ENUMERATION OF SINGULAR CURVES WITH PRESCRIBED TANGENCIES

By

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DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

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ABSTRACT

This thesis aims to study the enumerative geometry of curves via a purely topological method. My research can be divided into two parts, looking at the nature of the enumerative problems that we have studied. The first topic of my doctoral thesis is mainly concerned with the stable map and Gromov-Witten theory and the last three chapters are devoted to the study of singular curves inside a linear system.

We have revisited some preliminaries in Gromov-Witten theory and quantum cohomology in the first chapter of my thesis. In the second chapter of this thesis, we have described a possible generalization of the famous enumerative problem of counting plane rational degree d curves into $\mathbb{P}^3$. This generalization is motivated due to the work of Kleiman and Piene [30] and very recently by the work of T. Laarakker [41]. We considered counting problem for rational *planar* degree d curves in $\mathbb{P}^3$ i.e., a rational degree d curves in $\mathbb{P}^3$ whose image lies inside some $\mathbb{P}^2$. In this setting, we have proved a recursive formula analogous to the famous Kontsevitch's recursion formula for $\mathbb{P}^2$ which counts rational *planar* degree d curves in $\mathbb{P}^3$, intersecting r generic lines and passing through s points in general position inside $\mathbb{P}^3$ such that $r+2s=3d+2$. This study can be thought of as a family version of the classical counting problem in $\mathbb{P}^2$.

Both the third chapter and fourth chapter of this thesis are devoted to studying singular curves with various tangency constraint to certain smooth divisor. In particular, we have studied the geometry of singular curves (curves having A_k singularities) in $\mathbb{P}^2$ having higher order of contact (tangency) to a fixed-line $E \in \mathbb{P}^2$. The singularities that we have considered in this thesis are often degenerate than nodes; mostly we deal with A_k singularities. This study is mainly motivated by the classical study due to Caporaso-Harris [11]. In the third chapter, we have studied the enumeration of degree d curves in $\mathbb{P}^2$ having any number of A_k singularities for any k, that is tangent to a fixed-line in $\mathbb{P}^2$ with appropriate insertion condition. Although the fourth chapter is a sequel to the previous chapter, we have introduced the concept of higher-order contact (tangency) to a fixed-line E inside $\mathbb{P}^2$. Here we have studied the enumeration of degree d curves having nodes or a cusp as an underlying singularity that is tangent to a fixed-line inside $\mathbb{P}^2$ through correct number of generic points in $\mathbb{P}^2$. We also showed that this has an immediate connection to the last chapter of this thesis.

In the last chapter of this thesis, we have studied the counting problem of degree d curves in $\mathbb{P}^2$ having any two degenerate singularities of type A_k such that the total codimension of the two singularities can be at most 6. This problem is very classical in algebraic geometry and everything is known up to the total codimension 7, due to the work of many algebraic geometers. We use a topological method to study this question. We are expecting that we can extend this result up to total codimension 9 and we want to include the singularity of type D_k, E_k as well. Also, our method can very easily be generalized to other algebraic surfaces as well. One of the instances of it can be found in [3].

Contents

Summary

Enumeration of curves is classically an important subject in enumerative algebraic geometry. The last 30 years have been very crucial for the mathematical development of this subject immediately after the appearance of the technique called Gromov-Witten theory.

In my doctoral thesis, we have studied mainly two kinds of enumerative problems. The first enumerative problem deals with the study of genus-zero Gromov-Witten theory of degree *d planar* curves inside $\mathbb{P}^3$. The motivation for this work originated from the work due to Kleiman and Piene ([33]) and T.Laarakker ([41]) in a linear system setting. A *planar* curve is defined to be a curve in $\mathbb{P}^3$, whose image lies inside a $\mathbb{P}^2$. In this scenario we defined our moduli space $\overline{\mathscr{M}}_{0,k}^{\text{Planar}}(\mathbb{P}^3, d)$ as a fibre bundle over $\widehat{\mathbb{P}}^3$ (i.e., the space of all planes inside $\mathbb{P}^3$). Therefore the dimension of this moduli space is $3d+2$. Then we proved the genus zero Gromov-Witten theory in this case, i.e., we established an explicit formula to enumerate rational *planar* curves of degree d intersecting r generic lines and passing through s points in $\mathbb{P}^3$ in general position such that $r+2s = 3d+2$. In analogy to the historical development, this can be viewed as a family version of the famous question of enumerating rational curves in $\mathbb{P}^2$, that was studied by Kontsevich-Manin ([38]) and Ruan-Tian ([59]).

The second topic of my study involves the singular curves. Moreover, we impose additional tangency constraints on some divisor. In this direction, we have devoted our attention to the counting problems: (A) counting degree d curves with a certain number of A_k singularities on it and it is tangent to a fixed-line in $\mathbb{P}^2$ satisfying certain point insertions, (B) enumeration of degree d curves in $\mathbb{P}^2$ having two singularities of certain kind satisfying the appropriate enumerative constraint.

Historically, both kinds of enumerative problems are classically important. An extensive amount of work has been done from several algebraic geometers in this direction. The method we use to study these types of questions is purely topological.

Along the direction of the first type of problems, we have proved an explicit formula that enumerates the number of degree d curves in $\mathbb{P}^2$ having any number of A_k singularities on it and is tangent to a fixed-line in $\mathbb{P}^2$ passing through an appropriate number of generic points. This study was motivated by the work due to Caporaso-Harris [11]. This work can be regarded as the first step towards the generalization of Caporaso-Harris results to higher singularities.

Next, we extend our topological method and found the recursive formulas for $N(\mathsf{T}_1 \circ \cdot \circ \mathsf{T}_n)$, i.e., counts of curves with multiple tangency points. Next, we made the curve singular, furthermore, we imposed any order tangency at a point different from the point of singularities. We established a recursive formula to compute the number $N(A_1^{\delta} \circ \mathsf{T}_k)$ for $\delta \in \{1,2\}$ and $N(A_2 \circ \mathsf{T}_k)$. Note that when singularities are nodes, our numbers agree with the numbers from Caporaso-Harris. Similarly, when the singularity is cusp with tangency of order one, we can verify some of our numbers with the numbers computed by Ernstroem, Kennedy [15].

Next, we come to the second type of problem; in this direction, we have studied the enumeration of degree d curves in $\mathbb{P}^2$ having two singularities of type A_k up to codimension 6. Thus we have proved some recursive formula which enumerates the numbers $N(\mathscr{P}A_{k_1} \circ \mathscr{P}A_{k_2})$ such that $k_1 + k_2 \leq 6$. This work is motivated by the work due to Kazaryan [27]. Hence we can verify our numbers with the numbers that Kazaryan has obtained.

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Chapter 1

Introduction

1.1 Preliminaries

Enumerative geometry is a special branch of Algebraic Geometry. One of the broad focuses of this subject deals with various counts of geometric objects; could be counting curves, hypersurfaces, or certain types of sub-varieties satisfying certain kinds of constraints. The more modern language of this subject deals with certain types of sheaf counts.

In modern times, various groups of mathematicians from different branches of mathematics and physics have contributed to enumerative geometry with numerous different approaches. For example, in 1990, a major development had been done with ideas from string theory by various peoples such as Kontsevich-Manin, Witten, Vafa, and others. Gromov-Witten's theory and the related structures led the path of modern enumerative geometry. As a result, beautiful solutions to various classical, long-lasting open enumerative problems, are completely understood. For example, the problem of counting genus g curves in $\mathbb{P}^2$ passing through $3d-1+g$ general points in $\mathbb{P}^2$ are well understood.

Over the last 30 years, different branches of mathematics such as topology, symplectic or differential geometry, combinatorics have enlarged this fascinating subject. Our approach in this thesis entirely comes from ideas of topology.

Usually, numerous enumerative problems can be stated very easily, some of them can be stated without any prior background in enumerative algebraic geometry. On the contrary, the solutions to these problems require deep and technical knowledge from various branches in mathematics such as algebraic geometry, symplectic geometry, and differential topology.

Kontsevich's moduli space and the intersection theory play a prominent role in modern enumerative geometry, especially for various theoretical development of genus g maps into some homogeneous algebraic varieties. Plenty of solutions for enumerative problems was interpreted as intersection numbers inside the above moduli space. In this thesis, we have considered a similar moduli space and we have expressed our counts of geometric objects as the intersection numbers on this moduli space. It is closely related to the classical enumerative problem in $\mathbb{P}^2$. Another type of problem, that we have studied is inside a linear system. One of the problems in this setting concerns the study of singular curves with certain type contact (tangency) conditions to some smooth divisors with some appropriate insertions. Lastly, we have studied the enumeration of singular curves in $\mathbb{P}^2$ up to certain codimension, where the singularities are usually more degenerate than nodes. We can ask the simplest enumerative question as follows:

Question 1.1.1. *How many plane curves of degree d are there passing through* $\delta_d = \frac{d(d+3)}{2}$ *generic points in* $\mathbb{P}^2$*?*

Note that any degree d plane curve (in the complex projective plane) can be thought of as degree d homogeneous equation

$$\sum_{i,j,k} c_{ijk} x^i y^j z^k = 0$$

with $c_{i,j,k} \in \mathbb{C}$, not all $c_{i,j,k}$ are zero. Since any two equation as above if they differs by a multiplication of a non-zero scalar then they determines the same curve. Thus the space of all degree d curves in $\mathbb{P}^2$ is a δ_d dimensional projective space. Now for a generically chosen point $p \in \mathbb{P}^2$ the statement "degree d curves passing through the point p" corresponds to a hyperplane in $\mathbb{P}^{\delta_d}$. Hence for a generic choice of δ_d points there is a unique degree d curve satisfying the above constraint in the question provided the intersection of all the hyperplanes are transversal.

This type of question becomes very difficult when we ask the curve to have a certain type of singularities in it. For example, suppose we want to count the plane cubics having a "cusp" passing through 7 generic points in $\mathbb{P}^2$. Although, it has been known from a very ancient time that there are 24 cuspidal plane cubics through 7 generic points but proving this requires non-trivial technique. We will discuss a method in the last chapter to enumerate plane curves with various singularities. Numerous techniques usually tackle an enumerative geometric problem as follows :

Suppose, we want to study the intersection theory of certain kinds of geometric elements. For example, one may be interested in enumerating the number of elements of a set A, consisting of a certain fixed type of geometric objects satisfying a finite number of constraints μ_i. In general, one first constructs a space parametrizing all the families of such geometric objects, one might call this as parameter space or a moduli space, denoted by $\mathscr{M}$ depending on their choice and interest of the geometric objects, whose every point, in a precise sense corresponds to the elements of A.

For example, mostly we are intended to study the intersection theory of singular curves on the linear system.

Quite often, one might have constructed a bigger moduli space by including the “limiting objects” to make the moduli space compact so that it becomes a suitable space to study the intersection theory of certain kinds of geometric objects. Next, the constraints μ_i must be realized as the subspaces (hyperplanes, closed sub-schemes, sub stacks). Let $c_i \subset \mathscr{M}$ and their associated cycles $[c_i] \in H^*(\mathscr{M})$ in an appropriate cohomology theory which is known. Then, the knowledge of cohomology theory (intersection theory), i.e., $H^*(M)$ helps us in the computation of the intersection product of the cycles $[c_i]$. Thus, at least, integrating the final class yields a number, most often it is not the enumerative number that we want, we need to do more work to get the correct answer that we were looking for.

Sometimes the moduli space (compactified) is so suitable for certain kinds of geometric objects that it does not need any constraint to give an enumerative answer, i.e., the virtual dimension of the moduli space is *zero*. This was explored in the Gromov-Witten theory of Calabi-Yau-threefolds.

1.2 Enumerative Geometry and stable map

Over the last thirty years, there has been enormous progress in modern enumerative geometry inspired by the work of many people from both the Mathematics and Physics discipline. In this squeal, the mathematical breakthrough in modern enumerative geometry has begun with the famous work of Kontsevich and Manin. The idea of introducing the notion of stable maps, due to Kontsevich, turns out to be the suitable notation to solve the long-lasting open problem of counting rational curves in a smooth projective variety. The theory of stable maps provides the analog of Deligne-Mumford stable curves. The moduli space of stable maps denoted by$\overline{\mathscr{M}}_{g,n}(X,\beta)$, where X is a smooth projective variety and $\beta \in A_1(X)$ a homology class. A typical element of this space is a tuple $(C, x_1, \cdots, x_n, f)$, where C is a curve of arithmetic genus g with nodes as worst possible singularities, x_i are distinct smooth marked points on C, $f : C \longrightarrow X$ is an arbitrary map that satisfies certain stability conditions. The ideas and techniques developed by M. Kontsevich have revolutionized enumerative geometry: stable maps and quantum cohomology. Kontsevich’s celebrated formula, which solves a longstanding question of counting *rational* degree d curves inside $\mathbb{P}^2$. In [38], the authors proved the following:

Theorem 1.2.1. *(Kontsevich)*

Let us denote the number n_d defined by the number of plane rational curves of degree d pass through

$3d-1$ *given points in general position. Then* n_d *is given by*

$$n_d = \sum_{d_1+d_2=d} n_{d_1} n_{d_2} d_1 d_2 \left[\binom{3d-4}{3d_1-2} d_1 d_2 - \binom{3d-4}{3d_1-3} d_2^2 \right] \tag{1.1}$$

provided $d \geq 2$.

The above formula was discovered in a rather different context and it came as a beautiful surprise and lead the modern path for Gromov-Witten theory (a theory largely inspired by various ideas from physics).

1.3 Moduli space of stable maps and genus zero Gromov-Witten invariants

We will mainly consider enumerative problems concerning rational curves, we will only study stable maps and their moduli spaces with the assumption that the curves are of arithmetic genus zero. Let X be a homogeneous algebraic variety then the Gromov-Witten theory is well-understood in the literature. For the preliminaries, we will mainly follow ([13], [36], and [38]) and the references therein.

We recall the notion of the moduli spaces of stable maps, which will be the basic objects of study. We will only interested in the genus-zero Gromov-Witten theory of X, a complex smooth projective variety of dimension n. One can always characterize an irreducible rational curve since it can be parametrized by the projective line.

Definition 1.3.1. *Let* $u : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ *be a holomorphic map then the degree of the map u is defined as the degree of the direct image cycle* $u_*[\mathbb{P}^1]$*. For example, a constant map has degree zero.*

Moduli space of curves and stable map

In this section, we will briefly introduce the notion of moduli space of rational curves then we will concentrate on genus zero stable maps. We will always work over $\mathbb{C}$ as our base field. Note that every smooth, irreducible complex projective curve C of genus 0 is isomorphic to $\mathbb{P}^1$.

Definition 1.3.2. *A smooth n pointed genus* 0 *curve* $(C, p_1, \cdots, p_n)$ *is a projective, smooth rational curve C with a choice of n distinct smooth points* $p_1, \cdots, p_n \in C$*, called the marking of the curve.*

An isomorphism between two such curves $(C, p_1, \cdots, p_n)$ and $(C', p'_1, \cdots, p'_n)$ is an isomorphism $\rho : C \longrightarrow C'$ preserving the order of the markings, i.e., $\rho(p_i) = p_{i'}$.

Definition 1.3.3. *Let us consider the tuple $(C, p_1, \cdots, p_n)$ where C is a connected curve of arithmetic genus g with nodes as the worst possible singularities and $p_1, \cdots, p_n$ are distinct smooth points of C. The marked curve $(C, p_1, \cdots, p_n)$ is said to be stable if in the normalization of the curve any genus 0 component has at least three distinguish points, inverse images of nodes or marked point p_i.*

Note that the stability of the curve is guaranteed by the inequality $2g-2+n>0$.

Proposition 1.3.4. *For $n \geq 3$, there is a fine moduli space, denoted $\mathscr{M}_{0,n}$, for the problem of classifying stable n-pointed smooth rational curves up to isomorphism.*

Example 1.3.5. *For $n=3$, given any smooth rational curve with three markings (C, p_1, p_2, p_3), there is a unique isomorphism $(\mathbb{P}^1, 0, 1, \infty) \longrightarrow (C, p_1, p_2, p_3)$. That is, there exists only one isomorphism class, and consequently, $\mathscr{M}_{0,3}$ is a single point.*

The first non-trivial example of a moduli space of pointed rational curves is $\mathscr{M}_{0,4}$. Every four marked rational curves (C, p_1, p_2, p_3, p_4), is isomorphic to $(\mathbb{P}^1, 0, 1, \infty, p)$ for some unique $p \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. One can show that $\mathscr{M}_{0,4} \simeq \mathbb{P}^1$.

Theorem 1.3.6 (Knudsen [35])**.** *For $n \geq 3$, there exist a smooth projective variety $\overline{\mathscr{M}}_{0,n}$. It is a fine moduli space for n marked stable rational curves containing $\mathscr{M}_{0,n}$ as a dense open subset.*

Throughout this thesis, we mainly work with coarse moduli spaces. Onward, we will denote the coarse moduli space by $\overline{M}_{0,n}$.

Boundary divisors Boundary divisors are of particular interesting cycles of codimension 1.

Let us denote the set of n marks by $[n]$. Then a general point of the boundary divisor $D(A \mid B)$ for each partition $[n] = A \cup B$ with A, B disjoint and cardinality of $A, B \geq 2$, represents a curve with two twigs having A markings on one twig, and B marks on the other. Sometimes we will make abuse of notation, and we will denote by $(i,\ j \mid k,\ l)$ a general curve of a boundary divisor of $\overline{M}_{0,4}$ such that $i,\ j \in A$ and $k,\ l \in B$. If we denote the markings on $\overline{M}_{0,4}$ by $x_1,\ x_2,\ x_3,\ x_4$, one of the three boundary divisors of $\overline{M}_{0,4}$ we will denote as $(x_1,\ x_2 \mid x_3,\ x_4)$.

Stable maps

We will now focus on our main object of study: marked rational curves in complex projective space. An irreducible rational curve can be parametrized by the projective line; therefore the morphism of the form $u : \mathbb{P}^1 \longrightarrow \mathbb{P}^r$ will play a crucial role in our study.

Lemma 1.3.7. *A map with n markings on it is stable if and only if it has only a finite number of automorphisms.*

Remark 1.3.8. *The proof of the above lemma can be found in [37], Lemma 2.3.1.*

Theorem 1.3.9 (Cf. FP-Notes [18])**.** *There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ which parametrizes the isomorphism classes of stable degree d maps to $\mathbb{P}^r$ with n markings on it.*

Theorem 1.3.10 (Cf. FP-Notes [18])**.** *$\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a projective normal irreducible variety and it can be seen locally as isomorphic to a quotient of a smooth variety by the action of a finite group.*

The compactified moduli space $\overline{M}_{0,n}(\mathbb{P}^m, d)$ has the dimension

$$md + m + d + n - 3.$$

We will not discuss anything related to the construction of $\overline{M}_{0,n}(\mathbb{P}^m, d)$, [37] can be considered as one of the excellent references for the above.

Evaluation Map

For each marking p_i on the curve there is a natural evaluation map,

$$ev_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \longrightarrow \mathbb{P}^r$$

$$ev_i(C, p_1, \cdots, p_n, u) = u(p_i)$$

For example, let $H \subset \mathbb{P}^r$ be a hyperplane, then for each i the inverse image $ev_i^{-1}(H)$ is a divisor in $\overline{M}_{0,n}(\mathbb{P}^r, d)$. It consists of all maps whose i-th marking is mapped into H. For example, if $H^2 \in \mathbb{P}^2$ is a point class, then $ev_i^{-1}(H^2)$ is a codimension 2 divisor in $\overline{M}_{0,n}(\mathbb{P}^2, d)$.

Forgetful Map

Let us consider two sets of markings respectively A and B with $A \subset B$. Then there exist a natural forgetful map $f : \overline{M}_{0,B}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{0,A}(\mathbb{P}^r, d)$ which forgets the markings in the complement $B \setminus A$ in any order and this map factors through

$$\overline{M}_{0,n+1}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{0,n}(\mathbb{P}^r, d)$$

The forgetful maps are very crucial and they require special care in the study of stable map theory. While forgetting the markings it might happen that the resultant becomes an unstable map then we have to contract the unstable component so that the image of a forgetful map remains stable.

Lemma 1.3.11. *Let us assume that $n \geq 4$, then the forgetful map $\overline{M}_{0,n}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{0,4}$ is a flat morphism.*

Remark 1.3.12. *The illustration of the above lemma can be found in [37]. It is also been discussed that the above map is flat for $n \geq 3$.*

Boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$

The boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is made up from the curves whose domains are reducible curves. Let the n markings given by $\{p_1, \cdots, p_n\}$ and choose a partition such that $A \cup B = \{p_1, \cdots, p_n\}$ together with two non-negative integers d_A and d_B such that $d_A + d_B = d$. For the above choice of partition (where $|A| \geq 2$ if $d_A = 0$ and $|B| \geq 2$ if $d_B = 0$) there exists an irreducible divisor, denoted $D(A, B; d_A, d_B)$, called a boundary divisor. A typical point on this divisor represents a map u whose domain is a tree with two twigs, $C = C_A \cup C_B$, with the points of A in C_A and those of B in C_B, such that the restriction of u to C_A is a map of degree d_A and the restriction of u to C_B is of degree d_B. This is given by a picture as follows:

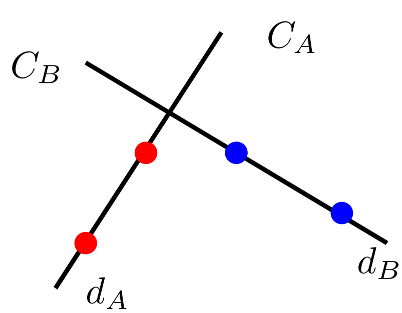

Figure 1.1: Boundary divisor

For example, $\overline{M}_{0,5}(\mathbb{P}^2, 2)$ has 42 boundary divisors.

Recursive structure of the boundary

Let us make the following Proposition without proof:

Proposition 1.3.13. *The boundary of $\overline{M}_{0,n}$ is a divisor with normal crossing.*

A typical element in $D(A, B; d_A, d_B)$ corresponds to a reducible curve with two twigs and distributes the corresponding markings among each other. Let us denote the point of intersection of the two twigs by p. Then the component of the reducible curve on A twig side corresponds to an element of $\overline{M}_{0,A\cup\{p\}}$, and similarly on B twig side gives an element in $\overline{M}_{0,B\cup\{p\}}$. Note that the stability of the reducible curve implies the stability of the resultant two-component curve.

Conversely, a typical element in $\overline{M}_{0,A\cup\{p\}} \times \overline{M}_{0,B\cup\{p\}}$ can reconstruct a reducible curve in $D(A,B;d_A,d_B)$ by identifying the two markings p,attaching the two curves in a node at p. In this way, we will get a canonical isomorphism

$$D(A,B;d_A,d_B) \longrightarrow \overline{M}_{0,A\cup\{p\}} \times \overline{M}_{0,B\cup\{p\}}$$

since each of the two components with fewer markings are stable and smooth, we can conclude that the boundary divisor $D(A,B;d_A,d_B)$ is irreducible as well as smooth.

Let us mention an important fact that may be useful later when we proof Kontsevitch's recursion formula for $\mathbb{P}^2$.

Lemma 1.3.14. *Let $Y \in \mathbb{P}^r$ be a sub variety. Then the inverse image $ev_i^{-1}(Y)$ has proper intersection with each of the boundary divisors $D(A,B;d_A,d_B)$. If Y has codimension k inside $\mathbb{P}^r$ then $ev_i^{-1}(Y) \cap D(A,B;d_A,d_B)$ is of codimension $k+1$ in $\overline{M}_{0,n}(\mathbb{P}^r,d)$.*

An immediate corollary of the above Lemma is the following

Corollary 1.3.15. *For any irreducible sub variety $Y \in \mathbb{P}^r$, the inverse image under evaluation map is irreducible in $\overline{M}_{0,n}(\mathbb{P}^r,d)$.*

Fundamental boundary relation

Assume that $n \geq 4$. Let us consider the composition of flat forgetful maps $\overline{M}_{0,n}(\mathbb{P}^r,d) \longrightarrow \overline{M}_{0,n} \longrightarrow \overline{M}_{0,4}$. Let $D(i,j\,;\,k,l)$ be the divisor in $\overline{M}_{0,n}(\mathbb{P}^r,d)$ defined as the inverse image of a divisor $(i,j\,;\,k,l)$ in $\overline{M}_{0,4}$. Then

$$D(i,j\,;\,k,l) = \sum D(A,B;d_A,d_B)$$

where the sum is taken over all possible partitions discussed above. Since $\overline{M}_{0,4} \simeq \mathbb{P}^1$, is path-connected, all three boundary divisors are equivalent. This yield a fundamental relation

$$\sum_{\substack{|A|\cup|B|=n \\ i,j\in A,\ j,k\in B \\ d_A+d_B=d}} D(A,B;d_A,d_B) = \sum_{\substack{|A|\cup|B|=n \\ i,k\in A,\ j,l\in B \\ d_A+d_B=d}} D(A,B;d_A,d_B) = \sum_{\substack{|A|\cup|B|=n \\ i,l\in A,\ j,k\in B \\ d_A+d_B=d}} D(A,B;d_A,d_B) \qquad (1.2)$$

We will discuss the consequences of the above relation below.

Gromov-Witten invariants

Various numerical invariants have been defined for a smooth algebraic variety. For example, Chern classes provide a rich structure to the theory. One such invariant, namely the Gromov-Witten invariant is defined as integrals on the moduli space of stable maps. These invariants have their origins in physics, by construction, it remains invariant under deformation of the complex structure of a given projective algebraic variety X. Although the Gromov-Witten invariants are defined from both algebraic and symplectic geometric techniques, in this thesis, we mostly concentrate on the algebraic side.

Let X be a smooth projective variety then the Chow groups $A_*(X)$ are well understood. So one can perform operations of intersection theory on it. There is a perfect pairing namely Poincaré duality which allows us to consider homology and cohomology classes simultaneously. Throughout, we will consider the coefficient ring as Q. It turns out for $X = \mathbb{P}^r$ the intersection ring is isomorphic to the cohomology ring of $\mathbb{P}^r$. However, $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a singular variety hence performing intersection theory on it is not easy.

We have the following natural maps:

$$\pi_1 : \overline{M}_{0,n}(\mathbb{P}^r, d) \longrightarrow (\mathbb{P}^r)^n$$

$$\pi_2 : \overline{M}_{0,n}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{0,n}.$$

The map π_1 sends a moduli point $u : (C, p_1, \cdots, p_n) \longrightarrow \mathbb{P}^r$ to the n tuple $(u(p_1), \cdots, u(p_n))$. Next, observe that even if $u : C \longrightarrow \mathbb{P}^r$ is a stable map, it need not be a stable curve in the sense of Deligne and Mumford [53]. Now, if $n \geq 3$, then successively contracting the unstable components of C gives a stable curve $\tilde{C}$. Then π_2 maps $u : (C, p_1, \cdots, p_n) \longrightarrow \mathbb{P}^r$ to the isomorphism class of $\tilde{C}$ [13].
The above maps give natural maps

$$\pi_1^* : H^*(\mathbb{P}^r, Q)^{\otimes n} \longrightarrow H^*(\overline{M}_{0,n}(\mathbb{P}^r, d), Q)$$

$$\pi_{2*} : H_*(\overline{M}_{0,n}(\mathbb{P}^r, d), Q) \longrightarrow H_*(\overline{M}_{0,n}, Q)$$

where our assumption on n is the same i.e $n \geq 3$. Poincaré duality and π_{2*} induces the Gysin map

$$\pi_{2!} : H^*(\overline{M}_{0,n}(\mathbb{P}^r, d), Q) \longrightarrow H^{2m+*}(\overline{M}_{0,n}, Q) \tag{1.3}$$

where $m = (r+1)d - r$. Then the Gromov-Witten class for the cohomology classes $\alpha_1, \cdots, \alpha_n \in H^*(\mathbb{P}^r, Q)$, defined as

$$I_{0,n,d}(\alpha_1, \cdots, \alpha_n) = \pi_{2!}(\pi_1^*(\alpha_1 \otimes \cdots \otimes \alpha_n)). \tag{1.4}$$

Hence we have Gromov-Witten invariant

$$\langle I_{0,n,d}\rangle(\alpha_1,\cdots,\alpha_n)=\int_{\overline{M}_{0,n}} I_{0,n,d}(\alpha_1,\cdots,\alpha_n). \tag{1.5}$$

Using (1.4), the above reduces to

$$\langle I_{0,n,d}\rangle(\alpha_1,\cdots,\alpha_n)=\int_{\overline{M}_{0,n}(\mathbb{P}^r,d)} \pi_1^*(\alpha_1,\cdots,\alpha_n). \tag{1.6}$$

Note that equation (1.6) make sense when $n<3$. Thus although Gromov-Witten classes require $n\geq 3$, Gromov-Witten invariants are defined for all $n\geq 0$. We will use the notation

$$N_{0,d}\langle\alpha_1,\cdots,\alpha_n\rangle=\langle I_{0,n,d}\rangle(\alpha_1,\cdots,\alpha_n).$$

Axioms for Gromov-Witten classes

In [38], Kontsevich and Manin proposed certain number of axioms for Gromov-Witten classes. From the algebraic point of view it has been shown that the Gromov-Witten classes satisfies these axiom in ([6], [44], [5], [4]) and in symplectic case it has been proved in ([59], [60], [45]). We will describe the axioms which are satisfied by the Gromov-Witten class without proving them. The proofs can be found in the above references including [13].

Let us assume that X is a smooth projective variety and $g=0,\ n\geq 3$. Let $\beta\in H_2(X;\mathbb{Z})$ be a homology class. If d is the number characterizing the homology class β, then we will denote $\langle I_{0,n,\beta}\rangle$ instead of $\langle I_{0,n,d}\rangle$. We now state the axioms for $\langle I_{0,n,\beta}\rangle$.

Linearity Axiom. The very first axiom asserts that $\langle I_{0,n,\beta}\rangle$ is linear in each variable. This is as expected since the sum of cycles is simply given their union.

Effective Axiom. This axiom ensures that $\langle I_{0,n,\beta}\rangle = 0$ if β is not an effective class.

Degree Axiom. Recall that the moduli space $\overline{M}_{0,n}(X,\beta)$ has the expected dimension $2\ dimX-6-\int_\beta \omega_X+n$. Let $\alpha_1,\cdots,\alpha_n\in H^*(X,Q)^{\otimes n}$, all α_i are homogeneous classes. Then the degree axiom simplifies that $I_{0,n,\beta}(\alpha_1,\cdots,\alpha_n)$ is a top degree class if and only if

$$\sum_{i=1}^n deg\alpha_i = 2\ \mathrm{dim}X-6-\int_\beta \omega_X+n.$$

Note that this axiom is valid for $n\geq 0$.

Equivariance Axiom. The symmetric group of n letters denoted by S_n has natural action on the cohomology groups $H^*(X,Q)^{\otimes n}$, $H^*(\overline{M}_{0,n},Q)$. Then this axiom guarantees that the map

$$I_{0,n,\beta}: H^*(X,Q)^{\otimes n}\longrightarrow H^*(\overline{M}_{0,n},Q)$$

is a S_n equivariant map. Thus for Gromov-Witten invariants, equivariance means

$$\langle I_{0,n,\beta}\rangle(\alpha_1,\cdots,\alpha_i,\alpha_{i+1},\cdots,\alpha_n)=(-1)^{deg\alpha_i\ deg\alpha_{i+1}}\langle I_{0,n,\beta}\rangle(\alpha_1,\cdots,\alpha_{i+1},\alpha_i,\cdots,\alpha_n).$$

Fundamental Class Axiom. Let $n\geq 4$. We have a natural map $\pi_n:\overline{M}_{0,n}\longrightarrow\overline{M}_{0,n-1}$ by forgetting the last marked point. Let $[X]\in H^0(X,Q)$ be the fundamental class of X, then the axiom asserts that

$$\langle I_{0,n,\beta}\rangle(\alpha_1,\cdots,\alpha_{n-1},[X])=\pi_n^*\langle I_{0,n,\beta}\rangle(\alpha_1,\cdots,\alpha_{n-1})$$

in other wards, this axiom implies that the above Gromov-Witten invariant can be non-zero if and only if $\langle I_{0,n,\beta}\rangle(\alpha_1,\cdots,\alpha_{n-1},[X])$ is a top degree class. This holds true whenever π_n is defined. This axiom is true if either $n\geq 4$ or $\beta\neq 0$ and $n\geq 1$.

Divisor Axiom. Let $n\geq 4$ and π_n is defined as above. For $\alpha_n\in H^2(X,Q)$ the degree axiom asserts that

$$\pi_{n*}\langle I_{0,n,\beta}\rangle(\alpha_1,\cdots,\alpha_{n-1},\alpha_n)=(\int_X\alpha_n)\langle I_{0,n,\beta}\rangle(\alpha_1,\cdots,\alpha_{n-1})$$

Point mapping Axiom. This axiom deals with the case $\beta=0$. For the genus zero case only, if α_i are homogeneous cohomology classes, then

$$\langle I_{0,n,0}\rangle(\alpha_1,\cdots,\alpha_n)=\begin{cases}\int_X(\alpha_1,\cup\cdots\cup,\alpha_n)\,[\overline{M}_{0,n}] & \text{if}\quad \sum_{i=1}^n\alpha_i=2\,dimX\\ 0 & \text{otherwise}\end{cases}\tag{1.7}$$

Splitting Axiom. Let us consider a splitting $n=n_1+n_2$. Given two stable curves $(C_1,p_1,\cdots,p_{n_1},p_{n_1+1})$ and $(C_2,\tilde{p}_1,\cdots,\tilde{p}_{n_2},\tilde{p}_{n_2+1})$ we can obtain C from $C_1\cup C_2$ by identifying p_{n_1+1} with $\tilde{p}_{n_2+1}$, a stable genus zero curve $(C,p_1,\cdots,p_{n_1},\tilde{p}_1,\cdots,\tilde{p}_{n_2})$. Hence we get the map

$$\delta:\overline{M}_{0,n_1+1}\times\overline{M}_{0,n_2+1}\longrightarrow\overline{M}_{0,n}$$

thus the splitting axiom yield the formula for $\delta^*\langle I_{0,n,0}\rangle(\alpha_1,\cdots,\alpha_n)$ as

$$\sum_{\beta=\beta_1+\beta_2}\sum_{i,j}g^{ij}\langle I_{0,n_1+1,0}\rangle(\alpha_1,\cdots,\alpha_{n_1},T_i)\otimes\langle I_{0,n,0}\rangle(T_j,\alpha_{n_1+1},\cdots,\alpha_n)$$

This means that our cohomology class for the diagonal in $H^*(X\times X,Q)$ is given by $\sum_{i,j}g_{i,j}T_i\otimes T_j$. Note that the above sum is finite due to Effective axiom.

Deformation Axiom. This axiom asserts that the Gromov-Witten classes should be invariant under the deformation of complex structures. This treatment is much more natural and suitable in the symplectic setting.

This set of axioms are extremely useful for doing computation of Gromov-Witten invariants although not always necessary.

Quantum Cohomology

Various techniques have been developed over thirty years to present Gromov-Witten theory in a systematic way and techniques should be helpful to predict the related structures. One of such is Quantum cohomology. Throughout we will only focus on genus-zero Gromov-Witten theory so we will be concentrating on small quantum cohomology, for a complete reference we will refer the reader to ([38], [37]). One can generalize the notion of cup product in the usual cohomology ring to the quantum product which forms the quantum cohomology ring. Quite often the structure constants of this ring encode the Gromov-Witten invariant. The quantum product is associative and the quantum cohomology ring has a unit. We will see surprisingly quantum associativity implies Kontsevich's recursion formula for $\mathbb{P}^2$, which is considered to be a breakthrough in this area.

Let us consider X be a homogeneous variety and choose a basis for the cohomology ring of it as $\{T_0 = Id, T_1, \cdots, T_r\}$. There is a natural intersection matrix defined by usual cup product namely

$$g_{ij} = \int_X T_i \cup T_j.$$

Let g^{ij} be the inverse of g_{ij}, then the product $T_i \cup T_j$ can be expressed as follows

$$\begin{aligned} T_i \cup T_j &= \sum_{k,l} \left(\int_X T_i \cup T_j \cup T_k \right) g^{kl} T_l \\ &= \sum_{k,l} \langle I_{0,3,\beta=0} \rangle (T_i \cup T_j \cup T_k) g^{kl} T_l \end{aligned}$$

Given a class $\alpha = \sum x_i T_i$, we define the following generating series

$$F(\alpha) = \sum_{m\geq 3} \sum_{\beta} \frac{1}{m!} \langle I_{0,n,\beta} \rangle (\gamma_1 \cdots \gamma_m)$$

Note that the series F becomes a formal power series in $Q[[x_0, \cdots, x_r]]$. The expanded form of F can be written as

$$F(x_0, \cdots, x_r) = \sum_{m_0+\cdots+m_r \geq 3} \langle I_{0,n,\beta} \rangle (T_0^{m_0} \cdots T_r^{m_r}) \frac{x_0^{m_0} \cdots x_r^{m_r}}{m_0! \cdots m_r!}.$$

Note that we always consider the variables corresponding to the codimension two or more classes as zero class. One can see that taking third order partial derivative of F with respect to x_i, x_j and x_k

$$F_{ijk} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k} = \sum_{m\geq 0} \sum_{\beta} \frac{1}{m!} \langle I_{0,n,\beta} \rangle (\alpha^m, T_i, T_j, T_k)$$

Now we will define the quantum product as the multiplication on $A^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} Q[[x_0, \cdots, x_r]]$ by defining

$$T_i * T_j = \sum_{k,l} F_{ijk} g^{kl} T_l$$

and extend the multiplication to $Q[[x_0,\cdots,x_r]]$ linearly.

Lemma 1.3.16. *The quantum multiplication is commutative, associative and the quantum cohomology ring has unit T_0.*

For a detailed review of quantum cohomology and its application we refer [18]. The central aspect of the quantum product is associative property. As a consequence of quantum associativity, we will get Kontsevich's recursion formula.

Theorem 1.3.17. *The quantum product $*$ is associative. That is,*

$$(T^i * T^j) * T^k = T^i * (T^j * T^k)$$

Proof. The theorem follows from the linear equivalence among the boundary divisors that we have seen in (1.2) and the splitting Lemma. Although quantum associativity is a formal consequence of the above but it has much more geometric significance in enumerative geometry, for example see ([71], [14]).

Let us now expand both sides of the associativity relation. Using the definition above left hand side is given by

$$(T^i * T^j) * T^k = (\sum_{e+f=r} F_{ije}T^f) * T^k = \sum_{e+f=r}\sum_{l+m=r} F_{ije}F_{fkl}T^m.$$

Next, expanding the right hand side in a similar way we get the following

$$T^i * (T^j * T^k) = \sum_{e+f=r}\sum_{l+m=r} F_{jke}F_{fil}T^m$$

hence the associativity gives us

$$\sum_{e+f=r}\sum_{l+m=r} F_{ije}F_{fkl} = \sum_{e+f=r}\sum_{l+m=r} F_{jke}F_{fil}$$

since T^i are linearly independent, so we can conclude that

$$\sum_{e+f=r} F_{ije}F_{fkl} = \sum_{e+f=r} F_{ije}F_{fkl}.$$

□

Following the notes [19], let us assume that $T_0,\cdots,T_m$ be a basis of the cohomology group $A^1(X)$ and let $T_{m+1},\cdots,T_r$ be a basis for other cohomology groups. Then the fundamental numbers associated to the Gromov-Witten invariant can be described as

$$N_\beta(n_{m+1},\cdots,n_r) = \langle I_{0,r-m,\beta}\rangle (T_{m+1}^{n_{m+1}}\cdots T_r^{n_r}), \qquad \forall\, n_i \geq 0$$

we can formally define these numbers to be zero if $\sum n_i(codim(T_i)-1) \neq dimX - 3 + \int_\beta c_1(T_X)$.

Enumerative Application

Let us now define a potential function as

$$F(\bar{x}) = F_{classical}(\bar{x}) + F_{quantum}(\bar{x}).$$

When $\beta = 0$, the classical term of the above potential takes the form

$$F_{classical}(\bar{x}) = \sum_{n_0+\cdots+n_r=3} \int_X \left(T_0^{n_0} \cdots T_r^{n_r}\right) \frac{x_0^{n_0}}{n_0!} \cdots \frac{x_r^{n_r}}{n_r!}$$

next, we describe $F_{quantum}(\bar{x})$ by using the properties (1.3),(2) and (3)

$$G(\bar{x}) = \sum_{n_{m+1}+\cdots+n_r \geq 0} \sum_{\beta \neq 0} N_\beta(n_{m+1}, \cdots, n_r) \prod_{i=1}^{m} e^{\left(\int_\beta c_1(T_X)\right)x_i} \prod_{i=m+1}^{r} \frac{x_i^{n_i}}{n_i!}$$

Note that the classical term only contains the numbers of the form $\int_X T_i \cup T_j \cup T_k$, where as the quantum expression has more interesting enumerative numbers. For projective plane take $T_0 = 1$, the class of a line is denoted by T_1 and T_2 be the class of a point. One can verify $g_{ij} = 1$ for $i + j = 2$ else 0 and the same is true for g^{ij}. Thus

$$T_i * T_j = F_{ij0}T_2 + F_{ij1}T_1 + F_{ij2}T_0.$$

Let us look at

$$(T_1 * T_1) * T_2 = F_{221}T_1 + F_{222}T_0 + F_{111}(F_{121}T_1 + F_{112}T_0) + F_{112}T_2$$

$$T_1 * (T_1 * T_2) = F_{122}T_1 + F_{121}(F_{111}T_1 + F_{112}T_0)$$

Next, equating the coefficients of T_0 from both the equation we conclude

$$F_{222} = F_{112}^2 - F_{111}F_{122} \tag{1.8}$$

Since the number for rational curves n_d is non-zero only if the curve passes through $3d - 1$ general points in projective plane, so

$$G(\bar{x}) = \sum_{d \geq 1} n_d \frac{e^{dx_0} x_1^{3d-1}}{(3d-1)!}$$

Note that it can very easily be shown by computing the partial derivatives of G and using the equation (1.8) we will get the identity in theorem (1.2.1).

Proof of Kontsevich's formula for rational plane curves

With the above notation, we are ready to prove the famous recursive formula due to Kontsevich for rational curves in $\mathbb{P}^2$.

Proof. Let us consider our moduli space $\overline{M}_{0,3d}(\mathbb{P}^n,d)$ with $3d$ marking on the curves namely, $p_1,\ p_2,\ q_1,\ \cdots,\ q_{3d-2}$. Let L_1 and L_2 be two lines and let $Q_1,\cdots,Q_{3d-2}$ points in $\mathbb{P}^2$ in general position. Let $\mathscr{Z}\subset\overline{M}_{0,3d}(\mathbb{P}^2,d)$ be a sub-variety corresponds to the passing through points and intersecting lines, i.e., $\mathscr{Z}$ defined by intersection of the inverse images of the above points and lines under the evaluation morphism. In particular as cycles

$$\mathscr{Z}=ev_1^*[L_1]\ \ ev_2^*[L_2]\ \ ev_3^*[Q_1]\ \ ev_4^*[Q_2]\ \ \mathscr{H}_p^{3d-4}$$

where $[H]$ is the hyper-plane class inside $\mathbb{P}^2$ and $\mathscr{H}_p$ denotes the class representing the homology class corresponding the curve passing through point and ev_i denotes the evaluation map at the ith marking. Now recall the fundamental equivalence (1.2), the result will follow from the following observation:

$$\mathscr{Z}\cap D(p_1,p_2;\ q_1,q_2)\ =\ \mathscr{Z}\cap D(p_1,q_1;\ p_2,q_2) \tag{1.9}$$

Note that the points and lines can be chosen in such a way that $\mathscr{Z}$ is a curve and it intersects the boundary transversely and it lies completely inside the open dense locus of $\overline{M}_{0,3d}(\mathbb{P}^2,d)$.

Let us consider the left-hand side of the equation (1.9). Here $(p_1,p_2\mid q_1,q_2)$ denotes the domain which is a wedge of two spheres with the marked points p_1 and p_2 on the first sphere and q_1 and q_2 on the second sphere. The domain $(p_1,q_1\mid p_2,q_2)$ is defined similarly. Since $\overline{M}_{0,4}\approx\mathbb{P}^1$ is path connected, any two points determine the same divisor.

Let us now consider the projection map

$$\pi:\overline{M}_{0,4}(\mathbb{P}^2,d)\longrightarrow\overline{M}_{0,4}.$$

Let us now intersect the cycle $\mathscr{Z}$ by pulling back the left hand side of (1.9), via π. We get the following:

$$\pi^*(p_1,p_2\mid q_1,q_2)\cdot\mathscr{Z}=n_d+\sum_{d=d_1+d_2,\ d_1,d_2>0}\binom{3d-4}{3d_1-1}n_{d_1}n_{d_2}d_1^3d_2 \tag{1.10}$$

where in the above expression, $d:=d_1+d_2,\ d_1,d_2>0$

Next, we will justify the right hand side. We will intersect $\mathscr{Z}$ by pulling back the right hand side of (1.9), via the map π. This yields the number

$$\pi^*(p_1,q_1\mid p_2,q_2)\cdot\mathscr{Z}=2n_d+\sum_{d=d_1+d_2,\ d_1,d_2>0}\binom{3d-4}{3d_1-1}n_{d_1}n_{d_2}d_1^2d_2^2 \tag{1.11}$$

Note that the equations (1.9), (1.10) and (1.11) give the formula (1.2.1).

□

1.4 Counting curves using Atiyah-Bott Localization

Another extremely powerful method to study the theory of counting rational curves (Gromov–Witten invariants) is the **Atiyah–Bott localization technique**. We will discuss here how to compute the rational curves in $\mathbb{P}^2$ using the Localization technique.

Let M be a smooth projective variety equipped with an action of a torus $\mathbb{T} = (\mathbb{C}^*)^r$ on it, then one can consider the *equivariant cohomology* denoted by $H^*_{\mathbb{T}}(M)$ as $H^*(E\mathbb{T} \times_{\mathbb{T}} M)$.

For example, if $M = \{pt\}$ the equivariant cohomology ring is the polynomial ring $H^*_{\mathbb{T}}(pt) = H^*(B\mathbb{T}) \simeq \mathbb{C}[\alpha_1, \alpha_2, \cdots, \alpha_r]$. where

$$\alpha_i = c_1(\gamma^*_{(\mathbb{P}^\infty)_i}), \quad 1 \leq i \leq r$$

(α_i's are known as $\mathbb{T}$ weights). All of the usual operations on cohomology (pullback, integration, Chern classes, etc) have equivariant analogs.

Theorem 1.4.1. *Suppose n-torus acts smoothly on a compact oriented even dimensional manifold M, and suppose that each connected component of the $\mathbb{T}$ fixed locus $M^{\mathbb{T}} \subset M$ is a compact orientable sub-manifold of M . Let $F_1, \cdots, F_N$ be the connected components of $M^{\mathbb{T}}$. The inclusion $i_{F_j} : F_j \longrightarrow M$ induces a homomorphism*

$$(i_{F_j})_* : H^*_{\mathbb{T}}(F_j) \longrightarrow H^*_{\mathbb{T}}(M)$$

Furthermore, the equivariant Euler class of the normal bundle $N_{F_j/M}$ in M is well-defined and invertible.
If $\alpha \in H^*_{\mathbb{T}}(M)$, then

$$\int_M \alpha = \sum_{F_j} \int_{F_j} \frac{\alpha_{|F_j}}{Euler^{\mathbb{T}}(N_{F_j/M})}$$

Remark 1.4.2. *According to the Atiyah–Bott localization formula, all of the information about the equivariant cohomology of M is contained in the equivariant cohomology of its $\mathbb{T}$-fixed locus.*

In our case, consider $\mathbb{P}^2$ with its usual action by $(\mathbb{C}^*)^3$.

$$T : (\mathbb{C}^*)^3 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad T(x,y,z) = (t_1x,\ t_2y,\ t_3z) \tag{1.12}$$

It induces an action on $\overline{M}_{0,n}(\mathbb{P}^2, d)$ (by post composing with the action), then one has to **control the fixed object** to apply similar extensions of the above theorem.

We see that $\{p_0 = [1:0:0],\ p_1 = [0:1:0],\ p_2 = [0:0:1]\}$ are the only $\mathbb{T}$-fixed points of $\mathbb{P}^2$. Now a $\mathbb{T}$-fixed point $[f:\mathbb{P}^1 \longrightarrow \mathbb{P}^2] \in \overline{M}_{0,n}(\mathbb{P}^2,d)$ can be described as a graph Γ (a tree with half legs) as follows:

- The vertices of Γ correspond to the connected components of $f^{-1}(p_0,p_1,p_2)$. The vertex v labeled with the $g(v)$, genus of the corresponding component.
- Each edge corresponds to a non-contracted component of the domain curve. The edge e labeled with the degree $d(e)$ of the restriction of f to this component.
- An edge is connected to a vertex whenever sub-components of the domain are incident.
- Half edge or leg occurs when the vertex corresponds to one or more to $\mathbb{T}$-contacted components.

satisfying $\sum_{v\in V(\Gamma)} g(v)+|E(\Gamma)|-V(\Gamma)+1=0$ and $\sum_{e\in E(\Gamma)} d(e)=d$.

Remark 1.4.3. *Due to Kontsevich, $\overline{M}_\Gamma$ has a induced course moduli space structure from $\overline{M}_{0,n}(\mathbb{P}^2,d)$.*

For example, let us compute the fix points of the standard $\mathbb{T}$ action on $\overline{M}_{0,1}(\mathbb{P}^2,2)$. Let us define

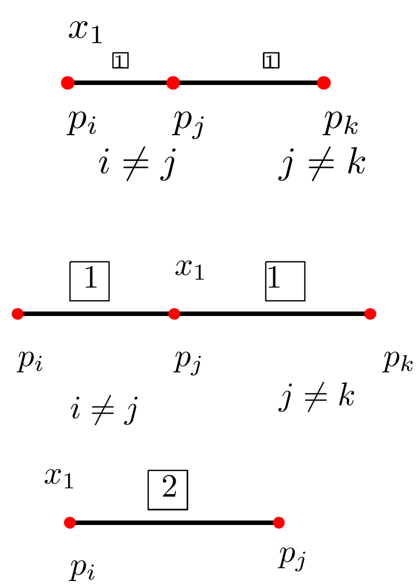

Figure 1.2: Stable torus fix points

Gromov-Witten invariants of $\mathbb{P}^n$

$$GW_{0,n}^{\mathbb{P}^n} := \int_{[\overline{M}_{0,n}(\mathbb{P}^n,d)]} ev_1^*(\lambda_1)\cup\cdots\cup ev_n^*(\lambda_n). \tag{1.13}$$

Where $\lambda_i \in H^*(\mathbb{P}^n;\mathbb{Z})$ and $ev_i : \overline{M}_{0,n}(\mathbb{P}^n,d) \longrightarrow \mathbb{P}^n$ be the evaluation map.

Since ev_i are equivariant with respect to the action of $\mathbb{T}$ on $\overline{M}_{0,n}(\mathbb{P}^2,d)$ and $\mathbb{P}^2$, so we have

$$ev_i^* : H_{\mathbb{T}}^*(\mathbb{P}^2) \longrightarrow H_{\mathbb{T}}^*(\overline{M}_{0,n}(\mathbb{P}^2,d)).$$

Let $H \in H^*_{\mathbb{T}}(\mathbb{P}^2)$ denotes the equivariant hyper-plane class. Then,

$$\text{Equivariant } GW^{\mathbb{P}^2}_{0,n} := \int_{[\overline{M}_{0,n}(\mathbb{P}^2,d)]_{\mathbb{T}}} ev_1^*(H_1^2) \cup \cdots \cup ev_n^*(H_n^2). \tag{1.14}$$

Let $i : \overline{M}_\Gamma \longrightarrow \overline{M}_{0,n}(\mathbb{P}^2,d)$ be the inclusion map. We now want to compute

$$\left\langle \underbrace{H^2 \cdots H^2}_{3d-1} \right\rangle^{\mathbb{P}^2}_{0,d} = \int_{[\overline{M}_{0,n}(\mathbb{P}^2,d)]_{\mathbb{T}}} ev_1^*(H_1^2) \cup \cdots \cup ev_{3d-1}^*(H_{3d-1}^2). \tag{1.15}$$

Then after applying Atiyah-Bott localization the above reduces to

$$= \sum_\Gamma \frac{1}{A_\Gamma} \int_{\overline{M}_\Gamma} \frac{i^*\left(ev_1^*(H^2) \cup \cdots \cup ev_{3d-1}^*(H^2)\right)}{Euler^{\mathbb{T}}(N_\Gamma)}. \tag{1.16}$$

Since for a topological space M with $\mathbb{T}$ action, $M \times_{\mathbb{T}} E\mathbb{T}$ is a bundle over $B\mathbb{T}$ with fibre M, where $E\mathbb{T}$ is the classifying space over $B\mathbb{T}$. Then we have the natural inclusion $i_M : X \longrightarrow M \times_{\mathbb{T}} E\mathbb{T}$ (it induces $i_M^* : H^*_{\mathbb{T}}(X) \longrightarrow H^*(X)$).

$$\begin{array}{ccc} M & \rightarrow & point \\ \downarrow i & & \downarrow i \\ M_T & \rightarrow & BT \end{array}$$

This implies

$$i^*_{point} \circ \int_{M_{\mathbb{T}}} = \int_M \circ \quad i_M^*.$$

Example: $\int_M P(c_k(E)) = i^*_{point}\left(\int_{M_T} P(c_k^{\mathbb{T}})\right)$, where P is a polynomial consisting of Chern numbers.

Next, we note that one can explicitly calculate the Euler class of the equivariant normal bundle at a fixed point as follows

$$\frac{1}{Euler^{\mathbb{T}}(N_\Gamma)} = e(E)e(F)e(V)$$

where

$$e(E) = \prod_{edges\ e} \frac{(-1)^{d(e)} d(e)^{2d(e)}}{d(e)!^2(\alpha_i - \alpha_j)^{2d(e)}} \prod_{a+b=d(e),\ k \neq i,j} \frac{1}{\frac{a}{d(e)}\alpha_i + \frac{b}{d(e)}\alpha_j - \alpha_k}$$

$$e(F) = \prod_{flag\ F} \prod_{j \neq \mu(F)} (\alpha_{\mu(F)} - \alpha_j) \prod_{g(v(F))>0\ or\ val(v(F)) \geq 3} \frac{1}{\omega_F - \psi_F} \prod_{g(v(F))=0\ and\ val(v(F))=2} \frac{1}{\omega_{F_1} + \omega_{F_1}}$$

$$e(V) = \prod_{Vertices\ v} \prod_{j \neq \mu(F)} c_{(\alpha_{\mu(F)} - \alpha_j)^{-1}} (\alpha_{\mu(F)} - \alpha_j)^{g(v)-1} \prod_{g(v)=0\ and\ val(v)=1} \omega_{F(v)}. \tag{1.17}$$

Note: Above formula has been explicitly worked out in the paper "Localization of Virtual class" by Graber and Pandharipande [23].

Remark 1.4.4. *Suppose we want to compute the number of rational lines in* $\mathbb{P}^2$ *passing through* 2 *points in general position via localization, then the equation* (1.16) *reduces to the sum of the following rational function:* $\sum \frac{{\alpha_0}^2{\alpha_1}^2}{(\alpha_0-\alpha_1)^2(\alpha_0-\alpha_2)(\alpha_1-\alpha_2)}$. *One can observe that there is some non-trivial magical cancellation is going on and finally it produces the number* 1.

1.5 Counting *planar* curves in $\mathbb{P}^3$

Let us denote the dual of $\mathbb{P}^3$ by $\hat{\mathbb{P}}^3$; this is the space of $\mathbb{P}^2$'s inside $\mathbb{P}^3$. An element of $\hat{\mathbb{P}}^3$ can be thought of as a non-zero linear functional $\eta : \mathbb{C}^4 \longrightarrow \mathbb{C}$ upto scaling (i.e., it is the projectivization of the dual of $\mathbb{C}^4$). Given such an η, we define the projectivization of its zero set as $\mathbb{P}^2_\eta$. Hence $\mathbb{P}^2_\eta \subset \mathbb{P}^3$. Now we will define a *planar* degree d curve in $\mathbb{P}^3$ as follows:

Definition 1.5.1. *A* ***planar*** *curve in* $\mathbb{P}^3$ *is defined to be a curve in* $\mathbb{P}^3$*, whose image lies inside a* $\mathbb{P}^2_\eta$, for some η.

Enumeration of **planar** curve has appeared in the literature. For example, any **planar** conic always lies inside a unique plane. Thus the number of **planar** conics intersecting 8 generic lines in $\mathbb{P}^3$ is 92, which is known from past decades. The more important number is 12960, which has been known for at least 150 years ago [62]. This number is equal to the number of **planar** cubics having a node and intersecting 11 general lines in $\mathbb{P}^3$.

Motivated by the study of natural generalizations of the enumerative problems studied in the papers of Kleiman and Piene ([33]) and T.Laarakker ([41]) in the linear system setting, we have studied the parallel question of counting stable rational maps into a family of moving target spaces. This can be viewed as a family version of the famous question of enumerating rational curves in $\mathbb{P}^2$, that was studied by Kontsevich-Manin ([38]) and Ruan-Tian ([59]). In [51], we study the following:

Theorem 1.5.2. *There is an explicit formula for counting rational degree d* ***planar*** *curves in* $\mathbb{P}^3$ *passing through s points and intersecting r lines in general position such that* $r+2s=3d+2$.

The above theorem is joint work with R. K. Singh and R. Mukherjee. We presented this work in chapter 2.

1.6 Counting curves in a Linear system

Enumerative Geometry deals with the enumeration of solutions when the polynomials come from various geometric situations and the intersection theory gives techniques to accomplish the enumeration. The classical study of various enumerative problems dates back to the eighteenth century. Hilbert's fifteenth problem lead the path from classical Schubert calculus to modern enumerative geometry. Numerous problem in this subject has been extensively studied by algebraic geometers. However, the modern development of enumerative geometry is strongly influenced by ideas and amusing predictions from physics. This subject has brought several branches of mathematics together and its interaction with other areas has been overwhelming over the past decades. In my doctoral thesis, my main focus is on various counting problems in complex projective surface $\mathbb{P}^2$.

In the last three chapters of my doctoral thesis, our main object of study is the geometry of singular curves. The nature of the constraint that we have studied in this direction are as follows :

- First, we deal with counting singular curves with certain contact (tangency) constraints to a fixed-line E in $\mathbb{P}^2$.
- Secondly, we study the enumeration of degree d curves in $\mathbb{P}^2$ with a certain type of singularities passing through an appropriate number of general points.

We note that our method is a topological method and this can be applied to a very general setup, which is a part of our future research.

A plane curve can be described by a homogeneous degree d polynomial in three variables x, y and z. It is of the form

$$F_d := \sum_{i+j+k=d} c_{ijk}\, x^i y^j z^k$$

where $c_{ijk} \in \mathbb{C}$. The set of all such polynomials forms a complex vector space of dimension $\binom{d+2}{2}$. Two such non-zero polynomials determine the same curve if and only if the polynomials are non-zero multiples of each other. Hence, therefore, the set of all such curves in $\mathbb{P}^2$ having degree d can be identified with the projective space of dimension $\delta_d := \binom{d+2}{2} - 1 = \frac{d(d+3)}{2}$.

Definition 1.6.1. *A n dimensional linear system is defined to be a family of sub-varieties* $\{ H_t \}_{t\in\mathbb{P}^r}$, *where each* H_t *is a degree d hypersurface in* $\mathbb{P}^r$, *i.e.,*

$$\{H_t = Z(\ t_0 S_0 + \cdots + t_r S_r\ )\ \}_{t\in\mathbb{P}^r}$$

for $S_0, \cdots, S_r \in H^0(\mathbb{P}^r,\ \mathscr{O}(d))$.

For example, a linear system of dimension 1 is the simplest family of varieties called the pencil. We will consider the linear system $\mathbb{P}(H^0(\mathbb{P}^r,\ \mathscr{O}(d)))$ through out the last three chapters of my thesis. We will denote

$$\mathscr{D} := \mathbb{P}(H^0(\mathbb{P}^r,\ \mathscr{O}(d))) \approx \mathbb{P}^{\delta_d}, \text{ where } \delta_d = \frac{d(d+3)}{2}.$$

Let us introduce a couple of definitions:

Definition 1.6.2. *Let* $f : \mathbb{P}^2 \longrightarrow \mathscr{O}(d)$ *be a holomorphic section. A point* $t \in f^{-1}(0)$ *is defined to be the singularity type* A_k *,* D_k *if there exists a local coordinate system* $(x,y) : (U,t) \longrightarrow (\mathbb{C}^2, 0)$ *such that* $f^{-1}(0) \cap U$ *is given by*

$$A_k : y^2 + x^{k+1} = 0. \qquad k \geq 0, \qquad D_k : y^2 x + x^{k-1} = 0 \qquad k \geq 4$$

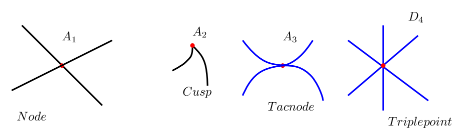

Figure 1.3: Local pictures of some singularities.

Usually in literature, t is a *smooth* point of $f^{-1}(0)$ if it is a singularity of type A_0, a *simple node* (or just node) if it is A_1 type singularity, a *cusp* if it is of the type A_2 and a *tacnode* if it is of the type A_3. In more common terminology, "a singularity of codimension k" refers to the number of independent conditions having that singularity imposes on the space of curves. More precisely, it is the expected codimension of the equisingular strata. Hence, an A_k singularity is a singularity of codimension k.

As we mentioned earlier, the question (1.1.1) becomes much more difficult when we ask the curve to have some non-degenerate singularity or it has some contact order to some divisor in the ambient algebraic variety. There has been a great deal of study related to curves with tangency, for example, there are 3264 plane conics that are tangent to all the given five conics in the plane. Steiner's original answer to this problem, 7776, was incorrect. The intersection consists of double lines, conics whose equation is the square of a linear equation. The first correct answer is due to de Jonquieres [cf. [31], p.469] in 1859.

At the outset of past decades, studying nodal curves is considered a classically important topic. Counting nodal curves is well understood by now. On the contrary, enumeration problems with singularities that are more degenerate than nodes are less explored and are an active area of recent research.

Now we are ready to state a general type of question in this setting:

Question 1.6.3. *Let $E \subset \mathbb{P}^2$ be a fixed smooth divisor. Let $(\delta_{F_1}, \cdots, \delta_{F_l})$ be a l-tuple, $(\delta_{E_1}, \cdots, \delta_{E_m})$ be a m-tuple and $(k_1, \cdots, k_n)$ be a n-tuple of non-negative integers. Let there be a total of $l+m+n$ points in $\mathbb{P}^2$ in general position out of which l number of points are outside E and $m+n$ number of points are in E. Then we will define the number $N_d(\mathfrak{X}_1^{\delta_{F_1}} \ldots \mathfrak{X}_l^{\delta_{F_l}} \mathfrak{X}_1^{\delta_{E_1}} \cdots \mathfrak{X}_m^{\delta_{E_m}} \mathsf{T}_{k_1} \ldots \mathsf{T}_{k_n})$, the number of degree d curves in $\mathbb{P}^2$, that passes through appropriate number of generic points, having δ_{F_i} number of singularities of type $\mathfrak{X}_i$ at l number of points outside E, which are denoted by $\mathfrak{X}_i^{\delta_{F_i}}$, δ_{E_i} number of singularities of type $\mathfrak{X}_i$ at m number points on the divisor E, which are denoted by $\mathfrak{X}_i^{\delta_{E_i}}$ and the curve is tangent to E of order k_i at n number of smooth points in E. What is $N_d(\mathfrak{X}_1^{\delta_{F_1}} \ldots \mathfrak{X}_l^{\delta_{F_l}} \mathfrak{X}_1^{\delta_{E_1}} \cdots \mathfrak{X}_m^{\delta_{E_m}} \mathsf{T}_{k_1} \ldots \mathsf{T}_{k_n})$?*

When all the singularities are *nodes* and the divisor E is a line, the above question can be summarized by the picture below.

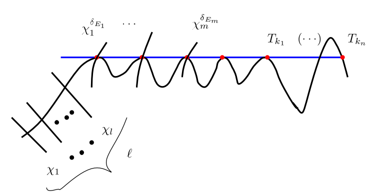

Figure 1.4: General question involving nodes and tangencies.

The above question in such generality is *open* till the date. However, in the literature, this question in some special situation classically understood which brought several branches of mathematics together. We will start by mentioning the recent motivating result by ([25],[65],[46]) which explains that there exists a universal polynomial in terms of Chern classes that count the numbers for the above problem. Very recently in [41], the author proved a generalization of the famous Göttsche Conjecture [65] for a relative effective divisor C on a smooth projective family of surfaces. Finally, they apply their method to calculate node polynomials for plane curves intersecting general lines in three-dimensional projective space.

Let us consider that $m, n = 0$ for the question (6.2.6), i.e, the number of a singular point lying on the divisor E and the number of points of tangency are zero then (6.2.6) reduces to the question of counting curves with various singularities which is an *open problem*. When all the singularities are nodes, this question has been extensively studied by algebraic geometers from various perspectives. In this direction, some beautiful results can be found in ([40],[8], [30]), etc. Next, when the singularities are more degenerate than nodes, there are only a few results available in the literature such results include amongst all ([15], [27], [29], [74]).

Main results of the papers ([2], [1], [3]) has some partial solution to the question (6.2.6) namely, for two singular points where the first singularity is a node and the second singularity is any singularity of codimension k i.e., the authors have proved recursive formulas for the numbers $N_d(A_1^{\delta} \circ \mathfrak{X})$ such that $k+\delta \leq 8$. Next, jointly with my advisor Ritwik Mukherjee and R. Singh, we are studying the problem of enumerating two singular points when both singularities can be any singularity such that the total codimension is 9. This work is in progress.

Next, when $m=0$, and the singularities are all nodes then the question (6.2.6) is completely solved by Caporaso-Harris ([11]). As recently as 2020, in [16], the authors found a recursive formula for the number of rational curves maximally tangent to a given divisor using the WDVV equation. In their recent paper in 2019 ([48]), D. McDuff and K. Siegel use methods from Symplectic Geometry to count rational curves with maximal tangencies to a divisor in a Symplectic Manifold.

Unfortunately, there is almost no progress when the singularities are more degenerate than nodes. There are a few results available in the literature when the singularity is a cusp, we will refer the reader to ([15], [77]). When l, m, and n all are non-zero and the singularities are nodes then there are some partial results scattered in the literature, and there is almost no result available for singularities more degenerate than cusp. Finally, we will describe our work which enables us to understand the question (6.2.6) for *higher singularities*.

Enumeration of singular curves with tangencies

We are now ready to state our main results from the last three chapters of this thesis.

Theorem 1.6.4. *Let k be a non-negative integer and $\delta_1, \delta_2, \ldots, \delta_k$ a collection of positive integers. Define*

$$\delta_d := \frac{d(d+3)}{2} \qquad \text{and} \qquad w_d := \delta_d - (1+\delta_1+2\delta_2+\ldots+k\delta_k).$$

Let $N_d^{\mathrm{T}}(A_1^{\delta_1}\ldots A_k^{\delta_k})$ denote the number of degree d-curves in $\mathbb{P}^2$, passing through w_d generic points, having δ_i (ordered) singularities of type A_i (for all i from 1 to k) that is tangent to a given line. Then we have an explicit recursive formula to compute $N_d^{\mathrm{T}}(A_1^{\delta_1}\ldots A_k^{\delta_k})$.

We presented this result in chapter 3 of this thesis. As a sequel of this result, in chapter 4 we have studied the following:

Theorem 1.6.5. *Let $E \in \mathbb{P}^2$ be a fixed line. Let us consider* $\mathbf{K} = (k_1, \cdots, k_n)$and $\mathbf{m} = (m_1, \cdots, m_n)$ *be two n tuples consisting of non-negative integers and $N_d^E(\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n},\ \mathbf{m})$ denotes the number of degree d curves in $\mathbb{P}^2$ tangent to E at n distinct points in E of order k_i for all $i = 1, \cdots, n$ and these tangency points are at the intersection of* **m** *generic lines, passing through $\delta_d - \mid \mathbf{K} \mid - \mid \mathbf{m} \mid$ generic points. Then we have established a recursive formula for $N_d^E(\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n},\ \mathbf{m})$ provided $d \geq |\mathbf{K}|$.*

The above theorem does not include any singularity on the curve, deals with only tangency constraints. Next, we will enumerate curves imposing certain types of singularities on them.

Theorem 1.6.6. *Let $k,\ m,\ \delta$ be three non-negative integers. For $0 \leq \delta \leq 2$, $N_d^E(A_1^\delta \circ T_k,\ m)$ denotes the number of degree d curves having δ nodes that are tangent to E of order k passing through $\delta_d - \delta - k - m$ generic points and the point of tangency is at the intersection of m generic lines. We obtain an explicit recursive formulas for $N_d^E(A_1^\delta \circ T_k,\ m)$, provided $d \geq \delta + k + 1$.*

Remark 1.6.7. *Note that while extending the above result for $\delta \geq 3$, we have the natural obstacle due to the occurrence of the triple point along with some branched condition to the line. We can only compute $N_d^E(A_1^3 \circ \mathsf{T}_2)$. In this case, there will be no branched condition for the triple point due to dimensional constraint.*

Theorem 1.6.8. *For two non-negative integers k and m, let $N_d^E(A_2 \circ T_k,\ m)$ be the number of degree d curves having a cusp that is tangent to E of order k passing through $\delta_d - k - 2 - m$ generic points and the point of tangency is at the intersection of m generic lines. Then we established an explicit recursive formula to compute $N_d^E(A_2 \circ T_k,\ m)$, provided $d \geq k + 3$.*

Remark 1.6.9. *An important consequence of the above study involving tangencies enables us to compute degree d tacnodal curves in $\mathbb{P}^2$. Enumeration of tacnodal curves using tangency may be considered as the most natural way amongst the techniques available in the literature.*

Finally, we want to count curves with two degenerate singularities, more degenerate than nodes. In the last chapter of my thesis, we have presented some of the proofs of an ongoing project that deals with the following question:

Question 1.6.10. *Let $\mathcal{O}(d) \longrightarrow \mathbb{P}^2$ be a holomorphic line bundle. Let us denote by $\mathcal{D} := \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(d))$ the space of all non-zero holomorphic sections up to scaling. What is $N(\mathfrak{X}_1 \circ \mathfrak{X}_2)$, the number of plane degree d curves, that belong to the linear system $\mathcal{D}$, passing through $\delta_d - (cd_{\mathfrak{X}_1} + cd_{\mathfrak{X}_2})$ points in general position and having two singularities of the type $\mathfrak{X}_1$, $\mathfrak{X}_2$ whose codimensions are $cd_{\mathfrak{X}_1}$, $cd_{\mathfrak{X}_2}$ respectively?*

We will be considering the total codimension of the singularities upto 6, i.e., $cd_{\mathfrak{X}_1}, + cd_{\mathfrak{X}_2} \leq 6$. However, we will only discuss the enumeration of $N(A_{k_1} \circ A_{k_2})$ such that $k_1 + k_2 \leq 6$.

Chapter 2

Enumeration of rational curves in a moving family of $\mathbb{P}^2$

2.1 Introduction

One of the most fundamental and studied problems in enumerative geometry is the following: what is N_d^δ, the number of degree d curves in $\mathbb{P}^2$ that have δ distinct nodes and pass through $\frac{d(d+3)}{2} - \delta$ generic points? The question was studied more than a hundred years ago by Zeuthen ([72]) and has been studied extensively in the last thirty years by Ran ([56], [57]), Vainsencher ([67]), Caporaso-Harris ([11]), Kazarian ([27]), Kleiman and Piene ([30]), Florian Block ([8]), Tzeng and Li ([65], [46]), Kool, Shende and Thomas ([40]) and Berczi ([7]) and amongst others, Fomin and Mikhalin ([17]). This question has been investigated from several perspectives and is very well understood.

The problem motivates a natural generalization considered by Kleiman and Piene in [33], where they study the enumerative geometry of singular curves in a moving family of surfaces. More recently, this question has been studied further by T.Laarakker in [41], where he obtains a formula for the following number: how many degree d curves are there in $\mathbb{P}^3$ whose image lies in a $\mathbb{P}^2$, that pass through $\frac{d(d+3)}{2} + 3 - \delta$ generic lines and have δ-nodes (provided $d \geq \delta$). This can be viewed as a family version of the classical problem of computing N_d^δ.

Motivated by the papers of Kleiman and Piene ([33]) and T.Laarakker ([41]), we have studied the parallel question of counting stable rational maps into a family of moving target spaces. This can be viewed as a family version of the famous question of enumerating rational curves in $\mathbb{P}^2$, that was studied by Kontsevich-Manin ([38]) and Ruan-Tian ([59]). The main result of this chapter is as follows:

Main Result 2.1.1. *Let $N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s)$ be the number of genus zero, degree d curves in $\mathbb{P}^3$, whose image lies in a $\mathbb{P}^2$, intersecting r generic lines and s generic points (where $r+2s = 3d+2$). We have a recursive formula to compute $N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s)$ for all $d \geq 2$.*

Remark 2.1.2. *Note that for $d = 1$, the corresponding question is classical Schubert calculus and there $r+2s = 4$ as opposed to 5.*

Remark 2.1.3. *We note that when $s = 3$ and $d \geq 2$, the number $N_d^{\mathbb{P}^3,\mathrm{Planar}}(3d-4,3)$ is the number of rational curves in $\mathbb{P}^2$ through $3d-1$ points; this is because 3 generic points in $\mathbb{P}^3$ determine a unique $\mathbb{P}^2$. We also note that when $s > 3$, $N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s)$ is zero, since 4 or more generic points do not lie in a plane.*

We have written a program to implement our formula; the program is available on our web page

https://sites.google.com/view/paulanantadulal/home.

In section2.4, we verify that the numbers we compute are logically consistent with those obtained by T. Laarakker in [41] till $d = 6$. This gives strong evidence to support the conjecture that his formulas for δ-nodal planar degree d curves in $\mathbb{P}^3$ are expected to be enumerative when $d \geq 1+[\frac{\delta}{2}]$ (as opposed to $d \geq \delta$ which is proved in [41]). Starting from $d = 7$, we can not use the result [41] to make any consistency check, since the corresponding nodal polynomial is not expected to be enumerative (due to the presence of double lines); this is explained in section2.4.

2.2 Notation

Let us define a **planar** curve in $\mathbb{P}^3$ to be a curve, whose image lies inside a $\mathbb{P}^2$. We will now develop some notation to describe the space of planar curves of a given degree d.

Let us denote the dual of $\mathbb{P}^3$ by $\hat{\mathbb{P}}^3$; this is the space of $\mathbb{P}^2$'s inside $\mathbb{P}^3$. An element of $\hat{\mathbb{P}}^3$ can be thought of as a non zero linear functional $\eta : \mathbb{C}^4 \longrightarrow \mathbb{C}$ upto scaling (i.e., it is the projectivization of the dual of $\mathbb{C}^4$). Given such an η, we define the projectivization of its zero set as $\mathbb{P}^2_\eta$. In other words,

$$\mathbb{P}^2_\eta := \mathbb{P}(\eta^{-1}(0)).$$

Note that this $\mathbb{P}^2_\eta$ is a subset of $\mathbb{P}^3$. Note that the space $\overline{M}_{0,0}(\mathbb{P}^n,d)$ has been constructed explicitly and it is shown to be a projective normal algebraic variety [18]. Since $\mathbb{P}^2_\eta \subset \mathbb{P}^3$ as a closed sub scheme hence, $\overline{M}_{0,0}(\mathbb{P}^2_\eta,d)$ has the induced course moduli structure from $\overline{M}_{0,0}(\mathbb{P}^3,d)$. Next, when $d \geq 2$, we define the moduli space of planar degree d curves into $\mathbb{P}^3$ as a fibre bundle over $\hat{\mathbb{P}}^3$. More precisely, we define

$$\pi : \overline{M}_{0,k}^{\mathrm{Planar}}(\mathbb{P}^3,d) \longrightarrow \hat{\mathbb{P}}^3$$

to be the fiber bundle, such that

$$\pi^{-1}([\eta]) := \overline{M}_{0,k}(\mathbb{P}^2_{\eta}, d).$$

Here we are using the standard notation to denote $M_{0,k}(X,\beta)$ to be the moduli space of genus zero stable maps, representing the class $\beta \in H_2(X,\mathbb{Z})$ and $\overline{M}_{0,k}(X,\beta)$ to be its stable map compactification. Since the dimension of a fiber bundle is the dimension of the base, plus the dimension of the fiber, we conclude that the dimension of $\overline{M}_{0,k}^{\text{Planar}}(\mathbb{P}^3,d)$ is $3d+2+k$.

Next, we note that the space of planes in $\mathbb{P}^3$ can also be thought of as the Grassmannian $\mathbb{G}(3,4)$. Let $\gamma_{3,4}$ denote the tautological three plane bundle over the Grassmannian. Since $\mathbb{G}(3,4)$ can be identified with $\hat{\mathbb{P}}^3$, we can think of $\gamma_{3,4}$ as a bundle over $\hat{\mathbb{P}}^3$.

When $d=1$, we define $\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,1)$ to be

$$\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,1) := \mathbb{P}(\gamma_{3,4}^*) \longrightarrow \hat{\mathbb{P}}^3.$$

We note that an element of $\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,1)$ is of the form (L,H), where L is a line in $\mathbb{P}^3$ and H is a plane containing L. Since a line is not contained in a unique plane, $\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,1)$ is not the same as the space of lines; infact we note that the space of lines is 4 dimensional, while the dimension of $\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,1)$ is 5.

We will now define a few numbers by intersecting cycles on $\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,d)$, the moduli space with zero marked points (this includes the case $d=1$; unless otherwise stated we always include the case $d=1$ in any of our statements). Let $\mathscr{H}_L$ and $\mathscr{H}_p$ denote the classes of the cycles in $\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,d)$ that corresponds to the subspace of curves passing through a generic line and a point respectively. We also denote a to be the standard generator of $H^*(\hat{\mathbb{P}}^3;\mathbb{Z})$. Let us now define

$$N_d^{\mathbb{P}^3,\text{Planar}}(r,s,\theta) := \langle a^{\theta},\ \overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,d) \cap \mathscr{H}_L^r \cap \mathscr{H}_p^s \rangle. \tag{2.1}$$

We formally define the the left hand side of (2.1) to be zero if $r+2s+\theta \neq 3d+2$ (since the right hand side of (2.1) doesn't make sense unless $r+2s+\theta = 3d+2$). Note that when $\theta=0$, $r+2s=3d+2$ and $d\geq 2$, $N_d^{\mathbb{P}^3,\text{Planar}}(r,s,0)$ is precisely equal to the number of rational planar degree d-curves in $\mathbb{P}^3$ intersecting r generic lines and s generic points.

For the notational convenience of the reader we will explain the behaviour of the natural parameter θ. Let $[x_1,x_2,x_3,x_4]$ denotes an arbitrary point in $\mathbb{P}^3$ then the linear equation $a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$ describes a plane inside $\mathbb{P}^3$, and $[a_1,a_2,a_3,a_4]$ denotes the corresponding point in $\hat{\mathbb{P}}^3$ (known as the dual of projective space $\mathbb{P}^3$), i.e., each plane in $\mathbb{P}^3$ corresponds to a point in $\hat{\mathbb{P}}^3$. In other words, the points of $\hat{\mathbb{P}}^3$ parametrize the space of planes inside $\mathbb{P}^3$. Thus

- For $\theta = 0$ our point $[a_1, a_2, a_3, a_4]$ is free to move inside $\hat{\mathbb{P}}^3$, i.e., when our parameter space is the whole $\hat{\mathbb{P}}^3$.

- For $\theta = 1$, suppose $[y_1, y_2, y_3, y_4]$ is a fixed point in $\hat{\mathbb{P}}^3$. Then the linear equation $a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 = 0$ determines a fixed plane inside $\hat{\mathbb{P}}^3$ (of course this also corresponds to a point in $\mathbb{P}^3$ and that is nothing but the point $[y_1, y_2, y_3, y_4]$ itself.) This tells us that our parameter space is reduced to some fixed plane (say $\hat{\mathbb{P}}^2$) contained inside $\hat{\mathbb{P}}^3$.

- For $\theta = 2$, we consider two generic planes (which are defined simultaneously by the two linear equations $a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 = 0$ and $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$) inside $\hat{\mathbb{P}}^3$ which intersects along a fixed line (say $\hat{\mathbb{P}}^1$). Hence the parameter space is now reduced to a line inside $\hat{\mathbb{P}}^3$.

- Now we can guess what $\theta = 3$ ought to be, we consider three generic planes inside $\hat{\mathbb{P}}^3$ which intersects at a fixed point and this corresponds to a fixed plane (say $\mathbb{P}^2$) inside $\mathbb{P}^3$. This corresponds to the old question, what is the number of degree d rational curves inside $\mathbb{P}^2$ passing through $3d - 1$ points.

- For $\theta \geq 3$, the parameter space becomes empty.

Next, we will define a number $B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta)$ by intersecting it on the product of two moduli spaces as follows:

$$\begin{aligned} B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta) \quad &:= \Big\langle \pi_1^*(\mathscr{H}_L^{r_1}\mathscr{H}_p^{s_1}) \cdot \pi_2^*(\mathscr{H}_L^{r_2}\mathscr{H}_p^{s_2})(\pi_2^* a)^{\theta} \cdot (\pi^*\Delta), \\ &\qquad [\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d_1) \times \overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d_2)] \Big\rangle. \end{aligned} \tag{2.2}$$

Here Δ denotes the class of the diagonal in $\hat{\mathbb{P}}^3 \times \hat{\mathbb{P}}^3$ and π_1 and π_2 are the obvious projection maps. Again, we formally define the left hand side of (2.2) to be zero, unless $r_1 + 2s_1 + r_2 + 2s_2 + \theta = 3d_1 + 3d_2 + 4$ (since in that case, the right hand side doesn't make sense).

2.3 Recursive Formula and its Proof

We are now ready to state our recursion formula

Lemma 2.3.1. *If $d=1$, then the number $N_d^{\mathbb{P}^3,\text{Planar}}(r,s,\theta)$ is given by*

$$N_1^{\mathbb{P}^3,\text{Planar}}(r,s,\theta)=\begin{cases} 0 & \text{if } (r,s,\theta)=(1,2,0),\\ 0 & \text{if } (r,s,\theta)=(3,1,0),\\ 0 & \text{if } (r,s,\theta)=(5,0,0),\\ 1 & \text{if } (r,s,\theta)=(0,2,1),\\ 1 & \text{if } (r,s,\theta)=(2,1,1),\\ 2 & \text{if } (r,s,\theta)=(4,0,1),\\ 1 & \text{if } (r,s,\theta)=(1,1,2),\\ 2 & \text{if } (r,s,\theta)=(3,0,2),\\ 0 & \text{if } (r,s,\theta)=(0,1,3),\\ 1 & \text{if } (r,s,\theta)=(2,0,3),\\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Lemma 2.3.2. *If $d=2$, then the number $N_d^{\mathbb{P}^3,\text{Planar}}(r,s,\theta)$ is given by*

$$N_2^{\mathbb{P}^3,\text{Planar}}(r,s,\theta)=\begin{cases} 92 & \text{if } (r,s,\theta)=(8,0,0),\\ 18 & \text{if } (r,s,\theta)=(6,1,0),\\ 4 & \text{if } (r,s,\theta)=(4,2,0),\\ 1 & \text{if } (r,s,\theta)=(2,3,0),\\ 34 & \text{if } (r,s,\theta)=(7,0,1),\\ 6 & \text{if } (r,s,\theta)=(5,1,1),\\ 1 & \text{if } (r,s,\theta)=(3,2,1),\\ 0 & \text{if } (r,s,\theta)=(1,3,1),\\ 8 & \text{if } (r,s,\theta)=(6,0,2),\\ 1 & \text{if } (r,s,\theta)=(4,1,2),\\ 0 & \text{if } (r,s,\theta)=(2,2,2),\\ 0 & \text{if } (r,s,\theta)=(0,3,2),\\ 1 & \text{if } (r,s,\theta)=(5,0,3),\\ 0 & \text{if } (r,s,\theta)=(3,1,3),\\ 0 & \text{if } (r,s,\theta)=(1,2,3),\\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Different approaches to some low degree computation

Note that any *planar* conic in $\mathbb{P}^3$ always lies inside a fixed plane. We will now describe the classical Schubert calculus method and use it to calculate *planar* conic and genus zero *planar* cubic in $\mathbb{P}^3$. We will mention a few other related questions which will help us to verify our answers. This low degree computation plays a motivational role in the early stage of this project. Suppose we want to count rational *planar* cubic in $\mathbb{P}^3$ intersecting 8 generic lines in $\mathbb{P}^3$. We note that the space of generic *planar* cubic has genus *one*. Thus we have to enumerate nodal cubic in $\mathbb{P}^3$ intersecting 8 generic lines in $\mathbb{P}^3$. Let us begin by the computation of the following question:

Question 2.3.3. *How many conics are there in $\mathbb{P}^3$ intersecting m generic lines and passing through n generic points in $\mathbb{P}^3$ such that $m+2n=8$?*

Let us now describe a few facts before approaching the solution of the above problem. Let us define

$$S := \{([\eta],\, q) \in \widehat{\mathbb{P}}^3 \times \mathbb{P}^3 : \eta(q) = 0\}.$$

We observe that S is a fiber bundle over $\widehat{\mathbb{P}}^3$ with the fiber $\mathbb{P}^2$. Set theoretically, it is a plane in $\mathbb{P}^3$ and a point p that lies on that plane. Let us now consider the following section of the line bundle induced by the evaluation map, i.e.,

$$\mathrm{ev} : \widehat{\mathbb{P}}^3 \times \mathbb{P}^3 \longrightarrow \hat{\gamma}^*_{\widehat{\mathbb{P}}^3} \otimes \gamma^*_{\mathbb{P}^3}, \qquad \text{given by} \qquad \{\mathrm{ev}([\eta],\, [q])\}(\eta \otimes q) := \eta(q),$$

where $\hat{\gamma}^*_{\widehat{\mathbb{P}}^3}$ and $\gamma^*_{\mathbb{P}^3}$ are dual of the tautological line bundles over $\widehat{\mathbb{P}}^3$ and $\mathbb{P}^3$ respectively (or equivalently $\mathscr{O}_{\widehat{\mathbb{P}}^3}(1)$ and $\mathscr{O}_{\mathbb{P}^3}(1)$ respectively). Thus we gather

$$S = \mathrm{ev}^{-1}(0). \tag{2.5}$$

Next, we consider the fibre bundle $E_d \longrightarrow \widehat{\mathbb{P}}^3$ over $\widehat{\mathbb{P}}^3$, such that the fibre over each $[\eta] \in \widehat{\mathbb{P}}^3$ is the space of degree d curves in $\mathbb{P}^2_\eta$. Next, we note that $\widehat{\mathbb{P}}^3$ is naturally isomorphic to $\mathbb{G}(3,4)$ via the annihilator map. Let us denote $\gamma_{\mathbb{G}(3,4)} \longrightarrow \mathbb{G}(3,4)$ to be the tautological three-plane bundle over the Grassmannian. Hence, via the above isomorphism, we gather

$$E_d \approx \mathbb{P}(\mathrm{Sym}^d \gamma^*_{\mathbb{G}(3,4)}) \longrightarrow \hat{\mathbb{P}}^3.$$

Hence, E_d is a fibre bundle over $\widehat{\mathbb{P}}^3$, whose fibers are isomorphic to $\mathbb{P}^{\frac{d(d+3)}{2}}$. A typical element of E_d will be denoted by $([f],\, [\eta])$; this implies that f is a homogeneous polynomial of degree d defined on the plane $\mathbb{P}^2_\eta$.

The cohomology ring structure of projective bundles

We now recall some basic facts about the cohomology ring of the above projective fiber bundle. Recall that via the annihilator map, $\widehat{\mathbb{P}}^3$ is isomorphic to $\mathbb{G}(3,4)$. With the help of this isomorphism, we can realize a (which is actually an element of $H^*(\widehat{\mathbb{P}}^3)$) as an element of $H^*(\mathbb{G}(3,4))$. We observe that

$$c(\gamma^*_{\mathbb{G}(3,4)}) = 1 + a + a^2 + a^3.$$

Next, using the splitting principle, we conclude that

$$c(\mathrm{Sym}^d \gamma^*_{\mathbb{G}(3,4)}) = 1 + c_1 a + c_2 a^2 + c_3 a^3, \qquad \text{where} \tag{2.6}$$

$$c_1 := \frac{d(d+1)(d+2)}{6}, \qquad c_2 := \frac{d(d+1)(d+2)(d+3)(d^2+2)}{6} \qquad \text{and}$$
$$c_3 := \frac{d(d+1)(d+2)(d+3)(d^2+2)(d^3+3d^2+12d+12)}{1296}. \tag{2.7}$$

Observe that $E_d = \mathbb{P}(\mathrm{Sym}^d \gamma^*_{\mathbb{G}(3,4)})$, is a $\mathbb{P}^{n-1}$ bundle, where

$$n := 1 + \frac{d(d+3)}{2}. \tag{2.8}$$

Hence, we conclude (by the Leray Hirsch Theorem) that the ring structure of E_d is given by

$$H^*(E_d) \approx \frac{\mathbb{Z}[a,\lambda]}{\langle a^4,\ \lambda^n + c_1 a\lambda^{n-1} + c_2 a^2 \lambda^{n-2} + c_3 a^3 \lambda^{n-3}\rangle}, \tag{2.9}$$

where $\gamma_{E_d} \longrightarrow \mathbb{P}(\mathrm{Sym}^d \gamma^*_{\mathbb{G}(3,4)})$ denotes the tautological line bundle and $\lambda := c_1(\gamma^*_{E_d})$.

Next, we will define a function $\phi(n,m)$ as

$$\phi(n,m) = \begin{cases} \langle \sigma_1^n \sigma_2^m,\ \mathbb{G}(2,4)\rangle, & \text{if } n+2m = 4 \\ 0 & \text{if } n+2m \neq 4 \text{ or } n<0 \text{ or } m<0. \end{cases}$$

where σ_1, σ_2 and $\sigma_{1,1}$ are the *Schubert cycles* and $\mathbb{G}(2,4)$ denotes the set of lines in $\mathbb{P}^3$.

We can tabulate the cohomology intersections of the above *Schubert cycles* as follows

.	1	σ_1	σ_2	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$
1	1	σ_1	σ_2	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$
σ_1	σ_1	$\sigma_2 + \sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0
σ_2	σ_2	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0
$\sigma_{1,1}$	$\sigma_{1,1}$	$\sigma_{2,1}$	0	$\sigma_{2,2}$	0	0
$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0	0
$\sigma_{2,2}$	$\sigma_{2,2}$	0	0	0	0	0

Table 2.1: Intersection table of the cohomology ring $G(2,4)$.

Proof. Let us consider the space of lines in $\mathbb{P}^3$ as the grassmanian $\mathbb{G}(2,4) \subset \mathbb{C}^4$. We realize that counting *planar* conics is same as counting a pair (η, C) where η is a 3 dimensional linear subspace of $\mathbb{C}^4$, i.e., $\eta : \mathbb{C}^4 \longrightarrow \mathbb{C}$ is a linear map. Let us consider the tautological 3 plane bundle over $\mathbb{G}(3,4)$ as earlier

$$\gamma_{\mathbb{G}(3,4)} := \{(\eta, v) \in \mathbb{G}(3,4) \times \mathbb{C}^4 : v \in \eta\}.$$

Then a conic C can be thought of as an element of $\mathbb{P}(sym^2(\gamma^*_{\mathbb{G}(3,4)}))$. Thus the dimension of the space of *planar* conics is $dim\ \mathbb{G}(3,4) + dim\ Sym^2\mathbb{C}^3 - 1 = 8$.

Let us denote σ_1, σ_2 and $\sigma_{1,1}$ are cohomology generator of $\mathbb{G}(2,4)$ (we refer the reader to [26]). It is proved in [73], every *planar* conic passing through a point in $\mathbb{P}^3$ in general position denoted by $\mathscr{M}_p$ and the homology class representing this class given by $\lambda\ a$ similarly a *planar* conic intersecting a generic line in $\mathbb{P}^3$ is represented by $\mathscr{M}_\ell = \lambda + 2a$, where $c_1(\gamma^*_{E_2}) = \lambda$, as defined earlier.

One can identify $\mathbb{G}(3,4)$ with $\widehat{\mathbb{P}}^3$ via annihilator of the map η. Thus we can identify the space of conics through a generic point as the zero set of the following bundle:

$$\phi_p : \mathbb{P}(Sym^2(\gamma^*_{\mathbb{G}(3,4)})) \longrightarrow \gamma^*_{E_2}$$

defined as

$$\{\phi_p([C])\}(C) = C(p)$$

Hence the homology class represented by the subspace M_p, space of conics passing through the point p is given by the Poincaré dual of $\lambda\ a$.

Next, we will describe the space of *planar* conic intersecting a generic line in $\mathbb{P}^3$, as the zero set of the line bundle

$$\widetilde{\phi} : \gamma^*_{E_2} \otimes \pi^*(\Lambda\gamma_{\mathbb{G}(3,4)})^{*\ 2} \longrightarrow \mathbb{P}(Sym^2(\gamma^*_{\mathbb{G}(3,4)}))$$

such that the homology class represented by the subspace of *planar* curves intersecting a line is given by the Poincare dual to

$$\lambda + 2a$$

where $\pi : \mathbb{P}(Sym^2(\gamma^*_{\mathbb{G}(3,4)})) \longrightarrow \mathbb{G}(3,4)$.

Now, for conics intersecting m general lines and passing through n generic points can be interpreted as follows:

$$\Big\langle\ \mathscr{M}_\ell^m\ \mathscr{M}_p^n\ a^\theta,\ \ [\overline{E_2}]\ \Big\rangle$$

provided this intersection is transverse intersection. Hence collecting all coefficients of $a^3\lambda^5$, we will get the numbers presented in the table (2.3.2). □

In a similar fashion, we can solve the following question as well

Question 2.3.4. *How many pairs involving line and conic are there in $\mathbb{P}^3$, where both the conic and the line lies inside a fix plane (i.e., in $\mathbb{P}^2_\beta$) such that the conic intersects k_2 lines and the line intersects k_1 lines with $k_1 + k_2 = 10$?*

Proof. Let us recall the notations introduced above. We can describe a pair consisting line and conic lies in a fixed plane as

$$\{([\ell],\ [C],\ [\eta]) \in \mathbb{G}(2,4) \times E_2 \ \mid\ (\ell,\ C) \text{ lies inside a fixed plane}\}$$

From the earlier discussion, a conic intersecting a line represented by the cycle $\mathscr{M}_\ell = \lambda + 2\,a$. Next, the space of lines represented by the class $\sigma_1 \in H^*(\mathbb{G}(2,4))$. Hence the number of the pair line and conic satisfying the above condition is

$$\langle e(\gamma^*_{\widehat{\mathbb{P}}^3} \otimes \gamma^*_{\mathbb{G}(2,4)})\, \mathscr{M}_\ell^{k_2}\, \sigma_1^{k_1},\ \mathbb{G}(2,4) \times E_2\rangle \ =\ \langle (a^2 + a\sigma_1 + \sigma_{1,1})\, \sigma_1^{k_1}\, (\lambda + 2a)^{k_2},\ \mathbb{G}(2,4) \times E_2\rangle$$

Therefore, using the ring structure and the above table we will get our required numbers. □

Next, we want to compute the number N_3^{pl}. In particular we want to study the following question:

Question 2.3.5. *How many nodal planar cubic are there through s number of points in $\mathbb{P}^3$ in general position and intersecting r generic lines in $\mathbb{P}^3$ such that $r + 2s = 11$?*

Proof. We will continue with the setup and notation described in the previous question. As before counting *planar* nodal cubic is the same as counting a pair (η, C) where η is the same as before and any *planar* cubic having a node in it. Now we can thought of *planar* cubic as an element of $E_3 \approx \mathbb{P}(sym^3(\gamma^*_{3,4}))$. Recall that E_3 is a $\mathbb{P}^9$ bundle over $\widehat{\mathbb{P}}^3$, hence the cohomology ring of E_3 can be computed from the equation (2.9). Let us consider

$$\mathscr{D}^{pl} := \{([f],\ [\eta],\ p) \in E_d \times \mathbb{P}^3 \mid \eta(p) = 0\}$$

This can be thought of as fibre product of the fibre bundles S and E_d over $\widehat{\mathbb{P}}^3$, i.e., the following commutative diagram

$$\begin{array}{ccc} \mathscr{D}^{pl} & \rightarrow & E_d \\ \downarrow \pi^* & & \downarrow \pi \\ S & \rightarrow & \widehat{\mathbb{P}}^3 \end{array}$$

Following analogous arguments as before, we note that the homology class corresponding to the condition that a *planar* cubic passing through a generic point in $\mathbb{P}^3$ can be represented by the class $[\mathscr{H}_p] = \lambda a$. Similarly, a *planar* cubic intersecting a generic line in $\mathbb{P}^3$ is represented by the class $[\mathscr{M}_\ell] = \lambda + 3a$, where $\mathscr{H}_p$, $\mathscr{H}_\ell$ are the classes corresponding to intersecting lines in $\mathbb{P}^3$ and passing through a generic point in $\mathbb{P}^3$. Let the cohomology class H is the hyperplane class of $\mathbb{P}^3$. Note that we have the following short exact sequence:

$$0 \longrightarrow W := \mathrm{Ker}\nabla\eta \longrightarrow T\mathbb{P}^3 \longrightarrow \gamma_{\mathbb{P}^3} \otimes \widehat{\gamma} \longrightarrow 0 \tag{2.10}$$

Hence the first and second Chern class of W can be obtained from the property $c(W)\ c(\gamma^*_{\mathbb{P}^3}\otimes\widehat{\gamma}^*) = c(T\mathbb{P}^3)$. Thus via splitting principle, we get $c_1(W) = 3a - H$, and the $c_2(W) = (3a-H)^2 - 3a(3a-H) + 3a^2$.

Therefore, the number of nodal *planar* curves in $\mathbb{P}^3$ passing through s points and intersecting r lines is the cardinality of the following set

$$\{([f],\ [\eta],\ p) \in \mathscr{D}^{pl} \mid f(p) = 0,\ \nabla f_p = 0, \eta \in S\} \cap \mu$$

where the class $[\mu] = \mathscr{H}_\ell^r \mathscr{H}_p^s \lambda^\theta$. Next, we note that the nodal condition can be seen as the section of following bundle:

$$\begin{aligned} \Psi^{pl}_{A_0} : \mathscr{D}^{pl} &\longrightarrow \mathscr{L}^{pl}_{A_0} := \gamma^*_{E_d} \otimes \gamma^{*\,3}_{\mathbb{P}^3} \\ \text{defined by}\{\Psi^{pl}_{A_0}([f],\ [\eta],\ p)\}(f) &= f(p) \\ \Psi^{pl}_{A_1} : \Psi^{pl\ -1}_{A_0} &\longrightarrow \mathscr{L}^{pl}_{A_1} := \gamma^*_{E_d} \otimes T^*W \otimes \gamma^{*\,3}_{\mathbb{P}^3} \\ \text{defined by}\{\Psi^{pl}_{A_1}([f],\ [\eta],\ p)\}(f) &= \nabla f_{|_p} \end{aligned}$$

Note that we can prove the transversality of the above two sections in a similar manner as we have obtained transversality of some sections in the last three chapters in this thesis. Hence, the number of a *planar* nodal curve can be interpreted as

$$\left\langle e(\mathscr{L}^{pl}_{A_0})\ \ e(\mathscr{L}^{pl}_{A_1})\ \ \mathscr{H}_L^r\ \ \mathscr{H}_P^s\ \ \lambda^n,\ \ [\mathscr{D}^{pl}] \right\rangle \tag{2.11}$$

Thus, collecting the coefficient of $a^3H^3\lambda^{n-1}$ we will get the following numbers: We observed that the

(d,r,s)	$(3,11,0)$	$(3,9,1)$	$(3,7,2)$
$N_3(A_1^{pl},\ r,\ s)$	12960	1392	34

Table 2.2: Number of planar cubics

numbers $N_3(A_1^{pl},\ r,\ s)$ are equal to the numbers $N_d^{\mathbb{P}^3,\ Planar}(r,s)$ for $d = 3$, calculated from the recursive formula (2.3.6). □

We are now ready to state our main theorem.

Theorem 2.3.6. *If $d \geq 3$, then*

$$N_d^{\mathbb{P}^3,\text{Planar}}(r,s,\theta) = \begin{cases} 0 & \textit{if}\ \ r+2s+\theta \neq 3d+2, \\ 0 & \textit{if}\ \ s > 3, \\ 0 & \textit{if}\ \ \theta > 3. \end{cases} \tag{2.12}$$

In the remaining case when $r+2s+\theta = 3d+2$, $s \leq 3$ and $\theta \leq 3$, we have

$$
\begin{aligned}
N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s,\theta) = {} & 2dN_d^{\mathbb{P}^3,\mathrm{Planar}}(r-2,s+1,\theta) \\
& + \sum_{r_1=0}^{r-3}\sum_{s_1=0}^{s}\sum_{d_1=1}^{d-1}\binom{r-3}{r_1}\binom{s}{s_1}d_1^2 d_2 \times \\
& \Big(d_2 B_{d_1,d_2}(r_1+1,s_1,r_2-2,s_2,\theta) - d_1 B_{d_1,d_2}(r_1,s_1,r_2-1,s_2,\theta)\Big),
\end{aligned}
\tag{2.13}
$$

where in the above expression, $d_2 := d-d_1$, $r_2 := r-r_1$ and $s_2 := s-s_1$. Furthermore, $\forall d_1,d_2 \geq 1$, we have

$$
B_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta) = \sum_{i=0}^{3} N_{d_1}^{\mathbb{P}^3,\mathrm{Planar}}(r_1,s_1,i) \times N_{d_2}^{\mathbb{P}^3,\mathrm{Planar}}(r_2,s_2,\theta+3-i). \tag{2.14}
$$

Remark 2.3.7. *We note that equations* (2.3)*,* (2.4)*,* (2.12)*,* (2.13) *and* (2.14) *allow us to compute $N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s,\theta)$ for all $d \geq 1$ and all $r,s,\theta \geq 0$.*

Proof of Theorem 2.3.6: We will start by proving equation (2.12). The first equation is true simply because we are pairing a cohomology class with a homology class of different dimensions (see the remark after equation (2.1)). Next, when $s > 3$, we note that there can not be any planar curves, because 4 generic points do not lie on a plane. Finally, we note that $a^\theta = 0 \in H^*(\hat{\mathbb{P}}^3;\mathbb{Z})$ if $\theta > 3$, which proves the last equality.

We now justify the main thing, which is equation (2.13). The idea is very similar to the idea used to compute the number of rational degree d curves in $\mathbb{P}^2$ (and also $\mathbb{P}^3$) that is given in [38], [47] and [18]. As in the case of counting curves in $\mathbb{P}^2$, let us first consider $\overline{M}_{0,4}$. This space is isomorphic to $\mathbb{P}^1$; hence we have the equivalence of the following divisors

$$
(x_1x_2|x_3x_4) \equiv (x_1x_3|x_2x_4). \tag{2.15}
$$

Here $(x_1x_2|x_3x_4)$ denotes the domain which is a wedge of two spheres with the marked points x_1 and x_2 on the first sphere and x_3 and x_4 on the second sphere. The domain $(x_1x_3|x_2x_4)$ is defined similarly. Since $\overline{M}_{0,4} \approx \mathbb{P}^1$ is path connected, any two points determine the same divisor.

Let us now consider the projection map

$$
\pi : \overline{M}_{0,4}^{\mathrm{Planar}}(\mathbb{P}^3,d) \longrightarrow \overline{M}_{0,4}.
$$

We define a cycle $\mathscr{Z}$ in $\overline{M}_{0,4}^{\mathrm{Planar}}(\mathbb{P}^3,d)$, given by

$$
\mathscr{Z} := \mathrm{ev}_1^*([H]) \cdot \mathrm{ev}_2^*([H]) \cdot \mathrm{ev}_3^*([L]) \cdot \mathrm{ev}_4^*([L]) \cdot \mathscr{H}_L^{r-3} \cdot \mathscr{H}_p^s \cdot a^\theta,
$$

where $[H]$ and $[L]$ denote the class of a hyperplane and a line in $\mathbb{P}^3$, ev_i denotes the evaluation map at the i^{th} marked point and a denotes the generator of $H^*(\hat{\mathbb{P}}^3;\mathbb{Z})$.

Let us now intersect the cycle $\mathscr{Z}$ by pulling back the left-hand side of (2.15), via π.

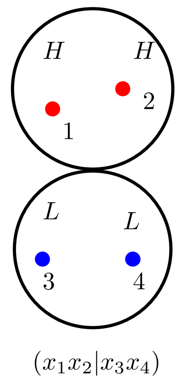

Figure 2.1: Left hand side of the recursion

We now note that

$$\pi^*(x_1x_2|x_3x_3)\cdot\mathscr{Z} = N_d^{\mathbb{P}^3,\text{Planar}}(r,s,\theta) + \sum_{r_1=0}^{r-3}\sum_{s_1=0}^{s}\sum_{d_1=1}^{d-1}\binom{r-3}{r_1}\binom{s}{s_1}d_1^3d_2\times B_{d_1,d_2}(r_1,s_1,r_2-1,s_2,\theta), \tag{2.16}$$

where in the above expression, $d_2 := d-d_1$, $r_2 := r-r_1$ and $s_2 := s-s_1$.

Next, we will compute the right hand side analogously.

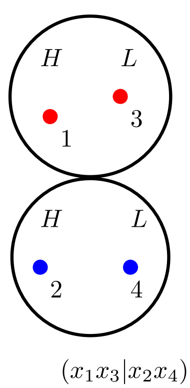

Figure 2.2: Right hand side of the recursion

Note that, using similar arguments gives

$$\pi^*(x_1x_3|x_2x_4)\cdot\mathscr{Z} = 2dN_d^{\mathbb{P}^3,\mathrm{Planar}}(r-2,s+1,\theta) + \sum_{r_1=0}^{r-3}\sum_{s_1=0}^{s}\sum_{d_1=1}^{d-1}\binom{r-3}{r_1}\binom{s}{s_1}d_1^2d_2^2\times B_{d_1,d_2}(r_1+1,s_1,r_2-2,s_2,\theta), \tag{2.17}$$

where as before $d_2 := d-d_1$, $r_2 := r-r_1$ and $s_2 := s-s_1$.

We now note that equations (2.15), (2.16) and (2.17), imply our desired recursive formula (2.13).

Next, we will justify the formula for $B_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta)$ (equation (2.14)). This follows immediately from the fact that the class of the diagonal is given by

$$\Delta_{\hat{\mathbb{P}}^3\times\hat{\mathbb{P}}^3} = \pi_1^*a^3+\pi_1^*a^2\cdot\pi_2^*a+\pi_1^*a\cdot\pi_2^*a^2+\pi_2^*a^3,$$

where a denotes the generator of $H^*(\hat{\mathbb{P}}^3;\mathbb{Z})$ and π_1,π_2 denote the respective projection maps. The formula now follows immediately from the definition of

$$B_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta)$$

(namely, equation (2.2)).

It remains to prove the two base cases of the recursion, namely Lemma 2.3 and Lemma 2.4.

Proof of Lemma 2.3.1: We recall that $\overline{M}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3,1)$ is defined to be the projective bundle $\mathbb{P}(\gamma^*_{\mathbb{G}(3,4)})\longrightarrow\hat{\mathbb{P}}^3$. Now, we note that the Chern classes of the rank three vector bundle $\gamma^*_{\mathbb{G}(3,4)}\longrightarrow\hat{\mathbb{P}}^3$ are given by

$$c_i(\gamma^*_{\mathbb{G}(3,4)}) = a^i\in H^{2i}(\hat{\mathbb{P}}^3;\mathbb{Z}).$$

The reason is explained in [73, Page 18]. Here a is the standard generator of $H^*(\hat{\mathbb{P}}^3;\mathbb{Z})$.

Next, we note that $a^i=0$ if $i>3$. Hence, the cohomology ring of $H^*(\mathbb{P}(\gamma^*_{\mathbb{G}(3,4)}))$ is given by

$$H^*(\mathbb{P}(\gamma^*_{\mathbb{G}(3,4)}))\approx\frac{\mathbb{Z}[a,\lambda]}{\langle\lambda^3+\lambda^2a+\lambda a^2+a^3\rangle} \tag{2.18}$$

where $\tilde{\gamma}\longrightarrow\mathbb{P}(\gamma^*_{\mathbb{G}(3,4)})$ is the tautological line bundle over the projectivized bundle $\mathbb{P}(\gamma^*_{\mathbb{G}(3,4)})$ and $\lambda := c_1(\tilde{\gamma}^*)\in H^*(\mathbb{P}(\gamma^*_{\mathbb{G}(3,4)}))$. This follows from [9, Page 270].

We now note the following two important facts: intersecting a generic line, corresponds to the cycle

$$\mathscr{H}_l = \lambda + a.$$

Furthermore, passing through a generic point, corresponds to the cycle

$$\mathscr{H}_p = \lambda a.$$

The reason for this can again be found in [73, Pages 18 and 19]. Hence, to compute $N_1^{\mathbb{P}^3,\text{Planar}}(r,s,\theta)$ we have to compute the expression

$$(\lambda+a)^r(\lambda a)^s a^\theta,$$

use the relationship

$$\lambda^3 = -(\lambda^2 a + \lambda a^2 + a^3)$$

and extract the coefficient of $\lambda^2 a^3$. This gives us all the numbers for various values of r, s and θ.

Proof of Lemma 2.3.2: First we note that every conic in $\mathbb{P}^3$ lies inside a unique plane (except a double line). Hence, let us consider the projective bundle

$$\mathbb{P}(\mathrm{Sym}^2(\gamma^*_{\mathbb{G}(3,4)})) \longrightarrow \hat{\mathbb{P}}^3.$$

This space $\mathbb{P}(\mathrm{Sym}^2(\gamma^*_{\mathbb{G}(3,4)}))$ is the space of conics in $\mathbb{P}^3$ and a plane that contains the conic. The space of all smooth conics form an open dense subspace of $\mathbb{P}(\mathrm{Sym}^2(\gamma^*_{\mathbb{G}(3,4)}))$. Hence, to compute the numbers $N_2^{\mathbb{P}^3,\text{Planar}}(r,s,\theta)$ (which is defined as an intersection number on $\overline{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,2)$), we can instead compute the relevant intersection number on $\mathbb{P}(\mathrm{Sym}^2(\gamma^*_{\mathbb{G}(3,4)}))$.

Next, we note that $\mathbb{P}(\mathrm{Sym}^2(\gamma^*_{\mathbb{G}(3,4)}))$ is a $\mathbb{P}^5$ bundle over $\hat{\mathbb{P}}^3$. The cohomology ring structure of the total space is given by

$$H^*(\mathbb{P}(\mathrm{Sym}^2(\gamma^*_{\mathbb{G}(3,4)}))) \approx \frac{\mathbb{Z}[a,\lambda]}{\langle \lambda^6 + 4\lambda^5 a + 10\lambda^4 a^2 + 20\lambda^3 a^3 \rangle}. \tag{2.19}$$

This follows from the *splitting principle* (see page 275 in [9]). We now note the following two important facts: intersecting a generic line, corresponds to the cycle

$$\mathscr{Z}_l = \lambda + 2a.$$

Furthermore, passing through a generic point, corresponds to the cycle

$$\mathscr{Z}_p = \lambda a.$$

The reason for this can again be found in [73, Pages 18 and 19]. Hence, to compute $N_2^{\mathbb{P}^3,\text{Planar}}(r,s,\theta)$ we have to compute the expression

$$(\lambda+2a)^r(\lambda a)^s a^\theta,$$

use the relationship given by the cohomology ring structure in (2.19), i.e.

$$\lambda^6 = -(4\lambda^5 a + 10\lambda^4 a^2 + 20\lambda^3 a^3)$$

and extract the coefficient of $\lambda^5 a^3$. This gives us all the numbers for various values of r, s and θ.

2.4 Low degree checks

We now describe concrete low degree checks that we have performed. Using our recursive formula, we have obtained the following number: Next, let $N_d^{\mathrm{Node},\delta}(r,s)$ denote the number of planar degree d

(d,r,s)	$(3,11,0)$	$(4,14,0)$	$(5,17,0)$	$(6,20,0)$
$N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s)$	12960	3727920	1979329280	1763519463360

Table 2.3: Planar curves having nodal singularity.

curves in $\mathbb{P}^3$ with δ (unordered) nodes intersecting r generic lines and s generic points. These numbers are computed in [41]. Using that, we get the following table. Finally, let us denote by $\mathrm{Red}_d^{\mathrm{Node},\delta}(r,s)$

(d,r,s,δ)	$(3,11,0,1)$	$(4,14,0,3)$	$(5,17,0,6)$	$(6,20,0,10)$
$N_d^{\mathrm{Node},\delta}(r,s)$	12960	4057340	2487128120	2681467886460

Table 2.4: Genus 0 Planar curves.

to be the number of reducible planar degree d curves in $\mathbb{P}^3$ with δ (unordered) nodes intersecting r generic lines and s generic points. This number can be computed using [41, Proposition 8.4]. Using that, we now note that in all the cases we have tabulated,

(d,r,s,δ)	$(3,11,0,1)$	$(4,14,0,3)$	$(5,17,0,6)$	$(6,20,0,10)$
$\mathrm{Red}_d^{\mathrm{Node},\delta}(r,s)$	0	329420	507798840	917948423100

Table 2.5: Number of reducible planar degree d curves in $\mathbb{P}^3$.

$$N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s) = N_d^{\mathrm{Node},\frac{(d-1)(d-2)}{2}}(r,s) - \mathrm{Red}_d^{\mathrm{Node},\frac{(d-1)(d-2)}{2}}(r,s).$$

This is positive evidence for the fact that T. Laarakker's formula for $N_d^{\mathrm{Node},\delta}(r,s)$ is actually enumerative when $d \geq 1+[\frac{\delta}{2}]$ (as opposed to $d \geq \delta$, which is proved in [41]). We also note that when $d=7$ and $\delta = \frac{(d-1)(d-2)}{2} = 15$, the formula for $N_d^{\mathrm{Node},\frac{(d-1)(d-2)}{2}}(r,s)$ is not expected to be enumerative because of an obvious geometric reason. To see why, suppose $s=0$ and $r=23$. Through the required $r=23$ lines, we can place a double line through 4 lines and through the remaining 19 lines we can place a quintic that intersects the line so that the double line and the quintic lie in a plane. This is a degenerate configuration, and hence $N_7^{\mathrm{Node},15}(23,0)$ is not expected to be enumerative.

This is analogous to the case of counting δ-nodal degree d curves in $\mathbb{P}^2$; let N_d^{δ} denote that number. A formula for this number can be explicitly found in [8] for instance. On the other hand, let $N_d^{\mathbb{P}^2}$ denote the number of rational degree d curves in $\mathbb{P}^2$ through $3d-1$ generic points. Till $d=6$, we can verify that $N_d^{\mathbb{P}^2}$ is logically consistent with the corresponding value of $N_d^{\frac{(d-1)(d-2)}{2}}$ (after subtracting the number of irreducible curves). From $d=7$, we can not make any such consistency check, because N_7^{15} is not enumerative; there are double lines that can pass through two of the 20 points and a quintic through the remaining 18 points. We also note that this fact is consistent with the Göttsche threshold of when the number N_d^{δ} is supposed to be enumerative; this is proved in [34]. Our computations give evidence to show that a similar bound is likely to be true for the case of planar curves in $\mathbb{P}^3$.

2.5 Application via WDVV

Let us recall that the Gromov-Witten potential for $\mathbb{P}^3$ is given by $F(\bar{x})$. Let us denote the quantum component of $F(\bar{x})$ is given by $G(x_0,x_1,x_2)=\sum N_{r,s}\, e^{dx_0}\,\frac{x_1^r}{r!}\,\frac{x_2^s}{s!}$, such that $r+2s=4d$, provided $d\geq 1$. This immediately produce the recursion formula for $N_{r,s}$, where number $N_{r,s}$ denotes the number of rational curves in $\mathbb{P}^3$ passing through s generic points and intersecting r general lines in $\mathbb{P}^3$ such that $r+2s=4d$. Analogously we can show that the quantum associativity of the quantum product gives

$$2F_{123}-F_{222}=F_{111}F_{222}-F_{112}F_{122} \tag{2.20}$$

The function $G(\bar{x})$ satisfies (2.20). Hence it gives the recursion formula for enumeration of rational curves in $\mathbb{P}^3$. Thus we have seen earlier for $\mathbb{P}^2$ and $\mathbb{P}^3$ if the potential function satisfies certain differential equation namely WDVV equation then the computation of recursive formula is a mater of computing partial derivatives. Broadly speaking the WDVV equation is equivalent to the recursion formula for computation of rational curves in $\mathbb{P}^2$ as well as in $\mathbb{P}^3$.

Let us continue with the definition of S as earlier. Then by Leray-Hirsch theorem we have $H^*(S;\mathbb{Z})=\frac{\mathbb{Z}[a,\lambda]}{\langle a^4=0,\ a^3+a^2\lambda+a\lambda^2+\lambda^3=0\rangle}$. Note that we have a projection map $\pi_s: S\longrightarrow \mathbb{P}^3$ that will induce a map

$$\pi s^*: H^*(\mathbb{P}^3)\longrightarrow H^*(S),\ \text{can be specified as}$$

$$H\longrightarrow\lambda,\quad H^2\longrightarrow\lambda^2,\quad H^3\longrightarrow a\,\lambda^2.$$

Let us denote the cohomology classes of the ring $H^*(S)$ are $T_{i,j}:=a^i\lambda^j,\ \forall\ 0\leq i\leq 3,\ 0\leq j\leq 2$ (T_{ij} for all i,j are basically the restriction of the cohomology classes from $\mathbb{P}^3$). Let $\gamma_{n+1},\cdots,\gamma_m\in$

$\{a^i \mid 0 \leq i \leq 3\}$. Then we can define the numbers defined in (2.1) as

$$I_{0,n,d} = \int_{[\overline{M}_{0,n}^{\text{Planar}}(\mathbb{P}^3,d)]} \prod_{k=1,\ j\neq 0}^{n} ev_k^*(T_{ij}) \prod_{k=n+1}^{m} ev_1^*(\gamma_k) \tag{2.21}$$

provided $\sum_{k=1}^{n} codim(T_{ij}) + \sum_{k=n+1}^{m} codim(\gamma_k) = 3d+2+n$.

Let us now consider the potential function for genus zero *planar* curves in $\mathbb{P}^3$ as follows:

$$\Phi^{pl}(t_{ij}) = \sum_{d=1}^{\infty} \sum_{r+2s+\theta=3d+2} N_d(r,s,\theta) \ \frac{t_{02}^r}{r!} \frac{t_{12}^s}{s!} \ \left(exp^{\ t_{01}}\right) \left(e^{d\sum_{0\leq i\leq 3} t_{i1}}\right). \tag{2.22}$$

Then we observed that the generating function $\Phi^{pl}(t_{ij})$ above satisfies the following differential equation

$$\Phi^{pl}_{t_{01}t_{01}t_{01}} \Phi^{pl}_{t_{31}t_{02}t_{02}} + \Phi^{pl}_{t_{02}t_{02}t_{02}} = \Phi^{pl}_{t_{01}t_{02}t_{01}} \Phi^{pl}_{t_{31}t_{01}t_{02}} + 2\,\Phi^{pl}_{t_{01}t_{02}t_{12}} \tag{2.23}$$

where $\Phi^{pl}_{t_{ij}}$ denotes the partial derivative of Φ^{pl} with respect to the variable t_{ij}. Note that the above equation implies the recursion that we have obtained in (2.3.6) by looking at the coefficient of $\frac{t_{02}^r}{(r-3)!} \frac{t_{12}^s}{s!} \ \left(exp^{\ t_{01}}\right) \ \left(e^{d\sum_{0\leq i\leq 3} t_{i1}}\right)$ from the equation (2.23).

However, we have not come across the analogous generating series in terms of Gromov-Witten class for *planar* curves yet. This observation is extremely encouraging for further development of any enumerative question which is a fiber bundle analog of some classical problem. We have not yet systematically defined "*planar* quantum product" hence, we are not able to say at this point the recursive formula (3.1) which we have obtained, is a consequence of some "*planar* associativity" or not. We want to analyze all these analogous questions to the classical questions in near future. We also want to study the fiber bundle analog of Dubrovin formalism [14] in the future.

Atiyah-Bott localization for *planar* curves :

Atiyah-Bott localization technique has been played a prominent role in the developments of the Gromov-Witten theory during past decades. We now want to study the result obtained [51], using the localization technique. We will mainly follow the technique introduced by Kontsevich [39] and Graber and Pandharipande [23]. We have been able to compute only some numbers from the list of *planar* lines (2.3). We will pursue this study in near future.

Chapter 3

Enumeration of curves with singularity and tangency

3.1 Introduction

A fundamental problem in enumerative geometry is to count curves with prescribed singularities. This question has been studied for a very long time starting with Zeuthen ([72]) more than a hundred years ago. It has been studied extensively in the last thirty years from various perspectives by numerous mathematicians including amongst others, Z. Ran ([56], [58]), I. Vainsencher ([67]), L. Caporaso and J. Harris ([11]), M. Kazarian ([27]), S. Kleiman and R. Piene ([30]), D. Kerner ([28] and [29]), F. Block ([8]), Y. J. Tzeng and J. Li ([65], [46]), M. Kool, V. Shende and R. Thomas ([40]), S. Fomin and G. Mikhalkin ([17]), G. Berczi ([7]) and S. Basu and R. Mukherjee ([2], [1] and [3]).

A closely related question is to enumerate curves with prescribed singularities that are tangent to a given line. This question also has a long history that can be traced back to Zeuthen. As early as 1848, Zeuthen computed the characteristic number of rational quartics in $\mathbb{P}^2$ tangent to a given line.

In the last thirty years an extensive amount of work has been done in enumerating curves that are tangent to a given line when the *prescribed singularities are nodes*. These include among others the results of Ran ([58]), I.Vainsencher([67]), Caporasso-Harris ([11]), R.Vakil([69], [70]), A. Gathmann ([20] and [21]) and C. Cadman and L. Chen ([10]).

Very recently, using methods of algebraic cobordism, Y. J. Tzeng has shown ([25]) that a universal formula exists for the characteristic number of curves in a linear system, that are tangent to a given line and that have prescribed singularities (more degenerate than nodes).

We now mention a result that we are aware of concerned with the tangency question in other spaces. In [12], Y. Cooper and R. Pandharipande study the Severi problem involving single tangency condition via the matrix elements in Fock space.

There is also an extensive body of work in the context of counting stable maps that are tangent to a given divisor (i.e. counting curves of a fixed genus tangent to a given divisor). This is done by carrying out the computation in the moduli space of stable maps. Some of the work in this field include among others the results of Z. Ran ([58]), I. Vainsencher([67]), R. Vakil([69], [70]), A. Gathmann ([20] and [21]) and C. Cadman and L. Chen ([10]), where the authors count rational or elliptic curves tangent to a given divisor. This question was also studied using the WDVV equation by N. Takahashi ([63]) in 2002 and it is still an active area of research interest; as recently as 2019, H. Fan and L. Wu found a recursive formula for the number of rational curves maximally tangent to a given divisor ([16]) using the WDVV equation. This problem is also of great interest in Symplectic Geometry. In their recent paper in 2019 ([48]), D. McDuff and K. Siegel use methods from Symplectic Geometry to count rational curves with maximal tangencies to a divisor in a Symplectic Manifold.

With so such work already done in the area of counting curves with tangencies, one might wonder what is there left to ask? Well, the question of counting curves with more degenerate singularities is a much more difficult question. For instance, no attempt has been made to extend the Caporaso-Harris formula to curves having cusps. The only result we are aware of for higher singularities is the result of L. Ernström and G. Kennedy ([15]), which is in the setting of stable maps. In that paper, the authors solve the question of enumerating genus-zero cuspidal curves in $\mathbb{P}^2$, that is tangent to a given line (in fact multiple lines). We are not aware of any further progress in extending those results to higher singularities.

With this background and motivation, we now state the main result of our paper. We will be studying curves in a linear system (in fact degree d curves in $\mathbb{P}^2$). Our main result is a recursive formula for the characteristic number of curves that are tangent to a given line and that have any number of prescribed singularities (of type A_k).

In this chapter, we will present our work on the enumerative problem of counting curves with tangency constraints. In this work, I have obtained a recursive formula for the characteristic number of curves that are tangent to a given line and that have prescribed singularities (of type A_k). Furthermore, till codimension eight we can obtain explicit formulas. The method we use is the method of dynamical intersection theory, similar to what is used in [74], [2], [1] and [3].

Remark 3.1.1. *Note that in the above study, we have considered any degenerate singularities; namely A_k type singularities (in particular it is more degenerate than nodes).*

Before stating the main result of the paper, let us make a couple of definitions:

Definition 3.1.2. *Let $f : \mathbb{P}^2 \longrightarrow \mathscr{O}(d)$ be a holomorphic section. A point $q \in f^{-1}(0)$ is of singularity type A_k if there exists a coordinate system $(x,y) : (U,q) \longrightarrow (\mathbb{C}^2, 0)$ such that $f^{-1}(0) \cap U$ is given by*

$$y^2 + x^{k+1} = 0.$$

In more common terminology, q is a *smooth* point of $f^{-1}(0)$ if it is a singularity of type A_0, a *simple node* (or just node) if its singularity type is A_1, a *cusp* if its type is A_2 and a *tacnode* if its type is A_3. We will frequently use the phrase "a singularity of codimension k". This refers to the number of independent conditions having that singularity imposes on the space of curves. More precisely, it is the expected codimension of the equisingular strata. Hence, an A_k singularity is a singularity of codimension k.

Next, given a non negative integer k and positive integers $\delta_1, \ldots, \delta_k$, let us define $N_d(A_1^{\delta_1} \ldots A_k^{\delta_k})$ to be the number of degree d-curves in $\mathbb{P}^2$, passing through $\frac{d(d+3)}{2} - (\delta_1 + 2\delta_2 + \ldots + k\delta_k)$ generic points having δ_i ordered singularities of type A_i.

Similarly, we define $N_d(A_1^{\delta_1} \ldots A_k^{\delta_k}; L_{A_i})$ to be the number of degree d-curves in $\mathbb{P}^2$, passing through $\frac{d(d+3)}{2} - (1 + \delta_1 + 2\delta_2 + \ldots + k\delta_k)$ generic points, having δ_j ordered singularities of type A_j (when $i \neq j$), $\delta_i - 1$ singularities of type A_i and another singularity of type A_i lying on a given line.

The main result of this paper is as follows:

Main Theorem 3.1.3. *Let k be a non negative integer and $\delta_1, \delta_2, \ldots, \delta_k$ a collection of positive integers. Define*

$$\delta_d := \frac{d(d+3)}{2} \qquad \text{and} \qquad w_d := \delta_d - (1 + \delta_1 + 2\delta_2 + \ldots + k\delta_k).$$

Let $N_d^{\mathrm{T}}(A_1^{\delta_1} \ldots A_k^{\delta_k})$ denote the number of degree d-curves in $\mathbb{P}^2$, passing through w_d generic points, having δ_i (ordered) singularities of type A_i (for all i from 1 to k) that is tangent to a given line. Then,

$$N_d^{\mathrm{T}}(A_1^{\delta_1} \ldots A_k^{\delta_k}) = 2(d-1)N_d(A_1^{\delta_1} \ldots A_k^{\delta_k}) - \sum_{i=1}^{i=k} \delta_i (i+1) N_d(A_1^{\delta_1} \ldots A_k^{\delta_k}; L_{A_i}), \tag{3.1}$$

for all $d \geq d_{\min}$, where

$$d_{\min} := k + (2\delta_1 + \delta_2 + \ldots + \delta_k).$$

Remark 3.1.4. *We note that the numbers $N_d(A_1^{\delta} A_k)$ are directly given in the papers of S. Basu and R. Mukherjee ([2], [1] and [3]) when $\delta + k \leq 8$. The results of those papers can be used to compute $N_d(A_1^{\delta} A_k; L_{A_i})$ when $\delta + k \leq 8$ with no further effort (since they obtain an equality on the level of* *cycles). Hence, using these numbers and using equation (3.1) we can obtain a complete formula for* $N_d^{\mathrm{T}}(A_1^{\delta}A_k)$ *when* $\delta + k \leq 8$. *The formulas for* $N_d^{\mathrm{T}}(A_k)_{1\leq k\leq 8}$, $N_d^{\mathrm{T}}(A_1A_k)_{1\leq k\leq 7}$ *and* $N_d^{\mathrm{T}}(A_1^{\delta})_{1\leq \delta\leq 8}$ *are listed explicitly in section 3.5.*

Remark 3.1.5. *Next, we note that in [27], M. Kazarian computes all the characteristic number of curves with upto seven singularities. We believe he obtains an equality on the level of cycles; hence we believe in principle his method can be used to compute the characteristic number of curves with singularities, where one of the singularity is required to lie on a line (till codimension seven). Hence, using equation (3.1), we can in principle obtain a formula for* $N_d^{\mathrm{T}}(A_1^{\delta_1}A_2^{\delta_2}\ldots A_k^{\delta_k})$ *provided the total codimension is seven.*

Remark 3.1.6. *When* $k=0$, *we will abbreviate* $N_d(A_1^{\delta_1}\ldots A_k^{\delta_k})$ *as* N_d *and we will abbreviate* $N_d^{\mathrm{T}}(A_1^{\delta_1}\ldots A_k^{\delta_k})$ *as* N_d^{T}. *We note that* N_d *is the number of degree* d *curves in* $\mathbb{P}^2$ *passing through* δ_d *generic points; hence* $N_d = 1$. *Similarly,* N_d^{T} *is the number of degree* d *curves in* $\mathbb{P}^2$ *passing through* $\delta_d - 1$ *generic points that is tangent to a given line. Hence, the* $k=0$ *case of equation* (3.1) *implies*

$$N_d^{\mathrm{T}} = 2(d-1).$$

Remark 3.1.7. *The bound* $d \geq d_{\min}$ *is imposed to ensure we get transversality of certain sections. However, this bound is not necessarily sharp; the bound is sufficient to get transversality, but it is not always necessary.*

Remark 3.1.8. *We are not completely certain about the fact whether Kazaryan's method can be used to compute the characteristic number of curves with singularities, where one of the singularity is required to lie on a line.*

3.2 Overview of the method

We use a topological method to compute the degenerate contribution to the Euler class, which is the main attraction of this work. Through out, we will heavily use one concrete fact from differential topology.

We now give an overview of the method we use. Our starting point will be the following classical fact from Differential Topology:

Theorem 3.2.1. *Let* $V \longrightarrow M$ *be an oriented vector bundle over a compact, oriented manifold* M *and* $s: M \longrightarrow V$ *a section that is transverse to zero. If the rank of* V *is equal to the dimension of* M*, then* *the signed cardinality of $s^{-1}(0)$ is the Euler class of V, evaluated on the fundamental class of M, i.e.,*

$$| \pm s^{-1}(0) | = \langle e(V), [M] \rangle$$

Remark 3.2.2. *Through out we will be working over the field of complex numbers. Thus in complex setting the signed cardinality of $s^{-1}(0)$ is nothing but the actual cardinality of the set.*

Remark 3.2.3. *We will express the tangency condition as the vanishing of a section of an appropriate vector bundle. However, the corresponding Euler class involves a degenerate contribution. The central aspect of this paper is how we compute the degenerate contribution to the Euler class. We use the method of "dynamic intersections" (cf. Chapter 11 in [19]) to compute the degenerate contribution to the Euler class.*

Let us now give a brief overview of how we will obtain the formula of the Main Theorem. As is typically the case in enumerative geometry, we will try to express our enumerative numbers as the zeros of a geometrically meaningful section of an appropriate vector bundle. We will consider the space of curves having the prescribed singularities; to keep the discussion simple let us for the moment assume there is exactly one prescribed singularity χ. Along with this we will also consider a fixed line L and a point p that lies on the line; this setup can be summarized by the following picture:

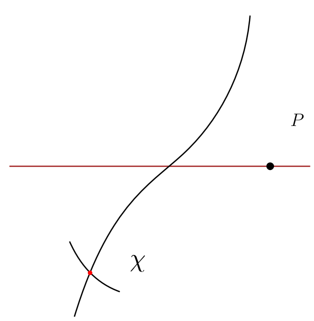

Figure 3.1: Point lies on the line.

We now impose the condition that the point p has to lie on the curve; furthermore the curve has to be tangent to L at p. This can be summarized by the following picture:
We will interpret these conditions we impose on p (namely the fact that it has to lie on the curve and be tangent to the line) as a section of an appropriate bundle. We will show that this section is transverse to zero. Hence, we expect that the Euler class of this bundle will give us the desired number of singular curves, tangent to a given line. This expectation is not true. This is because the section also vanishes

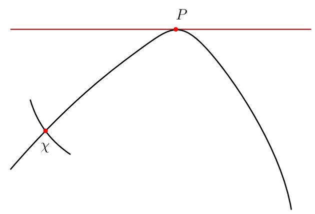

Figure 3.2: Point lies on the curve and is tangent to the line.

on a degenerate locus; namely when the singularity lies on the line and in fact becomes equal to p. This degenerate locus is summarized by the following picture:

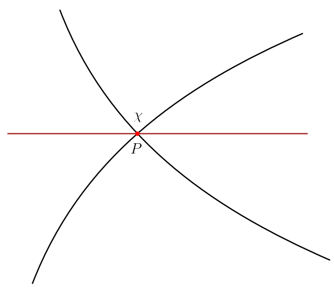

Figure 3.3: Degenerate Locus.

The central aspect of this chapter is how we compute the degenerate contribution to the Euler class. We use the method of "dynamic intersections" (cf. Chapter 11 in [19]) to compute the degenerate contribution to the Euler class. The details of this are carried out in this chapter.

3.3 Proof of Main Theorem

Let us denote $\mathscr{D}$ to be the space of non-zero homogeneous degree d-polynomials in three variables upto scaling, i.e.,

$$\mathscr{D} := \mathbb{P}(H^0(\mathbb{P}^2, \mathscr{O}(d))) \approx \mathbb{P}^{\delta_d}.$$

Hence, $\mathscr{D}$ can be identified with the space of degree d curves in $\mathbb{P}^2$ (not necessarily irreducible). Let

$$\gamma_{\mathscr{D}} \longrightarrow \mathscr{D} \qquad \text{and} \qquad \gamma_{\mathbb{P}^2} \longrightarrow \mathbb{P}^2$$

be the tautological line bundles over $\mathscr{D}$ and $\mathbb{P}^2$ respectively.

We will now prove our main theorem (i.e., we will prove (3.1)). Given non negative integer k

and positive integers $\delta_1,\ldots \delta_k$, let us define

$$M := \mathscr{D} \times (\mathbb{P}^2)^{\delta_1} \times \ldots \times (\mathbb{P}^2)^{\delta_k} \qquad \text{and}$$
$$\mathscr{S} := \{([f], q_1^1, \ldots q_{\delta_1}^1; \ldots; q_1^k, \ldots q_{\delta_k}^k) \in M : \ f \text{ has an } A_i \text{ singularity at } q_\eta^i, \ q_\eta^i \ \text{ are all distinct}\}. \tag{3.2}$$

We will show shortly that if $d \geq d_{\min}$, then $\mathscr{S}$ is a complex sub manifold of M, of dimension $w_d + 1$. Let us now make the following abbreviation:

$$\overline{q} := (q_1^1, \ldots q_{\delta_1}^1; \ldots; q_1^k, \ldots q_{\delta_k}^k) \in (\mathbb{P}^2)^{\delta_1} \times \ldots \times (\mathbb{P}^2)^{\delta_k}.$$

We now define the following two sections of line bundles over $\overline{\mathscr{S}} \times L$:

$$\psi_{\mathrm{ev}} : \overline{\mathscr{S}} \times L \longrightarrow \mathbb{L}_{\mathrm{ev}} := \gamma_{\mathscr{D}}^* \otimes \gamma_L^{*d}, \qquad \text{given by} \qquad \{\psi_{\mathrm{ev}}([f], \overline{q}, p)\}(f) := f(p) \qquad \text{and}$$
$$\psi_{\mathrm{T}} : \psi_{\mathrm{ev}}^{-1}(0) \longrightarrow \mathbb{L}_{\mathrm{T}} := \gamma_{\mathscr{D}}^* \otimes T^*L \otimes \gamma_L^{*d}, \qquad \text{given by} \qquad \{\psi_{\mathrm{T}}([f], \overline{q}, p)\}(f \otimes v) := \nabla f|_p(v).$$

Here γ_L denotes the tautological line bundle over L (which is the same as the restriction of the tautological line bundle $\gamma_{\mathbb{P}^2}$ to L).

Let us now define

$$\mathscr{B}_\eta^i := \{([f], \overline{q}, p) \in \overline{\mathscr{S}} \times L : q_\eta^i = p\} \qquad \text{and} \qquad \mathscr{B} := \bigcup \mathscr{B}_\eta^i.$$

We claim that restricted to $\mathscr{S} \times L - \mathscr{B}$, the sections ψ_{ev} and ψ_{T} are transverse to zero. We will prove that claim shortly.

Next, let μ be the subspace of curves in $\mathscr{D}$ that pass through w_d generic points and let

$$\pi_{\mathscr{D}} : M \times L \longrightarrow \mathscr{D}$$

be the projection map. Since the points are generic, the sub manifold $\pi_{\mathscr{D}}^{-1}(\mu)$ will intersect $\mathscr{S} \times L$ transversally (inside $M \times L$).

Remark 3.3.1. *Note that throughout this thesis, we will be using the following abuse of notation. If E is a bundle over M_1, we will consider that E is a bundle over $M_1 \times M_2$. The reason behind this is that we will be referring by E the pullback bundle $\pi_1^* \longrightarrow M_1 \times M_2$, where π_1 is the first projection map. Thus on a similar note, a cohomology class β in M_1, we will also say that β is a cohomology class in $M_1 \times M_2$; our intended meaning for the class being $\pi_1^* \beta$.*

Next, we note that if f is tangent to L at p, then

$$f(p)=0 \qquad \text{and} \qquad \nabla f|_p(v)=0 \qquad \forall\, v\in T_pL. \tag{3.3}$$

In other words,

$$\psi_{\mathrm{ev}}([f],\overline{q},p)=0 \qquad \text{and} \qquad \psi_{\mathrm{T}}([f],\overline{q},p)=0. \tag{3.4}$$

However, equation (3.4) is also satisfied on $\mathscr{B}$ (i.e., when one of the singular points q^i_η, happens to lie on the line L, i.e., one of the points q^i_η becomes equal to the tangency point p). Hence, our desired number $N^{\mathrm{T}}_d(A_1^{\delta_1}\ldots A_k^{\delta_k})$ is the number of solutions to

$$\psi_{\mathrm{ev}}([f],\overline{q},p)=0, \qquad \psi_{\mathrm{T}}([f],\overline{q},p)=0, \qquad ([f],\overline{q},p)\in\Big(\mathscr{S}\times L-\mathscr{B}\Big)\cap\Big(\pi_{\mathscr{D}}^{-1}\mu\Big). \tag{3.5}$$

We note that since the points are generic,

$$\Big(\mathscr{S}\times L-\mathscr{B}\Big)\cap\Big(\pi_{\mathscr{D}}^{-1}\mu\Big)=\Big(\overline{\mathscr{S}}\times L-\mathscr{B}\Big)\cap\Big(\pi_{\mathscr{D}}^{-1}\mu\Big)$$

Hence, we conclude that

$$\langle e(\mathbb{L}_{\mathrm{ev}})e(\mathbb{L}_{\mathrm{T}}),\, [\overline{\mathscr{S}}\times L]\cap[\pi_{\mathscr{D}}^{-1}\mu]\rangle = N^{\mathrm{T}}_d(A_1^{\delta_1}\ldots A_k^{\delta_k})+\mathscr{C}_{\mathscr{B}_\mu}, \tag{3.6}$$

where $\mathscr{C}_{\mathscr{B}_\mu}$ is the contribution of the section from the boundary $\mathscr{B}\cap\big(\pi_{\mathscr{D}}^{-1}\mu\big)$. We note that $\cap$ denotes intersection inside the space $M\times L$.

Next, we note that the left hand side of equation (3.6) is given by

$$\langle e(\mathbb{L}_{\mathrm{ev}})e(\mathbb{L}_{\mathrm{T}}),\, [\overline{\mathscr{S}}\times L]\cap[\pi_{\mathscr{D}}^{-1}\mu]\rangle = 2(d-1)N_d(A_1^{\delta_1}\ldots A_k^{\delta_k}). \tag{3.7}$$

We will now compute the quantity $\mathscr{C}_{\mathscr{B}_\mu}$. Let us first analyze the set $\mathscr{B}\cap\big(\pi_{\mathscr{D}}^{-1}\mu\big)$. This is the union of the sets

$$\mathscr{B}^i_\eta\cap\big(\pi_{\mathscr{D}}^{-1}\mu\big).$$

We now note that $\mathscr{B}^i_\eta\cap\big(\pi_{\mathscr{D}}^{-1}\mu\big)$ is the set of all degree d curves passing through the w_d generic points, having δ_j (ordered) singularities of type A_j (for all j from 1 to k) and where the $(q^i_\eta)^{\mathrm{th}}$ singular point lies on a line. Note that the $(q^i_\eta)^{\mathrm{th}}$ singular point corresponds to a singularity of type A_i. Hence,

$$|\mathscr{B}^i_\eta\cap\big(\pi_{\mathscr{D}}^{-1}\mu\big)| = N_d(A_1^{\delta_1}\ldots A_k^{\delta_k};L_{A_i}).$$

We claim that each point of $\mathscr{B}^i_\eta \cap \left(\pi_{\mathscr{D}}^{-1}\mu\right)$ vanishes with a multiplicity of $(i+1)$. Hence, the total contribution from the set $\mathscr{B} \cap \left(\pi_{\mathscr{D}}^{-1}\mu\right)$ to the Euler class is given by

$$\mathscr{C}_{\mathscr{B}_\mu} = \sum_{i=1}^{i=k} \delta_i (i+1) N_d(A_1^{\delta_1} \dots A_k^{\delta_k}; L_{A_i}). \tag{3.8}$$

Equations (3.6), (3.7) and (3.8) give us equation (3.1).

We will now prove the claims that we have made regarding transversality and multiplicity.

3.4 Transversality and Multiplicity

We will start by recalling a few facts about A_k singularities that are proved in [2], section 3. Let $\mathscr{U}$ be a neighbourhood of the origin in $\mathbb{C}^2$ and $f : \mathscr{U} \longrightarrow \mathbb{C}^2$ a holomorphic function. Let i, j be non-negative integers. We define

$$f_{ij} := \frac{\partial^{i+j} f}{\partial^i x \partial^j y}\bigg|_{(x,y)=(0,0)}.$$

Let us now define the following directional derivatives, which are functions of f_{ij}:

$$A_3^f := f_{30}, \qquad A_4^f := f_{40} - \frac{3f_{21}^2}{f_{02}}, \qquad A_5^f := f_{50} - \frac{10 f_{21} f_{31}}{f_{02}} + \frac{15 f_{12} f_{21}^2}{f_{02}^2},$$

$$A_6^f := f_{60} - \frac{15 f_{21} f_{41}}{f_{02}} - \frac{10 f_{31}^2}{f_{02}} + \frac{60 f_{12} f_{21} f_{31}}{f_{02}^2} + \frac{45 f_{21}^2 f_{22}}{f_{02}^2} - \frac{15 f_{03} f_{21}^3}{f_{02}^3} - \frac{90 f_{12}^2 f_{21}^2}{f_{02}^3},$$

$$A_7^f := f_{70} - \frac{21 f_{21} f_{51}}{f_{02}} - \frac{35 f_{31} f_{41}}{f_{02}} + \frac{105 f_{12} f_{21} f_{41}}{f_{02}^2} + \frac{105 f_{21}^2 f_{32}}{f_{02}^2} + \frac{70 f_{12} f_{31}^2}{f_{02}^2} + \frac{210 f_{21} f_{22} f_{31}}{f_{02}^2}$$
$$- \frac{105 f_{03} f_{21}^2 f_{31}}{f_{02}^3} - \frac{420 f_{12}^2 f_{21} f_{31}}{f_{02}^3} - \frac{630 f_{12} f_{21}^2 f_{22}}{f_{02}^3} - \frac{105 f_{13} f_{21}^3}{f_{02}^3} + \frac{315 f_{03} f_{12} f_{21}^3}{f_{02}^4} + \frac{630 f_{12}^3 f_{21}^2}{f_{02}^4},$$

and

$$A_8^f := f_{80} - \frac{28 f_{21} f_{61}}{f_{02}} - \frac{56 f_{31} f_{51}}{f_{02}} + \frac{210 f_{21}^2 f_{42}}{f_{02}^2} + \frac{420 f_{21} f_{22} f_{41}}{f_{02}^2} - \frac{210 f_{03} f_{21}^2 f_{41}}{f_{02}^3} + \frac{560 f_{21} f_{31} f_{32}}{f_{02}^2} -$$
$$\frac{840 f_{13} f_{21}^2 f_{31}}{f_{02}^3} - \frac{420 f_{21}^3 f_{23}}{f_{02}^3} + \frac{1260 f_{03} f_{21}^3 f_{22}}{f_{02}^4} - \frac{35 f_{41}^2}{f_{02}} + \frac{280 f_{22} f_{31}^2}{f_{02}^2} - \frac{280 f_{03} f_{21} f_{31}^2}{f_{02}^3} - \frac{1260 f_{21}^2 f_{22}^2}{f_{02}^3} +$$
$$\frac{105 f_{04} f_{21}^4}{f_{02}^4} - \frac{315 f_{03}^2 f_{21}^4}{f_{02}^5} + \frac{168 f_{21} f_{51} f_{12}}{f_{02}^2} + \frac{280 f_{31} f_{41} f_{12}}{f_{02}^2} - \frac{1680 f_{21}^2 f_{32} f_{12}}{f_{02}^3} - \frac{3360 f_{21} f_{22} f_{31} f_{12}}{f_{02}^3} +$$
$$\frac{2520 f_{03} f_{21}^2 f_{31} f_{12}}{f_{02}^4} + \frac{2520 f_{13} f_{21}^3 f_{12}}{f_{02}^4} - \frac{840 f_{21} f_{41} f_{12}^2}{f_{02}^3} + \frac{7560 f_{21}^2 f_{22} f_{12}^2}{f_{02}^4} - \frac{560 f_{31}^2 f_{12}^2}{f_{02}^3} - \frac{5040 f_{03} f_{21}^3 f_{12}^2}{f_{02}^5}$$
$$+ \frac{3360 f_{21} f_{31} f_{12}^3}{f_{02}^4} - \frac{5040 f_{21}^2 f_{12}^4}{f_{02}^5} \tag{3.9}$$

The procedure to obtain A_k^f is given in the proof of the following Proposition. We will now state a necessary and sufficient criteria for a curve to have a specific singularity of type $A_{k \geq 1}$.

Lemma 3.4.1. *Let $f = f(x,y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$ such that $f_{00} = 0$ and $\nabla f|_{(0,0)} \neq 0$. Then the origin is a smooth point of the curve.*

Lemma 3.4.2. *Let $f = f(x,y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$ such that $f_{00} = 0$, $\nabla f|_{(0,0)} = 0$ and $\nabla^2 f|_{(0,0)}$ is non-degenerate. Then the curve has a singularity of type A_1 at the origin.*

Remark 3.4.3. *Lemma 3.4.1 is also known as the Implicit Function Theorem and Lemma 3.4.2 is also known as the Morse Lemma.*

We now state a necessary and sufficient condition for a curve to have an $A_{k\geq 2}$ singularity. This can be thought of as a continuation of Lemma 3.4.2.

Lemma 3.4.4. *Let $f = f(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$ such that $f_{00} = 0$, $\nabla f|_{(0,0)} = 0$ and there exists a non-zero vector $\eta = (v_1, v_2)$ such that at the origin $\nabla^2 f(\eta, \cdot) = 0$, i.e., the Hessian is degenerate. Let $x := v_1 r + v_2 s, y := -\bar{v}_2 r + \bar{v}_1 s$ and f_{ij} be the partial derivatives with respect to the new variables x and y. Then, the curve $f^{-1}(0)$ has a singularity of type A_k at the origin if $f_{02} \neq 0$ and the directional derivatives A^f_i defined in* (3.13) *are zero for all $i \leq k$ and $A^f_{k+1} \neq 0$.*

Proof: The result follows from the following observation.

Observation 3.4.5. *Let $f = f(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$ such that $f(0,0)$, $\nabla f|_{(0,0)} = 0$ and there exists a non-zero vector $\eta = (v_1, v_2)$ such that at the origin $\nabla^2 f(v, \cdot) = 0$, i.e., the Hessian is degenerate. Let $x := v_1 r + v_2 s, y := -\bar{v}_2 r + \bar{v}_1 s$ and f_{ij} be the partial derivatives with respect to the new variables x and y. If $f_{02} \neq 0$, there exists a coordinate chart (u,v) centered around the origin in $\mathbb{C}^2$ such that*

$$f = \begin{cases} v^2, & \text{or} \\ v^2 + u^{k+1}, & \text{for some } k \geq 2. \end{cases} \tag{3.10}$$

In terms of the new coordinates we have $f_{00} = f_{10} = f_{01} = f_{20} = f_{11} = 0$ and $f_{02} \neq 0$. Here $\partial_x + 0\partial_y = (1,0)$ is the distinguished direction along which the Hessian is degenerate.

Proof of observation: Let the Taylor expansion of f in the new coordinates be given by

$$f(x,y) = \mathrm{A}_0(x) + \mathrm{A}_1(x)y + \mathrm{A}_2(x)y^2 + \dots.$$

By our assumption on f, $\mathrm{A}_2(0)\neq 0$. We claim that there exists a holomorphic function $\mathrm{B}(x)$ such that after we make a change of coordinates $y = y_1 + \mathrm{B}(x)$, the function f is given by

$$f = \hat{\mathrm{A}}_0(x) + \hat{\mathrm{A}}_2(x)y_1^2 + \hat{\mathrm{A}}_3(x)y_1^3 + \ldots$$

for some $\hat{\mathrm{A}}_k(x)$ (i.e., $\hat{\mathrm{A}}_1(x)\equiv 0$). To see this, we note that this is possible if $\mathrm{B}(x)$ satisfies the identity

$$\mathrm{A}_1(x) + 2\mathrm{A}_2(x)\mathrm{B} + 3\mathrm{A}_3(x)\mathrm{B}^2 + \ldots \equiv 0. \tag{3.11}$$

Since $\mathrm{A}_2(0)\neq 0$, $\mathrm{B}(x)$ exists by the Implicit Function Theorem.

Remark 3.4.6. *Moreover, it is unique if we require* $\mathrm{B}(0) = 0$.

Therefore, we can compute $\mathrm{B}(x)$ as a power series using (3.11) and then compute $\hat{\mathrm{A}}_0(x)$. Hence,

$$f = v^2 + \frac{A_3^f}{3!}x^3 + \frac{A_4^f}{4!}x^4 + \ldots, \quad \text{where} \quad v = \sqrt{(\hat{\mathrm{A}}_2 + \hat{\mathrm{A}}_3 y_1 + \ldots)}\,y_1, \tag{3.12}$$

satisfies (3.10).

Following the above procedure we find A_i^f for any i. For example,

$$A_3^f = f_{30}, \qquad A_4^f = f_{40} - \frac{3f_{21}^2}{f_{02}}, \qquad A_5^f = f_{50} - \frac{10 f_{21} f_{31}}{f_{02}} + \frac{15 f_{12} f_{21}^2}{f_{02}^2}, \ldots \tag{3.13}$$

and so on. We are now ready to prove the claim that the space of curves with prescribed singularities is a smooth manifold of the expected dimension, provided d is sufficiently large.

Lemma 3.4.7. *Let M and $\mathscr{S}$ be as defined in equation* (3.2)*. If $d \geq d_{\min}$, then $\mathscr{S}$ is a complex sub manifold of M, of dimension $w_d + 1$.*

Proof: We will prove this statement by considering an affine chart. Hence, let us consider the vector space $\mathscr{F}_d \approx \mathbb{C}^{\frac{d(d+3)}{2}+1}$ of polynomials in two variables of degree at most d. Let us denote $p_i := (x_i, y_i) \in \mathbb{C}^2$ and define

$$\mathscr{S}_{\text{affine}} := \{(f, p_1, p_2, \ldots, p_\delta) \in \mathscr{F}_d \times \left((\mathbb{C}^2)^\delta - \Delta\right) : f \text{ has an } A_{k_i}\text{-singularity at } p_i,\ p_i \text{ all distinct}\}.$$

Here Δ denotes the fat diagonal of $(\mathbb{C}^2)^\delta$ (i.e., if any two points are equal, they belong to the fat diagonal). We will show that $\mathscr{S}_{\text{affine}}$ is a smooth complex sub manifold of $\mathscr{F}_d \times \left((\mathbb{C}^2)^\delta - \Delta\right)$ of codimension c_d. In order to do that, we will describe $\mathscr{S}_{\text{affine}}$ locally as the zero set of certain holomorphic functions.

Let us suppose that

$$(f, \overline{p}) := (f, p_1, p_2, \ldots, p_\delta) \in \mathscr{S}_{\text{affine}}$$

Suppose f has an A_k singularity at $p_1 := (x_1, y_1)$, then we can use Lemma 3.4.4 to see that there exist a sufficiently small open sets $U_{p_1} \subset \mathscr{F}_d$ around f and $V_{p_1} \subset \mathbb{C}^2$ around p_1 such that on $U_{p_1} \times V_{p_1}$(possibly after making a linear change of coordinates) $f_{yy}(x_1, y_1)$, the second partial derivative of f with respect to y, evaluated at (x_1, y_1) is non zero. Let us now define

$$\hat{x} := x - \frac{f_{xy}(x_1, y_1)}{f_{yy}(x_1, y_1)} y \qquad \text{and} \qquad \hat{y} := y.$$

We note that $\hat{x}$ is well defined, since $f_{yy}(x_1, y_1) \neq 0$. We will now define $A_k^{f(x_1,y_1)}$ to be the expressions obtained in (3.13), where we replace f_{ij} with the $(i, j)^{\text{th}}$ partial derivative of f with respect to $\hat{x}$ and $\hat{y}$, evaluated at (x_1, y_1). As an example,

$$\begin{aligned} A_3^{f(x_1,y_1)} &= \Big(\partial_x - \frac{f_{xy}(x_1, y_1)}{f_{yy}(x_1, y_1)} \partial_y\Big)^3 f \\ &= \Big(f_{xxx} - 3\frac{f_{xy}}{f_{yy}} f_{xxy} + 3\Big(\frac{f_{xy}}{f_{yy}}\Big)^2 f_{xyy} + \Big(\frac{f_{xy}}{f_{yy}}\Big)^3 f_{yyy}\Big)\Big|_{(x_1,y_1)}. \end{aligned}$$

Since f has an A_{k_i} singularity at $p_i := (x_i, y_i)$, all are distinct points so we can assume (possibly after a linear change of co-ordinates) that $f_{yy}(x_i, y_i)$, the second partial derivative of f with respect to y, evaluated at (x_i, y_i) is non zero. Then repeated use of Lemma 3.4.4 will give us sufficiently small open neighbourhoods $U := \cap_i U_{p_i} \subset \mathscr{F}_d$ and $V := \prod_i V_{p_i} \subset \Big((\mathbb{C}^2)^\delta - \Delta\Big)$ so that we can define $A_{k_i}^{f(x_i,y_i)}$ to be the expressions as obtained in (3.4.4) for each i.
Next, let $\mathscr{U} := U \times V$ be a sufficiently small open neighbourhood of $(f, \overline{p})$ in $\mathscr{F}_d \times \Big((\mathbb{C}^2)^\delta - \Delta\Big)$. Let us define the function $\Phi : \mathscr{U} \longrightarrow \mathbb{C}^{c_d}$, given by

$$\begin{aligned} \Phi(f, \overline{p}) := \Big(&f(x_1, y_1), f_x(x_1, y_1), f_y(x_1, y_1), A_2^{f(x_1,y_1)}, \ldots, A_{k_1}^{f(x_1,y_1)}; \\ &f(x_2, y_2), f_x(x_2, y_2), f_y(x_2, y_2), A_2^{f(x_2,y_2)}, \ldots, A_{k_2}^{f(x_2,y_2)}; \ldots, ; \\ &f(x_\delta, y_\delta), f_x(x_\delta, y_\delta), f_y(x_\delta, y_\delta), A_2^{f(x_\delta,y_\delta)}, \ldots, A_{k_\delta}^{f(x_\delta,y_\delta)}\Big) \end{aligned}$$

We claim that $\mathbf{0}$ is a regular value of Φ. If we can show that, then our claim is proved.

To prove the claim, we will construct curves. Since the points p_i are all distinct, we will show that for different possibilities of points we can produce curves $\eta_i \in \mathscr{F}_d$ be such that

$$\eta_i(x_j, y_j) = \delta_{i,j}.$$

Remark 3.4.8. *There are plenty of ways one can construct such curves η_i. In practice it is enough to construct curves η_i such that $\eta_i(x_j, y_j) \neq 0$.*

We can easily construct such an η_i by taking product of all the $(x-x_j)$, except $(x-x_i)$ combined with $(y-y_j)$, i.e., for n distinct points

$$\eta_i := (z_{r_1}-z_1)\cdots(z_{r_{i-1}}-z_{i-1})\widehat{(z_{r_i}-z_i)}(z_{r_{i+1}}-z_{i+1})\cdots(z_{r_n}-z_n)$$

where

$$(z_{r_s}) = \begin{cases} (x,x_s) & if \quad x_i \neq x_s \\ (y,y_s) & if \quad y_i \neq y_s \end{cases}$$

Let us consider the point $p_i := (x_i,y_i)$. The curve f has an A_{k_i} singularity at p_i. As an example, if f has at least A_1 singularity at p_i then there are sufficiently small neighbourhoods around each p_i where $f(p_i), f_x(p_i), f_y(p_i)$ vanishes. So in this situation if we simply construct curves as follows:

$$\gamma^i_{00}(t) := f + t\eta_i^2,\ \ \gamma^i_{10}(t) := f + t(x-x_i)\eta_i^2,\ \ \gamma^i_{01}(t) := f + t(y-y_i)\eta_i^2.$$

So the above construction enables us

$$\{d\Phi|_{(f,\overline{p})}\}(\gamma^{i'}_{0,0}(0)) = (0,\cdots,\underbrace{(*,0,0)}_{\text{i th position}},\cdots,0)$$

$$\{d\Phi|_{(f,\overline{p})}\}(\gamma^{i'}_{1,0}(0)) = (0,\cdots,\underbrace{(0,*,0)}_{\text{i th position}},\cdots,0)$$

$$\{d\Phi|_{(f,\overline{p})}\}(\gamma^{i'}_{0,1}(0)) = (0,\cdots,\underbrace{(0,0,*)}_{\text{i th position}},\cdots,0)$$

then the above computation implies that $\mathbf{0}$ is a regular value as claimed.

Next, note that if f has singularity at least as degenerate as cusp at some point assuming that there is already A_1 singularity present at that point, then we can consider that f has a genuine cusp which is equivalent to $f_{20}f_{02}-f_{11}^2=0$ (determinant of Hessian vanishes). Since the cusp is a genuine cusp so without loss of generality we can assume that $f_{02}\neq 0$. So one can simply construct a curve $\gamma^i_{20}(t) := f + t(\hat{x}-x_i)^2\eta_i$ and considering $\gamma^i_{02}(t) := 0, \quad \gamma^i_{11}(t) := 0 \quad \forall i$, where $\hat{x}$ is defined below, for each point p_i.

Note that

$$\{d\Phi|_{(f,\overline{p})}\}(\gamma^{i'}_{2,0}(0)) = (0,\cdots,\underbrace{*}_{i_{20}\text{ th position}},0,0,\cdots,0)$$

one can observe that this computation proves the claim for cusp.

Finally, if $k_i \geq 2$, i.e., f has higher A_{k_i} singularities then we have made a linear change of coordinates so that the kernel of the Hessian is $\partial_x|_{p_i} + m\partial_y|_{p_i}$, where $m := \frac{-f_{xy}(x_i,y_i)}{f_{yy}(x_i,y_i)}$. Let us now define the curves

$$\gamma^i_{20}(t) := f + t(\hat{x} - x_i)^2\eta_i, \quad \ldots, \quad \gamma^i_{k_i 0}(t) := f + t(\hat{x} - x_i)^{k_i}\eta_i,$$

and considering $\gamma^i_{\alpha\beta}(t) := 0$ for $\alpha \neq k_i, \beta \neq 0$ for all i from 1 to δ. Here $\hat{x} := x + my$. We now note that

$$\{d\Phi|_{(f,\overline{p})}\}(\gamma^{i'}_{\alpha\beta}(0))$$

span the tangent space of $T_{\mathbf{0}}\mathbb{C}^{c_d}$. This proves the claim.

Lemma 3.4.9. *Restricted to $\mathscr{S} \times \mathrm{L} - \mathscr{B}$, the sections ψ_{ev} and ψ_{T} are transverse to zero.*

Proof: First, suppose

$$\psi_{\mathrm{ev}}([f], \overline{q}, p) = 0 \qquad \Longleftrightarrow \qquad f(p) = 0.$$

We will produce the following curve. Let us consider a curve η_{00} in $\mathscr{S}$ such that $\eta(p) \neq 0$. Consider

$$\gamma_{00}(t) := (f + t\eta_{00}, \overline{q}, p).$$

This proves transversality of the evaluation map.

Next, let us consider a curve η_{T} such that

$$\nabla\eta_{\mathrm{T}}|_p(v) \neq 0$$

if $v \in T_pL - 0$.

The construction of a curve η_{T} will follow from above discussion. Now consider the curve

$$\gamma_{\mathrm{T}}(t) := f + t\eta_{\mathrm{T}}.$$

This proves transversality of the section ψ_{T} is transverse to zero.

Finally, we are ready to prove the main theorem about the multiplicity.

Theorem 3.4.10. *Let $\mu \subset \mathscr{D}$ be the subspace of curves passing through w_d generic points and suppose*

$$([f], \overline{q}, p) \in \mathscr{S} \times L \cap \pi^{-1}_{\mathscr{D}}(\mu).$$

Suppose

$$\psi_{\mathrm{ev}}([f], \overline{q}, p) = 0, \qquad \psi_{\mathrm{T}}([f], \overline{q}, p) = 0, \qquad ([f], \overline{q}, p) \in \mathscr{B}^k_{\eta} \cap \left(\pi^{-1}_{\mathscr{D}}\mu\right). \tag{3.14}$$

Then the order of vanishing is $(k+1)$.

Remark 3.4.11. *Note that $\mathscr{S}\times L\cap\pi_{\mathscr{D}}^{-1}(\mu)$ is a smooth complex manifold of dimension* 2*. Hence it makes sense to talk about the order of vanishing of a section of a rank two bundle.*

Proof: Suppose $([f],\overline{q},p)$ satisfies equation (3.14). We will construct a neighbourhood of $([f],\overline{q},p)$ inside $\mathscr{S}\times L$. Since $([f],\overline{q},p)\in\mathscr{B}_{\eta}^{k}\cap\left(\pi_{\mathscr{D}}^{-1}\mu\right)$ and μ denotes a subspace of curves in $\mathscr{D}$ passing through w_d generic points, we conclude that f has an A_k singularity of p. Without loss of generality, we can take $p:=[0,0,1]\in\mathbb{P}^2$. Let us also assume that the line L passing through p is given by the equation

$$L:=\{[X,Y,Z]\in\mathbb{P}^2 : aX+bY=0\}, \tag{3.15}$$

where a and b are two fixed complex numbers. Let us now write down the Taylor expansion of f around the point p. Let us define

$$x:=\frac{X}{Z} \qquad \text{and} \qquad y:=\frac{Y}{Z}.$$

hence, we get that

$$f=\frac{f_{20}}{2}x^2+f_{11}xy+\frac{f_{02}}{2}y^2+\frac{f_{30}}{6}x^3+\ldots.$$

If f has an $A_{k\geq 2}$ singularity at p, we conclude that f_{02} or f_{20} can not both be zero; let us assume in that case $f_{02}\neq 0$. If f has an A_1 singularity at p, then after a linear change of coordinates, we can ensure that $f_{02}\neq 0$. Hence, in all the cases, we can assume without loss of generality that $f_{02}\neq 0$.

After making a suitable change of coordinates, the function f is given by

$$f=\hat{y}^2+\hat{x}^{k+1}.$$

After the change of coordinates, the line L in (3.15), will be given by

$$L:=\{[X,Y,Z]\in\mathbb{P}^2 : \hat{y}+M\hat{x}+E(\hat{y},\hat{x})=0\},$$

where E is second order and $M=\frac{a}{b}$; without loss of generality we are assuming $b\neq 0$. Since L is a generic line, we can assume this (i.e., we are assuming the line is not given by $x=0$). Let us now assume that k is even (i.e., $k+1$ odd). A solution to the equation $f=0$, close to $(0,0)$ is given by

$$\hat{y}=t^{k+1}, \qquad \hat{x}=t^2 \qquad t \text{ is small but non zero.}$$

Furthermore, every solution to $f=0$ is of this type. We now consider the second equation of evaluating the derivative along L. That gives us

$$\begin{aligned}(M\partial_{\hat{x}}+\partial_{\hat{y}})f &= Mf_{\hat{x}}+f_{\hat{y}}\\ &= 2\hat{y}+(k+1)M\hat{x}^{k}\\ &= 2t^{k+1}+(k+1)Mt^{2k}.\end{aligned}$$

Hence, the order of vanishing is $(k+1)$. If k is odd (i.e., $k+1$ is even), then there are two solutions. Each solution vanishes with order $\frac{k+1}{2}$; hence the total order of vanishing is $k+1$. In either case, the total order of vanishing is $k+1$.

3.5 Explicit Formulas

For the convenience of the reader, we will explicitly write down the formulas for $N_d^{\mathrm{T}}(A_k)_{1\leq k\leq 8}$, $N_d^{\mathrm{T}}(A_1A_k)_{1\leq k\leq 7}$ and $N_d^{\mathrm{T}}(A_1^{\delta})_{1\leq\delta\leq 8}$. These are obtained from Main Theorem (equation (3.1)); combined with the numbers given in the papers of S. Basu and R. Mukherjee ([2], [1] and [3]). We will then use these formulas to make low degree check in section 3.6. A mathematica program can be found in my website

$$https://sites.google.com/view/paulanantadulal$$

which evaluates the formulas for $N_d^{\mathrm{T}}(A_k)_{1\leq k\leq 8}$:

$$N_d^{\mathrm{T}}(A_1)=6d(d-1)(d-2), \qquad N_d^{\mathrm{T}}(A_2)=12(2d^3-8d^2+8d-1),$$
$$N_d^{\mathrm{T}}(A_3)=4(25d^3-146d^2+228d-84), \qquad N_d^{\mathrm{T}}(A_4)=120(3d^3-20d^2+36d-15),$$
$$N_d^{\mathrm{T}}(A_5)=36(35d^3-260d^2+524d-239), \qquad N_d^{\mathrm{T}}(A_6)=7(632d^3-5134d^2+11343d-5538),$$
$$N_d^{\mathrm{T}}(A_7)=24(651d^3-5702d^2+13602d-7002) \qquad \text{and}$$
$$N_d^{\mathrm{T}}(A_8)=288(190d^3-1778d^2+4533d-2436).$$

Next, the formulas for $N_d^{\mathrm{T}}(A_1A_k)_{1\leq k\leq 7}$ are:

$$\begin{aligned}
N_d^{\mathrm{T}}(A_1^2) &= 2(9d^5-45d^4+30d^3+123d^2-145d+6),\\
N_d^{\mathrm{T}}(A_1A_2) &= 12(d-3)(6d^4-18d^3-22d^2+67d-13),\\
N_d^{\mathrm{T}}(A_1A_3) &= 12(25d^5-171d^4+187d^3+774d^2-1535d+426),\\
N_d^{\mathrm{T}}(A_1A_4) &= 20(54d^5-414d^4+534d^3+2238d^2-5207d+1815),\\
N_d^{\mathrm{T}}(A_1A_5) &= 18(210d^5-1770d^4+2572d^3+11299d^2-29650d+11959),\\
N_d^{\mathrm{T}}(A_1A_6) &= 21(632d^5-5766d^4+9164d^3+42837d^2-123391d+55068),\\
N_d^{\mathrm{T}}(A_1A_7) &= 8(5859d^5-57177d^4+97677d^3+485874d^2-1509623d+725940).
\end{aligned}$$

Finally, the formulas for $N_d^{\mathrm{T}}(A_1^{\delta})_{3\leq\delta\leq 8}$ are:

$$\begin{aligned}
N_d^{\mathrm{T}}(A_1^3) &= 6(9d^7-63d^6+36d^5+549d^4-857d^3-1148d^2+2266d-300),\\
N_d^{\mathrm{T}}(A_1^4) &= 18(9d^9-81d^8+36d^7+1458d^6-2834d^5-8500d^4+22455d^3+13543d^2-49222d+10488),\\
N_d^{\mathrm{T}}(A_1^5) &= 6\big(81d^{11}-891d^{10}+270d^9+27270d^8-63450d^7-303912d^6\\
&\quad +1014807d^5+1348725d^4-6097876d^3-1168832d^2+12259248d-3513840\big),\\
N_d^{\mathrm{T}}(A_1^6) &= 1458d^{13}-18954d^{12}+2916d^{11}+882090d^{10}-2390310d^9-15901596d^8+64328418d^7\\
&\quad +130916898d^6-732619008d^5-395637750d^4+3855455766d^3\\
&\quad -418407408d^2-7418026440d+2643818400,\\
N_d^{\mathrm{T}}(A_1^7) &= 4374d^{15}-65610d^{14}+4317138d^{12}-13352850d^{11}-114293592d^{10}+543520530d^9\\
&\quad +1481762970d^8-9946281060d^7-8470208502d^6+95900422338d^5+1014814332d^4\\
&\quad -467415101124d^3+168796887984d^2+880782565392d-374053619520 \qquad \text{and}\\
N_d^{\mathrm{T}}(A_1^8) &= 13122d^{17}-223074d^{16}-34992d^{15}+19717992d^{14}-68543496d^{13}\\
&\quad -719400528d^{12}+3933317556d^{11}+13400193204d^{10}-105120249336d^9\\
&\quad -119845037160d^8+1587321808632d^7+150918108768d^6-13835625254910d^5\\
&\quad +5746599271062d^4+64281794069664d^3-38151916883064d^2\\
&\quad -120388035085920d+59358641529600.
\end{aligned}$$

3.6 Low degree checks

In this section, we will make some non trivial low degree checks by comparing our formulas with the results of others.

3.6.1 Verification with the Caporasso-Harris formula

We will start by verifying the numbers $N_d(A_1^{\delta})_{1\leq\delta\leq 8}$. We note that the Caporasso-Harris formula ([11]) computes $N_d(A_1^{\delta})^{\mathrm{T}}$ for any δ. We have verified that our formulas for $N_d(A_1^{\delta})_{1\leq\delta\leq 8}$ produce the same answer as the Caporasso-Harris formula for several values of d; we have written a C++ program to implement the Caporasso-Harris formula (which is available on request). The reader is invited to use the C++ program to check that it produces the same answer given by our formula (explicitly written down in section 3.5) for any specific value of d.

3.6.2 Verification with the results of Ran and Fomin-Mikhalkin

Here we will give a table of our numbers using the main recursive formula (3.1) which we have verified with the numbers of rational degree d curves in $\mathbb{P}^2$ tangent to a line, denoted by M_d^T, calculated earlier by Z.Ran [58], Fomin-Mikhalkin [17] and others.

Such as the number of **rational quartics** in $\mathbb{P}^2$ through 10 generic points can be verified with our formula as :

$$\begin{aligned} M^T_{d=4} &= N^{\mathrm{T}}(A_1^3)\mid_{d=4} - \text{reducible tangential quartics through 10 generic points.} \\ &= 2364 - \binom{10}{2}4 \qquad = \quad 2184. \end{aligned}$$

Let us denote by $N_{g,d}^{Irr,T}$ be the number of genus g degree d curves in $\mathbb{P}^2$ tangent to a line passing through $3d-2+g$ generic points. These numbers are computed in [69], [11], [58], [2] and others. Some of the numbers we tabulated as: Next, we display some of our numbers which are necessary to

$degree, genus$	$(3,0)$	$(4,0)$	$(4,1)$	$(4,2)$	$(4,3)$	$(5,1)$	$(5,2)$	$(5,3)$	$(6,2)$
$N_{g,d}^{Irr,T}$	36	2184	1010	144	6	424480	203616	49580	326594238

Table 3.1: Number of genus g degree d curves in $\mathbb{P}^2$ tangent to a line.

verify the above numbers, calculated from (3.1) for k numbers of nodes, i.e., $N_d^{\mathrm{T}}(A_1^k)$ as:

$degree, k$	$(3,1)$	$(4,3)$	$(4,2)$	$(4,1)$	$(4,0)$	$(5,5)$	$(5,4)$	$(5,2)$	$(6,8)$
$N_d^{\mathrm{T}}(A_1^k)$	36	2364	1010	144	6	424480	204246	49580	334237506

Table 3.2: Number of reducible curves.

Finally if we denote by $M^{Red,T}_{d_1,d_2}(A_1^s)$ to be the number of reducible tangential degree d_1 and d_2 curves in $\mathbb{P}^2$ having s numbers of nodes tangent to a given line passing through appropriate number of generic points i.e., degree d_1 curves and the degree d_2 curve which is tangent to a line having certain number of nodes(total degree $d = d_1 + d_2$ reduced curves) through appropriate numbers of generic points. Note that if $s = 0$ then $M^{Red,T}_{d_1,d_2}$ gives the numbers of reducible tangential curve (degree d_1 curve and degree d_2 component is tangent to line) without singularity.

We see that all the numbers tabulated above are $N^{Irr,T}_{g,d} = N_d^{\mathrm{T}}(A_1^k) - M^{Red,T}_{d_1,d_2}(A_1^s)$.

d_1, d_2, s	$(1,3,0)$	$(1,5,2)$	$(1,4,0)$	$(1,5,3)$
$M^{Red,T}_{d_1,d_2}$	4	6106	6	49580

Table 3.3: Two componented curves with tangency.

• Presently, we can verify our numbers with the help of recursion formula avaliable in [1], [2] upto codimension 8 numbers only.

3.6.3 Verification of $N_d^{\mathrm{T}}(A_2)$ using a result of Kazarian

In [27], Kazarian has computed the number $N_d(A_1A_2A_3)$, the characteristic number of degree d curves with one node, one cusp and one tacnode. According to Kazaryan's formula, that number is 2256 when $d = 4$. We will verify that number.

We note that $N_4(A_1A_2A_3)$ is the number of quartics through 8 points that have one node, one cusp and one tacnode. This can happen if the curve breaks into a cubic and a line, such that the cubic has a cusp and is tangent to the given line (and the entire configuration passes through 8 points). Since the cubic is tangent to the given line, it will intersect the curve at one more point. Let us now find out how many such configurations are there. First of all, we could place a line through 2 points and a cuspidal cubic through 6 points tangent to a given line. There are a total of

$$\binom{8}{2} \times N_3^{\mathrm{T}}(A_2)$$

such configurations. The other possibility is that we place a cuspidal cubic through 7 points and a line through one point that is tangent to this cuspidal cubic. We claim that the total number of such configurations (n) is

$$n = 3N_3(A_2).$$

We will justify this shortly. Using the values of $N_3^{\mathrm{T}}(A_2)$ and $N_3(A_2)$, we note that

$$\begin{aligned}\binom{8}{2} \times N_3^{\mathrm{T}}(A_2) + \binom{8}{7} \times n &= \binom{8}{2} \times N_3^{\mathrm{T}}(A_2) + \binom{8}{7} \times 3N_3(A_2) \\ &= \binom{8}{2} \times 60 + \binom{8}{7} \times 3 \times 24 \\ &= 2256.\end{aligned}$$

This agrees with the number predicted by Kazaryan's formula.

Let us now justify the value of n. Let us denote $\mathscr{D}_1$ and $\mathscr{D}_3$ to be the space of lines and space of cubics in $\mathbb{P}^2$ respectively. We note that $\mathscr{D}_1$ and $\mathscr{D}_3$ are isomorphic to $\mathbb{P}^2$ and $\mathbb{P}^9$ respectively.

Let us define

$$\mathscr{S} := \{([f], q) \in \mathscr{D}_3 \times \mathbb{P}^2 : \ f \text{ has an } A_2 \text{ singularity at } q\}.$$

For notational convenience, let us denote $\mathbb{P}^2_1$ and $\mathbb{P}^2_3$ to be two isomorphic copies of $\mathbb{P}^2$. With that notation, we define the following space

$$\mathscr{X} := \{([f_1], q_1, [f_3], q_3) \in \mathscr{D}_1 \times \mathbb{P}^2_1 \times \mathscr{D}_3 \times \mathbb{P}^2_3 : \ \ ([f_3], q_3) \in \overline{\mathscr{S}}, \ \ f_1(q_1) = 0, \ \ f_3(q_1) = 0\}. \quad (3.16)$$

Next, we note that over the space $\mathscr{X}$, we have the following short exact sequence of bundles

$$0 \longrightarrow \mathbb{L} := \mathrm{Ker}(\nabla f_1|_{q_1}) \longrightarrow T\mathbb{P}^2|_{q_1} \xrightarrow{\nabla f_1|_{q_1}} \gamma^*_{\mathscr{D}_1} \otimes \gamma^*_{\mathbb{P}^2} \longrightarrow 0. \quad (3.17)$$

Let us now define the following set

$$X := \{([f_1], q_1, [f_3], q_3) \in \mathscr{D}_1 \times \mathbb{P}^2_1 \times \mathscr{D}_3 \times \mathbb{P}^2_3 : ([f_1], q_1, [f_3], q_3) \in \mathscr{X}, \ \ \nabla f_3|_q(v) = 0, \ \ \forall v \in \mathbb{L}\}. \quad (3.18)$$

Let us now denote y_1, y_3, a_1 and a_3 to be the hyperplane classes of $\mathscr{D}_1$, $\mathscr{D}_3$, $\mathbb{P}^2_1$ and $\mathbb{P}^2_3$ respectively. We note that intersecting $[X]$ with y_3 corresponds to studying the subspace of cubics passing through a generic point and intersecting $[X]$ with y_1 corresponds to studying the subspace of lines passing through a generic point. Our aim is to count the configurations where the cubic passes through 7 points and the line passes through 1 point. Hence, let us intersect $[X]$ this with $y_1 y_3^7$. However, this intersection will also include the number of lines that pass through the given point and the cuspidal point of the cubic. By using the same argument as in the proof of Theorem 3.4.10, this configuration contributes with a multiplicity of 3. Hence,

$$[X]\cdot[y_1y_3^7] = n + 3N_3(A_2). \tag{3.19}$$

It remains to compute $[X]\cdot[y_1y_3^7]$. First we note that

$$\langle y_3^7,\ [\overline{\mathscr{S}}]\rangle = N_3(A_2). \tag{3.20}$$

Using equations (3.16), (3.18), (3.17) and (3.20), we conclude that

$$\begin{aligned}[X]\cdot[y_1y_3^7] &= \langle y_1y_3^7 e(\gamma_{\mathscr{D}_1}^*\otimes\gamma_{\mathbb{P}_1^2}^*)e(\gamma_{\mathscr{D}_3}^*\otimes\gamma_{\mathbb{P}_3^2}^{*3})e(\gamma_{\mathscr{D}_3}^*\otimes\gamma_{\mathbb{P}_3^2}^{*3}\otimes\mathbb{L}^*),\ \mathscr{D}_1\times\mathbb{P}_1^2\times\overline{\mathscr{S}}\rangle \\ &= 6\,N_3(A_2). \end{aligned} \tag{3.21}$$

Equations (3.19) and (3.21), we conclude that $n = 3\,N_3(A_2)$ as claimed.

3.6.4 Verification with the results of L. Ernström and G. Kennedy

We now verify a couple of numbers computed by L. Ernström and G. Kennedy (in [15]). In [15], the authors compute the number of rational cuspidal degree d curves, passing through $3d-3$ points and tangent to a given line. Let us denote this number to be C_d^{T}. The result of their computations (in [15]) gives us the following numbers:

$$C_3^{\mathrm{T}} = 60 \qquad \text{and} \qquad C_4^{\mathrm{T}} = 6912.$$

We now note the following fact about the numbers we have computed:

$$N_3^{\mathrm{T}}(A_2) = 60 \qquad \text{and} \qquad \frac{N_4^{\mathrm{T}}(A_1^2A_2)}{2} = 6912.$$

This is precisely as expected. The characteristic number of rational cuspidal cubics tangent to a given line (i.e. C_3^{T}) should precisely be equal to the characteristic number of degree 3 curves having a cusp, tangent to a given line (i.e. $N_3^{\mathrm{T}}(A_2)$). Secondly, the characteristic number of rational cuspidal quartics tangent to a given line (i.e. C_4^{T}) should precisely be equal to the characteristic number of degree 4 curves having two unordered nodes and one cusp, tangent to a given line (i.e. $\frac{N_4^{\mathrm{T}}(A_1^2A_2)}{2}$). Note that we divide by 2 since the nodes in $N_4^{\mathrm{T}}(A_1^2A_2)$ are ordered.

Remark 3.6.1. *It should be possible to generalize our method to get similar type of result for certain other types of singularities such as* D_k*,* E_6*,* E_7*, and* E_8*. We are not aware of any low degree checks involving tangency conditions with* D_k*,* E_6*,* E_7*, and* E_8 *singularities which will support any prediction about the formula involving those singularities. We also hope that this method can be employed to generalize these results for other complex surfaces. In [3], the authors have obtained the results in a compact complex surface essentially using the crucial ideas developed in [1] where they authors studied their result considering the compact complex surface to be* $\mathbb{P}^2$.

Chapter 4

Higher-order tangencies on the complex projective plane

4.1 Introduction

From a broad perspective, this chapter is a sequel to the previous chapter. In this chapter, we will be objectively focusing on the study of singular curves with a higher order of contact (we will always refer it by tangency condition) to a fixed smooth divisor $E \in \mathbb{P}^2$. The study of relative geometry is quite often considered to tackle the problems in absolute geometry. In the theory of absolute counting of curves, one natural extension would be relative curve count. The study of relative curve count dates back to Salmon [61]. The study of relative geometry and various enumerative questions concerning relative invariants has a long history that can be traced back to Zeuthen. As early as 1848, Zeuthen computed the characteristic number of rational quartics in $\mathbb{P}^2$ through 10 given points tangent to a given line. Around 19-th century, the study of the enumerative question "how many conics are there tangent to five conics?" was a very interesting incident in mathematics back then. In the stable map theory, counts of various relative invariants such as the study of relative Gromov-Witten invariants is an active area of research. Li ([42], [43]) had constructed the theory of relative Gromov-Witten invariants in full generality, i.e., for the curves of any genus in a projective manifold M with fixed local orders of contacts to some fixed hypersurface $Z \subset M$. One of the successful studies in this direction was due to A. Gathman [20], in his thesis, extending the idea of studying degeneration to hyperplanes in $\mathbb{P}^n$ to any arbitrary ample hypersurface. The main content of his thesis can be summarized by saying that he obtained a systematic approach to solve the following enumerative problem:

Problem 4.1.1. *Let Z be a smooth hypersurface of a complex projective manifold M. How one can compute the Gromov-Witten invariants of Z from those of M?*

In [20], the author studied relative Gromov-Witten invariants using this degeneration technique and one of his major development using this technique was to prove genus-zero Mirror symmetry for quintic threefold inside $\mathbb{P}^4$ using this idea. In other words, one can relate invariants of $\mathbb{P}^n$ to the invariants of the hypersurfaces. Although all of these works are extremely motivating and fascinating, it only studies the curves with at most nodal singularities. It would be interesting to study relative invariants in this setup, possibly including higher singularities. Unfortunately, there were some partial results concerning further development of the relative study with higher singularities.

Let us consider one classical enumerative problem on plane curves as follows:

Question 4.1.2. *Let $E \subset \mathbb{P}^2$ be a smooth plane curve, and consider $\alpha = (\alpha_1, \alpha_2, \cdots)$ and $\beta = (\beta_1, \beta_2, \cdots)$. How many degree d rational plane curves in $\mathbb{P}^2$ meet E at α_k "fixed" points with order of contact k and β_l "moving" points with order of contact l passing through $3d - 1 - \sum(k\alpha_k + (l-1)\beta_l)$ points in $\mathbb{P}^2$ in general position, if all contacts with E occur at uni-branched points?*

Note that when $\alpha = 0$ and $\beta = (3d, 0, \cdots)$ the above question reduces to the question (1.2.1), the solution to which is given by Kontsevich using the theory of Gromov-Witten invariants for $\mathbb{P}^2$.

In [10], the above question has been solved completely when E is a line using generalized Severi variety techniques. Implicitly, they have defined relative Gromov-Witten invariants in the process of generalizing the theory of Gromov-Witten invariants to higher genus.
In [68], the author then extended their idea when E is a smooth conic, later on, in [10], the authors had solved the question (6.2.2) when E is a smooth cubic for all α, β except for $(\alpha, \beta) = (0, e_{3d})$ using different technique.

After so much earlier developments in this direction one extremely natural question can be asked as follows:

Problem 4.1.3. *If E is smooth degree d_1 curve inside $\mathbb{P}^2$. How to enumerate degree d curves in $\mathbb{P}^2$ having one or more degenerate singularities (possibly more degenerate than nodes) that are tangent to E to some order passing through an appropriate number of generic points?*

So far, a few results are known. We have made a very brief survey in the previous chapter on this. We have seen that the above problem involving singularities more degenerate than nodes remains unexplored. An important relationship of this question is to count curves with higher singularities which is the main attraction of this thesis. We have seen that the study of nodal curves took more than 150 years in literature. Hence, enumerating curves with various higher singularities is an extremely difficult subject. In this thesis, we have attempted to study higher singularities by reducing them to an intermediate object; the study of singular curves with higher-order tangency to a certain divisor. We are not aware of any developments when E is singular, how to study relative invariants? We will now extend our method to higher-order tangencies to a fixed-line in $\mathbb{P}^2$. When the singularities are nodes we saw agreement with the result due to Caporaso-Harris. We have performed some low degree checks when the divisor is any degree d_1 curve inside $\mathbb{P}^2$, giving us consistent answers with the result of Gathman, Fan, and Yu ([20], [16]) that enables us to remark that our method extends suitably for any smooth divisor in $\mathbb{P}^2$. We intend to explore this question for singular divisors using our technique in the future.

4.2 Notations and Preliminaries

In the first section, we recall the singular curves mean that the degree d curve has some singularity in it. We specialized ourselves to only A_k singularities as earlier. Now we want to study the geometry of these singular curves that are tangent to any order to a fixed-line E in $\mathbb{P}^2$.

We will now introduce some more notations and definitions which will help us to present our result precisely. Let us denote $\mathscr{D}$ to be the space of non-zero homogeneous degree d-polynomials in three variables upto scaling, i.e.,

$$\mathscr{D} := \mathbb{P}(H^0(\mathbb{P}^2, \mathscr{O}(d))) \approx \mathbb{P}^{\delta_d}.$$

Let $\mathfrak{X}$ be a singularity of a given type. We will make abuse of notation and we will denote $\mathfrak{X}$ to be the space of curves and a marked point p such that the curve has a singularity of type $\mathfrak{X}$ at p. More precisely,

$$\mathfrak{X} := \{([f],\ p) \in \mathscr{D} \times \mathbb{P}^2 : f \ \text{ has a signularity of type } \mathfrak{X} \text{ at } p\}.$$

For example,

$$A_3 := \{([f],\ p) \in \mathscr{D} \times \mathbb{P}^2 : f \ \text{ has } A_3 \text{ singularity at } p\}.$$

Next, given n subsets $M_1, M_2, \ldots, M_n$ of $\mathscr{D} \times \mathbb{P}^2$, we define

$$M_1 \circ M_2 \circ \ldots \circ M_n := \{([f],\ p_1, \ldots, p_n) \in \mathscr{D} \times (\mathbb{P}^2)^n : ([f],\ p_1) \in M_1, \ldots, ([f],\ p_n) \in M_n, \\ \text{and} \quad p_1, \ldots, p_n \quad \text{are all distinct}\}.$$

For example, $A_1^3 \circ A_2$ denotes the space of curves with four distinct ordered points, where it has three ordinary nodes at three points and a cusp at the last point. Similarly, $A_1 \circ \overline{A}_2$ is the space of curves with two distinct ordered points, where it has a simple node at the first point and a singularity at least degenerate as cusp at the last point; the curve could have a tacnode at the second marked point.

For any n tuple $\mathbf{K} = (k_1, \cdots, k_n)$, where each k_i are non-negative integers, we will define the following:

$$|\mathbf{K}| = k_1 + \cdots + k_n$$

We will denote the sequence $(0, \cdots, 1, \cdots, 0)$ by e_n that is all the entries of e_n are zero except 1 at the n-th position (so that any n tuples $\mathbf{m}$ can be be expressed as $\sum_{i=1}^n m_i e_i$).

Let us now define our object of study. Let E be a fixed line in $\mathbb{P}^2$. Then our basic objects are degree d curves in $\mathbb{P}^2$. Given a non-negative integer δ, let us define

$$\mathsf{T}_k := \{([\tilde{f}],\ [f],\ p) \in \mathscr{D}_1 \times \mathscr{D}_d \times \mathbb{P}^2 : \text{f tangent to } \tilde{f} \text{ of order } k \text{ at } p\ \}$$

Then we can define the following space as

$$A_n \circ \mathsf{T}_k := \{([\tilde{f}],\ [f],\ q,\ p) \in \mathscr{D}_1 \times \mathscr{D}_d \times \mathbb{P}^2 \times \mathbb{P}^2 : ([f],\ q) \in \overline{A}_n, \\ \text{f tangent to } \tilde{f} \text{ of order } k \text{ at } p\ , p \neq q\}$$

Similarly, for $\delta \geq 2$ we can define

$$A_1^\delta \circ T_k := \{([\tilde{f}],\ [f],\ q_1, \cdots, q_\delta,\ p) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^\delta \times \mathbb{P}^2 : ([f],\ q_1, \cdots, q_\delta) \in \overline{A_1^\delta} \\ \text{f tangent to } \tilde{f} \text{ of order } k \text{ at } p\ , p \neq q_i,\ \forall\ i\}$$

Let us define $N^E(\mathfrak{X}^\delta \circ T_{k_1} \circ \cdots \circ T_{k_n})$ to be the number of degree d curves in $\mathbb{P}^2$ having δ number singularities of type $\mathfrak{X}$ of codimension $cd_{\mathfrak{X}}$ and it is tangent to E, a fixed line in $\mathbb{P}^2$ of order $\mathbf{K} = (k_1, \cdots, k_n)$ at n points of E at intersection of $m_i,\ \forall\ i = 1, \cdots, n$ generic lines, (this notion due to Caporaso-Harris [11], i.e., when $m_i = 0$ it corresponds to the tangency at unspecified points and $m_i = 1$ corresponds to tangency at specified points), passing through appropriate number of points in general position.

In the simplest case, when $\delta = 0$, then $N^E(T_{k_1} \circ \cdots \circ T_{k_n},\ \mathbf{m})$ denotes the number of degree d curves in $\mathbb{P}^2$ with the order of tangency $\mathbf{K}$ to $E \in \mathbb{P}^2$ at the intersection of $\mathbf{m}$ generic lines, passing through $\delta_d - |\mathbf{K}| - |\mathbf{m}|$ points in general position. Similarly if $\delta = 1$ and $\mathbf{K} = 0$ the notation $N^E(\mathfrak{X}) = N(\mathfrak{X}, E)$ denotes the singularity $\mathfrak{X}$ lies on the divisor E.

Next, we will describe the tangency condition locally with the help of the following Lemma:

Lemma 4.2.1. *Let $f = f(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$ such that $f_{00} = 0$, $\nabla f_{|(0,0)} \neq 0$ and there exists a non-zero vector $\eta = (v_1, v_2) \in Ker(\nabla \tilde{f}_{|(0,0)})$ such that at the origin $\nabla f(\eta) = 0$. Let $x := v_1 r + v_2 s, y := -v_2 r + v_1 s$ and f_{ij} be the partial derivatives with respect to the new variables x and y. Then, the curve $f^{-1}(0)$ represents tangency of order k to $\tilde{f}^{-1}(0)$ at the origin if $f_{01} \neq 0$ and the directional derivatives T_i defined as $\frac{f_{i0}}{i!}$ are zero for all $i \leq k$ and $\mathsf{T}_{k+1} \neq 0$.*

Proof of observation: Let us consider that the Taylor expansion of F is given by

$$F(x,y) = \mathrm{A}_0(x) + \mathrm{A}_1(x)y + \mathrm{A}_2(x)y^2 + \dots.$$

By our assumption on F, $\mathrm{A}_1(0) \neq 0$. We claim that there exists a holomorphic function $\mathrm{B}(x,y)$ such that after we make a change of coordinates $y = \mathrm{B}(x,y)y_1$, the function F is given by

$$F = \hat{\mathrm{A}}_0(x) + \mathrm{B}(x,y)y_1$$

for some $\mathrm{B}(x,y)$ (i.e., $\mathrm{B}(x,y) = f_{01} + f_{11}x + f_{02}y + \cdots$). Since $\mathrm{A}_1(0) \neq 0$, $\mathrm{B}(x,y)$ exists by the Implicit Function Theorem. Therefore, we can compute $\mathrm{B}(x,y)$ as a power series using (3.11) and then compute $\hat{\mathrm{A}}_0(x)$. Hence,

$$F(x,y) = \mathrm{B}(x,y)y_1 + \mathsf{T}_1 x + \frac{\mathsf{T}_2}{2!}x^2 + \frac{\mathsf{T}_3}{3!}x^3 + \dots, \text{ where } \frac{\mathsf{T}_i}{i!} = f_{i0} \tag{4.1}$$

Remark 4.2.2. *Note that* $\mathrm{B}(x,y)$ *is unique if we require* $\mathrm{B}(0,0) = 0$.

Now, since $\mathrm{B}(0,0) \neq 0$ we can further change of coordinate as $\hat{y} = \mathrm{B}(x,y)y_1$. Then the above reduces to

$$F(x,\hat{y}) = \hat{y} + f_{10}x + \frac{f_{20}}{2!}x^2 + \frac{f_{30}}{3!}x^3 \dots$$

4.3 Recursive formulas

In this section, we will state our results explicitly. This section is a part of an ongoing project. However, we will provide the proofs of weaker results than expected. Our main results can be summarized as follows:

Theorem 4.3.1. *Let* $\mathbf{K} = (k_1, \cdots, k_n)$, and $\mathbf{m} = (m_1, \cdots, m_n)$ *be two n tuples consisting of non-negative integers. The number $N_d^E(T_{k_1} \circ \cdots \circ T_{k_n}, \mathbf{m})$ denotes the number of degree d curves in $\mathbb{P}^2$ tangent to E* *at n distinct points in E of order* k_i, $\forall\, i = 1, \cdots, n$ *passing through* $\delta_d - |\,\mathbf{K}\,| - |\,\mathbf{m}\,|$ *general points and the tangency points are at the intersection of* $\mathbf{m}$ *generic lines. Then we have established a recursive formula for* $N^E_d(T_{k_1} \circ \cdots \circ T_{k_n},\ \mathbf{m})$ *provided* $d \geq |\,\mathbf{K}\,|$.

The above theorem describes the case when there are no singular points involved, i.e., all the points of our study are only tangency points.

Remark 4.3.2. *Note that the numbers obtained from the above formula can be verified with the result of Caporaso-Harris[11]. Let us mention an important point, our underlying geometry on curves with so many tangency conditions has no relation with the result obtained [11]. Also, note that in [11], the numbers for curves having tangency condition are bi-product to some other question namely, counting nodal curves question. So, in that perspective, this result is different from the result [11].*

Next, we will invoke singular points along with the tangency point.

Theorem 4.3.3. *Let* $\mathfrak{X} = A_1$ *or* A_2 *be two singularities of codimension* 1 *and* 2 *respectively. Let* k, m_1 *and* $\ell = (n_1, \cdots, n_\delta)$ *be a tuple of non-negative integers. Then for* $\delta \in \{1, 2\}$ *we established explicit recursive formulas for* $N^E(A_1^\delta \circ T_k,\ \ell,\ m)$ *and* $N^E(A_2 \circ T_k,\ n_1,\ m)$ *respectively provided* $d \geq d_{min}$, *where* $d_{min} := cd_{\mathfrak{X}} + k + 1$.

Theorem 4.3.4. *Let* m_1, m_2 *are two non-negative integers. Let us denote by* $N_d(A_1^L \circ A_1^L,\ m_1,\ m_2)$ *the number of degree d curves in* $\mathbb{P}^2$ *having two nodes lying on the same line passing through* $\delta_d - 4$ *points in general position, both the nodes are at intersection of* m_1 *and* m_2 *generic lines. We have an explicit formula to compute* $N_d(A_1^L \circ A_1^L,\ m_1,\ m_2)$ *provided* $d \geq 2$.

Remark 4.3.5. *As an important corollary of the above theorem, we can compute curves of degree d having a tacnode passing through a right number of generic points in* $\mathbb{P}^2$. *Note that this study does not invoke cuspidal curves.*

Remark 4.3.6. *As we have pointed out the fact in the introduction that when the divisor is singular, we have no answer to these above questions.*

Let us now briefly describe the procedure to compute our numbers and what are the actual difficulties we face along the way. Suppose we want to count the singular curve in $\mathbb{P}^2$ with some higher tangencies. Then one of the main difficulties is to study the type of the resultant singularity when the point of singularity and the point of tangency of certain order collide with each other. For example, suppose we want to enumerate $N^E(A_1 \circ \mathsf{T}_2)$, i.e., the number of degree d curves in $\mathbb{P}^2$ having a node tangent to a fixed-line to order two.

$$A_1 \circ \mathsf{T}_2 := \{([\tilde{f}],\ [f],\ q,\ p) \in \mathscr{D}_1 \times \mathscr{D} \times (\mathbb{P}^2)^2 \mid ([f],\ q) \in \overline{A}_1,\ f \text{ is tangent to } \tilde{f} \text{ of order } 2, p \neq q\}.$$

Our first step would be to find some suitable space that we already have studied and whose closure contains $A_1 \circ \mathsf{T}_2$ as a subset. Here an obvious candidate is a space $A_1 \circ \mathsf{T}_1$ which we have studied earlier. Let us consider μ to be the homology class representing the enumerating constraint, i.e., the constraint which fixes the line and it should pass through $\delta_d - 2$ generic points in $\mathbb{P}^2$. Now we will make use of the lemma (4.2.1) and we can express the number $N^L(A_1 \circ \mathsf{T}_2)$ as the cardinality of the set

$$\{([\tilde{f}],\ [f],\ q,\ p) \in \overline{A_1 \circ \mathsf{T}_1} \mid \nabla^2 f_p(v \otimes v) = 0,\ v \in Ker(\nabla \tilde{f}_p)\} \cap \mu. \tag{4.2}$$

Note that $\overline{A_1 \circ \mathsf{T}_1} = \overline{\overline{A}_1 \circ T_1}$. We now have to study the following boundary

$$\mathscr{B} = \{([\tilde{f}],\ [f],\ q,\ p) \in A_1 \circ \mathsf{T}_1 \mid q = p\}.$$

Next, we will interpret the condition $\nabla^2 f_p(v \otimes v) = 0$ as a section of a vector bundle $\mathscr{L}_{\mathsf{T}_2}$ over the space $\overline{A_1 \circ T_1}$ and we will show that the induce section is transverse to the zero set. Thus the number

$$\left\langle e(\mathscr{L}_{\mathsf{T}_2}),\ [\overline{A_1 \circ T_1}] \cap [\mu] \right\rangle = N_d^L(A_1 \circ \mathsf{T}_2)\ +\ C_{B\cap\mu}.$$

The main difficulty is to compute $C_{B\cap\mu}$ in the above formula. Note that the left-hand side can be easily calculated using the properties of vector bundles. So the main obstacle to solve these type of questions is to understand $\mathscr{B}$ and the excess contribution of it to the Euler class. In this thesis, almost all the problems that we have studied have analogous difficulties. We have used a topological method as indicated in the second chapter to approach the above issue.

4.4 Proof of the recursive formulas

We are now ready to prove the recursive formulas that we have stated in section (4.3). We will begin by proving the following:

Theorem 4.4.1. *Let k, m be two non-negative integers and $N_d^E(T_k,\ m)$ denotes the number of degree d plane curves tangent to the divisor E to order k passing through $\delta_d - k - m$ points in general position* *at the intersection of m generic lines. When E is a line in $\mathbb{P}^2$ we have*

$$N_d^E(T_k,\, m) = \begin{cases} (k+1)(d-k) & \text{if } m=0 \\ d-k & \text{if } m=1 \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

provided $d \geq k$.

Proof. Let us define

$$\mathsf{T}_k := \{([f_d],\, [\tilde{f}],\, p) \in \mathscr{D}_d \times \tilde{\mathscr{D}} \times \mathbb{P}^2 : f_d \text{ is tangent to } \tilde{f} \text{ at } p \text{ to order } k\}$$

Let us consider μ to be the generic cycle representing the following cycle

$$[\mu] = y_d^{\delta_{d-k}} y_1^{\delta_{d_1}} a^m$$

We now define the sections of the following bundles:

$$\begin{aligned} \psi_{\mathsf{T}_0} : \mathscr{D}_d \times \tilde{\mathscr{D}} \times \mathbb{P}^2 \longrightarrow \mathscr{L}_{\mathsf{T}_0} := \gamma_{\mathscr{D}_d}^* \otimes \gamma_{\tilde{\mathscr{D}}} \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes \gamma_{\mathbb{P}^2}^{*d_1} \text{ is defined by} \\ \{\psi_{\mathsf{T}_0}([f_d],\, [\tilde{f}],\, p)\}(f_d, \tilde{f}) = (f_d(p), \tilde{f}(p)) \\ \psi_{\mathsf{T}_1} : \psi_{\mathsf{T}_0}^{-1}(0) \longrightarrow \mathscr{L}_{\mathsf{T}_1} := \gamma_{\mathscr{D}_d}^* \otimes L^* \otimes \gamma_{\mathbb{P}^2}^* \text{ is defined by} \\ \{\psi_{\mathsf{T}_1}([f_d],\, [\tilde{f}],\, p)\}(f_d \otimes v) = \nabla f_d(v) \\ \cdots \\ \cdots \\ \psi_{\mathsf{T}_k} : \psi_{\mathsf{T}_{k-1}}^{-1}(0) \longrightarrow \mathscr{L}_{\mathsf{T}_k} := \gamma_{\mathscr{D}_d}^* \otimes L^{*k} \otimes \gamma_{\mathbb{P}^2}^* \text{ is defined by} \\ \{\psi_{\mathsf{T}_k}([f_d], [\tilde{f}],\, p)\}(f_d \otimes \underbrace{v \otimes \cdots \otimes v}_{k}) = \nabla^k f_d(\underbrace{v \otimes \cdots \otimes v}_{k}) \end{aligned} \tag{4.4}$$

we will show shortly that these sections above are all transverse to zero.

Hence

$$\langle e(\mathscr{L}_{\mathsf{T}_k}), [\overline{\mathsf{T}_{k-1}}] \cap [\mu] \rangle = N_d^E(\mathsf{T}_k,\, m) + C_{B\cap\mu} \tag{4.5}$$

where $C_{B\cap\mu}$ denote the contribution from the points of the boundary $B \cap \mu$ to the Euler class. We now claim that $B \cap \psi_{\mathsf{T}_i}^{-1}(0) = \phi$ for all $i \geq 1$. Since we are only interested in the components of the boundary where the sections vanish. Now suppose that the point degenerates to a singular point then we can see that the corresponding vertical derivative for tangency (which is the derivative along the divisor) can not vanish. Hence plugging $C_{B\cap\mu} = 0$ and computing the left-hand side of the equation

(4.5) we will get our desired formula.

Let us now explain how to compute the left-hand side of the equation (4.5). We will denote $y := c_1(\gamma^*_{\mathscr{D}_d}), y_1 := c_1(\gamma^*_{\tilde{\mathscr{D}}})$ and $a := c_1(\gamma^*_{\mathbb{P}^2})$. Recall that μ represents the subspace of curves passing through $y^{\delta_d-k}y_1^{\delta_{d_1}}$ generic points; hence μ represents the homology class Poincaré dual to $y^{\delta_d-k}y_1^{\delta_{d_1}}$. Hence, the left-hand side of equation (4.5) is equal to

$$\langle e(\mathscr{L}_{\mathsf{T}_k}), [\overline{\mathsf{T}_{k-1}}]\cap[\mu]\rangle = \langle y\, e(\mathscr{L}_{\mathsf{T}_{k-1}}),\ [\overline{\mathsf{T}_{k-1}}]\cap[\mu]\rangle +$$

$$\langle k\, y_1\, e(\mathscr{L}_{\mathsf{T}_{k-1}}),\ [\overline{\mathsf{T}_{k-1}}]\cap[\mu]\rangle + (d-2k)\langle a\, e(\mathscr{L}_{\mathsf{T}_{k-1}}),\ [\overline{\mathsf{T}_{k-1}}]\cap[\mu]\rangle \quad \forall\, k\ \geq 1.$$

Claim 4.4.2. *Let us consider* $([f_d],[f_1],p) \in \mathsf{T}_{k-1}$. *Then there exist points* $([f^t_d],[\tilde{f}^t],p(t)) \in \mathsf{T}_k$ *sufficiently close to* $([f_d],[f_1],p)$, *such that*

$$\psi_{\mathsf{T}_k}([f_d],[\tilde{f}],p) = 0$$

where v is a nonzero vector belongs to $T_p\tilde{f}^{-1}(0)$. *Furthermore, every such solutions satisfies the condition*

$$\psi_{\mathsf{T}_{k+1}}([f_d],[\tilde{f}],p) \neq 0.$$

Proof. We will prove the above working in an affine setting. Let $p = [0:0:1]$ and in this affine coordinate system the Taylor series expansion of the curve is

$$F(x,y) = \mathrm{A}_0(x) + \mathrm{A}_1(x)y + \mathrm{A}_2(x)y^2 + \dots.$$

Next, if the curve is tangent to $f_1^{-1}(0)$ at p_t to certain order then after a suitable change of coordinate we can express the curve as

$$F(x,\hat{y}) = \hat{y} + f_{t_{10}}x + \frac{f_{t20}}{2!}x^2 + \frac{f_{t30}}{3!}x^3 \dots \tag{4.6}$$

Now since $([f^t_d],[f^t_1],p_t) \in \mathsf{T}_{k-1}$, then the equation (4.6) reduces to

$$F(x_t,\hat{y}_t) = \hat{y}_t + \frac{f_{k0}}{k!}x_t^k + \dots$$

By choosing $p_t = (x_t, 0)$, we gather from above equation

$$\frac{f_{t_{k-1}0}}{k-1!} = \frac{f_{t_{k0}}}{k!}x_t + O(x_t^2) \tag{4.7}$$

So the above solution shows that T_k contributes to the Euler class with multiplicity one. $\square$

Let us now prove the corresponding transversality claims. We will consider the affine setup where $p=[0:0:1]$. Let us consider ∂_x and ∂_y to be the standard basis vectors for $T\mathbb{P}^2_{|_p}$. Next, let us look at the following short exact sequence

$$0 \longrightarrow L = Ker(\nabla \tilde{f}) \longrightarrow T\mathbb{P}^2 \longrightarrow \gamma^*_{\tilde{\mathscr{D}}} \otimes \gamma^{*d_1}_{\mathbb{P}^2} \longrightarrow 0$$

then there exist a nonzero vector $v \in Ker(\nabla \tilde{f}_{|_p})$. We can certainly assume that $v=[\partial_x]$. Let us now consider the polynomial

$$\rho_{k0} = X^k Z^{d-k}.$$

We observe that

$$\rho_{k0}(p)=0,\ \nabla^i \rho_{k0}(p)=0,\ \forall\ 1 \leq i \leq k-1,\ \ \nabla^k \rho_{k0}(p) \neq 0.$$

Consider the curve $\gamma_{k0} : (-\varepsilon, \varepsilon) \longrightarrow \mathscr{D}_d \times \tilde{\mathscr{D}} \times \mathbb{P}^2$ defined by

$$\gamma_{k0}(t) = (f_d + t\rho_{k0},\ \tilde{f},\ p)$$

Note that $\{\nabla \psi_{\mathsf{T}_k}([f_d],\ [\tilde{f}],\ p)\}(\gamma'_{k0}(0)) = \nabla^k \rho_{k0}(p) \neq 0$ holds for any k. Hence the transversality follows. □

Next, we will study the case when there are multiple points of tangency involved. Before going into the proof of the general case let us concentrate on some particular computations as below:

Computation of the number $N(\mathsf{T}_1 \circ \mathsf{T}_2,\ m_1,\ m_2)$

Let us define the space

$$\mathsf{T}_1 \circ \mathsf{T}_2 := \{([\tilde{f}],\ [f],\ q_1,\ q_2) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^2 : \text{f tangent to } \tilde{f} \text{ of order one and two at } q_1,\ q_2 \text{, where } q_1 \neq q_2.\}$$

Let us consider μ to be the generic cycle representing the following cycle

$$[\mu] = y_d^{\delta_d - 2} y_1^{\delta_{d_1}} a_1^{m_1} a_2^{m_2}.$$

Then $N(T_1 \circ T_2)$ is the cardinality of the set

$$\{([\tilde{f}],\ [f],\ q_1,\ q_2) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^2 : ([\tilde{f}],\ [f],\ q_1,\ q_2) \in \overline{\mathsf{T}_1 \circ \mathsf{T}_1}$$
$$\nabla^2 f(v,v)_{|_{q_2}} = 0, \nabla^3 f(v,v,v)_{|_{q_2}} \neq 0, v \in T_{q_2} L\} \cap \mu.$$

We now consider section

$$\psi_{T_2} : \overline{\mathsf{T}_1 \circ \mathsf{T}_1} \longrightarrow \mathscr{L}_{T_2} := \gamma_{\mathscr{D}}^* \otimes T^*L^{\otimes 2} \otimes \gamma_{\mathbb{P}^2}^{*d}.$$

We will show that the above section is transverse to zero. Hence

$$\langle e(\mathscr{L}_{T_2}), [\overline{\mathsf{T}_1 \circ \mathsf{T}_1}] \cap [\mu] \rangle = N^E(T_1 \circ T_2) + C_{\mathscr{B}\cap\mu}$$

where

$$\mathscr{B} := \{([\tilde{f}],\ [f], q_1, q_2) \in B : q_1 = q_2\}.$$

Hence, $C_{\mathscr{B}\cap\mu} = N^E(T_3)$. We have shown that it contributes to the Euler class with multiplicity 2.

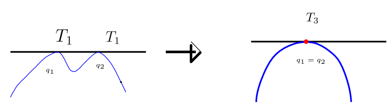

Figure 4.1: When two T_1's collide to each other.

Computation of the number $N_d(\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_4,\ m_1,\ m_2,\ m_3)$

Let us defined the space

$$\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_3 := \{([\tilde{f}],\ [f],\ q_1,\ q_2, q_3) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^3 : f \text{ tangent to } \tilde{f} \text{ of order one, two and three at } q_1,\ q_2 \text{ and } q_3 \text{ respectively }, q_1 \neq q_2 \neq q_3\}.$$

Let us consider μ to be the generic cycle representing the following cycle

$$[\mu] = y_d^{\delta_d - 7} y_1^{\delta_{d_1}} a_1^{m_1} a_2^{m_2} a_3^{m_3}.$$

Then $N(\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_4,\ m_1,\ m_2,\ m_3)$ is the cardinality of the set

$$\{([\tilde{f}],\ [f],\ q_1,\ q_2,\ q_3) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^3 : ([f_1], [f], q_1, q_2, q_3) \in \overline{\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_3}, \nabla^4 f(v^{\otimes 4})_{|q_3} = 0\} \cap \mu.$$

We now consider section

$$\psi_{T_4} : \overline{\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_3} \longrightarrow \mathscr{V}_{T_4} := \gamma_{\mathscr{D}}^* \otimes T^*L^{\otimes 4} \otimes \gamma_{\mathbb{P}^2}^{*d}.$$

We will show that the above section is transverse to zero. Hence

$$\langle e(\mathscr{V}_{T_4}), [\overline{\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_3}] \cap [\mu] \rangle = N^E(T_1 \circ T_2 \circ \mathsf{T}_4,\ m_1,\ m_2,\ m_3) + C_{\mathscr{B}\cap\mu}$$

where

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2.$$

We will describe

$$\mathscr{B}_1 := \{([\tilde{f}],\ [f], q_1, q_2, q_3) \in B : q_1 = q_3\}$$

and

$$\mathscr{B}_2 := \{([\tilde{f}],\ [f], q_1, q_2, q_3) \in B : q_2 = q_3\}.$$

We will show that $\mathscr{B}_1 \approx \mathsf{T}_5$ and $\mathscr{B}_2 \approx \mathsf{T}_6$. Furthermore, we will show that the contribution from T_5 to the Euler class is 2 and T_6 contributes 3 to the Euler class. Plugging all these we will get the final recursion for $N(\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_4,\ m_1,\ m_2,\ m_3)$. Here we omit the proofs of all the claims that we have made since we will prove the general statement in one go.

Proof of the Theorem (4.3.3) **for $\delta = 0$ case**

Proof. We will now prove our formula for $N_d^E(\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n},\ \mathbf{m})$, when E is a fixed line in $\mathbb{P}^2$. Let us now define

$$\begin{aligned} \mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n} := \{([f_d],\ [\tilde{f}],\ p_1, \cdots, p_n) \in \mathscr{D}_d \times \tilde{\mathscr{D}} \times (\mathbb{P}^2)^n : ([f_d],\ [\tilde{f}],\ p_1, \cdots, p_n) \\ \in \overline{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n}},\ \ f_d \ \text{ is tangent to } \tilde{f} \text{ at } p_i \text{ to order } k_i, \forall\, i = 1, \cdots, n\}. \end{aligned}$$

Let us consider μ to be the generic cycle representing the following cycle

$$[\mu] = y_d^{\delta_d - k_1 \cdots - (k_n - 1) - m_1 \cdots - m_n} y_1^{\delta_{d_1}} a_1^{m_1} \cdots a_n^{m_n}$$

We will define the section corresponding to the k_n-th order tangency to the divisor at the last marked point as:

$$\psi_{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n}} : \psi^{-1}_{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n - 1}}(0) \longrightarrow \mathscr{L}_{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n}} := \gamma^*_{\mathscr{D}_d} \otimes L^{*k_n} \otimes \gamma^*_{\mathbb{P}^2}$$

the section is defined as

$$\{\psi_{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n}}([f_d], [\tilde{f}], p_1, \cdots, p_n)\}(f_d \otimes \underbrace{v \otimes \cdots \otimes v}_{k_n}) = \nabla^{k_n} f_d(\underbrace{v \otimes \cdots \otimes v}_{k_n}). \tag{4.8}$$

We will prove very shortly that this section is transverse to zero.

Let us now define $\mathscr{B} = \overline{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n - 1}} - \mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n - 1}$.

Hence

$$\langle e(\mathscr{L}_{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n}}),\ [\overline{\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n - 1}}] \cap [\mu] \rangle = N_d^E(\mathsf{T}_{k_1} \circ \cdots \circ \mathsf{T}_{k_n},\ \mathbf{m}) + C_{B \cap \mu} \tag{4.9}$$

where $C_{B\cap\mu}$ denote the excess contribution from the boundary $B\cap\mu$ to the Euler class. We have seen that $B\cap\psi_{\mathsf{T}_{k_i}}^{-1}(0)=\phi$ for all $i\geq 1$ when there is only one tangency point involved. Now we will study the boundary when there is more tangency point involved. We will define

$$\mathscr{B}_{i_1,\cdots,i_n} := \{([f_d],[\tilde{f}],p_1,\cdots,p_n)\in \mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}:\ \ p_{i_1}=\cdots=p_{i_n}\}.$$

Let us now consider $\mathscr{B}_{i,n},\ \forall\ i=1,\cdots,n.$

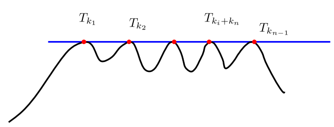

Figure 4.2: when T_{k_i} and T_{k_n} are collide to each other.

We claim that $\mathscr{B}_{i,n}\approx\overline{\mathsf{T}}_{k_{i+n}}$, where $\mathscr{B}_{i,n}$ indicates i-th point collides to the last point. We will justify this claim shortly. Note that $\psi_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}$ section vanishes on $\overline{\mathsf{T}}_{k_{i+n}}$, hence it also vanishes on $\overline{\mathsf{T}}_{k_{i+n}}\cap\mu$. We claim that the excess contribution from each of the points of $\mathscr{B}_{i,n}\cap\mu$ is $(i+1)\ N_d^E(\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{n+i},\ \mathbf{m})$. Thus the total contribution to the Euler class from the two pointed boundary is

$$\sum_{i=1}^{n}(i+1)\ N_d^E(\mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_{n+i}},\ \mathbf{m}).$$

Next, we will consider the three pointed boundary namely, $\mathscr{B}_{i,j,n}$. We claim that $\mathscr{B}_{i,j,n}\cap\psi_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}^{-1}=\phi$. In fact $\mathscr{B}_{i_1,\cdots,i_n}\cap\psi_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}^{-1}=\phi$ for all $n\geq 3$. That is when three or more tangency points collides to each other then there is no contribution to the Euler class from the boundary points. Hence plugging $C_{B\cap\mu}$ and unwinding the left-hand side of the equation (4.9) we will get our desired formula.

The computation on the left-hand side of the equation (4.9) follows in from similar computation as in the equation (4.5).

$$\begin{aligned}
&\langle e(\mathscr{L}_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}),[\overline{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n-1}}]\cap[\mu]\rangle\\
&=\langle y\ e(\mathscr{L}_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}),\ [\overline{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_{n-1}}}]\cap[\mu]\rangle+\\
&\langle k_n\ y_1\ e(\mathscr{L}_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}),\ [\overline{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_{n-1}}}]\cap[\mu]\rangle+\\
&(d-2k_n)\langle a\ e(\mathscr{L}_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}),\ [\overline{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_{n-1}}}]\cap[\mu]\rangle\quad\forall\ k\ \geq 1
\end{aligned}$$

Finally, we will prove the closure, multiplicity, and transversality claim that we indicated earlier.

Claim 4.4.3. *Let us consider for each* $1\leq i\leq n-1,\ ([f_d],[f_1],p_1,\cdots,\hat{p}_i,\cdots,p_n)$ $\in\mathsf{T}_{k_1}\circ\cdots\circ\hat{\mathsf{T}}_{k_i},\cdots\circ\mathsf{T}_{k_{n+i}}$*. Then there exist points* $([f_d^t],[f_1^t],p_1(t),\cdots,p_n(t))$

$\in \mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_n}$ *sufficiently close to* $([f_d],[f_1],p_1,\cdots,\hat{p}_i,\cdots,p_n)$, *such that*

$$\psi_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}([f_d],[\tilde{f}],p_1,\cdots,p_n)=0$$

where v is a nonzero vector belongs to $T_{p_n}\tilde{f}^{-1}(0)$. Furthermore every such solutions satisfies the condition

$$\psi_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_{n+1}}}([f_d],[\tilde{f}],p_1,\cdots,p_n)\neq 0.$$

Remark 4.4.4. *Note that the above claim explains the following situation when three pointed tangency question:*

- *if the first and the third tangency point collide then we will get the following* $\overline{\mathsf{T}_{k_1}\circ\mathsf{T}_{k_2}\circ\mathsf{T}_{k_3}}\supset\overline{\mathsf{T}_{k_2}\circ\mathsf{T}_{k_3+k_1+1}}$
- *similarly if the second and the third tangency point collide then we will get the following* $\overline{\mathsf{T}_{k_1}\circ\mathsf{T}_{k_2}\circ\mathsf{T}_{k_3}}\supset\overline{\mathsf{T}_{k_1}\circ\mathsf{T}_{k_3+k_2+1}}$

the second part of the Lemma says that the nearby curve that found as small solution those can not lie in $\mathsf{T}_{k_1}\circ\mathsf{T}_{k_2}\circ\mathsf{T}_{k_3+1}$

Proof. We will concentrate on the affine setting. Let $p_n=[0:0:1]$ and in this affine coordinate system the Taylor series expansion of the curve is

$$F(x,y)=\mathrm{A}_0(x)+\mathrm{A}_1(x)y+\mathrm{A}_2(x)y^2+\ldots.$$

Now, in this affine setup we will consider the directional derivative $v=\partial_x$ where $v\in T_{p_n}\tilde{f}^{-1}(0)$. Next, if the curve is tangent to $\tilde{f}^{-1}(0)$ at $p_n(t)$ to certain order then after a suitable change of coordinate we can express the curve as

$$F(x,\hat{y})=\hat{y}+f_{t_{10}}x+\frac{f_{t_{20}}}{2!}x^2+\frac{f_{t_{30}}}{3!}x^3\ldots \tag{4.10}$$

Now since $([f_d^t],[\tilde{f}^t],p_1(t),\cdots,p_n(t))\in\pi_n^*(\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n})$, then the equation (4.10) reduces to

$$F(x_t,\hat{y}_t)=\hat{y}_t+\frac{f_{t_{k_n+1,0}}}{k_n+1!}x_t^{k_n+1}+\frac{f_{t_{k_n+2,0}}}{k_n+2!}x_t^{k_n+2}+\ldots \tag{4.11}$$

since the tangency condition on the last point implies that

$$f_d^t(p_n(t))=0,\nabla f^t_{d_{|_{p_n(t)}}}(v)=0,\ldots,\nabla^{k_n}f^t_{d_{|_{p_n(t)}}}(v^{\otimes k_n})=0.$$

The curve $([f_d],[f_1],p_1,\cdots,\hat{p}_i,\cdots,p_n)$ lies in $\mathsf{T}_{k_1}\circ\cdots\circ\hat{\mathsf{T}}_{k_i}\circ\cdots\circ\mathsf{T}_{k_{n+i+1}}$ so $f_{k_{n+i+1}}\neq 0$ and hence in the new coordinate system $f_{t_{k_{n+i+1}}}\neq 0$. Next, we will treat the equation (4.11) as primary equation to analyze the collision of T_i condition to the T_n-th condition. We will distinguish the resultant curve in equation (4.11) by denoting $\tilde{F}$, so that there will be no notational confusion.

Note that the T_i condition amounts to

$$\tilde{F}(p_i(t))=0,\nabla\tilde{F}_{|_{p_i(t)}}(v)=0,\ldots,\nabla^{k_i}\tilde{F}_{|_{p_i(t)}}(v^{\otimes k_i})=0.$$

Then the above implies to solve the following:

$$\begin{aligned}\hat{y}_t+\frac{f_{t_{k_n+1,0}}}{k_n+1!}x_t^{k_n+1}+\frac{f_{t_{k_n+2,0}}}{k_n+2!}x_t^{k_n+2}+\ldots&=0\\ \frac{f_{t_{k_n+1,0}}}{k_n!}x_t^{k_n}+\frac{f_{t_{k_n+2,0}}}{k_n+1!}x_t^{k_n+1}+\ldots&=0\\ &\vdots\\ \frac{f_{t_{k_n+1,0}}}{k_n-i!}x_t^{k_n-i}+\ldots+\frac{f_{t_{k_n+i+1,0}}}{k_n!}x_t^{k_n}+\ldots&=0\end{aligned}\tag{4.12}$$

From the above equations we can solve

$$\frac{f_{t_{k_n+i,0}}}{k_n+i!}=-\frac{f_{t_{k_n+i+1,0}}}{k_n+i+1!}\;x_t\;+O(x_t^2)$$

Likewise we can solve for other equations using the above solution. Finally, if we plug in the iterative solution from the equation (4.12) in the equation (4.10), then we get the small solution as

$$f_{t_{k_n+i}}=Ax_t^{i+1}+O(x_t^{i+2})\;\;\forall\, i\geq 1,\;\;\text{where}\;\;A=(-1)^{i+1}\frac{(i+1)\;P(k)\;f_{t_{k_n+i+1,0}}}{(k_n)(k_n+1)\cdots(k_n+i)}x_t^{i+1},\tag{4.13}$$
and $P(k)$ is a non-homogeneous polynomial of degree i.

Since both $(x,y)\neq 0$, therefore, we can assume $x\neq 0$ but small and $A\neq 0$ giving us the required small solution. □

Corollary 4.4.5. *Let $\psi_{\mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_n}}:\psi^{-1}_{\mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_n}}(0)\longrightarrow\mathscr{L}_{\mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_n}}$ as introduced before. The rank of this vector bundle is same as the dimension of $\triangle_{i,n}(\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n})$ and $\psi_{\mathsf{T}_{k_1}\circ\cdots\circ\mathsf{T}_{k_n}}$ a generic smooth section. Suppose that the curve with ordered tangency $([f_d],[f_1],p_1,\cdots,\hat{p}_i,\cdots,p_n)\in\mathsf{T}_{k_1}\circ\cdots\circ\hat{\mathsf{T}}_{k_i}\circ\cdots\circ\mathsf{T}_{k_{n+i}}$. Then the section*

$$\pi_i^*\psi_{\mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_n}}:\psi^{-1}_{\mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_n}}(0)\longrightarrow\mathscr{L}_{\mathsf{T}_{k_1},\cdots,\mathsf{T}_{k_n}}$$

vanishes around $([f_d],[f_1],p_1,\cdots,\hat{p}_i,\cdots,p_n)$ with the multiplicity $(i+1)$.

Proof. This follows from the equation (4.13). □

Let us now prove the corresponding transversality claims. We will mainly follow the earlier affine setup where $p_n = [0:0:1]$. As before let us consider ∂_x and ∂_y to be the standard basis vectors for $T\mathbb{P}^2_{|p_n}$. We will assume as before $v = [\partial_x]$. Let us now consider the polynomial

$$\rho_{k_1\cdots k_n} = \Big(\frac{Z(p_1)}{X(p_1)}X - Z\Big)^{k_1} Z^{d-k_1} + \cdots + \Big(\frac{Z(p_{n-1})}{X(p_{n-1})}X - Z\Big)^{k_n - 1} Z^{d-k_n+1} X^k\, Z^{d-k_n}.$$

We observe that

$$\rho_{k_1\cdots k_n}(p_i) = 0, i = 1, \cdots, n-1, \;\; \cdots, \nabla^{k_i}\rho_{k_1\cdots k_n}(p_i) = 0,\; i = 1, \cdots, n-1 \text{ but } \nabla^k \rho_{k_1\cdots k_n}(p_n) \neq 0.$$

Consider the curve $\gamma_{k_1\cdots k_n} : (-\varepsilon, \varepsilon) \longrightarrow \mathscr{D}_d \times \tilde{\mathscr{D}} \times (\mathbb{P}^2)^n$ defined by

$$\gamma_{k_1,\cdots,k_n}(t) = (f_d + t\rho_{k_1\cdots k_n}, \tilde{f}, p_1, \cdots, p_n).$$

Note that $\{\nabla \psi_{\mathsf{T}_{k_n}}([f_d],[\tilde{f}], p_1, \cdots, p_n)\}(\gamma'_{k_1\cdots k_n}(0)) = \nabla^{k_n}\rho_{k_1\cdots k_n}(p_n) \neq 0$ holds for any k_i and any of the n points. Hence the transversality follows. □

4.5 Proof of the recursive formulas involving singularity

In this section we will study the higher-order contact with the divisor E when the underlying curve is singular i.e, the curve may have node or cusp as singularities in it. Next, we will proceed to the cases when $\delta \neq 0$.

Computation of the number $N_d^E(A_1 \circ \mathsf{T}_k,\, n_1,\, m)$ for all k

Note that this question can be seen as a special case of the main result [11] when $n_1 = 0$. However, for $n_1 \neq 0$ this can be obtained via their method with some effort. We will compute these numbers using a topological method. Our numbers are consistent with those obtained [11].

Theorem 4.5.1. *Let $N_d^E(A_1 \circ \mathsf{T}_k,\, n_1,\, m)$ denotes the number of degree d curves having a node tangent to E of order k passing through $\delta_d - (k+1) - n_1 - m$ generic points where the node is at the intersection of n_1 and the tangency is at m generic lines. Then the recursive formula for $N_d(A_1 \circ \mathsf{T}_k,\, n_1,\, m)$ is given by*

$$N_d^E(A_1 \circ \mathsf{T}_k,\, n_1,\, m) = \begin{cases} 2(d-1)N_d(A_1,\, n_1) - 2N_d(A_1; E,\, n_1) & \text{for } k=1,\, m=0 \\ N_d(A_1,\, n_1) - 2N_d(A_1; E,\, n_1+1) & \text{for } k=1,\, m=1 \\ 0 & \text{for } k=1,\, m\geq 2 \\ \text{for } k \geq 2, & \\ & \\ N_d^E(A_1 \circ \mathsf{T}_{k-1},\, n_1,\, m) + (d-2k)\, N_d^E(A_1 \circ \mathsf{T}_{k-1},\, n_1,\, m+1) & \\ -N_d^E(A_1^{\mathsf{T}_{k-1}},\, n_1 + m) & \end{cases} \tag{4.14}$$

provided $d \geq k+1$.

Proof. Let us define the space

$$A_1 \circ \mathsf{T}_k := \{([f],\ [\tilde{f}],\ q,\ p) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^2 : ([f],\ q) \in A_1$$
$$f \text{ tangent to } \tilde{f} \text{ of order } k,\ q \neq p\}.$$

We will apply theorem (3.1.3), equation (3.1) when $k = 1$.

Next, we will focus on the case $k \geq 2$. Let us consider μ to be the generic cycle representing the following cycle

$$[\mu] = y_d^{\delta_d - k - 1} y_1^{\delta_{d_1}} a^{n_1} a_1^m.$$

We will abbreviate $N_d^E(A_1 \circ \mathsf{T}_k,\ n_1,\ m)$ as $N_d^E(A_1 \circ T_k)$. Then the number $N_d^E(A_1 \circ T_k)$ is the cardinality of the set

$$\{([f],\ [\tilde{f}],\ q,\ p) \in A_1 \circ \overline{\mathsf{T}}_{k-1} : \nabla^k f(v^{\otimes k})_{|_p} = 0, v \in T_p L\} \cap \mu.$$

We now consider the section

$$\Psi_{T_k} : A_1 \times \overline{\mathsf{T}}_{k-1} \longrightarrow \mathscr{L}_{T_k} := \gamma_{\mathscr{D}}^* \otimes T^* L^{\otimes k} \otimes \gamma_{\mathbb{P}^2}^{*d}.$$

We will show that the above section is transverse to the zero set which is equivalent to show that the the line bundle $\mathscr{L}_{T_k}$ is $k+1$ ample. Let us now define

$$\mathscr{B} = \overline{A_1 \circ \overline{\mathsf{T}}_{k-1}} - A_1 \circ \overline{\mathsf{T}}_{k-1}.$$

Hence

$$\left\langle e(\mathscr{L}_{T_k}),\ [\overline{A_1 \circ \overline{\mathsf{T}}_{k-1}}] \cap [\mu] \right\rangle = N_d^E(A_1 \circ T_k)\ +\ C_{\mathscr{B} \cap \mu} \tag{4.15}$$

where the excess contribution $C_{\mathscr{B} \cap \mu}$ comes from the points of $\mathscr{B} \cap \mu$ to the Euler class. We will prove it in [50]. We note that only the points of $\mathscr{B}$, where the section vanishes is relevant for us. Hence we are only interested to the component of $\mathscr{B} \cap \mu$ where q becomes equal to p. Next, let us define

$$\mathscr{B}(q,p) := \{([f],\ [\tilde{f}],\ q,\ p) \in B : q = p\}.$$

We claim that $\mathscr{B}(q,\ p) \approx A_1^{\mathsf{T}_{k-1}}$. Geometrically, $\mathscr{B}(q,p)$ denotes the component of the boundary where the nodal point and the point of tangency collide with each other. So we can expect the following thing

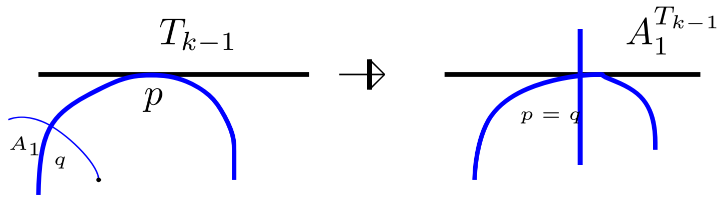

Figure 4.3: Nodal curve and is tangent to the line of any order.

to happen:

We claim that the excess contribution from $A_1^{\mathsf{T}_{k-1}}$ to the Euler class is 1. We plug in the value for $\mathscr{B} \cap \mu$ in (4.15) we get the final recursion.

Let us now prove the corresponding transversality claims. We will consider the affine setup as before. Let us assume the tangency point $p = [0:0:1]$ and $q = [x_1 : y_1 : 1]$. Let us consider ∂_x and ∂_y to be the standard basis vectors for $T\mathbb{P}^2_{|_p}$. Next, let us look at the following short exact sequence

$$0 \longrightarrow L = Ker(\nabla \tilde{f}) \longrightarrow T\mathbb{P}^2 \longrightarrow \gamma^*_{\tilde{\mathscr{D}}} \otimes \gamma^{*d_1}_{\mathbb{P}^2} \longrightarrow 0.$$

Then there exist a nonzero vector $v \in Ker(\nabla \tilde{f}_{|_p})$. Without loss of generality we will assume that $v = [\partial_x]$. Next, consider the polynomial

$$\rho_{1k} = \Big(\frac{Z(q)}{X(q)}X - Z\Big)^2 X^k Z^{d-k-2}.$$

We observe that

$$\rho_{1k}(p) = 0,\ \nabla^i \rho_{1k}(p) = 0,\ \forall\ 1 \leq i \leq k-1\ \cdots, \nabla^k \rho_{1k}(p) \neq 0.$$

Consider the curve $\gamma_{1k} : (-\varepsilon, \varepsilon) \longrightarrow \mathscr{D}_d \times \tilde{\mathscr{D}} \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\gamma_{1k}(t) = (f_d + t\rho_{1k}, \tilde{f}, q, p)$$

Note that $\{\nabla \psi_{\mathsf{T}_k}([f_d], [\tilde{f}], q, p)\}(\gamma'_{1k}(0)) = \nabla^k \rho_{1k}(p) \neq 0$ holds for any k. Hence the transversality follows. □

Now we will consider the enumerating curves with tangency when the curve has two nodes.

Computation of the number $N_d^E(A_1^2 \circ \mathsf{T}_k,\ n_1,\ m)$ for all k

Theorem 4.5.2. *Let $N_d^E(A_1^2 \circ \mathsf{T}_k,\ n_1,\ m)$ denotes the number of degree d curves having two nodes tangent to E of order k passing through $\delta_d - (k+2) - n_1 - m$ generic points where the last node is* *at the intersection of* n_1 *and the tangency point is at m generic lines respectively. Then the recursive formula for* $N_d(A_1^2 \circ \mathsf{T}_k,\ n_1,\ m)$ *is given by*

$$N_d^E(A_1^2 \circ \mathsf{T}_k,\ n_1,\ m) = \begin{cases} 2(d-1)N_d(A_1^2,\ n_1) - 4N_d(A_1 \circ A_1; E,\ n_1) & \text{for } k=1,\ m=0 \\ N_d(A_1^2,\ n_1) - 4N_d(A_1 \circ A_1; E,\ n_1+1) & \text{for } k=1,\ m=1 \\ 0 & \text{for } k=1,\ m\geq 2 \\ \text{for } k \geq 2, & \\ & \\ N_d^E(A_1^2 \circ \mathsf{T}_{k-1},\ n_1,\ m) \ +\ (d-2k)\ N_d^E(A_1^2 \circ \mathsf{T}_{k-1},\ n_1,\ m+1) & \\ -2\ N_d^E(A_1 \circ A_1^{\mathsf{T}_{k-1}},\ n_1+m) & \end{cases} \tag{4.16}$$

provided $d \geq k+3$.

Proof. Let us recall the space

$$A_1^2 \circ \mathsf{T}_k := \{([f],\ [\tilde{f}],\ q_1, q_2,\ p) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^3 : ([f],\ q_1, q_2) \in A_1^2$$
$$\text{f tangent to } \tilde{f} \text{ of order } k,\ q_1 \neq q_2 \neq p\}.$$

Note that when $k=1$, we will apply the theorem (3.1.3), equation (3.1). Hence we only focus on the calculation of the number $N_d^E(A_1^2 \circ \mathsf{T}_k)$ (here we are using an abuse of notation for $N_d^E(A_1^2 \circ \mathsf{T}_k,\ n_1,\ m)$) when $k \geq 2$.

Let us consider μ to be the generic cycle representing the following cycle

$$[\mu] = y_d^{\delta_d - k-2} y_1^{\delta_{d_1}} a^{n_1} a_1^m.$$

Then $N(A_1^2 \circ T_k)$ is the cardinality of the set

$$\{([f],\ [\tilde{f}],\ q_1,\ q_2,\ p) \in A_1^2 \circ \overline{\mathsf{T}}_{k-1} : \nabla^k f(v^{\otimes k})_{|p} = 0, v \in T_p L\} \cap \mu.$$

We now consider the following section

$$\Psi_{T_k} : A_1^2 \circ \overline{\mathsf{T}}_{k-1} \longrightarrow \mathscr{L}_{T_k} := \gamma_{\mathscr{D}}^* \otimes T^* L^{\otimes k} \otimes \gamma_{\mathbb{P}^2}^{*d}.$$

We will show that the above section is transverse to the zero set which is equivalent to show that the the line bundle $\mathscr{L}_{T_k}$ is $k+2$ ample. Let us now define

$$\mathscr{B} = \overline{A_1^2 \circ \overline{\mathsf{T}}_{k-1}} - A_1^2 \circ \overline{\mathsf{T}}_{k-1}.$$

Hence

$$\Big\langle e(\mathscr{L}_{T_k}),\ [\overline{A_1^2 \circ \overline{\mathsf{T}}_{k-1}}] \cap [\mu] \Big\rangle = N_d^E(A_1^2 \circ T_k) + C_{\mathscr{B} \cap \mu} \tag{4.17}$$

where $C_{\mathscr{B}\cap\mu}$ denotes the excess degenerate contribution of the section Ψ_{T_k} to the Euler class from the points of $\mathscr{B}\cap\mu$. We note that only the points of $\mathscr{B}$, where the section vanishes is relevant for us. Hence we are only interested to the component of $\mathscr{B}\cap\mu$ where q_i becomes equal to p. Next, let us define as before

$$\mathscr{B}(q_1,\ p) := \{([f],\ [\tilde{f}],\ q_1,\ q_2,\ p) \in B : q_1 = p\}.$$

We will prove it in [50] that

$$\mathscr{B}(q_1, p) \approx A_1 \circ A_1^{\mathsf{T}_{k-1}}. \tag{4.18}$$

Geometrically, $\mathscr{B}(q_1, p)$ denotes the component of the boundary where the nodal point and the point of tangency collide with each other. So we can expect the following thing to happen:

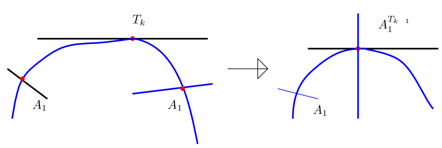

Figure 4.4: Curve having two nodes tangent to the line of any order.

Similarly we can show that the boundary contribution $\mathscr{B}(q_2,\ p) \approx A_1 \circ A_1^{\mathsf{T}_{k-1}}$ to the Euler class is 1. Next we will explain the computation of $N(A_1 \circ A_1^{\mathsf{T}_k},\ m)$. We will abbreviate $N(A_1 \circ A_1^{\mathsf{T}_k},\ m)$ as $N(A_1 \circ A_1^{\mathsf{T}_k})$ for notational simplicity.

Proposition 4.5.3. *For a positive integers $k \geq 1$, the number $N(A_1 \circ A_1^{\mathsf{T}_k})$ is given by*

$$\langle e(\mathscr{L}_{T_k}),\ [A_1 \circ A_1^{\mathsf{T}_{k-1}}] \cap\ [\mu] \rangle \ = \ N_d^E(A_1 \circ A_1^{\mathsf{T}_k}) \ + \ N_d^E(A_3^{T_{k-1}}) \tag{4.19}$$

provided $d \geq k+3$

Assuming our claim (4.18) and the proposition (4.5.3) we will show that the total boundary contribution is

$$2\ N_d^E(A_1 \circ A_1^{\mathsf{T}_{k-1}},\ m) \tag{4.20}$$

with multiplicity 1 to the Euler class. Thus plugging in the value of $C_{\mathscr{B}\cap\mu}$ from (4.20) we get the final recursion formula.

Let us now prove the corresponding transversality claims. We will consider the affine setup as before. Let us assume the tangency point $p = [0:0:1]$ and $q_i = [x_i : y_i : 1]$, $\forall\ i = 1,2$. Next, we consider the polynomial

$$\rho_{2k} = \Big(\frac{Z(q_1)}{X(q_1)}X - Z\Big)^2 \Big(\frac{Z(q_2)}{X(q_2)}X - Z\Big)^2 X^k Z^{d-k-4}.$$

We observe that

$$\rho_{2k}(q_i) = 0,\ \nabla_{q_i}\rho_{2k} = 0\ \forall\ i = 1,2.$$

$$\rho_{2k}(p) = 0,\ \nabla^i\rho_{2k}(p) = 0,\ \forall\ i = 1,\cdots,k-1,\ \nabla^k\rho_{2k}(p) \neq 0.$$

Consider the curve $\gamma_{2k} : (-\varepsilon,\varepsilon) \longrightarrow \mathscr{D}_d \times \tilde{\mathscr{D}} \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\gamma_{2k}(t) = (f_d + t\rho_{2k}, \tilde{f}, q_1, q_2, p).$$

Note that $\{\nabla\psi_{\mathsf{T}_k}([f_d],[\tilde{f}],q_1,q_2,p)\}(\gamma'_{2k}(0)) = \nabla^k\rho_{2k}(p) \neq 0$ holds for any k. Hence the transversality follows. □

Next, we will study the case where there are singularities that are more degenerate than nodes involved.

Computation of the number $N_d^E(A_2 \circ \mathsf{T}_k,\ n_1,\ m)$ for all k

Theorem 4.5.4. *Let $N_d^E(A_2 \circ \mathsf{T}_k,\ n_1,\ m)$ denotes the number of degree d curves having a cusp tangent to E of order k passing through $\delta_d - (k+2) - n_1 - m$ generic points where the cusp and the tangency is at the intersection of $n_1,\ m$ generic lines respectively. Then the recursive formula for $N_d(A_2 \circ \mathsf{T}_k,\ n_1,\ m)$ is given by*

$$N_d^E(A_2 \circ \mathsf{T}_k,\ n_1,\ m) = \begin{cases} 2(d-1)N_d(A_2,\ n_1) - 3N_d(A_2;E,\ n_1) & \textit{for } k=1,\ m=0 \\ N_d(A_2,\ n_1) - 3N_d(A_2;E,\ n_1+1) & \textit{for } k=1,\ m=1 \\ 0 & \textit{for } k=1,\ m\geq 2 \\ \textit{for } k\geq 2, & \\ & \\ N_d^E(A_2 \circ \mathsf{T}_{k-1},\ n_1,\ m) + (d-2k)\,N_d^E(A_2 \circ \mathsf{T}_{k-1},\ n_1,\ m+1) & \\ -\,2\ N_d^E(A_2^{\mathsf{T}_{k-1}},\ n_1+m) & \end{cases} \tag{4.21}$$

provided $d \geq k+2$.

Remark 4.5.5. *We note that the case $k=1$, i.e., the first order tangency problem is included in theorem* (3.1.3)*, equation* (3.1)*. Thus we will prove the above theorem for $k \geq 2$.*

Proof. Let us define the space

$$A_2 \circ \mathsf{T}_k := \{([f],\ [\tilde{f}],\ q,\ p) \in \mathscr{D}_1 \times \mathscr{D}_d \times (\mathbb{P}^2)^2 : ([f],\ q) \in A_2$$
$$f \text{tangent to } \tilde{f} \text{ of order } k,\ q \neq p\}.$$

Let us consider μ to be the generic cycle representing the following cycle

$$[\mu] = y_d^{\delta_d - k - 2} y_1^{\delta_{d_1}} a^{n_1} a_1^m.$$

We will abbreviate $N_d^E(A_2 \circ \mathsf{T}_k,\ n_1,\ m)$ as $N_d^E(A_2 \circ T_k)$. Then the number $N_d^E(A_2 \circ T_k)$ is the cardinality of the set

$$\{([f],\ [\tilde{f}],\ q,\ p) \in A_2 \circ \overline{\mathsf{T}}_{k-1} : \nabla^k f(v^{\otimes k})_{|_p} = 0, v \in T_pL\} \cap \mu.$$

We now consider the following section

$$\Psi_{_{T_k}} : A_2 \circ \overline{\mathsf{T}}_{k-1} \longrightarrow \mathscr{V}_{T_k} := \gamma_{\mathscr{D}}^* \otimes T^*L^{\otimes k} \otimes \gamma_{\mathbb{P}^2}^{*d}.$$

We will show that the above section is transverse to the zero set which is equivalent to show that the the line bundle $\mathscr{V}_{T_k}$ is $k+2$ ample. Let us now define

$$\mathscr{B} = \overline{A_2 \circ \overline{\mathsf{T}}_{k-1}} - A_2 \circ \overline{\mathsf{T}}_{k-1}.$$

Hence

$$\left\langle e(\mathscr{V}_{T_k}), [\overline{A_2 \circ \overline{\mathsf{T}}_{k-1}}] \cap [\mu] \right\rangle = N_d^E(A_2 \circ T_k,\ n_1,\ m) + C_{\mathscr{B} \cap \mu} \tag{4.22}$$

where $C_{\mathscr{B} \cap \mu}$ denotes the excess contribution of the section $\Psi_{_{T_k}}$ to the Euler class from the points of $\mathscr{B} \cap \mu$. We will prove it in [50]. We note that only the points of $\mathscr{B}$, where the section vanishes is relevant for us. Hence we are only interested to the component of $\mathscr{B} \cap \mu$ where q becomes equal to p. Next, let us define where

$$\mathscr{B}(q,\ p) := \{([f],\ [\tilde{f}],\ q,\ p) \in B : q = p\}.$$

We claim that $\mathscr{B}(q,\ p) \approx A_2^{\mathsf{T}_{k-1}}$. Geometrically, $\mathscr{B}(q,\ p)$ denotes the component of the boundary where the cuspidal point and the point of tangency collide to each other. So we can expect the following thing to happen:

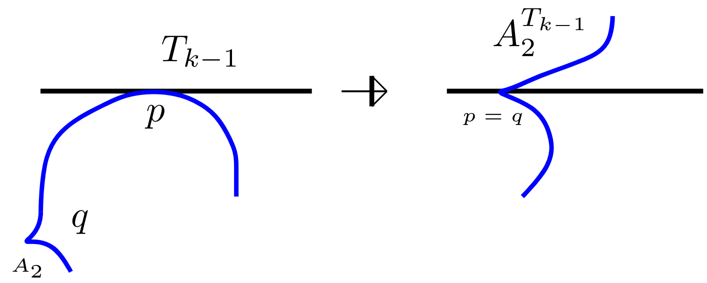

Figure 4.5: When the cuspidal point and the tangency point collide.

Furthermore, we claim that the contribution from $A_2^{\mathsf{T}_{k-1}}$ to the Euler class is 2. We plug in the value for $\mathscr{B} \cap \mu$ in (4.22) we get the final recursion. Note that the numbers for $k \geq 2$ are all new. We are not aware of any previous result in the past which calculates these numbers by any method.

Let us now prove the corresponding transversality claims. We will consider the affine setup as before. Let us assume the tangency point $p = [0 : 0 : 1]$ and $q = [x_1 : y_1 : 1]$ be the cuspidal point. Let us consider ∂_x and ∂_y to be the standard basis vectors for $T\mathbb{P}^2_{|_p}$. Next, let us look at the following short exact sequence

$$0 \longrightarrow L = Ker(\nabla \tilde{f}) \longrightarrow T\mathbb{P}^2 \longrightarrow \gamma^*_{\tilde{\mathscr{D}}} \otimes \gamma^{*d_1}_{\mathbb{P}^2} \longrightarrow 0.$$

Then there exist a nonzero vector $v \in Ker(\nabla \tilde{f}_{|_p})$. Without loss of generality we can assume that $v = [\partial_x]$. Let us now consider the polynomial

$$\rho_{A_2} = \Big(X - X(q)\Big) \; \Big(Y - Y(q)\Big) \; X^k \, Z^{d-k-2}.$$

We observe that

$$\rho_{A_2}(q) = 0, \; \nabla_{|_q} \rho_{A_2} = 0, det(\nabla^2 f)_{|_q} = 0$$

$$\rho_{A_2}(p) = 0, \; \nabla \rho_{A_2}(p) = 0, \cdots, \nabla^{k-1} \rho_{A_2}(p) = 0, \text{ but } \; \nabla^k \rho_{A_2}(p) \neq 0.$$

Consider the curve $\gamma_{2k} : (-\varepsilon, \varepsilon) \longrightarrow \mathscr{D}_d \times \tilde{\mathscr{D}} \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\gamma_{A_2}(t) = (f_d + t\rho_{A_2}, \; \tilde{f}, \; q, \; p).$$

Note that $\{\nabla \psi_{\mathsf{T}_k}([f_d], \; [\tilde{f}], \; q, \; p)\}(\gamma'_{A_2}(0)) = \nabla^k \rho_{A_2}(p) \neq 0$ holds for any k. Hence the transversality follows. □

Theorem 4.5.6. *Let $N_d(A_1^L \circ A_1^L, \; \ell, \; m)$ be same as defined earlier. The polynomial to compute this number is given by*

$$N(A_1^L \circ A_1^L, \;\; m) = \begin{cases} (d-2)(9d-25) & \textit{when } m \; = 0 \\ 3d - 8 & \textit{when } m \; = 1 \\ 0 & \textit{when } m \; \geq 2 \end{cases} \tag{4.23}$$

Proof. Let us recall the space

$$A_1^L \circ A_1^L := \{([\tilde{f}],\, [f],\, p,\, q) \in \mathcal{D}_1 \times \mathcal{D} \times (\mathbb{P}^2)^2 \mid f \text{ has two distinct nodes on } \tilde{f},\ p \neq q\}.$$

Since we know the number $N^E(A_1 \circ \mathsf{T}_1)$ from the theorem (4.14), as a special case we can compute $N(A_1^L \circ \mathsf{T}_1)$, i.e., the number of degree d curves in $\mathbb{P}^2$ having a node that is tangent to a fixed-line L passing through $\delta_d - 3$ generic points, where the node lies on the line. The corresponding space can be visualized by the following picture:

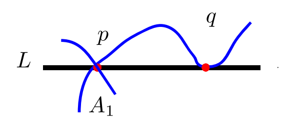

Figure 4.6: Node lying on a line at p and tangency at q.

Next, we will compute the number $N(A_1^L \circ A_1^L)$ by studying the section corresponding to the nodal condition over the space $A_1^L \circ \mathsf{T}_1$ at the tangency point. Let us consider the cycle $[\mu]$ corresponding to the insertion conditions Poincaré dual to $y_1^2\, y^{\delta_d - 3}\, a^2$. The number $N(A_1^L \circ A_1^L)$ is the cardinality of a set as follows

$$N(A_1^L \circ A_1^L) = |\, \{\, ([\tilde{f}],\, [f],\, p,\, q) \in \overline{A_1^L \circ \mathsf{T}_1} \;\mid\; \nabla f|_q(w) = 0,\ w \notin T_q L\} \cap \mu\, |\,.$$

Now, we will express the nodal condition on top of tangency i.e., we have to express the vanishing of the derivative along the normal direction as a section of the following bundle

$$\Psi_{A_1} : \overline{A_1^L \circ \mathsf{T}_1} \longrightarrow \mathcal{L}_{A_1} := \gamma_{\mathcal{D}}^* \otimes (T\mathbb{P}^2/TL)^* \otimes \gamma_{\mathbb{P}^2}^{*\,d}$$
$$\text{defined by } \{\Psi_{A_1}([\tilde{f}],\, [f],\, p,\, q)\}(f \otimes w) \;=\; \nabla_{|_q} f(w).$$

We will show that the above section is transverse to the zero set which is equivalent to showing that the line bundle $\mathcal{L}_{A_1}$ is $k+3$ ample. Let us now define

$$\mathcal{B} = \overline{A_1^L \circ \mathsf{T}_1} - A_1^L \circ \mathsf{T}_1.$$

Hence

$$\langle\, e(\mathcal{L}_{A_1}),\, [\overline{A_1^L \circ \mathsf{T}_1}] \cap [\mu] \rangle = N_d(A_1^L \circ A_1^L,\, \ell,\, m) \;+\; C_{\mathcal{B} \cap \mu} \tag{4.24}$$

where $C_{\mathcal{B}\cap\mu}$ denotes the boundary contribution of the section Ψ_{A_1} to the Euler class from the points of $\mathcal{B} \cap \mu$. We note that only the points of $\mathcal{B}$, where the section vanishes is relevant for us. Hence, we are only interested to the component of $\mathscr{B}\cap\mu$ where q becomes equal to p. Next, let us define where

$$\mathscr{B}(q,\ p) := \{([f],\ [\tilde{f}],\ q,\ p) \in B : q = p\}$$

We claim that $\mathscr{B}(q,\ p) \approx BA_1^{\mathsf{T}_1}$. Geometrically, $\mathscr{B}(q,\ p)$ denotes the component of the boundary where the nodal point and the point of tangency on L collide to each other. So we can expect the following thing to happen:

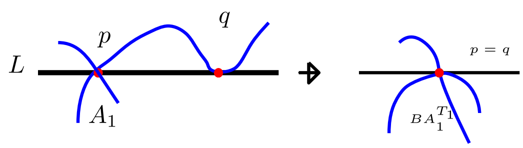

Figure 4.7: When the nodal point and the tangency point on L collide to each other.

Hence the contribution from $\mathscr{B}\cap\mu \approx BA_1^{\mathsf{T}_1}$. Furthermore, we will show that the excess contribution from the points of $BA_1^{\mathsf{T}_1}\cap\mu$ to the Euler class is 1. Then we plug in the value for $\mathscr{B}\cap\mu$ in (4.24) we get the final recursion.

Let us now prove the transversality claim. We will continue with the setup as earlier. We will assume as before $p = [x_1 : y_1 : 1]$ be the nodal point and $q = [0 : 0 : 1]$ as tangency point and $v = [\partial_x]$. Let us now consider the polynomial

$$\rho_{A_1^L} = \Big(X - X(p)\Big)^2 X^k Z^{d-k-2}.$$

We observe that

$$\rho_{A_1^L}(q) = 0,\ \nabla\rho_{A_1^L}(q) = 0, \cdots, \nabla^{k-1}\rho_{A_1^L}(q) = 0,\ \text{but}\ \ \nabla^k\rho_{A_1^L}(q) \neq 0.$$

Consider the curve $\gamma_{2k} : (-\varepsilon, \varepsilon) \longrightarrow \mathscr{D}_d \times \tilde{\mathscr{D}} \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\gamma_{A_1^L}(t) = (f_d + t\rho_{A_1^L},\ \tilde{f},\ q,\ p)$$

Note that $\{\nabla\psi_{A_1}([f_d],\ [\tilde{f}],\ q,\ p)\}(\gamma'_{A_1^L}(0)) = \nabla^k\rho_{A_1^L}(q) \neq 0$ holds for any k. Hence the transversality follows.

Claim 4.5.7. *Let us consider* $([f],\ [\tilde{f}],\ q,\ p) \in BA_1^{\mathsf{T}_1}$. *Then there exist* $([f_t],\ [\tilde{f}_t],\ q_t,\ p_t) \in A_1^L \circ \mathsf{T}_1$ *such that*

$$\{\Psi_{A_1}([\tilde{f}],\ [f],\ p,\ q)\}(f \otimes w) = 0$$

where $w \notin T_qL$. *Moreover, the section vanishes with multiplicity* 1.

Assuming the above multiplicity claim, the proof is complete. ∎

Now we are ready to illustrate the most exciting consequence of the above study. As a corollary of the above theorem we will prove the following:

Claim 4.5.8. *Let $N(A_3,\, m)$ be the number of degree d curves in $\mathbb{P}^2$ having a tacnode passing through $\delta_d - 3 - m$ generic points at intersection of m generic lines. The polynomial to compute this number is given by*

$$N(A_3,\, m) = \begin{cases} 168 - 192d + 50d^2 & \textit{when } m = 0 \\ -48 + 25d & \textit{when } m = 1 \\ 5 & \textit{when } m = 2 \\ 0 & \textit{when } m \geq 3 \end{cases} \tag{4.25}$$

Proof. Let us recall

$$A_1^L \circ A_1^L := \{([\tilde{f}],\, [f],\, p,\, q) \in \tilde{\mathscr{D}} \times \mathscr{D} \times (\mathbb{P}^2)^2 \mid f \text{ has two distinct nodes on } \tilde{f},\; p \neq q\}.$$

Then the cycle representing $A_1^L \circ A_1^L$ denoted by $[Z]$ in the ambient space $\tilde{\mathscr{D}} \times \mathscr{D} \times (\mathbb{P}^2)^2$. Next, we will consider the space of all lines in $\mathbb{P}^2$ and two distinct points such that two points comes together along the line as

$$\nabla_L = \{([\tilde{f}],\, p,\, q) \in \tilde{\mathscr{D}} \times (\mathbb{P}^2)^2 \mid p = q\}.$$

Thus therefore we can think of the class $[\tilde{Z}]$ Poincaré dual to $(a_1 + a_2 - y_1)$ inside $\tilde{\mathscr{D}} \times (\mathbb{P}^2)^2$. Then we want to look at the intersection of the two cycles Z, $\tilde{Z}$ inside $\tilde{\mathscr{D}} \times \mathscr{D} \times (\mathbb{P}^2)^2$ that produces curves with tacnode, i.e., geometrically we are looking at collision of two points p, q along the line L. Geometrically the following happens

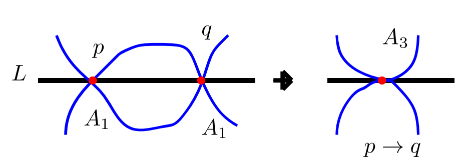

Figure 4.8: Two node on the same line collide into a tacnode.

Claim 4.5.9. *Let the cycles $[Z]$, $[\tilde{Z}]$ as above. Then the intersection of $[Z]$ and $[\pi_f^* \tilde{Z}]$ inside $\tilde{\mathscr{D}} \times \mathscr{D} \times (\mathbb{P}^2)^2$ is transverse, where $\pi_f : \tilde{\mathscr{D}} \times \mathscr{D} \times (\mathbb{P}^2)^2 \longrightarrow \tilde{\mathscr{D}} \times (\mathbb{P}^2)^2$ be the projection map.*

The above lemma follows from the regular value technique as before. Hence the proof. ∎

Remark 4.5.10. *We have also studied the enumeration of* $N_d^E(A_1^3 \circ \mathsf{T}_2)$*, i.e., the number of degree* d *curves having three nodes, tangent to order two to a fixed-line in* $\mathbb{P}^2$ *passing through* $\delta_d - 5$ *points in general position. Which may enable us to study some other curve counting problems having higher singularities. The number* $N_d^E(A_1^3 \circ \mathsf{T}_k)$ *for* $k \geq 3$ *requires an understanding of curves with a triple point that are tangential to some order to the line. At this moment we do not know how to deal with these obvious obstacles. We hope to explore the above and the questions concerning multiple tangency point constraints in recent future.*

4.6 Some explicit polynomials

In this section, we will explicitly write down some of the polynomials that we have obtained from our recursive formulas. We have written a Mathematica program to compute all these numbers which is available on my web page

https://sites.google.com/view/paulanantadulal

These polynomials are often useful for various nontrivial low-degree checks. When there are two or more tangency points involved and the divisor is the line for all the tangency conditions, we have the following

$$N_d^E(\mathsf{T}_1 \circ \mathsf{T}_1) = 24 - 20d + 4d^2$$

$$N_d^E(\mathsf{T}_1 \circ \mathsf{T}_2) = 72 - 42d + 6d^2$$

$$N_d^E(\mathsf{T}_2 \circ \mathsf{T}_2 \circ \mathsf{T}_2) = -9072 + 3942d - 567d^2 + 27d^3$$

$$N_d^E(\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_4) = -15120 + 5730d - 720d^2 + 30d^3$$

$$N_d^E(\mathsf{T}_1 \circ \mathsf{T}_2 \circ \mathsf{T}_3 \circ \mathsf{T}_4) = 2059200 - 723120d + 94920d^2 - 5520d^3 + 120d^4$$

Similarly when the curve has the singularity A_1^δ or A_2 tangent to order k to the given line in $\mathbb{P}^2$, for some k the corresponding polynomials are given by

$$N_d^E(A_1 \circ \mathsf{T}_2) = 3 + 33d - 36d^2 + 9d^3$$
$$N_d^E(A_1 \circ \mathsf{T}_3) = 16 + 64d - 60d^2 + 12d^3$$
$$N_d^E(A_1^2 \circ \mathsf{T}_2) = 120 - 798d + 441d^2 + 180d^3 - 162d^4 + 27d^5$$
$$N_d^E(A_1^2 \circ \mathsf{T}_3) = 168 - 1620d + 732d^2 + 360d^3 - 252d^4 + 36d^5$$
$$N_d^E(A_2 \circ \mathsf{T}_1) = -12 + 96d - 96d^2 + 24d^3$$
$$N_d^E(A_2 \circ \mathsf{T}_2) = -64 + 248d - 180d^2 + 36d^3$$

4.7 Checks with existing results

In this section, we will verify our results with the earlier existing results.

4.7.1 Consistent checks with Caporaso-Harris and Fan-Wu

We note that the Caporasso-Harris formula ([11]) computes
$N^E(A_1^\delta \circ T_{k_1} \circ T_{k_2} \circ \cdots \circ T_{k_n})$ for any δ when the divisor E is line. When the divisor is a line we can directly verify that our formula for the above produces the same answer as the Caporasso-Harris formula for several values of d; we have written a program to implement our formula (which is available on request). The reader is invited to use the program to check that it produces the same answer given by Caporaso-Harris.

Very recently, in [16], the authors have obtained the recursive formula to compute degree d rational curves in $\mathbb{P}^2$ with maximal order of contact with a smooth divisor at a specific point. When our divisor is smooth degree $d_1 \geq 2$ curve in $\mathbb{P}^2$, we have verified that our numbers agree with the numbers calculated in [16].

4.7.2 Curves with cusp satisfying tangency condition

In [15], the authors have obtained a formula which computes the degree d rational cuspidal curves in $\mathbb{P}^2$ passing through a points in general positions tangent to b general lines and tangent to c general lines a specific point on it such that $a+b+2c = 3d-2$. Actually, their formula tells us the number of the above types of curves lying on some cycles in $\mathbb{P}^2$. Now in our recursive formula containing the singularity cusp directly produces the numbers tabulated in the paper [15] till rational cubic. We also verify the number for rational quartics tangent to a line. A little more thought is required to get this number from our formula, which we have explained in the previous chapter.

4.7.3 Curves with higher-order tangency to a smooth cubic and other hypersurfaces

Counting relative invariants even in $\mathbb{P}^2$ turns out to be a very difficult problem when the degree of the divisor becomes large. For example, when E is a smooth cubic [10], the authors have a solution to the question 6.2.2 for all (α,β) except for $(0,e_{3d})$. In his thesis, A. Gathman [20] had studied the problem of counting relative invariants for hypersurfaces. He had shown that absolute invariants can be calculated using those relative invariants. So we can use his program "GROWI" to calculate relative numbers when the divisor $E \in P^2$ could be any smooth curve of degree $d_1 \geq 2$. We have seen the agreement of these numbers with ours. When the smooth divisor E is a line we will we will tabulate some initial numbers that we have checked from "GROWI" as follows:

$d,\ d_1$	$N^E(A_1 \circ T_2)$	$N^E(A_1 \circ T_3)$	$N^E(A_1 \circ T_4)$
$d=3,\ d_1=1$	21	0	0
$d=4,\ d_1=1$	135	80	0
$d=5,\ d_1=1$	393	336	195

Table 4.1: Numbers from the GROWI program.

Chapter 5

Two pointed singularities

5.1 Introduction

We have seen earlier that counting plane curves having certain singularities (possibly more degenerate than nodes) is a very hard problem. Let us state an enumerative question in this regard:

Question 5.1.1. *Let $L \longrightarrow X$ be a holomorphic line bundle over a compact complex surface and $\mathscr{D} := \mathbb{P}H^0(X,L) \approx \mathbb{P}^{\delta_L}$ as defined earlier. What is $N(A_1^{\delta} \circ \mathfrak{X})$, the number of curves in X, that belong to the linear system $H^0(X,L)$, passing through $\delta_L - (k+\delta)$ points in general position and having δ distinct nodes and one singularity of type $\mathfrak{X}$ whose codimension is k?*

The above question has already been studied by numerous mathematicians using several different techniques. It has been observed that this question itself becomes increasingly difficult when the total codimension i.e., $\delta + k$ increases, as well as $\mathfrak{X}$, becomes more and more degenerate. In [3], Basu and Mukherjee have given explicit formulas for the following numbers $N(A_1^{\delta} \circ \mathfrak{X})$, the number of degree d curves having δ different nodes and one singularity of type $\mathfrak{X}_k$, such that they also pass through the required number of generic points where the total codimension is at most 8, i.e., $\delta + k \leq 8$. In this section, our main aim is to obtain recursive formulas for the number of degree d curves in $\mathbb{P}^2$ having singularities of type $\mathfrak{X}_{k_1}$ and $\mathfrak{X}_{k_2}$ passing through an appropriate number of generic points, such that the total codimension $k_1 + k_2 \leq 6$. For simplicity, we denote it by $N(\mathfrak{X}_1 \circ \mathfrak{X}_2)$.

In the chapter 3, we have introduced the notion of A_k singularities, now we will define another type of singularities that we will encounter in this thesis.

Definition 5.1.2. *Let $f : \mathbb{P}^2 \longrightarrow \mathscr{O}(d)$ be a holomorphic section. A point $q \in f^{-1}(0)$ is of singularity type D_k, E_6, E_7, E_8 or X_9 if there exists a coordinate system $(x,y) : (U,q) \longrightarrow (\mathbb{C}^2,0)$ such that*

$f^{-1}(0) \cap U$ *is given by*

$$D_k : y^2x + x^{k-1} = 0 \qquad k \geq 4,$$

$$E_6 : y^3 + x^4 = 0, \qquad E_7 : y^3 + yx^3 = 0, \qquad E_8 : y^3 + x^5 = 0,$$

$$X_9 : x^4 + y^4 = 0.$$

This question has strong consequences and a deep relationship with various developments in both Algebraic geometry and symplectic geometry. Till 2010, there was an extensive amount of work has been done to study δ nodal curves on an algebraic surface. It includes a century's worth of outstanding mathematicians amongst them Kazaryan [27], Z. Ran [58], Caporaso-Harris [11], Ravi Vakil [69], Ionel-Parker [24], Tehrani-Zinger [64], [40].

Let us now ask a sufficiently general question as follows:

Question 5.1.3. *What is* $N(\mathfrak{X}_1 \circ \cdots \circ \mathfrak{X}_n)$*, the number of curves in* X*, that belong to the linear system* $H^0(X,L)$*, passing through* $\delta_L - (cd_{\mathfrak{X}_1} + \cdots + cd_{\mathfrak{X}_n})$ *points in general position and having n number of singularities of the type* $\mathfrak{X}_1, \cdots \mathfrak{X}_n$ *whose codimensions are* $cd_{\mathfrak{X}_1}, \cdots, cd_{\mathfrak{X}_n}$ *respectively?*

Note that when the total codimension $\sum_i cd_{\mathfrak{X}_{k_i}} \leq 7$ then the above question has solved by Kazaryan. A few results are scattered around the literature for the above question with a total codimension higher than 7. We are aware of two such results due to A. Weber, M. Mikosz, and P.Pragacz ([55], [49]) where they solve the question for codimension 8 with one singular point by extending the method of Kazaryan. For a broad overview of this subject, we refer the reader to [32].

5.1.1 Relation to Gromov-Witten invariants

Let us consider another interesting and classical enumerative problem; namely enumeration of curves with fixed genus. To start with, we can focus on genus zero curves (known as rational curves), which we have already seen in chapter 1 in this thesis. Modern enumerative geometry gets accelerated due to a systematic understanding of such kinds of problems. However, the breakthrough in this setting comes from physics motivation. For example, the solution to counting rational curves, the theory of Gromov-Witten invariants, and quantum cohomology due to Kontsevich-Manin and independently by Ruan-Tian. After that study of genus g curves via stable map theory revolutionized this subject due to many interesting works of Algebraic geometers and Symplectic geometers. Recently this subject is also fascinating for Tropical geometers. In this setting, counting curves with higher singularities (i.e., genus g curves with higher singularities) is a much more difficult question and this type of question has not been much explored. Pandharipande [54] and later Kock [36] computed rational curves in $\mathbb{P}^2$ with cusp. Latter on Zinger, studied counting rational curves with cusp and some selected higher singularities using the Symplectic geometric method. This question is indirectly related to the counting problem in a linear system that we are studying.

Counting rational cubic through 8 generic points in $\mathbb{P}^2$ is the same as the number of nodal cubics through 8 generic points in $\mathbb{P}^2$. Next, we can think of rational quartics through 11 generic points in $\mathbb{P}^2$ is the same as irreducible quartics with three unordered nodes through 11 points in general position. Hence, we can subject our obtained numbers to a related question in the stable map setting. In most all the chapters of this thesis, we have been able to verify our numbers using the above phenomena.

5.2 Setup and Notations

The notation $\mathfrak{X}$ throughout this thesis represents a singularity type (of the type A_k, D_k or E_k) of a degree d curve at some point in $\mathbb{P}^2$. We also use the notation $\mathfrak{X}$ to denote the space of degree d curves having a singularity type $\mathfrak{X}$ at some point of $\mathbb{P}^2$. In set theoretic notation, we can express $\mathfrak{X}$ as

$$\mathfrak{X} := \{([f], p) \in \mathscr{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathfrak{X} \text{ at } p\}.$$

Let us continue following the notation as earlier chapters of this thesis and [1]. In a similar manner, we can define the space of degree d curves having two arbitrary singularities, say $\mathfrak{X}_1$ and $\mathfrak{X}_2$,

$$\mathfrak{X}_1 \circ \mathfrak{X}_2 := \{([f], p, q) \in \mathscr{D} \times \mathbb{P}^2 \times \mathbb{P}^2 : ([f], p) \in \mathfrak{X}_1, ([f], q) \in \mathfrak{X}_2,\ p \neq q\}.$$

Let us now define a projectivised vector bundle over $\mathbb{P}^2$ as $\pi : \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{P}^2$, where the fibre over each point p is a tangent vector at the point p, known as projectivised tangent bundle over $\mathbb{P}^2$. It turns out studying the space of curves with the above singularities becomes a hard question to tackle, on the other hand, we will define an auxiliary object $\mathscr{P}\mathfrak{X}$ which makes this question somewhat a tractable question. We can now define the space of curves with a singularity of the above type where certain directional derivatives vanish along some specified direction. In particular, we will define

$$\mathscr{P}A_k := \{([f], l_p) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 : ([f], p) \in A_k,\ \nabla^2 f|_p(v, \cdot) = 0 \quad \forall v \in l_p\} \quad \text{if } k \geq 2,$$

$$\mathscr{P}D_4 := \{([f], l_p) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 : ([f], p) \in D_4,\ \nabla^3 f(v, v, v) = 0 \quad \forall v \in l_p\},$$

$$\mathscr{P}D_k := \{([f], l_p) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 : ([f], p) \in D_k,\ \nabla^3 f(v, v, \cdot) = 0 \quad \forall v \in l_p\} \quad \text{if } k \geq 5,$$

$$\mathscr{P}E_k := \{([f], l_p) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 : ([f], p) \in E_k,\ \nabla^3 f(v, v, \cdot) = 0 \quad \forall v \in l_p\} \quad \text{if } k = 6, 7, 8.$$

$$\mathscr{P}D_k^{\vee} := \{([f], p) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 : ([f], p) \in D_k,\ \nabla^3 f|_p(v, v, v) = 0,\ \nabla^3 f|_p(v, v, w) \neq 0,$$
$$\forall\ v \in l_q - 0 \text{ and } w \in (T_p\mathbb{P}^2)/l_p - 0\}, \quad \text{if } k > 4.$$

As an example, $\mathscr{P}A_2$ is the space of degree d curves with a marked point p and a marked direction $v \in l_p$, such that the curve has a cusp at p and v belongs to the kernel of the Hessian.

Next, the projection map $\pi : \mathscr{P}A_k \longrightarrow A_k$ is one to one for all $k \geq 2$. Similarly, the projection map $\pi : \mathscr{P}D_4 \longrightarrow D_4$ is three to one. Next, we will define

$$\mathscr{P}A_1 := \{([f],\ l_p) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 : ([f], p) \in A_1,\ \nabla^2 f|_p(v, v) = 0, \quad \forall v \in l_p\}$$

Space $\mathscr{P}A_1$ is the space of curves with a marked point p and a marked direction $v \in l_p$, such that it has a node at p and the second derivative along with v vanishes. Hence, the projection map $\pi : \mathscr{P}A_1 \longrightarrow A_1$ is a two to one map since there are two such marked directions.

Now we can define the following:

$$\mathscr{P}\mathfrak{X}_1 \circ \mathscr{P}\mathfrak{X}_2 := \{([f],\ l_p,\ l_q) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2_1 \times \mathbb{P}T\mathbb{P}^2_2 : ([f],\ l_p) \in \mathscr{P}\mathfrak{X}_1, ([f],\ l_q) \in \mathscr{P}\mathfrak{X}_2,\ p \neq q\}.$$

Before, we proceed further, let us make some conventions regarding certain notations. If $S \subset \mathscr{D} \times \mathbb{P}^2$ and let $\pi : \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \to \mathscr{D} \times \mathbb{P}^2$ be a map. Then $\hat{S} := \pi^{-1}(S)$ and $\operatorname{codim}(S) = \operatorname{codim}(\hat{S})$.

- $\dot{S}$ represents a specific direction in a particular fiber attached to S.
- $\hat{S}$ represents attaching a fiber over S.
- $\hat{\hat{S}}$ represents attaching fibers in the same direction over S (possibly at the same point).
- $\hat{\dot{S}}$ denotes attaching a fiber at the first position and fixing a specific direction and a dot at the second position denotes attaching a fiber, where the direction is not fixed.
- If $S \subset \mathscr{D} \times \mathbb{P}^2 \times \mathbb{P}^2 \implies \hat{\hat{S}} \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$.

- If $S \subset \mathscr{D} \times \mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2 \implies \hat{} S \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$.
- If $S \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}^2 \implies S\hat{} \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$.

If $S_1 \circ S_2 \subset \mathscr{D} \times \mathbb{P}^2 \times \mathbb{P}^2$. Then

- $\hat{S}_1 \circ S_2 \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}^2$.
- $S_1 \circ \hat{S}_2 \subset \mathscr{D} \times \mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$.
- $\hat{S}_1 \circ \hat{S}_2 \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$.

If $S_1 \subset \mathscr{D} \times \mathbb{P}^2$ and $S_2 \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2$. Then

- $\hat{S}_1 \circ S_2 \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$.
- $S_2 \circ \hat{S}_1 \subset \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$.

We now set up some notation for the cohomology classes that we will encounter in this thesis repeatedly. First, let us define

$$c_1 := c_1(L), \;\; x_i := c_i(T^*\mathbb{P}^2), \;\; \lambda := c_1(\hat{\gamma}^*), \;\; y := c_1(\gamma_{\mathscr{D}}^*)$$

where $\gamma_{\mathscr{D}} \longrightarrow \mathscr{D}$ and $\hat{\gamma} \longrightarrow \mathbb{P}TX$ are the tautological line bundles. These are all cohomology classes in $\mathbb{P}^2$, $\mathbb{P}T\mathbb{P}^2$ and $\mathscr{D}$; by pulling them back via relevant projection maps, they define cohomology classes in $\mathscr{D} \times \mathbb{P}^2$ and $\mathscr{D} \times \mathbb{P}T\mathbb{P}^2$ respectively. For notational simplification, we will denote the constraints for intersection number in $\mathscr{D} \times \mathbb{P}(T\mathbb{P}^2)$ by $\mu := (n, m_1, m_2, \theta)$, where the class of $[\mu]$ is Poincare dual to $y^n c_1^{m_1} x_1^{m_2} \lambda^{\theta}$. For example, $\mu_{n+1,\theta+1}$ would mean the constraint has type $(n+1, m_1, m_2, \theta+1)$. Then we will use abuse of notation by denoting the constraint for intersection number in $\mathscr{D} \times T\mathbb{P}^2$ as $\mu := (n, m_1, m_2)$.

We will now state and prove some important lemmas involving certain properties of the underlying singularities of the plane curve. We can now define some numbers for a singularity $\mathfrak{X}$ or $\mathscr{P}\mathfrak{X}$ of codimension k as follows

$$N(A_1^{\delta} \circ \mathfrak{X};\, n_1,\, m_1,\, m_2) := \langle y^{\delta_d - k - n_1 - m_1 - 2m_2 - \delta} c_1^{m_1} x_1^{m_1} x_2^{m_2},\;\; [\overline{A_1 \circ \mathfrak{X}}] \rangle,$$
$$N(A_1^{\delta} \circ \mathscr{P}\mathfrak{X};\, n_1,\, m_1,\, m_2, \theta) := \langle y^{\delta_d - k - n_1 - m_1 - 2m_2 - \delta - \theta} c_1^{m_1} x_1^{m_1} x_2^{m_2} \lambda^{\theta},\;\; [\overline{A_1^{\delta} \circ \mathscr{P}\mathfrak{X}}] \rangle.$$

Lemma 5.2.1. *Let $f = f(x,y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $f_{00}, \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$ and there does not exist a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^3 f(w,w,\cdot) = 0$. Then the curve $f^{-1}(0)$ has a $\mathscr{D}_4$ singularity at the origin.*

Lemma 5.2.2. *Let $f = f(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $f_{00}, \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^3 f(w,w,\cdot) = 0$. Let $x = w_1 r + w_2 s,\ y = -\overline{w}_2 r + \overline{w}_1 s$ and f_{ij} be the partial derivatives with respect to the new variables x and y. Then, the curve $f^{-1}(0)$ has a $\mathscr{D}_k$-node at the origin (for $5 \leq k \leq 7$) if $f_{12} \neq 0$ and the directional derivatives $\mathscr{D}_i^f$ defined in (5.3) are zero for all $i \leq k$ and $\mathscr{D}_{k+1}^f \neq 0$.*

The proof of these above Lemmas follows the similar arguments that we have used to prove (4.1) in chapter 3; it involves the study of the Taylor expansion of the curve in a local coordinate system with appropriate change of coordinates. For the sake of completeness, we will prove the Lemma (5.2.2).

Proof. Let us consider the Taylor series expansion of f near the origin is given by

$$F(x,y) = \frac{f_{12}}{2!}xy^2 + \frac{f_{03}}{3!}y^3 + \frac{f_{40}}{4!}x^4 + \cdots \tag{5.1}$$

since by hypothesis we have $f_{30} = 0, f_{21} = 0$ and $f_{12} \neq 0$. Now we will make change of coordinate by $x = \tilde{x} + B(y)$ so that we can express the function as $F = \tilde{x}F_1(\tilde{x}, y)$. This implies that we can kill of all powers of y. Assuming such $B(y)$ exist, we can write the expression for $F_1(\tilde{x}, y)$ as

$$F_1(\tilde{x}, y) = A_0(\tilde{x}) + A_1(\tilde{x})y + A_2(\tilde{x})y^2 + \cdots \tag{5.2}$$

where we claim that $B(y)$ is a holomorphic function and we will show the existence of $B(y)$ by Implicit function theorem. Recall that we want to describe the equation (5.1) as $F = \tilde{x}F_1(\tilde{x}, y)$ by the change of variable $x = \tilde{x} + B(y)$. This is equivalent to saying that we need to produce a B such that $F(B(y), y) = 0$. Next, we note that plugging in $x = \tilde{x} + yG(y)$ in the equation (5.1) we have the following

$$0 = F(yG(y), y) = \frac{f_{12}}{2!}y^3 G(y) + \frac{f_{03}}{3!}y^3 + \frac{f_{40}}{4!}y^4 G(y)^4 + \frac{f_{31}}{3!}y^4 G(y)^3 + \cdots$$

Now this implies that

$$\frac{f_{12}}{2!}G(y) + \frac{f_{03}}{3!} + \frac{f_{40}}{4!}yG(y)^4 + \frac{f_{31}}{3!}yG(y)^3 + \cdots = 0$$

Then by the Implicit function theorem, $G(y)$ exist since $f_{12} \neq 0$, whence $B(y) = yG(y)$.

We will now make another change of coordinate $y = \tilde{y} + C(\tilde{x})$ so that

$$F_1 = \hat{A}_0(\tilde{x}) + \hat{A}_2(\tilde{x})y^2 + \hat{A}_3(\tilde{x})y^3 + \cdots$$

i.e. $\hat{A}_1(\tilde{x}) \equiv 0$ since $\hat{A}_2(0) \neq 0$, this is possible. So we left with the following expression:

$$F = \tilde{x}(\hat{A}_0(\tilde{x}) + \hat{A}_2(\tilde{x})y^2 + \hat{A}_3(\tilde{x})y^3 + \cdots)$$

Let us now consider $\hat{A}_0(\tilde{x} = \frac{\mathscr{D}_6}{4!}\tilde{x}^3 + \frac{\mathscr{D}_7}{5!}\tilde{x}^4 + \cdots$ then F is given by

$$F = \tilde{x}(\hat{y}_1^2 + \frac{\mathscr{D}_6}{4!}\tilde{x}^3 + \frac{\mathscr{D}_7}{5!}\tilde{x}^4 + \cdots)$$

where $\hat{y}_1^2 = \hat{A}_2(\tilde{x})y^2 + \hat{A}_3(\tilde{x})y^3 + \cdots$. Thus we notice that if $\hat{A}_0(\tilde{x}) \equiv 0$ then $F = \tilde{x}\hat{y}_1^2$. Otherwise, we will choose a smallest integer k such that $\mathscr{D}_{k+1} \neq 0$. Now let us make the change of variable by

$$x_1 = \sqrt[k-1]{\frac{\mathscr{D}_{k+1}}{(k-1)!}\tilde{x}^{k-1} + \frac{\mathscr{D}_{k+2}}{(k)!}\tilde{x}^k + \cdots}$$

with $\tilde{x} = fx_1 + O(x_1^2)$ and $f = \left(\frac{(k-1)!}{\mathscr{D}_{k+1}}\right)^{k-1}$. Thus in this new change of coordinate gives us

$$F = (fx_1 + x_1^2 h)\hat{y}_1^2 + x_1^{k-1}$$

for some holomorphic function $h(x_1, \hat{y}_1)$. Now one can define $y_1 = \hat{y}_1\sqrt{f + x_1 h}$ then we get

$$F = y_1^2 x_1 + x_1^{k-1}$$

as intended. Let us recall $f_{ij} := \frac{\partial^{i+j} f}{\partial^i x \partial^j y}|_{(x,y)=(0,0)}$ where $f = f(x,y)$ is a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}^2$. Then we can define

$$D_6^f = f_{40},\ D_7^f = f_{50} - \frac{5f_{31}^2}{3f_{12}},\ D_8^f = f_{60} + \frac{5f_{03}f_{31}f_{50}}{3f_{12}^2} - \frac{5f_{31}f_{41}}{f_{12}} - \frac{10f_{03}f_{31}^3}{3f_{12}^3} + \frac{5f_{22}f_{31}^2}{f_{12}^2} \tag{5.3}$$

□

5.3 Overview of our method

In this chapter, our objective is to study the enumeration of curves with certain singularities. The crucial aspect of the method we used in this thesis is due to A.Zinger. The fact that a curve has a certain singularity of the type we have encountered before means that certain derivatives vanish (an example of this fact is the Implicit Function Theorem and a version of the Morse Lemma). We usually interpret these vanishing derivatives as the section of some appropriate bundle. Thus, our intersection numbers are the zeros of a section of some bundle restricted to the *open* part of a variety (or manifold). Note that the region we typically have to restrict ourselves to the open part of a manifold because we are mainly interested in enumerating curves with more than one singular point; hence we have to consider the space of curves with a collection of marked points, where all the marked points are distinct.

Next, we observe that if the line bundle is sufficiently ample, then the sections corresponding to taking certain derivatives will be transverse to the zero set restricted to the open part.

Remark 5.3.1. *In our case, the open part of our variety/ manifold will always going to be smooth.*

Next, we evaluate the Euler class of this bundle on the fundamental class of the variety/manifold. We may hope that this number is our required number. As one might expect, and it is always the case that we will encounter some extra contribution from the boundary. This is because the section will usually be going to vanish on the boundary and hence give an excess (degenerate) contribution to the Euler class.

Remark 5.3.2. *Any algebraic variety defines a homology class since the singularities have at least real codimension two. This follows from the standard results from differential topology, namely that any singular space whose singularities are of real codimension two or more (i.e., a pseudo cycle) defines a homology class.*

As we have seen the central part of the problem is therefore to study the degenerate locus. It turns out any enumerative problem involving degenerate singularity posses this phenomenon.

The most famous and well-studied method of computing degenerate contributions to the Euler class goes under the name global excess intersection theory, which is developed in Fulton's book [19].

In this thesis, we use a *local intersection theory* to compute degenerate contributions to the Euler class due to Aleksey Zinger [75]. This approach never deals with taking any blowups. It mainly involves perturbation of the relevant section *smoothly* and counting (possibly up to a sign) the number of zeros of the section near the degenerate loci.

As we have mentioned earlier that we will heavily make use of the fact (3.2.1) from differential topology. However, all most all the time we can not directly use it since we will be typically studying some spaces that have non-smooth closure. When the set of singular points has real codimension two or more, we will be using the following:

Theorem 5.3.3. *Let $X \subset \mathbb{P}^n$ be a compact, smooth algebraic variety and let $Y \subset X$ be a smooth subvariety, not necessarily closed. Let $V \longrightarrow X$ be an oriented vector bundle having rank that is exactly equal to the dimension of Y. The following are equivalent:*

- *The closure of Y inside X is an algebraic variety and it defines a homology class.*
- *Let $s : X \longrightarrow V$ be a smooth generic section, then the zero set of s intersect Y transversely and intersect $\overline{Y} - Y$ nowhere.*
- *The number of zeros of s inside Y, counted with appropriate signs, is the Euler class of the bundle V evaluated on the homology class $[\overline{X}]$, i.e.,*

$$|\pm s^{-1}(0) \cap X| = \langle e(V), [\overline{X}] \rangle$$

Let us give a brief idea of enumerating curves with some singularity using our method. Suppose we want to compute the number $N(\mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_m)$. Then we will first find some $\mathscr{P}\mathfrak{X}_j$ for which $N(\mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_j)$ has already been calculated and most importantly which contains $\mathscr{P}\mathfrak{X}_m$ in the closure. In particular, we want $\mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_m$ to be a subset of $\overline{\mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_j}$. Then we have to describe the closure of $\mathscr{P}\mathfrak{X}_j$ and $\mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_j$ explicitly as

$$\begin{aligned}
\overline{\mathscr{P}\mathfrak{X}_j} &= \mathscr{P}\mathfrak{X}_j \sqcup \overline{\mathscr{P}\mathfrak{X}_m} \cup \mathscr{B}_1 \\
\overline{\overline{\mathscr{P}\mathfrak{X}_l} \circ \mathscr{P}\mathfrak{X}_j} &= \mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_j \sqcup \overline{\mathscr{P}\mathfrak{X}_l} \circ (\overline{\mathscr{P}\mathfrak{X}_j} - \mathscr{P}\mathfrak{X}_j) \cup \mathscr{B}_2 \\
\overline{\overline{\mathscr{P}\mathfrak{X}_l} \circ \mathscr{P}\mathfrak{X}_j} &= \mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_j \sqcup \overline{\mathscr{P}\mathfrak{X}_l} \circ (\overline{\mathscr{P}\mathfrak{X}_m} \cup \mathscr{B}_1) \cup \mathscr{B}_2
\end{aligned}$$

where $\mathscr{B}_1, \mathscr{B}_2$ may contain one or more degenerate singularities known as degenerate locus which is the most difficult part of our method. The central essence of our method is that we will explicitly describe the degenerate locus and we will calculate the multiplicity to which it contributes to the Euler class. Note that $\overline{\mathscr{P}\mathfrak{X}_l \circ \mathscr{P}\mathfrak{X}_j} = \overline{\overline{\mathscr{P}\mathfrak{X}_l} \circ \mathscr{P}\mathfrak{X}_j}$. Our main focus of this project is to compute $\mathscr{B}_2$ explicitly which in turn same as studying the collision of two singularities more degenerate than nodes with at most total codimension of the two singularities.

Definition 5.3.4. *A holomorphic line bundle $L \longrightarrow M$ over a compact complex manifold M is said to be k ample if $L \approx L_1^{\otimes n} \otimes \xi \longrightarrow M$ for some $n \geq k$, where $L_1 \longrightarrow M$ is a very ample line bundle and $\xi \longrightarrow M$ is a line bundle such that the linear system $H^0(M, \xi)$ is base point free.*

All the results that we obtain in this thesis are valid satisfying some ampleness conditions. The ampleness criterion is imposed to prove that the sections we encounter are transverse to the zero set.

However, the bound that we impose is not at all optimal bound. Note that the intersection numbers that we compute need not always be enumerative, i.e. each curve appears with a multiplicity of one in the linear system. Thus ampleness plays a crucial role to study certain intersection number is enumerative or not.

5.4 Recursive formulas

We are now ready to state the recursive formulas that we have obtained till now. We will only concentrate on the new recursive formulas that we have obtained up to codimension 6.

Theorem 5.4.1. *Let the total codimension of the two singular points is* 2, *then*

$$N(\mathscr{P}A_1 \circ \mathscr{P}A_1,\ \mu,\ \tilde{\mu}) = N(\hat{A}_1 \circ \mathscr{P}A_1,\ \mu,\ \tilde{\mu}) + N(\hat{A}_1 \circ \mathscr{P}A_1,\ \mu_{n+1},\ \tilde{\mu}) + 2\,N(\hat{A}_1 \circ \mathscr{P}A_1,\ \mu_{p\theta+1},\ \tilde{\mu})$$
$$+\boldsymbol{3}\,N(\hat{\ddot{\mathscr{P}}}D_4,\ \mu+\tilde{\mu})$$

provided the line bundle is sufficiently $(2d+2)$*-ample.*

Theorem 5.4.2. *Let the total codimension of the two singular points is* 3, *then*

$$N(\mathscr{P}A_1 \circ \mathscr{P}A_2,\ \mu,\ \tilde{\mu}) = N(\mathscr{P}A_1 \circ \mathscr{P}A_1,\ \mu,\ \tilde{\mu}) + N(\mathscr{P}A_1 \circ \mathscr{P}A_1,\ \mu,\ \tilde{\mu}_{\tilde{n}+1}) + N(\mathscr{P}A_1 \circ \mathscr{P}A_1,\ \mu,\ \tilde{\mu}_{\tilde{m}+1})$$
$$-\boldsymbol{4}N(\ddot{\mathscr{P}}A_3,\ \mu+\tilde{\mu}) - \boldsymbol{2}N(\hat{\ddot{\mathscr{P}}}D_4,\ \mu+\tilde{\mu}) - N(\ddot{\hat{\mathscr{P}}}D_4,\ \mu+\tilde{\mu}) - \boldsymbol{3}N(\hat{\hat{\mathscr{P}}}D_4,\ \mu+\tilde{\mu})$$

provided the line bundle is sufficiently $(2d+3)$*-ample.*

Theorem 5.4.3. *Let the total codimension of the two singular points is* 4, *then*

$$N(\mathscr{P}A_2 \circ \mathscr{P}A_2,\ \mu,\ \tilde{\mu}) = N(\mathscr{P}A_1 \circ \mathscr{P}A_2,\ \mu,\ \tilde{\mu}) +$$
$$N(\mathscr{P}A_1 \circ \mathscr{P}A_2,\ \mu_{n+1},\ \tilde{\mu} + N(\mathscr{P}A_1 \circ \mathscr{P}A_2,\ \mu_{m+1},\ \tilde{\mu}) - \boldsymbol{3}N(\ddot{\mathscr{P}}A_4,\ \mu+\tilde{\mu})$$
$$-\ N(\hat{\ddot{\mathscr{P}}}D_5,\ \mu+\tilde{\mu}) - \boldsymbol{2}N(\hat{\ddot{\mathscr{P}}}\check{D}_5,\ \mu+\tilde{\mu}) -\ N(\ddot{\hat{\mathscr{P}}}D_5,\ \mu+\tilde{\mu}) - \boldsymbol{2}N(\ddot{\hat{\mathscr{P}}}\check{D}_5,\ \mu+\tilde{\mu})$$

provided the line bundle is sufficiently $(2d+4)$*-ample.*

Theorem 5.4.4. *Let the total codimension of the two singular points is* 5, *then*

$$N(\mathscr{P}A_2 \circ \mathscr{P}A_3,\ \mu,\ \tilde{\mu}) = N(\mathscr{P}A_2 \circ \mathscr{P}A_2,\ \mu,\ \tilde{\mu}) + N(\mathscr{P}A_2 \circ \mathscr{P}A_2,\ \mu,\ \tilde{\mu}_{\tilde{n}+1}) + 3N(\mathscr{P}A_2 \circ \mathscr{P}A_2,\ \mu,\ \tilde{\mu}_{\theta+1})$$
$$-\boldsymbol{3}N(\ddot{\mathscr{P}}A_5,\ \mu+\tilde{\mu}) - \boldsymbol{16}N(\hat{\ddot{\mathscr{P}}}E_6,\ \mu+\tilde{\mu})$$

provided the line bundle is sufficiently $(2d+5)$*-ample.*

Theorem 5.4.5. *Let the total codimension of the two singular points is* 6*, then*

$$N(\mathscr{P}A_3\circ\mathscr{P}A_3,\ \mu,\ \tilde{\mu}) = N(\mathscr{P}A_2\circ\mathscr{P}A_3,\ \mu,\ \tilde{\mu})+N(\mathscr{P}A_2\circ\mathscr{P}A_3,\ \mu_{n+1},\ \tilde{\mu})$$

$$+3\,N(\mathscr{P}A_2\circ\mathscr{P}A_3,\ \mu_{p\theta+1},\ \tilde{\mu})-\boldsymbol{4}\,N(\ddot{\mathscr{P}}A_6,\ \mu+\tilde{\mu})-\boldsymbol{5}\,N(\ddot{\mathscr{P}}\check{\check{D}}_6,\ \mu+\tilde{\mu})-\boldsymbol{6}\,N(\ddot{\mathscr{P}}\check{D}_6,\ \mu+\tilde{\mu})$$

provided the line bundle is sufficiently $(2d+6)$*-ample.*

Remark 5.4.6. *Note that the various numbers upto codimension* 6*, for example the numbers corresponding to the one pointed singularities, i.e.,* $N(\mathscr{P}A_k)$*,* $N(\mathscr{P}D_k)$ *and* $N(\mathscr{P}E_k)$ *such that* $k\leq 6$ *and the formulas for two pointed singularities i.e.,* $N(A_1\circ\mathscr{P}A_k)$*,* $N(A_1\circ\mathscr{P}D_k)$ *such that* $k\leq 5$ *can be found in [1], [6] and [3].*

5.5 Proof of recursive formulas

In this section, we will prove the recursive formulas that we have obtained assuming the corresponding closure and multiplicity claims. We will prove the closure and multiplicity claims in our upcoming paper [52]. Assuming these technicalities we will present the main ideas to prove our formulas. We will start with a known theorem in [2]

Theorem 5.5.1. *Let* $\mu=(n,m,r,p\theta)$ *and* $\tilde{\mu}=(\tilde{n},\tilde{m},\tilde{r},\tilde{\theta})$ *tuples of non negative integer. Then*

$$N(\hat{A}_1\circ\mathscr{P}A_1,\ \mu,\ \tilde{\mu}) = \begin{cases} 0, \text{ if } p\theta=0\\ N(A_1\circ\mathscr{P}A_1,\ \mu,\ \tilde{\mu}), \text{ if } p\theta=1\\ N(A_1\circ\mathscr{P}A_1,\ \mu_{n+1,p\theta-1},\ \tilde{\mu})-N(A_1\mathscr{P}A_1,\ \mu_{m_1+1,p\theta-2},\ \tilde{\mu})\end{cases}$$

provided the line bundle is sufficiently $(2d+2)$*-ample.*

Proof. The proof can be found in [2], Theorem (1.2). ☐

Theorem 5.5.2. *Let us consider* $\mu=(n,m,r)$ *and* $\tilde{\mu}=(\tilde{n},\tilde{m},\tilde{r},\theta)$ *tuples of non negative integer. Then*

$$N(A_2\circ\mathscr{P}A_2,\ \mu,\ \tilde{\mu}) =$$

$$2N(A_1\circ\mathscr{P}A_2,\ \mu,\ \tilde{\mu})+2N(A_1\circ\mathscr{P}A_2,\ \mu_{n+1},\ \tilde{\mu})+$$

$$2N(A_1\circ\mathscr{P}A_2,\ \mu_{m_1+1},\ \tilde{\mu})+\boldsymbol{3}N(\mathscr{P}A_4,\ \mu+\tilde{\mu})+\boldsymbol{3}N(\mathscr{P}D_4,\ \mu+\tilde{\mu})$$

provided the line bundle is sufficiently $(2d+4)$*-ample.*

Remark 5.5.3. *Note that the explicit formula for* $N(A_1\circ\mathscr{P}D_4)$ *has been studied [1]. The nontrivial closure of space* $A_1\circ\mathscr{P}D_4$ *has been analyzed there. Thus we can find a recursive formula to compute* $N(A_2\circ\mathscr{P}D_4)$ *by studying the section corresponding to the det*$(\nabla^2 f)$ *over the space* $\overline{A_1\circ\mathscr{P}D_4}$*. W can analogously follow the proof of the above theorem to complete the recursive formula for* $N(A_2\circ\mathscr{P}D_4)$*.*

Proof of theorem 5.5.2

We have defined the space

$$\overline{A}_1 \circ \mathscr{P}A_2 := \{([f], q_1, l_{q_2}) \in \mathscr{D} \times X \times \mathbb{P}TX : f \text{has a singularity of type } A_1 \text{ at } q_1, \text{ type } A_2 \text{ at } q_2, \\ \nabla^2 f(v,.) = 0, q_1 \neq q_2, \forall v \in l_{q_2}\}$$

Let us denote $[\mu\tilde{\mu}]$ to be the generic cycle representing the homology class Poincaré dual to

$$\tilde{c}_1^{\tilde{n}} \tilde{x}_1^{\tilde{m}_1} \tilde{x}_2^{\tilde{m}_2} c_1^n x_1^{m_1} x_2^{m_2} \lambda^{\theta} y^{\delta_d - (\tilde{n}+n+\tilde{m}_1+m_1+2\tilde{m}_2+2m_2+\theta+3)}$$

We will now define the intersection number considering the following line bundle that is induced by the determinant of the hessian map, namely:

$$\psi_{A_2} = \overline{A}_1 \circ \mathscr{P}A_2 \longrightarrow \mathscr{L}_{det} := \gamma_{\mathscr{D}}^{*\otimes 2} \otimes \Lambda^2 T^* X^{\otimes 2} \otimes L^{\otimes 2}$$

defined by

$$\psi_{A_2}([f],\ q_1,\ l_{q_2})(f \otimes v \otimes w) = \det \begin{pmatrix} \nabla^2 f(v,v) & \nabla^2 f(v,w) \\ \nabla^2 f(w,v) & \nabla^2 f(w,w) \end{pmatrix}$$

We will prove shortly that L is sufficiently $(2d+4)$-ample, then this section is transverse to the zero set. Next, let us define

$$\mathscr{B} := \overline{\overline{A}_1 \circ \mathscr{P}A_2} - \overline{A}_1 \circ \mathscr{P}A_2.$$

Hence

$$\langle e(\mathscr{L}_{det}), \quad [\overline{A_1 \circ \mathscr{P}A_2}] \cap [\mu\tilde{\mu}] \rangle = N(A_2\mathscr{P}A_2, \mu, \tilde{\mu}) + C_{\mathscr{B} \cap \mu\tilde{\mu}} \tag{5.4}$$

where $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ denotes the contribution of the section to the Euler class from the points of $\mathscr{B} \cap \mu\tilde{\mu}$. Here we are using the result from [3], i.e we know the explicit description of the boundary $\mathscr{B} \cap \mu\tilde{\mu}$. Although the whole boundary is not relevant while computing the contribution to the Euler class; only the points at which section vanishes are relevant.

Hence we conclude that using the result [3], Equation(3), Lemma 6.3, we know

$$\mathscr{B}(q_1, q_2) := \{([f], q_1, l_{q_2}) \in \mathscr{B} : q_1 = q_2\} = \overline{\mathscr{P}A_4} \cup \overline{\mathscr{P}D_4}$$

Hence the total contribution from boundary to the Euler class is

$$m_1\, N(\mathscr{P}A_4, \mu, \tilde{\mu}) + m_2\, N(\mathscr{P}D_4, \mu, \tilde{\mu})$$

we will now justify the multiplicities m_1 and m_2 respectively by proving the following claims

Claim 5.5.4. *If* $([f],q_1,l_{q_1}) \in \overline{\mathscr{P}A_4} \cap \psi_{A_2}^{-1}(0)$*, then this section vanishes on* $([f],q_1,l_{q_1})$ *with a multiplicity 3.*

Proof of the claim 5.5.4

It is shown in [2], Lemma 6.11

$$\{([f],q_1,l_{q_1}) \in \overline{A_1 \circ \mathscr{P}A_2}\} \supset \Delta\overline{\mathscr{P}A_4}$$

and

$$\{([f],q_1,l_{q_1}) \in \overline{A_1 \circ \mathscr{P}A_3}\} \cap \Delta\overline{\mathscr{P}A_5} = \phi$$

Furthermore, if $([f],q_1,l_{q_1}) \in \Delta\overline{\mathscr{P}A_4}$ then all the small solutions $([f_t],q_1^t,l_{q_2^t}) \in \overline{A_1 \circ \mathscr{P}A_2}$ are constructed as follows:

$$\begin{aligned} \mathscr{A}_2^{f_t} = 0, \quad \hat{\hat{y}} = 0, \quad x_t \neq 0 \quad \text{(but small)} \\ \mathscr{A}_3^{f_t} = \frac{\mathscr{A}_5^{f_t}}{20} x_t^2 + O(x_t^3) \\ \mathscr{A}_4^{f_t} = -\frac{2\mathscr{A}_5^{f_t}}{5} x_t + O(x_t^2) \end{aligned} \tag{5.5}$$

Next, using the above-constructed solutions we see that the multiplicity of the equation

$$\det \begin{pmatrix} \nabla^2 f(v,v) & \nabla^2 f(v,w) \\ \nabla^2 f(w,v) & \nabla^2 f(w,w) \end{pmatrix} = \varepsilon \tag{5.6}$$

is 3, which justifies $m_1 = 3$.

In a similar manner we can justify $m_2 = 3$. Hence, the total contribution from all the components of type $\mathscr{B}(q_1,l_{q_2})$ equals

$$3\,N(\mathscr{P}A_4,\mu,\tilde{\mu}) + 3\,N(\mathscr{P}D_4,\mu,\tilde{\mu})$$

Thus plugging in the value for $\mathscr{B}(q_1,l_{q_2})$ in the equation (5.4) we get the final recursion formula.

Now let us now justify the transversality. Without loss of generality let us assume that $q_2 = [0:0:1]$ and $q_1 = [X_1 : Y_1 : 1]$. Since we have $q_1 \neq q_2$ so X_1, Y_1 both can not be zero; let us assume that X_1 is nonzero. The consider the polynomial

$$\eta_{det} := (X - X_1)^2\, X^2\, Z^{d-4}$$

we note the following properties of η_{det}:

$$\begin{aligned} \eta_{det}(q_i) = 0, \quad \forall i = 1,2 \\ \nabla\eta_{det}(q_i) = 0, \quad \forall i = 1,2 \\ (f_{t20}f_{t02} - f_{t11}^2)_{|q_1} \neq 0 \end{aligned} \tag{5.7}$$

Now consider that the curve $\gamma : (-\varepsilon,\varepsilon) \longrightarrow \mathscr{D}$, given by

$$\gamma(t) = ([f + t\eta_{det}],\ q_1,\ q_2)$$

We see that because of equation (5.7), the curve $\gamma(t)$ lies in $A_1 \circ \overline{\mathscr{P}A_2}$. We now note that

$$\{\nabla\psi_{A_2|_{([f],\ q_1,\ q_2)}}\}(\gamma'(0)) = (f_{t_{20}}f_{t_{02}} - f_{t_{11}}^2)_{|_{q_1}} \tag{5.8}$$

then we observe that the right-hand side of the above equation is nonzero, hence the section ψ_{A_2} is transverse to zero. This completes the proof.

Proof of the theorem 5.4.1: computation of $N(\mathscr{P}A_1 \circ \mathscr{P}A_1, \tilde{\mu}, \mu)$

Let us recall that

$$\begin{aligned}\mathscr{P}A_1 \circ \overline{\hat{A}}_1 := \{([f],\ l_q,\ l_p) \in \mathscr{D} \times \mathbb{P}T\mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2 : & f \text{ has a singularity of type } \mathscr{P}A_1 \text{ at } q,\\ & ([f],\ l_p) \in \overline{\hat{A}}_1,\quad p \neq q\}.\end{aligned}$$

Let $[\mu\tilde{\mu}]$ be a generic cycle, representing the class

$$[\mu\tilde{\mu}] = y^{\delta_d - (n_1+m_1+2m_2+\theta+\tilde{n}_1+\tilde{m}_1+2\tilde{m}_2+p\theta+2)} c_1^{n_1} \tilde{c_1}^{\tilde{n}_1} \tilde{\lambda}^{p\theta} \lambda^{\theta} x_1^{m_1} x_2^{m_2} \tilde{x}_1^{\tilde{m}_1} \tilde{x}_2^{\tilde{m}_2}$$

We now define a section of the following line bundle

$$\begin{aligned}&\Psi_{\mathscr{P}\mathrm{A}_1} : \mathscr{P}A_1 \circ \overline{\hat{A}}_1 \longrightarrow \mathbb{L}_{\mathscr{P}\mathrm{A}_1} := \gamma_{\mathscr{D}}^* \otimes \hat{\gamma}^{*2} \otimes \gamma_{\mathbb{P}^2}^{*d}\\ &\text{defined by } \{\Psi_{\mathscr{P}\mathrm{A}_1}([f], l_{q_1}, l_{q_2})\}([f], v, v) = \nabla f_{|_p}(v, v)\end{aligned} \tag{5.9}$$

where $\hat{\gamma} \longrightarrow \mathbb{P}T\mathbb{P}^2$ be the tautological line bundle. We will show shortly that the above section is transverse to zero set with some ampleness condition. Next, let us define

$$\mathscr{B} := \overline{\mathscr{P}A_1 \circ \hat{A}_1} - \mathscr{P}A_1 \circ \overline{\hat{A}_1}$$

Hence

$$\langle e(\mathbb{L}_{\mathscr{P}A_1}),\ [\overline{\mathscr{P}A_1 \circ \hat{A}_1}] \cap [\mu\tilde{\mu}]\rangle = \tag{5.10}$$

$$N(\mathscr{P}A_1 \circ \mathscr{P}A_1, \tilde{\mu}, \mu) + C_{\mathscr{B}\cap\mu\tilde{\mu}}. \tag{5.11}$$

where the notation $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ denotes the excess contribution of the section $\Psi_{\mathscr{P}\mathrm{A}_1}$ to the Euler class from the points of $\mathscr{B} \cap \mu\tilde{\mu}$. We now describe $\mathscr{B}(l_q,\ l_p)$ as before. We only need to consider the component of $\mathscr{B}$ where two points $q,\ p$ become equal, i.e.,

$$\mathscr{B}(l_q,\ l_p) := \{([f],\ l_q,\ l_p) \in \mathscr{B} \mid q = p\}$$

Let us define a few things. Let v be a fixed non zero vector that belongs to l_p. Let us define the subsets W_1 and W_2 as

$$W_1 := \{([f],\, l_q,\, l_p) \in \overline{\mathscr{P}A_1 \circ \hat{A}_1} : \nabla^2 f|_{q_p} \not\equiv 0\},$$
$$W_2 := \{([f],\, l_q,\, l_p) \in \overline{\mathscr{P}A_1 \circ \hat{A}_1} : \nabla^2 f|_{q_p} \equiv 0\}, \tag{5.12}$$

We claim that

$$\mathscr{B}(l_q, l_p) \cap W_1 \approx \overline{\hat{\ddot{\mathscr{P}}} A_3} \cap W_1, \tag{5.13}$$

$$\mathscr{B}(l_q, l_p) \cap W_2 \approx \overline{\hat{\overline{D}}}_4 \cap W_2 \tag{5.14}$$

We can now explicitly describe $\mathscr{B}$, i.e., using the equations (5.13), (5.14) we can conclude that

$$\mathscr{B} \approx \overline{\hat{\ddot{\mathscr{P}}} A_3} \cup \overline{\hat{\overline{D}}}_4 \tag{5.15}$$

We claim that the contribution to the Euler class from each of the points of $\overline{\hat{\ddot{\mathscr{P}}} A_3} \cap \mu\tilde{\mu}$ and $\overline{\hat{\overline{D}}}_4 \cap \mu\tilde{\mu}$ are 0 and 3 respectively. Note that when we intersect $\overline{\hat{\ddot{\mathscr{P}}} A_3}$ with $\mu\tilde{\mu}$, we will get an isolated set of a finite number of points. Hence, our section $\Psi_{\mathscr{P}\mathrm{A}_1}$ will not vanish there. Thus it does not contribute to the Euler class. Next, let us make the following claim

Claim 5.5.5. *If* $([f],\, l_q,\, l_p) \in \overline{\hat{\overline{D}}}_4 \cap \Psi^{-1}_{\mathscr{P}\mathrm{A}_1}(0)$*, then this section vanishes on* $([f],\, l_q,\, l_p)$ *with a multiplicity 3.*

Therefore the total contribution from the boundary to the Euler class is

$$3\, N(\overline{\hat{\overline{D}}}_4,\; \mu + \tilde{\mu}). \tag{5.16}$$

Plugging this contribution in the equation (5.10) we get the final formula as (5.4.1).

We will now show that the section $\Psi_{\mathscr{P}A_1}$ is transverse to the zero set.

Note that we want to prove $\mathscr{P}A_1 \circ \mathscr{P}A_1$ is a smooth complex submanifold of $\mathscr{D} \times (\mathscr{P}T\mathbb{P}^2)^2$ (provided $d \geq 2\delta + 1$). We will prove even a stronger statement: we will show that $\mathscr{P}A_1 \circ \mathscr{P}A_1$ is a smooth complex submanifold of $\mathscr{D} \times (\mathscr{P}T\mathbb{P}^2)^2$ and the section $\Psi_{\mathscr{P}A_1}$ are transverse to zero. Our desired claim follows immediately from this statement since $\mathscr{P}A_1 \circ \mathscr{P}A_1$ is an open subset of $\overline{\mathscr{P}A_1 \circ \mathscr{P}A_1}$.

Let us now begin by showing that $\Psi_{\mathscr{P}A_1}$ is transverse to zero if $d \geq 2\delta + 1$. Suppose

$$\{\Psi_{\mathscr{P}A_1}([f],\, q_1,\, q_2)\}(f \otimes v) = 0.$$

Without loss of generality, we can assume $q_2 = [0:0:1]$ and $q_1 = [X_1 : Y_1 : 0]$. Since all the q_i are distinct, we conclude that X, Y both can not be zero. Let ∂_x and ∂_y be the standard basis vectors for $T\mathbb{P}^2|_{q_2}$ (corresponding to the first two coordinates). Hence

$$l_{q_2} = [a_1\partial_x + a_2\partial_y] \in \mathbb{P}T\mathbb{P}^2|_{q_2}$$

for some complex numbers a_1, a_2 not both of them are zero. Without loss of generality, we can assume $l_{q_2} = [\partial_x]$. Let us now consider the homogeneous degree d polynomial, given by

$$\rho_{20} := (X - X_1)^2\, X^2\, Z^{d-4}$$

and consider the corresponding curve $\gamma_{20}(t)$. We now note

$$\{\{\nabla\Psi_{\mathscr{P}A_1}|_{([f],\, l_{q_1},\, l_{q_2})}\}(\gamma_{20}'(0))\}(f\otimes\partial_x\otimes\partial_x) = \nabla^2\rho_{20|_{[0,0,1]}}(\partial_x,\partial_x) \neq 0$$

Thus we conclude that the section is transverse to zero. Hence the theorem is complete assuming the equations (5.16) and the claim5.5.5.

Proof of the Theorem 5.4.2

We have defined the space

$$\mathscr{P}A_1\circ\overline{\mathscr{P}A}_1 := \{([f],\, l_{q_1},\, l_{q_2}) \in \mathscr{D}\times(\mathbb{P}T\mathbb{P})^2 : ([f], l_{q_1}) \in \mathscr{P}A_1, \quad ([f],\, l_{q_2}) \in \mathscr{P}A_1, \quad q_1 \neq q_2\}.$$

Let $[\mu\tilde{\mu}]$ be a generic cycle representing the homology class Poincaré dual to

$$c_1^{n_1}\tilde{c}_1^{\tilde{n}_1}x_1^{m_1}\tilde{x}_1^{\tilde{m}_1}x_2^{m_2}\tilde{x}_2^{\tilde{m}_2}\lambda^{\theta}\tilde{\lambda}^{p\theta}y^{\delta_d-(n_1+\tilde{n}_1+m_1+\tilde{m}_1+2m_2+2\tilde{m}_2+\theta+p\theta+3)}.$$

We now define a section of the following line bundle

$$\Psi_{\mathscr{P}\mathrm{A}_2} : \mathscr{P}A_1\circ\overline{\mathscr{P}\mathrm{A}}_1 \longrightarrow \mathbb{L}_{\mathscr{P}\mathrm{A}_2} := \gamma_{\mathscr{D}}^*\otimes\hat{\gamma}^*\otimes(\pi^*T\mathbb{P}^2/\hat{\gamma})^*\otimes\gamma_{\mathbb{P}^2}^{*d}, \qquad \text{given by}$$

$$\{\Psi_{\mathscr{P}\mathrm{A}_2}([f],\, l_{q_1},\, l_{q_2})\}(f\otimes v\otimes w) := \nabla^2 f|_{q_2}(v,w).$$

We will prove that this section is transverse to zero provided it satisfies the ampleness condition. Next, let us define

$$\mathscr{B} := \overline{\mathscr{P}A_1\circ\overline{\mathscr{P}\mathrm{A}}_1} - \mathscr{P}A_1\circ\overline{\mathscr{P}\mathrm{A}}_1.$$

Hence

$$\langle e(\mathbb{L}_{\mathscr{P}\mathrm{A}_2}),\ [\overline{\mathscr{P}A_1\circ\overline{\mathscr{P}\mathrm{A}}_1}]\cap[\mu\tilde{\mu}]\rangle = N(\mathscr{P}\mathrm{A}_1\mathscr{P}\mathrm{A}_2,\tilde{\mu},\mu) + C_{\mathscr{B}\cap\mu\tilde{\mu}}. \tag{5.17}$$

where $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ denotes the excess contribution of the section $\Psi_{\mathscr{P}\mathrm{A}_2}$ to the Euler class from the points of $\mathscr{B}\cap\mu\tilde{\mu}$. We now give an explicit description of $\mathscr{B}$. As before, we only need to consider the component of $\mathscr{B}$ where two points q_i become equal. Define $\mathscr{B}(l_{q_1},\, l_{q_2})$ as before. We will study the boundary locus $\mathscr{B}(l_{q_1},\, l_{q_2})$ as we do in the previous computation and we will show in [52] that

$$\mathscr{B}(l_{q_1}, l_{q_2}) \approx \overline{\ddot{\mathscr{P}}A_3} \cup \overline{\hat{\ddot{\mathscr{P}}}D_4} \cup \overline{\ddot{\hat{\mathscr{P}}}D_4} \cup \overline{\hat{\hat{D}}_4}. \tag{5.18}$$

For example, we observe the following:

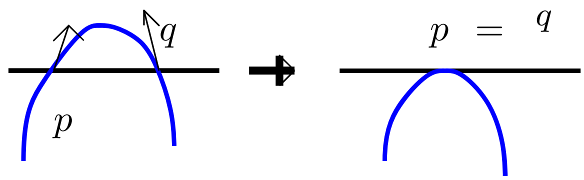

Figure 5.1: When two $\mathscr{P}A_1$ collide we get $\mathscr{P}A_3$.

Furthermore, we also prove that the excess contribution from $\overline{\ddot{\mathscr{P}}\mathrm{A}_3}\cap\mu\tilde{\mu}$ to the Euler class is 4. Similarly we will prove that the contribution from $\overline{\hat{\ddot{\mathscr{P}}}\mathscr{D}_4}\cap\mu\tilde{\mu}$ and $\overline{\ddot{\hat{\mathscr{P}}}\mathscr{D}_4}\cap\mu\tilde{\mu}$ are 2 and 1 respectively, while the contribution from $\hat{\hat{\mathscr{D}}}_4\cap\mu$ is 2 and the contribution from $\hat{\hat{D}}_5\cap\mu$ is 1. Hence the total contribution from all the components of type $\mathscr{B}(l_q, l_p)$ equals

$$4N(\ddot{\mathscr{P}}A_3, \mu+\tilde{\mu}) + 2N(\hat{\ddot{\mathscr{P}}}D_4, \mu+\tilde{\mu}) + N(\ddot{\hat{\mathscr{P}}}D_4, \mu+\tilde{\mu}) + 2N(\hat{\hat{D}}_4, \mu+\tilde{\mu}) + 2N(\hat{\hat{D}}_5, \mu+\tilde{\mu}). \tag{5.19}$$

Plugging in the degenerate contribution from equation (5.19) to the equation (5.17), hence giving us Theorem 5.4.2.

Next, we want to prove $\mathscr{P}A_1\circ\mathscr{P}A_2$ is a smooth complex sub-manifold of $\mathscr{D}\times(\mathbb{P}T\mathbb{P}^2)^2$ (provided $d\geq 2\delta+2$). We will prove a slightly stronger statement: we prove that $\mathscr{P}A_1\circ\mathscr{P}A_2$ is a smooth complex sub manifold of $\mathscr{D}\times(\mathbb{P}T\mathbb{P}^2)^2$ and the section $\Psi_{\mathscr{P}A_2}$ and are transverse to zero. Our desired claim follows immediately from this statement since $\mathscr{P}A_1\circ\mathscr{P}A_2$ is an open subset of $\overline{\mathscr{P}A_1\circ\mathscr{P}A_2}$.

We will begin by showing that $\Psi_{\mathscr{P}A_2}$ is transverse to zero if $d\geq 2\delta+2$. Suppose

$$\{\Psi_{\mathscr{P}A_2}([f],\, q_1,\, q_2)\}(f\otimes v\otimes w)=0.$$

Without loss of generality, we can assume $q_2=[0:0:1]$ and $q_1=[X_1:Y_1:0]$. Since all the q_i are distinct, we conclude that X, Y both can not be zero. Let ∂_x and ∂_y be the standard basis vectors for $T\mathbb{P}^2|_{q_2}$ (corresponding to the first two coordinates). Hence

$$l_{q_2}=[a_1\partial_x+a_2\partial_y]\in\mathbb{P}T\mathbb{P}^2|_{q_2}$$

for some complex numbers a_1, a_2, both of them can not be zero. Without loss of generality, we can assume $l_{q_2} = [\partial_x]$. Let us now consider the homogeneous degree d polynomial, given by

$$\rho_{11} := (X - X_1)^2 \, X \, Y \, Z^{d-4}$$

and consider the corresponding curve $\gamma_{11}(t)$ analogously as before. Then the transversality follows from the computation of the derivative of the section $\Psi_{\mathscr{P}A_2}$ along the curve $\gamma_{11}(t)$ as before.

Proof of the Theorem 5.4.3

We have defined the space

$$\overline{\mathscr{P}A_1} \circ \mathscr{P}\mathrm{A}_2 := \{([f],\, l_{q_1},\, l_{q_2}) \in \mathscr{D} \times (\mathbb{P}T\mathbb{P}^2)^2 : ([f],\, l_{q_1}) \in \overline{\mathscr{P}A_1}, \quad ([f],\, l_{q_2}) \in \mathscr{P}\mathrm{A}_2, \quad q_1 \neq q_2\}.$$

Let $[\mu\tilde{\mu}]$ be a generic cycle representing the homology class Poincaré dual to

$$c_1^{n_1} \tilde{c}_1^{\tilde{n}_1} x_1^{m_1} \tilde{x}_1^{\tilde{m}_1} x_2^{m_2} \tilde{x}_2^{\tilde{m}_2} \lambda^{\theta} \tilde{\lambda}^{p\theta} y^{\delta_d - (n_1 + \tilde{n}_1 + m_1 + \tilde{m}_1 + 2m_2 + 2\tilde{m}_2 + \theta + p\theta + 4)}.$$

We now define a section of some bundle as follows:

$$\Psi_{\mathscr{P}\mathrm{A}_2} : \overline{\mathscr{P}A_1} \circ \mathscr{P}\mathrm{A}_2 \longrightarrow \mathbb{L}_{\mathscr{P}\mathrm{A}_2} := \gamma_{\mathscr{D}}^* \otimes \hat{\gamma}^* \otimes (\pi^* T\mathbb{P}^2/\hat{\gamma})^* \otimes \gamma_{\mathbb{P}^2}^{*d}, \qquad \text{given by}$$

$$\{\Psi_{\mathscr{P}\mathrm{A}_2}([f],\, l_{q_1},\, l_{q_2})\}(f \otimes v \otimes w) := \nabla^2 f|_{q_1}(v, w).$$

We will shortly prove that this section is transverse to zero set satisfying the ampleness condition. Next, let us define

$$\mathscr{B} := \overline{\overline{\mathscr{P}A_1} \circ \mathscr{P}\mathrm{A}_2} - \overline{\mathscr{P}A_1} \circ \mathscr{P}\mathrm{A}_2.$$

Hence

$$\langle e(\mathbb{L}_{\mathscr{P}\mathrm{A}_2}),\ [\overline{\overline{\mathscr{P}A_1} \circ \mathscr{P}\mathrm{A}_2}] \cap [\mu\tilde{\mu}] \rangle = N(\mathscr{P}\mathrm{A}_2 \circ \mathscr{P}\mathrm{A}_2,\ \mu,\ \tilde{\mu}) + C_{\mathscr{B} \cap \mu\tilde{\mu}}. \tag{5.20}$$

as before, the notation $C_{\mathscr{B} \cap \mu\tilde{\mu}}$ denotes the excess contribution of the section $\Psi_{\mathscr{P}\mathrm{A}_2}$ to the Euler class from the points of $\mathscr{B} \cap \mu\tilde{\mu}$. We now give an explicit description of $\mathscr{B}$. As before, we only need to consider the component of $\mathscr{B}$ where two points become equal. Define $\mathscr{B}(l_q,\, l_p)$ as before. We will study the boundary locus analogously to the previous computation and we claim that

$$\mathscr{B}(l_{q_1},\, l_{q_2}) \approx \overline{\ddot{\mathscr{P}}\mathrm{A}_4} \cup \overline{\hat{\ddot{\mathscr{P}}}\mathscr{D}_5} \cup \overline{\ddot{\hat{\mathscr{P}}}\mathscr{D}_5} \cup \hat{\ddot{\mathscr{P}}}\check{D}_5 \cup \ddot{\hat{\mathscr{P}}}\check{D}_5. \tag{5.21}$$

Furthermore, we show that the contribution from $\overline{\ddot{\mathscr{P}}A_4} \cap \mu\tilde{\mu}$ to the Euler class is 3. Similarly the contribution from $\overline{\hat{\ddot{\mathscr{P}}}D_5} \cap \mu\tilde{\mu}, \overline{\ddot{\hat{\mathscr{P}}}D_5} \cap \mu\tilde{\mu}$ and $\overline{\hat{\ddot{\mathscr{P}}}\check{D}_5} \cap \mu\tilde{\mu}$ are 2, 1 and 1 respectively, while the contribution from $\hat{\ddot{\mathscr{P}}}\check{D}_5 \cap \mu\tilde{\mu}$ is 2. Hence the total degenerate contribution from all the components of type $\mathscr{B}(l_q,\, l_p)$ equals

$$\begin{aligned} 3\,N(\mathscr{P}\ddot{A}_4,\, \mu+\tilde{\mu}) \,+\, 2\,N(\mathscr{P}\ddot{\hat{D}}_5,\, \mu+\tilde{\mu}) \,+\, N(\mathscr{P}\hat{\ddot{D}}_5,\, \mu+\tilde{\mu}) \\ +\, 2\,N(\, \hat{\ddot{\mathscr{P}}}\check{D}_5,\, \mu+\tilde{\mu}) \,+\, N(\, \hat{\ddot{\mathscr{P}}}\check{D}_5\, ,\mu+\tilde{\mu}). \end{aligned} \tag{5.22}$$

Then plugging the above for $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ in the equation (5.20) giving us Theorem 5.4.3.

Let us now prove the above transversality and multiplicity claims. Let us continue with the same setup as earlier computation. Let $q_1 = [0:0:1]$ and $q_2 = [X_2 : Y_2 : 1]$. Consider the polynomial

$$\tilde{\zeta}_{11} := (X - X_2)^2\, X\, Y\, Z^{d-4}$$

and consider the corresponding curve as $\tilde{\gamma}_{11}(t)$ in a similar fashion as we do for the computation of $N(\mathscr{P}\mathrm{A}_2 \circ \mathscr{P}\mathrm{A}_2, \mu, \tilde{\mu})$. We note that the section $\psi_{\mathscr{P}A_2}$ is transverse to zero by computing the derivative of the section along $\gamma_{11}(t)$. Hence, $\mathscr{P}A_2 \circ \mathscr{P}A_2$ is a smooth complex submanifold of $\mathscr{D} \times (\mathbb{P}T\mathbb{P}^2)^2$. Hence this completes the proof assuming the equation (5.21), the multiplicity claim (5.22).

Proof of the Theorem 5.4.4

We have defined the space

$$\mathscr{P}A_2 \circ \overline{\mathscr{P}A_2} := \{([f],\, l_{q_1},\, l_{q_2}) \in \mathscr{D} \times (\mathbb{P}T\mathbb{P}^2)^2 : ([f],\, l_{q_1}) \in \mathscr{P}A_2, \quad ([f],\, l_{q_2}) \in \overline{\mathscr{P}A_2}, \quad q_1 \neq q_2\}.$$

Let $[\mu\tilde{\mu}]$ be a generic cycle representing the homology class Poincaré dual to

$$c_1^{n_1} \tilde{c}_1^{\tilde{n}_1} x_1^{m_1} \tilde{x}_1^{\tilde{m}_1} x_2^{m_2} \tilde{x}_2^{\tilde{m}_2} \lambda^{\theta} \tilde{\lambda}^{p\theta} y^{\delta_d - (n_1 + \tilde{n}_1 + m_1 + \tilde{m}_1 + 2m_2 + 2\tilde{m}_2 + \theta + p\theta + 5)}.$$

We now define a section of the following bundle

$$\begin{aligned} &\Psi_{\mathscr{P}\mathrm{A}_3} : \overline{\mathscr{P}A_2 \circ \overline{\mathscr{P}\mathrm{A}_2}} \longrightarrow \mathbb{L}_{\mathscr{P}\mathrm{A}_3} := \gamma_{\mathscr{D}}^* \otimes \hat{\gamma}^{*3} \otimes \gamma_{\mathbb{P}^2}^{*d}, \qquad \text{given by} \\ &\{\Psi_{\mathscr{P}\mathrm{A}_3}([f],\, l_{q_1},\, l_{q_2})\}(f \otimes v^{\otimes 3}) := \nabla^3 f|_{q_1}(v, v, v). \end{aligned}$$

We will prove that whenever the ampleness condition is satisfied, the above section is transverse to the zero set. Next, let us define

$$\mathscr{B} := \overline{\mathscr{P}A_2 \circ \overline{\mathscr{P}\mathrm{A}_2}} - \mathscr{P}A_2 \circ \overline{\mathscr{P}\mathrm{A}_2}.$$

Hence

$$\langle e(\mathbb{L}_{\mathscr{P}\mathrm{A}_2}),\ \ [\overline{\mathscr{P}A_2 \circ \overline{\mathscr{P}\mathrm{A}_2}}] \cap [\mu] \rangle = N(\mathscr{P}A_2 \circ \mathscr{P}A_3, \mu, \tilde{\mu}) + C_{\mathscr{B} \cap \mu}. \tag{5.23}$$

As before, the notation $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ denotes the excess contribution of the section $\Psi_{\mathscr{P}\mathrm{A}_3}$ to the Euler class from the points of $\mathscr{B} \cap \mu\tilde{\mu}$. We now give an explicit description of $\mathscr{B}$. As before, we only need to consider the component of $\mathscr{B}$ where two points become equal. Note that the whole boundary is not relevant while computing the contribution to the Euler class; only the points at which the section vanishes is relevant. Define $\mathscr{B}(l_{q_1},\ l_{q_2})$ as before. We will analogously study the boundary locus as before and we claim that

$$\mathscr{B}(l_{q_1}, l_{q_2}) \approx \overline{\ddot{\mathscr{P}}\mathrm{A}_5} \,\cup\, \overline{\hat{\ddot{\mathscr{P}}}E_6} \tag{5.24}$$

Furthermore, we show that the contribution from $\overline{\ddot{\mathscr{P}}\mathrm{A}_5} \cap \mu$ is 3. The contribution from $\overline{\hat{\ddot{\mathscr{P}}}\mathscr{D}_6} \cap \mu$ and $\overline{\hat{\ddot{\mathscr{P}}}\mathscr{D}_6} \cap \mu$ are 0 respectively, while the contribution from $\hat{\ddot{\mathscr{P}}}E_6 \cap \mu$ is 16. Hence the total boundary contribution from all the components of type $\mathscr{B}(l_{q_1},\ l_{q_2})$ equals

$$3\, N(\ddot{\mathscr{P}}\mathrm{A}_5,\ \mu + \tilde{\mu}) \ +\ 16\, N(\hat{\ddot{\mathscr{P}}}E_6,\ \mu + \tilde{\mu}) \tag{5.25}$$

Then plugging the above for $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ in the equation (5.23) giving us Theorem (5.4.4).
We now need to show that $\mathscr{P}A_2 \circ \mathscr{P}A_3$ is a smooth complex sub manifold of $\mathscr{D} \times (\mathbb{P}T\mathbb{P}^2)^2$ and whence $\psi_{\mathscr{P}A_3}$ is transverse to the zero set. Let us now continue with the earlier setup. Let $q_2 = [0:0:1]$ and $q_1 = [X_1 : Y_1 : 1]$. Next, consider the polynomial

$$\tilde{\zeta}_{30} := (X - X_1)^2\, X^3\, Z^{d-5}$$

and consider the corresponding curve $\tilde{\gamma}_{30}(t)$ as

$$\tilde{\gamma}_{30}(t) = (f + t\tilde{\zeta}_{30},\ l_{q_1},\ l_{q_2}).$$

Then by computing the derivative of the section $\psi_{\mathscr{P}A_2}$ along $\tilde{\gamma}_{30}(t)$ we can conclude the section $\psi_{\mathscr{P}A_3}$ is transverse to the zero set. Hence the proof is complete assuming the (5.24) and the multiplicity claim (5.25).

Proof of the Theorem 5.4.5

We have defined the space

$$\overline{\mathscr{P}A_2} \circ \mathscr{P}A_3 := \{([f],\ l_{q_1},\ l_{q_2}) \in \mathscr{D} \times (\mathbb{P}T\mathbb{P}^2)^2 : ([f],\ l_{q_1}) \in \overline{\mathscr{P}A_2}, \quad ([f], l_{q_2}) \in \mathscr{P}A_3, \quad q_1 \neq q_2\}.$$

Let $[\mu\tilde{\mu}]$ be a generic cycle representing the homology class Poincaré dual to

$$c_1^{n_1}\tilde{c}_1^{\tilde{n}_1}x_1^{m_1}\tilde{x}_1^{\tilde{m}_1}x_2^{m_2}\tilde{x}_2^{\tilde{m}_2}\lambda^{\theta}\tilde{\lambda}^{p\theta}y^{\delta_d-(n_1+\tilde{n}_1+m_1+\tilde{m}_1+2m_2+2\tilde{m}_2+\theta+p\theta+6)}.$$

We now define a section of the following line bundle

$$\Psi_{\mathscr{P}\mathrm{A}_3}:\overline{\overline{\mathscr{P}A}_2\circ\mathscr{P}\mathrm{A}_3}\longrightarrow\mathbb{L}_{\mathscr{P}\mathrm{A}_3}:=\gamma_{\mathscr{D}}^{*}\otimes\hat{\gamma}^{*3}\otimes\gamma_{\mathbb{P}^2}^{*d},\qquad\text{given by}$$

$$\{\Psi_{\mathscr{P}\mathrm{A}_3}([f],\,l_{q_1},\,l_{q_2})\}(f\otimes v^{\otimes 3}):=\nabla^3 f|_{q_1}(v,v,v).$$

We will prove that this section is transverse to zero set satisfying some ampleness condition. Next, let us define

$$\mathscr{B}:=\overline{\overline{\mathscr{P}A}_2\circ\mathscr{P}\mathrm{A}_3}-\overline{\mathscr{P}A}_2\circ\mathscr{P}\mathrm{A}_3.$$

Hence

$$\langle e(\mathbb{L}_{\mathscr{P}\mathrm{A}_3}),\;[\overline{\overline{\mathscr{P}A}_2\circ\mathscr{P}A_3}]\cap[\mu\tilde{\mu}]\rangle=N(\mathscr{P}A_3\circ\mathscr{P}A_3,\,\mu,\,\tilde{\mu})+C_{\mathscr{B}\cap\mu\tilde{\mu}}.\tag{5.26}$$

where $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ denotes the degenerate contribution to the Euler class from the points of $\mathscr{B}\cap\mu\tilde{\mu}$ We now describe the boundary $\mathscr{B}$ explicitly. As before, we only interested to the component of $\mathscr{B}$ where two points q_i become equal. Next, define $\mathscr{B}(l_{q_1},\,l_{q_2})$ as before. We will follow the exact same path to study $\mathscr{B}(l_{q_1},l_{q_2})$ and we will claim that

$$\mathscr{B}(l_{q_1},l_{q_2})\approx\overline{\ddot{\mathscr{P}}A_6}\,\cup\,\overline{\ddot{\mathscr{P}}D_6^{\vee}}\,\cup\,\overline{\ddot{\mathscr{P}}D_6^{\vee\vee}}\tag{5.27}$$

Furthermore, we show that the degenerate contribution from $\overline{\ddot{\mathscr{P}}\mathrm{A}_6}\cap\mu\tilde{\mu}$ to the Euler class is 4. Analogously, the contribution from $\ddot{\mathscr{P}}D_6^{\vee\vee}\cap\mu\tilde{\mu}$ is 5, while the contribution from $\ddot{\mathscr{P}}D_6^{\vee}\cap\mu\tilde{\mu}$ is 6. Hence the total contribution from all the components of type $\mathscr{B}(l_{q_1},\,l_{q_2})$ equals

$$4\,N(\ddot{\mathscr{P}}\mathrm{A}_6,\,\mu+\tilde{\mu})\;+\;5\,N(\ddot{\mathscr{P}}D_6^{\vee\vee},\,\mu+\tilde{\mu})\;+\;6\,N(\ddot{\mathscr{P}}D_6^{\vee},\,\mu+\tilde{\mu})\tag{5.28}$$

Hence, plugging in the value of $C_{\mathscr{B}\cap\mu\tilde{\mu}}$ in the equation (5.26) giving us Theorem 5.4.5.

Let us now continue with the same setup as earlier computation. Let $q_1=[0:0:1]$ and $q_2=[X_2:Y_2:1]$. Consider the polynomial

$$\xi_{30}:=(X-X_2)^2\,X^3\,Z^{d-5}$$

and consider the corresponding curve as $\tilde{\gamma}_{30}(t)$ in a similar fashion as we do for the earlier computations. We note that the section $\psi_{\mathscr{P}A_3}$ is transverse to zero by computing the derivative of the section along $\gamma_{30}(t)$. Hence, $\mathscr{P}A_3\circ\mathscr{P}A_3$ is a smooth complex sub manifold of $\mathscr{D}\times(\mathbb{P}T\mathbb{P}^2)^2$. Thus the proof is complete assuming (5.27), and the multiplicity claim (5.28).

Remark 5.5.6. *Note that if* θ *or* $p\theta$ *is equaled to zero then we are not in the projectivised space anymore. Hence, no direction is involved. Therefore our recursive formula for* θ or $p\theta = 0$ *will recover the corresponding codimension* 6 *numbers computed earlier by Kazaryan [27].*

Chapter 6

Conclusion and future research

6.1 Conclusion

Our area of research is enumerative geometry. Gromov-Witten's theory and the related structures lead the path of modern enumerative geometry. As a result beautiful solutions to various classical enumerative problems some of which were long-lasting open problems, are completely understood. For example, the problem of counting genus g curves in $\mathbb{P}^2$ passing through $3d-1+g$ general points in $\mathbb{P}^2$ is well understood.

Our research interest can be divided into two parts looking at the nature of the enumerative problems. Broadly speaking my interest can be localized by saying that my focus is on counting singular curves some times it might have a large order of contact with some divisor in a linear system and Gromov-Witten theory and its related structures. All of these questions have a long history that traced back to the eighteenth century or even before. For example, one can look at ([72],[30]), for a more detailed overview let me refer the reader to ([32]).

We conclude our doctoral thesis as follows. We consider the setting is degree d curves in $\mathbb{P}^3$, that are rational and are contained in a plane. The analogous problem in $\mathbb{P}^2$ was open until Kontsevich's breakthrough result revolutionized the field. The setting we choose is one higher dimensional count, but even more importantly, can be interpreted as a "family version" of the "static version" studied by Kontsevich. More precisely, we studied

Theorem 6.1.1 (A.Paul, R.Mukherjee and R. Singh [51]). *There is an explicit recursive formula for counting rational degree d curves are there in $\mathbb{P}^3$ whose image lies in some $\mathbb{P}^2$ (known as planar curves) passing through s points and intersecting r lines in general position such that $r+2s=3d+2$.*

We have discussed this in greater detail in the second chapter.

Next, in the third and fourth chapters we have considered the study of plane curves with singularities in a linear system. Here we consider curves that may be singular, and/or are tangent to a divisor, in a possibly singular way, as well as possibly satisfy some constraints on the directional derivatives in specific directions. To deal with such situations we use differential topological methods.

In this direction we have obtained our first result (3.1). Next, we consider another twist on the setting. Namely, here we require the rational curve to have higher tangency to a fixed divisor. This higher tangency is encoded by an integer vector k. Recursions for such counts are more difficult to prove. Here we have presented (4.3.1).

Finally we consider the enumeration of curves with two higher singularities, the permitted singularities are required to have some vanishing directional derivatives in specified directions. We have studied the counting problem of degree d curves in $\mathbb{P}^2$ having any two degenerate singularities of type A_k such that the total codimension of the two singularities can be at most 6, i,e., (5.4.1) to (5.4.5).

6.2 Future Research

Let us define a **planar** curve in $\mathbb{P}^3$ to be a curve, whose image lies inside a $\mathbb{P}^2$. In [51], jointly with R. Mukherjee and R.K. Singh, we have studied the enumeration of genus-zero planar curves in a moving family of $\mathbb{P}^2$. Moreover, we can say have studied the parallel question of counting stable rational maps into a family of moving target spaces. This question can be thought of as a family version of the famous question of enumerating rational curves. We have used the famous WDVV equation to obtain the above result. Now in recent times, the localization technique became one of the most powerful methods to study enumerative problems. Nemours problems have been successfully studied via localization in enumerative geometry where the WDVV equation does not provide any useful information. The first proof of genus-zero **Mirror symmetry** was proved using the localization technique by A. Givental [20].

Now we want to study the result obtained in [51], using the localization technique. This enables us to compute the numbers for *planar* curves directly. This project will be a re-derivation of a previously known result using a completely unrelated technique that has no relation with the previous method.

After knowing all these results for the genus-zero case, we want to study the following:

Question 6.2.1. *How many genus one degree d curves are there in $\mathbb{P}^2$ that are tangent to a fixed* *divisor at multiple points in $\mathbb{P}^2$ passing through an appropriate number of generic points?*

6.2.1 Stable maps and relative invariants

A great body of work has been done to understand Gromov-Witten invariants and the underlying geometric structures that it carries, yet Gromov-Witten invariants are *unclear* in general. A parallel interesting study would be the stable maps satisfying certain incidence conditions to some fixed or "variable" divisor. These questions are of great interest known as the relative Gromov-Witten invariant theory. A. Gathman ([20],[22]), in his thesis, started solving the genus 0 and 1 relative invariant count, and no further development is known. He had shown that the study of a relative invariant is important by showing that the absolute invariants can be calculated using those relative invariants.

Counting relative invariants even in $\mathbb{P}^2$ turns out to be a very difficult problem when the degree of the divisor becomes large. Let us ask the following question:

Question 6.2.2. *Let $E \subset \mathbb{P}^2$ be a smooth divisor, and consider $\alpha = (\alpha_1, \alpha_2, \cdots)$ and $\beta = (\beta_1, \beta_2, \cdots)$. How many degree d rational plane curves in $\mathbb{P}^2$ meet E at α_k "fixed" points with order of contact k and β_l "moving" points with order of contact l passing through $3d - 1 - \sum(k\alpha_k + (l-1)\beta_l)$ points in $\mathbb{P}^2$ in general position, if all contacts with D occur at unibranched points?*

In [10], the above question has been solved completely when E is a line, in fact, implicitly they have defined relative Gromov-Witten invariants in the process of generalizing the theory of Gromov-Witten invariants to higher genus. Then Vakil [68] extended the result when E is conic in the stable map setting. After that Cadman and Chen extended the result for a smooth cubic [10] for all (α, β) except for the case $(0, e_{3d})$ (i.e., except for maximal tangency). The above question itself is not yet completed for divisors of a large degree and any order of contact.

6.2.2 Stable maps with higher singularities

We have seen an extensive body of work that has been studied involving singularities in a linear system. Analogously one can think of singularity questions in the setting of stable maps. In particular one can ask the following question:

Question 6.2.3. *How many genus g degree d curves are there in $\mathbb{P}^2$ "having a A_k, $k \geq 2$" singularity passing through appropriate number of points in $\mathbb{P}^2$ in general position?*

It turns out the above problem is extremely difficult provided the concept of having a $A_k,\ k \geq 2$ singularity in this setting is understood. There are only a few results are known such as ([76],[77]) solves the question (6.2.3) for cusp, D_4 and E_6. The question of enumerating genus g curves involving higher singularities remains unexplored.

6.2.3 Counting singular curves in a linear system

We will now describe some developments in the setting of a linear system. We will begin by the following question:

Question 6.2.4. *Let $L \longrightarrow X$ be a holomorphic line bundle over a compact complex surface and $\mathscr{D} := \mathbb{P}H^0(X,L) \approx \mathbb{P}^{\delta_L}$ be the space of non zero holomorphic sections upto scaling. Let $(\delta_1, \cdots, \delta_n)$ are n-tuples of non-negative integers. What is $N(\mathfrak{X}_1^{\delta_1} \cdots \mathfrak{X}_n^{\delta_n})$, the number of curves in X, that belong to the linear system $H^0(X,L)$, passing through $\delta_L - (\delta_1 cd_{\mathfrak{X}_1} + \cdots + \delta_n cd_{\mathfrak{X}_n})$ generic points and having n number of singularities of type $\mathfrak{X}_i$ for $i = 1, \cdots, n$ whose co-dimensions are $cd_{\mathfrak{X}_i}$ respectively.*

We will briefly mention the development of the above question a little later. Let us fix a complex compact surface to be $\mathbb{CP}^2$ and define the concept of tangency to some divisor. Then we will describe the most general question one can ask in this setting and subsequently, we will discuss the corresponding development.

Let us fix a divisor $E \in \mathbb{P}^2$. We now define what do we mean by saying a curve in $\mathbb{P}^2$ is tangent to E to a certain order, i.e

Definition 6.2.5. *Let us consider that the divisor E is a fixed line in $\mathbb{P}^2$. Let $f : \mathbb{P}^2 \longrightarrow \mathscr{O}(d)$ be a holomorphic section. The curve is tangent to L of order k at the point $q \in f^{-1}(0)$, then there exists a coordinate system $(x,y) : (U,q) \longrightarrow (\mathbb{C}^2, 0)$ such that $f^{-1}(0) \cap \mathscr{U}$ is given by*

$$y + x^{k+1} = 0$$

Now we are ready to state the most general question one can ask in this setting:

Question 6.2.6. *Let us consider $(\delta_{F_1}, \cdots, \delta_{F_l})$ be a l-tuple, $(\delta_{E_1}, \cdots, \delta_{E_m})$ be a m-tuple and $(k_1, \cdots, k_n)$ be a n-tuple of non-negative integers. Let there are a total of $l+m+n$ points in $\mathbb{P}^2$ in general position out of which l number of points are outside E and $m+n$ number of points are in E. Then we will define the number*

$N_d(\mathfrak{X}_1^{\delta_{F_1}} \ldots \mathfrak{X}_l^{\delta_{F_l}} \mathfrak{X}_1^{\delta_{E_1}} \cdots \mathfrak{X}_m^{\delta_{E_m}} \mathsf{T}_{k_1} \ldots \mathsf{T}_{k_n})$, the number of degree d curves in $\mathbb{P}^2$, that passes through

appropriate number of generic points, having δ_{F_i} number of singularities of type $\mathfrak{X}_i$ at l number points outside E, which are denoted by $\mathfrak{X}_i^{\delta_{F_i}}$, δ_{L_i} number of singularities of type $\mathfrak{X}_i$ at m number points on the divisor E, which are denoted by $\mathfrak{X}_i^{\delta_{E_i}}$ and the curve is tangent to E of order k_i at n number of smooth points in E.
What is $N_d(\mathfrak{X}_1^{\delta_{F_1}} \dots \mathfrak{X}_l^{\delta_{F_l}} \mathfrak{X}_1^{\delta_{E_1}} \cdots \mathfrak{X}_m^{\delta_{E_m}} \mathsf{T}_{k_1} \dots \mathsf{T}_{k_n})$?

Remark 6.2.7. *Note that the above question for $l = 0$ implies that we want to count curves with singularities and tangency conditions in addition to that the singularities lie on the divisor. It turns out the idea of solving the question* (6.2.6) *for $l = 0$ gives a very good understanding of the question when $m, n = 0$. In other words, counting curves with degenerate singularities is a difficult problem and the geometric behavior of this question is not systematic at all. However, for $l = 0$ case behaves nicely and the solution implies the enumeration of curves with singularities which I intend to explore further.*

The above question in such generality is *open* till the date. However, in the literature, this question in some special situation classically understood which brought several branches of mathematics together. We will now describe the development of the above question in some special situations. Along the way, we will describe our contribution to this question.

We will start by mentioning the recent motivating result by ([25],[66]) which explains that there exists a universal polynomial in terms of chern classes that count the numbers for the above problem. Very recently in [41], the author proved a generalisation of the famous Göttsche Conjecture for a relative effective divisor C on a smooth projective family of surfaces. Finally, they apply their method to calculate node polynomials for plane curves intersecting general lines in three-dimensional projective space.

Let us consider that $m, n = 0$ for the question (6.2.6), i.e, the number of a singular point lying on the divisor E and the number of points where tangency occurs are zero then (6.2.6) reduces to the question of counting curves with various singularities which is an *open problem*. When all the singularities are nodes then this question has been extensively studied by algebraic geometers from various perspectives. In this direction, some beautiful results can be found in ([40],[8], [30]), etc. Next, when the singularities are more degenerate than nodes there are only a few results available in the literature such results include amongst all ([15], [27], [29], [74]).

Main results of the papers ([2], [1], [3]) has some partial solution to the question (6.2.6) namely, for two singular points and the first singularity is a node and the second singularity is any singularity of codimension k i.e, we have a solution for $N_d(A_1^{\delta}\mathfrak{X})$ such that $k+\delta \leq 8$. So jointly with my advisor Ritwik Mukherjee and R. Singh we are working on the problem (6.2.4) for two singular points when both singularities can be any singularity such that the total codimension is 9. This work is in progress and we have shorted out this question till the total codimension is 6. Next codimensional stratum i.e, $7,8$ and 9 codimensional study is in progress.

Next, when $m=0$, and the singularities are all nodes then the question (6.2.6) is completely solved by Caporaso-Harris ([11]). As recently as 2020, [16], the authors found a recursive formula for the number of rational curves maximally tangent to a given divisor using the WDVV equation. This problem is also of great interest in Symplectic Geometry. In their recent paper in 2019 ([48]), D. McDuff and K. Siegel use methods from Symplectic Geometry to count rational curves with maximal tangencies to a divisor in a Symplectic Manifold.

Unfortunately, there is almost no progress when the singularities are more degenerate than nodes. There are some results in the literature when the singularity is a cusp, we will refer the reader to ([15], [77]). When $l,m,$ and n all are nonzero and the singularities are nodes then there are some partial results scattered in the literature, and there is almost no result available for singularities more degenerate than nodes. Finally, we want to understand the question (6.2.6) for *higher singularities*.

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