_1$.

Next, let us assume $\delta \geq 2$ and consider $\mathcal{B}(q_{i_1}, q_{i_2}, q_{\delta+1})$. We claim that

$$\mathcal{B}(q_{i_1}, q_{i_2}, q_{\delta+1}) \approx \overline{A_1^{\delta-2} \circ A_3} \tag{2.25}$$

for all distinct pairs (i_1, i_2). We also claim that the contribution from each of the points of

$$\mathcal{B}(q_{i_1}, q_{i_2}, q_{\delta+1}) \cap \mu$$

is 4. We will justify both these claims shortly. Hence the sum total of the contribution as we vary over all (i_1, i_2) is precisely $\binom{\delta}{2}\mathrm{B}_2$, where B_2 is as defined in eq. (2.17).

Finally, let us assume $\delta \geq 3$ and consider $\mathcal{B}(q_{i_1}, q_{i_2}, q_{i_3}, q_{\delta+1})$. We claim that

$$\mathcal{B}(q_{i_1}, q_{i_2}, q_{i_3}, q_{\delta+1}) \approx \overline{A_1^{\delta-3} \circ A_5} \cup \overline{A_1^{\delta-3} \circ D_4} \tag{2.26}$$

for all distinct triples (i_1, i_2, i_3). Note that $\overline{A_1^{\delta-3} \circ A_5} \cap \mu$ is empty, since the sum of their dimensions is one less than the dimension of the ambient space where we are intersecting them. Hence, we get no contribution from $\overline{A_1^{\delta-3} \circ A_5} \cap \mu$. Finally, we claim that the contribution from each of the points of $\overline{A_1^{\delta-3} \circ D_4} \cap \mu$ is 18. Hence the sum total of the contribution as we vary over all (i_1, i_2, i_3) is precisely $\binom{\delta}{3}\mathrm{B}_3$, where B_3 is as defined in eq. (2.17).

Let us now prove the claims regarding transversality and degenerate contributions to the Euler class. We will start by proving transversality. We are going to show that if $d \geq 2\delta + 1$, then the sections Ψ_{A_0} and Ψ_{A_1}, defined on $A_1^\delta \circ \mathcal{S}_{\mathcal{D}}$ and $\Psi_{A_0}^{-1}(0)$, respectively, are transverse to zero. Since $A_1^{\delta+1}$ is precisely the zero set of the sections Ψ_{A_0} and Ψ_{A_1}, this implies that $A_1^{\delta+1}$ is a smooth complex submanifold of $\mathcal{S}_{\mathcal{D}_{\delta+1}}$.

Let us begin by showing that Ψ_{A_0} is transverse to zero if $d \geq 2\delta + 1$. We will be using induction on δ to prove our claim. We note that $\delta = 0$ is the base case of the induction (i.e. showing that A_1 is a smooth submanifold of $\mathcal{S}_{\mathcal{D}}$). Hence, if we are showing that $A_1^{\delta+1}$ is a smooth manifold, then we can assume by the induction hypothesis that A_1^δ is already a smooth submanifold of $\mathcal{S}_{\mathcal{D}_\delta}$. Hence, $A_1^\delta \circ \mathcal{S}_{\mathcal{D}}$ is also a smooth submanifold of $\mathcal{S}_{\mathcal{D}_{\delta+1}}$ (locally, $A_1^\delta \circ \mathcal{S}_{\mathcal{D}}$ is simply a product of A_1^δ with $\mathbb{P}^2$). Let us now assume that $([f], [\eta], q_1, \ldots, q_\delta, q_{\delta+1})$ belongs $A_1^\delta \circ \mathcal{S}_{\mathcal{D}}$ and that

$$\Psi_{A_0}([f], [\eta], q_1, \ldots, q_{\delta+1}) = 0.$$

Without loss of generality, let us assume that $[\eta]$ determines the plane where the last coordinate is zero, $q_{\delta+1}$ is the point where only the third coordinate is nonzero, and the rest are zero, that is,

$$\mathbb{P}^2_\eta \approx \{[X, Y, Z, W] \in \mathbb{P}^3 : W = 0\}, \text{ and } q_{\delta+1} := [0, 0, 1, 0].$$

Assume that the remaining points are given by

$$q_i := [X_i, Y_i, Z_i, 0] \text{ for } i = 1 \text{ to } \delta.$$

For simplicity, we can assume that all Z_i are nonzero. Furthermore, since all the q_i are distinct, we conclude that X_i and Y_i can not both be zero; for simplicity let us assume X_i is nonzero for each i (from 1 to δ). Consider the homogeneous degree d polynomial, given by

$$\rho_{00} := \Big(X - \frac{X_1}{Z_1}Z\Big)^2\Big(X - \frac{X_2}{Z_2}Z\Big)^2 \ldots \cdot \Big(X - \frac{X_\delta}{Z_\delta}Z\Big)^2 Z^{d-2\delta}.$$

We note the following facts about ρ_{00}:

$$\rho_{00}(q_i) = 0 \,\forall\, i = 1 \text{ to } \delta, \tag{2.27}$$

$$\nabla \rho_{00}|_{q_i} = 0 \,\forall\, i = 1 \text{ to } \delta, \text{ and} \tag{2.28}$$

$$\rho_{00}(q_{\delta+1}) \neq 0. \tag{2.29}$$

Now consider the curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{S}_{\mathcal{D}_{\delta+1}}$, given by

$$\gamma(t) := ([f + t\rho_{00}], [\eta], q_1, \ldots, q_{\delta+1}).$$

Because of eq. (2.27) and eq. (2.28), we conclude that this curve lies in $\overline{A}_1^\delta \circ \mathcal{S}_{\mathcal{D}}$. We now note that

$$\{\{\nabla \Psi_{A_0}|_{([f],[\eta],q_1,\ldots,q_{\delta+1})}\}(\gamma'(0))\}(f) = \rho_{00}(q_{\delta+1}). \tag{2.30}$$

Using eq. (2.29), we conclude that the right hand side of eq. (2.30) is nonzero, whence Ψ_{A_0} is transverse to zero. Next, let us prove transversality for the section Ψ_{A_1}. Consider the polynomials,

$$\rho_{10} := \Big(X - \frac{X_1}{Z_1}Z\Big)^2\Big(X - \frac{X_2}{Z_2}Z\Big)^2 \ldots \cdot \Big(X - \frac{X_\delta}{Z_\delta}Z\Big)^2 XZ^{d-2\delta-1}, \text{ and}$$
$$\rho_{01} := \Big(X - \frac{X_1}{Z_1}Z\Big)^2\Big(X - \frac{X_2}{Z_2}Z\Big)^2 \ldots \cdot \Big(X - \frac{X_\delta}{Z_\delta}Z\Big)^2 YZ^{d-2\delta-1}.$$

We note that ρ_{10} and ρ_{01} satisfy eq. (2.27) and eq. (2.28) (with ρ_{00} replaced with ρ_{10} and ρ_{01} respectively). Furthermore,

$$\rho_{10}(q_{\delta+1}) = 0, \text{ and } \rho_{01}(q_{\delta+1}) = 0. \tag{2.31}$$

Construct the curves

$$\gamma_{10}(t) := ([f + t\rho_{10}], [\eta], q_1, \ldots, q_{\delta+1}), \text{ and } \gamma_{01}(t) := ([f + t\rho_{01}], [\eta], q_1, \ldots, q_{\delta+1}).$$

Because of eq. (2.27) and eq. (2.28) (with ρ_{00} replaced with ρ_{10} and ρ_{01} respectively) and eq. (2.31), these curves lie inside $\Psi_{A_0}^{-1}(0)$. We now note that

$$\{\{\nabla\Psi_{A_1}|_{([f],[\eta],q_1,\ldots,q_{\delta+1})}\}(\gamma_{10}'(0))\}(f) = \lambda Z^{d-2\delta-1}\nabla X|_{[0,0,1,0]}, \text{ and}$$
$$\{\{\nabla\Psi_{A_1}|_{([f],[\eta],q_1,\ldots,q_{\delta+1})}\}(\gamma_{01}'(0))\}(f) = \lambda Z^{d-2\delta-1}\nabla Y|_{[0,0,1,0]},$$

where

$$\lambda := (-X_1/Z_1)^2 \cdot (-X_2/Z_2)^2 \ldots (-X_\delta/Z_\delta)^2.$$

Since $\nabla X|_{[0,0,1,0]}$ and $\nabla Y|_{[0,0,1,0]}$ are two linearly independent vectors of $T\mathbb{P}^2_\eta|_{[0,0,1,0]}$, we conclude that Ψ_{A_1} is transverse to zero.

Remark 2.6.1. We have actually proved something stronger than transversality. We have shown that the linearizations $\nabla\Psi_{A_0}$ and $\nabla\Psi_{A_1}$, restricted to the tangent space of $\mathcal{D}$ are surjective. This follows from the way we are proving the transversality claim; we produce a curve $\gamma_{ij}(t)$ that only changes the curve $[f]$. It keeps the plane $[\eta]$ and the points $q_1, \ldots, q_{\delta+1}$ unchanged. This fact will be used later on.

Let us now justify the closure and multiplicity claims. We will start by giving the reason for eq. (2.23) and eq. (2.24). Let us start by explaining why eq. (2.23) is true. For the convenience of the reader, let us explicitly rewrite eq. (2.23), i.e.,

$$\left(\overline{A_1^{\delta}\circ \mathcal{S}_{\mathcal{D}}} - A_1^{\delta}\circ \mathcal{S}_{\mathcal{D}}\right) \cap \Delta_{i,\delta+1} \approx \overline{A_1^{\delta-1}\circ A_1} \ \ \forall \ i = 1 \text{ to } \delta. \tag{2.32}$$

Let us first recall what is $A_1^{\delta}\circ \mathcal{S}_{\mathcal{D}}$; it is the space of curves with δ distinct ordered points on the curve (namely $q_1, \ldots, q_\delta$), such that the curve has an A_1 singularity at those δ points and one more point $q_{\delta+1}$ such that it is free (it may or may not lie on the curve). However, $q_{\delta+1}$ is distinct from all the other δ points. The left hand side of eq. (2.32) comprises of those elements of the closure, where $q_{\delta+1}$ becomes equal to q_i. That is clearly the right hand side of eq. (2.32), namely $\overline{A_1^{\delta-1}\circ A_1}$.

Let us now justify eq. (2.24). This is the same as the argument given in the proof [1, Lemma 6.3 (1), Page 685] and [1, Corollary 6.6, Page 689], we will explain the arguments to keep the discussion of the proof of Theorem 2.4.3 self contained. Let

$$([f], [\eta], q_1, \ldots, q_{\delta-1}, q_\delta, q_\delta) \in A_1^{\delta-1}\circ A_1.$$

We remind the reader here that the right hand side of the above expression is to be thought of as a subset of $\overline{A_1^{\delta}\circ \mathcal{S}_{\mathcal{D}}}$. Let $([f_t], [\eta_t], q_1(t), \ldots, q_{\delta-1}(t), q_\delta(t), q_{\delta+1}(t))$ be a nearby element. Let us consider an affine chart where we send the point $q_{\delta+1}(t)$ to $(0,0)$, and write down the Taylor expansion of f_t. Doing that, we conclude that

$$f_t(x,y) = f_{t_{00}} + f_{t_{10}}x + f_{t_{01}}y + \frac{f_{t_{20}}}{2}x^2 + f_{t_{11}}xy + \frac{f_{t_{02}}}{2}y^2 + \ldots \tag{2.33}$$

We note that as per our hypothesis, $([f], [\eta], q_\delta) \in A_1$, i.e. f has an A_1 singularity at the point q_δ. Hence, we conclude that $f_{20}f_{02} - f_{11}^2 \neq 0$. However, since f_t is close to f, and $q_{\delta+1}(t)$ is close to $q_{\delta+1}$, we conclude that

$$f_{t_{20}}f_{t_{02}} - (f_{t_{11}})^2 \neq 0. \tag{2.34}$$

To compute the contribution from the degenerate locus, we basically need to solve for the number of small solutions (x, y) to the set of equations:

$$f_t(x,y) = f_{t00} + f_{t10}x + f_{t01}y + \frac{f_{t20}}{2}x^2 + f_{t11}xy + \frac{f_{t02}}{2}y^2 + \ldots = \varepsilon_{00}, \tag{2.35}$$

$$f_{t_x}(x,y) = f_{t10} + f_{t20}x + f_{t11}y + \ldots = \varepsilon_{10}, \text{ and} \tag{2.36}$$

$$f_{t_y}(x,y) = f_{t01} + f_{t11}x + f_{t02}y + \ldots = \varepsilon_{01}, \tag{2.37}$$

where ε_{ij} is a small perturbation. By eq. (2.34) and the Inverse Function Theorem, we conclude that we can solve for small (x, y) using eq. (2.36) and eq. (2.37) and plug it in eq. (2.35). The right hand side of eq. (2.35) is basically the evaluation map written in local coordinates. Hence, the number of solutions to eq. (2.35) is precisely equal to

$$\langle e(\mathcal{L}_{A_0}),\ [\overline{A_1^{\delta-1} \circ A_1}] \cap \mu\rangle$$

Next, we need to consider the case when

$$([f], [\eta], q_1, \ldots, q_{\delta-1}, q_\delta, q_\delta) \in A_1^{\delta-1} \circ A_2.$$

Let $([f_t], [\eta_t], q_1(t), \ldots, q_{\delta-1}(t), q_\delta(t), q_{\delta+1}(t))$ be a nearby element. As before, let us consider an affine chart sending the point $q_{\delta+1}(t)$ to $(0, 0)$, and write down the Taylor expansion of f_t. Since $([f], [\eta], q_\delta) \in \overline{A}_2$, we conclude that f_{t00}, f_{t10}, f_{t01} and $f_{t20}f_{t02} - f_{t11}^2$ are small. Furthermore, since $([f], [\eta], q_{\delta+1}) \in A_2$, we conclude that f_{t20} and f_{t02} can not both be zero. Let us assume $f_{t02} \neq 0$. Hence, writing down the Taylor expansion, we conclude that

$$f_t(x,y) = f_{t00} + f_{t10}x + f_{t01}y + P_0(x) + P_1(x)y + P_2(x)y^2 + \ldots, \quad \text{where } P_2(0) \neq 0.$$

We will now make a change of coordinates; let us define

$$\hat{y} := y + B(x),$$

where $B(x)$ is a function to be determined. We claim that there exists a unique holomrphic $B(x)$ such that after plugging it in $\hat{f}_t(x, y)$ we get

$$f_t(x, y(x, \hat{y})) = f_{t00} + f_{t10}x + f_{t01}y - f_{t01}B(x) + \hat{P}_0(x) + \hat{P}_2(x)\hat{y}^2 + \hat{P}_3(x)\hat{y}^3 + \ldots$$

In other words, we want $\hat{P}_1(x) \equiv 0$. This is possible if $B(x)$ satisfies the equation

$$P_1(x) + 2P_2(x)B(x) + 3P_3(x)B(x)^2 + \ldots = 0. \tag{2.38}$$

Since $P_2(0) = \frac{\hat{f}_{t02}}{2} \neq 0$, $B(x)$ exists by the Implicit Function Theorem and we can compute $B(x)$ explicitly as a power series using eq. (2.38) and then compute $\hat{A}_0(x)$. Hence,

$$f_t(x, y(x, \hat{y})) = f_{t00} + f_{t10}x + f_{t01}y - f_{t01}B(x) + \varphi(x, \hat{y})\hat{y}^2 + \frac{\mathcal{B}_2^{f_t}}{2!}x^2 + \frac{\mathcal{B}_3^{f_t}}{3!}x^3 + \ldots, \tag{2.39}$$

where

$$\mathcal{B}_2^{f_t} := \hat{f}_{t20} - \frac{\hat{f}_{t11}^2}{\hat{f}_{t02}}, \qquad \mathcal{B}_3^{f_t} := \frac{\hat{f}_{t30}}{6} + \frac{f_{t11}^2\hat{f}_{t12}}{\hat{f}_{t02}^2} \neq 0, \quad \text{and} \quad \varphi(0,0) \neq 0. \tag{2.40}$$

For notational convenience, let us define

$$\hat{f}_t(x, \hat{y}) := f_t(x, y(x, \hat{y})), \tag{2.41}$$

i.e. $\hat{f}$ is basically f written in the new coordinates (namely x and $\hat{y}$). The contribution to the Euler class is the number of solutions to the set of equations

$$\hat{f}_t(x, \hat{y}) = \varepsilon_{00}, \quad \hat{f}_{t_x}(x, \hat{y}) = \varepsilon_{10}, \quad \text{and } \hat{f}_{t_{\hat{y}}}(x, \hat{y}) = \varepsilon_{01}, \tag{2.42}$$

where $(\mathcal{B}_2^{\hat{f}_t}, x, \hat{y})$ are small. Let us now try to solve eq. (2.42). By eliminating $\hat{y}$ and $\mathcal{B}_2^{\hat{f}_t}$ from the last two equations and plugging it into the first equation, we get that we need to solve for

$$\frac{\mathcal{B}_3^{f_t}}{12}x^3 + \mathrm{O}(x^4) = \varepsilon. \tag{2.43}$$

The number of solutions to eq. (2.43) is clearly 3, since $\mathcal{B}_3^{f_t}$ is non zero. This proves that the contribution from $A_1^{\delta-1} \circ A_1$ is given by eq. (2.24).

Next, let us justify eq. (2.25). Without loss of generality, it suffices to justify it when $i_1 := \delta - 1$ and $i_2 := \delta$. Hence, we need to show that

$$\{([f], [\eta], q_1, \ldots, q_{\delta+1}) \in \overline{A_1^\delta \circ \mathcal{S}_{\mathcal{D}}} : q_{\delta-1} = q_\delta = q_{\delta+1}\} = \overline{A_1^{\delta-2} \circ A_3}. \tag{2.44}$$

Before proceeding further, let us make a simple observation. Notice that the left hand side of eq. (2.44) is the same as

$$\{([f], [\eta], q_1, \ldots, q_{\delta+1}) \in \overline{A_1^\delta} : q_{\delta-1} = q_\delta\}. \tag{2.45}$$

Hence, an equivalent way of stating eq. (2.44) is

$$\{([f], [\eta], q_1, \ldots, q_\delta) \in \overline{A_1^\delta} : q_{\delta-1} = q_\delta\} = \overline{A_1^{\delta-2} \circ A_3}. \tag{2.46}$$

Following [1, Eq. (6.4), Page 685], we conclude that

$$\Big(\{([f], [\eta], q_1, \ldots, q_{\delta-1}, q_\delta, q_{\delta+1}) \in \overline{A_1^\delta \circ \mathcal{S}_{\mathcal{D}}} : q_{\delta-1} = q_\delta = q_{\delta+1}\}\Big) \cap \Big(A_1^{\delta-2} \circ A_2\Big) = \emptyset. \tag{2.47}$$

Eq. (2.47) is saying that if two nodes come together, then the singularity has to be more degenerate than a cusp. Hence, the singularity has to be at least as degenerate as a tacnode (since $\overline{A}_2 = A_2 \cup \overline{A}_3$). Hence, the left hand side of eq. (2.44) is a subset of its right hand side. We will now prove the converse. We will simultaneously prove the following four statements:

$$\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\overline{A_1^\delta\circ\mathcal{S}_{\mathcal{D}}}:q_{\delta-1}=q_\delta=q_{\delta+1}\}\supset A_1^{\delta-2}\circ A_3, \tag{2.48}$$

$$\Big(\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\overline{A_1^\delta\circ A_1}:q_{\delta-1}=q_\delta=q_{\delta+1}\}\Big)\cap\Big(A_1^{\delta-2}\circ A_3\Big)=\emptyset, \tag{2.49}$$

$$\Big(\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\overline{A_1^\delta\circ A_1}:q_{\delta-1}=q_\delta=q_{\delta+1}\}\Big)\cap\Big(A_1^{\delta-2}\circ A_4\Big)=\emptyset, \tag{2.50}$$

$$\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\overline{A_1^\delta\circ A_1}:q_{\delta-1}=q_\delta=q_{\delta+1}\}\supset A_1^{\delta-2}\circ A_5. \tag{2.51}$$

Since $\overline{A_1^\delta\circ\mathcal{S}_{\mathcal{D}}}$ is a closed set, eq. (2.48) implies that the right hand side of eq. (2.44) is a subset of its left hand side. Before we prove the above four statements, let us explain intuitively the significance of each of the statements.

The first statement, eq. (2.48) is saying that every tacnode is in the closure of two nodes (we remind the reader that the left hand side of eq. (2.48) is same as the expression given by eq. (2.45)). Geometrically, figure 2.1 explains the meaning of eq. (2.48).

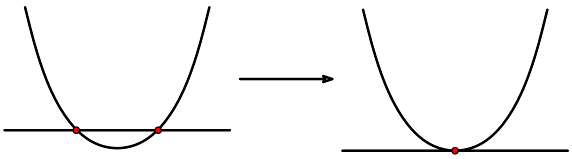


Figure 2.1: Two nodes colliding into a tacnode

The second statement, eq. (2.49) is saying that in the closure of three nodes, we get a singularity more degenerate than a tacnode. The third statement, eq. (2.50) is saying that in the closure of three nodes, we get a singularity more degenerate than an A_4 singularity. Finally, eq. (2.51) is saying that every A_5 singularity is in the closure of three nodes. Geometrically, figure 2.2 explains the meaning of eq. (2.51) We are now ready to prove the above statements. Let us prove the following two claims:

Claim 2.6.2. Let $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-2},q_\delta)\in A_1^{\delta-2}\circ A_3$. Then there exists points

$$\big([f_t],[\eta_t],q_1(t),\ldots,q_{\delta-2}(t);q_{\delta-1}(t),q_\delta(t),q_{\delta+1}(t)\big)\in A_1^{\delta-2}\circ\mathcal{S}_{\mathcal{D}}^3$$

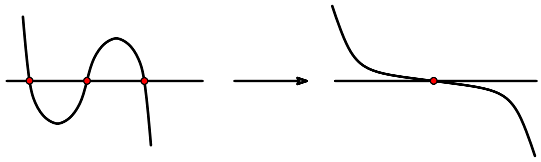


Figure 2.2: Three nodes colliding into an A_5-singularity

sufficiently close to $([f],[\eta],q_1,\dots,\dots,q_{\delta-2};q_\delta,q_\delta,q_\delta)$, such that

$$f_t(q_i(t))=0, \quad \nabla f_t|_{q_i(t)}=0 \ \text{ for } \ i=\delta-1 \text{ and } \delta. \tag{2.52}$$

Furthermore, *every* such solution satisfies the condition

$$\Big(f_t(q_{\delta+1}(t)),\ \nabla f_t|_{q_{\delta+1}(t)}\Big)\neq(0,0), \tag{2.53}$$

that is, $\big([f_t],[\eta_t],q_1(t),\dots,q_{\delta-2}(t);q_{\delta-1}(t),q_\delta(t),q_{\delta+1}(t)\big)\notin A_1^\delta\circ A_1$. In fact, if

$$([f],[\eta],q_1,\dots,\dots,q_\delta)\in A_1^{\delta-2}\circ A_4,$$

then there does not exist any point

$$\big([f_t],[\eta_t],q_1(t),\dots,q_{\delta-2}(t);q_{\delta-1}(t),q_\delta(t),q_{\delta+1}(t)\big)\in A_1^{\delta-2}\circ\mathcal{S}_{\mathcal{D}}^3$$

sufficiently close to $([f],[\eta],q_1,\dots,\dots,q_{\delta-2};q_\delta,q_\delta,q_\delta)$, such that

$$f_t(q_i(t))=0, \quad \nabla f_t|_{q_i(t)}=0 \ \text{ for } \ i=\delta-1,\delta, \text{ and } \delta+1. \tag{2.54}$$

Claim 2.6.3. Let $([f],[\eta],q_1,\dots,\dots,q_{\delta-2},q_\delta)\in A_1^{\delta-2}\circ A_5$. Then there exists points

$$\big([f_t],[\eta_t],q_1(t),\dots,q_{\delta-2}(t);q_{\delta-1}(t),q_\delta(t),q_{\delta+1}(t)\big)\in A_1^{\delta-2}\circ A_1^3$$

sufficiently close to $([f],[\eta],q_1,\dots,\dots,q_{\delta-2};q_\delta,q_\delta,q_\delta)$.

Remark 2.6.4. We note Claim 2.6.2 proves eq. (2.48), eq. (2.49) and eq. (2.50) simultaneously. We also note that Claim 2.6.3 proves eq. (2.51).

Remark 2.6.5. Before proceeding with the proof of the claims, let us make a shorthand notation. We denote

$$O(|(x_1, x_2, \ldots, x_n)|^k)$$

to be a **holomorphic** function (in the variables $x_1, \ldots, x_n$), defined in a neighborhood of the origin in $\mathbb{C}^n$, whose order of vanishing is at least k (i.e., all the terms of degree lower than k are absent in the Taylor expansion of the function around the origin). We say that such an expressions is of order k. For example, $x_1^4 + x_1 x_2 x_3^2 + x_2^2 x_3^3$ is a term of order 4 and we will denote it by $O(|(x_1, x_2, x_3)|^4)$. Note that we are always dealing with holomorphic functions. Hence, suppose a function (in say one variable) is of type $O(|x|^2)$, it means, its Taylor expansion is of the type

$$f(x) = a_2 x^2 + a_3 x^3 + \ldots .$$

It does not mean that there are terms of type $x\overline{x}$ (although the $|x|^2$ in the $O(|x|^2)$ might suggest that). Henceforth, it will be understood that $O(|x|^n)$ and $O(x^n)$ mean the same thing in our paper (the latter is the standard notation in one variable).

Proof of claims 2.6.2 and 2.6.3: Let us define

$$\mathbb{C}^2_z := \{(x, y, z) \in \mathbb{C}^3 : z = 0\}.$$

We will now work in an affine chart where we send the plane $\mathbb{P}^2_{\eta_t}$ to $\mathbb{C}^2_z$ and the point $q_\delta(t) \in \mathbb{P}^2_{\eta_t}$ to $(0, 0, 0) \in \mathbb{C}^2_z$. Using this chart, let us write down the Taylor expansion of f_t around the point $(0, 0)$, namely

$$f_t(x, y) = \frac{f_{t_{20}}}{2} x^2 + f_{t_{11}} xy + \frac{f_{t_{02}}}{2} y^2 + \ldots$$

Note that since eq. (2.52) holds (for $i = \delta$), we conclude that $f_{t_{00}}$, $f_{t_{10}}$ and $f_{t_{01}}$ are zero.

Next, since $([f], [\eta], q_\delta) \in A_3$, we conclude that $f_{t_{20}}$ and $f_{t_{02}}$ can not both be zero; let us assume $f_{t_{02}} \neq 0$. Hence, $f_t(x, y)$ can be re-written as

$$f_t(x, y) = P_0(x) + P_1(x)y + P_2(x)y^2 + \dots,$$

where $P_2(0) \neq 0$. We will now make a change of coordinates; let us define

$$\hat{y} := y - B(x)$$

where $B(x)$ is a function that is to be determined. We claim that there exists a unique holomorphic $B(x)$ (vanishing at the origin) such that after plugging it in $f_t(x, y)$ we get

$$f_t(x, y(x, \hat{y})) = \hat{P}_0(x) + \hat{P}_2(x)\hat{y}^2 + \hat{P}_3(x)\hat{y}^3 + \dots$$

In other words, we want $\hat{P}_1(x) \equiv 0$. This is possible if $B(x)$ satisfies the equation

$$P_1(x) + 2P_2(x)B(x) + 3P_3(x)B(x)^2 + \dots = 0. \tag{2.55}$$

Since $P_2(0) = \frac{f_{t02}}{2} \neq 0$, $B(x)$ exists by the Implicit Function Theorem and we can compute $B(x)$ explicitly as a power series using eq. (2.55) and then compute $\hat{A}_0(x)$. Hence,

$$f_t(x, y(x, \hat{y})) = \varphi(x, \hat{y})\hat{y}^2 + \frac{\mathcal{B}_2^{f_t}}{2!}x^2 + \frac{\mathcal{B}_3^{f_t}}{3!}x^3 + \frac{\mathcal{B}_4^{f_t}}{4!}x^4 + \mathcal{R}(x)x^5,$$

where

$$\mathcal{B}_2^{f_t} := f_{t20} - \frac{f_{t11}^2}{f_{t02}}, \quad \text{and} \quad \mathcal{B}_3^{f_t} := \frac{f_{t30}}{6} + \frac{f_{t11}^2 f_{t12}}{f_{t02}^2}, \dots\dots, \qquad \varphi(0,0) \neq 0.$$

Also $\mathcal{R}(x)$ is a holomorphic function defined in a neighborhood of the origin.

Since $([f], [\eta], q_\delta) \in A_3$, we conclude that $\mathcal{B}_2^{f_t}$ and $\mathcal{B}_3^{f_t}$ are small (close to zero) and $\mathcal{B}_4^{f_t}$ is nonzero. Let us make a further change of coordinates and denote

$$\hat{\hat{y}} := \sqrt{\varphi(x,\hat{y})}\hat{y}.$$

Note that we can choose a branch of the square root since $\varphi(0,0) \neq 0$. Next, for notational convenience, let us now define

$$\hat{f}_t(x,\hat{\hat{y}}) := f_t(x, y(x, \hat{y}(\hat{\hat{y}})))), \tag{2.56}$$

i.e., $\hat{f}_t$ is basically f_t written in the new coordinates (namely x and $\hat{\hat{y}}$). Hence,

$$\hat{f}_t(x,\hat{\hat{y}}) = \hat{\hat{y}}^2 + \frac{\mathcal{B}_2^{f_t}}{2!}x^2 + \frac{\mathcal{B}_3^{f_t}}{3!}x^3 + \frac{\mathcal{B}_4^{f_t}}{4!}x^4 + \mathcal{R}(x)x^5.$$

We will now solve eq. (2.52) for $i = \delta - 1$. We note that this is amounts to solving for the set of equations

$$\hat{f}_t(u,v) = 0,\ \hat{f}_{t_x}(u,v) = 0,\ \text{and}\ \ \hat{f}_{t_{\hat{\hat{y}}}}(u,v) = 0, \tag{2.57}$$

where (u, v) is small but not equal to $(0, 0)$, **and** requiring $\hat{f}_t$ to have $\delta - 2$ more nodes (all distinct from each other and distinct from $(0, 0)$ and (u, v)). The solutions to eq. (2.57) are given by

$$\begin{aligned} v &= 0 \\ \mathcal{B}_2^{f_t} &= \frac{\mathcal{B}_4^{f_t}}{12}u^2 + 4u^3\mathcal{R}(u) + 2u^4\mathcal{R}'(u),\ \text{and} \\ \mathcal{B}_3^{f_t} &= -\frac{\mathcal{B}_4^{f_t}}{2}u - 18u^2\mathcal{R}(u) - 6u^3\mathcal{R}'(u). \end{aligned} \tag{2.58}$$

To see how, we first use the third equation of eq. (2.57) to get $v = 0$. Then we use the second and first equations of eq. (2.57) to get the value of $\mathcal{B}_2^{f_t}$ and $\mathcal{B}_3^{f_t}$. This concludes the proof of the claim when $\delta = 2$.

Let us now assume $\delta > 2$. Let f_t be a curve obtained from eq. (2.58). We note that this curve has two nodes; it has one node at $(0,0)$ and another node at the point $(\hat{x},\hat{y}) = (u,0)$. Let us call this point $q_\delta(t)$. We will also call $(0,0)$ as $q_{\delta+1}(t)$. We now need to produce a third node. We recall that the original curve f had a node at q_1. Hence, the curve f_t evaluated at q_1 and the derivatives evaluated at q_1 are small (but not necessarily zero). We claim that we can find a curve g_t close to f_t such that g_t will continue to have nodes at $q_{\delta+1}(t)$ and $q_\delta(t)$ and it will also have a node at q_1. Let us see why this is true. For that, we first make a digression.

Let us first consider the following situation:

Lemma 2.6.6. *Let $F : \mathbb{C}^m \times \mathbb{C}^k \longrightarrow \mathbb{C}^k$ be a holomorphic function such that whenever $F(\overline{x},\overline{y}) = 0$, the differential $dF|_{(\overline{x},\overline{y})}$ restricted to the tangent space $T\mathbb{C}^m|_{\overline{x}}$ is surjective. Suppose $(\overline{x},\overline{y})$ is a point such that*

$$F(\overline{x},\overline{y}) = \varepsilon,$$

where ε is sufficiently small. Then there exists a $\overline{x}_\varepsilon$, close to $\overline{x}$, such that

$$F(\overline{x}_\varepsilon,\overline{y}) = 0$$

Proof: Follows from the Implicit Function Theorem.

Let us now continue and make a similar statement for sections of Vector Bundles.

Lemma 2.6.7. *Let $\psi : X \times Y \longrightarrow V$ be a holomorphic section of a rank k vector bundle (and the dimension of Y is also equal to k). Suppose whenever $\psi(x,y) = 0$, the vertical derivative $\nabla\psi|_{(x,y)}$ restricted to the tangent space $TX|_x$ is surjective. Suppose (x,y) is a point such that $\psi(x,y)$ is small (to be made sense by choosing a metric on the vector bundle). Then there exists a x_ε, close to x, such that*

$$\psi(x_\varepsilon,y) = 0$$

Proof: Follows from Lemma 2.6.6, by unwinding definitions and writing down the section in local coordinates.

We now continue with our earlier discussion. We need to show that f_t can be modified a little bit to retain the first two nodes and also produce a third node at q_1. First we note that

$$([f_t], q_\delta(t), q_{\delta+1}(t), q_1) \in A_1^2 \circ \mathcal{S}_{\mathcal{D}}.$$

The last point q_1 does not lie on the curve, but both the evaluation map and the vertical derivative give us something small. We have already shown that for the evaluation map and the vertical derivative, the lineraization of the section, restricted to the Tangent space of the space of curves (namely $\mathcal{D}$) is surjective (see Remark 2.6.1). Hence, using Lemma 2.6.7, we conclude that we can find a curve g_t close to f_t such that g_t has nodes at $q_{\delta+1}(t)$ and $q_\delta(t)$ and also has a node at q_1. We can now similarly produce another node at q_2 and so on till $q_{\delta-2}$. This completes the proof of eq. (2.52).

Next, let us prove eq. (2.53), i.e., we have to show that in a neighborhood of a tacnode, we can not have a curve with three distinct nodes. More precisely, we need to show that there can not be any solutions to the set of equations

$$\hat{f}_t(u_1, v_1) = 0, \quad \hat{f}_{t_x}(u_1, v_1) = 0, \quad \hat{f}_{t_{\hat{y}}}(u_1, v_1) = 0, \tag{2.59}$$

$$\hat{f}_t(u_2, v_2) = 0, \quad \hat{f}_{t_x}(u_2, v_2) = 0, \quad \hat{f}_{t_{\hat{y}}}(u_2, v_2) = 0, \tag{2.60}$$

where $(0,0)$, (u_1, v_1), (u_2, v_2) are all distinct (but small). Let us try to solve for the above set of equations. Let us first explicitly write down $\hat{f}_t(x, \hat{y})$ as

$$\hat{f}_t(x, \hat{y}) = \hat{y}^2 + \frac{\mathcal{B}_2^{f_t}}{2!}x^2 + \frac{\mathcal{B}_3^{f_t}}{3!}x^3 + \frac{\mathcal{B}_4^{f_t}}{4!}x^4 + \frac{\mathcal{B}_5^{f_t}}{5!}x^5 + \frac{\mathcal{B}_6^{f_t}}{6!}x^6 + \ldots \tag{2.61}$$

To begin with, we unwind eq. (2.59) using the expression for $\hat{f}_t$ as given by eq. (2.61) and solve for $\mathcal{B}_2^{f_t}$ and $\mathcal{B}_3^{f_t}$ in terms of $u_1, v_1, \mathcal{B}_4^{f_t}, \mathcal{B}_5^{f_t}$ and $\mathcal{B}_6^{f_t}$. We then plug in these values for

$\mathcal{B}_2^{f_t}$ and $\mathcal{B}_3^{f_t}$ in eq. (2.61) and plug it in eq. (2.60). Now we can solve for $\mathcal{B}_4^{f_t}$ and $\mathcal{B}_5^{f_t}$ in terms of $\mathcal{B}_6^{f_t}$ and then plugging back those values in the previous expressions for $\mathcal{B}_2^{f_t}$ and $\mathcal{B}_3^{f_t}$, gives us their values in terms of $\mathcal{B}_6^{f_t}$. Doing that, we get

$$\begin{aligned}
v_1, v_2 &= 0,\\
\mathcal{B}_2^{f_t} &= \frac{1}{360}\mathcal{B}_6^{f_t} u_1^2 u_2^2 + O(|(u_1,u_2)|^5),\\
\mathcal{B}_3^{f_t} &= -\frac{1}{60}\mathcal{B}_6^{f_t}(u_1^2 u_2 + u_1 u_2^2) + O(|(u_1,u_2)|^4),\\
\mathcal{B}_4^{f_t} &= \frac{1}{30}\mathcal{B}_6^{f_t}(u_1^2 + 4u_1 u_2 + u_2^2) + O(|(u_1,u_2)|^3), \text{ and}\\
\mathcal{B}_5^{f_t} &= -\frac{1}{3}\mathcal{B}_6^{f_t}(u_1 + u_2) + O(|(u_1,u_2)|^2), \qquad (2.62)
\end{aligned}$$

where $O(|(u_1,u_2)|^n)$ is as defined in Remark 2.6.5. Hence, $\mathcal{B}_4^{f_t}$ is close to zero, which is a contradiction, since $([f],[\eta],q_\delta) \in A_3$. Since $\mathcal{B}_5^{f_t}$ is also close to zero, we get the last part of the Claim 2.6.2 (i.e., eq. (2.50)). Finally, we note that the solutions constructed in eq. (2.62) immediately prove Claim 2.6.3 (in fact these are the *only* possible solutions). This finishes the proof of Claim 2.6.2 and Claim 2.6.3. □

Next, we claim that each point of $(A_1^{\delta-2} \circ A_3) \cap \mu$ contributes 4 to the Euler class in eq. (2.20). Using eq. (2.58), we conclude that the multiplicity is the number of small solutions $(x, \hat{\hat{y}}, u)$ to the following set of equations

$$\begin{aligned}
\hat{f}_t(x,\hat{\hat{y}}) &:= \hat{\hat{y}}^2 + \frac{\mathcal{B}_2^{f_t}}{2!}x^2 + \frac{\mathcal{B}_3^{f_t}}{3!}x^3 + \frac{\mathcal{B}_4^{f_t}}{4!}x^4 + \mathcal{R}(x)x^5 = \varepsilon_0,\\
\hat{f}_{tx}(x,\hat{\hat{y}}) &:= \mathcal{B}_2^{f_t}x + \frac{\mathcal{B}_3^{f_t}}{2}x^2 + \frac{\mathcal{B}_4^{f_t}}{12}x^3 + 5x^4\mathcal{R}(x) + \mathcal{R}'(x)x^5 = \varepsilon_1,\\
\hat{f}_{t\hat{\hat{y}}}(x,\hat{\hat{y}}) &:= 2\hat{\hat{y}} = \varepsilon_2,\\
\mathcal{B}_2^{f_t} &= \frac{\mathcal{B}_4^{f_t}}{12}u^2 + 4u^3\mathcal{R}(u) + 2u^4\mathcal{R}'(u), \text{ and}\\
\mathcal{B}_3^{f_t} &= -\frac{\mathcal{B}_4^{f_t}}{2}u - 18u^2\mathcal{R}(u) - 6u^3\mathcal{R}'(u),
\end{aligned}$$

where $(\varepsilon_0,\varepsilon_1,\varepsilon_2) \in \mathbb{C}^3$ is small and generic. Let us write $u := h + x$ and Taylor expansion of $\mathcal{R}(x+h)$ and $\mathcal{R}'(x+h)$ around $h=0$, that is,

$$\mathcal{R}(x+h) = \mathcal{R}(x) + h\mathcal{R}'(x) + \frac{h^2}{2}\mathcal{R}''(x) + \dots, \text{ and}$$
$$\mathcal{R}'(x+h) = \mathcal{R}'(x) + h\mathcal{R}''(x) + \dots \tag{2.63}$$

Hence, substituting the values of $\mathcal{B}_2^{f_t}$, $\mathcal{B}_3^{f_t}$, $\mathcal{R}(x+h)$ and $\mathcal{R}'(x+h)$, we conclude that we need to find the number of small solutions (x,h) to the following set of equations

$$\frac{(x^2h^2)\Big(\mathcal{B}_4^{f_t} + O(|(x,h)|)\Big)}{24} = \varepsilon_3, \text{ and} \tag{2.64}$$
$$\frac{(xh)\Big(\mathcal{B}_4^{f_t}h - \mathcal{B}_4^{f_t}x + O(|(x,h)|^2)\Big)}{12} = \varepsilon_1, \tag{2.65}$$

where $\varepsilon_3 := \varepsilon_0 - \frac{\varepsilon_2^2}{4}$. We claim that we can set ε_1 to be 0; that is justified in section 2.6.1. Assuming that claim, we use eq. (2.65) to solve for x in terms of h and conclude that

$$x = h + O(h^2). \tag{2.66}$$

This is because $x=0$ and $h=0$ can not be solutions to eq. (2.65) (since if we plug it back in eq. (2.64), we will get 0 and not ε_3). Plugging in the value of x from eq. (2.66) into eq. (2.64), we get

$$\frac{\mathcal{B}_4^{f_t}}{24}h^4 + O(h^5) = \varepsilon_3. \tag{2.67}$$

Eq. (2.67) clearly has 4 solutions.

Finally, we need to justify eq. (2.26) and the corresponding contribution to the Euler class. More precisely, we are going to show that

$$\{([f],[\eta],q_1,\dots,q_{\delta+1}) \in \overline{A_1^{\delta}\circ \mathcal{S}_{\mathcal{D}}} : q_{\delta-2}=q_{\delta-1}=q_\delta=q_{\delta+1}\} = \overline{A_1^{\delta-3}\circ A_5} \cup \overline{A_1^{\delta-3}\circ D_4}. \tag{2.68}$$

Just like eq. (2.44) is equivalent to eq. (2.46), we similarly conclude that eq. (2.68) can be equivalently stated as

$$\{([f],[\eta],q_1,\ldots,q_\delta)\in\overline{A_1^\delta}:q_{\delta-2}=q_{\delta-1}=q_\delta\}=\overline{A_1^{\delta-3}\circ A_5}\cup\overline{A_1^{\delta-3}\circ D_4}. \tag{2.69}$$

Let us define

$$\begin{aligned}W_1&:=\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\mathcal{S}_{\mathcal{D}_{\delta+1}}:f(q_{\delta+1})=0,\ \nabla f|_{q_{\delta+1}}=0,\ \ \nabla^2 f|_{q_{\delta+1}}\neq 0\},\\ W_2&:=\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\mathcal{S}_{\mathcal{D}_{\delta+1}}:f(q_{\delta+1})=0,\ \nabla f|_{q_{\delta+1}}=0,\ \ \nabla^2 f|_{q_{\delta+1}}=0\}.\end{aligned} \tag{2.70}$$

In order to prove eq. (2.68), it suffices to show that

$$\Big(\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\overline{A_1^\delta\circ\mathcal{S}_{\mathcal{D}}}:q_{\delta-2}=q_{\delta-1}=q_\delta=q_{\delta+1}\}\Big)\cap W_1 = \Big(\overline{A_1^{\delta-3}\circ A_5}\Big)\cap W_1 \quad \text{and} \tag{2.71}$$

$$\Big(\{([f],[\eta],q_1,\ldots,q_{\delta+1})\in\overline{A_1^\delta\circ\mathcal{S}_{\mathcal{D}}}:q_{\delta-2}=q_{\delta-1}=q_\delta=q_{\delta+1}\}\Big)\cap W_2=\overline{A_1^{\delta-3}\circ D_4}. \tag{2.72}$$

Note that the right hand side of eq. (2.72) is a subset of W_2; hence we didn't write down $\cap W_2$ on the right hand side of eq. (2.72). Let us first justify eq. (2.71). Eq. (2.49) and eq. (2.50), show that the the left hand side of (2.71) is a subset of its right hand side. Furthermore, eq. (2.51) shows that the right hand side of (2.71) is a subset of its left hand side; hence eq. (2.71) is true.

We will now prove eq. (2.72). Eq. (2.49) shows that the left hand side of eq. (2.72) is a subset of its right hand side. Hence, what remains is to show that the right hand side of eq. (2.72) is a subset of its left hand side. Before we start the proof of that assertion, let us give an intuitive idea about the significance of that statement. The statement is saying that every triple point is in the closure of three nodes. To summarize, the geometric sig-

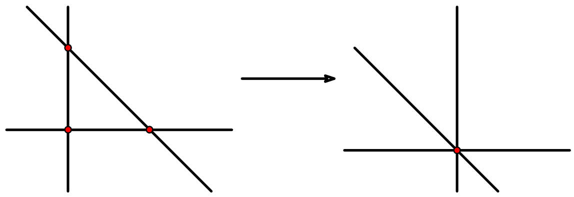


Figure 2.3: Three nodes colliding into a triple point

nificance of eq. (2.71) is given by figure 2.2 while the geometric significance of eq. (2.72) is given by figure 2.3. Eq. (2.68) says that these are the **only** two pictures that can occur.

Let us now prove eq. (2.72). We will prove the following claim:

Claim 2.6.8. Let $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-3},q_\delta) \in A_1^{\delta-3}\circ D_4$. Then, there exists points

$$\big([f_t],[\eta_t],q_1(t),\ldots,q_{\delta-3}(t);q_{\delta-2}(t),q_{\delta-1}(t),q_\delta(t),q_{\delta+1}(t)\big) \in A_1^{\delta-3}\circ\mathcal{S}_{\mathcal{D}}^4$$

sufficiently close to $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-3};q_\delta,q_\delta,q_\delta,q_\delta)$ such that

$$f_t(q_i(t))=0, \quad \nabla f_t|_{q_i(t)}=0 \ \text{ for } \ i=\delta-2,\delta-2, \ \text{and } \delta. \tag{2.73}$$

Remark 2.6.9. We note that claim 2.6.8 implies that the right hand side of eq. (2.72) is a subset of the left hand side.

Proof: Following the setup of the proof of claim 2.6.2, we will now work in an affine chart, where we send the plane $\mathbb{P}^2_{\eta t}$ to $\mathbb{C}^2_z$ and the point $q_\delta(t)\in\mathbb{P}^2_{\eta t}$ to $(0,0,0)\in\mathbb{C}^2_z$. Using this chart, let us write down the Taylor expansion of f_t around the point $(0,0)$, namely

$$f_t(x,y)=\frac{f_{t_{20}}}{2}x^2+f_{t_{11}}xy+\frac{f_{t_{02}}}{2}y^2+\frac{f_{t_{30}}}{6}x^3+\frac{f_{t_{21}}}{2}x^2y+\frac{f_{t_{12}}}{2}xy^2+\frac{f_{t_{03}}}{6}y^3+\ldots \tag{2.74}$$

Since $([f],[\eta],q_\delta)\in D_4$, we conclude that $f_{t_{20}}$, $f_{t_{11}}$ and $f_{t_{02}}$ are all small (close to zero). Let us now construct solutions to eq. (2.73). Let us assume that the points $q_{\delta-1}(t)$ and $q_{\delta-2}(t)$ are sent to $(x_1,y_1,0)$ and $(x_2,y_2,0)$ under the affine chart we are considering.

Hence, constructing solutions to eq. (2.73) is same as constructing solutions to the set of equations

$$f_t(x_1, y_1) = 0, \qquad f_{t_x}(x_1, y_1) = 0, \qquad f_{t_y}(x_1, y_1) = 0, \text{ and} \tag{2.75}$$

$$f_t(x_2, y_2) = 0, \qquad f_{t_x}(x_2, y_2) = 0, \qquad f_{t_y}(x_2, y_2) = 0, \tag{2.76}$$

where $(0,0), (x_1, y_1)$ and (x_2, y_2) are all distinct (but close to each other).

Next, let us define

$$g_t(x, y) := x f_{t_x}(x, y) + y f_{t_y}(x, y) - 2 f_t(x, y). \tag{2.77}$$

The quantity $g(x, y)$ is similarly defined with f_t replaced by f. We note that solving eq. (2.75) and eq. (2.76) is equivalent to solving

$$g_t(x_1, y_1) = 0, \qquad f_{t_x}(x_1, y_1) = 0, \qquad f_{t_y}(x_1, y_1) = 0 \qquad \text{and} \tag{2.78}$$

$$g_t(x_2, y_2) = 0, \qquad f_{t_x}(x_2, y_2) = 0, \qquad f_{t_y}(x_2, y_2) = 0, \tag{2.79}$$

where $(0,0), (x_1, y_1)$ and (x_2, y_2) are all distinct (but close to each other). We now note that $g_t(x, y)$ and $f_t(x, y)$ have exactly the same cubic term in the Taylor expansion. Furthermore, $g_t(x, y)$ has no quadratic term.

Let us now study the cubic term of the Taylor expansion of f carefully. Let us assume first $f_{30} \neq 0$. Since $([f], [\eta], q) \in D_4$, we conclude that the cubic term factors into three **distinct** linear factors. Hence, the cubic term can be written as

$$\frac{f_{30}}{6}(x - P_1(0)y)(x - P_2(0)y)(x - P_3(0)y), \tag{2.80}$$

where $P_1(0), P_2(0)$ and $P_3(0)$ are all distinct. Note that $P_1(0), P_2(0)$ and $P_3(0)$ are explicit expressions involving the coefficients f_{ij}. If $f_{30} = 0$, then the cubic term will be of the type

$$\frac{f_{21}}{2}y(x-P_1(0)y)(x-P_2(0)y),$$

where $P_1(0)$ and $P_2(0)$ are distinct and f_{21} is nonzero. We will assume that $f_{30} \neq 0$; the case $f_{30} = 0$ can be dealt with similarly. Hence, g_t (or equivalently f_t) can be written as

$$g_t(x,y) = \frac{f_{t30}}{6}(x-P_1y)(x-P_2y)(x-P_3y) + O(|(x,y)|^4), \tag{2.81}$$

where P_i are the same as $P_i(0)$, but with the f_{ij} replaced by $f_{t_{ij}}$. For notational simplicity, we denoted these quantities by the letter P_i and not $P_i(t)$.

Let us now make a change of coordinates

$$x := \hat{x} + O(|(\hat{x},\hat{y})|^2), \text{ and } y := \hat{y} + O(|(\hat{x},\hat{y})|^2) \tag{2.82}$$

such that

$$g_t = \frac{f_{t30}}{6}(\hat{x}-P_1\hat{y})(\hat{x}-P_2\hat{y})(\hat{x}-P_3\hat{y}). \tag{2.83}$$

Hence, $g_t = 0$ has three distinct solutions, given by $\hat{x} = P_i\hat{y}$ for $i = 1, 2$ and 3. Converting back in terms of x, we conclude that the solutions to $g_t(x,y) = 0$ (where (x,y) is small but nonzero) are given by

$$y = u, \text{ and } x = P_iu + E_i(u), \tag{2.84}$$

where $E_i(u)$ is a second order term in u (and u is small but nonzero).

Next, for notational simplicity we will denote $f_{t_{02}}$ by the letter w. Let us consider the solution $y := u$ and $x = P_1u + E_1(u)$ of the equation $g_t(x,y) = 0$. Plugging this in

$f_{t_x}(x,y)=0$ and $f_{t_y}(x,y)=0$ and solving for $P_1 f_{t_{11}}$ and $P_1^2 f_{t_{20}}$, we conclude that

$$\begin{aligned} P_1 f_{t_{11}} &= \frac{P_1 f_{t_{30}}}{6}(P_1-P_2)(P_1-P_3)u - w + O(|(u,w)|^2), \text{ and} \\ P_1^2 f_{t_{20}} &= -\frac{P_1 f_{t_{30}}}{3}(P_1-P_2)(P_1-P_3)u + w + O(|(u,w)|^2). \end{aligned} \tag{2.85}$$

Let us now consider a second solution to $g_t(x,y)=0$ (where (x,y) is small but nonzero). This will be given by $y := v$ and $x := P_2 v + E_2(v)$, where v is small but nonzero (or the analogous thing with P_2 replaced by P_3). Using eq. (2.85) to express the values of $f_{t_{11}}$ and $f_{t_{20}}$ in terms of u and w and then using $f_{t_x}(x,y)=0$, we conclude that

$$w = \frac{f_{t_{30}}}{6}\Big(P_1^3 - 2P_1^2P_2 - P_1^2P_3 + 2P_1P_2P_3\Big)u + \frac{f_{t_{30}}}{6}\Big(P_1^2P_3 - P_1^2P_2\Big)v + O(|(u,w)|^2). \tag{2.86}$$

Similarly, using eq. (2.85) to express the values of $f_{t_{11}}$ and $f_{t_{20}}$ in terms of u and w and then using $f_{t_y}(x,y)=0$, we conclude that

$$w = \frac{f_{t_{30}}}{6}\Big(-P_1^2P_2 + P_1P_2P_3\Big)u + \frac{f_{t_{30}}}{6}\Big(-P_1P_2^2 + P_1P_2P_3\Big)v + O(|(u,w)|^2). \tag{2.87}$$

Equating the right hand sides of eq. (2.86) and eq. (2.87), we conclude that

$$\frac{f_{t_{30}}}{6}P_1(P_1-P_2)(P_1-P_3)u - \frac{f_{t_{30}}}{6}P_1(P_1-P_2)(P_2-P_3)v + O(|(u,v,w)|^2) = 0. \tag{2.88}$$

From eq. (2.88), we can further conclude that

$$P_1 v = \Big(\frac{P_1-P_3}{P_2-P_3}\Big)(P_1 u) + O(|(u,w)|^2). \tag{2.89}$$

Finally, substituting the value for v from eq. (2.89) into w in eq. (2.86), we get that

$$w = -\frac{f_{t30}}{3} P_1 P_2 (P_1 - P_3) u + O(|(u, w)|^2) \Rightarrow w = -\frac{f_{t30}}{3} P_1 P_2 (P_1 - P_3) u + O(|u|^2). \tag{2.90}$$

Plugging the value of w from eq. (2.90) in eq. (2.85), we conclude that

$$\begin{aligned} f_{t_{11}} &= \frac{f_{t30}}{6}(P_1 + P_2)(P_1 - P_3)u + O(|u|^2), \text{ and} \\ f_{t_{20}} &= -\frac{f_{t30}}{3}(P_1 - P_3)u + O(|u|^2). \end{aligned}$$

Hence, solutions to eq. (2.75) and eq. (2.76) exist, given by

$$\begin{aligned} (x_1, y_1) &= (P_1 u + E_1(u), u), \\ (x_2, y_2) &= \Big(P_2 \frac{(P_1 - P_3)}{(P_2 - P_3)} u + E_2(u), \ \frac{(P_1 - P_3)}{(P_2 - P_3)} u + E_4(u) \Big), \\ f_{t_{11}} &= \frac{f_{t30}}{6}(P_1 + P_2)(P_1 - P_3)u + E_5(u), \\ f_{t_{20}} &= -\frac{f_{t30}}{3}(P_1 - P_3)u + E_6(u), \\ f_{t_{02}} &= -\frac{f_{t30}}{3} P_1 P_2 (P_1 - P_3)u + E_7(u), \end{aligned} \tag{2.91}$$

where u is small and nonzero and the E_i are all second order terms. Furthermore, there are **exactly** 6 distinct solutions, that corresponds to (P_1, P_2) being replaced with (P_i, P_j), where the (P_i, P_j) are ordered (or alternatively, we can think of this this way; the (P_i, P_j) is unordered as far as the construction of f_t is concerned, but we can permute the values of (x_1, y_1) and (x_2, y_2)). This proves Claim 2.6.8, and hence proves eq. (2.26). □

Let us now justify the multiplicity. We claim that each point of $(A_1^{\delta-3} \circ D_4) \cap \mu$ contributes 18 to the Euler class in eq. (2.20). As we just explained, there are exactly 6 distinct solutions to eq. (2.75) and eq. (2.76); we will call each distinct solution of eq. (2.75) and eq. (2.76) a **branch** of a neighborhood of $A_1^{\delta-3} \circ D_4$ inside $\overline{A_1^{\delta}}$. Since there are 6 branches, it suffices to show that the multiplicity from each branch is 3 (in which case the total contribution to the Euler class will be 18). Let us now compute the multiplicity from each branch.

Let us consider the branch given by eq. (2.91). The multiplicity from this branch is the number of small solutions (x, y, u) to the following set of equations

$$f_t(x, y) = \varepsilon_0, \; f_{t_x}(x, y) = \varepsilon_1, \; \text{and} \; f_{t_y}(x, y) = \varepsilon_2 \tag{2.92}$$

where $(\varepsilon_0, \varepsilon_1, \varepsilon_2) \in \mathbb{C}^3$ is small but generic, and $f_{t_{20}}, f_{t_{11}}$ and $f_{t_{02}}$ are as given in eq. (2.91). We claim that we can set ε_1 and ε_2 to be zero; this is justified in section 2.6.1. Hence, we need to find the number of small solutions (x, y, u) to the set of equations

$$f_t(x, y) = \varepsilon_0, \; f_{t_x}(x, y) = 0 \; \text{and} \; f_{t_y}(x, y) = 0.$$

This is same as the number of small solutions (x, y, u) to the set of equations

$$g_t(x, y) = -2\varepsilon_0, \tag{2.93}$$

$$f_{t_x}(x, y) = 0, \; \text{and} \; f_{t_y}(x, y) = 0, \tag{2.94}$$

where $g_t(x, y)$ is as defined in eq. (2.77). Let us start by solving only the two equations in eq. (2.94). Plugging in the values for $f_{t_{20}}, f_{t_{11}}$ and $f_{t_{02}}$ as given in eq. (2.91) and solving the equation $f_{t_x}(x, y) = 0$, we conclude that

$$\Big(-2x + (P_1 + P_2)y\Big)\Big(u + O(|u|^2)\Big) = \\ \frac{(3x^2 - 2(P_1 + P_2 + P_3)xy + (P_1P_2 + P_1P_3 + P_2P_3)y^2)}{(P_3 - P_1)} + O(|(x, y)|^3) \tag{2.95}$$

Similarly, plugging in the values for $f_{t_{20}}, f_{t_{11}}$ and $f_{t_{02}}$ as given in eq. (2.91) and solving the equation $f_{t_y}(x, y) = 0$, we conclude that

$$\Big((P_1 + P_2)x - 2P_1P_2y\Big)\Big(u + O(|u|^2)\Big) =$$

$$-\frac{((P_1+P_2+P_3)x^2-2(P_1P_2+P_1P_3+P_2P_3)xy+3P_1P_2P_3y^2)}{(P_3-P_1)}+O(|(x,y)|^3). \tag{2.96}$$

Multiplying eq. (2.95) by $(P_1+P_2)x-2P_1P_2y$ and multiplying eq. (2.96) by $(-2x+(P_1+P_2)y)$, we conclude that

$$\Big(x-P_1y\Big)\Big(x-P_2y\Big)\Big((P_1+P_2-2P_3)x-(2P_1P_2-P_1P_3-P_2P_3)y\Big)+O(|x,y|^4)= u^2O(|(x,y)|^2). \tag{2.97}$$

Let us now solve eq. (2.97). Let us make a change of coordinates

$$x=\hat{x}+O(|(\hat{x},\hat{y})|^2),\ \text{and}\ y=\hat{y}+O(|(\hat{x},\hat{y})|^2)$$

such that eq. (2.97) can be rewritten as

$$\Big(\hat{x}-P_1\hat{y}\Big)\Big(\hat{x}-P_2\hat{y}\Big)\Big((P_1+P_2-2P_3)\hat{x}-(2P_1P_2-P_1P_3-P_2P_3)\hat{y}\Big)= u^2O(|(\hat{x},\hat{y})|^2) \tag{2.98}$$

Using eq. (2.98), we solve for $\hat{x}$ in terms for $\hat{y}$ and u and convert back to x and y to conclude that the only possible solutions are given by

$$x=P_1y+E_8(y,u),\ \text{or}\ x=P_2y+E_9(y,u),\ \text{or}$$
$$(P_1+P_2-2P_3)x=(2P_1P_2-P_1P_3-P_2P_3)y+E_{10}(y,u), \tag{2.99}$$

such that $E_i(y,0)=O(|y|^2)$, for $i=8,9$ and 10. Plugging the solutions obtained in eq. (2.99) into eq. (2.95), solving for y in terms of u and then plugging that back into eq. (2.99) to express x in terms of u, we conclude that the only possible solutions to eq. (2.94) are given by

$$(x,y) = \Big(P_1 u + \widetilde{E}_1(u), u + \widehat{E}_1(u)\Big), \text{ or} \tag{2.100}$$

$$(x,y) = \Big(P_2\Big(\frac{P_1 - P_3}{P_2 - P_3}\Big)u + \widetilde{E}_2(u), \Big(\frac{P_1 - P_3}{P_2 - P_3}\Big)u + \widehat{E}_2(u)\Big), \text{ or} \tag{2.101}$$

$$(x,y) = \Big(\frac{(2P_1P_2 - P_1P_3 - P_2P_3)}{3(P_2 - P_3)}u + \widetilde{E}_3(u), \frac{(P_1 + P_2 - 2P_3)}{3(P_2 - P_3)}u + \widehat{E}_3(u)\Big), \tag{2.102}$$

where $\widetilde{E}_i(u)$ and $\widehat{E}_i(u)$ are second order terms (for $i = 1, 2$ and 3). From eq. (2.91), we conclude that the solutions in eq. (2.100) and eq. (2.101) with $\widetilde{E}_i(u)$ replaced by $E_i(u)$ and $\widehat{E}_i(u)$ replaced by 0 (for $i = 1$ and 2) is a solution to eq. (2.94). Since the solutions in eq. (2.100) and eq. (2.101) are the **only** solutions to eq. (2.94), we conclude that $\widetilde{E}_i(u) = E_i(u)$ and $\widehat{E}_i(u) = 0$ (for $i = 1$ and 2). Hence, if we plug the solutions obtained from eq. (2.100) and eq. (2.101) into $f_t(x,y)$ (or equivalently $g_t(x,y)$), we will get 0 and not ε_0. Hence, we reject the solutions given by eq. (2.100) and eq. (2.101).

It remains to consider the solution given by eq. (2.102). Plugging in the expression for x and y from eq. (2.102) into $g_t(x,y)$ gives us

$$g_t(x,y) = \Big(\frac{(P_2 - P_1)^2(P_3 - P_1)^2}{162(P_2 - P_3)}\Big)u^3 + O(u^4). \tag{2.103}$$

From eq. (2.103), we conclude that $g_t(x,y) = -2\varepsilon_0$ has 3 solutions. This justifies the multiplicity and concludes the proof of theorem 2.4.3. □

Local degree of a smooth map

It remains to show why we could set ε_2 to be 0 in eq. (2.65) and set $(\varepsilon_1, \varepsilon_2)$ to be $(0,0)$ in eq. (2.92). Let us first recall the definition of the local degree of a smooth map around a given point. We will follow the discussion and the theory developed in [24].

Let us begin with the proposition 2.1.2 of [24]. The statement is as follows:

Proposition 2.6.10. *Let $f \in C^2(\bar{\Omega}, \mathbb{R}^n)$ where Ω is an open subset of $\mathbb{R}^n$ and let $b \notin f(\partial\Omega)$. Let ρ_0 be the distance between b and $f(\partial\Omega)$ with $\rho_0 > 0$. Let $b_1, b_2 \in B(b; \rho_0)$, the ball of radius ρ_0 with center b. If b_1, b_2 both are regular values of f, then* $\deg(f, \Omega, b_1) = \deg(f, \Omega, b_2)$ *where* $\deg(f, \Omega, y)$ *represent the degree of f at y (i.e. the number of solutions to the equation $f(x) = y$ in Ω).*

Let us first justify the assertion for eq. (2.65). Let U be an open ball in $\mathbb{C}^2$ with center $(0, 0)$ and radius r, where r is sufficiently small and positive real number. Consider the map $\varphi : U \longrightarrow \mathbb{C}^2$, given by

$$\begin{aligned}\varphi(x, h) &= (\varphi_1(x, h), \varphi_2(x, h)) \\ &:= \Big(\frac{(x^2h^2)\Big(\mathcal{B}_4^{f_t} + O(|(x,h)|)\Big)}{24}, \frac{(xh)\Big(\mathcal{B}_4^{f_t}h - \mathcal{B}_4^{f_t}x + O(|(x,h)|^2)\Big)}{12}\Big).\end{aligned}$$

Before proceeding further, let us first prove the following claim:

Claim 2.6.11. If $\varepsilon \neq 0$, then the point $(\varepsilon, 0)$ is a regular value of φ defined as above.

Proof. Let us assume $\varphi(x, h) = (\varepsilon, 0)$. Using the fact that $\varphi_2(x, h) = 0$, we conclude that

$$x(h) = h + O(h^2).$$

Plugging in this value of x in $\varphi_1(x, h)$, we conclude that

$$h^4\Big(\frac{\mathcal{B}_4^{f_t}}{24} + O(h)\Big) = \varepsilon. \tag{2.104}$$

Note that if h is sufficiently small, then $\frac{\mathcal{B}_4^{f_t}}{24} + O(h)$ is non-zero, since $\mathcal{B}_4^{f_t}$ is non-zero. We also note that since ε is non-zero, eq. (2.104) implies that h is non-zero.

Next, let us compute the determinant of the differential of φ at $(x(h), h)$. It is given by

$$M := \det \begin{pmatrix} \varphi_{1_x} & \varphi_{1_h} \\ \varphi_{2_x} & \varphi_{2_h} \end{pmatrix} \Big|_{(x(h),h)} = h^5\Big(\frac{(\mathcal{B}_4^{f_t})^2}{72} + O(h)\Big) \tag{2.105}$$

Using eq. (2.104) and eq. (2.105), we conclude that

$$\begin{aligned} M &= h^4 \cdot h\Big(\frac{(\mathcal{B}_4^{f_t})^2}{72} + O(h)\Big) \\ &= \varepsilon \frac{h\Big(\frac{(\mathcal{B}_4^{f_t})^2}{72} + O(h)\Big)}{\Big(\frac{\mathcal{B}_4^{f_t}}{24} + O(h)\Big)}. \end{aligned} \tag{2.106}$$

Since $\mathcal{B}_4^{f_t}$ is non-zero, h is small and non-zero, and ε is non-zero, we conclude from eq. (2.106) that M is non-zero. Hence, $(\varepsilon, 0)$ is a regular value of φ. $\square$

Next, we note that if S is a non empty subset of $\mathbb{C}^2$, then the distance function $d_S : \mathbb{C}^2 \longrightarrow \mathbb{R}$ is a continuous function. Hence, the set

$$\begin{aligned} V &:= (d_{\varphi(\partial U)} - d_X)^{-1}(0, \infty) \\ &= \{(\varepsilon_1, \varepsilon_2) \in \mathbb{C}^2 \mid d_{\varphi(\partial U)}(\varepsilon_1, \varepsilon_2) > d_X(\varepsilon_1, \varepsilon_2)\} \end{aligned}$$

is an open subset of $\mathbb{C}^2$, where the function d_X denotes the distance from the X-axis. Note that $d_X(\varepsilon_1, \varepsilon_2) = |\varepsilon_2|$ and this distance is achieved by taking the distance from the point $(\varepsilon_1, \varepsilon_2)$ to the point $(\varepsilon_1, 0)$ on the X-axis.

Now, we will show that $V \cap \varphi(U) \neq \emptyset$. Note that

$$\partial U = \{(x, h) \in \mathbb{C}^2 : |x|^2 + |h|^2 = r^2\}.$$

Observe that ∂U is compact; so is $\varphi(\partial U)$. Therefore it is closed in $\mathbb{C}^2$. Hence, $d_{\varphi(\partial U)}(\varepsilon, 0) = 0$ if and only if $(\varepsilon, 0) \in \varphi(\partial U)$. We conclude that $(\varepsilon, 0) \in V$ if and only if $(\varepsilon, 0) \notin \varphi(\partial U)$. Now, let $(\varepsilon, 0) \in \varphi(\partial U)$. Let us assume $\varphi(x, h) = (\varepsilon, 0)$ with $|x|^2 + |h|^2 = r^2$ and $\varepsilon \neq 0$.

We conclude from $\varphi_2(x,h) = 0$ and eq. (2.104) that

$$x(h) = h + O(h^2) \qquad \text{and} \qquad h^4\Big(\frac{\mathcal{B}_4^{f_t}}{24} + O(h)\Big) = \varepsilon.$$

Now using the fact $|x|^2 + |h|^2 = r^2$, we conclude that $|\varepsilon| = \dfrac{|\mathcal{B}_4^{f_t}|}{96} r^4 + O(r^5)$. Hence we get either $\varepsilon = 0$ or $|\varepsilon| = \dfrac{|\mathcal{B}_4^{f_t}|}{96} r^4 + O(r^5)$. Note that $\dfrac{|\mathcal{B}_4^{f_t}|}{96} r^4 + O(r^5) \neq 0$ as $\mathcal{B}_4^{f_t} \neq 0$ and r is sufficiently small. So, $(\varepsilon, 0) \in V$ for all non-zero ε with $|\varepsilon| < \dfrac{|\mathcal{B}_4^{f_t}|}{96} r^4 + O(r^5)$ (i.e. for all $|\varepsilon|$ sufficiently small). From eq. (2.67) we conclude that the system $\varphi(x,h) = (\varepsilon, 0)$ has solutions in U, where ε is small but non-zero. Hence $(\varepsilon, 0) \in V \cap \varphi(U)$ for some non-zero ε with $|\varepsilon| < \dfrac{\mathcal{B}_4^{f_t}}{96} r^4 + O(r^5)$. Hence, $V \cap \varphi(U)$ is non empty.

Let $(\varepsilon_1, \varepsilon_2) \in V \cap \varphi(U)$. Therefore by definition of V, $\rho_0 := d_{\varphi(\partial U)}(\varepsilon_1, \varepsilon_2) > |\varepsilon_2| \geq 0$. Now, $\varphi(\partial U)$ is a closed subset of $\mathbb{C}^2$ and $d_{\varphi(\partial U)}(\varepsilon_1, \varepsilon_2) > 0$ together implies that $(\varepsilon_1, \varepsilon_2) \notin \varphi(\partial U)$. According to Proposition 2.6.10, $\deg(\varphi, U, (a,b)) = \deg(\varphi, U, (\varepsilon_1, 0))$ for all regular values (a,b) of φ that belong to $B(\rho_0; (\varepsilon_1, \varepsilon_2))$. That is, the number of solutions in U to both the equations $\varphi(x,h) = (a,b)$ and $\varphi(x,h) = (\varepsilon_1, 0)$ are the same, provided (a,b) is a regular value of φ. This justifies our claim in eq. (2.65).

Let us now justify the assertion for eq. (2.92). The argument is similar to the previous argument. We just need to prove the following claim:

Claim 2.6.12. Let $U \subseteq \mathbb{C}^3$ be a small open neighborhood of $(0,0,0)$ and $\varphi : U \longrightarrow \mathbb{C}^3$ be given by

$$\varphi(x,y,u) := \Big(f_t(x,y), f_{t_x}(x,y), f_{t_y}(x,y)\Big),$$

where f_t is as given in eq. (2.74) and $f_{t_{20}}, f_{t_{11}}, f_{t_{02}}$ are as given in eq. (2.91). Let $\widehat{U} \subseteq \mathbb{C}$ be a small open neighborhood of 0. If ε is a generic point of $\widehat{U}$, then $(\varepsilon, 0, 0)$ is a regular value of φ.

Proof. Let $(x,y,u) \in U$ such that $\varphi(x,y,u) = (\varepsilon, 0, 0)$. We note that

$$\det \begin{pmatrix} f_{t_x}(x,y) & f_{t_{xx}}(x,y) & f_{t_{yx}}(x,y) \\ f_{t_y}(x,y) & f_{t_{xy}}(x,y) & f_{t_{yy}}(x,y) \\ f_{t_u}(x,y) & f_{t_{xu}}(x,y) & f_{t_{yu}}(x,y) \end{pmatrix} = f_{t_u}(x,y) \cdot \det \begin{pmatrix} f_{t_{xx}}(x,y) & f_{t_{yx}}(x,y) \\ f_{t_{xy}}(x,y) & f_{t_{yy}}(x,y) \end{pmatrix}. \tag{2.107}$$

This is because $f_{t_x}(x,y)$ and $f_{t_y}(x,y)$ are both equal to zero. We now note that f_t has an A_1 singularity at $(0,0)$; hence determinant of Hessian of f_t does not vanish at $(0,0)$. Since (x,y) is small, we conclude that the determinant of the Hessian of f_t at (x,y) is non-zero. Hence, if the right hand side of eq. (2.107) is zero, then $f_{t_u}(x,y)$ has to be zero. We claim this is not possible for a generic ε. To see why this is so, note that the solution to the equation $\varphi(x,y,u) = (\varepsilon,0,0)$ with $\varepsilon \in \widehat{U}$ is given by eq. (2.102). After plugging the value of (x,y) obtained in eq. (2.102) to the expression of $f_t(x,y)$, we conclude from eq. (2.103) that

$$f_{t_u}(x,y) = -\Big(\frac{3(P_2-P_1)^2(P_3-P_1)^2}{324(P_2-P_3)}\Big)u^2 + O(u^3).$$

Note that $f_{t_u}(x,y)$ is a power series of u which is not identically zero in a small open subset of $\mathbb{C}$ containing the origin, and hence it has only finitely many zeros. We conclude that $(\varepsilon,0,0)$ is a regular value of φ for all but a finite set of ε; in particular, for a generic ε, $(\varepsilon,0,0)$ is a regular value of φ. □

2.6.2 Proof of Theorem 2.4.4: computation of $N(A_1^{\delta}\mathcal{P}A_1)$ when $0 \leq \delta \leq 2$.

We will now justify our formula for $N(A_1^{\delta}\mathcal{P}A_1, r, s, n_1, n_2, n_3, \theta)$, when $0 \leq \delta \leq 2$. If $\theta = 0$, then the formula follows from eq. (2.12).

Let us now assume $\theta > 0$. Recall that

$$A_1^\delta \circ \overline{\hat{A}}_1 := \{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \mathcal{S}_{\mathcal{D}_\delta} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} : ([f],[\eta],l_{q_{\delta+1}}) \in \overline{\hat{A}}_1, \\ f \text{ has a singularity of type } A_1 \text{ at } q_1,\ldots,q_\delta \text{ with } q_1,\ldots,q_{\delta+1} \text{ are all distinct}\}.$$

Let μ be a generic cycle given by

$$\mu = a^{n_1}\lambda^{n_2}(\pi_{\delta+1}^* H)^{n_3}(\pi_{\delta+1}^*\lambda_W)^\theta \mathcal{H}_L^r \mathcal{H}_p^s.$$

We now define a section of the following bundle

$$\Psi_{\mathcal{P}A_1} : A_1^\delta \circ \overline{\hat{A}}_1 \longrightarrow \mathcal{L}_{\mathcal{P}A_1} := \gamma_{\mathcal{D}}^* \otimes \gamma_W^{*2} \otimes \gamma_{\mathbb{P}^3}^{*d}, \quad \text{given by} \\ \{\Psi_{\mathcal{P}A_1}([f],q_1,\ldots,q_\delta,l_{q_{\delta+1}})\}(f\otimes v^{\otimes 2}) := \nabla^2 f|_{q_{\delta+1}}(v,v).$$

We will show shortly that this section is transverse to zero. Next, let us define

$$\mathcal{B} := \overline{A_1^\delta \circ \overline{\hat{A}}_1} - A_1^\delta \circ \overline{\hat{A}}_1.$$

Hence

$$\langle e(\mathcal{L}_{\mathcal{P}A_1}), \ [\overline{A_1^\delta \circ \overline{\hat{A}}_1}] \cap \widetilde{\mu}\rangle = N(A_1^\delta \mathcal{P}A_1, r, s, n_1, n_2, n_3, \theta) + \mathcal{C}_{\mathcal{B}\cap\mu}, \tag{2.108}$$

where as before, $\mathcal{C}_{\mathcal{B}\cap\mu}$ denotes the contribution of the section to the Euler class from $\mathcal{B}\cap\mu$. When $\delta = 0$, the boundary $\mathcal{B}$ is empty. Hence, plugging in $\mathcal{C}_{\mathcal{B}\cap\mu} = 0$, and unwinding the left hand side of eq. (2.108) gives us the formula of Theorem 2.4.4 for $\delta = 0$.

Let us now assume $\delta > 0$. Given k distinct integers $i_1, i_2, \ldots, i_k \in [1, \delta+1]$, let $\Delta_{i_1,\ldots,i_k}$ be as defined in the proof of Theorem 2.4.3. Let us define

$$\hat{\Delta}_{i_1,\ldots,i_k} := \pi^{-1}(\Delta_{i_1,\ldots,i_k}),$$

where $\pi : \mathcal{S}_{\mathcal{D}_\delta} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} \longrightarrow \mathcal{S}_{\mathcal{D}_{\delta+1}}$ is the projection map. Let us define

$$\mathcal{B}(q_{i_1}, \ldots, q_{i_{k-1}}, l_{q_{\delta+1}}) := \mathcal{B} \cap \hat{\Delta}_{i_1,\ldots,i_{k-1},\delta+1}.$$

Let us now consider $\mathcal{B}(q_i, l_{q_{\delta+1}})$. We claim that,

$$\mathcal{B}(q_i, l_{q_{\delta+1}}) \approx \overline{A_1^{\delta-1} \circ \hat{A}_3}, \tag{2.109}$$

where $\mathcal{B}(q_i, l_{q_{\delta+1}})$ is identified as a subset of $\mathcal{S}_{\mathcal{D}_{\delta-1}} \times_{\widehat{\mathbb{P}}^3} \mathbb{P}W_{\mathcal{D}}$ in the obvious way (namely via the inclusion map where the $(\delta+1)^{\text{th}}$ point is equal to the i^{th} point). We will justify that shortly. Let us now intersect $\overline{A_1^{\delta-1} \circ \hat{A}_3}$ with μ. This will be an isolated set of finite points. Hence, the section $\Psi_{\mathcal{P}A_1}$ will not vanish on $\overline{A_1^{\delta-1} \circ \hat{A}_3} \cap \mu$. Hence it does not contribute to the Euler class.

Next, let us consider $\mathcal{B}(q_{i_1}, q_{i_2}, l_{q_{\delta+1}})$. We claim that

$$\mathcal{B}(q_{i_1}, q_{i_2}, l_{q_{\delta+1}}) \approx \overline{A_1^{\delta-2} \circ \hat{A}_5} \cup \overline{A_1^{\delta-2} \circ \widehat{D}_4}. \tag{2.110}$$

The set $\overline{A_1^{\delta-2} \circ \hat{A}_5} \cap \mu$ is empty since the sum total of the dimensions of these two varieties is one less than the dimension of the ambient space. Next, we note that the section $\Psi_{\mathcal{P}A_1}$ vanishes everywhere on $\overline{A_1^{\delta-2} \circ \widehat{D}_4}$; hence it also vanishes on $\overline{A_1^{\delta-2} \circ \widehat{D}_4} \cap \mu$. We claim that the contribution from each of the points of $\mathcal{B}(q_{i_1}, q_{i_2}, l_{q_{\delta+1}}) \cap \mu$ is 6. Hence the total contribution from all the components of type $\mathcal{B}(q_{i_1}, q_{i_2}, l_{q_{\delta+1}})$ is

$$6\binom{\delta}{2} N(A_1^{\delta-2}\widehat{D}_4, n_1, n_2, n_3, \theta).$$

Plugging this in eq. (2.108) gives us the formula of theorem 2.4.4.

Let us now justify the transversality, closure and multiplicity claims. We will follow the setup of theorem 2.4.3. Suppose

$$\Psi_{\mathcal{P}A_1}([f], [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}}) = 0.$$

As before, we assume η determines the plane where the last component is zero and $q_{\delta+1} := [0,0,1,0]$. Let us consider $T\mathbb{P}^2_\eta|_{[q_{\delta+1}]}$. Let ∂_x and ∂_y be the standard basis vectors for $T\mathbb{P}^2_\eta|_{[q_{\delta+1}]}$ (corresponding to the first two coordinates). Hence

$$l_{q_{\delta+1}} = [a\partial_x + b\partial_y] \in \mathbb{P}T\mathbb{P}^2_\eta|_{[q_{\delta+1}]}$$

for some complex numbers a, b not both of which are zero. Without loss of generality, we can assume $l_{q_{\delta+1}} = [\partial_x]$. Let us now consider the polynomial

$$\rho_{20} := \Big(X - \frac{X_1}{Z_1}Z\Big)^2\Big(X - \frac{X_2}{Z_2}Z\Big)^2 \ldots \cdot \Big(X - \frac{X_\delta}{Z_\delta}Z\Big)^2 X^2 Z^{d-2\delta-2},$$

and consider the corresponding curve $\gamma_{20}(t)$. We now note

$$\{\{\nabla\Psi_{\mathcal{P}A_1}|_{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}})}\}(\gamma_{20}'(0))\}(f\otimes\partial_x\otimes\partial_x) = \lambda Z^{d-2\delta-2}\nabla^2 X|_{[0,0,1,0]}(\partial_x,\partial_x).$$

Since $\nabla^2 X|_{[0,0,1,0]}(\partial_x,\partial_x)$ is nonzero, we conclude that the section is transverse to zero.

Next, let us justify the closure claims. Let us start with eq. (2.109). This statement is saying that when two nodes collide, we get a tacnode. Hence, the proof of eq. (2.109) is same as the proof of eq. (2.25).

Next, let us consider eq. (2.110). Again, this statement is saying what happens what happens when three nodes collide. Hence, the proof of eq. (2.110) is same as the proof of eq. (2.26).

It remains to justify the contribution from the points of $\overline{A_1^{\delta-2}\circ\widehat{D}_4}\cap\mu$. We will use the solutions constructed in eq. (2.91). Using the expression for $f_{t_{20}}$, we note that the multiplicity from each branch is the number of small solutions u to the equation

$$-\frac{f_{t_{30}}}{3}(P_1 - P_3)u + E_6(u) = \varepsilon.$$

This is 1. Since there are 6 branches, the total multiplicity is 6. □

2.6.3 Proof of Theorem 2.4.6: computation of $N(A_1^{\delta}\mathcal{P}A_2)$ when $0 \leq \delta \leq 2$.

We will justify our formula for $N(A_1^{\delta}\mathcal{P}A_2, r, s, n_1, n_2, n_3, \theta)$, when $0 \leq \delta \leq 2$. Recall that

$$A_1^{\delta} \circ \overline{\mathcal{P}A}_1 := \{([f], [\eta], q_1, \ldots, q_{\delta}, l_{q_{\delta+1}}) \in \mathcal{S}_{\mathcal{D}_{\delta}} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} : ([f], [\eta], l_{q_{\delta+1}}) \in \overline{\mathcal{P}A}_1,$$
$$f \text{ has a singularity of type } A_1 \text{ at } q_1, \ldots, q_{\delta} \text{ with } q_1, \ldots, q_{\delta+1} \text{ are all distinct}\}.$$

Let μ be a generic cycle given by

$$\mu := a^{n_1}\lambda^{n_2}(\pi_{\delta+1}^* H)^{n_3}(\pi^*\lambda_W)^{\theta}\mathcal{H}_L^r\mathcal{H}_p^s.$$

Recall that as per the hypothesis of Theorem 2.4.6, if $\delta = 2$ then $\theta = 0$. We now define a section of the following line bundle

$$\Psi_{\mathcal{P}A_2} : A_1^{\delta} \circ \overline{\mathcal{P}A}_1 \longrightarrow \mathbb{L}_{\mathcal{P}A_2} := \gamma_{\mathcal{D}}^* \otimes \gamma_W^* \otimes (W/\gamma_W)^* \otimes \gamma_{\mathbb{P}^3}^{*d}, \quad \text{given by}$$
$$\{\Psi_{\mathcal{P}A_2}([f], q_1, \ldots, q_{\delta}, l_{q_{\delta+1}})\}(f \otimes v \otimes w) := \nabla^2 f|_{q_{\delta+1}}(v, w).$$

We will show shortly that this section is transverse to zero. Next, let us define

$$\mathcal{B} := \overline{A_1^{\delta} \circ \overline{\mathcal{P}A}_1} - A_1^{\delta} \circ \overline{\mathcal{P}A}_1.$$

Hence

$$\langle e(\mathbb{L}_{\mathcal{P}A_2}), \ [\overline{A_1^{\delta} \circ \overline{\mathcal{P}A}_1}] \cap \widetilde{\mu}\rangle = N(A_1^{\delta}\mathcal{P}A_2, r, s, n_1, n_2, n_3, \theta) + \mathcal{C}_{\mathcal{B}\cap\mu}. \tag{2.111}$$

Define $\mathcal{B}(q_{i_1}, \ldots q_{i_k}, l_{q_{\delta+1}})$ as before. For simplicity, let us set $(i_1, i_2, \ldots, i_k) := (\delta - k, \ldots, \delta - 1, \delta)$. Before we describe $\mathcal{B}(q_{i_1}, q_{i_2}, l_{q_{\delta+1}})$, let us define a few things. Let v be a fixed nonzero vector that belongs to $l_{q_{\delta+1}}$. Let us define W_1, W_2, W_3 and W_4 as

$$\begin{aligned}
W_1 &:= \{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \overline{\mathcal{P}A}_1} : \nabla^2 f|_{q_{\delta+1}} \not\equiv 0\},\\
W_2 &:= \{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \overline{\mathcal{P}A}_1} : \nabla^2 f|_{q_{\delta+1}} \equiv 0\},\\
W_3 &:= \{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \overline{\mathcal{P}A}_1} : \nabla^3 f|_{q_{\delta+1}}(v,v,v) \neq 0\},\\
W_4 &:= \{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \overline{\mathcal{P}A}_1} : \nabla^3 f|_{q_{\delta+1}}(v,v,v) = 0\}.
\end{aligned} \tag{2.112}$$

We claim that

$$\mathcal{B}(q_\delta, l_{q_{\delta+1}}) \cap W_1 \approx \overline{A_1^{\delta-1} \circ \mathcal{P}A_3} \cap W_1, \tag{2.113}$$

$$\mathcal{B}(q_\delta, l_{q_{\delta+1}}) \cap W_2 \approx \overline{A_1^{\delta-1} \circ \widehat{D}_4}, \tag{2.114}$$

$$\mathcal{B}(q_{\delta-1}, q_\delta, l_{q_{\delta+1}}) \cap W_1 \subset \overline{A_1^{\delta-2} \circ \mathcal{P}A_5} \cap W_1, \tag{2.115}$$

$$\mathcal{B}(q_{\delta-1}, q_\delta, l_{q_{\delta+1}}) \cap (W_2 \cap W_4) \approx \overline{A_1^{\delta-2} \circ \mathcal{P}D_4} \cap W_1, \text{ and} \tag{2.116}$$

$$\mathcal{B}(q_{\delta-1}, q_\delta, l_{q_{\delta+1}}) \cap (W_2 \cap W_3) \subset \overline{A_1^{\delta-2} \circ \widehat{D}_5}. \tag{2.117}$$

Notice that eq. (2.115) and eq. (2.117) say that the left hand side is a subset of the right hand side (unlike the other three equations, which assert equality of sets). We now note that eq. (2.113) and eq. (2.114), imply that

$$\mathcal{B}(q_{i_1}, l_{q_{\delta+1}}) \approx \overline{A_1^{\delta-1} \circ \mathcal{P}A_3} \cup \overline{A_1^{\delta-1} \circ \widehat{D}_4}, \tag{2.118}$$

while eq. (2.115), eq. (2.116) and eq. (2.117) imply that

$$\mathcal{B}(q_{i_1}, q_{i_2}, l_{q_{\delta+1}}) \subset \overline{A_1^{\delta-2} \circ \mathcal{P}A_5} \cup \overline{A_1^{\delta-2} \circ \mathcal{P}D_4} \cup \overline{A_1^{\delta-2} \circ \widehat{D}_5}. \tag{2.119}$$

We claim that the contribution to the Euler class from each of the points of $\overline{A_1^{\delta-1} \circ \mathcal{P}A_3} \cap \mu$, $\overline{A_1^{\delta-1} \circ \widehat{D}_4} \cap \mu$ and $\overline{A_1^{\delta-2} \circ \mathcal{P}D_4} \cap \mu$ are $2, 3$ and 4 respectively.

Next, we note that for dimensional reasons, the intersection of $\overline{A_1^{\delta-1} \circ \mathcal{P}A_5}$ with μ is empty. The intersection of $\mathcal{B}(q_{\delta-1}, q_\delta, l_{q_{\delta+1}}) \cap W_1$ with μ is also empty by eq. (2.115), and hence does not contribute to the Euler class. Finally, let us consider the component corresponding to the left hand side of eq. (2.117); this is where we will use $\theta = 0$. Let us consider the projection map

$$\pi : \mathcal{S}_{\mathcal{D}_\delta} \times_{\widehat{\mathbb{P}}^3} \mathbb{P}W_D \longrightarrow \mathcal{S}_{\mathcal{D}_{\delta+1}}.$$

We recall that

$$\overline{A_1^{\delta-2} \circ \widehat{D}_5} = \pi^{-1}(\overline{A_1^{\delta-2} \circ D_5}).$$

Since $\theta = 0$, we note that μ is the pullback of a class ν, that is,

$$\mu = \pi^*(\nu).$$

Hence, the intersection of μ with $\overline{A_1^{\delta-2} \circ \widehat{D}_5}$ is in one to one correspondence with the intersection of ν with $\overline{A_1^{\delta-2} \circ D_5}$. But the degree of the cohomology class ν is one more than the dimension of the cycle $\overline{A_1^{\delta-2} \circ D_5}$. Hence, the intersection of $\overline{A_1^{\delta-2} \circ D_5}$ with ν is empty, and hence the intersection of μ with $\overline{A_1^{\delta-2} \circ \widehat{D}_5}$ is empty. As a result, the intersection of $\mathcal{B}(q_{\delta-1}, q_\delta, l_{q_{\delta+1}}) \cap (W_2 \cap W_3)$ with μ is also empty by eq. (2.117). Therefore the total contribution from all the components of type $\mathcal{B}(q_{i_1}, l_{q_{\delta+1}})$ equals

$$2\binom{\delta}{1} N(A_1^{\delta-1}\mathcal{P}A_3, r, s, n_1, n_2, n_3, \theta) + 3\binom{\delta}{1} N(A_1^{\delta-1}\widehat{D}_4, r, s, n_1, n_2, n_3, \theta),$$

while the total contribution from all the components of type $\mathcal{B}(q_{i_1}, q_{i_2}, l_{q_{\delta+1}})$ equals

$$4\binom{\delta}{2} N(A_1^{\delta-2}\mathcal{P}D_4, r, s, n_1, n_2, n_3, \theta).$$

Plugging this in eq. (2.111) gives us the formula of theorem 2.4.6.

Let us now prove the claim about transversality. This follows from following the setup of proof of transversality in Theorem 2.4.4. We consider the polynomial

$$\rho_{11} := \Big(X - \frac{X_1}{Z_1}Z\Big)^2\Big(X - \frac{X_2}{Z_2}Z\Big)^2 \ldots \cdot \Big(X - \frac{X_\delta}{Z_\delta}Z\Big)^2 XYZ^{d-2\delta-2},$$

and the corresponding curve $\gamma_{11}(t)$. Transversality follows by computing the derivative of the section $\Psi_{\mathcal{P}A_2}$ along the curve $\gamma_{11}(t)$ as before.

Next, let us justify the closure and multiplicity claims. We will start by justifying eq. (2.118). It suffices to justify eq. (2.113) and eq. (2.114). Let us rewrite these two equations explicitly, namely

$$\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_1} : q_\delta = q_{\delta+1}\} \cap W_1 = \overline{A_1^{\delta-1} \circ \mathcal{P}A_3} \cap W_1, \tag{2.120}$$

and

$$\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_1} : q_\delta = q_{\delta+1}\} \cap W_2 = \overline{A_1^{\delta-1} \circ \widehat{D}_4}. \tag{2.121}$$

Since $\widehat{D}_4$ is a subset of W_2, we did not write $\cap W_2$ on the right hand side of eq. (2.121).

Let us now start the proof of eq. (2.120). Let us first explain why the left hand side of eq. (2.120) is a subset of its right hand side. To see that, first we note that $\mathcal{P}A_1$ is a subset of $\overline{\hat{A}}_1$. Since we have shown while proving eq. (2.25) and eq. (2.44) that when two nodes collide we get a tacnode in eq. (2.44), we conclude that

$$\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \hat{A}_1} : q_\delta = q_{\delta+1}\} = \overline{A_1^{\delta-1} \circ \hat{A}_3}.$$

Hence, we conclude that

$$\begin{aligned} &\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_1} : q_\delta = q_{\delta+1}\} \subset \overline{A_1^{\delta-1} \circ \hat{A}_3} \\ \text{i.e., } &\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_1} : q_\delta = q_{\delta+1}\} \cap W_1 \subset \overline{A_1^{\delta-1} \circ \hat{A}_3} \cap W_1. \end{aligned} \tag{2.122}$$

Suppose $([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}})$ belongs to the left hand side of eq. (2.122). Since $([f],[\eta],l_{q_{\delta+1}}) \in \overline{\mathcal{P}A}_1$, we conclude that

$$\nabla^2 f|_{q_{\delta+1}}(v,v) = 0 \;\; \forall \;\; v \in l_{q_{\delta+1}}.$$

Since $([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}})$ is a subset of the right hand side of eq. (2.122), we conclude that the Hessian $\nabla^2 f|_{q_{\delta+1}}$ is not identically zero, but it has a non trivial kernel. We claim that v is in the kernel of the Hessian. To see why, let us assume that the nonzero vector $\tilde{v}$ is in the kernel of the Hessian, that is, $\nabla^2 f|_{q_{\delta+1}}(\tilde{v},\cdot) = 0$. Let w be any other vector, linearly independent from $\tilde{v}$. Since the Hessian is not identically zero and the vector space is two dimensional, we conclude that $\nabla^2 f|_{q_{\delta+1}}(w,w) \neq 0$. Hence, writing the vector $v := \lambda_1\tilde{v} + \lambda_2 w$ and using $\nabla^2 f|_{q_{\delta+1}}(v,v) = 0$, we conclude that $\lambda_2 = 0$. Hence, v belongs to the kernel of the Hessian. But we also note that if $([f],[\eta],l_q) \in \mathcal{P}A_3$ and $\nabla^2 f|_q(v,\cdot) = 0$, then $\nabla^3 f|_q(v,v,v) = 0$. Hence, we can improve eq. (2.122) and conclude that the left hand side of eq. (2.120) is a subset of its right hand side.

Let us now prove the converse. We will simultaneously prove the following two statements

$$\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_1} : q_\delta = q_{\delta+1}\} \supset A_1^{\delta-1} \circ \mathcal{P}A_3, \tag{2.123}$$

and

$$\Big(\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_2} : q_\delta = q_{\delta+1}\}\Big) \cap \Big(A_1^{\delta-1} \circ \mathcal{P}A_3\Big) = \emptyset. \tag{2.124}$$

We will prove the following claim:

Claim 2.6.13. Let $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-1},l_{q_\delta}) \in A_1^{\delta-1} \circ \mathcal{P}A_3$. Then there exists points

$$\big([f_t],[\eta_t],q_1(t),\ldots,q_{\delta-2}(t);q_{\delta-1}(t),q_\delta(t),l_{q_{\delta+1}(t)}\big) \in A_1^\delta \circ \mathcal{P}A_1 \tag{2.125}$$

sufficiently close to $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-1};q_\delta,l_{q_\delta})$. Furthermore, *every* such solution satisfies the condition

$$\nabla^2 f|_{q_{\delta+1}}(v, w) \neq 0, \tag{2.126}$$

if v is a nonzero vector that belongs to $l_{q_{\delta+1}(t)}$ and w is a nonzero vector that belongs to $T\mathbb{P}^2_\eta|_{q_{\delta+1}(t)}/l_{q_{\delta+1}(t)}$. In other words,

$$\big([f_t], [\eta_t], q_1(t), \ldots, q_{\delta-1}(t), q_\delta(t), l_{q_{\delta+1}(t)}\big) \notin A_1^\delta \circ \mathcal{P}A_2.$$

Remark 2.6.14. We note that Claim 2.6.13 simultaneously proves eq. (2.123) and eq. (2.124).

Proof: Following the setup of the proofs of Claim 2.6.2 and Claim 2.6.8, we will now work in an affine chart, where we send the plane $\mathbb{P}^2_{\eta t}$ to $\mathbb{C}^2_z$ and the point $q_\delta(t) \in \mathbb{P}^2_{\eta t}$ to $(0,0,0) \in \mathbb{C}^2_z$. We also choose coordinates, such that $\partial_x \in l_{q_{\delta+1}(t)}$. Using this chart, let us write down the Taylor expansion of f_t around the point $(0,0)$, namely

$$f_t(x, y) = f_{t_{11}} xy + \frac{f_{t_{02}}}{2} y^2 + \frac{f_{t_{30}}}{6} x^3 + \frac{f_{t_{21}}}{2} x^2 y + \frac{f_{t_{12}}}{2} x y^2 + \frac{f_{t_{03}}}{6} y^3 + \ldots$$

Since $([f_t], [\eta_t], l_{q_\delta(t)}) \in \mathcal{P}A_1$, we conclude that $f_{t_{20}}$ is zero. Next, let us consider the Taylor expansion of f (not f_t). We note that $([f], [\eta], l_{q_\delta}) \in \mathcal{P}A_3$. This means that f_{11} and f_{02} can not both be zero (since that would mean the Hessian is identically zero). If $f_{02} = 0$ and $f_{11} \neq 0$, then it implies that $([f], [\eta], l_{q_\delta}) \in \hat{A}_1$ (and hence does not belong to $\mathcal{P}A_3$). Therefore, $f_{02} \neq 0$, and hence we conclude that $f_{t_{02}} \neq 0$. Finally, since $([f], [\eta], l_{q_\delta}) \in \mathcal{P}A_3$, we conclude that f_{11} and f_{30} are zero; hence $f_{t_{11}}$ and $f_{t_{30}}$ are small (close to zero). We will mainly follow the Proof of claim 2.6.2. Since $f_{t_{02}} \neq 0$ we can make the same change of coordinates $\hat{y} := y + B(x)$ as in the Proof of claim 2.6.2 and write f_t as

$$f_t(x, y(x, \hat{y})) = \varphi(x, \hat{y})\hat{y}^2 + \frac{\mathcal{B}_2^{f_t}}{2!} x^2 + \frac{\mathcal{B}_3^{f_t}}{3!} x^3 + \frac{\mathcal{B}_4^{f_t}}{4!} x^4 + \mathcal{R}(x) x^5,$$

where

$$\mathcal{B}_2^{f_t} := -\frac{f_{t_{11}}^2}{f_{t_{02}}}, \qquad \mathcal{B}_3^{f_t} := f_{t_{30}} - \frac{3f_{t_{11}}f_{t_{21}}}{f_{t_{02}}} + \frac{3f_{t_{11}}^2 f_{t_{12}}}{f_{t_{02}}^2} - \frac{3f_{t_{11}}^3 f_{t_{03}}}{f_{t_{02}}^3}, \ldots, \qquad \varphi(0,0) \neq 0. \tag{2.127}$$

Also $\mathcal{R}(x)$ is a holomorphic function defined in a neighborhood of the origin.

Since $([f], [\eta], q_\delta) \in \mathcal{P}A_3$, we conclude that $\mathcal{B}_2^{f_t}$ and $\mathcal{B}_3^{f_t}$ are small (close to zero) and $\mathcal{B}_4^{f_t}$ is nonzero. Let us make a further change of coordinates and denote

$$\hat{\hat{y}} := \sqrt{\varphi(x, \hat{y})}\hat{y}.$$

Note that we can choose a branch of the square root since $\varphi(0,0) \neq 0$. Next, for notational convenience, let us now define

$$\hat{f}_t(x, \hat{\hat{y}}) := f_t(x, y(x, \hat{y}(\hat{\hat{y}})))), \tag{2.128}$$

i.e., $\hat{f}_t$ is basically f_t written in the new coordinates (namely x and $\hat{\hat{y}}$). Hence,

$$\hat{f}_t(x, \hat{\hat{y}}) = \hat{\hat{y}}^2 + \frac{\mathcal{B}_2^{f_t}}{2!}x^2 + \frac{\mathcal{B}_3^{f_t}}{3!}x^3 + \frac{\mathcal{B}_4^{f_t}}{4!}x^4 + \mathcal{R}(x)x^5.$$

We now note that constructing the points on the left hand side of eq. (2.125) amounts to solving the set of equations

$$\hat{f}_t = 0, \ \hat{f}_{t_x} = 0, \ \text{and} \ \hat{f}_{t_{\hat{\hat{y}}}} = 0, \tag{2.129}$$

where $(x, \hat{\hat{y}})$ is small but not equal to $(0,0)$.

We will now construct solutions to eq. (2.129). The solutions to eq. (2.129) are given by

$$\hat{\hat{y}} = 0, \ \mathcal{B}_2^{f_t} = \frac{\mathcal{B}_4^{f_t}}{12}x^2 + O(|x|^3), \ \text{and} \ \mathcal{B}_3^{f_t} = -\frac{\mathcal{B}_4^{f_t}}{2}x + O(|x|^2). \tag{2.130}$$

Now we use the expression of $\mathcal{B}_2^{f_t}, \mathcal{B}_3^{f_t}$ and conclude from eq. (2.130) that

$$\frac{f_{t_{11}}^2}{f_{t_{02}}} = -\frac{\mathcal{B}_4^{f_t}}{12}x^2 + O(|x|^3), \text{ and} \tag{2.131}$$

$$f_{t_{30}} = -3\mathcal{B}_4^{f_t}x + O(|x|^2). \tag{2.132}$$

Hence, there are two solutions to eq. (2.131), given by

$$f_{t_{11}} = \Big(\sqrt{\frac{-f_{t_{02}}\mathcal{B}_4^{f_t}}{12}}\Big)x + O(|x|^2), \text{ or } f_{t_{11}} = -\Big(\sqrt{\frac{-f_{t_{02}}\mathcal{B}_4^{f_t}}{12}}\Big)x + O(|x|^2), \tag{2.133}$$

where $\surd$ denotes a branch of the square root. Hence, there are **exactly** two solutions to eq. (2.129), given by

$$x = u, \quad f_{t_{11}} = \pm\Big(\sqrt{\frac{-f_{t_{02}}\mathcal{B}_4^{f_t}}{12}}\Big)u + O(|u|^2) \tag{2.134}$$

and $\hat{\hat{y}} = 0$ and $f_{t_{30}}$ as given by eq. (2.132), where we plug in the expressions for x and $f_{t_{11}}$ as given by eq. (2.134) to express them in terms of u (the exact expressions in terms of u are not so important, hence we have not written that out explicitly). This proves Claim 2.6.13. Since eq. (2.134) are the **only** solutions and $\mathcal{B}_4^{f_t} \neq 0$, we also conclude that eq. (2.126) is true. □

It remains to compute the multiplicity. We claim the each point of $(A_1^{\delta-1} \circ \mathcal{P}A_3) \cap \mu$ contributes 2 to the Euler class in eq. (2.111). Using eq. (2.134) we conclude that the multiplicity from each branch is the number of small solutions u to the equation

$$\Big(\sqrt{\frac{-f_{t_{02}}\mathcal{B}_4^{f_t}}{12}}\Big)u + O(|u|^2) = \varepsilon \qquad \text{and} \qquad -\Big(\sqrt{\frac{-f_{t_{02}}\mathcal{B}_4^{f_t}}{12}}\Big)u + O(|u|^2) = \varepsilon.$$

This number is 1 in each case, and hence the total multiplicity is 2. □

Next, let us justify eq. (2.121). Let us first explain why the left hand side of eq. (2.121) is a subset of its right hand side. If $([f], [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}}) \in W_2$, then it means that $\nabla^2 f|_{q_{\delta+1}} = 0$. Hence, it means that $([f], [\eta], l_{q_{\delta+1}}) \in \overline{\widehat{D}_4}$. Hence, the left hand side of eq. (2.121) is a subset of its right hand side.

Let us now prove eq. (2.121). Before that, let us introduce a new space. Let us define

$$\widehat{D}_4^{\#} := \{([f],[\eta],l_q) \in \widehat{D}_4 : \nabla^3 f|_q(v,v,v) \neq 0 \ \text{ if } \ v \in l_q - 0\}.$$

Note that $\overline{\widehat{D}_4^{\#}} = \overline{\widehat{D}}_4$. We will now simultaneously prove the following two statements:

$$\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_1} : q_\delta = q_{\delta+1}\} \supset A_1^{\delta-1} \circ \widehat{D}_4^{\#}, \tag{2.135}$$

and

$$\Big(\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_2} : q_\delta = q_{\delta+1}\}\Big) \cap \Big(A_1^{\delta-1} \circ \widehat{D}_4^{\#}\Big) = \emptyset. \tag{2.136}$$

We will prove the following claim:

Claim 2.6.15. Let $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-1},l_{q_\delta}) \in A_1^{\delta-1} \circ \widehat{D}_4^{\#}$. Then there exists points

$$\big([f_t],[\eta_t],q_1(t),\ldots,q_{\delta-2}(t);q_{\delta-1}(t),q_\delta(t),l_{q_{\delta+1}(t)}\big) \in A_1^\delta \circ \mathcal{P}A_1 \tag{2.137}$$

sufficiently close to $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-1};q_\delta,l_{q_\delta})$. Furthermore, *every* such solution satisfies the condition

$$\nabla^2 f|_{q_{\delta+1}}(v,w) \neq 0, \tag{2.138}$$

if v is a nonzero vector that belongs to $l_{q_{\delta+1}(t)}$ and w is a nonzero vector that belongs to $T\mathbb{P}^2_\eta|_{q_{\delta+1}(t)}/l_{q_{\delta+1}(t)}$. In other words,

$$\big([f_t],[\eta_t],q_1(t),\ldots,q_{\delta-1}(t),q_\delta(t),l_{q_{\delta+1}(t)}\big) \notin A_1^\delta \circ \mathcal{P}A_2.$$

Remark 2.6.16. We note that claim 2.6.15 proves eq. (2.135) and eq. (2.136) simultaneously (since $\overline{\widehat{D}_4^{\#}} = \overline{\widehat{D}}_4$).

Proof: Following the setup of the proofs of Claim 2.6.2, Claim 2.6.8 and Claim 2.6.13, we will now work in an affine chart, where we send the plane $\mathbb{P}^2_{\eta_t}$ to $\mathbb{C}^2_z$ and the point $q_\delta(t) \in \mathbb{P}^2_{\eta_t}$ to $(0,0,0) \in \mathbb{C}^2_z$. We also choose coordinates, such that $\partial_x \in l_{q_{\delta+1}(t)}$. Using this chart, let us write down the Taylor expansion of f_t around the point $(0,0)$, namely

$$f_t(x,y) = f_{t_{11}}xy + \frac{f_{t_{02}}}{2}y^2 + \frac{f_{t_{30}}}{6}x^3 + \frac{f_{t_{21}}}{2}x^2y + \frac{f_{t_{12}}}{2}xy^2 + \frac{f_{t_{03}}}{6}y^3 + \ldots$$

Since $([f_t],[\eta_t],l_{q_\delta(t)}) \in \mathcal{P}A_1$, we conclude that $f_{t_{20}}$ is zero. Next, since $([f],[\eta],l_{q_\delta}) \in \widehat{D}_4$, we conclude that f_{20}, f_{11} and f_{02} are zero; hence $f_{t_{11}}$ and $f_{t_{02}}$ are small (close to zero). Hence, constructing points on the right hand side of eq. (2.137) amounts to finding solutions to the set of equations

$$f_t = 0,\ f_{t_x} = 0 \text{ and } f_{t_y} = 0, \tag{2.139}$$

where (x,y) is small but not equal to $(0,0)$. Let us define

$$g_t(x,y) = -2f_t(x,y) + xf_{t_x}(x,y) + yf_{t_y}(x,y).$$

We note that $f_t(x,y)$ and $g_t(x,y)$ have the same cubic term in the Taylor expansion. Furthermore, $g_t(x,y)$ does not contain any quadratic term. Since $([f],[\eta],l_{q_\delta}) \in \widehat{D}_4$, we conclude that $f_{t_{30}} \neq 0$. Let

$$x := \hat{x} + E_1(\hat{x},\hat{y}), \text{ and } y := \hat{y} + E_2(\hat{x},\hat{y})$$

be a change of coordinates (where E_1 and E_2 are second order terms), such that

$$g_t = \frac{f_{t_{30}}}{6}(\hat{x} - P_1\hat{y})(\hat{x} - P_2\hat{y})(\hat{x} - P_3\hat{y})$$

There are three solutions to $g_t = 0$, given by $\hat{y} = u$ and $\hat{x} = P_i\hat{u}$, for $i = 1,2$ and 3. Converting back in terms of x and y, we conclude that the solutions to $g_t = 0$ are given by

$$y = u, \text{ and } x = P_i u + O(|u|^2).$$

Let us consider the solution $x = P_1 u + O(|u|^2)$; the other two cases can be dealt with similarly. We plug this solution into the equations $f_{t_x} = 0$ and $f_{t_y} = 0$ and solve for $f_{t_{11}}$ and $f_{t_{02}}$ in terms of u. Doing that, we get the solutions to eq. (2.139) are given by

$$\begin{aligned} y &= u, \\ x &= P_1 u + O(|u|^2), \\ f_{t_{11}} &= -\frac{f_{t_{30}}}{6}(P_1 - P_2)(P_1 - P_3)u + O(|u|^2), \text{ and} \\ f_{t_{02}} &= \frac{f_{t_{30}}}{3} P_1 (P_1 - P_2) u + O(|u|^2). \end{aligned} \tag{2.140}$$

Two more similar solutions corresponding to $x = P_2 u + O(|u|^2)$ and $x = P_3 u + O(|u|^2)$. This proves the first assertion of claim 2.6.15. Furthermore, since $f_{t_{30}} \neq 0$ and P_1, P_2 and P_3 are distinct, we conclude using eq. (2.140) that $f_{t_{11}} \neq 0$; this proves eq. (2.138). □

It remains to compute the multiplicity. We claim the each point of $(A_1^{\delta-1} \circ \widehat{D}_4^{\#}) \cap \mu$ contributes 3 to the Euler class in eq. (2.111). Using eq. (2.140) we conclude that the multiplicity from each branch is the number of small solutions u to the equation

$$-\frac{f_{t_{30}}}{6}(P_1 - P_2)(P_1 - P_3)u + O(|u|^2) = \varepsilon.$$

This number is 1, and hence the total multiplicity is 3. Finally, we note that since μ is a generic cycle all points of $(A_1^{\delta-1} \circ \widehat{D}_4) \cap \mu$ will actually belong to $(A_1^{\delta-1} \circ \widehat{D}_4^{\#}) \cap \mu$. □

Before proceeding further, note that we have proved

$$\Big(\{([f], [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}}) \in \overline{A_1^\delta \circ \mathcal{P}A_1} : q_{\delta-1} = q_\delta = q_{\delta+1}\}\Big) \cap \Big(A_1^{\delta-1} \circ \widehat{D}_4^{\#}\Big) = \emptyset. \tag{2.141}$$

To see why that is so, our proof of the claim shows that the family we constructed can not have a third node.

Next, let us prove eq. (2.115), eq. (2.116) and eq. (2.117) (that is, we will analyze what happens when three points come together). Let us start with the proof of eq. (2.115). Let us show that

$$\Big(\{([f],[\eta],q_1,\ldots,q_\delta,l_{q_{\delta+1}})\in\overline{A_1^\delta\circ\mathcal{P}A_1}:q_{\delta-1}=q_\delta=q_{\delta+1}\}\Big)\cap\Big(A_1^{\delta-1}\circ\mathcal{P}A_4\Big)=\emptyset. \tag{2.142}$$

We note that eq. (2.142) immediately implies eq. (2.115). In order to prove eq. (2.142), it suffices to prove the following claim:

Claim 2.6.17. Let $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-2},l_{q_\delta})\in A_1^{\delta-2}\circ\mathcal{P}A_4$. Then there does not exist any point

$$\big([f_t],[\eta_t],q_1(t),\ldots,q_{\delta-2}(t);q_{\delta-1}(t),q_\delta(t),l_{q_{\delta+1}(t)}\big)\in A_1^\delta\circ\mathcal{P}A_1 \tag{2.143}$$

sufficiently close to $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-1};q_\delta,l_{q_\delta})$.

Proof: Let us continue with the setup of claim 2.6.13. As before, since $f_{t_{02}}\neq 0$, we can make a change of coordinates $\hat{y}:=y+B(x)$ and write f_t as

$$f_t(x,y(x,\hat{y}))=\varphi(x,\hat{y})\hat{y}^2+\frac{\mathcal{B}_2^{f_t}}{2!}x^2+\frac{\mathcal{B}_3^{f_t}}{3!}x^3+\frac{\mathcal{B}_4^{f_t}}{4!}x^4+\frac{\mathcal{B}_5^{f_t}}{5!}x^5+\frac{\mathcal{B}_6^{f_t}}{6!}x^6+\mathcal{R}(x)x^7,$$

where $\mathcal{B}_k^{f_t}$ are as defined in eq. (2.127), $\varphi(0,0)\neq 0$ and $\mathcal{R}(x)$ is a holomorphic function defined in a neighborhood of the origin. Let us make a further change of coordinates and denote

$$\hat{\hat{y}}:=\sqrt{\varphi(x,\hat{y})}\hat{y}.$$

as in the Proof of claim 2.6.13. Let us denote the polynomial f_t by $\hat{f}_t$ which is a polynomial in two variables x and $\hat{\hat{y}}$. Hence,

$$\hat{f}_t(x, \hat{y}) = \hat{y}^2 + \frac{\mathcal{B}_2^{f_t}}{2!}x^2 + \frac{\mathcal{B}_3^{f_t}}{3!}x^3 + \frac{\mathcal{B}_4^{f_t}}{4!}x^4 + \frac{\mathcal{B}_5^{f_t}}{5!}x^5 + \frac{\mathcal{B}_6^{f_t}}{6!}x^6 + \mathcal{R}(x)x^7.$$

We claim that there does not exist any solutions to the set of equations

$$\hat{f}_t(u_1, v_1) = 0, \quad \hat{f}_x(u_1, v_1) = 0, \quad \hat{f}_{\hat{y}}(u_1, v_1) = 0, \text{ and} \tag{2.144}$$
$$\hat{f}_t(u_2, v_2) = 0, \quad \hat{f}_x(u_2, v_2) = 0, \quad \hat{f}_{\hat{y}}(u_2, v_2) = 0, \tag{2.145}$$

where (u_1, v_1) and (u_2, v_2) and $(0, 0)$ are all distinct, but close to each other.

We now note that the only solutions to eq. (2.144) and eq. (2.145) is given by

$$\begin{aligned}
& v_1, v_2 = 0, \\
\mathcal{B}_2^{f_t} &= \frac{1}{360}\mathcal{B}_6^{f_t} u_1^2 u_2^2 + O(|(u_1, u_2)|^5), \\
\mathcal{B}_3^{f_t} &= -\frac{1}{60}\mathcal{B}_6^{f_t}(u_1^2 u_2 + u_1 u_2^2) + O(|(u_1, u_2)|^4), \\
\mathcal{B}_4^{f_t} &= \frac{1}{30}\mathcal{B}_6^{f_t}(u_1^2 + 4u_1u_2 + u_2^2) + O(|(u_1, u_2)|^3), \\
\mathcal{B}_5^{f_t} &= -\frac{1}{3}\mathcal{B}_6^{f_t}(u_1 + u_2) + O(|(u_1, u_2)|^2).
\end{aligned} \tag{2.146}$$

To see why this is so, we simply note that eq. (2.144) and eq. (2.145) are the same as eq. (2.59) and eq. (2.60); hence, the argument is exactly the same as how we justified eq. (2.62) is the solution to eq. (2.144) and eq. (2.145).

We now note that v_1, v_2 are both zero; hence u_1 and u_2 are both nonzero, but small. Hence, $\mathcal{B}_5^{f_t}$ is close to zero. This is a contradiction, since $([f], [\eta], l_{q_\delta}) \in \mathcal{P}A_4$.

Next, let us prove (2.116). We will prove the following claim:

Claim 2.6.18. Let $([f], [\eta], q_1, \ldots, \ldots, q_{\delta-2}, l_{q_\delta}) \in A_1^{\delta-2} \circ \mathcal{P}D_4$. Then there exists points

$$\big([f_t], [\eta_t], q_1(t), \ldots, q_{\delta-3}(t); q_{\delta-2}(t), q_{\delta-1}(t), q_\delta(t), l_{q_{\delta+1}(t)}\big) \in A_1^\delta \circ \mathcal{P}A_1 \tag{2.147}$$

sufficiently close to $([f],[\eta],q_1,\ldots,\ldots,q_{\delta-2};q_{\delta-1},q_{\delta-1},l_{q_\delta})$. Furthermore, *every* such solution satisfies the condition

$$\nabla^2 f|_{q_{\delta+1}}(v,w)\neq 0, \tag{2.148}$$

if v is a nonzero vector that belongs to $l_{q_{\delta+1}(t)}$ and w is a nonzero vector that belongs to $T\mathbb{P}^2_\eta|_{q_{\delta+1}(t)}/l_{q_{\delta+1}(t)}$. In other words,

$$\big([f_t],[\eta_t],q_1(t),\ldots,q_{\delta-3}(t);q_{\delta-2}(t),q_{\delta-1}(t),q_\delta(t),l_{q_{\delta+1}(t)}\big)\notin A_1^\delta\circ\mathcal{P}A_2.$$

Proof: Following the setup of the proof of Claim 2.6.15, let us write down the Taylor expansion of f_t around the point $(0,0)$, namely

$$f_t(x,y)=f_{t_{11}}xy+\frac{f_{t02}}{2}y^2+\frac{f_{t30}}{6}x^3+\frac{f_{t21}}{2}x^2y+\frac{f_{t12}}{2}xy^2+\frac{f_{t03}}{6}y^3+\ldots$$

Since $([f_t],[\eta_t],l_{q_\delta(t)})\in\mathcal{P}A_1$, we conclude that $f_{t_{20}}$ is zero. Next, since $([f],[\eta],l_{q_\delta})\in\mathcal{P}D_4$, we conclude that f_{11}, f_{02} and f_{30} are zero; hence $f_{t_{11}}$, $f_{t_{02}}$ and $f_{t_{30}}$ are small (close to zero). Constructing points on the right hand side of eq. (2.147) amounts to finding solutions to the set of equations

$$f_t(x_1,y_1)=0,\qquad f_{t_x}(x_1,y_1)=0,\qquad f_{t_y}(x_1,y_1)=0,\ \text{and} \tag{2.149}$$

$$f_t(x_2,y_2)=0,\qquad f_{t_x}(x_2,y_2)=0,\qquad f_{t_y}(x_2,y_2)=0, \tag{2.150}$$

where $(0,0)$, (x_1,y_1) and (x_2,y_2) are all distinct (but close to each other). As before, we define

$$g_t(x,y):=xf_{t_x}(x,y)+yf_{t_y}(x,y)-2f_t(x,y).$$

We note that g_t has no quadratic term and has the same cubic term as f_t. The cubic term of f can be written as either $\frac{f_{03}}{6}(y-P_1(0)x)(y-P_2(0)x)y$ (if $f_{03}\neq 0$) or it can be written as $\frac{xy}{2}(f_{21}x+f_{12}y)$ (if $f_{03}=0$). We will assume the former case; the latter case can be dealt with similarly. Hence, we can write g_t as

$$g_t(x,y) = \frac{f_{t_{03}}}{6}(y - P_1 x)(y - P_2 x)(y - P_3 x) + E(x,y),$$

where E is a fourth order term. Let us assume that P_3 is close to zero. We also note that since $f_{t_{21}} \neq 0$, hence P_1 and P_2 are both nonzero. Using the equation $g_t = 0$, let us consider the solution

$$x = u, \text{ and } y = P_1 u + O(|u|^2).$$

Let us now use $f_{t_x}(x,y) = 0$ and solve for $f_{t_{11}}$ in terms of u. Doing that, we get

$$f_{t_{11}} = \frac{f_{t_{03}}}{6}(P_1^2 - P_1 P_2 - P_1 P_3 + P_2 P_3)u + O(|u|^2).$$

Plugging in this value of $f_{t_{11}}$ into the equation f_{t_y} and solving for $f_{t_{02}}$, we get that

$$f_{t_{02}} = \frac{f_{t_{03}}}{6}\Big(- 2P_1 + 2P_2 + 2P_3 - \frac{2P_2 P_3}{P_1}\Big)u + O(|u|^2).$$

Let us now try to produce a second node. We will justify shortly that $x = v$ and $y = P_2 v + O(|v|^2)$ is a not a possible solution. Hence, let us consider $x = v$ and $y = P_3 v + O(|v|^2)$. Plugging this into $f_{t_y}(x,y) = 0$ and solving for u in terms of v, we conclude that

$$u = \Big(\frac{P_1(P_3 - P_2)}{(P_1 - P_2)(P_1 - 2P_3)}\Big)v + O(|v|^2).$$

Plugging in this value for u into $f_{t_x}(x,y) = 0$ and solving for P_3, we conclude that

$$P_3 = O(|v|).$$

Plugging in the value of P_3 into u and then plugging that back into $f_{t_{11}}$ and $f_{t_{02}}$, we conclude that

$$\begin{aligned} u &= \frac{P_2}{P_2 - P_1} v + O(|v|^2), \\ f_{t_{11}} &= -\frac{f_{t_{03}}}{6} P_1 P_2 v + O(|v|^2), \\ f_{t_{02}} &= \frac{f_{t_{03}}}{3} P_2 v + O(|v|^2). \end{aligned}$$

There are four ways to construct such solutions (interchange (P_1, P_3), with (P_2, P_3)). Furthermore, we can permute the nodal points. From the expression for $f_{t_{11}}$ we see that the order of vanishing is 1; hence the total multiplicity is 4.

It remains to show why we reject the solution $x = v$ and $y = P_2 v + O(|v|^2)$. If we take that solution, then we plug it in $f_{t_x} = 0$, then solving for u (in terms of v), we conclude that

$$u = \Big(\frac{P_3 - P_2}{P_1 - P_3}\Big) v + O(|v|^2).$$

Plugging this into f_{t_y}, we conclude that

$$f_{t_y} = \frac{f_{t_{03}}}{3} \Big(\frac{(P_1 - P_2)^2 (P_3 - P_2)}{P_1}\Big) v^2 + O(|v|^2).$$

This is clearly nonzero, if v is small and nonzero. Hence, we reject the solution corresponding to $x = v$ and $y = P_2 v + O(|v|^2)$. This completes the proof.

Finally, let us justify eq. (2.117). This follows from eq. (2.141). This completes the proof of Theorem 2.4.6. □

2.6.4 Proof of Theorem 2.4.8: computation of $N(A_1^{\delta}\mathcal{P}A_3)$ when $0 \leq \delta \leq 1$.

We will justify our formula for $N(A_1^{\delta}\mathcal{P}A_3, r, s, n_1, n_2, n_3, \theta)$, when $0 \leq \delta \leq 1$. Recall that

$$A_1^{\delta} \circ \overline{\mathcal{P}A}_2 := \{([f], [\eta], q_1, \ldots, q_{\delta}, l_{q_{\delta+1}}) \in \mathcal{S}_{\mathcal{D}_{\delta}} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} : ([f], [\eta], l_{q_{\delta+1}}) \in \overline{\mathcal{P}A}_2,$$
$$f \text{ has a singularity of type } A_1 \text{ at } q_1, \ldots, q_{\delta} \text{ with } q_1, \ldots, q_{\delta+1} \text{ are all distinct}\}.$$

Let μ be a generic cycle given by

$$\mu := a^{n_1}\lambda^{n_2}(\pi_{\delta+1}^* H)^{n_3}(\pi^*\lambda_W)^{\theta}\mathcal{H}_L^r\mathcal{H}_p^s.$$

We now define a section of the following bundle

$$\Psi_{\mathcal{P}A_3} : A_1^{\delta} \circ \overline{\mathcal{P}A}_2 \longrightarrow \mathbb{L}_{\mathcal{P}A_3} := \gamma_{\mathcal{D}}^* \otimes \gamma_W^{*3} \otimes \gamma_{\mathbb{P}^3}^{*d}, \text{ given by}$$
$$\{\Psi_{\mathcal{P}A_3}([f], [\eta], q_1, \ldots, q_{\delta}, l_{q_{\delta+1}})\}(f \otimes v^{\otimes 3}) := \nabla^3 f|_{q_{\delta+1}}(v, v, v).$$

Analogous to [1, Lemma 6.1], we conclude that for $d \geq 4$,

$$\overline{\mathcal{P}A}_2 = \mathcal{P}A_2 \cup \overline{\mathcal{P}A}_3 \cup \overline{\widehat{D}}_4. \tag{2.151}$$

Furthermore, analogous to [1, Lemma 6.3] we conclude that for $d \geq 4$,

$$\overline{A_1^{\delta} \circ \mathcal{P}A_2} = (\overline{A_1^{\delta}} \circ \mathcal{P}A_2) \cup \overline{A_1^{\delta}} \circ (\overline{\mathcal{P}A}_2 - \mathcal{P}A_2) \cup \overline{A_1^{\delta-1}} \circ (\Delta\overline{\mathcal{P}A}_4 \cup \Delta\overline{\widehat{D}}_5). \tag{2.152}$$

Let us define

$$\mathcal{B} := \overline{A_1^{\delta} \circ \overline{\mathcal{P}A}_2} - A_1^{\delta} \circ (\mathcal{P}A_2 \cup \overline{\mathcal{P}A}_3).$$

We will show shortly that the section $\Psi_{\mathcal{P}A_3}$ vanishes on the points of $A_1^\delta \circ \mathcal{P}A_3$ transversally. Hence,

$$\langle e(\mathbb{L}_{\mathcal{P}A_3}), \ [\overline{A_1^\delta \circ \overline{\mathcal{P}A}_2}] \cap \widetilde{\mu} \rangle = N(A_1^\delta \mathcal{P}A_3, n_1, n_2, n_3, \theta) + \mathcal{C}_{\mathcal{B}\cap\mu}. \tag{2.153}$$

We now give an explicit description of $\mathcal{B}$. Let us first define

$$\mathcal{B}_0 := \{([f], [\eta], q_1, \dots q_\delta, l_{q_{\delta+1}}) \in \mathcal{B} : q_1, q_2 \dots q_{\delta+1} \text{ are all distinct}\}.$$

In other words, $\mathcal{B}_0$ is that component of the boundary, where all the points are still distinct. By eq. (2.151), we conclude that

$$\mathcal{B}_0 = \overline{A_1^\delta} \circ \overline{\widehat{D}}_4.$$

If we intersect $\mathcal{B}_0$ with μ then we will get a finite set of points. Since the representative μ is generic, we conclude that the third derivative along v will not vanish, i.e., the section $\Psi_{\mathcal{P}A_3}$ will not vanish on those points. Hence, $\mathcal{B}_0 \cap \mu$ does not contribute to the Euler class.

Next, let us consider the components of $\mathcal{B}$ where one (or more) of the q_i become equal to the last point $q_{\delta+1}$. Define $\mathcal{B}(q_{i_1}, \dots q_{i_k}, l_{q_\delta})$ as before. Analogous to the proof of [1, Lemma 6.3], we conclude that

$$\mathcal{B}(q_1, l_{q_{\delta+1}}) \approx \overline{A_1^{\delta-1} \circ \mathcal{P}A_4} \cup \overline{A_1^{\delta-1} \circ \widehat{D}_5}.$$

Furthermore, analogous to the proof of [1, Corollary 6.13, Page 700], we conclude that the contribution to the Euler class from each of the points of $\overline{A_1^{\delta-1} \circ \mathcal{P}A_4} \cap \mu$ is 2. Finally, we note that the section $\Psi_{\mathcal{P}A_3}$ does not vanish on $\overline{A_1^{\delta-1} \circ \widehat{D}_5} \cap \mu$, since μ is generic. Hence, the total contribution from all the components of type $\mathcal{B}(q_{i_1}, l_{q_{\delta+1}})$ equals

$$2\binom{\delta}{1} N(A_1^{\delta-1}\mathcal{P}A_4, n_1, n_2, n_3, \theta).$$

Plugging in this in eq. (2.153) gives us the formula of theorem 2.4.8.

It just remains to prove the transversality claim. This follows from following the setup of proof of transversality in Theorem 2.4.6. We consider the polynomial

$$\rho_{30} := \Big(X - \frac{X_1}{Z_1}Z\Big)^2\Big(X - \frac{X_2}{Z_2}Z\Big)^2 \dots \cdot \Big(X - \frac{X_\delta}{Z_\delta}Z\Big)^2 X^3 Z^{d-2\delta-3},$$

and the corresponding curve $\gamma_{30}(t)$. Transversality follows by computing the derivative of the section $\Psi_{\mathcal{P}A_3}$ along the curve $\gamma_{30}(t)$ as before. □

2.6.5 Proof of Theorem 2.4.9: computation of $N(\mathcal{P}A_4)$.

We will now justify our formula for $N(\mathcal{P}A_4, r, s, n_1, n_2, n_3, \theta)$. Let μ be a generic cycle given by

$$\mu := a^{n_1}\lambda^{n_2}(\pi_{\delta+1}^* H)^{n_3}(\pi^*\lambda_W)^\theta \mathcal{H}_L^r \mathcal{H}_p^s.$$

Let $v \in \gamma_W$ and $w \in \pi^*W/\gamma_W$ be two fixed nonzero vectors. Let us introduce the following abbreviation:

$$f_{ij} := \nabla^{i+j} f|_q(\underbrace{v, \cdots v}_{i \text{ times}}, \underbrace{w, \cdots w}_{j \text{ times}}).$$

We now define a section of the following bundle

$$\Psi_{\mathcal{P}A_4} : \overline{\mathcal{P}A}_3 \longrightarrow \mathbb{L}_{\mathcal{P}A_4} := \gamma_{\mathcal{D}}^{*2} \otimes \gamma_W^{*4} \otimes (W/\gamma_W)^{*2} \otimes \gamma_{\mathbb{P}^3}^{*2d}, \text{ given by}$$
$$\{\Psi_{\mathcal{P}A_4}([f], l_q)\}(f^{\otimes 2} \otimes v^{\otimes 4} \otimes w^{\otimes 2}) := f_{02} A_4^f,$$

where $A_4^f := f_{40} - \frac{3f_{21}^2}{f_{02}}$. Analogous to [1, Lemma 6.1], we conclude that

$$\overline{\mathcal{P}A}_3 = \mathcal{P}A_3 \cup \overline{\mathcal{P}A}_4 \cup \overline{\mathcal{P}D}_4. \tag{2.154}$$

Hence, let us define

$$\mathcal{B} := \overline{\mathcal{P}A_3} - \mathcal{P}A_3 \cup \overline{\mathcal{P}A_4}.$$

We will show shortly that the section $\Psi_{\mathcal{P}A_4}$ vanishes on the points of $\mathcal{P}A_4$ transversally. Hence,

$$\langle e(\mathbb{L}_{\mathcal{P}A_4}),\ \ [\overline{\mathcal{P}A_3}] \cap \widetilde{\mu}\rangle = N(A_1^\delta \mathcal{P}A_4, r, s, n_1, n_2, n_3, \theta) + \mathcal{C}_{\mathcal{B}\cap\mu}. \tag{2.155}$$

Let us now study the boundary $\mathcal{B}$. By eq. (2.154), we conclude that

$$\mathcal{B} \cap \mu = \overline{\mathcal{P}D}_4 \cap \mu.$$

Since the representative μ is generic, we conclude that the directional derivative f_{21} will not vanish on those points. Since $f_{02} = 0$ on $\mathcal{B}$, we conclude that

$$f_{02}A_4^f = f_{02}f_{40} - 3f_{21}^2 \neq 0$$

if $f_{21} \neq 0$. Hence, the section $\Psi_{\mathcal{P}A_4}$ will not vanish on $\mathcal{B} \cap \mu$. Hence, the total boundary contribution is zero and eq. (2.155) gives us the formula of Theorem 2.4.9.

It remains to prove the claim regarding transversality. This follows from following the setup of proof of transversality in Theorem 2.4.8. We consider the polynomial

$$\rho_{40} := X^4 Z^{d-4},$$

and the corresponding curve $\gamma_{40}(t)$. Transversality follows by computing the derivative of the section $\Psi_{\mathcal{P}A_4}$ along the curve $\gamma_{40}(t)$ as before. □

2.6.6 Proof of Theorem 2.4.10: computation of $N(\mathcal{P}D_4)$.

We will now justify our formula for $N(\mathcal{P}D_4, r, s, n_1, n_2, n_3, \theta)$. Let μ be a generic cycle given by

$$\mu := a^{n_1}\lambda^{n_2}(\pi_{\delta+1}^*H)^{n_3}(\pi^*\lambda_W)^{\theta}\mathcal{H}_L^r\mathcal{H}_p^s.$$

As before, let $v \in \gamma_W$ and $w \in \pi^*W/\gamma_W$ be two fixed nonzero vectors. Define a section of the following bundle

$$\Psi_{\mathcal{P}D_4} : \overline{\mathcal{P}A_3} \longrightarrow \mathbb{L}_{\mathcal{P}D_4} := \gamma_{\mathcal{D}}^* \otimes (W/\gamma_W)^{*2} \otimes \gamma_{\mathbb{P}^3}^{*d}, \text{ given by}$$
$$\{\Psi_{\mathcal{P}D_4}([f], l_q)\}(f \otimes w^{\otimes 2}) := \nabla^2 f|_q(w, w).$$

We recall eq. (2.154), namely

$$\overline{\mathcal{P}A_3} = \mathcal{P}A_3 \cup \overline{\mathcal{P}A_4} \cup \overline{\mathcal{P}D_4}.$$

We now define

$$\mathcal{B} := \overline{\mathcal{P}A_3} - (\mathcal{P}A_3 \cup \overline{\mathcal{P}D_4}).$$

We will show that the section $\Psi_{\mathcal{P}D_4}$ vanishes on the points of $\mathcal{P}D_4$ transversally. Hence,

$$\langle e(\mathbb{L}_{\mathcal{P}D_4}),\ [\overline{\mathcal{P}A_3}] \cap \widetilde{\mu}\rangle = N(\mathcal{P}D_4, r, s, n_1, n_2, n_3, \theta) + \mathcal{C}_{\mathcal{B}\cap\mu}. \tag{2.156}$$

By definitions, the section $\Psi_{\mathcal{P}D_4}$ does not vanish on $\mathcal{P}A_4 \cap \mu$. Hence, the total boundary contribution is zero and eq. (2.156) gives us the formula of Theorem 2.4.10.

It remains to prove the claim regarding transversality. This follows from following the setup of proof of transversality in Theorem 2.4.9. We consider the polynomial

$$\rho_{02} := Y^2 Z^{d-2},$$

and the corresponding curve $\gamma_{02}(t)$. Transversality follows by computing the derivative of the section $\Psi_{\mathcal{P}D_4}$ along the curve $\gamma_{02}(t)$ as before. □

2.7 Enumeration of planar cubics having one node via Caporaso-Harris approach

In [9], Caporaso and Harris computed the characteristic number of plane curves having any number of nodes. In this section, we intend to apply their idea in planar version setup, and count the number of planar nodal cubics intersecting 11 generic lines. The main idea of Caporaso-Harris is to describe the cycle explicitly which is given by the space of curves that passes through a point on a fixed line. We first discuss step by step how they computed the number of nodal cubics in plane using their idea, and then, implement it to the planar version setup.

Before proceeding further, let us first setup some notations. If α is a sequence of non-negative integers such that all but finitely many terms are zero, then define

$$|\alpha| = \sum_i \alpha_i,$$
$$I\alpha = \sum_i i\alpha_i.$$

Let L be a fixed line in $\mathbb{P}^2$. For any d, δ and two such sequence α, β of non-negative integers such that all but finitely many terms are zero satisfying $I\alpha + I\beta = d$, define $V^{d,\delta}(\alpha, \beta)$ to be the closure of the locus of reduced δ-nodal plane curves of degree d, not containing L, with total of $|\alpha| + |\beta|$ points of intersection with L, among which α_i

specified points on L each of having order of contact i, and β_i unspecified points on L each of having order of contact i. The space $V^{d,\delta}(\alpha,\beta)$ is known as *generalized Severi variety*, and its dimension is

$$\frac{d(d+1)}{2} - \delta + |\beta|.$$

For $p \in L$, if $\mathcal{H}_p$ be the cycle given by the curves passing through p in $V^{d,\delta}(\alpha,\beta)$, then Caporaso and Harris express $\mathcal{H}_p$ in terms of $V^{d',\delta'}(\alpha',\beta')$, where $d' \leq d$, $0 \leq \delta' \leq \delta$. Repeating this process successively, a recursive formula of the degree of $V^{d,\delta}(\alpha,\beta)$ can be obtained.

If $\mathcal{H}$ and $\mathcal{H}_p$ denote the hyperplane in $V^{d,\delta}(\alpha,\beta)$ given by the curves that pass through a point in $\mathbb{P}^2$ and L, respectively, then for any d, δ, α, β,

$$\mathcal{H}^k \cdot [V^{d,\delta}(\alpha,\beta)] = \mathcal{H}_p \cdot \mathcal{H}^{k-1} \cdot [V^{d,\delta}(\alpha,\beta)], \tag{2.157}$$

where $k \geq 1$. Geometrically, the equality states that the cycle given by the curves that pass through k generic points is the same as the cycle given by the curves that pass through $k-1$ generic points in $\mathbb{P}^2$ and one generic point on L. This will be useful as we proceed further.

Note that we want to count the number of nodal cubic in $\mathbb{P}^2$ that passes through 8 generic points. Assume $d = 3$, $\delta = 1$, $\alpha = \{0\}$, and $\beta = \{3\}$. Then it is the same amount of asking the degree of $V^{d,\delta}(\alpha,\beta)$, which is same as $\mathcal{H}^8 \cdot [V^{d,\delta}(\alpha,\beta)]$.

Step 1: To compute $\mathcal{H}^8 \cdot [V^{3,1}(\{0\},\{3\})]$, it is enough to compute $\mathcal{H}_p \cdot \mathcal{H}^7 \cdot [V^{3,1}(\{0\},\{3\})]$ by eq. (2.157). From [9, Theorem 1.3], we conclude that

$$\mathcal{H}_p \cdot [V^{3,1}(\{0\},\{3\})] = [V^{3,1}(\{1\},\{2\})].$$

Now intersect the above cycle by $\mu_1 = \mathcal{H}^7$ to get equality of numbers, and the number in the left hand side is our desired number. Thus it is enough to compute $\mu_1 \cdot [V^{3,1}(\{1\},\{2\})]$.

Step 2: In this case, take $\mu_2 = \mathcal{H}^6$. By eq. (2.157), it is enough to consider $\mathcal{H}_p \cdot \mu_2 \cdot$

$[V^{3,1}(\{1\},\{2\})]$. Again by [9, Theorem 1.3], we conclude that

$$\mathcal{H}_p \cdot [V^{3,1}(\{1\},\{2\})] = [V^{3,1}(\{2\},\{1\})].$$

Similarly, intersecting μ_2 in the above cycle, we get equality of desired number.

Step 3: Similar process apply here. In this case, take $\mu_3 = \mathcal{H}^5$. The cycle relationship here is given by

$$\mathcal{H}_p \cdot [V^{3,1}(\{2\},\{1\})] = [V^{3,1}(\{3\},\{0\})] + 2[V^{2,0}(\{0\},\{2\})].$$

To get the desired equality of numbers, we intersect both the sides of the cycle relationship by μ_3. Note that here the second term of the right hand side is enumerative number involving conics (not cubics).

Step 4: In this case, take $\mu_4 = \mathcal{H}^4$. Repeating application of [9, Theorem 1.3] yields that

$$\mathcal{H}_p \cdot [V^{3,1}(\{3\},\{0\})] = [V^{2,1}(\{0\},\{2\})] + 2[V^{2,0}(\{0\},\{0,1\})] + 3[V^{2,0}(\{1\},\{0\})].$$

Similarly, intersect the above cycle with μ_4 to get equality of numbers.

Final calculation: Observe that the number $\mathcal{H}^8 \cdot [V^{d,\delta}(\alpha,\beta)]$ can be written as the combinations of the enumerative numbers involving conics. Thus the question of counting cubics transforms into the question of counting conics. Again the numbers involving conics can be derived by repeating the arguments of Caporaso-Harris. To avoid the complexity, we compute the numbers by alternative means instead of repeating the process. The second term of step 3 is about to the number of smooth conic that passes through 5 generic points, and the answer is 1. The first term of step 4 is about to the number of nodal conic that passes through 4 generic points, and the answer is 3. The second term of step 4 is about to the number of smooth conic that passes through 4 generic points and tangent to a given line, and the answer is 2. The third term of step 4 is about to the number of smooth conic that passes through 4 generic points and one more specific points on a given line, and the answer is 1.

Therefore, the final answer is

$$(2\times 1)+(1\times 3)+(2\times 2)+(3\times 1)=12.$$

Remark 2.7.1. Note that the counting question about cubics reduces to that of conics. This is actually question of counting cubic where the cubics are the union of a conic and the fixed line L. In that sense, it is question of counting pairs that consist of conic and the fixed line L.

We now implement the above idea to count planar nodal cubics in $\mathbb{P}^3$. In the case of plane curves, the line L was fixed, and the enumeration of nodal cubic is the same amount of asking to count the number of pairs, consist of the fixed line L and a nodal cubic, such that the nodal cubic passes through 8 generic points. Alternatively, we can ask about the number of pairs, consisting of a line L and a nodal cubic, such that the nodal cubic passes through 8 generic points, and the line passes through 2 generic points. We implement the latter in the planar version setup. Note that we allow to move the line as oppose to fix it in the original version of Caporaso-Harris approach.

Before proceeding further, let us first set up some notations. The space of planar degree d curves by $\mathcal{D}_d$, and the space of planar degree d curves having at least one node by $\mathcal{N}_d$. We start by defining $Y_0 := \mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{N}_d$. Let us denote the class of degree d curves that intersect a line by $\mathcal{H}_{L,d}$. Also, denote the class of pairs consisting of a line and a degree d curve that lie on a same plane such that the curve passes through a point on the line by $\widetilde{\mathcal{H}}_{L,d}$. Note that there are two natural projection maps from Y_0. One is on $\mathcal{D}_1$ and the other one is on $\mathcal{N}_d$; we denote the maps by π_1, π_2, respectively.

Now we are all set to count nodal planar cubics that intersect 11 generic lines. As mentioned earlier, we count the number of pairs, consisting of lines and planar nodal cubic, such that both lie in a common plane, and the line intersects through 2 generic lines and the nodal cubic intersects 11 generic lines. That is, we want to compute $\pi_1^*\mathcal{H}_{L,1}^2 \cdot \pi_2^*\mathcal{H}_{L,3}^{11} \cdot [Y_0]$.

Abusing notation, we denote $\mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{N}_3$ by the same letter Y_0.

Also, define the space

$$W_1 :=\{([\eta],[f_1],[f_3],p_1) \in Y_0 \times \mathbb{P}^3 \mid ([\eta],p_1) \in \mathcal{S}, f_3(p_1)=0\}.$$

Let $\mu'_{10} := \pi_1^*\mathcal{H}_{L,1}^2 \cdot \pi_2^*\mathcal{H}_{L,3}^{10}$. Then

$$\pi_1^*\mathcal{H}_{L,1}^2 \cdot \pi_2^*\mathcal{H}_{L,3}^{11} \cdot [Y_0] = H_1^2\mu'_{10} \cdot [W_1].$$

It is enough to find the right hand side.

Step 1′: Let us define the space

$$Y_1 := \{([\eta],[f_1],[f_3],p_1) \in W_1 \mid f_1(p_1)=0\}.$$

It can be shown that the following cycle relationship holds.

$$[f_1(p_1)=0] \cdot [W_1] = [Y_1]. \tag{2.158}$$

This cycle relation is analogous to the cycle relation of step 1 in the case of plane curves. Now, intersecting both side of eq. (2.158) by $H_1\mu'_{10}$, we get the equality of numbers.

Therefore,

$$\begin{aligned} H_1\mu'_{10} \cdot [f_1(p_1)=0] \cdot [W_1] &= H_1\mu'_{10} \cdot (\lambda_1 + H_1) \cdot [W_1] \\ &= H_1^2\mu'_{10} \cdot [W_1] + \lambda_1 H_1 \mu'_{10} \cdot [W_1]. \end{aligned}$$

Using eq. (2.158), we conclude that

$$H_1^2\mu'_{10} \cdot [W_1] = H_1\mu'_{10} \cdot [Y_1] - \lambda_1 H_1\mu'_{10} \cdot [W_1].$$

Remark 2.7.2. Observe that the number $\lambda_1 H_1\mu'_{10} \cdot [W_1]$ is the degenerate contribution to the calculation of $H_1^2\mu'_{10} \cdot [W_1]$. We will get more degenerate contributions in the subsequent steps. A similar process may apply to obtain the values of these degenerate contributions. Here we want to compute only $\pi_1^*\mathcal{H}_{L,1}^2 \cdot \pi_2^*\mathcal{H}_{L,3}^{11} \cdot [Y_0]$ by assuming the degenerate contributions. Hence, instead of applying the approach of Caporaso-Harris again, we directly compute them by obtaining the degrees of the Euler classes of appropriate vector bundles.

An Euler class computation yields $\lambda_1 H_1 \mu'_{10} \cdot [W_1] = 6120$.

Now, our goal is to compute $H_1\mu'_{10} \cdot [Y_1]$. Similarly, define

$$W_2 := \{([\eta], [f_1], [f_3], p_1, p_2) \in Y_1 \times \mathbb{P}^3 \mid ([\eta], p_2) \in \mathcal{S}, f_3(p_2) = 0\}.$$

If we denote the cycle $\pi_1^*\mathcal{H}_{L,1}^2 \cdot \pi_2^*\mathcal{H}_{L,3}^9$ by μ'_9, then

$$H_1\mu'_{10} \cdot [Y_1] = H_1 H_2^2 \mu'_9 \cdot [W_2].$$

As before, our goal is to find $H_1 H_2^2 \mu'_9 \cdot [W_2]$.

Step 2′: Similarly define the space

$$Y_2 := \{([\eta], [f_1], [f_3], p_1, p_2) \in W_2 \mid f_1(p_2) = 0\}.$$

The cycle relationship, that is analogous to the cycle relation in step 2, is

$$[f_1(p_2) = 0] \cdot [W_2] = [Y_2] + [\Delta_{12}] \cdot [W_2], \tag{2.159}$$

where $[\Delta_{12}]$ is the cycle that corresponds to the fact that $p_1 = p_2$. We now intersect both the sides of eq. (2.159) with $H_1 H_2 \mu'_9$ to get the equality of numbers.

Therefore,

$$H_1 H_2 \mu'_9 \cdot [f_1(p_2) = 0] \cdot [W_2] = (\lambda_1 + H_2) H_1 H_2 \mu'_9 \cdot [W_2]$$

$$= \lambda_1 H_1 H_2 \mu_9' \cdot [W_2] + H_1 H_2^2 \mu_9' \cdot [W_2].$$

Using eq. (2.159), we conclude that

$$H_1 H_2^2 \mu_9' \cdot [W_2] = H_1 H_2 \mu_9' \cdot [Y_2] + [\Delta_{12}] H_1 H_2 \mu_9' \cdot [W_2] - \lambda_1 H_1 H_2 \mu_9' \cdot [W_2].$$

Again, an Euler class computation yields $\lambda_1 H_1 H_2 \mu_9' \cdot [W_2] = 4176$.
Again,

$$\begin{aligned}[\Delta_{12}] H_1 H_2 \mu_9' \cdot [W_2] &= \langle H_1^2 \mu_9', [Y_1] \rangle \\ &= 3432.\end{aligned}$$

Our goal is to compute $H_1 H_2 \mu_9' \cdot [Y_2]$. As before, we define the space W_3 and Y_3. If the cycle $\pi_1^* \mathcal{H}_{L,1}^2 \cdot \pi_2^* \mathcal{H}_{L,3}^8$ is denoted by μ_8', then

$$H_1 H_2 \mu_9' \cdot [Y_2] = H_1 H_2 H_3^2 \mu_8' \cdot [W_3].$$

Step 3′: Let us define

$$\begin{aligned}S := \{([\eta], [f_1], [f_2], p_1, p_2, p_3) \in \mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{D}_2 \times \left(\mathbb{P}^3\right)^3 \mid \eta(p_j) = 0, \\ f_1(p_j) = 0 \text{ for all } 1 \leq j \leq 3\}.\end{aligned}$$

The cycle relationship, that is analogous to the cycle relation in step 3, is

$$[f_1(p_3) = 0] \cdot [W_3] = [Y_3] + [\Delta_{12}] \cdot [W_3] + [\Delta_{23}] \cdot [W_3] + [S]. \tag{2.160}$$

As before, we intersect both sides of eq. (2.160) with $H_1 H_2 H_3 \mu_8'$, and a similar calculation yields

$$H_1 H_2 H_3^2 \mu_8' \cdot [W_3] = H_1 H_2 H_3 \mu_8' \cdot [Y_3] + H_1 H_2 H_3 \mu_8' [\Delta_{12}] \cdot [W_3] + H_1 H_2 H_3 \mu_8' [\Delta_{23}][W_3]$$

$$+ H_1H_2H_3\mu_8' \cdot [S] - \lambda_1 H_1H_2H_3\mu_8' \cdot [W_3].$$

In the similar manner,

$$\begin{aligned} H_1H_2H_3\mu_8'[\Delta_{23}] \cdot [W_3] &= \langle H_1H_2^2\mu_8', [Y_2]\rangle \\ &= 3144. \end{aligned}$$

Similarly, $H_1H_2H_3\mu_8'[\Delta_{12}] \cdot [W_3] = 3144.$

Again,

$$\begin{aligned} H_1H_2H_3\mu_8' \cdot [S] &= \langle \pi_1^*\mathcal{H}_{L,1}^2 \cdot \pi_2^*\mathcal{H}_{L,3}^8, [\mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{D}_2]\rangle \\ &= 2168. \end{aligned}$$

It is also immediate to check that

$$\begin{aligned} \lambda_1 H_1H_2H_3\mu_8' \cdot [W_3] &= 3 \times \langle \lambda_1 H_1H_2\mu_8', [Y_2]\rangle \\ &= 2016. \end{aligned}$$

Our goal is to compute $H_1H_2H_3\mu_8' \cdot [Y_3]$. As before, we define the space W_4. If we denote the cycle $\pi_1^*\mathcal{H}_{L,1}^2 \cdot \pi_2^*\mathcal{H}_{L,3}^7$ by μ_7', then

$$H_1H_2H_3\mu_8' \cdot [Y_3] = H_1H_2H_3H_4^2\mu_7' \cdot [W_4].$$

Step 4′: Before Proceeding further, let us define the following spaces.

$$\begin{aligned} U := \{([\eta], [f_1], [f_2], p_1, p_2, p_3, p_4) \in \mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{N}_2 \times \left(\mathbb{P}^3\right)^4 \mid \eta(p_j) = 0, \\ f_1(p_j) = 0 \text{ for all } 1 \leq j \leq 4\}, \end{aligned}$$

$$T := \{([\eta],[f_1],[f_2],p_1,p_2,p_3,p_4) \in \mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{N}_2 \times \left(\mathbb{P}^3\right)^4 \mid \eta(p_j)=0, \\ f_1(p_j)=0 \text{ for all } 1 \leq j \leq 4, \text{ and } f_1 \text{ is tangent to } f_2\},$$

$$V_i := \{([\eta],[f_1],[f_2],p_1,p_2,p_3) \in \mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{D}_2 \times \left(\mathbb{P}^3\right)^3 \mid f_2(p_i)=0 \text{ and} \\ \eta(p_j)=0, f_1(p_j)=0 \text{ for all } j=1,2,3,\}.$$

Here the analogous cycle relationship is

$$[W_4]\cdot[f_1(p_4)=0] = 2[T]+[U]+[\Delta_{14}][V_1]+[\Delta_{24}][V_2]+ \\ [\Delta_{34}][V_3]+[\Delta_{14}][W_4]+[\Delta_{24}][W_4]+[\Delta_{34}][W_4]. \tag{2.161}$$

As before, we intersect both sides of eq. (2.161) with $H_1H_2H_3H_4\mu_7'$ and a similar calculation yields

$$H_1H_2H_3H_4^2\mu_7'\cdot[W_4] = 2H_1H_2H_3H_4\mu_7'\cdot[T]+H_1H_2H_3H_4\mu_7'\cdot[U]+ \\ \binom{3}{1}H_1H_2H_3H_4\mu_7'\cdot[\Delta_{34}][V_3]+\binom{3}{1}H_1H_2H_3H_4\mu_7'\cdot[\Delta_{34}][W_4]- \\ \lambda_1H_1H_2H_3H_4\mu_7'\cdot[W_4].$$

The following numbers can be obtained by Euler class computations.

$$H_1H_2H_3H_4\mu_7'\cdot[T] = 1416,$$
$$H_1H_2H_3H_4\mu_7'\cdot[U] = 1960,$$
$$H_1H_2H_3H_4\mu_7'\cdot[\Delta_{34}][V_3] = 888,$$
$$H_1H_2H_3H_4\mu_7'\cdot[\Delta_{34}][W_4] = 2064,$$
$$\lambda_1H_1H_2H_3H_4\mu_7'\cdot[W_4] = 264.$$

Final Calculation: All the calculations from step 4 yield that

$$H_1 H_2 H_3 \mu_8' \cdot [Y_3] = H_1 H_2 H_3 H_4^2 \mu_7' \cdot [W_4] = 13384.$$

Also, from step 3, we conclude that

$$H_1 H_2 \mu_9' \cdot [Y_2] = H_1 H_2 H_3^2 \mu_8' \cdot [W_3] = 19824.$$

From step 2, it follows that

$$H_1 \mu_{10}' \cdot [Y_1] = H_1 H_2^2 \mu_9' \cdot [W_2] = 19080.$$

Finally, from step 1, it can be concluded that

$$\pi_1^* \mathcal{H}_{L,1}^2 \cdot \pi_2^* \mathcal{H}_{L,3}^{11} \cdot [Y_0] = H_1^2 \mu_{10}' \cdot [W_1] = 12960.$$

Hence, the number of nodal planar cubics that intersect 11 generic lines in $\mathbb{P}^3$ is 12960 which agrees with the answer obtained form theorem 2.4.1.

Remark 2.7.3. Our aim is to study the enumerative geometry of planar nodal curves via Caporaso-Harris approach. We hope that the calculation of planar nodal cubics is a starting point of implementing the idea of Caporaso-Harris in this setup.

2.8 Some low degree checks

By alternative means, we now compute some of the numbers that are stated in terms of recursive formulas in this chapter.

2.8.1 Computation of $N(\mathcal{P}A_3, r, s, 0, 0, 0, 0)$ when $d = 3$, $r = 9$ and $s = 0$

First, we note that a planar cubic has a A_3-singularity if and only if the cubic is union of a line and a conic such that both the line and conic lie on the same plane, and the line is tangent to the conic. That is, the question of counting planar cubics, having a tacnode, reduces to the question of counting the pairs that consist of planar lines and planar conics with appropriate restrictions. Since we have to deal lines and conics simultaneously, the space of planar degree d curves by $\mathcal{D}_d$ instead of $\mathcal{D}$, the notation which was used earlier in this chapter. We now describe the space of line and conic that lies on same plane and the line is tangent to the conic. If $([f_1], [\eta], q) \in \mathcal{S}_{\mathcal{D}_1}$, i.e., a planar line together with a point on the line, then we obtain the following short exact sequence of vector bundles over $\mathcal{S}_{\mathcal{D}_1}$.

$$0 \longrightarrow \ker(\nabla f_1|_q) \longrightarrow W \xrightarrow{\nabla f_1|_q} \gamma^*_{\mathcal{D}_1} \otimes \gamma^*_{\mathbb{P}^3} \longrightarrow 0 \tag{2.162}$$

Let us denote the total space $\ker(\nabla f_1|_q)$ by $\mathbb{L} \longrightarrow \mathcal{S}_{\mathcal{D}_1}$. Then the space of lines and conics such that both the line and conic lie on the same plane, and the line is tangent to the conic is given by

$$\mathcal{T} := \{([f_1], [f_2], [\eta], q) : \ ([f_1], [\eta], q) \in \mathcal{S}_{\mathcal{D}_1}, ([f_2], [\eta], q) \in \mathcal{S}_{\mathcal{D}_2}, \nabla f_1|_q(u) = 0 \, \forall u \in \mathbb{L}\}.$$

Interpreting $\mathcal{T}$ as a zero of section of some appropriate vector bundle over the space $\mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{D}_2 \times_{\widehat{\mathbb{P}}^3} \mathcal{S}$, the Euler class of the bundle will be

$$\tau := e(\gamma^*_{\mathcal{D}_1} \otimes \gamma^*_{\mathbb{P}^3}) e(\gamma^*_{\mathcal{D}_2} \otimes \gamma^{*2}_{\mathbb{P}^3}) e(\gamma^*_{\mathcal{D}_2} \otimes \gamma^{*2}_{\mathbb{P}^3} \otimes \mathbb{L}^*).$$

Denote the Chern classes of $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathbb{L}^*$ by λ_1, λ_2, and κ, respectively. Then eq. (2.162) yields

$$\kappa = \lambda_1 + h - c_1(W),$$

where $c_1(W) = 3H - a$ as given by eq. (2.21).

We are now all set to compute $N(\mathcal{P}A_3, r, 0)$. Observe that the dimension of the space $\mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{D}_2 \times_{\widehat{\mathbb{P}}^3} \mathcal{S}$ is 12. Now define

$$\begin{aligned} R_i &:= \langle (\lambda_1 + a)^i (\lambda_2 + 2a)^{9-i} \tau, [\mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{D}_2 \times_{\widehat{\mathbb{P}}^3} \mathcal{S}] \rangle \\ &= \deg((\lambda_1 + a)^i (\lambda_2 + 2a)^{9-i} \tau) \text{ in } \mathcal{D}_1 \times_{\widehat{\mathbb{P}}^3} \mathcal{D}_2 \times_{\widehat{\mathbb{P}}^3} \mathcal{S} \\ &= \text{ Coefficient of } \lambda_1^2 \lambda_2^5 H^3 a^3 \text{ in } (\lambda_1 + a)^i (\lambda_2 + 2a)^{9-i} (a + H) \tau. \end{aligned}$$

Now suppose we want to count $N(\mathcal{P}A_3, r, s, 0, 0, 0, 0)$ when $d = 3$, $r = 9$, and $s = 0$. That is the tacnodal cubics intersect 9 generic lines; in other words, a planar line and a planar conic together intersect total of 9 generic lines. Hence the 9 generic lines divide into two sets, one for which the conic intersects and another for which the line intersects. Suppose the line intersect i lines and the conic intersect $9 - i$ lines in general position, then the number of such configuration is exactly R_i. But i lines can be chosen among 9 lines in $\binom{9}{i}$ ways. Continuing this way we obtain

$$\begin{aligned} & N(\mathcal{P}A_3, 9, 0, 0, 0, 0, 0) \\ &= \sum_{i=0}^{9} \binom{9}{i} R_i \\ &= \left(\binom{9}{1} \times 184\right) + \left(\binom{9}{2} \times 184\right) + \left(\binom{9}{3} \times 104\right) + \left(\binom{9}{4} \times 24\right) \\ &= 20040. \end{aligned}$$

This agrees with the corresponding number obtained from Theorem 2.4.8.

2.8.2 Verification with T. Laraakker's result

Next we note that in [30, Appendix A, Page 32], T. Laraakker has explicitly written down the formulas for $N(A_1^{\delta+1}, 0, 0)$. We have verified that our formulas agree with his.

2.8.3 Verification with R. Mukherjee and R. Singh's result

We now verify some of the numbers obtained by R. Mukherjee and R. Singh in [35]. In [35], the authors compute $C_d^{\text{Planar},\mathbb{P}^3}(r,s)$, the number of planar genus zero degree d curves in $\mathbb{P}^3$ intersecting r lines and passing through s points having a cusp (where $r+2s = 3d+1$). Let us compare this with $N_d(A_1^\delta A_2, r, s)$, the number of planar degree d curves in $\mathbb{P}^3$, passing through r lines and passing through s points, that have δ (ordered) nodes and one cusp (where $r+2s = \dfrac{d(d+3)}{2} + 1 - \delta$). For $d=3$, and $\delta = 0$, this number should be the same as the characteristic number of genus zero planar cubics in $\mathbb{P}^3$ with a cusp, i.e., $C_d(r,s)$. We have verified that is indeed the case. We tabulate the numbers for the readers convenience:

$$C_3(10,0) = 17760, \quad C_3(8,1) = 2064, \quad C_3(6,2) = 240 \quad \text{and} \quad C_3(4,3) = 24.$$

These numbers are the same as $N_d(A_1^\delta A_2, r, s)$ for $d=3$ and $\delta = 0$.

Next, we note that when $d=4$ and $\delta=2$, the number $\frac{1}{\delta!}N_d(A_1^\delta A_2, r, s)$ is same as the characteristic number of genus zero planar quartics in $\mathbb{P}^3$ with a cusp, i.e., $C_d(r,s)$. We have verified that fact. The numbers are

$$C_4(13,0) = 10613184, \quad C_4(11,1) = 760368, \quad C_4(9,2) = 49152, \quad C_4(7,3) = 2304.$$

These numbers are the same as $\frac{1}{2!}N_d(A_1^\delta A_2, r, s)$ for $d=4$ and $\delta=2$. We have to divide out by a factor of $\delta!$ because in the definition of $N_d(A_1^\delta A_2, r, s)$, the nodes are ordered.

Chapter 3

Gromov-Witten invariants of projective spaces

Let X be a smooth projective variety over $\mathbb{C}$. In order to solve some enumerative problem of curves in X, we first need to find a suitable space that parametrizes the curves in X. The parametrizing space may not always be compact; however, compactness is needed to solve enumerative problems. We need to compactify by adding some extra points to that space. The standard moduli space in enumerative geometry of curves is the *moduli space of stable maps*. For example, if n, g are non-negative integers and $\beta \in H_2(X)$ are fixed, then $\overline{M}_{g,n}(X, \beta)$, the moduli space of stable maps, is a compactification of the space of all tuple $(C, p_1, p_2, \cdots, p_n, f)$, where

- C is a smooth curve of genus g,
- $p_1, p_2, \cdots, p_n$ are all distinct points of C,
- $f : C \longrightarrow X$ is a morphism such that $f_*([C]) = \beta$.

The compactification is obtained by allowing C to be connected nodal curve with several irreducible components.

The construction of this moduli space is discussed in section 3.1. In Section 3.2, we define numerical invariants, with the help of the intersection theory of the moduli space of stable maps, which are known as the *Gromov-Witten invariants*. We also study some of the properties of these invariants in the subsequent sections.

3.1 Stable Curves and Stable Maps

We will start this section by defining stable curves.

Definition 3.1.1. An n-**pointed quasi-stable curve** is a tuple $(C, p_1, p_2, \cdots, p_n)$, where

(i) C is a projective, connected, reduced curve with at worst nodes as singularities;

(ii) $p_1, p_2, \cdots, p_n$ are n distinct non-singular points of C which we call as the **marked points** of C.

A morphism $\phi : (C, p_1, p_2, \cdots, p_n) \to (C', p'_1, p'_2, \cdots, p'_n)$ between n-pointed quasi-stable curves is a morphism $\phi : C \to C'$ such that $\phi(p_i) = p'_i$ for all $i = 1, 2, \cdots, n$. The genus of a quasi-stable curve $(C, p_1, p_2, \cdots, p_n)$ is the arithmetic genus $g = h^1(C, \mathcal{O}_C)$. An n-pointed quasi-stable curve is said to be **stable** if it has finite automorphism group.

A point on a quasi-stable curve is called a **special point** if it is either a marked point or a nodal point. A quasi-stable curve $(C, p_1, p_2, \cdots, p_n)$ is stable if and only if the following conditions hold for each irreducible component $E \subset C$:

(i) If E has genus 0, then it contains at least three special points,

(ii) If E has genus 1, then it contains at least one special point.

Note that for a smooth curve of genus $g = 0$, the group of automorphism has dimension 3 whereas the group has dimension one if $g = 1$ and the group is finite if $g > 1$. Hence a finite automorphism group can be obtained by fixing three and one special points for the case of genus zero and genus one irreducible components, respectively.

The set of isomorphism classes of n-pointed stable curves of genus g is denoted by $\overline{M}_{g,n}$, and the subset consisting of the isomorphism class of stable n-pointed smooth curves of genus g is denoted by $M_{g,n}$.

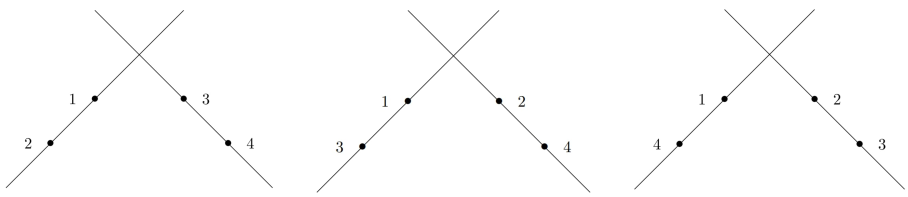
The spaces $M_{g,n}$ and $\overline{M}_{g,n}$ are not always the same. We illustrate this fact by some examples. Any smooth curve of genus zero is always isomorphic to $\mathbb{P}^1$ and the automorphism group of $\mathbb{P}^1$ is the projective linear group $PGL(2, \mathbb{C})$ which has dimension 3.

Example 3.1.2. It is immediate that for any three point $p_1, p_2, p_3 \in \mathbb{P}^1$, there always exists a unique automorphism of $\mathbb{P}^1$ which sends p_1, p_2, p_3 to $0, 1, \infty$ respectively. Hence, $M_{0,3}$ is a point. In this case, $\overline{M}_{0,3} = M_{0,3}$.

We now give an example where these two spaces are not the same.

Example 3.1.3. Consider the moduli space $M_{0,4}$. Let us first consider four distinct points p_1, p_2, p_3, p_4 (say) in $\mathbb{P}^1$. Since there is a unique automorphism of $\mathbb{P}^1$ sending p_1, p_2, p_3 to $0, 1, \infty$ respectively, image of p_4 under this automorphism is automatically determined and different from $0, 1$ and ∞. The space $M_{0,4}$ is the same as $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which is certainly not compact. Its compactification is isomorphic to $\mathbb{P}^1$, that is, $\overline{M}_{0,4} \cong \mathbb{P}^1$.

Those three extra points, which appear in the compactification of $M_{0,4}$, are represented by the following three marked curves:



$(1\,2 \mid 3\,4)$ $(1\,3 \mid 2\,4)$ $(1\,4 \mid 2\,3)$

Figure 3.1: Codimension 1 strata in $\overline{M}_{0,4}$.

We call these curves as the boundary divisors of $\overline{M}_{0,4}$.

Note that if $n \leq 2 - 2g$, the moduli space $\overline{M}_{g,n}$ is empty due to the stability condition. Hence, we will always impose the inequality $2g - 2 + n > 0$ to get at least a non-empty moduli space.

The arithmetic genus, $P_a(C)$, of a curve C is the genus of a 'smoothing' of C. This can be computed by the following formula:
If C has δ nodes and the normalization of C has n components of topological genera $g_1, \ldots, g_n$, respectively, then

$$p_a(C) = \sum_{i=1}^{n} (g_i - 1) + \delta + 1.$$

Observe that, both the topological and arithmetic genus coincide in the case of a smooth curve by Serre duality.

Construction 3.1.4. If $2g-2+n > 0$, a n-pointed stable curve can be associated to each n-pointed quasi-stable curve $(C, p_1, \ldots, p_n)$ of genus g, which is called the **stabilization** of $(C, p_1, \ldots, p_n)$ and is denoted by $s\,(C, p_1, \ldots, p_n)$. The construction of $s\,(C, p_1, \ldots, p_n)$ is as follows:

(i) If any genus 0 irreducible component of C has only one special point, that component is collapsed to a point.

(ii) If any genus 0 irreducible component of C has exactly two special points, that component is also collapsed and, those two special points have to be identified.

Any component having the above two properties is called unstable component. We say that $s\,(C, p_1, \ldots, p_n)$ is obtained from $(C, p_1, \ldots, p_n)$ by killing the unstable components.

Let S be an algebraic scheme over $\mathbb{C}$. A *family of n-pointed, genus g stable curves* over S is a flat, projective morphism $\pi : \mathcal{C} \to S$ together with n sections $p_1, \ldots p_n$ such that for each closed point s of S, the fiber $(\mathcal{C}_s, p_1(s), \ldots, p_n(s))$ is a n-pointed stable curve of genus g. An isomorphism of two such family over the same base S is an isomorphism of domains which respects the corresponding sections. For each scheme S over $\mathbb{C}$, let $\overline{\mathcal{M}}_{g,n}(S)$ be the set of isomorphism classes of family of n-pointed, genus g stable curves over S. This assignment results a moduli functor

$$\overline{\mathcal{M}}_{g,n} : \text{Schemes}/\mathbb{C} \to \text{Sets}$$

from the category of schemes over $\mathbb{C}$ to the category of Sets as follows. Deligne-Mumford [13] and Knudsen [27] proved that $\overline{\mathcal{M}}_{g,n}$ is representable by a Deligne-Mumford stack. More precisely, we have the following theorem:

Theorem 3.1.5. *Assume g, n are non-negative integers with $2g - 2 + n > 0$. The space $\overline{M}_{g,n}$ of stable curves is a proper, irreducible, smooth and separated Deligne-Mumford stack of dimension $3g - 3 + n$. Also, $M_{g,n}$ is a dense open substack of $\overline{M}_{g,n}$.*

Proof. See [13]. □

There is a natural morphism

$$\overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}.$$

The morphism can be described explicitly as follows.
Given a stable curve $(C, p_1, \ldots, p_n, p_{n+1})$, consider the quasi-stable curve $(C, p_1, \ldots, p_n)$, by forgetting the last marked point. Then consider its stabilization $s\,(C, p_1, \ldots, p_n)$, by killing the unstable components described in Construction 3.1.4, this will be the required element of $\overline{M}_{g,n}$. The space $\overline{M}_{g,n+1}$ is often called the **universal curve** over $\overline{M}_{g,n}$.

Repeating the above process, we will deduce a natural forgetful morphism

$$\overline{M}_{g,n} \longrightarrow \overline{M}_{g,I},$$

where $I \subseteq \{1, 2, \ldots, n\}$. This morphism is obtained by forgetting the marked point labeled outside I and stabilizing the resulting quasi-stable curve.

Let us now move on to the moduli space of stable maps. Here instead of taking curves, we consider morphisms from curves to a fixed target space.

Definition 3.1.6. Let X be a smooth projective variety. An n**-pointed quasi-stable map to** X of genus g is a tuple $(C, p_1, p_2, \ldots, p_n, f)$, where

(i) $(C, p_1, p_2, \ldots, p_n)$ is a n-pointed quasi-stable curve of arithmetic genus g,

(ii) $f : C \longrightarrow X$ is a morphism.

A morphism $\phi : (C, p_1, p_2, \ldots, p_n, f) \longrightarrow (C', p'_1, p'_2, \ldots, p'_n, f')$ of n-pointed quasi-stable map is a morphism $\phi : (C, p_1, p_2, \ldots, p_n) \longrightarrow (C', p'_1, p'_2, \ldots, p'_n)$ with the property that $f' \circ \phi = f$. A tuple $(C, p_1, p_2, \ldots, p_n, f)$ is called **stable** if it has finite automorphism group. The stability condition is equivalent to the following conditions:

(i) If f is constant on a genus 0 component, it contains at least three special points.

(ii) If f is constant on a genus 1 component, it contains at least one special point.

Let $H_2^+(X, \mathbb{Z})$ be the semigroup of the homology classes of algebraic curves modulo torsion. For any quasi-stable map $(C, p_1, p_2, \ldots, p_n, f)$ to X, $f_*([C]) \in H_2^+(X, \mathbb{Z})$. The set of all isomorphism classes of n-pointed stable maps to X of genus g and of class $\beta \in H_2^+(X, \mathbb{Z})$ is called the moduli space of stable maps and is denoted by $\overline{M}_{g,n}(X, \beta)$. The subset of $\overline{M}_{g,n}(X, \beta)$ consisting of all stable maps with smooth domain is denoted by $M_{g,n}(X, \beta)$.

Remark 3.1.7. It is worth to note that for a stable map $(C, p_1, p_2, \ldots, p_n, f)$, the underlying curve $(C, p_1, p_2, \ldots, p_n)$ need not be stable. In particular, if X is a point, then the n-pointed quasi-stable curve $(C, p_1, p_2, \ldots, p_n)$ is necessarily stable. Moreover, in this case, $\overline{M}_{g,n}(X, \beta)$ and $\overline{M}_{g,n}$ are the same. In this sense, $\overline{M}_{g,n}(X, \beta)$ is a generalization of $\overline{M}_{g,n}$ when the target variety is smooth and projective of positive dimension.

Note that $\overline{M}_{g,n}(X, 0)$ is just $\overline{M}_{g,n} \times X$. In particular, $\overline{M}_{0,3}(X, 0) = X$. Hence, $\overline{M}_{g,n}(X, \beta)$ is empty if both $\beta = 0$ and $2g - 2 + n \leq 0$. Hence we always assume either $\beta \neq 0$ or $2g - 2 + n > 0$ to get at least a non-empty moduli space of stable maps.

Construction 3.1.8. Assume $\beta \neq 0$ or $n > 2-2g$. A n-pointed stable map to X can be associated to each n-pointed quasi-stable map $(C, p_1, \ldots, p_n, f)$ of genus g, which is called the **stabilization** of $(C, p_1, \ldots, p_n, f)$ and is denoted by $s\,(C, p_1, \ldots, p_n, f)$. This can be obtained by applying Construction 3.1.4 (i) and (ii) to the components of the underlying quasi-stable curve $(C, p_1, p_2, \ldots, p_n)$ on which f is constant. The resulting map is a stable map obtained by killing the unstable components as before in Construction 3.1.4.

As before, for any algebraic scheme S over $\mathbb{C}$, a family of stable maps over S can be defined. More precisely, a family of n-pointed, genus g stable maps to X over S is a tuple $(\pi : \mathcal{C} \longrightarrow S, p_1, p_2, \ldots, p_n, f : \mathcal{C} \longrightarrow X)$ where

(i) $(\pi : \mathcal{C} \longrightarrow S, p_1, p_2, \ldots, p_n)$ is a family of stable curves over S,

(ii) $f : \mathcal{C} \longrightarrow X$ is a morphism.

As before, a morphism between two families of n-pointed, genus g stable maps to X over S is defined in an obvious way. For each schemes S over $\mathbb{C}$, let $\overline{\mathcal{M}}_{g,n}(X, \beta)(S)$ be the set of all isomorphism classes of families of n-pointed, genus g stable maps to X over S and of class β. This association results a moduli functor

$$\overline{\mathcal{M}}_{g,n}(X, \beta) : \text{Schemes}/\mathbb{C} \rightarrow \text{Sets}$$

from the category of schemes over $\mathbb{C}$ to the category of Sets.

Theorem 3.1.9. *Let X be a smooth projective variety and $\beta \in H_2^+(X, \mathbb{Z})$. Assume either $\beta \neq 0$ or $n > 2-2g$. Then the moduli functor $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is representable by $\overline{M}_{g,n}(X, \beta)$ which is a proper Deligne-Mumford stack.*

Proof. See [6], [17]. □

For detailed discussion and construction of moduli space of stable maps, we refer to [6], [17], [11].

Remark 3.1.10. In general, $\overline{M}_{g,n}(X,\beta)$ is not smooth, connected, irreducible, reduced or equidimensional.

3.1.1 Natural structures

The representability of the moduli functor naturally leads us to find morphisms between various moduli spaces and the space X. Some of them are listed below.

Let X be a projective variety with $\beta \in H_2^+(X,\mathbb{Z})$. Let S be an algebraic scheme.

(i) For $1 \leq i \leq n$, the natural transformation $\overline{\mathcal{M}}_{g,n}(X,\beta)(S) \longrightarrow \mathrm{Mor}(S,X)$, given by $[(\pi : \mathcal{C} \longrightarrow S, p_1, p_2, \ldots, p_n, f : \mathcal{C} \longrightarrow X)] \mapsto p_i$, yields a natural morphism

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \longrightarrow X,$$

given by

$$[(C, p_1, p_2, \ldots, p_n, f)] \mapsto f(p_i).$$

This map is called evaluation map at the i-th marked point.

(ii) Assume $n > 2-2g$. The natural transformation $\overline{\mathcal{M}}_{g,n}(X,\beta)(S) \longrightarrow \mathrm{Mor}(S,\overline{M}_{g,n})$, given by forgetting the target X and the morphism to X, yields a natural morphism

$$\overline{M}_{g,n}(X,\beta) \longrightarrow \overline{M}_{g,n},$$

given by

$$[(C, p_1, p_2, \ldots, p_n, f)] \mapsto [s(C, p_1, p_2, \ldots, p_n)].$$

This map is called the forgetful map that forgets the target.

(iii) Assume either $\beta \neq 0$ or $n > 3-2g$. There is another natural forgetful map

$$\pi_i : \overline{M}_{g,n}(X,\beta) \longrightarrow \overline{M}_{g,n-1}(X,\beta),$$

given by

$$[(C, p_1, p_2, \ldots, p_n, f)] \mapsto [s(C, p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n, f)],$$

which is obtained by forgetting the i^{th} marked point and stabilizing the resulting quasi-stable map to X. The process of stabilization is described in Construction 3.1.8. This morphism is called the forgetful map obtained by forgetting the last marked point. The morphism

$$\pi_n : \overline{M}_{g,n}(X, \beta) \longrightarrow \overline{M}_{g,n-1}(X, \beta)$$

is often called the universal curve over $\overline{M}_{g,n-1}(X, \beta)$.

Composing the forgetful map several times, we obtain a natural forgetful map

$$\overline{M}_{g,n}(X, \beta) \longrightarrow \overline{M}_{g,I}(X, \beta),$$

where I is a subset of $\{1, 2, \ldots, n\}$.

3.2 Gromov-Witten invariants

The idea of Gromov-Witten invariants is to consider the intersection theory of the space of stable maps to X, rather than X itself. To relate this moduli space with enumerative geometry, if $W_1, W_2, \ldots, W_n$ be the sub-varieties of X, then consider the product $\prod_{i=1}^{n} \text{ev}_i^*([W_i])$. If the product has dimension 0, the degree of the product is a rational number that a gives a geometric interpretation of the space of curves passing through $W_1, W_2, \ldots$ and W_n simultaneously. The curve counting question now translates to computing the intersection product on the moduli space of stable maps to X. The appropriate intersection product will not always give the answer to the enumerative problem we are looking for. This is due to the contributions, that come from the extra stuffs considered at the time of compactification, to the intersection product.

In order to shape this idea, we first need to consider the dimension of the moduli space $\overline{M}_{g,n}(X, \beta)$. As mentioned in Remark 3.1.10, the moduli space $\overline{M}_{g,n}(X, \beta)$ has several components of different dimensions. However, there is a so-called **virtual dimension** or **expected dimension** given by

$$\text{vdim}\, \overline{M}_{g,n}(X, \beta) := -K_X \cdot \beta + (\dim X - 3)(1 - g) + n, \tag{3.1}$$

from the deformation theory of stable maps. Here K_X denotes the canonical divisor of X. For any point $[(C, p_1, p_2, \ldots, p_n, f)]$ of $\overline{M}_{g,n}(X, \beta)$, the local dimension of it is at least equal to the virtual dimension. The local dimension is equal to the expected dimension if the deformation theory is unobstructed at that point, and the point is also a smooth point of the moduli space in this case. For example, the equality happens if the cohomology group $H^1(C, f^*TX)$ vanishes, where TX is the tangent bundle of X.

After having information about the virtual dimension, one needs to take into account the fundamental class of $\overline{M}_{g,n}(X, \beta)$. This usually does not exist in general. However, there is a naturally defined virtual fundamental class $[\overline{M}_{g,n}(X, \beta)]^{\text{virt}} \in A_k(\overline{M}_{g,n}(X, \beta), \mathbb{Q})$, where $k = \text{vdim}\, \overline{M}_{g,n}(X, \beta)$. If the deformation theory is unobstructed at each point of $\overline{M}_{g,n}(X, \beta)$, then the virtual fundamental class is the usual one. For example, this happens when $g = 0$ and X is a homogeneous variety. For more detailed discussion about this case, we refer to [17]. For detailed discussion of the construction of the virtual fundamental classes we refer to [4], [5], [11], [31], and for the intersection theory of Deligne-Mumford stacks, see [43].

Definition 3.2.1. Consider the moduli space $\overline{M}_{g,n}(X, \beta)$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are arbitrary classes in $A^*(X)$. The **Gromov-Witten invariant** is denoted by $I^g_\beta(\gamma_1, \gamma_2, \ldots, \gamma_n)$, and is defined by

$$I^g_\beta(\gamma_1, \gamma_2, \ldots, \gamma_n) := \int_{[\overline{M}_{g,n}(X,\beta)]^{\text{virt}}} \text{ev}_1^*(\gamma_1) \ldots \text{ev}_n^*(\gamma_n).$$

Here integration means the evaluation of the homogeneous component of the product $\mathrm{ev}_1^*(\gamma_1)\dots\mathrm{ev}_n^*(\gamma_n)$ of codimension equal to vdim $\overline{M}_{g,n}(X,\beta)$ on the virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}}$. Observe that $I^g_\beta(\gamma_1,\gamma_2,\dots,\gamma_n)=0$ unless

$$\sum_{i=1}^{n} \mathrm{codim}\,\gamma_i = \mathrm{vdim}\,\overline{M}_{g,n}(X,\beta), \tag{3.2}$$

where codim $\alpha = d$ if $\alpha \in A^d(X)$.

As discussed earlier in this section, Gromov-Witten invariants does not always have enumerative significance. However, there are a few cases when it gives the answer to the enumerative questions. In particular, it happens if the deformation theory is unobstructed at each point of moduli space.

Example 3.2.2. Let X be a homogeneous variety ($X=\mathbb{P}^r$ is a special case). Assume $V_1,\dots,V_n \in A^*(X)$. Then the Gromov-Witten invariant $I^0_\beta(V_1,\dots,V_n)$ is the number of degree β rational curves that passes through the constraints which are represented by the classes $V_1,\dots,V_n$ (see [17, Lemma 14]). As a special case, if H denotes the cohomology generator of $\mathbb{P}^2$, then $I^0_1(H^2,H^2)$ is 1 as there is only one line in $\mathbb{P}^2$ that passes through 2 general points.

Remark 3.2.3. Here the Gromov-Witten invariants are not defined in the most general setting. There are more general tautological classes in $A^*(\overline{M}_{g,n}(X,\beta),\mathbb{Q})$, and we can similarly define invariants using them. The invariants defined here are called the primary Gromov-Witten invariants. In this thesis, we will deal only with the primary one. For the general case, we refer to [11].

3.3 Axioms of Gromov-Witten invariants

We will now give some basic relations among Gromov-Witten invariants. Though we call the relations axioms, they are not axioms. We briefly sketch the proof of each of the axioms. For detailed discussion see [6], [11].

Proposition 3.3.1. *Let* $\pi : \overline{M}_{g,n+1}(X,\beta) \longrightarrow \overline{M}_{g,n}(X,\beta)$ *be the universal curve over* $\overline{M}_{g,n}(X,\beta)$*. Let* $\gamma_1, \gamma_2, \ldots, \gamma_n \in A^*(X)$ *and* $\alpha \in A^*(\overline{M}_{g,n+1}(X,\beta))$*. Then*

$$\pi_*(\mathrm{ev}_1^*(\gamma_1)\ldots\mathrm{ev}_n^*(\gamma_n)\cdot\alpha\cdot[\overline{M}_{g,n+1}(X,\beta)]^{virt}) = \mathrm{ev}_1^*(\gamma_1)\ldots\mathrm{ev}_n^*(\gamma_n)\cdot\pi_*\left(\alpha\cdot[\overline{M}_{g,n+1}(X,\beta)]^{virt}\right).$$

Proof. This is immediate. First, observe that π commutes with the evaluation maps. The rest follows from the projection formula. □

All the properties stated below will be corollary to Proposition 3.3.1. The first property states that if at least one of the marked points does not have an insertion, then the corresponding Gromov-Witten invariant vanishes.

Property 1 (The **fundamental class axiom).** *Assume either* $\beta \neq 0$ *or* $n > 2-2g$*. Let* $\gamma_1, \gamma_2, \ldots, \gamma_n \in A^*(X)$*. Then*

$$I_\beta^g(\gamma_1, \gamma_2, \ldots, \gamma_n, 1) = 0.$$

Proof. Replace α by 1 in Proposition 3.3.1. Note that $\pi_*\left([\overline{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}\right) = 0$ for dimensional reason. The result follows. □

Property 2 (The **divisor axiom).** *Assume either* $\beta \neq 0$ *or* $n > 2-2g$*. Let* $\gamma_1, \gamma_2, \ldots, \gamma_n \in A^*(X)$*. If* $\gamma \in A^1(X)$*, then*

$$I_\beta^g(\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma) = (\beta\cdot\gamma) I_\beta^g(\gamma_1, \gamma_2, \ldots, \gamma_n).$$

Proof. In this case, we replace α by $\mathrm{ev}_{n+1}^*(\gamma)$ in Proposition 3.3.1. By axiom IV of definition 7.1 in [6], which is proved in [4], it follows that

$$\pi^*[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}} = [\overline{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}.$$

Again for any $[(C, p_1, p_2, \ldots, p_n, f)] \in \overline{M}_{g,n}(X, \beta)$,

$$\pi_* \left(\mathrm{ev}^*_{n+1}(\gamma)\pi^* \left([(C, p_1, p_2, \ldots, p_n, f)]\right)\right) = (\beta \cdot \gamma)[(C, p_1, p_2, \ldots, p_n, f)]$$

as there are total of $\beta \cdot \gamma$ choices for the point p_{n+1}. □

One can produce stable maps by gluing finite number of stable maps in the following way:

Let $g_1, g_2 \geq 0$ be such that $g = g_1 + g_2$ and I_1, I_2 be two disjoint subset of $\{1, 2 \ldots, n\}$ such that $I_1 \sqcup I_2 = \{1, 2 \ldots, n\}$, where $\sqcup$ means disjoint union. Also assume $\beta, \beta_1, \beta_2 \in H_2^+(X, \mathbb{Z})$ such that $\beta = \beta_1 + \beta_2$. This gives rise to an injective map

$$\overline{M}_{g_1, I_1 \cup \{\dagger\}}(X, \beta_1) \times_X \overline{M}_{g_2, I_2 \cup \{\ddagger\}}(X, \beta_2) \longrightarrow \overline{M}_{g,n}(X, \beta),$$

where the fiber product is given by two evaluation maps to X, one evaluation is given by $\dagger^{\text{th}}$ marked point of the first moduli space whereas the second one is given by $\ddagger^{\text{th}}$ marked point of the second moduli space. The image in $\overline{M}_{g,n}(X, \beta)$ is denoted by $D(g_1, \beta_1, I_1 \mid g_1, \beta_1, I_1)$. This substack carries a virtual fundamental class which is induced by the virtual fundamental classes of $\overline{M}_{g_1, I_1 \cup \{\dagger\}}(X, \beta_1)$ and $\overline{M}_{g_2, I_2 \cup \{\ddagger\}}(X, \beta_2)$, respectively. The virtual dimension of the substack is

$$\begin{aligned} &\mathrm{vdim}\, \overline{M}_{g_1, I_1 \cup \{\dagger\}}(X, \beta_1) + \mathrm{vdim}\, \overline{M}_{g_2, I_2 \cup \{\ddagger\}}(X, \beta_2) - \dim X \\ &= \mathrm{vdim}\, \overline{M}_{g,n}(X, \beta) - 1 \end{aligned}$$

Hence, $D(g_1, \beta_1, I_1 \mid g_2, \beta_2, I_2)$ is of codimension 1 in $\overline{M}_{g,n}(X, \beta)$. They are called as **boundary divisors** usually.

Remark 3.3.2. In the similar way, one can define boundary divisors of the moduli space of stable curves. For example, all the extra curves in Example 3.1.3 are boundary divisors.

We are now in a position to describe the splitting axiom. Let $\{T_i\}_{i=0}^p$ be a homogeneous basis of $A^*(X)$ with $T_0 = 1$ and (g_{ij}) be the intersection matrix given by $g_{ij} = \int_X T_i \cdot T_j$, $0 \leq i, j \leq p$. If (g^{ij}) is the inverse of (g_{ij}), then the class of diagonal Δ_X inside $X \times X$ is given by

$$[\Delta_X] = \sum g^{ij} T_i \otimes T_j \tag{3.3}$$

Property 3 (Splitting axiom). *If* $\gamma_1, \gamma_2, \ldots, \gamma_n \in A^*(X)$*, then*

$$\int_{[D(g_1,\beta_1,I_1|g_2,\beta_2,I_2)]^{virt}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) = \sum g^{ij} I_{\beta_1}^{g_1}(\{\gamma_k\}_{k\in I_1}, \mathrm{ev}_\dagger^*(T_i)) \cdot I_{\beta_2}^{g_2}(\{\gamma_k\}_{k\in I_2}, \mathrm{ev}_\ddagger^*(T_j)).$$

Proof. Clearly,

$$[D(g_1, \beta_1, I_1 \mid g_2, \beta_2, I_2)]^{\mathrm{virt}} = [\overline{M}_{g_1, I_1\cup\{\dagger\}}(X, \beta_1)]^{\mathrm{virt}} [\overline{M}_{g_2, I_2\cup\{\ddagger\}}(X, \beta_2)]^{\mathrm{virt}} [\Delta_X].$$

The rest follows. □

Property 4 (The **reduction axiom**). *If* $\gamma_1, \gamma_2, \ldots, \gamma_n \in A^*(X)$*, then*

$$I_\beta^g(\gamma_1, \gamma_2, \ldots, \gamma_n) = \sum g^{ij} I_\beta^{g-1}(\gamma_1, \gamma_2, \ldots, \gamma_n, T_i, T_j).$$

Proof. Consider the natural morphism $\pi : \overline{M}_{g-1,n+2}(X, \beta) \longrightarrow \overline{M}_{g,n}(X, \beta)$, given by

$$[(C, p_1, p_2, \ldots, p_{n+2}, f)] \longmapsto [(C, p_1, p_2, \ldots, p_n, f)],$$

together with the condition

$$f(p_{n+1}) = f(p_{n+2}).$$

This condition will increase the arithmetic genus of the curve by 1.

Note that $\pi_* \left([\Delta_X][\overline{M}_{g-1,n+2}(X, \beta)]^{\mathrm{virt}}\right) = [\overline{M}_{g,n}(X, \beta)]^{\mathrm{virt}}$. The proof now follows from the expression of $[\Delta_X]$ in (3.3), and the fact that π commutes with up to n^{th} evaluation maps. □

Chapter 4

Rational and elliptic Gromov-Witten invariants of del Pezzo surfaces

In this chapter, we focus on some explicit calculations of Gromov-Witten invariants of some special surfaces which are known as *del Pezzo* surface. Our main aim is to compute elliptic $(g = 1)$ Gromov-Witten invariants of a del Pezzo surface. To compute it, we need to know the rational $(g = 0)$ invariants first. To make this chapter self-contained, we first compute the rational Gromov-Witten invariants, which were computed individually by Kontsevich-Manin [28] and Göttsche-Pandharipande [19]. After that, we move towards computing elliptic Gromov-Witten invariants of the same space.

4.1 Del Pezzo surfaces

Definition 4.1.1. A smooth projective algebraic surface is said to be a **del Pezzo** surface if the anti-canonical divisor $-K_X$ is ample.

The degree of a del Pezzo surface X is defined to be the self-intersection number

$$d_X := -K_X \cdot -K_X = K_X \cdot K_X.$$

The degree d_X can vary between 1 and 9. The following theorem is about the classification of del Pezzo surfaces.

Theorem 4.1.2 (Classification of del Pezzo surfaces)**.** *Let X be a del Pezzo surface. Then* *exactly one of the following holds:*

(a) X *is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$ *with* $d_X = 8$.

(b) X *is isomorphic to* $\mathbb{P}^2$ *blown up at* k *general points with* $0 \leq k \leq 8$. *In this case,* $d_X = 9 - k$.

Proof. See [33]. □

Remark 4.1.3. In the above theorem, by general points we mean:

(i) None of the three points lie on a line,

(ii) None of the six points lie on a conic,

(iii) No eight of them lie on a singular conic, and none of the points is the singular point.

We proceed with the above classification of del Pezzo surfaces. If X has degree $9 - k$ and is not $\mathbb{P}^1 \times \mathbb{P}^1$, then we have the blowup morphism $Bl : X \to \mathbb{P}^2$. In this case, the exceptional divisors of X are denoted by $E_1, \ldots, E_k$, and the pull-back of the class of a hyperplane in $\mathbb{P}^2$ via Bl is denoted by L. We have

$$H^2(X, \mathbb{Z}) = \mathbb{Z}\langle L, E_1, \ldots, E_k \rangle,$$

and the relations are given by $L^2 = 1, \;\; E_i^2 = -1, \;\; L \cdot E_i = E_i \cdot E_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. The anti-canonical divisor is given by $-K_X = 3L - E_1 - \ldots - E_k$. The cohomology ring, $H^*(X, \mathbb{Z})$ is also generated by $L, E_1, \ldots, E_k$ with relations mentioned above together with $L^3 = 0$, $E_i^3 = 0$ for all $1 \leq i \leq k$.

If $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $e_1 = \mathrm{pr}_1^*[\mathrm{pt}]$ and $e_2 = \mathrm{pr}_2^*[\mathrm{pt}]$, where pr_i is the natural projection map onto i^{th} component, and [pt] is the hyperplane class of a point in $\mathbb{P}^1$. Then

$$H^*(X, \mathbb{Z}) \cong \frac{\mathbb{Z}[e_1, e_2]}{\langle e_1^2, e_2^2 \rangle},$$

and $-K_X = 2e_1 + 2e_2$.

Note that the Chow ring and the cohomology ring of a del Pezzo surface are isomorphic. Since we assume cohomology ring is graded by complex degree, the cohomology ring and the Chow ring of a del Pezzo surface is used interchangeably throughout this chapter. Moreover, '$\cdot$' is used for both the cup product in cohomology as well as cap product between a homology and a cohomology class. This make sense because of the presence of poincaré duality between homology and cohomology groups.

4.2 Rational Gromov-Witten invariants

Let X be a del Pezzo surface and $\beta \in H_2^+(X, \mathbb{Z})$. Throughout this section, we deal with the moduli space $\overline{M}_{0,n}(X, \beta)$ for non-negative integers n and compute Gromov-Witten invariants $I_\beta^0(\gamma_1 \dots, \gamma_n)$ for any choice of classes $\gamma_1, \dots, \gamma_n \in A^*(X)$. Note that if $\gamma_i = 1$ for some i, by fundamental axiom, $I_\beta^0(\gamma_1 \dots, \gamma_n) = 0$ and if $\gamma_i \in A^1(X)$, by divisor axiom, it is enough to compute the corresponding $n-1$ pointed Gromov-Witten invariant obtained by removing i-th marked point. Therefore, it reduces to consider the case when all the γ_i's are given by the class of points.

If all the γ_i's are given by the class of points, $I_\beta^0(\gamma_1 \dots, \gamma_n)$ is zero unless $n = \kappa_\beta - 1$, where $\kappa_\beta := -K_X \cdot \beta$. This is because of the dimensional reason given in eq. (3.2). It reduces to the case when $n = \kappa_\beta - 1$ and all the γ_i's are given by the class of points. Our aim is to compute $I_\beta^0(\{[\mathrm{pt}_i]\}_{i=1}^{\kappa_\beta - 1})$, and we denote it by $N_\beta^{(0)}$ for convenience.

Theorem 4.2.1. *If $\kappa_\beta \geq 4$, the value of $N_\beta^{(0)}$ is determined by the following recursion relation:*

$$N_\beta^{(0)} = \sum_{\beta_1 + \beta_2 = \beta} \left[\binom{\kappa_\beta - 4}{\kappa_{\beta_1} - 2} \kappa_{\beta_2} - \binom{\kappa_\beta - 4}{\kappa_{\beta_1} - 1} \kappa_{\beta_1} \right] \kappa_{\beta_1} (\beta_1 \cdot \beta_2) N_{\beta_1}^{(0)} N_{\beta_2}^{(0)}.$$

The base case of the recursion is the following:

If X is a blow up at k general points, then $N_\beta^{(0)} = 1$ for $\beta = -E_1, -E_2, \dots, -E_k$ and L.

In this case, if $\beta = dH - \sum_{i=1}^{k} a_i E_i$ *with* $0 \leq \kappa_\beta \leq 3$*, then for* $1 \leq i \leq k$*, the number* $N_\beta^{(0)}$ *is given by the following recursion:*

$$d^2 a_i N_\beta^{(0)} = \left(d^2 - (a_i - 1)^2\right) N_{\beta+E_i}^{(0)} + \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 1}{\kappa_{\beta_1} - 1} (\beta_1 \cdot \beta_2)(\beta_1 \cdot H)(\beta_1 \cdot E_i)\Big[(\beta_2 \cdot H)(\beta_2 \cdot E_i) - (\beta_1 \cdot H)(\beta_1 \cdot E_i)\Big].$$

If X *is* $\mathbb{P}^1 \times \mathbb{P}^1$*, then* $N_\beta^{(0)} = 1$ *for* $\beta = e_1$ *and* e_2*.*

Before proceeding to the proof of Theorem 4.2.1, we need to analyse few general facts which will be useful for the subsequent sections.

We first need to know the class of diagonal Δ_X in $X \times X$. Let $\{T_0, \ldots, T_p\}$ be a homogeneous basis of $H^*(X, \mathbb{Z})$. If X is a blow up at k general points, then $p = k + 2$ and $T_0 = 1$, $T_1 = L$, $T_{i+1} = -E_i$ for $1 \leq i \leq k$ and $T_{k+2} = [\text{pt}]$. If $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $p = 3$ with $T_0 = 1$, $T_1 = e_1$, $T_2 = e_2$, $T_3 = [\text{pt}]$. Also the intersection matrix, introduced in Section 3.3, is given by the relations satisfied by the generators of $H^*(X, \mathbb{Z})$. Hence the expression of $[\Delta_X]$ can be derived immediately from eq. (3.3). Also for $\gamma_1, \gamma_2 \in H^*(X, \mathbb{Z})$,

$$\sum_{i,j=0}^{p} g^{ij} (\gamma_1 \cdot T_i)(\gamma_2 \cdot T_j) = \gamma_1 \cdot \gamma_2. \tag{4.1}$$

In Chapter 3, we have described several relations satisfied by various Gromov-Witten invariants, but mostly under the assumption either $\beta \neq 0$ or $n > 2-2g$. We need to analyse the case when $\beta = 0$. Note that $\overline{M}_{0,n}(X, 0) \cong \overline{M}_{0,n} \times X$. In particular, $\overline{M}_{0,n}(X, 0) \cong X$ for $n = 3$.

Lemma 4.2.2. *For any classes $\gamma_1, \ldots, \gamma_n \in A^*(X)$,*

$$I_0^0(\gamma_1, \ldots, \gamma_n) = \begin{cases} \int_X \gamma_1 \cdot \gamma_2 \cdot \gamma_3 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [18, pp. 982]. □

Assume $n \geq 4$. There is a natural forgetful morphism

$$\pi : \overline{M}_{0,n}(X, \beta) \longrightarrow \overline{M}_{0,4}$$

which is obtained by composing the forgetful morphisms introduced in Section 3.1.1. More precisely, π forget the target and all but first four marked points simultaneously. Again, we discussed in Example 3.1.3 that $\overline{M}_{0,4} \cong \mathbb{P}^1$ and both $(1\,2 \mid 3\,4)$ and $(1\,3 \mid 2\,4)$ are boundary divisors of $\overline{M}_{0,4}$. Also any two points of $\mathbb{P}^1$ are rationally equivalent.

Theorem 4.2.3 (Gluing Theorem)**.** *Both the boundary divisors $(1\,2 \mid 3\,4)$ and $(1\,3 \mid 2\,4)$ are rationally equivalent, that is,*

$$[(1\,2 \mid 3\,4)] = [(1\,3 \mid 2\,4)]$$

in $A^1(\overline{M}_{0,4})$.

Proof. See [28]. □

By pulling back of the above relation via π, we have

$$\pi^*\left([(1\,2 \mid 3\,4)]\right) = \pi^*\left([(1\,3 \mid 2\,4)]\right) \tag{4.2}$$

in $A^1(\overline{M}_{0,n}(X, \beta))$.

Note that

$$\pi^*\left([(1\,2 \mid 3\,4)]\right) = \sum D(0, \beta_1, A \cup \{1,2\} \mid 0, \beta_2, B \cup \{3,4\}), \tag{4.3}$$

where the sum is over all possible choice of $\beta_1, \beta_2 \in H_2^+(X,\mathbb{Z})$ and $A, B \subseteq \{5, 6 \ldots, n\}$ such that $\beta_1 + \beta_2 = \beta$ and $A \sqcup B = \{5, 6 \ldots, n\}$. The divisor $D(g_1, \beta_1, A \mid g_2, \beta_2, B)$ is defined in Chapter 3.

Similarly,

$$\pi^*\left([(1\,3 \mid 2\,4)]\right) = \sum D(0, \beta_1, A \cup \{1,3\} \mid 0, \beta_2, B \cup \{2,4\}), \tag{4.4}$$

where the sum is over all possible choice described above.

Proof of Theorem 4.2.1 : Consider the above discussion for $n = \kappa_\beta$. If μ is the class of a point and $\xi_X = -K_X$, then define

$$\mathcal{Z} := \mathrm{ev}_1^*(\xi_X) \cdot \mathrm{ev}_1^*(\xi_X) \cdot \mathrm{ev}_3^*(\mu) \cdot \ldots \cdot \mathrm{ev}_{\kappa_\beta}^*(\mu).$$

The class ξ_X is used since it is ample, and hence numerically effective. Multiplying both side of eq. (4.2) by $\mathcal{Z}$, we deduce

$$\pi^*\left([(1\,2 \mid 3\,4)]\right) \cdot \mathcal{Z} = \pi^*\left([(1\,3 \mid 2\,4)]\right) \cdot \mathcal{Z} \tag{4.5}$$

Observe that the total codimension of each of the individual cycles in the above equality is equal to vdim $\overline{M}_{0,\kappa_\beta}(X,\beta)$. Hence their degrees are the same. The expressions for $\pi^*\left([(1\,2 \mid 3\,4)]\right)$ and $\pi^*\left([(1\,3 \mid 2\,4)]\right)$, given in eq. (4.3) and eq. (4.4), respectively, will help us to find the degree of each of the individual piece. Let $\mu_1 = \mu_2 = \xi_X$, and $\mu_3 = \ldots = \mu_{\kappa_\beta} = [pt]$ be the class of a point.

Let us first consider the left hand side. By splitting axiom, we have

$$\int_{\overline{M}_{0,\kappa_\beta}(X,\beta)} \pi^*\left([(1\,2 \mid 3\,4)]\right)\cdot \mathcal{Z} = \sum_{\substack{\beta_1+\beta_2=\beta \\ A,B \\ i,j}} g^{ij} I^0_{\beta_1}(\{\mu_\alpha\}_{\alpha\in A}, T_i) I^0_{\beta_2}(\{\mu_\alpha\}_{\alpha\in B}, T_j),$$

where sum is over disjoint sets A, B satisfying

$$A\sqcup B=\{1,\ldots,\kappa_\beta\},\quad A\cap\{1,2,3,4\}=\{1,2\} \text{ and } B\cap\{1,2,3,4\}=\{3,4\}.$$

Note that if $\beta_1, \beta_2 > 0$, by divisor axiom the only non-trivial term occurs precisely when $|A| = \kappa_{\beta_1}+1$ and $|B| = \kappa_{\beta_2}-1$. The limiting case $\beta_2 = 0$ does not yield anything, however, $\beta_1 = 0$ will have non-trivial contribution to the sum and in this case, the nontrivial contribution occurs when $A = \{1,2\}$, $B = \{3,4,\ldots,\kappa_\beta\}$, $T_i = T_0$, $T_j = T_p$. From eq. (4.1), we conclude that

$$\begin{aligned}&\int_{\overline{M}_{0,\kappa_\beta}(X,\beta)} \pi^*\left([(1\,2 \mid 3\,4)]\right)\cdot \mathcal{Z} \\ &= N^{(0)}_\beta + \sum_{\substack{\beta_1+\beta_2=\beta \\ A\sqcup B=\{5,\ldots\kappa_\beta\}}} (\beta_1\cdot\beta_2)\kappa^2_{\beta_1} I^0_{\beta_1}(\{\mu_i\}_{i\in A}) I^0_{\beta_2}(\mu_3,\mu_4,\{\mu_i\}_{i\in B}).\end{aligned}$$

The non-trivial contribution occurs precisely when $|A| = \kappa_{\beta_1}-1$ and $|B| = \kappa_{\beta_1}-3$. Hence,

$$\int_{\overline{M}_{0,\kappa_\beta}(X,\beta)} \pi^*\left([(1\,2 \mid 3\,4)]\right)\cdot \mathcal{Z} = N^{(0)}_\beta + \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta-4}{\kappa_{\beta_1}-1} (\beta_1\cdot\beta_2)\kappa^2_{\beta_1} N^{(0)}_{\beta_1} N^{(0)}_{\beta_2}.$$

We now move on to the right hand side of (4.5). In the similar fashion, we use splitting axiom, and deduce

$$\int_{\overline{M}_{0,\kappa_\beta}(X,\beta)} \pi^*\left([(1\,3 \mid 2\,4)]\right)\cdot \mathcal{Z} = \sum_{\substack{\beta_1+\beta_2=\beta \\ A,B \\ i,j}} g^{ij} I^0_{\beta_1}(\{\mu_\alpha\}_{\alpha\in A}, T_i) I^0_{\beta_2}(\{\mu_\alpha\}_{\alpha\in B}, T_j),$$

where sum is over disjoint sets A, B satisfying

$$A \sqcup B = \{1, \dots, \kappa_\beta\}, \quad A \cap \{1,2,3,4\} = \{1,3\} \text{ and } B \cap \{1,2,3,4\} = \{2,4\}.$$

It follows from lemma 4.2.2 that (β_1, β_2) can neither be $(\beta, 0)$ nor be $(0, \beta)$. As before, for the rest of the choices of (β_1, β_2) none of i and j be 0 because of the fundamental axiom. Only choices of T_i and T_j, which will survive in the sum, are divisors, and hence we can apply the divisor axiom now. From eq. (4.1), we conclude that

$$\begin{aligned} &\int_{\overline{M}_{0,\kappa_\beta}(X,\beta)} \pi^*\left([(1\,3 \mid 2\,4)]\right) \cdot \mathcal{Z} \\ &= \sum_{\substack{\beta_1+\beta_2=\beta \\ A\sqcup B=\{5,\dots\kappa_\beta\}}} (\beta_1 \cdot \beta_2)\kappa_{\beta_1}\kappa_{\beta_2} I^0_{\beta_1}(\mu_3, \{\mu_i\}_{i\in A}) I^0_{\beta_2}(\mu_4, \{\mu_i\}_{i\in B}). \end{aligned}$$

The non-trivial contribution occurs precisely when $|A| = \kappa_{\beta_1} - 2$ and $|B| = \kappa_{\beta_1} - 2$. Hence,

$$\int_{\overline{M}_{0,\kappa_\beta}(X,\beta)} \pi^*\left([(1\,3 \mid 2\,4)]\right) \cdot \mathcal{Z} = \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 4}{\kappa_{\beta_1} - 2} (\beta_1 \cdot \beta_2)\kappa_{\beta_1}\kappa_{\beta_2} N^{(0)}_{\beta_1} N^{(0)}_{\beta_2}.$$

Now equating both degrees we obtain the recursion. We are not proving the base recursion here. For a proof, we refer to [19]. □

4.3 Elliptic Gromov-Witten invariants

We now study enumerative geometry of elliptic curves, i.e., curves with genus $g = 1$. Throughout this section, the moduli space we will be concerned with is $\overline{M}_{1,n}(X, \beta)$ for non-negative integers n. For any choice $\gamma_1, \dots, \gamma_n$ from $A^*(X)$, we compute the Gromov-Witten invariant $I^1_\beta(\gamma_1, \dots, \gamma_n)$. As discussed in the beginning of the previous section, it is enough to consider the case when $n = \kappa_\beta$, and all the γ_i's are given by the class of points. Therefore, our aim is to compute $I^1_\beta(\{[\mathrm{pt}_i]\}_{i=1}^{\kappa_\beta})$, which we denote by $N^{(1)}_\beta$ for notational simplicity.

The main idea of computing ration Gromov-Witten invariants in the previous section is the equality of divisors in $\overline{M}_{0,4}$ (gluing theorem). That equality of cycles pulls back to an equality of cycles in $\overline{M}_{0,\kappa_\beta}(X,\beta)$, and we intersect with correct cycles to get equality of numbers. In this case, an analogous statement to the gluing theorem is due to Getzler ([18]), but in codimension 2 instead of divisor relation. We pull back the above relation to some appropriate moduli space, and intersect it with correct cycle in order to get the equality of numbers. This will produce a recursive formula to compute $N_\beta^{(1)}$.

First, let us consider the space $\overline{M}_{1,4}$, the moduli space of genus one curves with four marked points. We are interested in certain S_4 invariant codimension 2 boundary strata in $\overline{M}_{1,4}$ which we list in Figure 4.1. For a complete list of generators of the S_4 invariant subspace of $H^4(\overline{M}_{1,4},\mathbb{Q})$, we refer to [18, Section 1]. In the figure, we draw the topological type and the marked point distribution of the generic curve in each strata. We use the same nomenclature as in [18], except for $\delta_{0,0}$ which was denoted by δ_β there (to avoid confusion between notations). These strata are denoted by the dual graph of the generic curve in [18].

These strata define cycles in $H^4(\overline{M}_{1,4},\mathbb{Q})$. Now define the following cycle in $H^4(\overline{M}_{1,4},\mathbb{Q})$, given by

$$\mathcal{R} := -2\delta_{2,2} + \frac{2}{3}\delta_{2,3} + \frac{1}{3}\delta_{2,4} - \delta_{3,4} - \frac{1}{6}\delta_{0,3} - \frac{1}{6}\delta_{0,4} + \frac{1}{3}\delta_{0,0}. \tag{4.6}$$

The main result of [18] is the following:

Theorem 4.3.1. *The cycle $\mathcal{R} = 0$ in $H^4(\overline{M}_{1,4},\mathbb{Q})$.*

Proof. See [18, Theorem 1.8]. □

This will subsequently be referred to as Getzler's relation. In [36], Pandharipande has shown that this relation, in fact, comes from a rational equivalence, i.e., $\mathcal{R} = 0$ in $A^2(\overline{M}_{1,4},\mathbb{Q})$.

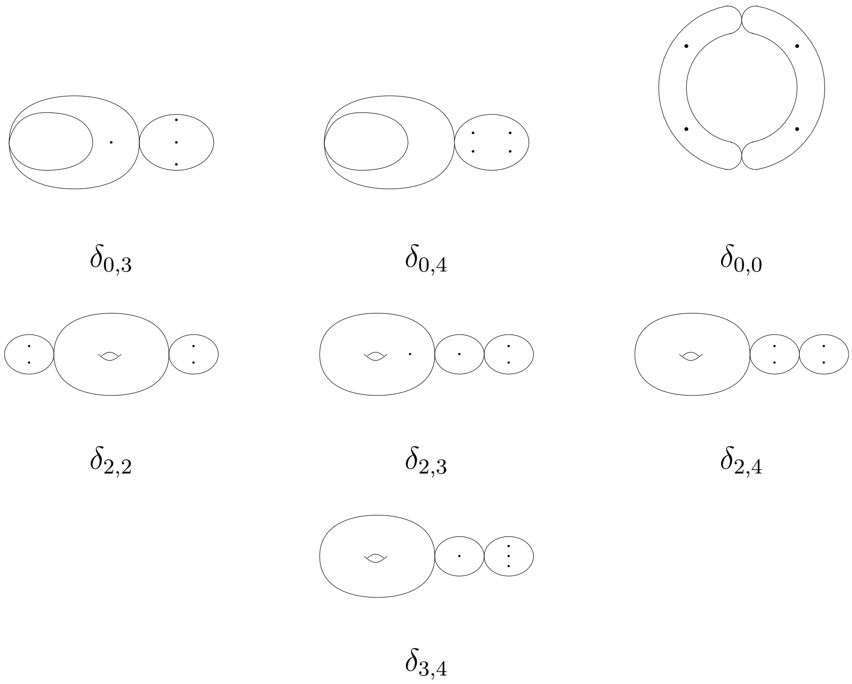


Figure 4.1: Codimension 2 strata in $\overline{M}_{1,4}$.

The main idea of the relation described above is based on two facts. First, the dimension of $H^4(\overline{M}_{1,4}, \mathbb{Q})^{S_4}$ is 7 and second, $H^4(\overline{M}_{1,4}, \mathbb{Q})^{S_4}$ is generated by total of 9 strata. The null space of 7×9 intersection matrix of the generators as given in [18, Theorem 1.2], has two relations among which one is given in [18, Lemma 1.1], and the other one is $\mathcal{R}$.

We will continue with all the notations introduced earlier in this chapter. We also need to know the elliptic Gromov-Witten invariants when $\beta = 0$.

Lemma 4.3.2. *For any classes $\gamma_1, \ldots, \gamma_n \in A^*(X)$,*

$$I_0^1(\gamma_1, \ldots, \gamma_n) = \begin{cases} \int_X c_1(X) \cdot \gamma_1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Here $c_1(X)$ denotes the first Chern class of the tangent bundle of X.

Proof. See [18, pp. 983]. □

Theorem 4.3.3. *Let us define the following four quantities:*

$$T_1 := \sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2}-1, \kappa_{\beta_3}-1} 2\kappa_{\beta_2}\kappa_{\beta_3}^2(\beta_1\cdot\beta_2) \Big(\big(4\kappa_{\beta_1}+\kappa_{\beta_2}-2\kappa_{\beta_3}\big)(\beta_2\cdot\beta_3) - 3\kappa_{\beta_2}(\beta_1\cdot\beta_3)\Big) N_{\beta_1}^{(1)} N_{\beta_2}^{(0)} N_{\beta_3}^{(0)},$$

$$T_2 := \sum_{\beta_1+\beta_2=\beta} \Bigg[\binom{\kappa_\beta - 2}{\kappa_{\beta_1}-1} 4\kappa_{\beta_2}^2 \Big(2\kappa_{\beta_1}\kappa_{\beta_2} - \kappa_{\beta_2}^2 - 3d_X(\beta_1\cdot\beta_2)\Big) + \binom{\kappa_\beta - 2}{\kappa_{\beta_1}} 2\kappa_{\beta_2}\Big(d_X(\beta_1\cdot\beta_2)\big(4\kappa_{\beta_1}+\kappa_{\beta_2}\big) + 2\kappa_{\beta_1}\kappa_{\beta_2}\big(2\kappa_{\beta_1}-\kappa_{\beta_2}\big)\Big)\Bigg] N_{\beta_1}^{(1)} N_{\beta_2}^{(0)},$$

$$T_3 := -\frac{1}{12}\sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1}-1} \kappa_{\beta_2}^2(\beta_1\cdot\beta_2)\Big[\kappa_{\beta_1}^2\big(\kappa_{\beta_1}-2\kappa_{\beta_2}-6(\beta_1\cdot\beta_2)\big) + \kappa_{\beta_2}(\beta_1\cdot\beta_1)\big(4\kappa_{\beta_1}+\kappa_{\beta_2}\big)\Big] N_{\beta_1}^{(0)} N_{\beta_2}^{(0)},$$

$$T_4 := -\frac{1}{12}\kappa_\beta^3\Big(\big(2+b_2(X)\big)\kappa_\beta - d_X\Big) N_\beta^{(0)}.$$

If $\kappa_\beta \geq 2$, the number $N_\beta^{(1)}$ satisfies the following recursive relation:

$$6d_X^2 N_\beta^{(1)} = T_1 + T_2 + T_3 + T_4. \tag{4.7}$$

The base case of the recursion is the following:

If X is a blow up at k general points, then

$$N_L^{(1)} = 0 \quad \text{and} \quad N_{E_i}^{(1)} = 0 \quad \forall i = 1 \text{ to } k.$$

If $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, then

$$N_{e_1}^{(1)} = 0 \quad \text{and} \quad N_{e_2}^{(1)} = 0.$$

Proof. If $n \geq 4$, we have natural forgetful morphism

$$\pi : \overline{M}_{1,n}(X,\beta) \longrightarrow \overline{M}_{1,4}$$

obtained by forgetting all but first four marked points and the target simultaneously. Let $n = \kappa_\beta + 2$, and $\mu \in H^4(X,\mathbb{Q})$ be the class of a point. Also assume $\xi_X = -K_X$. Define

$$\mathcal{Z} := \mathrm{ev}_1^*(\xi_X) \cdot \ldots \cdot \mathrm{ev}_4^*(\xi_X) \cdot \mathrm{ev}_5^*(\mu) \cdot \ldots \cdot \mathrm{ev}_{\kappa_\beta+2}^*(\mu).$$

As before we can choose ξ_X as it is ample and numerically effective. Since $\mathcal{R} = 0$ by Getzler's relation, we conclude that

$$\int_{[\overline{M}_{1,\kappa_\beta+2}(X,\beta)]^{\mathrm{virt}}} \pi^*\mathcal{R} \cdot \mathcal{Z} = 0. \tag{4.8}$$

For a cycle δ in $H^*(\overline{M}_{g,n}(X,\beta),\mathbb{Q})$, we introduce the following notation

$$N^{\delta}_{\beta,X}(\mu_1,\ldots,\mu_n) = \int_{[\overline{M}_{1,n}(X,\beta)]^{\mathrm{virt}}} \delta \cdot \mathrm{ev}_1^*(\mu_1)\cdots\mathrm{ev}_n^*(\mu_n).$$

Let $\mu_1 = \ldots = \mu_4 = \xi_X$, and $\mu_5 = \ldots = \mu_{\kappa_\beta+2} = [pt]$ be the class of points. If $\delta = \pi^*\delta_{2,2}$, by the splitting axiom

$$\begin{aligned} N^{\pi^*\delta_{2,2}}_{\beta,X} =& N^{\delta}_{\beta,X}(\mu_1,\ldots,\mu_{\kappa_\beta+2}) \\ =& \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C \\ i,j,k,l}} g^{ij}g^{kl} \quad I^1_{\beta_1}(T_i,T_k,\{\mu_\alpha|\alpha\in A\}) \\ & \times I^0_{\beta_2}(T_j,\{\mu_\alpha|\alpha\in B\}) \times I^0_{\beta_3}(T_l,\{\mu_\alpha|\alpha\in C\}), \end{aligned}$$

where sum is over disjoint sets A, B, C satisfying

$$A \sqcup B \sqcup C = \{1,\ldots,\kappa_\beta+2\}, \quad |B\cap\{1,2,3,4\}| = |C\cap\{1,2,3,4\}| = 2.$$

Note that if $\beta_1, \beta_2, \beta_3 > 0$, by the divisor axiom, the only non-trivial terms occur when $|A| = \kappa_{\beta_1}, |B| = \kappa_{\beta_2} + 1, |C| = \kappa_{\beta_3} + 1$. The limiting case $\beta_1 = 0$ does not yield anything, however $\beta_2 = 0$ or $\beta_3 = 0$ have non-trivial contributions to the sum. When $\beta_3 = 0, \beta_1, \beta_2 > 0$, the non-trivial contribution occurs precisely when $|C| = 2, T_l = [X]$, $|A| = \kappa_{\beta_1} - 1, T_k = [pt]$, and $|B| = \kappa_{\beta_2} + 1$. Finally, when $\beta_2 = \beta_3 = 0$, the only non-zero term occurs when $|B| = |C| = 2, T_l = T_j = [X]$ and $T_k = T_i = [pt]$. we obtain the following expression

$$
\begin{aligned}
N_{\beta,X}^{\pi^*\delta_{2,2}} = 3(\xi_X \cdot \xi_X)^2 N_\beta^{(1)} + 3 \sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2} - 1, \kappa_{\beta_3} - 1} (\beta_2 \cdot \xi_X)^2 (\beta_3 \cdot \xi_X)^2 \cdot \\
(\beta_1 \cdot \beta_2)(\beta_1 \cdot \beta_3) N_{\beta_1}^{(1)} N_{\beta_2}^{(0)} N_{\beta_3}^{(0)} \\
+ 6 \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1} (\xi_X \cdot \xi_X)(\beta_1 \cdot \beta_2)(\beta_2 \cdot \xi_X)^2 N_{\beta_1}^{(1)} N_{\beta_2}^{(0)}.
\end{aligned}
\tag{4.9}
$$

Next, let us consider the cycle $\delta_{2,3}$. We now have

$$
N_{\beta,X}^{\pi^*\delta_{2,3}} = \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C \\ i,j,k,l}} g^{ij} g^{kl} I_{\beta_1}^1 (T_i, \{\mu_\alpha | \alpha \in A\}) \times I_{\beta_2}^0 (T_j, T_k, \{\mu_\alpha | \alpha \in B\}) \times I_{\beta_3}^0 (T_l, \{\mu_\alpha | \alpha \in C\}),
$$

where the sum is over sets A, B, C satisfying

$$
A \sqcup B \sqcup C = \{1, \ldots, \kappa_\beta + 2\}, \quad |A \cap \{1,2,3,4\}| = |B \cap \{1,2,3,4\}| = 1.
$$

All the cases are similar to the previous calculation except, when $\beta_2 = 0$. In this case we can either have $|B| = 1, |A| = \kappa_{\beta_1}$, $T_i = [pt]$ and $T_j = [X]$; or $|B| = 1, |C| = \kappa_{\beta_3}$, $T_k = [X]$ and $T_l = [pt]$. We deduce

$$
N_{\beta,X}^{\pi^*\delta_{2,3}} = 12 \sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2} - 1, \kappa_{\beta_3} - 1} (\beta_1 \cdot \xi_X)(\beta_2 \cdot \xi_X)(\beta_3 \cdot \xi_X)^2 \cdot
$$

$$(\beta_1 \cdot \beta_2)(\beta_2 \cdot \beta_3) N^{(1)}_{\beta_1} N^{(0)}_{\beta_2} N^{(0)}_{\beta_3}$$

$$+ 12 \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1}} (\beta_1 \cdot \xi_X)(\beta_2 \cdot \xi_X) \Big((\xi_X \cdot \xi_X)(\beta_1 \cdot \beta_2) + (\beta_1 \cdot \xi_X)(\beta_2 \cdot \xi_X) \Big) N^{(1)}_{\beta_1} N^{(0)}_{\beta_2}$$

$$+ 12 \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1} (\beta_1 \cdot \xi_X)(\beta_2 \cdot \xi_X)^3 N^{(1)}_{\beta_1} N^{(0)}_{\beta_2}. \tag{4.10}$$

Moving on to $\delta_{2,4}$ we have

$$N^{\pi^*\delta_{2,4}}_{\beta,X} = \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C \\ i,j,k,l}} g^{ij} g^{kl} I^1_{\beta_1}(T_i, \{\mu_\alpha | \alpha \in A\}) \times I^0_{\beta_2}(T_j, T_k, \{\mu_\alpha | \alpha \in B\}) \times I^0_{\beta_3}(T_l, \{\mu_\alpha | \alpha \in C\}),$$

where the sum is over sets A, B, C satisfying

$$A \sqcup B \sqcup C = \{1, \ldots, \kappa_\beta + 2\}, \quad |B \cap \{1,2,3,4\}| = |C \cap \{1,2,3,4\}| = 2.$$

There is no contribution when $\beta_2 = 0$, however, we have a non-trivial contribution when $\beta_1 = 0$. We obtain

$$N^{\pi^*\delta_{2,4}}_{\beta,X} = 6 \sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2} - 1, \kappa_{\beta_3} - 1} (\beta_2 \cdot \xi_X)^2 (\beta_3 \cdot \xi_X)^2 \cdot (\beta_1 \cdot \beta_2)(\beta_2 \cdot \beta_3) N^{(1)}_{\beta_1} N^{(0)}_{\beta_2} N^{(0)}_{\beta_3}$$

$$+ 6 \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1}} (\beta_2 \cdot \xi_X)^2 (\xi_X \cdot \xi_X)(\beta_1 \cdot \beta_2) N^{(1)}_{\beta_1} N^{(0)}_{\beta_2}$$

$$+ 6 \sum_{\beta_1+\beta_2=\beta} \left(-\frac{1}{24}\right) \binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1} (\xi_X \cdot \beta_1)^3 (\beta_2 \cdot \xi_X)^2 (\beta_1 \cdot \beta_2) N^{(0)}_{\beta_1} N^{(0)}_{\beta_2}$$

$$+ 6 \left(-\frac{1}{24}\right) (\xi_X \cdot \beta)^3 (\xi_X \cdot \xi_X) N^{(0)}_{\beta}. \tag{4.11}$$

For $\delta_{3,4}$, we have

$$N_{\beta,X}^{\pi^*\delta_{3,4}} = \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C \\ i,j,k,l}} g^{ij}g^{kl}I^1_{\beta_1}(T_i,\{\mu_\alpha|\alpha\in A\}) \times I^0_{\beta_2}(T_j,\{\gamma_k,\mu_\alpha|\alpha\in B\}) \times I^0_{\beta_3}(T_l,\{\mu_\alpha|\alpha\in C\}),$$

where the sum is over sets A, B, C satisfying

$$A\sqcup B\sqcup C = \{1,\ldots,\kappa_\beta+2\},\quad |B\cap\{1,2,3,4\}|=1,\quad |C\cap\{1,2,3,4\}|=3.$$

The calculation is similar to the previous cases, so we omit the details. We obtain

$$\begin{aligned} N_{\beta,X}^{\pi^*\delta_{3,4}} = 4 &\sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta-2}{\kappa_{\beta_2}-1,\kappa_{\beta_3}-1}(\beta_2\cdot\xi_X)(\beta_3\cdot\xi_X)^3(\beta_1\cdot\beta_2)\cdot \\ &\qquad (\beta_2\cdot\beta_3)N^{(1)}_{\beta_1}N^{(0)}_{\beta_2}N^{(0)}_{\beta_3} \\ &+4\sum_{\beta_1+\beta_2=\beta}\binom{\kappa_\beta-2}{\kappa_{\beta_1}}(\beta_2\cdot\xi_X)^3(\beta_1\cdot\xi_X)N^{(1)}_{\beta_1}N^{(0)}_{\beta_2} \\ &+4\sum_{\beta_1+\beta_2=\beta}\binom{\kappa_\beta-2}{\kappa_{\beta_1}-1}(\beta_2\cdot\xi_X)^4N^{(1)}_{\beta_1}N^{(0)}_{\beta_2} \\ &+4\sum_{\beta_1+\beta_2=\beta}\binom{\kappa_\beta-2}{\kappa_{\beta_1}-1}\left(-\frac{1}{24}\right)(\xi_X\cdot\beta_1)^2(\beta_2\cdot\xi_X)^3(\beta_1\cdot\beta_2)N^{(0)}_{\beta_1}N^{(0)}_{\beta_2} \\ &+4\left(-\frac{1}{24}\right)(\xi_X\cdot\xi_X)(\beta\cdot\xi_X)^3N^{(0)}_\beta. \end{aligned} \tag{4.12}$$

The remaining cycles all have 2 genus zero components; so the calculations are simpler. We will first consider $\delta_{0,3}$:

$$N_{\beta,X}^{\pi^*\delta_{0,3}} = \frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta \\ A,B \\ i,j,k,l}} g^{ij}g^{kl}I^0_{\beta_1}(T_i,T_j,T_k,\{\mu_\alpha|\alpha\in A\}) \times I^0_{\beta_2}(T_l,\{\mu_\alpha|\alpha\in B\}),$$

where the sum is over sets A, B satisfying

$$A \sqcup B = \{1, \ldots, \kappa_\beta + 2\}, \quad |A \cap \{1, 2, 3, 4\}| = 1.$$

The factor of $\frac{1}{2}$ appears since the dual graph of a generic curve in $\delta_{0,3}$ has an automorphism of order 2. Neither $\beta_1 = 0$, nor $\beta_2 = 0$ has any non-trivial contribution; so it is straight forward to see that

$$N_{\beta,X}^{\pi^*\delta_{0,3}} = \sum_{\beta_1+\beta_2=\beta} 2\binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1}(\beta_1 \cdot \xi_X)(\beta_2 \cdot \xi_X)^3(\beta_1 \cdot \beta_2)(\beta_1 \cdot \beta_1)N_{\beta_1}^{(0)}N_{\beta_2}^{(0)}. \quad (4.13)$$

The calculation for $\delta_{0,4}$ is a bit more subtle.

$$N_{\beta,X}^{\pi^*\delta_{0,4}} = \frac{1}{2} \sum_{\substack{\beta_1+\beta_2=\beta \\ A,B \\ i,j,k,l}} g^{ij} g^{kl} I_{\beta_1}^0(T_i, T_j, T_k, \{\mu_\alpha | \alpha \in A\}) \times I_{\beta_2}^0(T_l, \{\mu_\alpha | \alpha \in B\}),$$

where the sum is over sets A, B satisfying

$$A \sqcup B = \{1, \ldots, \kappa_\beta + 2\}, \quad A \cap \{1, 2, 3, 4\} = \emptyset.$$

Contribution from $\beta_2 = 0$ is 0. When $\beta_1 = 0$, we must have $A = \emptyset$ which leads to

$$\begin{aligned} N_{\beta,X}^{\pi^*\delta_{0,4}} =& \frac{1}{2} \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1}(\beta_2 \cdot \xi_X)^4(\beta_1 \cdot \beta_2)(\beta_1 \cdot \beta_1)N_{\beta_1}^{(0)}N_{\beta_2}^{(0)} \\ &+ \frac{1}{2}(2 + b_2(X))(\beta \cdot \xi_X)^4 N_\beta^{(0)}. \end{aligned} \quad (4.14)$$

Finally, let us consider the cycle $\delta_{0,0}$.

$$N_{\beta,X}^{\pi^*\delta_{0,0}} = \frac{1}{2} \sum_{\substack{\beta_1+\beta_2=\beta \\ A,B \\ i,j,k,l}} g^{ij} g^{kl} I_{\beta_1}^0(T_i, T_k, \{\mu_\alpha | \alpha \in A\})$$

$$\times I^0_{\beta_2}(T_j, T_l, \{\mu_\alpha | \alpha \in B\}),$$

where the sum is over sets A, B satisfying

$$A \sqcup B = \{1, \ldots, \kappa_\beta + 2\}, \quad |A \cap \{1,2,3,4\}| = 2.$$

A similar calculation yields

$$N^{\pi^*\delta_{0,0}}_{\beta,X} = \frac{3}{2} \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1} (\beta_1 \cdot \xi_X)^2 (\beta_2 \cdot \xi_X)^2 (\beta_1 \cdot \beta_2)^2 N^{(0)}_{\beta_1} N^{(0)}_{\beta_2}. \tag{4.15}$$

Plugging all the values as given in $((4.9) - (4.15))$ to eq. (4.8), we obtain the desired formula given in eq. (4.7). □

Remark 4.3.4. In particular, if we consider X is $\mathbb{P}^2$, then the formula for computing the rational Gromov-Witten invariants is the same as the formula obtained by Kontsevich-Manin [28, eq. (5.17)], and in the case of elliptic invariants, the recursion obtained by us matches with the recursion that was obtained by Getzler [18, eq. (0.1)].

4.4 Enumerative applications of Gromov-Witten invariants

It is mentioned in Chapter 3 that Gromov-Witten invariants do not necessarily have enumerative significance always. They can even be negative rational numbers; for example, elliptic Gromov-Witten invariants of $\mathbb{P}^3$ are negative rational. It is proved in [17, Lemma 14] that if X is a projective variety (more generally a homogeneous variety), the rational Gromov-Witten invariants are enumerative. If X is a del Pezzo surface, then Ravi Vakil [42, Section 4] proved that all genus g Gromov-Witten invariants of X are enumerative. In particular, $N^{(0)}_\beta$ is the number of rational curves of degree β in X that pass through

$\kappa_\beta - 1$ general points in X. Also, $N_\beta^{(1)}$ is the number of elliptic curves of degree β in X that pass through κ_β general points in X.

Chapter 5

Summary and conclusions

In this thesis, we consider two different enumerative geometric problems which are generalizations of classical enumerative geometry of curves in $\mathbb{CP}^2$. Being generalizations of classical problems, they are important on their own.

In Chapter 2, enumerative geometry of singular curves of $\mathbb{CP}^3$ in a moving family of $\mathbb{CP}^2$ is considered. A curve in $\mathbb{CP}^3$ is said to be *planar* if it lies inside some $\mathbb{CP}^2$ in $\mathbb{CP}^3$. The space of planar curves of a fixed degree (say, d) in $\mathbb{CP}^3$ is a fiber bundle over the space of all hyperplane in $\mathbb{CP}^3$, that is, the Grassmannian, $\mathbb{G}(3,4)$ of 3 planes in $\mathbb{C}^4$. Thus enumerative geometry of planar curves is a fiber bundle analog of enumerative geometry of plane curves. Let us define

$$N_d^{\text{planar},\mathbb{P}^3}(A_1^\delta \mathfrak{X}; r, s)$$

to be the number of planar degree d curves in $\mathbb{P}^3$, intersecting r lines and passing through s points, and having δ distinct nodes and one singularity of type $\mathfrak{X}$, where $r + 2s = \frac{d(d+3)}{2} + 3 - (\delta + c_{\mathfrak{X}})$, and $c_{\mathfrak{X}}$ is the codimension of the singularity $\mathfrak{X}$. Kleiman and Piene (2004) obtained a formula for $N_d(A_1^\delta; r, s)$, when $\delta \leq 8$, and very recently, Ties Larakker (2018) has obtained a formula for $N_d^{\text{Planar},\mathbb{P}^3}(A_1^\delta; r, s)$ for all δ. In this thesis, we extend the above results to the singularities of higher codimension. An explicit formula for $N_d(A_1^\delta \mathfrak{X}, r, s)$ is obtained, when $\delta + c_{\mathfrak{X}} \leq 4$, provided $d \geq d_{\text{min}}$, where $d_{\text{min}} := c_{\mathfrak{X}} + 2\delta$.

Next, we consider enumerative geometry of elliptic curves in del Pezzo surfaces using Gromov-Witten theory. Enumerative geometry has made an impressive progress in the past three decades using this theory. Certain numerical invariants can be defined using the intersection theory of moduli space of stable maps; these are known as the so-called Gromov-Witten invariants. In Chapter 3, Gromov-Witten invariants are defined, and some of the properties is also studied. Kontsevich-Manin [28], and later Göttchse-Pandharipande [19] computed rational ($g = 0$) Gromov-Witten invariants of del Pezzo surfaces, and found their enumerative significance. In 1997, Getzler [18], computed elliptic ($g = 1$) Gromov-Witten invariants of $\mathbb{CP}^2$, and showed that such invariants are enumerative as well. Shortly after, vakil [42] proved that genus $g\,(\geq 0)$ Gromov-Witten invariants are enumerative, that is, to find the answer of genus g curve counting questions in del Pezzo surfaces, it is enough to find the corresponding Gromov-Witten invariant. In Chapter 4 of the thesis, we extend the result of Getzler to del Pezzo surface, and find a recursive formula to compute the elliptic Gromov-Witten invariants of a given del Pezzo.

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Appendix A

List of formulas

we write down some explicit formulas for counting $N_d^{\text{Planar},\mathbb{P}^3}(A_1^{\delta}\mathfrak{X}; r, s)$, the number of planar degree d curves in $\mathbb{P}^3$, intersecting r lines and passing through s points, and having δ distinct nodes and one singularity of type $\mathfrak{X}$, where $r+2s = \frac{d(d+3)}{2}+3-(\delta+c_{\mathfrak{X}})$, and $c_{\mathfrak{X}}$ is the codimension of the singularity $\mathfrak{X}$. We will denote this number by $N(A_1^{\delta}\mathfrak{X}; r, s)$ for the sake of simplicity of the notations. In Chapter 2, we compute $N(A_1^{\delta}\mathfrak{X}; r, s)$ recursively when $0 \leq \delta + c_{\mathfrak{X}} \leq 4$. Here we write down the formulas in terms of degree for the convenience of the reader. Note that $N(r, s)$ denote the number of degree d planar curves in $\mathbb{P}^3$, intersecting r lines and passing through s points, where $r + 2s = \frac{d(d+3)}{2} + 3$.

$$N(r,s) = \begin{cases} \frac{1}{324}d(d^2-1)(d+2)\,(d^2+4d+6)\,(2d^3+6d^2+13d+3) & \text{if } s=0, \\ \frac{1}{36}d(d^2-1)(d+2)\,(2d^2+8d+3) & \text{if } s=1, \\ \frac{1}{3}d(d-1)(d+4) & \text{if } s=2, \\ 1 & \text{if } s=3. \end{cases}$$

$$N(A_1, r, s) = \begin{cases} \frac{1}{108}d(d^2-1)^2(d+2)(d+3)\big(2d^4+4d^3+d^2-10d-6\big) & \text{if } s=0, \\ \frac{1}{12}d(d-1)^2(d+3)\,(2d^4+6d^3-9d^2-3d-2) & \text{if } s=1, \\ d(d-1)^2\,(d^2+3d-6) & \text{if } s=2, \\ 3(d-1)^2 & \text{if } s=3. \end{cases}$$

$$N(A_2,r,s) = \begin{cases} \frac{1}{27}d(d^2-1)(d^2-4)\big(2d^6+12d^5+11d^4 \\ \qquad -30d^3-49d^2-18\big) & \text{if } s=0, \\ \frac{1}{3}d(d-1)(d-2)\left(2d^5+12d^4+d^3-54d^2+9d+6\right) & \text{if } s=1, \\ 4d(d-1)(d-2)\left(d^2+3d-8\right) & \text{if } s=2, \\ 12(d-1)(d-2) & \text{if } s=3. \end{cases}$$

$$N(A_3,r,s) = \begin{cases} \frac{1}{162}d(d-1)(d-2)\big(50d^8+408d^7+539d^6-2556d^5 \\ \qquad -6625d^4+762d^3+10050d^2-11232d+8208\big) & \text{if } s=0, \\ \frac{1}{18}(d-2)(d-1)\big(50d^6+258d^5-485d^4-2241d^3 \\ \qquad +2172d^2+1512d-648\big) & \text{if } s=1, \\ \frac{2}{3}d(d-2)(d+5)\left(25d^2-96d+84\right) & \text{if } s=2, \\ 2\left(25d^2-96d+84\right) & \text{if } s=3. \end{cases}$$

$$N(A_4,r,s) = \begin{cases} \frac{5}{27}(d-1)(d-3)\big(6d^9+50d^8+41d^7-445d^6-715d^5 \\ \qquad +1529d^4+2720d^3-7902d^2+7164d-2160\big) & \text{if } s=0, \\ \frac{5}{3}(d-3)\big(6d^7+26d^6-105d^5-231d^4 \\ \qquad +765d^3-107d^2-762d+360\big) & \text{if } s=1, \\ 20d(d-3)(3d-5)\left(d^2+3d-12\right) & \text{if } s=2, \\ 60(d-3)(3d-5) & \text{if } s=3. \end{cases}$$

$$N(D_4,r,s) = \begin{cases} \frac{5}{36}(d-1)(d-2)^2(d+4)\big(2d^7+12d^6-d^5-66d^4-91d^3 \\ \qquad +234d^2-270d+108\big) & \text{if } s=0, \\ \frac{5}{4}(d-2)^2\left(2d^6+12d^5-15d^4-102d^3+85d^2+90d-48\right) & \text{if } s=1, \\ 15d(d-2)^2\left(d^2+3d-12\right) & \text{if } s=2, \\ 45(d-2)^2 & \text{if } s=3. \end{cases}$$

$$N(A_1^2,r,s)=\begin{cases}\frac{1}{108}d(d^2-1)(d^2-4)\big(6d^8+30d^7-25d^6-255d^5-142d^4\\ \qquad +333d^3+629d^2+18d+198\big) & \text{if } s=0,\\ \frac{1}{12}d(d-1)(d-2)\big(6d^7+30d^6-55d^5-297d^4+190d^3\\ \qquad +537d^2-69d-78\big) & \text{if } s=1,\\ d(d-1)(d-2)\left(d^2+3d-8\right)\left(3d^2-3d-11\right) & \text{if } s=2,\\ 3(d-1)(d-2)\left(3d^2-3d-11\right) & \text{if } s=3.\end{cases}$$

$$N(A_1A_2,r,s)=\begin{cases}\frac{1}{27}d(d-1)(d-2)(d-3)\big(6d^9+60d^8+155d^7-186d^6\\ \qquad -1288d^5-1422d^4+641d^3+1512d^2-2034d+1836\big) & \text{if } s=0,\\ \frac{1}{3}(d^2-1)(d-2)(d-3)\big(6d^6+36d^5-37d^4-338d^3\\ \qquad +123d^2+438d-144\big) & \text{if } s=1,\\ 4d(d-2)(d-3)(d+5)\left(3d^3-6d^2-11d+18\right) & \text{if } s=2,\\ 12(d-3)\left(3d^3-6d^2-11d+18\right) & \text{if } s=3.\end{cases}$$

$$N(A_1A_3,r,s)=\begin{cases}\frac{1}{54}(d-1)(d-3)\Big(50d^{11}+358d^{10}-489d^9-6967d^8\\ -3139d^7+40955d^6+40482d^5-112250d^4-131080d^3\\ \qquad +436176d^2-402480d+120960\Big) & \text{if } s=0,\\ \frac{1}{6}(d-3)\big(50d^9+158d^8-1471d^7-2389d^6+14857d^5\\ \qquad +2359d^4-41156d^3+7912d^2+41808d-19440\big) & \text{if } s=1,\\ 2d(d-3)\left(d^2+3d-12\right)\left(25d^3-71d^2-122d+280\right) & \text{if } s=2,\\ 6(d-3)\left(25d^3-71d^2-122d+280\right) & \text{if } s=3.\end{cases}$$

$$N(A_1^3,r,s) = \begin{cases} \frac{1}{108}d(d-1)(d-2)\Big(18d^{12}+108d^{11}-315d^{10}-2664d^9 \\ \quad +470d^8+21919d^7+19103d^6-58136d^5-106948d^4 \\ \quad +7039d^3+129360d^2-165798d+110700\Big) & \text{if } s=0, \\ \frac{1}{12}(d-1)(d-2)\Big(18d^{10}+54d^9-567d^8-1179d^7 \\ \quad +6383d^6+7774d^5-25775d^4-20197d^3 \\ \quad +26955d^2+20802d-8640\Big) & \text{if } s=1, \\ d(d-2)(d+5)\Big(9d^6-54d^5+9d^4+423d^3 \\ \quad -458d^2-829d+1050\Big) & \text{if } s=2, \\ 3\left(9d^6-54d^5+9d^4+423d^3-458d^2-829d+1050\right) & \text{if } s=3. \end{cases}$$

$$N(A_1^2A_2,r,s) = \begin{cases} \frac{1}{9}(d-1)(d-3)\Big(6d^{13}+36d^{12}-159d^{11}-1124d^{10} \\ \quad +1209d^9+12169d^8+664d^7-52991d^6-39896d^5 \\ \quad +127254d^4+129112d^3-452904d^2 \\ \quad +413280d-120960\Big) & \text{if } s=0, \\ (d-3)\Big(6d^{11}+12d^{10}-249d^9-236d^8+3653d^7 \\ \quad +367d^6-20186d^5+6389d^4+38600d^3 \\ \quad -7828d^2-42896d+19680\Big) & \text{if } s=1, \\ 12d(d-3)\left(d^2+3d-12\right)\big(3d^5-12d^4-30d^3 \\ \quad +125d^2+82d-280\big) & \text{if } s=2, \\ 36(d-3)\left(3d^5-12d^4-30d^3+125d^2+82d-280\right) & \text{if } s=3. \end{cases}$$

$$N(A_1^4, r, s) = \begin{cases} \frac{1}{36}(d-1)(d-3)\Big(18d^{15} + 90d^{14} - 747d^{13} - 3843d^{12} \\ \quad +11660d^{11} + 63140d^{10} - 75352d^9 - 486678d^8 + 73143d^7 \\ \quad +1773729d^6 + 1150606d^5 - 4123550d^4 - 3282032d^3 \\ \quad +12893256d^2 - 11795040d + 3404160\Big) & \text{if } s = 0, \\ \frac{1}{4}(d-3)\Big(18d^{13} + 18d^{12} - 945d^{11} - 261d^{10} + 18590d^9 \\ -4254d^8 - 164328d^7 + 80206d^6 + 653953d^5 - 362481d^4 \\ \quad -1051128d^3 + 245636d^2 + 1215312d - 554880\Big) & \text{if } s = 1, \\ 3d(d-3)\left(d^2 + 3d - 12\right)\Big(9d^7 - 45d^6 - 135d^5 + 801d^4 \\ \quad +691d^3 - 4671d^2 - 1386d + 7880\Big) & \text{if } s = 2, \\ 9(d-3)\big(9d^7 - 45d^6 - 135d^5 + 801d^4 + 691d^3 \\ \quad -4671d^2 - 1386d + 7880\big) & \text{if } s = 3. \end{cases}$$