Membrane Paradigm for Large-D Black Holes in AdS/dS Background

By

PARTHAJIT BISWAS

PHYS11201704029

National Institute of Science Education and Research, Bhubaneswar

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Chairman - Prof. Sudipta Mukherji   Date: 26/08/2021

S. Mukherji

Guide / Convener - Dr. Sayantani Bhattacharyya   Date: 24-08-2021

Sayantani B

Examiner - Dr. Alok Laddha   Date: 25-08-2021

Alok

Member 1 - Dr. Yogesh Kumar Srivastava   Date: 27-08-2021

Yogesh

Member 2 - Dr. Chethan N. Gowdigere   Date: 27/8/21

Chethan

Member 3 - Dr. Shamik Banerjee   Date: 26.08.2021

Shamik Banerjee

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I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date : 24.08.2021

Place : NISER, Bhubaneswar

Dr. Sayantani Bhattacharyya (Guide)

Sayantani B.

STATEMENT BY AUTHOR

Parthajit Biswas.

Parthajit Biswas

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Parthajit Biswas

Parthajit Biswas

List of Publications

Publications arising from the thesis

1. ** "*The Large-D black hole dynamics in AdS/dS backgrounds*" Sayantani Bhattacharyya, Parthajit Biswas, Bidisha Chakrabarty, Yogesh Dandeker and Anirban Dinda, JHEP 10 (2018) 033.
   ** (Subsections [9.2] and [9.3] of this paper - where we have given some checks of our results, have been used by one of the co-authors Yogesh Dandekar in his thesis)

2. ** "*Black holes in presence of cosmological constant : second order in 1/D*" Sayantani Bhattacharyya, Parthajit Biswas and Yogesh Dandekar, JHEP 10 (2018) 171.
   ** (Subsections [4.2] and [4.3] of this paper - where we have given some checks of our results, have been used by one of the co-authors Yogesh Dandekar in his thesis)

3. "*A leading order comparison between fluid - gravity and membrane - gravity dualities*" Sayantani Bhattacharyya, Parthajit Biswas and Milan Patra, JHEP 05 (2019) 022.

4. "*Stress Tensor for Large-D membrane at Subleading Orders*" Parthajit Biswas, JHEP 07 (2020) 110.

Other publications (not included in thesis)

1. "*Fluid - gravity and membrane - gravity dualities : Comparison at subleading orders*" Sayantani Bhattacharyya, Parthajit Biswas, Anirban Dinda and Milan Patra, JHEP 05 (2019) 054.

Presentations at conferences and workshops

1. December 2017, National String Meeting, NISER Bhubaneshwar, India.
   Presented **Talk** titled *"Large-D Black Holes in AdS/dS"*.

2. January 2018, Kavli Asian Winter School, ICTS Bangalore, India.
   Presented **Poster** titled *"The Large-D Black Hole Dynamics in AdS/dS"*.

3. February 2018, String Field Theory and String Phenomenology, HRI Allahabad, India. Presented **Talk** titled *"Large-D Black Hole Membrane Dynamics"*.

4. January 2019, Kavli Asian Winter School, Sogang University, Seoul, Korea.
   Presented **Talk** (Gong Show) & **Poster** titled *"Comparison between Fluid-Gravity and Membrane-Gravity Dualities"*.

5. December 2019, National String Meeting, IISER Bhopal, India.
   Presented **Talk** titled *"Fluid-Gravity and Membrane-Gravity Dualities : A Comparison"*.

Parthajit Biswas

DEDICATIONS

To My Parents

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Contents

Summary

In this thesis, we have discussed a particular limit (*spacetime dimensions $D \to \infty$ limit*) of Einstein's equation and demonstrated some of the simplifications it offers. In picturesque terms, the effect of taking $D \to \infty$ limit, can be thought of as to concentrate the gravitational effect of the black hole within a thin region of thickness of the order $\mathcal{O}\left(\frac{1}{D}\right)$ outside the horizon, leaving a hole in an otherwise undistorder background geometry. The surface of the hole can then be thought of as a membrane in that background geometry with properties obtained by integrating Einstein's equation near the horizon. This is what had previously been done for flat background spacetime in the papers [1–3] and known in the literature as *Large-D membrane paradigm*.

In chapter 2 and chapter 3 of this thesis, we have generalized the large-D program in arbitrary background spacetime, in particular to AdS/dS spacetime up to second subleading order in $\frac{1}{D}$ expansion.

In chapter 4 of this thesis, we have constructed a stress tensor on the membrane world volume up to second subleading order in $\frac{1}{D}$ expansion and demonstrated that the membrane equation derived in chapter 3 follows from the conservation equation of this stress tensor. This had previously been done up to first subleading order in $\frac{1}{D}$ expansion in the paper [4].

There exists another perturbative technique namely *Fluid-Gravity Correspondence*, which can be used to generate solutions of Einstein's equation in presence of negative cosmological constant. In chapter 5 of this thesis, we have compared these two perturbative techniques namely *Large-D membrane paradigm* and *Fluid-Gravity Correspondence*, and found that there is a regime in the parameter space where both these two techniques can be applied simultaneously, and in this overlap regime, we have found a perfect match between these two perturbative techniques up to first subleading order on both sides.

List of Tables

Chapter 1

Introduction

1.1 Large-D Membrane Paradigm

Most of the theories in physics have some parameters - some of them have continuous one, such as coupling constant, some other have discrete one, such as number of fields - these parameters can be varied from their actual values maintaining the consistency of the theory. It often happens in physics that the theories become simplified at the edge of the allowed values of these parameters. So, it is a fruitful strategy to try to solve the equation at this limit and then correct the solution order by order in a perturbative expansion.

General Theory of Relativity, in absence of any matter, described by Einstein-Hilbert action $\mathcal{L} = \sqrt{-g}R$ lacks any adjustable parameter. The only natural parameter one can think of is the dimensions of spacetime D. General Theory of Relativity is well defined in any dimensions $D \geq 4$ and also retains one of its most basic objects namely black hole. One might hope that the limit $D \to \infty$ results in a convenient simplification and possibly also a novel reformulation of the theory, at least for some phenomena. This strategy is somewhat similar in spirit with that of 't Hooft [5] who introduced a parameter N in the Yang Mills theory by replacing $SU(3)$ gauge group by $SU(N)$.

There might be several usefulness of this perturbative technique. Firstly, it is always good to have a new technique. Secondly, Einstein's equation in vacuum

$$R_{AB} = 0 \tag{1.1}$$

look innocuous, but, it is almost impossible to find exact solution of these coupled, non linear, partial differential equations for any phenomenon of interest unless there is a substantial amount of symmetry. Whatever we know about any physical situation, for example, collision of two black holes and its subsequent merger is due to numerics. However, the numerics involved is very much challenging, "Large-D" technique might give some analytic handle on the problem.

The first systematic study of the large dimensional limit of General Relativity has been done by Emparan and collaborators [1, 6–8]. Consider Schwarzschild-Tangherlini black hole solution [9] with Schwarzschild radius r_0 in D spacetime dimensions.

$$ds^2 = -\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right) dt^2 + \frac{dr^2}{\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right)} + r^2 d\Omega_{D-2}^2 \tag{1.2}$$

Now, if we take $r > r_0$ and keep it fixed then in the limit $D \to \infty$ the term $\left(\frac{r_0}{r}\right)^{D-3} \to 0$, so, the solutions (1.2) reduces to flat space solution. But, if we take $r = r_0\left(1 + \frac{R}{D-3}\right)$ and keep R fixed, then, in the limit $D \to \infty$ the term $\left(\frac{r_0}{r}\right)^{D-3} \to e^{-R}$. It follows that the tail of the black hole extends a distance of order $\frac{R}{D-3}$ outside the horizon, this will be referred as membrane region.

Emparan and collaborators have computed quasinormal mode (QNMs) frequencies of (1.2) in the limit when spacetime dimensions is very large [7, 10]. They have shown that there are two sets of quasi normal modes(QNMs)

- *Fast, non-decoupled* QNMs with frequencies of the order $\mathcal{O}\left(\frac{D}{r_0}\right)$. Most of the QNMs are in this category.
- *Slow, decoupled* QNMs with frequencies of the order $\mathcal{O}\left(\frac{1}{r_0}\right)$. There are only a few of them.

These slow, decoupled QNMs have support only in the thin region around the horizon. This result at the linear level suggests that this might be possible to construct a fully non-linear theory of the slow decoupled QNMs. This effective theory for black holes at large dimensions has been worked out in the papers [2, 11, 12]

In picturesque terms, the effect of taking $D >> 1$ can be thought of as to concentrate the gravitational effect of the black hole within a thin sliver of thickness of the order $\mathcal{O}\left(\frac{1}{D}\right)$ outside the horizon, leaving a hole in an otherwise undistorder background geometry. The surface of the hole then can be thought of as a membrane in that background with properties obtained by integrating Einstein's equation near the horizon. This is what has been done for flat background spacetime in the papers [2, 3].

In Chapter 2 and Chapter 3 of this thesis, we will generalize the large-D program in arbitrary background spacetime, in particular to AdS/dS spacetime. In last couple of years, there have been some interesting developments in Large-D program. It has been generalised for Einstein-Maxwell system in [13–15], for higher curvature gravity in [16–26]. Effective equation for special case of stationary membrane has been worked out in [27, 28]. Black hole physics become simplified at large dimensions due to the existence of a parametrically separated length scale $\frac{r_0}{D}$ other than the horizon length scale r_0. For Black branes there is another interesting length scale which is $\frac{r_0}{\sqrt{D}}$ as has been discussed in [12, 29]. The works that first successfully used $\frac{1}{D}$ as a perturbation parameter are [30, 31], although, the systematic study of black hole physics at large dimensions did not begin until the work of Emparan *et al.* [1].

Effective theory has been extended in several different directions - see [32, 33] for deformed boundary metrics, see [34–36] for effective theories at higher orders in $\frac{1}{D}$, see [21, 37–41] for effective theories for finite black holes.

Large-D technique has been used for the analysis of black holes collision in [42–44], for Greggory-Laflamme instability in [29,35,45] and for turbulence in [22,46,47]. See [48–54] for further developments. There is a recent review by Emparan and Herzog [55] about Large-D program, its application and its future prospects.

1.2 Stress Tensor for the Large-D membrane

It is a very natural question to ask - what is the gravitational radiation for any arbitrary membrane motion? The computation of radiation is a bit complicated. The explicit result for the metric corrections (see chapters 2 and 3) are valid for points whose distance from the horizon S obeys the inequality $S << r_0$, where r_0 is horizon length scale. So it would not be possible to read off the radiation by simply putting S to be very large in the explicit expressions. But, when $S >> \frac{r_0}{D}$ the solution reduces to a small fluctuations around the background spacetime. So, both the linearized approximation and the Large-D approximation are valid in the regime

$$\frac{r_0}{D} << S << r_0 \tag{1.3}$$

We can use Large-D approximation to calculate the effective linearized solution in the overlap regime then continue it using linearized approximation till infinity to get the radiation. There is a elegant way to implement the second step; first, calculate the Brown-York stress tensor of the linearized solution on the membrane

$$8\pi T^{(out)}_{AB} = K^{(out)}_{AB} - K^{(out)}\mathfrak{p}^{(out)}_{AB}\Big|_{\psi=1} \tag{1.4}$$

Where, $K^{(out)}_{AB}$ and $\mathfrak{p}^{(out)}_{AB}$ are respectively extrinsic curvature and the projector on the membrane world volume (see (4.69) for definitions). Then subtract from it $T^{(in)}_{AB}$ - which can be determined from the variation of a 'boundary counterterm'. Final expression of the stress tensor on the membrane is given by

$$T_{AB} = T^{(out)}_{AB} - T^{(in)}_{AB} \tag{1.5}$$

$T^{(out)}_{AB}$ and $T^{(in)}_{AB}$ are both tangential to membrane world volume and therefore, can equally well be regarded as stress tensor $T^{(out)}_{\mu\nu}$ and $T^{(in)}_{\mu\nu}$ - which entirely live on the membrane world volume [1].

[1]Here, {A,B} denote full spacetime index whereas, $\{\mu, \nu\}$ denote membrane world volume index

It turns out that

$$T^{(in)}_{\mu\nu} = -\frac{1}{\sqrt{-g^{(ind)}}}\frac{\delta}{\delta g^{\mu\nu}_{(ind)}} S_{(in)} \tag{1.6}$$

Where,

$$S_{(in)} = -\frac{1}{8\pi}\int \sqrt{-g^{(ind)}}\left[\sqrt{\mathcal{R}} + \frac{1}{2}\Big(\frac{\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}}\Big) + \mathcal{O}\left(\frac{1}{D}\right)\right] \tag{1.7}$$

here, $g^{(ind)}_{\mu\nu}$, $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are respectively intrinsic metric, intrinsic Ricci tensor and Ricci scalar.

This procedure yields a stress tensor on the membrane $T_{\mu\nu}$ [4] which is conserved and moreover, it satisfies a crucial identity $T_{\mu\nu}K^{\mu\nu} = 0$ order by order. Membrane equations follows from the conservation of the stress tensor. The stress tensor acts as the effective source for the radiation. To calculate the radiation, one needs to convolute the source against a retarded Green's function. Though the stress tensor is substantial, in fact, it is of the order $\mathcal{O}(D)$, the radition sourced by the membrane is of the order $\frac{1}{D^D}$ that is non perturbative in $\frac{1}{D}$ expansion. The radiation being non perturbative, follows from the property of the Green's function in large dimensions [4].

We have computed the stress tensor at the second subleading order in Chapter 4. Our main motivation for undertaking this very tedious calculation comes from the paper [56] where the authors tried to give a 'finite-D' completion of the large-D stress tensor. Here, we very briefly discuss the finite-D program.

The part of the stress tensor [4] that contribute to the leading order membrane equation is given by

$$16\pi T_{\mu\nu} = \mathcal{K}P_{\mu\nu} - 2\sigma_{\mu\nu} + (\mathcal{K}_{\mu\nu} - \mathcal{K}g_{\mu\nu}) \tag{1.8}$$

Where, $\sigma_{\mu\nu}$ is the shear tensor of the velocity field u_μ and $\mathcal{P}_{\mu\nu} = g^{ind}_{\mu\nu} + u_\mu u_\nu$ is the projector orthogonal to the membrane velocity. Now, if we consider (1.8) to be exact stress tensor at any finite-D then there is an inconsistency. Normal component of the conservation of the stress tensor gives the following identity

$$\mathcal{K}^{\mu\nu}T_{\mu\nu} = 0 \tag{1.9}$$

The stress tensor (1.8) does not satisfy the above condition exactly. This implies the stress tensor (1.8) does not even give consistent dynamics at finite-D. In [56], the authors have tried to cure the problem, they have proposed a finite-D completion of large-D stress tensor

$$16\pi T_{\mu\nu} = \tilde{\mathcal{K}}P_{\mu\nu} - 2\sigma_{\mu\nu} + (\mathcal{K}_{\mu\nu} - \mathcal{K}g_{\mu\nu}) \tag{1.10}$$

Where,

$$\tilde{\mathcal{K}} = \frac{\mathcal{K}^2 - \mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} + 2\mathcal{K}^{\mu\nu}\sigma_{\mu\nu}}{\mathcal{K} + u\cdot\mathcal{K}\cdot u} \tag{1.11}$$

It is not difficult to show that $\tilde{K}$ reduces to K at the large-D limit. So, the improved stress tensor reduces to large-D stress tensor at large-D limit, nevertheless, (1.10) satisfies the condition (1.9) at finite-D exactly.

This finite-D stress tensor exhibits some appealing properties. For example, the thermodynamics of static spherical membrane in flat as well as in AdS spacetime, obtained via this finite-D completion agrees exactly with their dual black holes even in finite dimension.

The motion of a probe membrane in Poincare Patch AdS sources linearized gravitational radiation and so a corresponding boundary stress tensor. The resultant boundary stress tensor, in the long wavelength limit, is a hydrodynamic stress tensor for a boundary conformal fluid. When expanded in derivative expansion, this boundary stress tensor gives answer that matches with that of the fluid gravity answer at zero and first derivative order even at finite-D. But, there is a mismatch in the second derivative order in finite-D. Finite-D stress tensor has been constructed from the membrane stress tensor which was known up to first order in $\frac{1}{D}$ expansion. Membrane stress tensor at the second order in $\frac{1}{D}$ expansion will help to write a further improved finite-D stress tensor. Mainly motivated by this, we have calculated the membrane stress tensor at the second subleading order in Chapter 4.

1.3 Comparison between 'Fluid-Gravity' and 'Membrane-Gravity' dualities

Fluid-Gravity correspondence [57–63] is another perturbative technique that can generate solutions of Einstein's equation in a perturbative series expansion in number of *derivatives* in presence of negative cosmological constant. Solutions generated using derivative expansion are 'black-hole' type solutions (i.e., spacetime with singularity shielded behind the horizon) that are in one to one correspondence with the solutions of relativistic Navier-Stokes equations. On the other hand, solutions generated using *Large-D technique* are also similar 'black hole' type solutions, but dual to the dynamics of a codimension-one membrane embedded in the asymptotic geometry.

It is natural to ask whether it is possible to apply both the perturbation techniques simultaneously in any regime(s) of the parameter space of the solutions, and if so, how the two solutions compare in those regimes. In chapter 5, we have tried to answer these two questions. In a nutshell, our final result is only what is expected.

- It is possible to apply both the perturbation techniques simultaneously. Further, in the regime where both D is large and derivatives are small in an appropriate sense, we could treat $\left(\frac{1}{D}\right)$ and ∂_μ (with respect to some length scale) as two independent small parameters, with no constraint on their ratio.

- In other words, if the metric dual to hydrodynamics is further expanded in inverse powers of dimension, it matches with the metric dual to membrane-dynamics, again expanded in terms of derivatives.

However, this matching is not at all manifest. We could see it only after some appropriate gauge or coordinate transformation of one solution to the other. The whole subtlety of our computation lies in finding the appropriate coordinate transformation.

The ‘large-D’ expansion technique, as described in chapter 2 and 3, generates the dynamical black brane geometry in a ‘split form’ where the full metric could always be written as a sum of pure AdS metric and something else. In other words, the black brane spacetime, constructed through ‘large-D’ approximation would always admit a very particular point-wise map to pure AdS geometry.

On the other hand, the spacetime dual to fluid dynamics does not require any such map for its perturbative construction and apparently there is no guarantee that the particular map used in ‘large-D’ technique, would also exist for the dynamical black brane geometries, constructed in ‘derivative expansion’.

In chapter 5, we have shown that the ‘hydrodynamic metric’[2] indeed could be ‘split’ as required through an explicit computation up to first order in derivative expansion. This map could be constructed in any number of dimension and is independent of the ‘large -D’ approximation. After determining this map, we have matched these two different gravity solutions up to the first subleading order on both sides.

One interesting outcome of this exercise is the matching of the dual theories of both sides. It essentially reduces to a rewriting of hydrodynamics in a large number of dimensions, in terms of the dynamics of the membrane. After implementing the correct gauge transformation, we finally get a field redefinition of the fluid variables (i.e., fluid velocity and the temperature) in terms of membrane velocity and its shape[3]. We hope such a rewriting would lead to some new ways to view fluid and membrane dynamics and more ambitiously to a new duality between fluid and membrane dynamics in a large number of dimensions, where gravity has no role to play (See [46], [56] for a similar discussion on such field redefinition and rewriting of fluid equations though in [46] the authors have taken the

[2]In this thesis, the black brane solution dual to fluid dynamics would always be referred to as the ‘hydrodynamic metric’.

[3]Truly speaking, what we have actually worked with is the reverse of what we have stated here, i.e., we determined the membrane velocity and the shape in terms of fluid variables, up to corrections of order $\mathcal{O}\left(\frac{1}{D},\ \partial^2\right)$. This is just for convenience. The relations we found are easily invertible within perturbation.

large D limit in a little different way than ours).

Chapter 2

Large-D membrane paradigm in AdS/dS at leading order

This chapter is based on [65].

As discussed in the introduction 1.1, in the large-D limit the dynamics is confined in the near horizon region, therefore, it does not care much about the asymptotic spacetime. This implies that the whole 'large-D' programme of solving Einstein's equation could easily be extended to situations where the asymptotic spacetime is not exactly flat. Membrane-Gravity correspondence is expected to hold for such cases, but now the membrane will be a codimension-one hypersurface in some non flat asymptotic geometry. In particular, this construction should be applicable in presence of cosmological constant [29].

In [3] and [14], the analysis is strictly applicable for asymptotically flat spacetime, though the answer has been expressed in 'background-covariant' form. In [3], where the authors have calculated membrane equations and metric corrections up to second subleading order, the covariance has also been implemented in the complicated intermediate steps.

Here we have extended the analysis of [2, 14] in such a way that the background covariance is manifest in every steps. We have also included cosmological constant which might have any sign.

The main motivation for including cosmological constant is the following. There exists another perturbative technique namely 'Fluid-Gravity' correspondence [58] which can be used to generate black hole solutions of Einstein's equation in presence of negative cosmological constant. Fluid-Gravity correspondence is true in any dimension, in particular, in large dimensions. We would eventually like to see how these two perturbative techniques can be compared? We will discuss about this in section 4.

The organization of this chapter is as follows. In section 2.1, we have described the initial set up of the problem, the main equation that we would like to solve for and the scheme of our perturbation technique. In section 2.2, we described how in our scheme, different quantities scale with the dimension D, the perturbation parameter. In section 2.3, we have described how we could guess the leading ansatz. Next in a small section 2.4, we described how our approach becomes manifestly covariant with respect to the embedding geometry of the membrane. In section 2.5, we briefly explained the algorithm we used to solve for the first subleading correction. In section 2.6, and section 2.7, we have derived and presented the first subleading correction to the metric and the equation governing the dual membrane and the velocity field. Then, in section 2.8, we have performed several checks on our ansatz. We have matched our solution with Schwarzschild AdS/dS black hole/brane and then with rotating black hole solution up to the required order in an expansion in $\left(\frac{1}{D}\right)$. Finally, in section 2.9, we have ended with discussions. We have several appendices with the details of all computation.

2.1 Set up

In this section, we will describe the basic set up of the problem and also the final goal in terms of equations. We will also present the final solution in schematic form that we will eventually determine. The two derivative action we will be working with is the Einstein-Hilbert action with cosmological constant

$$\mathcal{S} = \int \sqrt{-G}\,[R - \Lambda] \tag{2.1}$$

here Λ is assumed to scale with dimension D as follows [1].

$$\Lambda = [(D-1)(D-2)]\,\lambda, \qquad \lambda \sim \mathcal{O}(1) \tag{2.2}$$

The equation of motion we get by varying (2.1) with respect to the metric is

$$E_{AB} \equiv R_{AB} - \left(\frac{R-\Lambda}{2}\right) G_{AB} = 0 \tag{2.3}$$

Our goal, as mentioned before, is to find new 'black hole type' solutions (i.e. solutions with event horizon) of equation (2.3) in a power series expansion in $\frac{1}{D}$. Schematically, the solution will have the form

$$G_{AB} = g_{AB} + \sum_{k=0}^{\infty} \left(\frac{1}{D}\right)^k G^{(k)}_{AB} \tag{2.4}$$

[2] Here, g_{AB} is also a smooth solution to the same equation eq.(2.3). In the previous section, we have referred g_{AB} as the 'background' metric. The $G^{(k)}_{AB}$'s, on the other hand, are not smooth and their forms are such that the full metric G_{AB} would have horizon, and possibly singularities behind it. The full non linear dynamics of the decoupled QNMs are captured by $G^{(k)}_{AB}$'s. Since, the decoupled QNMs have support only in the membrane region, the $G^{(k)}_{AB}$'s should vanish exponentially as we go away from the horizon which implies that the g_{AB} is the asymptotic metric.

As explained in [2,3,14], our final solution will be parametrized by a codimension-one membrane, embedded in the background spacetime, with a velocity field on it. However, the velocity field and the curvature of this membrane are not independent data. We can solve for $G^{(k)}_{AB}$'s provided the velocity field and the extrinsic curvature of the membrane

[1] See section (2.5) for motivation of this choice

[2] In the later sections, we will often use the notation $G^{[k]}_{AB}$ to denote the solution corrected up to order $\mathcal{O}\left(\frac{1}{D^k}\right)$

$$G^{[k]}_{AB} = g_{AB} + \sum_{m=0}^{m=k} \frac{1}{D^m} G^{(m)}_{AB}$$

together satisfy some integrability condition. We would view this integrability condition as the dynamical equation for the codimension-one membrane. This leads to a 'membrane-gravity' duality in the sense that corresponding to every solution of the membrane equation we will be able to find a solution of the equation (2.3) in an expansion in $\left(\frac{1}{D}\right)$.

We will determine $G^{(k)}_{AB}$'s in such a way that if we view the membrane as a codimension-one hypersurface in the full spacetime G_{AB}, it becomes the event horizon of the metric G_{AB} and the velocity field on it reduces to its null generators [3, 14].

2.2 Scaling with D

Roughly speaking, Einstein's equation in D dimension are a set of $\frac{D(D+1)}{2}$ equations for $\frac{D(D+1)}{2}$ components of the metric tensor (modulo coordinate redefinition freedom). So, a naive large D limit would imply that both the number of equations as well as number of variables are blowing up with the perturbation parameter.

To get rid of this problem, we will implicitly assume that the large part of the metric is fixed by some symmetry and the metric is dynamical along some finite directions. In other words, we will assume the following form of the metric.

$$dS^2 = G_{AB}\, dX^A dX^B = \tilde{G}_{ab}(\{x^a\})\, dx^a dx^b + f(\{x^a\}) d\Omega^2 \tag{2.5}$$

Here, $\tilde{G}_{ab}(\{x^a\}),\ \{a, b\} = \{0, 1, \cdots, p\}$ is a finite $(p+1)$ dimensional dynamical metric, $d\Omega^2$ is the line element of the infinite $(D-p-1)$ dimensional symmetric space and $f(\{x^a\})$ is some arbitrary function of $\{x^a\}$.

Since, the metric is dual to the membrane embedded in the background spacetime g_{AB} with a velocity field along the membrane, the symmetry of the metric must be there in the membrane as well as in the velocity field and in the background. This will imply that the dual membrane is dynamical only along the finite x^a directions and simply wrap the symmetric space (with metric Ω_{AB}). Similarly, the velocity field will have components only along the finite x^a directions and also the non zero components will not depend on the coordinates of the Ω space. The same feature (i.e. no component along the symmetry directions as well as all the non-zero components depend only on $\{x^a\}$) would be true for any vector constructed out of membrane data. Similarly for tensors, the components along the symmetry directions would be proportional to the metric of the symmetric space Ω_{AB}.

In such cases, we could very easily see that the divergence of any vector or one form would be D times higher than the order of the quantity itself [2, 3, 14]. In fact such a rule would be true for any generic tensor with arbitrary number of indices. If $T_{A_1 A_2 \cdots A_n}$ is a generic tensor of order $\mathcal{O}\left(\frac{1}{D}\right)^k$ maintaining the symmetry of (2.5), then its divergence is of order $\mathcal{O}\left(\frac{1}{D}\right)^{k-1}$.

$$T_{A_1 A_2 \cdots A_n} \sim \mathcal{O}\left(\frac{1}{D}\right)^k \Rightarrow g^{A_p A_q} \nabla_{A_p} T_{A_1, A_2, \cdots A_q \cdots} \sim \mathcal{O}\left(\frac{1}{D}\right)^{k-1} \tag{2.6}$$

If the background metric g_{AB} admits a decomposition of the form (2.5), then Riemann tensor, Ricci tensor and Ricci scalar evaluated on g_{AB} will be of order $\mathcal{O}(1)$, $\mathcal{O}(D)$ and $\mathcal{O}(D^2)$ respectively.

$$R_{ABCD}|_{\text{on } g_{AB}} \sim \mathcal{O}(1), \quad R_{AB}|_{\text{on } g_{AB}} \sim \mathcal{O}(D), \quad R|_{\text{on } g_{AB}} \sim \mathcal{O}(D^2) \tag{2.7}$$

This implies that the Einstein's tensor evaluated on g_{AB} would be of order $\mathcal{O}(D^2)$[3] and as we want g_{AB} to solve (2.3), it justifies our choice of the scaling for the cosmological constant with D as given in (2.2).

However, we would not require any details of the decomposition as given in (2.5) [3]. The only aspect of it that we will use is the scaling law (2.6).

[3]Such a scaling is true for any generic case. It is always possible to choose special background where equation (2.7) is not true. A different choice of the D dependence for cosmological constant Λ would have led to such a 'non-generic' background.

2.3 Leading Ansatz

In our calculation, $G^{(0)}_{AB}$ is the leading ansatz that captures the nonlinear dynamics of the decoupled QNMs at the leading order. Any perturbation theory works provided we have a good guess of the leading answer. In this sense, we can carry on with our program provided we know the correct form of $G^{(0)}_{AB}$ that solves the equation (2.3) at the leading order in $\frac{1}{D}$ expansion. Now, we will describe how we can guess the form of the leading ansatz.

2.3.1 The form of the leading ansatz

As mentioned before, our solutions are characterized by two parameters namely the shape of a codimension-one hypersurface in the background spacetime and a unit normalized velocity field u_μ along the membrane[4].

We will first construct a smooth function ψ in the background spacetime such that $(\psi = 1)$ is the equation of the membrane. Next, we will construct a smooth one form $(O = O_A\, dX^A)$, defined everywhere in the background, such a way that the projection of $(-O^A)$ on the membrane reduces to u^μ. We will determine our final solution in terms of the membrane shape ψ and the one form field O. Note that, at this stage, there is a large ambiguity in the construction of ψ and O. The conditions that they have to reduce to something specific on $\psi = 1$ surface is certainly not enough to determine them completely. We will fix these ambiguity with some convenient choices (see subsection 2.5.2 for a detailed discussion on this point)

At this point, the simplest structure we could imagine for $G^{(0)}_{AB}$ (without involving any

[4]Throughout the thesis, we use Greek letters to denote indices along the membrane world volume as embedded in the background metric g_{AB}, whereas capital Latin letters denote full spacetime indices. Velocity field u_μ is unit normalized with respect to the induced metric on the membrane (denoted as $g^{(ind)}_{\mu\nu}$)

$$u_\mu u_\nu g^{\mu\nu}_{(ind)} = -1$$

derivative of ψ and O_A) is the following

$$G^{(0)}_{AB} = F\, O_A O_B \Rightarrow G_{AB} = g_{AB} + F\, O_A O_B + \mathcal{O}\left(\frac{1}{D}\right) \tag{2.8}$$

here, F is any arbitrary scalar function of ψ and $(O \cdot O)$ [5]. The inverse of the metric G_{AB} is given by

$$G^{AB} = g^{AB} - \left(\frac{F}{1+F(O\cdot O)}\right) O^A O^B + \mathcal{O}\left(\frac{1}{D}\right) \tag{2.9}$$

Here, all raising and lowering are with respect to the metric g_{AB}.

Now, firstly we want $\psi = 1$ to be the horizon when embedded in the full metric G_{AB}. This implies $(\partial_A\psi)(\partial_B\psi)G^{AB} = 0$ on $\psi = 1$. We will impose this condition order by order $\frac{1}{D}$ expansion. At leading order, we have

$$\begin{aligned}
&\left[d\psi\cdot d\psi - \left(\frac{F}{1+F(O\cdot O)}\right)(O\cdot d\psi)^2\right]_{\psi=1} = \mathcal{O}\left(\frac{1}{D}\right)\\
&\left[\frac{F}{1+F(O\cdot O)}\right]_{\psi=1} = \left[\frac{1}{O\cdot n}\right]^2_{\psi=1} + \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{2.10}$$

$$\text{where}\;\; n_A = \frac{\partial_A\psi}{\sqrt{d\psi\cdot d\psi}}$$

Secondly, we want our velocity vector field to be the null generator of the horizon

$$t^A = G^{AB} n_A|_{\psi=1}$$

Also, by definition the velocity field is given by the projection of $(-O^A)$ along the membrane. This in turn, implies

$$\begin{aligned}
&\left[\Pi^A_B O^B + G^{AB} n_B\right]_{\psi=1} = 0, \quad \text{where}\;\; \Pi^A_B = \text{projector} = \delta^A_B - n^A n_B\\
\Rightarrow &\left[O^A - (O\cdot n)n^A + n^A - O^A\left(\frac{F}{1+F(O\cdot O)}\right)(O\cdot n)\right]_{\psi=1} = \mathcal{O}\left(\frac{1}{D}\right)\\
\Rightarrow &\left[\left(1 - \frac{1}{O\cdot n}\right)\left(O^A - (O\cdot n)n^A\right)\right]_{\psi=1} = \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{2.11}$$

[5]Throughout the thesis '$\cdot$' denotes contraction with respect to the background metric g_{AB}

We have used (2.10) to go to the third line from the second line. From equation (2.11) it follows that

$$(O \cdot n)|_{\psi=1} = 1 + \mathcal{O}\left(\frac{1}{D}\right) \tag{2.12}$$

On the other hand, the velocity field on the membrane (viewed as a hypersurface in the background spacetime g_{AB}) is normalized to minus one which implies

$$\Pi^{AB} O_A O_B = -1 \tag{2.13}$$

From (2.12) and (2.13) it follows that O is a null one-form with respect to g_{AB} at leading order in $\left(\frac{1}{D}\right)$ expansion

$$g^{AB} O_A O_B = \mathcal{O}\left(\frac{1}{D}\right) \tag{2.14}$$

We will sometimes express O_A as

$$\begin{gathered} O_A = n_A - u_A \\ \text{where,} \;\; u_A = -\Pi_A^B O_B, \;\; \Pi_B^A = \delta_B^A - n^A n_B \end{gathered} \tag{2.15}$$

Here u_A, by construction, is always along the membrane and it will be the velocity vector field u_μ, when expressed in terms of the intrinsic coordinates of the membrane. From our analysis so far, we could see that the simplest form of $G^{(0)}_{AB}$ is the following.

$$G^{(0)}_{AB} = F \; O_A O_B = F \; (n_A - u_A)(n_B - u_B)$$

Now, if we do not want any derivative at the zeroth order F could only be a function of ψ since $(O \cdot O)$ is zero at the leading order. We also want F to be vanishing outside the thin membrane region of thickness of order $\mathcal{O}\left(\frac{1}{D}\right)$ around $\psi = 1$ surface. This would be ensured provided $F(\psi) \propto \psi^{-D}$. Now, if we substitute the fact that O is null at leading order in the equation (2.10), we find

$$F|_{\psi=1} = 1 + \mathcal{O}\left(\frac{1}{D}\right)$$

This fixes the proportionality constant in F to be one. So, the final expression of the leading ansatz[6] we get, is the following

$$G^{(0)}_{AB} = \psi^{-D} O_A O_B \tag{2.16}$$

This ansatz metric will solve equation (2.3) at leading order provided the following conditions are satisfied [2, 14]

$$\begin{aligned} &G^{(0)}_{AB} = \psi^{-D} O_A O_B \\ &\sqrt{g^{AB}(\partial_A\psi)(\partial_B\psi)}|_{\psi=1} = \frac{K}{D} + \mathcal{O}\left(\frac{1}{D}\right) \\ &g^{AB}\nabla_A O_B = K + \mathcal{O}(1) \end{aligned} \tag{2.17}$$

Here, K is the trace of the extrinsic curvature of the membrane which is a $\mathcal{O}(D)$ quantity. The membrane $\psi = 1$ is viewed as a codimension-one hypersurface embedded in the background spacetime g_{AB} and ∇_A denotes covariant derivative with respect to the background metric g_{AB}.

2.3.2 When ansatz solves the leading equation

Now we will demonstrate how $G^{(0)}_{AB}$ as given in equation (2.17) satisfies the equation (2.3) at leading order. We will simply evaluate the Einstein's equation on the metric $g_{AB} + G^{(0)}_{AB}$ and will see that the leading order (which turns out to be $\mathcal{O}(D^2)$) piece vanishes after using the conditions mentioned in (2.17)

Before getting into the details, we will first simplify the equation (2.3) by subtracting the trace of the equation

$$\begin{aligned} &R_{AB} - \left(\frac{R}{2}\right) G_{AB} = -\left[\frac{(D-2)(D-1)\lambda}{2}\right] G_{AB} \\ &\Rightarrow R = D(D-1)\lambda \\ &\Rightarrow \mathcal{E}_{AB} \equiv R_{AB} - (D-1)\lambda\, G_{AB} = 0 \end{aligned} \tag{2.18}$$

[6] We would like to emphasize that what we have presented here should *not* be thought of as a derivation for the ansatz metric. In the end, this is a 'guess' and our perturbation technique is developed around this starting ansatz. This guess could also be motivated from the fact that the final solution, in a very small region of size of $\mathcal{O}\left(\frac{1}{D}\right)$, looks like a D dimensional Schwarzschild black hole solution with a local radius and boost velocity [3, 14]

Now, we will evaluate R_{AB} on the metric $G^{[0]}_{AB} = g_{AB} + G^{(0)}_{AB}$. Details of the calculation are in appendix (A.2). Here we simply quote the final result.

$$
\begin{aligned}
R_{AB}|_{G^{[0]}_{AB}} = \psi^{-D}\left(\frac{DN}{2}\right)&\Big\{\left[DN - (\nabla\cdot O)\right](n_A O_B + n_B O_A) + (K - DN)O_A O_B\Big\}\\
&+ \left(\frac{\psi^{-2D}}{2}\right)\Big\{DN\left[DN - (\nabla\cdot O)\right]O_A O_B\Big\}\\
&+ \bar{R}_{AB} + \mathcal{O}(D)
\end{aligned}
\tag{2.19}
$$

where,

- $\bar{R}_{AB}$ is the Ricci tensor evaluated on the background metric g_{AB}
- ∇_A denotes the covariant derivative with respect to g_{AB}
- K is the trace of the extrinsic curvature of the membrane as embedded in the background spacetime with metric g_{AB} : $\quad K \equiv \nabla_A\, n^A$
- N is the norm of the one form $d\psi$: $\quad N \equiv \sqrt{(\partial_A\psi)(\partial_B\psi)g^{AB}}$

From (2.7), it follows that $\bar{R}_{AB} \sim \mathcal{O}(D)$. So, the leading equation reduces to

$$
\begin{aligned}
&\left[\psi^{-D}(K - DN) + \psi^{-2D}(DN - \nabla\cdot O)\right]O_A O_B\\
&+ \psi^{-D}(DN - \nabla\cdot O)(n_A O_B + n_B O_A) = \mathcal{O}(1)
\end{aligned}
\tag{2.20}
$$

As, O_A and n_A are two independent vectors in the background spacetime, equation (2.20) implies

$$
\begin{aligned}
(\nabla\cdot O - DN)_{\psi=1} &= \mathcal{O}(1)\\
(K - DN)_{\psi=1} &= \mathcal{O}(1)
\end{aligned}
\tag{2.21}
$$

Equation (2.20) is simply the conditions mentioned previously in equation (2.17). At leading order, the RHS of equation (2.3), which contains the effect of cosmological constant, does not contribute. Note also that the two equations in (2.20) together imply that

$$
(\nabla\cdot u)_{\psi=1} = \mathcal{O}(1) \tag{2.22}
$$

Where, u is defined in equation (2.15)[7]

2.4 Covariance w.r.t. 'background' metric

We could recast all the calculations in a manifestly covariant form with respect to the background metric g_{AB}. In fact, this feature is already there in the previous section (see equation (2.19)). The expression of R_{AB} involves partial derivatives of the metric. However, the expression in (2.19) have only covariant derivatives with respect to the background metric g_{AB}. In [3], the authors have argued this point from a physical point of view.

Here, we will see how it follows algebraically. This follows from the fact that though the Christoffel symbols are not tensors their differences are and therefore, the Christoffel symbols of the full metric G_{AB} could always be written as the Christoffel symbols of the background metric g_{AB} plus some correction which will have a form of a tensor with respect to the background metric. Then this feature could very easily be extended for the construction of the Riemann tensor and also for the Ricci tensor of the full metric G_{AB}.

The general form of our metric is given by

$$G_{AB} = g_{AB} + \chi_{AB}$$

Let $\hat{\Gamma}^A_{BC}$ and Γ^A_{BC} denote the Christoffel symbols corresponding to the metric g_{AB} and G_{AB} respectively

$$\begin{aligned}\Gamma^A_{BC} &= \frac{1}{2}G^{AC'}\left(\partial_C\, G_{C'B} + \partial_B\, G_{C'C} - \partial_{C'}\, G_{BC}\right)\\ &= \hat{\Gamma}^A_{BC} + \frac{1}{2}G^{AC'}\left(\nabla_C\, \chi_{C'B} + \nabla_B\, \chi_{C'C} - \nabla_{C'}\, \chi_{BC}\right)\end{aligned} \tag{2.23}$$

Here, ∇_A denotes the covariant derivative with respect to the background metric g_{AB}. We define the Ricci Tensor, R_{AB}, of the full metric by the following expression.

$$R_{AB} = \partial_k \Gamma^k_{AB} - \partial_B \Gamma^k_{Ak} + \Gamma^k_{km}\Gamma^m_{AB} - \Gamma^k_{Bm}\Gamma^m_{Ak}$$

[7] as explained in section(2.2), if we naively use the rules for counting order in $\left(\frac{1}{D}\right)$ expansion, $(\nabla \cdot u)$ should have been of $\mathcal{O}(D)$

Using equation (2.23), we could very easily rewrite it in the following form.

$$R_{AB} = \bar{R}_{AB} + \nabla_k \left[\delta\Gamma^k_{AB}\right] - \nabla_B \left[\delta\Gamma^k_{Ak}\right] + \left[\delta\Gamma^k_{km}\right]\left[\delta\Gamma^m_{AB}\right] - \left[\delta\Gamma^k_{Bm}\right]\left[\delta\Gamma^m_{Ak}\right] \tag{2.24}$$

where $\bar{R}_{AB}$ is the Ricci Tensor of the background and $\left[\delta\Gamma^A_{BC}\right]$ is the tensor appearing in the second term of equation (2.23)

$$\left[\delta\Gamma^A_{BC}\right] = \frac{1}{2} G^{AC'} \left(\nabla_C\, \chi_{C'B} + \nabla_B\, \chi_{C'C} - \nabla_{C'}\, \chi_{BC}\right) \tag{2.25}$$

Equations(2.23) and (2.24) are the main equations that we will use to determine the sub-leading order corrections to the ansatz metric in a manifestly covariant form.

2.5 General strategy for the first subleading correction

Once the leading ansatz $G^{(0)}_{AB}$, the function ψ and the one-form O are well-defined everywhere in the background with metric g_{AB}, we can describe the strategy to determine the subleading corrections to the metric i.e., the $G^{(k)}_{AB}$ for $k > 0$. In this chapter, our goal is to determine $G^{(1)}_{AB}$. Our method is essentially same as the one described in [3]. The purpose of this section is to mainly set up the notation and convention. We shall omit any detailed justification or 'all order proof', for the statements. Interested reader should refer to [3] for a thorough discussion.

2.5.1 Summary of the algorithm

We already know that if we evaluate Ricci tensor on $G^{[0]}_{AB} = g_{AB} + G^{(0)}_{AB}$, the leading piece is of order $\mathcal{O}(D^2)$. This leading piece vanishes provided O_A and ψ satisfy equations (2.21). Clearly after imposing equation (2.21), the leading non-vanishing piece in R_{AB} would be of order $\mathcal{O}(D)$. To cancel this piece up to corrections of order $\mathcal{O}(1)$ we add the new terms in the metric - $\left(\frac{1}{D}\right) G^{(1)}_{AB}$. Therefore, to begin with, $\left(\frac{1}{D}\right) G^{(1)}_{AB}$ will have the most general form that could contribute to the equation of motion (2.18) at order $\mathcal{O}(D)$. Also, any term in equation of motion that involves product of two components of $G^{(1)}_{AB}$ (i.e., non-linear in $G^{(1)}_{AB}$) will contribute at most at order $\mathcal{O}(1)$. Since in this chapter, we are interested only at order $\mathcal{O}(D)$, we have to treat $G^{(1)}_{AB}$ simply as a linear perturbation on $G^{[0]}_{AB}$. Then at order $\mathcal{O}\left(D\right)$, the equation of motion (2.18) will have two pieces. One piece will take the form of a linear differential operator acting on different (and so far unknown) components of $G^{(1)}_{AB}$ and the second piece will involve the $\mathcal{O}(D)$ piece coming from $G^{[0]}_{AB}$. The first piece will have an universal structure at all orders and we shall call it as 'homogeneous piece' or H_{AB}. The second part will be termed as 'source' (S_{AB}) . Schematically,

$$\mathcal{E}_{AB} \sim H_{AB} + S_{AB}$$

Our solution procedure will essentially be an 'inversion' of the universal differential operator in H_{AB}.

We shall determine $G^{(1)}_{AB}$ completely in terms of the function ψ and the one-form O, that are directly related to the basic data of our construction - the membrane and the velocity field. One advantage of our formalism is that we never need to choose any specific coordinate system on the membrane or for the background g_{AB}.

2.5.2 Subsidiary condition

Note that, so far, all the conditions on ψ and O are imposed only along the membrane. We want ψ to be one on the membrane hypersurface and the projection of O onto the membrane to reduce to the velocity field u_μ. The gravity equation (2.3) at leading order (see equation (2.21)) imposes some more constraints on ψ and O, but still they needed to be satisfied only at $(\psi = 1)$. Therefore, there is a large ambiguity in the construction of the function ψ and the one-form O. In this subsection, we shall fix this ambiguity with a certain convenient choice, which, following [2, 3, 14], we shall refer to as 'subsidiary conditions'.[8].

[8] The subsidiary conditions we have chosen in this thesis are different from what has been used in [2,3,14]. We found this choice most convenient because the metric correction at the first subleading order takes the

Subsidiary condition on ψ is chosen as follows.

$$\nabla^2 \psi^{-D} = 0 \quad \text{everywhere} \tag{2.26}$$

It could be shown that equation (2.26) is enough to determine ψ in an expansion in $\left(\frac{1}{D}\right)$ around the membrane ($\psi = 1$) [4]. Also we could easily see that (2.26) is consistent with the second equation (2.21)(See appendix (A.6)).

Now, we shall describe how we fixed the ambiguity in the definition of O_A. Unlike ψ, since O_A is a vector in the background with D components, we need D equations to fix it completely. From the construction of $G^{(0)}_{AB}$ we know that on the membrane, O^A is a null vector and $O \cdot n = 1$, where n_A in the unit normal to the membrane. Firstly, note that, once we have imposed equation (2.26), $\psi =$ constant surfaces and therefore the unit normal to them are well-defined everywhere. Therefore, we could easily lift these two conditions on O, which are initially imposed only on the membrane, to everywhere in the background. In terms of equation what we mean is the following

$$O \cdot O = 0 \ \text{ and } \ O \cdot n = 1 \quad \text{everywhere} \tag{2.27}$$

Equation (2.27) gives two scalar conditions on O. We still need $(D-2)$ equations through which we would be able to determine the remaining $(D-2)$ components of O_A, everywhere in the background . To fix them, we use the following differential equation.

$$\begin{gathered} P^B_A (O \cdot \nabla) O^A = 0 \quad \text{everywhere} \\ \text{where } P^B_A \equiv \delta^B_A - n_A O^B - O_A n^B + O_A O^B, \end{gathered} \tag{2.28}$$

Since, P^A_B is the projector to the subspace orthogonal to both n and O, equation (2.28) is effectively a collection of $(D-2)$ equations as required[9]. Equations (2.27) and (2.28)

simplest form. As we shall see, with this subsidiary condition, it simply vanishes and the first non-trivial correction appears only at the second subleading order.

[9] Because of equation (2.27) $O_A(O \cdot \nabla)O^A$ and $n_A(O \cdot \nabla)O^A$ are already determined.

$$O_A(O \cdot \nabla)O^A = 0, \;\; n_A(O \cdot \nabla)O^A = -O_A(O \cdot \nabla)n^A$$

together fix the ambiguities in all components of O, everywhere in the background.

It is possible to rewrite the subsidiary condition on O in a more geometric form. From equations (2.27) and (2.28), it follows that

$$(O \cdot \nabla)O^A = \left[n_B(O \cdot \nabla)O^B\right] O^A \quad \text{everywhere} \tag{2.29}$$

Equation (2.29) simply implies that throughout the background geometry, O^As are the tangent vectors to the null geodesics passing through the membrane.

In course of analysis we shall often define a u_A field everywhere in the background[10].

$$u_A \equiv -\Pi^B_A O_B \;\text{ where }\; \Pi^A_B \equiv \text{Projector on constant } \psi \text{ slices} = \delta^A_B - n^A n_B \tag{2.30}$$

Note that as a consequence of equation (2.27), u_A turns out to be a unit normalized time-like vector, which is orthogonal to n_A by construction.

$$g^{AB}u_A u_B = -1, \;\; g^{AB}u_A n_B = 0$$

From equation (2.27), it follows that $O \cdot n = O \cdot u = 1$ or $O_A = n_A - u_A$ everywhere. Also the projector P_{AB} of equation (2.28) is actually a projector orthogonal to both n_A and u_A and therefore could equivalently be expressed as

$$P_{AB} = g_{AB} - n_A n_B + u_A u_B$$

2.5.3 Choice of gauge

We shall choose a gauge such that

$$O^A G^{(1)}_{AB} = 0 \tag{2.31}$$

Note that our leading ansatz also satisfies this same gauge.

[10]Equation (2.30) apparently looks very similar to equation (2.15). However the main difference is that equation (2.30) is true for any constant ψ slices whereas equation (2.15) was specifically applied to the membrane i.e., $(\psi = 1)$.

After imposing equation (2.31), the most general structure for $G^{(1)}_{AB}$ is the following

$$
\begin{split}
&G^{(1)}_{AB} = \mathcal{S}_1 O_A O_B + \left(\frac{1}{D}\right)\mathcal{S}_2 P_{AB} + [O_A\mathcal{V}_B + O_B\mathcal{V}_A] + \mathcal{T}_{AB}\\
&\text{where}\\
&\qquad u^A\mathcal{V}_A = n^A\mathcal{V}_A = 0;\quad u^A\mathcal{T}_{AB} = n^A\mathcal{T}_{AB} = 0;\;\; g^{AB}\mathcal{T}^{(1)}_{AB} = 0
\end{split}
\tag{2.32}
$$

Here, the unknown scalar, vector and the tensors, $[\mathcal{S}_i,\;\; i = \{1,2\}]$, $\mathcal{V}_A$, $\mathcal{T}_{AB}$ are all of order $\mathcal{O}(1)$ and have explicit dependence on ψ as well as the derivatives of ψ and O.
Note the extra factor of $\left(\frac{1}{D}\right)$ in the term proportional to P_{AB}. This is because, by definition, $G^{(1)}_{AB}$ is the collection of those terms in the metric that contribute to the gravity equation at order $\mathcal{O}(D)$. As we shall see below, the term proportional to P_{AB} will contribute one extra factor of D in some terms of the gravity equation (the ones that involve a trace of the metric tensor). In other words, unless we suppress this term by an extra factor of $\left(\frac{1}{D}\right)$, it will contribute and mess-up the matching and solving of the equations at order $\mathcal{O}(D^2)$.

2.5.4 The form of explicit ψ dependence

We know that within the region where the metric correction is nontrivial, $(\psi - 1)$ is of order $\mathcal{O}\left(\frac{1}{D}\right)$. Therefore we would define a new order $\mathcal{O}(1)$ variable $R \equiv D(\psi - 1)$ to parametrize the explicit ψ dependence of the unknown scalar, vector and the tensor functions in equation (2.32). In terms of equation, we mean the following.

$$
\begin{split}
&\mathcal{S}_1 = \sum_n f_n(R)\,\mathfrak{s}_n,\quad \mathcal{S}_2 = \sum_n h_n(R)\,\mathfrak{s}_n\\
&\mathcal{V}_A = \sum_n v_n(R)\;[\mathfrak{v}_n]_A\quad \mathcal{T}_{AB} = \sum_n t_n(R)\;[\mathfrak{t}_n]_{AB}\\
&R \equiv D(\psi - 1)
\end{split}
\tag{2.33}
$$

Here $f_n(R),\; v_n(R),\; t_n(R)$ and $h_n(R)$ are functions that do not involve any explicit factors of D. The other expressions, $\mathfrak{s}_n$, $[\mathfrak{v}_n]_A$, $[\mathfrak{t}_n]_{AB}$ are the different scalar, vector and the tensor structures of order $\mathcal{O}(1)$, involving the derivatives of n_A and O_A that could appear at order $\mathcal{O}(1)$. The upper limit for the sum over n will generically be different in scalar, vector and tensor sector. These structures, by construction will not have any explicit dependence on ψ, since all such explicit dependence at this order will be captured by the function $f_n,\ v_n,\ t_n$ and h_n. However these structures will depend on ψ implicitly through the derivatives of n_A and O_A. But note that this will be a 'slow' dependence in $\left(\frac{1}{D}\right)$ expansion. More precisely, if we compute the variations of $\mathfrak{s}_n$, $\mathfrak{v}_n$ or $\mathfrak{t}_n$ in the direction of $\partial_A\psi$ it will always be of $\mathcal{O}(1)$, whereas the variations of $f_n(R)$, $h_n(R)$, $v_n(R)$ and $t_n(R)$, will be of order $\mathcal{O}(D)$. This is the reason, we could treat these structures, $\mathfrak{s}_n$, $[\mathfrak{v}_n]_A$ and $[\mathfrak{t}_n]_{AB}$ effectively as constants when we are doing the leading order computation with $G^{(1)}_{AB}$. See the next subsection for details.

2.5.5 Structure of 'Homogeneous piece'

In this subsection we shall list the detailed form of the homogeneous piece. As mentioned before, the homogeneous piece could be computed by simply linearizing the gravity equations (2.18) around $G^{[0]}_{AB}$, where the gauge-fixed form of the linear perturbation is given by $G^{(1)}_{AB}$. (See appendix A.1 for the details of the computation)

For convenience, we shall decompose the homogeneous piece into four parts.

$$H_{AB} = H^{scalar}_{AB} + H^{vector}_{AB} + H^{tensor}_{AB} + H^{trace}_{AB} \tag{2.34}$$

where

$$H^{scalar}_{AB} = \left(\frac{DN^2}{2}\right)\sum_n \mathfrak{s}_n\left(f''_n + f'_n\right)\left[n_B O_A + n_A O_B - \left(1-\psi^{-D}\right)O_B O_A\right] \tag{2.35}$$

$$\begin{aligned} H^{vector}_{AB} = &\left(\frac{N}{2}\right)\sum_n \left(\nabla\cdot\mathfrak{v}_n\right)\left[v'_n\left(n_A O_B + n_B O_A\right) - \psi^{-D} v_n O_B O_A\right] \\ &+\left(\frac{DN^2}{2}\right)\sum_n \left(v''_n + v'_n\right)\Bigg\{\Big(u_B\left[\mathfrak{v}_n\right]_A + u_A\left[\mathfrak{v}_n\right]_B\Big) \\ &\qquad + \psi^{-D}\Big(O_B\left[\mathfrak{v}_n\right]_A + O_A\left[\mathfrak{v}_n\right]_B\Big)\Bigg\} \end{aligned} \tag{2.36}$$

$$
\begin{aligned}
H_{AB}^{tensor} = & -\left(\frac{DN^2}{2}\right)\sum_n \left[t_n''(1-\psi^{-D})+t_n'\right][\mathfrak{t}_n]_{AB} \\
& +\left(\frac{N}{2}\right)\sum_n t_n'\Big(n_B\left(\nabla_C[\mathfrak{t}_n]^C_A\right)+A\leftrightarrow B\Big)
\end{aligned}
\tag{2.37}
$$

$$
H_{AB}^{trace} = -\left(\frac{DN^2}{4}\right)\sum_n \mathfrak{s}_n\Big\{2h_n''\, n_A n_B + h_n'\left[\psi^{-D}(n_A n_B - u_A u_B)+\psi^{-2D}O_B O_A\right]\Big\}
\tag{2.38}
$$

Here X' for any function $X(R)$ denotes $\frac{dX}{dR}$.

From the explicit expressions of H_{AB}, it follows that

$$
\left(\frac{1}{D}\right)\Pi^{AB}H_{AB} = \mathcal{O}(1)
\tag{2.39}
$$

where, Π_{AB} is the projector perpendicular to $(\psi = 1)$ hypersurface as embedded in the background.

It turns out that we could easily decouple these homogeneous parts of the $\mathcal{E}_{AB}$ by taking the following linear combination of the components.

$$
\begin{aligned}
& P^A_C H_{AB} P^B_{C'} - \frac{P_{CC'}}{D}\left(P^{AB}H_{AB}\right) \\
& = -\left(\frac{DN^2}{2}\right)\sum [\mathfrak{t}_n]_{CC'}\left[t_n''\left(1-\psi^{-D}\right)+t_n'\right]
\end{aligned}
\tag{2.40}
$$

$$
u^A H_{AB} P^B_C = -\left(\frac{DN^2}{2}\right)\sum_n (1-\psi^{-D})(v_n'+v_n'')[\mathfrak{v}_n]_C
\tag{2.41}
$$

$$
\begin{aligned}
u^A H_{AB} u^B = & -\left(\frac{DN^2}{2}\right)(1-\psi^{-D})\sum_n \mathfrak{s}_n\left[f_n''+f_n'-\left(\frac{\psi^{-D}}{2}\right)h_n'\right] \\
& -\left(\frac{N}{2}\right)\psi^{-D}\sum_n v_n(\nabla\cdot\mathfrak{v}_n)
\end{aligned}
\tag{2.42}
$$

$$
O^A O^B H_{AB} = -\frac{DN^2}{2}\sum_n h_n''\mathfrak{s}_n
\tag{2.43}
$$

Note that given equation (2.39), equations (2.40), (2.41) and (2.42) are simply the different components of $\left(\Pi_A^{A'}\Pi_B^{B'}H_{A'B'}\right)$ at leading non-trivial order in $(1/D)$ expansion.

2.5.6 Structure of 'Source'

In general the source S_{AB} will depend on all the coordinates, through some explicit dependence on ψ and also through different derivatives of O_A and n_A. As before, we can classify the ψ dependence of S_{AB} as 'slow' and 'fast'. The 'fast' pieces are those whose derivatives in the directions of increasing will have a factor of D, (i.e., the dependence on ψ is through $R \equiv D(\psi - 1)$). These are the parts which have been treated exactly at a given order. All other variations of the source terms, both along and away from the membrane hypersurface, are 'slow' (i.e., the derivatives are suppressed by a factor of $\left(\frac{1}{D}\right)$ compared to the 'fast' dependence) and therefore could effectively be treated as constants while solving for the next correction to the metric i.e, $G^{(1)}_{AB}$. This is why we simply invert the homogeneous piece H_{AB} assuming it to be an ordinary differential operator in the 'fast' variable R. See [2] and [14] for a more detailed explanation.

As we have seen in the previous subsection, the projected components of the homogeneous piece $(\Pi_A^{A'}\Pi_B^{B'}H_{A'B'})$ could be viewed as ordinary second order differential operator in the 'fast' variable R, acting on the unknown functions appearing in the metric correction. It follows that to determine the unknown functions $f(R),\ v(R)$ and $t(R)$, it is enough to solve the projected components of the gravity equations (2.18)

$$\Pi_A^{A'}\Pi_B^{B'}\mathcal{E}_{A'B'} = 0$$

The traceless piece of the projected $\mathcal{E}_{AB}$ leads to the following set of second order inhomogeneous differential equations for three sets of the unknown functions, $f_n(R),\ v_n(R)$ and $t_n(R)$.

$$\sum_n \frac{d}{dR}\left[\left(e^R-1\right)t'_n\right]\ [\mathfrak{t}_n]_{AB} = \left(\frac{2\,e^R}{DN^2}\right)\left[P^C_A P^{C'}_B - P_{AB}\left(\frac{P^{CC'}}{D}\right)\right] S_{CC'}$$

$$(1-e^{-R})\sum_n \frac{d}{dR}\left[e^R v'_n\right]\ [\mathfrak{v}_n]_A = \left(\frac{2\,e^R}{DN^2}\right)\left[u^B S_{BC} P^C_A\right] \tag{2.44}$$

$$(1-e^{-R})\sum_n \frac{d}{dR}\left[e^R f'_n - \frac{h_n}{2}\right]\ \mathfrak{s}_n = \left(\frac{2\,e^R}{DN^2}\right)\left(u^A S_{AB} u^B\right) - \sum_n v_n\left(\frac{\nabla\cdot\mathfrak{v}_n}{DN}\right)$$

In equation (2.44), we have also used the fact that $\left[\psi^{-D} = e^{-R} + \mathcal{O}\left(\frac{1}{D}\right)\right]$.

The equation for $h(R)$ is given by the $\mathcal{E}_{AB}$ with both indices projected in the direction of O.

$$O^A O^B \mathcal{E}_{AB} = 0 \Rightarrow \sum_n h''_n \mathfrak{s}_n = \left(\frac{2}{DN^2}\right)\left[O^A\ S_{AB}\ O^B\right] \tag{2.45}$$

Note that the last two equations in (2.44) will admit regular solutions at $\psi = 1$ only if

$$\begin{aligned} &\left[u^B S_{BC} P^C_A\right]_{R=0} = 0 \\ &\left[\left(\frac{2}{DN^2}\right)\left(u^A S_{AB} u^B\right) - \sum_n v_n\left(\frac{\nabla\cdot\mathfrak{v}_n}{DN}\right)\right]_{R=0} = 0 \end{aligned} \tag{2.46}$$

We shall see that both of these conditions will be true as a consequence of our membrane equation. In fact in [2] this is the regularity condition that has been used to determine the membrane equation.

2.5.7 Boundary condition

Since our differential operator (in R) is second order, we need two sets of boundary conditions to fix the integration constants. One of these is the 'normalizability'. In our construction it must be true that the metric is non-trivial only in a thin region of thickness $\mathcal{O}\left(\frac{1}{D}\right)$ around the membrane $\psi = 1$. This defines the normalizability conditions on the metric functions $f_n(R),\ v_n(R),\ t_n(R)$ and $h_n(R)$; in R coordinates they must vanish exponentially as $R \to \infty$ (recall $R = D\,(\psi - 1)$), so that outside the 'membrane region' the metric is that of the background. This 'normalizability' fixes one integration constant in each of the three differential equations in (2.44). It turns out that for equation (2.45) both the zero modes are non-normalizable or in other words in this case the 'normalizability' condition is enough to fix $h_n(R)$.

The other integration constant is fixed by the condition on the horizon. For f_n and v_n, it is fixed by our definition of the horizon itself. We want $\psi = 1$ to be the exact equation for the horizon of this geometry and u^A to be the null generator of the horizon. This implies that the following 'all order' equation on the horizon

$$u^A G_{AB}|_{\psi=1} = n_B \tag{2.47}$$

Note that by construction at any order the metric will take the form

$$G_{AB} = g_{AB} + f\ O_A O_B + (V_A\ O_B + V_B\ O_A) + h\ P_{AB} + t_{AB}$$

where $O_A = n_A - u_A$, $V \cdot O = V \cdot n = 0$, $O^A t_{AB} = n^A t_{AB} = 0$, $P^{AB} t_{AB} = 0$ Contracting this metric with u^A we find

$$u^A G_{AB} = u_B + f\ O_B + V_B$$

Now (2.47) fixes the values of f and V_A on $\psi = 1$ or equivalently $R = 0$.

$$\begin{aligned} f\ |_{\psi=1} = 1\ &\Rightarrow\ f_n(R=0) = 0, \\ V_A\ |_{\psi=1} = 0\ &\Rightarrow\ v_n(R=0) = 0 \end{aligned} \tag{2.48}$$

For the tensor sector i.e., the function $t_n(R)$, the other integration constant could be fixed by demanding the solution is regular at the horizon.

2.5.8 Solution in the form of integral

Once the boundary conditions are fixed, we can explicitly invert the differential operators and could write the solutions for $f_n(R)$, $v_n(R)$, $t_n(R)$, and $h_n(R)$ in terms of some definite integrals of the source. In this subsection, we shall present these formulas explicitly. As mentioned before in subsection (2.5.6), we could always rewrite source S_{AB} at any given order as some functions of 'fast' variable R multiplied by the 'slowly' varying scalar, vector or tensor structures relevant for that order. In other words the RHS of the three equations in (2.44) could be expressed as

$$\begin{aligned}
\text{RHS of 1st eqn} &= \left(\frac{2e^R}{N^2}\right)\sum_n [\mathfrak{t}_n]_{AB}\, S_n^{\text{tensor}}(R)\\
\text{RHS of 2nd eqn} &= \left(\frac{2e^R}{N^2}\right)\sum_n [\mathfrak{v}_n]_{A}\, S_n^{\text{vector}}(R)\\
\text{RHS of 3rd eqn} &= \left(\frac{2e^R}{N^2}\right)\sum_n [\mathfrak{s}_n]\, S_n^{\text{scalar}}(R) - \left(\frac{1}{DN}\right)\sum_n v_n(R)\,(\nabla\cdot\mathfrak{v}_n)
\end{aligned} \tag{2.49}$$

Similarly RHS of (2.45) could be written as

$$RHS = \left(\frac{2}{N^2}\right)\sum_n S_n^{\text{trace}}(R)\,\mathfrak{s}_n \tag{2.50}$$

Now we can explicitly write the solution for $G^{(1)}_{AB}$ in terms of definite integral of the source.

$$\begin{aligned}
&G^{(1)}_{AB} = \mathcal{S}_1 O_A O_B + \left(\frac{1}{D}\right)\mathcal{S}_2 P_{AB} + [O_A\mathcal{V}_B + O_B\mathcal{V}_A] + \mathcal{T}_{AB}\\
&\text{where,}\\
&\qquad u^A\mathcal{V}_A = n^A\mathcal{V}_A = 0;\quad u^A\mathcal{T}_{AB} = n^A\mathcal{T}_{AB} = 0;\quad g^{AB}\mathcal{T}_{AB} = 0
\end{aligned} \tag{2.51}$$

where,

$$\begin{aligned}
\mathcal{T}_{AB} &= -\left(\frac{2}{N^2}\right)\sum_n [\mathfrak{t}_n]_{AB}\int_R^\infty \left(\frac{dy}{e^y-1}\right)\left(\int_0^y dx\,\left[e^x\, S_n^{\text{tensor}}(x)\right]\right)\\
\mathcal{V}_A &= -\left(\frac{2}{N^2}\right)\sum_n [\mathfrak{v}_n]_A \int_R^\infty dy\, e^{-y}\left[\int_0^y dx\left(\frac{e^{2x}}{e^x-1}\right)S_n^{\text{vector}}(x)\right] + e^{-R}\mathcal{K}_A^{\text{vector}}\\
\mathcal{S}_2 &= \left(\frac{2}{N^2}\right)\sum_n \mathfrak{s}_n \int_R^\infty dy\left[\int_y^\infty dx\, S_n^{\text{trace}}(x)\right]\\
\mathcal{S}_1 &= -\left(\frac{2}{N^2}\right)\sum_n \mathfrak{s}_n \int_R^\infty dy\, e^{-y}\left[\int_0^y dx\left(\frac{e^{2x}}{e^x-1}\right)S_n^{\text{scalar}}(x)\right]\\
&\quad + \left(\frac{1}{2}\right)\int_R^\infty dz\, e^{-z}\left[-\mathcal{S}_2 + 2\int_0^z \left(\frac{dx}{1-e^{-x}}\right)\left(\frac{\nabla\cdot\mathcal{V}}{DN}\right)\right] + e^{-R}\mathcal{K}^{\text{scalar}}
\end{aligned} \tag{2.52}$$

Here $\mathcal{K}_s$ and $\mathcal{K}_v$ are two constants added so that $S_1|_{R=0} = \mathcal{V}_A|_{R=0} = 0$

$$\begin{aligned}
\mathcal{K}^{\text{scalar}} &= \left(\frac{2}{N^2}\right)\sum_n \mathfrak{s}_n \int_0^\infty dy\ e^{-y}\left[\int_0^y dx\left(\frac{e^{2x}}{e^x-1}\right)S_n^{\text{scalar}}(x)\right]\\
&\quad-\left(\frac{1}{2}\right)\int_0^\infty dz\ e^{-z}\left[-\mathcal{S}_2+2\int_0^z\left(\frac{dx}{1-e^{-x}}\right)\left(\frac{\nabla\cdot\mathcal{V}}{DN}\right)\right]\\
\mathcal{K}_A^{\text{vector}} &= \left(\frac{2}{N^2}\right)\sum_n [\mathfrak{v}_n]_A\int_0^\infty dy\ e^{-y}\left[\int_0^y dx\left(\frac{e^{2x}}{e^x-1}\right)S_n^{\text{vector}}(x)\right]
\end{aligned}\tag{2.53}$$

2.5.9 Constraint and membrane equation

Consider the following combinations of different components of H_{AB}.

1. $(n^B - \psi^{-D}O^B)\ H_{BC}\ P_A^C = \frac{N}{2}(1-e^{-R})\sum_n \nabla^B(\mathfrak{t}_n)_{BA}\ t'_n$
2. $(n^B - \psi^{-D}O^B)\ H_{BC}\ u^C = \left(\frac{DN}{2}\right)\sum_n\left(\frac{\nabla\cdot\mathfrak{v}_n}{D}\right)\left[v'_n(1-e^{-R}) - v_n e^{-R}\right]$

Note that the above combinations have at most one R derivative of the unknown functions. Clearly the same feature would be true if we take the above combinations on the components of $\mathcal{E}_{AB}$, since the source S_{AB} does not involve any of the unknown functions. Hence these combinations could be viewed as equations that restrict the 'initial conditions' (defined on any constant R slice) for the second order differential equations (see (2.44)) controlling the 'R-evolution' of the unknown functions. It follows that the 'constraint' equations in our case has the following form

$$\begin{aligned}
\mathcal{C} &\equiv (n^B - \psi^{-D}O^B)\ \mathcal{E}_{BC}\ u^C\\
C_A &\equiv (n^B - \psi^{-D}O^B)\ \mathcal{E}_{BC}\ P_A^C
\end{aligned}\tag{2.54}$$

In terms of source S_{AB} and the unknown metric functions, the above two constraints will take the following structure [11]

$$\begin{aligned}
\mathcal{C} &= (n^B - e^{-R}O^B)\, S_{BC}\, u^C + \left(\frac{DN}{2}\right)\sum_n \left(\frac{\nabla\cdot\mathfrak{v}_n}{D}\right)\left[v_n'(1-e^{-R}) - v_n e^{-R}\right] \\
C_A &= (n^B - e^{-R}O^B)\, S_{BC}\, P_A^C + \frac{N}{2}(1-e^{-R})\sum_n \nabla^B(\mathfrak{t}_n)_{BA}\, t_n'
\end{aligned} \tag{2.55}$$

Now it is known that if the constraint is satisfied along one slice and the dynamical equations are satisfied everywhere, then the constraint is automatically satisfied along all hypersurfaces [68]. In [3], this theorem has been explicitly verified for the constraint equations listed above in equations (2.55). Because of this theorem, we are allowed to impose the constraints (2.55) only on $\psi = 1$ hypersurface and do not worry about how these equations are solved away from the membrane. So at order $\mathcal{O}(D)$, the final form of the membrane equations

$$\begin{aligned}
\mathcal{C}|_{R=0} &= u^B\, S_{BC}\, u^C|_{R=0} \\
C_A|_{R=0} &= u^B\, S_{BC}\, P_A^C|_{R=0}
\end{aligned} \tag{2.56}$$

In deriving equation (2.56) we have used the fact that $O^A = n^A - u^A$ and $v_n(R=0) = 0$ because of our boundary condition. We also used the fact that $\mathcal{T}_{AB}^{(1)}$ is regular at $R = 0$ due to choice of integration limits (see equation (2.52)) and thus the term involving unknown tensor metric correction in C_A vanishes at $R = 0$.

Equations (2.56) are the genuine membrane equations that do not involve any of the unknown functions and therefore only constrain our membrane data. Also note that these

[11] We know that given the foliation of the spacetime with $\psi = const$ hypersurfaces, the equations of gravity could be decomposed into dynamical and constraint equations [68]. The constraint equations are the ones where one of the indices of the Einstein's equation is projected along the normal to the foliating hypersurfaces. In [3], this theory has been used and explained in detail in the context of our large D expansion. Along with the two combinations we mentioned in equations (2.54) one more constraint equation appears in [3], whose abstract form is the following

$$\mathcal{A} \equiv (1-\psi^{-D})\, O^A O^B\, \mathcal{E}_{AB} - P^{AB}\, \mathcal{E}_{AB}$$

However, we shall not analyze this combination here since it will not be required to obtain the final gravity solution and the membrane equations are the combinations that appear in the RHS of the first two equations in (2.44) and the regularity of the solutions also demand the vanishing of these constraints on $R = 0$.

The fact that given a solution to these constraint equations along the membrane, we can always solve the other dynamical equations (i.e. the other components of the $\mathcal{E}_{AB}$), by inverting the linear differential operator appearing in H_{AB}, establishes the 'membrane-gravity duality' that we have mentioned in the introduction.

2.6 The first subleading correction: $G^{(1)}_{AB}$

In this section we shall describe how we calculate the first subleading correction to the metric along with the coupled equations of motion for the membrane and the velocity field along it. As described in the previous section, at this order the source S_{AB} will simply be determined by evaluating the Ricci Tensor R_{AB} on the metric $G^{[0]}_{AB} = g_{AB} + G^{(0)}_{AB} = g_{AB} + \psi^{-D} O_A O_B$. The details of the computation of the Ricci Tensor are presented in the appendix (A.2). For convenience we quote the final answer for the source at first subleading order.

$$
\begin{split}
S_{AB} = e^{-R}\left(\frac{K}{2}\right)\Bigg[e^{-R} O_B O_A \left(\left(\hat{\nabla}\cdot u\right)_{R=0} - \frac{R}{K}\left(\hat{\nabla}\cdot E\right)_{R=0}\right) \\
+ (n_A O_B + n_B O_A)\left(\left(\hat{\nabla}\cdot u\right)_{R=0} - \frac{R}{K}\left(\hat{\nabla}\cdot E\right)_{R=0}\right) \\
+ (O_B P^C_A + O_A P^C_B)\left(\frac{\hat{\nabla}^2 u_C}{K} - \frac{\hat{\nabla}_C K}{K} + u^D K_{DC} - (u\cdot\hat{\nabla})u_C\right)_{R=0}\Bigg]
\end{split}
\tag{2.57}
$$

Where, $\hat{\nabla}$ is defined as follows, for any general tensor with n indices $W_{A_1 A_2 \cdots A_n}$

$$
\hat{\nabla}_A W_{A_1 A_2 \cdots A_n} = \Pi^C_A\, \Pi^{C_1}_{A_1} \Pi^{C_2}_{A_2} \cdots \Pi^{C_n}_{A_n} \left(\nabla_C W_{C_1 C_2 \cdots C_n}\right) \tag{2.58}
$$

Here, K_{AB} is the extrinsic curvature of the $\psi = 1$ hypersurface viewed as a submanifold in the background spacetime g_{AB}, defined as

$$
K_{AB} = \Pi^C_A \nabla_C n_B \tag{2.59}
$$

2.6.1 Constraint equation

In the previous section we have described how we could determine the constraint equations on the membrane by taking appropriate combination of the components of the source terms evaluated at $\psi = 1$. In this subsection, we shall first evaluate those combinations on S_{AB} and determine the constraints on the membrane data at the first subleading order. Note that at leading order there was only one scalar constraint on the membrane data

$$\nabla \cdot u \sim \mathcal{O}(1)$$

It turns out that at first subleading order we shall have one scalar and one vector equation. This matches with the number of free data we have on the membrane: the shape of the membrane (scalar function) and the unit normalized velocity field on it (the vector function).

Constraint in the vector sector

First we shall describe the constraint equation in the direction perpendicular to u_A. We shall refer to this as 'Vector constraint'.

$$\begin{aligned} & u^B S_{BC} P^C_A = \mathcal{O}(1) \\ \Rightarrow\ & \frac{K}{2} P^C_A \left[\frac{\hat{\nabla}^2 u_C}{K} - \frac{\hat{\nabla}_C K}{K} + u^D K_{DC} - (u \cdot \hat{\nabla}) u_C \right] = \mathcal{O}(1) \\ \Rightarrow\ & P^C_A \left[\frac{\hat{\nabla}^2 u_C}{K} - \frac{\hat{\nabla}_C K}{K} + u^D K_{DC} - (u \cdot \hat{\nabla}) u_C \right] = \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{2.60}$$

Note that in equation (2.60), all derivatives and all the indices (both contracted and free) are projected along the hypersurface ($\psi = 1$). Now it is easy to rewrite the constraint equation as an equation intrinsic to the membrane.

$$\mathcal{P}^\nu_\mu \left[\frac{\bar{\nabla}^2 u_\nu}{\mathcal{K}} - \frac{\bar{\nabla}_\nu \mathcal{K}}{\mathcal{K}} + u^\alpha \mathcal{K}_{\alpha\nu} - (u \cdot \bar{\nabla}) u_\nu \right] = \mathcal{O}\left(\frac{1}{D}\right) \tag{2.61}$$

Where, $\mathcal{P}_{\mu\nu} = g^{(ind)}_{\mu\nu} + u_\mu u_\nu$, $g^{(ind)}_{\mu\nu}$ denotes the induced metric on the membrane ($\psi = 1$ hypersurface) and $\bar{\nabla}$ is the covariant derivative with respect to $g^{(ind)}_{\mu\nu}$. The velocity field u_μ is the pull back of the bulk velocity field u_A and $\mathcal{K}_{\mu\nu}$ is the pull back of the extrinsic curvature of the membrane onto the hypersurface [12] and $\mathcal{K}$ is the trace of the extrinsic curvature.

Constraint in the scalar sector

Now we shall describe the constraint equation in the scalar sector, i.e.,the constraint in the direction of u_A.

$$0 = u^B S_{BC}\, u^C = \frac{K}{2}\left[\hat{\nabla}\cdot u\right] \tag{2.63}$$

As before, this equation also could be written purely in terms of the intrinsic data of the membrane.

$$\left[\hat{\nabla}\cdot u\right]_{\psi=1} = \bar{\nabla}\cdot u \tag{2.64}$$

where $\bar{\nabla}$ denotes the covariant derivative with respect to the intrinsic metric of the hypersurface $(\psi = 1)$ viewed as a membrane embedded in the background.

We finally find

$$\bar{\nabla}\cdot u \sim \mathcal{O}\left(\frac{1}{D}\right) \tag{2.65}$$

2.6.2 Dynamical equation

In this section we shall give details of the dynamical equations. It turns out that given our subsidiary condition and after imposing the scalar and vector constraint equations, the sources for all dynamical equation simply vanish leading to the vanishing of $G^{(1)}_{AB}$.

[12]In terms of equations, u_μ and $\mathcal{K}_{\mu\nu}$ is defined as

$$u_\mu = \left(\frac{\partial X^A}{\partial y^\mu}\right) u_A, \quad \mathcal{K}_{\mu\nu} = \left(\frac{\partial X^M}{\partial y^\mu}\right)\left(\frac{\partial X^N}{\partial y^\nu}\right) K_{MN} \tag{2.62}$$

where X^M denotes the coordinates of the full spacetime and y^μ denotes coordinates on the membrane.

Tensor sector

From the first equation of (2.44) we get the relevant differential equation for the 'tensor-type' correction at the first subleading order.

$$D \sum_n \left[\left(1 - e^{-R}\right) t''_n + t'_n\right] \; [\mathfrak{t}_n]_{AB} = \left(\frac{2}{N^2}\right) \left[P^C_A P^{C'}_B - P_{AB} \left(\frac{P^{CC'}}{D}\right)\right] S_{CC'} \tag{2.66}$$

But from equation (2.57) we could simply see that

$$P^C_A P^{C'}_B S_{CC'} = 0$$

In the language of equation(2.49) it implies that $S^{tensor}_n(R)$ vanishes for all (n). Substituting this in the first equation of (2.52) we find $\mathcal{T}^{(1)}_{AB}$ is zero.

Vector sector

From the second equation of (2.44) we get the relevant differential equation for the 'vector-type' correction at the first subleading order.

$$De^{-R}(1 - e^{-R}) \sum_n \frac{d}{dR} \left[e^R v'_n\right] \; [\mathfrak{v}_n]_A = \left(\frac{2}{N^2}\right) \left[u^B S_{BC} P^C_A\right] \tag{2.67}$$

Note that the RHS of equation (2.67) implicitly depends on ψ. However the dependence is 'slow', in the sense as one goes away from $(\psi = 1)$ hypersurface, the variation of the RHS is suppressed by a factor of $\left(\frac{1}{D}\right)$. Thus, at this order, we need to evaluate the RHS only at $(\psi = 1)$ hypersurface.

Now from equations (2.60) and (2.61) it follows that

$$\left[u^B S_{BC} P^C_A\right]_{\psi=1} = 0$$

In the language of equation(2.49) it implies that $S^{vector}_n(R)$ vanishes for all (n). Substituting this in the second equation of (2.52) we find $\mathcal{V}^{(1)}_A$ is zero.

Scalar sector

In the scalar sector there are two unknown functions $h(R)$ and $f(R)$ and therefore we need two equations. Clearly equation (2.45) and the last equation of (2.44) are the relevant equations here.

$$
\begin{aligned}
&O^A O^B \mathcal{E}_{AB} = 0 \Rightarrow \sum_n h''_n \mathfrak{s}_n = \left(\frac{2}{DN^2}\right)\left[O^A \, S_{AB} \, O^B\right] \\
&\text{and} \\
&De^{-R}(1-e^{-R})\sum_n \frac{d}{dR}\left[e^R f'_n - \frac{h_n}{2}\right] \mathfrak{s}_n \\
&= \left(\frac{2}{N^2}\right)\left(u^A S_{AB} u^B\right) - e^{-R}\sum_n v_n \left(\frac{\nabla\cdot \mathfrak{v}_n}{N}\right)
\end{aligned}
\tag{2.68}
$$

Now since $P^{AB}S_{AB}$ vanishes, the boundary conditions (see section (2.5.7) ensure that $h_n(R)$ is zero for every n. Given that $h_n(R)$ is zero and there is no correction in the vector sector (implying $v_n^{(1)}(R)$ is zero for every n) the second equation of (2.68) reduces to

$$
De^{-R}(1-e^{-R})\sum_n \frac{d}{dR}\left[e^R f'_n\right] \mathfrak{s}_n = \left(\frac{2}{N^2}\right)\left(u^A S_{AB} u^B\right) \tag{2.69}
$$

Now, following the same logic as we have used in 'Vector sector' , the RHS of equation (2.69) is simply the scalar constraint equation and therefore vanishes. Now the boundary conditions ensures that $f_n(R) = 0$ for every n.

2.7 Final metric and membrane equation

In this section we shall simply summarize our final result i.e., the metric and the membrane equation of motion up to the first subleading order. As we have seen in the previous section, given our subsidiary condition, the next to leading correction to the metric vanishes.

$$
G_{AB} = g_{AB} + \psi^{-D} O_A O_B + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{2.70}
$$

where the scalar function ψ and O_A are defined everywhere in the background (with metric g_{AB}) through the following equations

$$\begin{aligned}
&O^C\partial_C\psi = \sqrt{\partial\psi\cdot\partial\psi}, \quad O\cdot O = 0\\
&\nabla^2\psi^{-D} = 0\\
&(O\cdot\nabla)O_A = \left[\left(\frac{\partial_C\psi}{\sqrt{\partial\psi\cdot\partial\psi}}\right)(O\cdot\nabla)O^C\right]O_A
\end{aligned} \tag{2.71}$$

Clearly the asymptotic form of the full spacetime is given by the metric g_{AB}, which we have referred to as 'background'. ∇ is the covariant derivative with respect to g_{AB}.

The equations (2.71) are enough to fix ψ and O everywhere provided the shape of the $(\psi = 1)$ hypersurface and the one form field O_A on $(\psi = 1)$ hypersurface are given. We have referred to these two pieces of information as 'membrane data' . It turns out that (2.70) is a solution of the gravity equation provided the membrane data satisfy the following equation of motion

$$\begin{aligned}
&\mathcal{P}^\nu_\mu\left\{\bar\nabla^2 u_\nu - \bar\nabla_\nu\mathcal{K} + \mathcal{K}\left[u_\alpha\mathcal{K}^\alpha_\nu - (u_\alpha\bar\nabla^\alpha)u_\nu\right]\right\} = \mathcal{O}(1)\\
&\bar\nabla_\alpha u^\alpha = \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{2.72}$$

Equation (2.72) is an equation intrinsic to the membrane, in the sense that all raising and lowering of indices and the covariant derivatives are defined with respect to the induced metric on the membrane - a hypersurface embedded in the background g_{AB}. All the indices now can take $(D-1)$ values. $\mathcal{K}_{\mu\nu}$ is the extrinsic curvature tensor , viewed as a tensor structure defined on the membrane only. $\mathcal{K}$ is the trace of $\mathcal{K}_{\mu\nu}$. The velocity field u_μ is the projection of the one form O_A along the hypersurface. And $\mathcal{P}^\nu_\mu$ is the projector perpendicular to the velocity field u_μ. Like the extrinsic curvature tensor, this projector is also defined only along the membrane worldvolume.

Equations (2.70), (2.71) and (2.72) together are the final result of this chapter.

2.8 Checks: matching with known exact solutions

In this section we shall perform several checks on our solution for the metric and the equation of motion for the membrane. We know of few exact static and stationary black hole / brane solutions of the equation (2.3) in arbitrary dimension. Now our effective membrane equation (2.72) and the metric (2.70) are valid as long as the number of dimensions is very large. Clearly static and stationary exact solutions are special cases which must solve our equation and must match with our metric in the appropriate limit. In this section we shall show this matching explicitly for three different exact solutions in Asymptotically AdS space.

2.8.1 Schwarzschild Black Brane in AdS

In Kerr-Schild form AdS black brane is given by

$$dS^2 = dS^2_{\text{Poincare}} + r^{-(D-3)}\left(dt + \frac{dr}{r^2}\right)^2 \tag{2.73}$$

where dS^2_{Poincare} is the line element in Poincare patch AdS space.

$$dS^2_{\text{Poincare}} = \frac{dr^2}{r^2} - r^2 dt^2 + r^2 d\vec{x}^2_{D-2} \tag{2.74}$$

For the black brane geometry (2.73), the hypersurface $r = 1$ is the horizon and the null generator of the horizon is given by

$$l^A \partial_A = \ \partial_t$$

It follows that the dual membrane is given by the same surface $r = 1$, however viewed as a hypersurface embedded in the AdS space with metric dS^2_{Poincare} and the velocity field along the horizon is simply $u = -dt$. The induced metric on the membrane

$$dS^2_{\text{induced}} = -\ dt^2 \ + \ d\vec{x}^2_{D-2}$$

We can easily see that this velocity field u is divergence free along the membrane. It is very easy to compute the extrinsic curvature tensor for this configuration. The non-zero components of extrinsic curvature and trace of extrinsic curvature are given by

$$\mathcal{K}_{ij} = \delta_{ij}, \ \ \mathcal{K}_{tt} = -1, \ \ \mathcal{K} = D - 1 \ \text{ Where } \{i \ = 1, ..., \ D \ - \ 2\} \tag{2.75}$$

All the components of the derivatives of the velocity field on the membrane vanishes

$$\bar{\nabla}_\mu u_\nu = 0, \ \ \{\mu = t, i\} \tag{2.76}$$

Substituting equations (2.75) and (2.76) in the membrane equation (2.72) and using the fact that $P^t_t = P^t_i = 0$, we see that it is satisfied up to the required order.

Next we shall match the form of the metric. For this we need to read off ψ and u_A in such a way that

1. $\psi \ = \ 1$ surface is same as the $r \ = \ 1$ surface. In other words if we consider ψ as a function of r, then $\psi(r = 1) = 1$.
2. $u^A|_{r=1} = l^A$
3. Both ψ and the u^A satisfy the subsidiary conditions (2.26) and (2.29).

The normalized form of u is easy to guess.

$$u_A \, dx^A = -r \, dt \tag{2.77}$$

Translation symmetry in t and all i directions guarantees that ψ must be a function of r alone and it follows that the subsidiary condition on u is trivially satisfied (since any vector in the space perpendicular to $n \sim dr$ and $u \sim dt$ must vanish because of the symmetry). Now we shall solve for ψ in an expansion in $\left(\frac{1}{D}\right)$. Let us start by expanding ψ around the horizon $r = 1$.

$$\psi(r) = 1 + \left(a_{10} + \frac{a_{11}}{D}\right)(r-1) + a_{20}(r-1)^2 + \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{2.78}$$

Here $a_{10},\ a_{11},\ a_{20}$ are constants (to be determined by solving the subsidiary condition (2.26)) and we have also used the fact that within the 'membrane region' $(r-1) \sim \mathcal{O}\left(\frac{1}{D}\right)$. Substituting (2.78) in (2.26)) and solving order by order we find

$$\begin{aligned}\psi(r) &= 1 + \left(1 - \frac{1}{D}\right)(r-1) + \mathcal{O}\left(\frac{1}{D}\right)^3 \\ &= r - \left(\frac{r-1}{D}\right) + \mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{2.79}$$

Note that equations (2.79) and (2.77) imply that in the 'membrane region'

$$\begin{aligned}\frac{dr}{r} - r\ dt &= O_A dx^A \\ \frac{r^{-(D-3)}}{r^2} &= \psi^{-D} + \left(\frac{1}{D}\right)^2\end{aligned} \tag{2.80}$$

From equations (2.80) it follows that the metric of AdS Schwarzschild black brane is same as the one we determined in equation (2.70) up to correction of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

2.8.2 Schwarzschild Black Hole in Global AdS

In Kerr-Schild form, the global AdS black hole is given by

$$dS^2 = dS^2_{Global} + \left(\frac{r^{-(D-3)}}{1+r^2}\right)\left(\sqrt{1+r^2}\ dt + \frac{dr}{\sqrt{1+r^2}}\right)^2 \tag{2.81}$$

where dS^2_{Global} is given by

$$dS^2_{Global} = \frac{dr^2}{1+r^2} - (1+r^2)dt^2 + r^2 d\Omega^2_{D-2} \tag{2.82}$$

Horizon of this black hole spacetime (2.81) is located at the zero of the function $f(r) = 1 + r^2 - r^{-(D-3)}$. If horizon is at $r = r_0 \Rightarrow f(r_0) = 0,\ \ r_0 \neq 1$.

The null generator of the horizon is given by

$$l^A \partial_A = \frac{1}{\sqrt{1+r_0^2}}\partial_t$$

It follows that our membrane is given by the hypersurface $r = r_0$ embedded in the AdS space with metric as given by dS^2_{Global} and the velocity field along the horizon is simply

$u = -\sqrt{1+r_0^2}\, dt$. The induced metric on the membrane

$$ds^2_{\text{induced}} = -(1+r_0^2)dt^2 + r_0^2 d\Omega^2_{D-2}$$

We can easily see that this velocity field u is divergence free along the membrane. It is very easy to compute the extrinsic curvature tensor for this configuration. The non-zero component of the extrinsic curvature and the trace of extrinsic curvature are given by

$$\mathcal{K}_{tt} = -\sqrt{2},\ \ \mathcal{K}_{ab} = \sqrt{2}\,\Omega_{ab},\ \ \mathcal{K} = \frac{1}{\sqrt{2}} + (D-2)\sqrt{2} \tag{2.83}$$

Where Ω_{ab} is the metric on $(D-2)$ dimensional unit sphere

All the components of the derivatives of the velocity field on the membrane vanishes

$$\bar{\nabla}_\mu u_\nu = 0,\ \ \{\mu = t, a\} \tag{2.84}$$

Substituting equations (2.83) and (2.84) in the membrane equation (2.72) and using the fact that $P^t_t = P^t_a = 0$, we see that it is satisfied up to the required order.

Next we shall match the form of the metric. As in previous subsubsection we have to read off appropriate ψ and u_A defined everywhere in Global AdS space.

- Since the spacetime is static and also maintains spherical symmetry, ψ must be a function of r only. This implies $n_A dx^A \propto dr$. After normalization $n_A dx^A = \frac{dr}{\sqrt{1+r^2}}$.

- It follows that the normalized u has the form

$$u_A\, dx^A = -\sqrt{1+r^2}\, dt \ \text{ or } \ O_A\, dx^A = \left(\sqrt{1+r^2}\, dt + \frac{dr}{\sqrt{1+r^2}}\right) \tag{2.85}$$

  It is easy to see that this u will satisfy all the subsidiary condition as a consequence of the symmetry.

- Now ψ has to satisfy the subsidiary condition,

$$\nabla^2 \psi^{-D} = 0 \tag{2.86}$$

To solve the equation (2.86) we have to repeat the same procedure as we have done in the previous subsubsection. Now the only difference is that the background is not AdS-Poincare but global AdS and the covariant derivatives are also modified accordingly. This calculation is a bit complicated and the details are given in appendix (A.3)

$$\begin{aligned}\psi(r) &= 1 + \frac{\log 2}{D} + \left(\frac{1}{D}\right)^2 \left[\frac{(\log 2)^2}{2}\right] + \left(1 + \frac{\log 2 \text{ - } 2}{D}\right)(r-1) + \mathcal{O}\left(\frac{1}{D}\right)^3 \\ &= r\left(1 + \frac{\log 2}{D}\right) - (r-1)\frac{2}{D} + \left(\frac{1}{D}\right)^2 \frac{(\log 2)^2}{2} + \mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{2.87}$$

Here also (2.87) imply that in the 'membrane region'

$$\frac{r^{-(D-3)}}{1+r^2} = \psi^{-D} + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{2.88}$$

As in the previous subsection from equations (2.80) and (2.85) it follows that the metric of AdS Schwarzschild black hole is same as the one we determined in equation (2.70) up to correction of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

2.8.3 Rotating Black Hole in AdS

The explicit form of Kerr de-Sitter metric in $D = 2n+1$ dimensions([69], [70]) in Kerr-Schild form is given by

$$\begin{aligned}dS^2 &= dS^2_{\text{Ads}} + \frac{2M}{U}(k_A dx^A)^2 \\ G_{AB}dx^A dx^B &= g_{AB}dx^A dx^B + \frac{2M}{U}k_A k_B dx^A dx^B\end{aligned} \tag{2.89}$$

where,

$$\begin{aligned}dS^2_{\text{AdS}} = &-W(1+r^2)dt^2 + F dr^2 + \sum_{i=1}^{n} \frac{r^2 + a_i^2}{1-a_i^2}(d\mu_i^2 + \mu_i^2 d\phi_i^2) \\ &- \frac{1}{W(1+r^2)}\left(\sum_{i=1}^{n} \frac{(r^2+a_i^2)\mu_i d\mu_i}{1-a_i^2}\right)^2\end{aligned} \tag{2.90}$$

$$W = \sum_{i=1}^{n} \frac{\mu_i^2}{1-a_i^2}; \quad F = \frac{r^2}{1+r^2}\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}; \quad U = \sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{n}(r^2+a_j^2) \tag{2.91}$$

$$k_A dx^A = W dt + F dr - \sum_{i=1}^{n} \frac{a_i \mu_i^2}{1 - a_i^2} d\phi_i \tag{2.92}$$

g_{AB} is actually the metric of global AdS, but written in some rotating coordinates. The coordinate transformation that will bring it back to standard form (the one presented in equation (2.81)) is given in [69]. However we shall continue to work in the coordinates as given in equation (2.90). One of the advantage of using these coordinates is that the horizon of the black hole spacetime in these rotating coordinates is given by constant r slices, where the value of the constant is determined from the zero of the following function.

$$\frac{U}{F} - 2M = 0 \tag{2.93}$$

For convenience of computation we shall scale the parameter M in the following way

$$M = \prod_{i=1}^{n} (1 + a_i^2)$$

so that the horizon lies at $r = 1$, which would be the equation of our membrane. The induced metric on the membrane

$$dS^2_{\text{induced}} = -2\,W\,dt^2 + \sum_{i=1}^{n} \frac{1 + a_i^2}{1 - a_i^2} (d\mu_i^2 + \mu_i^2 d\phi_i^2) - \frac{1}{2\,W} \left(\sum_{i=1}^{n} \frac{(1 + a_i^2)\mu_i d\mu_i}{1 - a_i^2} \right)^2 \tag{2.94}$$

It turns out that k_A is null with respect to both the metric G_{AB} and g_{AB}. The null generator of the horizon is given by

$$l^A \partial_A = \frac{1}{\sqrt{2}} \left(\sum_{j=1}^{n} \frac{\mu_j^2}{1 + a_j^2} \right)^{-\frac{1}{2}} \left(\partial_t + 2 \sum_{i=1}^{n} \frac{a_i}{1 + a_i^2}\, \partial_{\phi_i} \right) \tag{2.95}$$

From here it follows that the velocity field along the horizon is given by

$$u_A dx^A = -\sqrt{2} \left(\sum_{j=1}^{n} \frac{\mu_j^2}{1 + a_j^2} \right)^{-\frac{1}{2}} \left(\sum_{i=1}^{n} \frac{\mu_i^2}{1 - a_i^2} (dt - a_i\, d\phi_i) \right) \tag{2.96}$$

Once we have the explicit form of the equation of the membrane and the velocity field, each term of (2.72) are computable.

Now as we have explained before, our $\left(\frac{1}{D}\right)$ expansion is valid provided the spacetime satisfies some large symmetry and is dynamical or non-trivial only in a finite number of directions. The metric in (2.89) will belong to this class, if only a finite number of rotation parameters a_i's are non-zero. But if we turn on arbitrary (though finite) number of a_i's, it turns out that explicit computation is very tedious for this complicated metric. So we have used *Mathematica* (version 9.0) here and to be explicit we have used two non-zero rotation parameters. We have first checked that this velocity field and the extrinsic curvature of the membrane do satisfy our membrane equation (2.72) up to the required order.

The next job is to check whether the spacetime metric (2.89) matches with equation (2.70) up to correction of order $\mathcal{O}\left(\frac{1}{D}\right)^2$. Now we know that k_A is exactly null with respect to g_{AB}. Clearly k_A is the most natural candidate for the null vector O_A we have in our metric.

Suppose

$$k_A = \mathcal{A}\, O_A$$

where $\mathcal{A}$ is some unknown function of r at the moment. Now note that the metric (2.89) will be precisely of the form (2.70) provided we identify

$$\mathcal{A}^2\left(\frac{2M}{U}\right) \rightarrow \left[\psi^{-D} + \mathcal{O}\left(\frac{1}{D}\right)^2\right]$$

The above equation along with the fact that $\mathcal{A}$ is a function of r , will imply that ψ also depends only on r. The unit normal to $\psi =$ constant slices is then given by

$$n_A dx^A = \sqrt{F}\, dr$$

Now $O_A\, n^A = 1$ implies $k_A n^A = \mathcal{A}$. Therefore once we know the explicit expression of n_A, we can fix $\mathcal{A}$. It turns out

$$\mathcal{A} = \sqrt{F}$$

However just identifying $\left[\mathcal{A}^2\left(\frac{2M}{U}\right)\right]$ with ψ^{-D} is not enough for the matching of the two metrics. We also have to see whether these ψ and O_A satisfy our subsidiary conditions. The above identification will be consistent with our subsidiary condition (2.26) provided

$$\begin{aligned}
&\nabla^2\left[\mathcal{A}^2\left(\frac{2M}{U}\right)\right] = \mathcal{O}(1)\\
&(k\cdot\nabla)k_A \propto \left[k_A + \mathcal{O}\left(\frac{1}{D}\right)\right], \quad n^A k_A = 1
\end{aligned} \tag{2.97}$$

[13] Here ∇ is defined with respect to the background metric g_{AB} and all raising and lowering of indices have been done using g_{AB}. In *Mathematica* we have explicitly verified this condition for two nonzero rotation parameters.

2.9 Discussions

In this chapter, we have used 'large D' techniques to find new dynamical 'black hole' solutions of Einstein's equation in presence of cosmological constant. The solutions are determined in an expansion in $\left(\frac{1}{D}\right)$ and are in 'one-to-one' correspondence with a dynamical membrane (characterized by its shape and a velocity field on it) embedded in the asymptotic geometry (which could be AdS or dS).

The method we have used is manifestly covariant with respect to this asymptotic geometry (which we have referred to as 'background'). We do not need to choose any coordinate system for the background geometry at any point of our derivation. The same calculation works for both global AdS and Poincare patch. The form of the final answer also remains invariant. However, they are different solutions with different asymptotic geometries and horizon topologies and this fact is encoded in the various covariant derivatives that appear in the final solution. These covariant derivatives are always defined with respect to the background.

[13]We already know that $k_A k^A = 0$. Now as along as $n_A k^A = 1$, we could always express k_A as $k_A = n_A - u_A$ such that $u\cdot u = -1$ and $n\cdot u = 0$ everywhere

We have applied this method to calculate the metric and the governing equation for the dual dynamical membrane up to the first subleading correction. Then we have performed several checks for our universal coordinate independent answer, by specializing to different coordinate systems.

- We matched them against the known exact and static solutions - Schwarzschild black hole/brane and Myers-Perry black holes for both asymptotically AdS and dS spaces.
- We have linearized our membrane equations and matched them against the known spectrum of black hole/brane QNMs in AdS space and black hole QNMs in dS space. This linearized analysis is not included in this thesis, see [65] for details.
- We have taken a special scaling limit of our equations and recovered the dual effective hydrodynamic equations that was determined in [29] for the AdS black branes in large number of dimensions. This analysis is also not included in this thesis, see [65] for details.

This calculation has been extended to Einstein-Maxwell system in presence of cosmological constant in [15].

Chapter 3

Large-D membrane paradigm in AdS/dS at subleading order

This chapter is based on [64].

In this chapter, we would like to extend the calculation of chapter 2 to the second subleading order. The key motivation is two-fold. Firstly, from the result of chapter 2, we know that at the first subleading order the background curvature does not appear explicitly in any of the equation or the solution. However, it should appear explicitly at second subleading order (which, very roughly speaking, captures the effect of two derivatives on the background). Secondly, from the experience of the 'flat space computation', it is expected that at this order, we should see the entropy production from a dynamical black hole.

However, in this chapter, we shall confine ourselves only to the computation of the membrane equation of motion and the metric correction up to the second subleading order in $\left(\frac{1}{D}\right)$ expansion. We leave the 'study of entropy production' for future.

The organization of this chapter is as follows. In section 3.1, we have described the basic set-up of our problem in terms of equations and also the final result for the metric corrections and the membrane equations. Next in section 3.2, we have given a sketch of the computation, which turns out to be quite tedious in this case. Many of the details we have collected in the appendices. In section 3.3, we have performed several checks of our results. Finally in section 3.4, we end with some discussions and future directions.

3.1 Set up and final result

In this section, we shall briefly define the basic set-up of our problem in terms of equations. It is essentially an extension of section 2.1. So we shall be very brief here.

Our aim is to solve Einstein's equation (2.18) up to second subleading order in $\frac{1}{D}$ expansion. Schematically our solution will take the form

$$G_{AB} = g_{AB} + G^{(0)}_{AB} + \left(\frac{1}{D}\right) G^{(1)}_{AB} + \left(\frac{1}{D}\right)^2 G^{(2)}_{AB} + \cdots \tag{3.1}$$

Here g_{AB} is the background metric and $G^{(0)}_{AB}$ is the leading ansatz given by (2.16). We shall determine the metric corrections in terms of ψ and O_A (defined in subsection 2.5.2) and their derivatives.

As we have discussed in chapter 2, Einstein's equation could be solved provided the extrinsic curvature of the $\psi = 1$ hypersurface (viewed as a hypersurface embedded in the background spacetime) and the velocity field u^A together satisfy some constraint equations on the horizon. We have determined the form of the constraint equation at the leading order in chapter 2. The constraint equation at the leading order is given by eq.(2.61) and eq.(2.65). Here, we are just rewriting the equation

$$\begin{gathered}\mathcal{P}^{\nu}_{\mu}\left[\frac{\bar{\nabla}^2 u_\nu}{\mathcal{K}} - \frac{\bar{\nabla}_\nu \mathcal{K}}{\mathcal{K}} + u_\alpha \mathcal{K}^{\alpha}_{\nu} - (u\cdot\bar{\nabla})u_\nu\right] = \mathcal{O}\left(\frac{1}{D}\right), \;\; \bar{\nabla}\cdot u = \mathcal{O}\left(\frac{1}{D}\right) \\ \text{where} \;\; \mathcal{P}_{\mu\nu} = g^{(ind)}_{\mu\nu} + u_\mu u_\nu\end{gathered} \tag{3.2}$$

Here $g^{(ind)}_{\mu\nu}$ denotes the induced metric on the membrane ($\psi = 1$ hypersurface) and $\bar{\nabla}$ is the covariant derivative with respect to $g^{(ind)}_{\mu\nu}$. The velocity field u_μ is the pull back of the bulk velocity field u_A and $\mathcal{K}_{\mu\nu}$ is the pull back of the extrinsic curvature of the membrane onto the hypersurface (see eq.(2.62) for definitions) and $\mathcal{K}$ is the trace of the extrinsic curvature.

As discussed in chapter 2, for every solution of the above constraint equations we could determine $G^{(1)}_{AB}$. It turns out that $G^{(1)}_{AB}$ simply vanishes given our choice of subsidiary conditions.

In this chapter our goal is to find corrections to equation (3.2) to the next order in $\left(\frac{1}{D}\right)$ expansion and also $G^{(2)}_{AB}$.

But before getting into any details of the computation, we shall first present our final result i.e., the subleading correction to the membrane equation (3.2) and the second subleading order metric correction $G^{(2)}_{AB}$. The metric correction would take the following form.

$$
\begin{split}
G^{(2)}_{AB} = &\left[O_A O_B \left(\sum_{n=1}^{2} f_n(R)\, \mathfrak{s}_n\right) + t(R)\, \mathfrak{t}_{AB} + v(R)\, \left(\, \mathfrak{v}_A O_B + \mathfrak{v}_B O_A\right)\right]\\
&\text{where } \ R \equiv D(\psi - 1), \ \ P_{AB} = g_{AB} - n_A n_B + u_A u_B\\
&\text{and, } \ n^A\, \mathfrak{v}_A = u^A\, \mathfrak{v}_A = 0, \ \ n^A\, \mathfrak{t}_{AB} = u^A\, \mathfrak{t}_{AB} = 0, \ \ g^{AB}\, \mathfrak{t}_{AB} = 0
\end{split}
\tag{3.3}
$$

where,

$$
\begin{split}
\mathfrak{t}_{AB} = &P^C_A P^D_B \Bigg[\bar{R}_{FCDE} O^E O^F + \frac{K}{D}\left(K_{CD} - \frac{\hat{\nabla}_C u_D + \hat{\nabla}_D u_C}{2}\right)\\
&\quad - P^{EF}(K_{EC} - \hat{\nabla}_E u_C)(K_{FD} - \hat{\nabla}_F u_D)\Bigg]\\
\mathfrak{v}_A = &P^B_A \Bigg[\frac{K}{D}\left(n^D u^E O^F \bar{R}_{FBDE}\right) + \frac{K^2}{2D^2}\left(\frac{\hat{\nabla}_B K}{K} + (u\cdot\hat{\nabla})u_B - 2\, u^D K_{DB}\right)\\
&\quad - P^{FD}\left(\frac{\hat{\nabla}_F K}{D} - \frac{K}{D}(u^E K_{EF})\right)\left(K_{DB} - \hat{\nabla}_D u_B\right)\Bigg]\\
\mathfrak{s}_1 = &\,u^E u^F n^D n^C \bar{R}_{CEFD} + \left(\frac{u\cdot\hat{\nabla}K}{K}\right)^2 + \frac{\hat{\nabla}_A K}{K}\left[4\, u^B K^A_B - 2\left[(u\cdot\hat{\nabla})u^A\right] - \frac{\hat{\nabla}^A K}{K}\right]\\
&- (\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B) - (u\cdot K\cdot u)^2 - \left[(u\cdot\hat{\nabla})u_A\right][(u\cdot\hat{\nabla})u^A] + 2\left[(u\cdot\hat{\nabla})u^A\right](u^B K_{BA})\\
&\quad - 3\,(u\cdot K\cdot K\cdot u) - \frac{K}{D}\left(\frac{u\cdot\hat{\nabla}K}{K} - u\cdot K\cdot u\right)\\
\mathfrak{s}_2 = &\frac{K^2}{D^2}\Bigg[-\frac{K}{D}\left(\frac{u\cdot\hat{\nabla}K}{K} - u\cdot K\cdot u\right) - 2\,\lambda - (u\cdot K\cdot K\cdot u) + 2\left(\frac{\hat{\nabla}_A K}{K}\right)u^B K^A_B - \left(\frac{u\cdot\hat{\nabla}K}{K}\right)^2\\
&+ 2\left(\frac{u\cdot\hat{\nabla}K}{K}\right)(u\cdot K\cdot u) - \left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right) - (u\cdot K\cdot u)^2 + n^B n^D u^E u^F \bar{R}_{FBDE}\Bigg]
\end{split}
\tag{3.4}
$$

Where, $\bar{R}_{ABCD}$ is the Riemann tensor[1] of the background metric g_{AB} and $\hat{\nabla}$ is defined as follows: for any general tensor with n indices $W_{A_1A_2\cdots A_n}$

$$\hat{\nabla}_A W_{A_1A_2\cdots A_n} = \Pi_A^C\, \Pi_{A_1}^{C_1}\Pi_{A_2}^{C_2}\cdots\Pi_{A_n}^{C_n}\left(\nabla_C W_{C_1C_2\cdots C_n}\right), \quad \text{with} \quad \Pi_{AB} = g_{AB} - n_A n_B \tag{3.5}$$

$$\begin{aligned} t(R) &= -\,2\left(\frac{D}{K}\right)^2 \int_R^\infty \frac{y\,dy}{e^y-1} \\ v(R) &= 2\left(\frac{D}{K}\right)^3\left[\int_R^\infty e^{-x}dx\int_0^x \frac{y\,e^y}{e^y-1}dy \;-\; e^{-R}\int_0^\infty e^{-x}dx\int_0^x\frac{y\,e^y}{e^y-1}dy\right] \end{aligned} \tag{3.6}$$

$$\begin{aligned} f_1(R) &= -2\left(\frac{D}{K}\right)^2\int_R^\infty x\,e^{-x}dx + 2\,e^{-R}\left(\frac{D}{K}\right)^2\int_0^\infty x\,e^{-x}dx \\ f_2(R) &= \left(\frac{D}{K}\right)\left[\int_R^\infty e^{-x}dx\int_0^x\frac{v(y)}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx\int_0^x\frac{v(y)}{1-e^{-y}}dy\right] \\ &\quad -\left(\frac{D}{K}\right)^4\left[\int_R^\infty e^{-x}dx\int_0^x\frac{y^2\,e^{-y}}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx\int_0^x\frac{y^2\,e^{-y}}{1-e^{-y}}dy\right] \end{aligned} \tag{3.7}$$

As we can see that our solution is parametrized by the shape of the constant ψ hypersurfaces (encoded in its extrinsic curvature K_{AB}) along with the velocity field u^A. However, because of our subsidiary conditions if we know K_{AB} and u^A along one constant ψ hypersurface, they are determined everywhere else. In this sense, the real data in our class of solutions are to be provided only along one simple surface; the most natural choice of which is the horizon or the hypersurface $\psi = 1$.

As we have mentioned before, we cannot choose any arbitrary shape of the membrane and velocity field as our initial data. The metric, presented above, would solve Einstein's equation (2.18) only if the data satisfy some constraint - the equation (3.2) with subleading

[1] Riemann tensor is defined by the relation

$$[\nabla_A, \nabla_B]\omega_C = \bar{R}_{ABC}{}^{D}\omega_D$$

corrections.

$$\left[\frac{\bar{\nabla}^2 u_\alpha}{\mathcal{K}} - \frac{\bar{\nabla}_\alpha \mathcal{K}}{\mathcal{K}} + u^\beta \mathcal{K}_{\beta\alpha} - u\cdot\bar{\nabla} u_\alpha\right]\mathcal{P}^\alpha_\gamma + \Bigg[-\frac{u^\beta \mathcal{K}_{\beta\delta}\mathcal{K}^\delta_\alpha}{\mathcal{K}} + \frac{\bar{\nabla}^2\bar{\nabla}^2 u_\alpha}{\mathcal{K}^3} - \frac{(\bar{\nabla}_\alpha\mathcal{K})(u\cdot\bar{\nabla}\mathcal{K})}{\mathcal{K}^3}$$
$$- \frac{(\bar{\nabla}_\beta\mathcal{K})(\bar{\nabla}^\beta u_\alpha)}{\mathcal{K}^2} - \frac{2\mathcal{K}^{\delta\sigma}\bar{\nabla}_\delta\bar{\nabla}_\sigma u_\alpha}{K^2} - \frac{\bar{\nabla}_\alpha\bar{\nabla}^2\mathcal{K}}{\mathcal{K}^3} + \frac{\bar{\nabla}_\alpha(\mathcal{K}_{\beta\delta}\mathcal{K}^{\beta\delta}\mathcal{K})}{K^3} + 3\frac{(u\cdot\mathcal{K}\cdot u)(u\cdot\bar{\nabla}u_\alpha)}{\mathcal{K}}$$
$$- 3\frac{(u\cdot\mathcal{K}\cdot u)(u^\beta\mathcal{K}_{\beta\alpha})}{\mathcal{K}} - 6\frac{(u\cdot\bar{\nabla}\mathcal{K})(u\cdot\bar{\nabla}u_\alpha)}{\mathcal{K}^2} + 6\frac{(u\cdot\bar{\nabla}\mathcal{K})(u^\beta\mathcal{K}_{\beta\alpha})}{\mathcal{K}^2} + 3\frac{u\cdot\bar{\nabla}u_\alpha}{D-3}$$
$$- 3\frac{u^\beta\mathcal{K}_{\beta\alpha}}{D-3} - \frac{(D-1)\lambda}{\mathcal{K}^2}\left(\frac{\bar{\nabla}_\alpha\mathcal{K}}{\mathcal{K}} - 2u^\sigma\mathcal{K}_{\sigma\alpha} + 2(u\cdot\bar{\nabla})u_\alpha\right)\Bigg]\mathcal{P}^\alpha_\gamma = \mathcal{O}\left(\frac{1}{D}\right)^2$$

$$\bar{\nabla}\cdot u - \frac{1}{2\mathcal{K}}\left(\bar{\nabla}_{(\alpha}u_{\beta)}\bar{\nabla}_{(\gamma}u_{\delta)}\mathcal{P}^{\beta\gamma}\mathcal{P}^{\alpha\delta}\right) = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{3.8}$$

Where $\bar{\nabla}$ is the covariant derivative with respect to $g^{(ind)}_{\mu\nu}$, the induced metric on $\psi = 1$ hypersurface. $\mathcal{K}_{\mu\nu}$ and u_μ are defined in (2.62). $\bar{\nabla}_{(\alpha}u_{\beta)}$ is defined as

$$\bar{\nabla}_{(\alpha}u_{\beta)} \equiv \bar{\nabla}_\alpha u_\beta + \bar{\nabla}_\beta u_\alpha$$

3.2 Sketch of the computation

It turns out that though the computation to determine the second order metric correction is tedious, conceptually it is a straightforward extension of what has been done in chapter 2. Therefore in this section, we shall omit most of the derivations and mention only those where there are some differences from 2.

We shall follow the same convention as in chapter 2. In particular our choice of gauge is also the same, namely

$$O^B G^{(2)}_{AB} = 0$$

With this gauge choice the second order correction could be parametrized as

$$\begin{split} G^{(2)}_{AB} = \bigg(O_A O_B \sum_n f_n(R)\,\mathfrak{s}_n + \frac{1}{D} P_{AB} \sum_n h_n(R)\,\mathfrak{s}_n + \sum_n t_n(R)\,[\mathfrak{t}_n]_{AB} \\ + \sum_n v_n(R)\,\big([\mathfrak{v}_n]_A O_B + [\mathfrak{v}_n]_B O_A \big) \bigg) \end{split} \tag{3.9}$$

$$\text{where,}\ \ R \equiv D(\psi - 1),\ \ P_{AB} = g_{AB} - n_A n_B + u_A u_B$$

$$\text{and,}\ \ n^A[\mathfrak{v}_n]_A = u^A[\mathfrak{v}_n]_A = 0,\ \ n^A[\mathfrak{t}_n]_{AB} = u^A[\mathfrak{t}_n]_{AB} = 0,\ \ g^{AB}[\mathfrak{t}_n]_{AB} = 0$$

Here $\mathfrak{s}_n$, $[\mathfrak{v}_n]_A$, $[\mathfrak{t}_n]_{AB}$ are different independent scalar, vector and tensor structures, constructed out of the membrane data.

Evaluating Einstein's equation (2.18) on $\left[G_{AB} = g_{AB} + G^{(0)}_{AB} + \left(\frac{1}{D}\right) G^{(1)}_{AB} + \left(\frac{1}{D}\right)^2 G^{(2)}_{AB} + \mathcal{O}\left(\frac{1}{D}\right)^3\right]$ up to order $\mathcal{O}(1)$, we get a set of coupled, ordinary but inhomogeneous differential equation for the unknown functions in equation (3.9). Boundary conditions for these differential equations are set by the following physical conditions.

1. The surface $(\psi = 1)$ or $(R = 0)$ is the event horizon and therefore a null hypersurface to all orders.

2. u^A is the null generator of this event horizon to all orders.

3. Bulk metric G_{AB} to all orders approaches g_{AB} as $R \to \infty$.

These conditions translate to the following constraints on the unknown functions.

$$\begin{split} f_n(R=0) = v_n(R=0) = 0, \quad h_n(R=0) = t_n(R=0) = \text{finite}, \\ \lim_{R\to\infty} f_n(R) = \lim_{R\to\infty} h_n(R) = \lim_{R\to\infty} v_n(R) = \lim_{R\to\infty} t_n(R) = 0 \end{split} \tag{3.10}$$

The homogeneous part H_{AB} (i.e., the part that acts like a differential operator on the space of unknown functions appearing in $G^{(2)}_{AB}$) is universal. It will have the same form as in the 'first order' calculation and we do not need to recalculate it. For convenience, here we shall quote the results for the homogeneous part as derived in chapter 2.

$$\begin{split} H_{AB} \equiv H^{(1)} O_A O_B + H^{(2)}(n_A O_B + n_B O_A) + H^{(3)} n_A n_B + H^{(tr)} P_{AB} \\ + \left(O_A P^C_B + O_B P^C_A\right) H^{(V_1)}_C + \left(n_A P^C_B + n_B P^C_A\right) H^{(V_2)}_C + H^{(T)}_{AB} \end{split} \tag{3.11}$$

where,

$$
\begin{aligned}
H^{(1)} &= -\frac{N^2}{2}(1-e^{-R})\sum_n \mathfrak{s}_n\left(f_n''+f_n'\right) - \frac{N}{2}e^{-R}\sum_n \frac{(\nabla\cdot\mathfrak{v}_n)}{D}v_n + \frac{N^2}{4}e^{-R}(1-e^{-R})\sum_n \mathfrak{s}_n h_n' \\
H^{(2)} &= \frac{N^2}{2}\sum_n \mathfrak{s}_n\left(f_n''+f_n'\right) + \frac{N}{2}\sum_n \frac{(\nabla\cdot\mathfrak{v}_n)}{D}v_n' - \frac{N^2}{4}e^{-R}\sum_n \mathfrak{s}_n h_n' \\
H^{(3)} &= -\frac{N^2}{2}\sum_n \mathfrak{s}_n h_n'' \\
H^{(tr)} &= 0 \\
H_C^{(V_1)} &= -\frac{N^2}{2}(1-e^{-R})\sum_n \left(v_n''+v_n'\right)[\mathfrak{v}_n]_C \\
H_C^{(V_2)} &= \frac{N^2}{2}\sum_n \left(v_n''+v_n'\right)[\mathfrak{v}_n]_C + \frac{N}{2D}\sum_n t_n'\left(\nabla_D[\mathfrak{t}_n]^D_C\right) \\
H_{AB}^{(T)} &= -\frac{N^2}{2}\sum_n \left[t_n''(1-e^{-R})+t_n'\right][\mathfrak{t}_n]_{AB}
\end{aligned}
\tag{3.12}
$$

Here for any R dependent function, $X'(R)$ denotes $\frac{dX(R)}{dR}$.

The 'source' parts of these equations are determined by evaluating the Einstein's equation on the first order corrected metric. By construction the order $\mathcal{O}(D^2)$ and order $\mathcal{O}(D)$ pieces of these equations will vanish and first non-zero contribution, relevant for the computation of this chapter , will be of $\mathcal{O}(1)$.

From the above discussion it follows that the key part of the computation is to determine the source term, which we denote here by S_{AB}. Since $G^{(1)}_{AB}$ vanishes, just like in chapter 2, here also the source will be given by E_{AB} calculated on $\left(g_{AB}+G^{(0)}_{AB}\right)$, however the complication lies in the fact that the calculation has to be carried out up to order $\mathcal{O}(1)$. Here we are presenting the final result for the source. See appendix B.1 for the details. For convenience, we shall decompose S_{AB} into its different components.

$$
\begin{aligned}
S_{AB} \equiv\ & S^{(1)} O_A O_B + S^{(2)}(n_A O_B + n_B O_A) + S^{(3)} n_A n_B + S^{(tr)} P_{AB} \\
& + \left(O_A P^C_B + O_B P^C_A\right) S^{(V_1)}_C + \left(n_A P^C_B + n_B P^C_A\right) S^{(V_2)}_C + S^{(T)}_{AB} \\
\text{where}\quad & O^A S^{(T)}_{AB} = n^A S^{(T)}_{AB} = 0, \quad S^{(T)}_{AB} P^{AB} = 0 \quad \text{and} \quad P_{AB} \equiv g_{AB} + u_A u_B - n_A n_B
\end{aligned}
\tag{3.13}
$$

The explicit expression for the different components are the following.

$$
\begin{aligned}
S^{(1)} &= e^{-2R}\left(\frac{K}{2}\right) E^{\text{scalar}} + \left(e^{-R} - e^{-2R}\right)\mathfrak{s}_1 + e^{-2R}\left(\frac{R^2}{2}\right)\left(\frac{D}{K}\right)^2 \mathfrak{s}_2 - R\left(\frac{e^{-2R}}{2}\right)\left(\hat{\nabla}\cdot E^{\text{vector}}\right)_{R=0} \\
S^{(2)} &= e^{-R}\left[-\mathfrak{s}_1 + \left(\frac{K}{2}\right) E^{\text{scalar}}\right]_{R=0} - R\left(\frac{e^{-R}}{2}\right)\left(\hat{\nabla}\cdot E\right)_{R=0} + e^{-R}\left(\frac{R^2}{2}\right)\left[\left(\frac{D^2}{K^2}\right)\mathfrak{s}_2\right]_{R=0} \\
S^{(V_1)}_C &= \frac{e^{-R}}{2}\left[K E^{\text{vector}}_C - 2\,R\left(\frac{D}{K}\right)\mathfrak{v}_C\right], \qquad S^{(T)}_{AB} = e^{-R}\,\mathfrak{t}_{AB} \\
S^{(3)} &= S^{tr} = 0, \quad S^{(V_2)}_C = 0
\end{aligned}
\tag{3.14}
$$

Where

$$
E^{\text{scalar}} = \left[\left(\hat{\nabla}\cdot u\right)\Big|_{\psi=1} - \frac{1}{2K}\left[\hat{\nabla}_{(A}u_{B)}\hat{\nabla}_{(C}u_{D)}P^{BC}P^{AD}\right]\right] \tag{3.15}
$$

$$
\begin{aligned}
E^{\text{vector}}_C =\ & \left[\frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K} + u^B K_{BA} - u\cdot\hat{\nabla} u_A\right] P^A_C \\
& + \Bigg[-\frac{u^B K_{BD} K^D_A}{K} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_A}{K^3} - \frac{(\hat{\nabla}_A K)(u\cdot\hat{\nabla}K)}{K^3} - \frac{(\hat{\nabla}_B K)(\hat{\nabla}^B u_A)}{K^2} \\
& - \frac{2K^{DE}\hat{\nabla}_D\hat{\nabla}_E u_A}{K^2} - \frac{\hat{\nabla}_A\hat{\nabla}^2 K}{K^3} + \frac{\hat{\nabla}_A(K_{BD}K^{BD}K)}{K^3} + 3\frac{(u\cdot K\cdot u)(u\cdot\hat{\nabla}u_A)}{K} \\
& - 3\frac{(u\cdot K\cdot u)(u^B K_{BA})}{K} - 6\frac{(u\cdot\hat{\nabla}K)(u\cdot\hat{\nabla}u_A)}{K^2} + 6\frac{(u\cdot\hat{\nabla}K)(u^B K_{BA})}{K^2} \\
& + 3\frac{u\cdot\hat{\nabla}u_A}{D-3} - 3\frac{u^B K_{BA}}{D-3} - \frac{(D-1)\lambda}{K^2}\left(\frac{\hat{\nabla}_A K}{K} - 2u^D K_{DA} + 2(u\cdot\hat{\nabla})u_A\right)\Bigg] P^A_C
\end{aligned}
\tag{3.16}
$$

See equation (3.4) for the definitions of $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{v}_C$, $\mathfrak{t}_{AB}$.

$\hat{\nabla}$ is defined as follows: for any general tensor with n indices $W_{A_1A_2\cdots A_n}$

$$
\hat{\nabla}_A W_{A_1A_2\cdots A_n} = \Pi^C_A\,\Pi^{C_1}_{A_1}\Pi^{C_2}_{A_2}\cdots\Pi^{C_n}_{A_n}\left(\nabla_C W_{C_1C_2\cdots C_n}\right) \tag{3.17}
$$

The final set of coupled differential equations that we have to solve is simply

$$H_{AB} + S_{AB} = 0 \tag{3.18}$$

As explained in chapter 2, the homogeneous part H_{AB} could be decoupled after taking its appropriate projection on different directions. Similar projections applied on S_{AB} will generate the sources for the scalar, vector, tensor and the trace sectors.

However, just as in the first order calculation, there is an 'integrability' condition. Note that $H^{(1)}$ and $H^{(V_1)}_C$ vanish at $R = 0$[2] . Hence consistency demands that $S^{(1)}$ and $S^{(V_1)}_C$ should also vanish on $R = 0$. In other words, these set of equations could be consistently solved only if on the horizon the velocity field u_A and the extrinsic curvature of the $\psi = 1$ membrane (viewed as a hypersurface embedded in the background) together satisfy the following equations.

$$\begin{aligned} S^{(1)}|_{R=0} &= \left(\frac{K}{2}\right) E^{\text{scalar}}|_{R=0} = 0 \\ S^{(V_1)}_C|_{R=0} &= \left(\frac{K}{2}\right) E^{\text{vector}}_C|_{R=0} = 0 \end{aligned} \tag{3.19}$$

By appropriate pull-back these equations could be recast as an intrinsic equation on the hypersurface and they generate the next order correction to the constraint equation (3.2). We have described them in equations (3.8).

Once the constraint equations are satisfied, we could see that in the source S_{AB} only two scalar structures ($\mathfrak{s}_1$ and $\mathfrak{s}_2$), one vector structure ($\mathfrak{v}_C$) and one tensor structure ($\mathfrak{t}_{AB}$) appear. So altogether we have 6 unknown functions (2 functions for the scalar sector, 2 in the trace sector, 1 in the vector sector and 1 in the tensor sector).

The decoupled ODEs for different unknown metric functions:

- Scalar sector:

  For $h_n(R)$: $H^{(3)} + S^{(3)} = 0$ for $f_n(R)$: $H^{(1)} + S^{(1)} = 0, \quad n = 1, 2$

[2] To see the vanishing of $H^{(1)}$ at $R = 0$ we have to use the fact that $v_n(R)$ vanishes at $R = 0$ as a consequence of our boundary condition. See equation (3.10)

- Vector sector:

  For $v(R)$: $H_C^{(V_1)} + S_C^{(V_1)} = 0$

- Tensor sector:

  For $t(R)$: $H_{AB}^{(T)} + S_{AB}^{(T)} = 0$

Now we shall give the explicit form of the equations sector by sector.

Tensor sector:

Here the explicit form of the equation is as follows

$$t''(1-e^{-R}) + t' = \frac{2}{N^2}\, e^{-R} = 2\left(\frac{D}{K}\right)^2 e^{-R} \tag{3.20}$$

We can integrate this equation. After imposing

$$t(R=0) = \text{finite} \ \text{ and } \ \lim_{R\to\infty} t(R) = 0$$

we find the result as presented in the first equation of (3.6).

Vector sector:

Here the explicit form of the equation is as follows

$$(1-e^{-R})\frac{d}{dR}(e^R v') + 2\left(\frac{D}{K}\right)^3 R = 0 \tag{3.21}$$

After imposing

$$v(R=0) = 0 \ \text{ and } \ \lim_{R\to\infty} v(R) = 0$$

we find the result as presented in the second equation of (3.6).

Trace sector:

The equations for $h_n(R)$ is simply given by

$$-\frac{N^2}{2}\sum_n h_n''\mathfrak{s}_n = 0 \tag{3.22}$$

Integrating this differential equation with the boundary condition (3.10), we found correction in the trace sector vanishes i.e., $h_n(R) = 0$

Scalar sector:

The equations for $f_1(R)$ and $f_2(R)$ are given by

$$\begin{aligned} e^{-R}(1-e^{-R})\,\frac{d}{dR}\left[e^R f_1'\right] &= 2\left(\frac{D}{K}\right)^2 e^{-R}(1-e^{-R}) \\ e^{-R}(1-e^{-R})\,\frac{d}{dR}\left[e^R f_2'\right] &= -\left(\frac{D}{K}\right) e^{-R}\, v(R) + \left(\frac{D}{K}\right)^4 R^2\, e^{-2R} \end{aligned} \tag{3.23}$$

To derive the second equation we have used the fact (see appendix B.2.2 for derivation)

$$\frac{\nabla\cdot\mathfrak{v}}{D} = \mathfrak{s}_2 \tag{3.24}$$

After imposing

$$f_n(R=0) = 0 \;\text{ and }\; \lim_{R\to\infty} f_n(R) = 0, \quad n = 1, 2$$

we find the result as presented in the third and the fourth equation of (3.6).

3.3 Checks

In this section we shall perform several checks on our calculation. Roughly the checks could be of two types. The first is the internal consistency of our solutions and the systems of equations, i.e, to verify that if we simply substitute our solution in the system of equations (3.18), each and every component of it vanishes up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)$. The details of it would be presented in subsection 3.3.1.

The second type of checks are the ones where we shall take several limits and match our results with some answers, known previously. One trivial check in this category that we have tried on every stage of our computation is to match with the known results in asymptotically flat case [3], by setting the cosmological constant Λ to zero. The corrected constraint equation (3.8) manifestly matches with equation no (4.5) and (4.12) respectively of [3], if we set Λ to zero. At this stage it is difficult to match the two metrics even after setting Λ to zero, since our subsidiary conditions are different from that of [3] and we leave it for future.

The other significant check that we have performed is the matching of the spectrum of linearized fluctuation derived from our constraint equations to that of the Quasi-Normal modes already calculated in [10]. This linearized calculation is not included in this thesis, see [64] for details.

3.3.1 Check for internal consistency

In this subsection, we shall explicitly verify that our solution for the metric along with the membrane equations constraining the membrane data, does satisfy equation (3.18) i.e., each of its components vanishes up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)$.

Let $\mathcal{E}_{AB}$ denote the LHS of equation (3.18).

$$\mathcal{E}_{AB} \equiv H_{AB} + S_{AB}$$

From the list of the decoupled ODEs (see the discussion below equation (3.18)) it is clear that the 4 of the 7 independent components of $\mathcal{E}_{AB}$ must be satisfied since we have solved for the metric functions by integrating them. These components are

$$u^A u^B \mathcal{E}_{AB}, \;\; O^A O^B \mathcal{E}_{AB}, \;\; u^A P^C_B \mathcal{E}_{AC}, \;\; P^C_A P^{C'}_B \left[\mathcal{E}_{CC'} - \left(\frac{\mathcal{E}}{D-2}\right) P_{CC'}\right]$$

where $\mathcal{E}$ denotes the projected trace of $\mathcal{E}_{AB}$ i.e., $\mathcal{E} = P^{AB}\mathcal{E}_{AB}$

From the explicit expressions of H_{AB} it is clear that $u^A H_{AB} u^B = H^{(1)}$ and $u^A H_{AC} P^C_B = H^{(V_1)}_B$ vanish at $\psi = 1$ and membrane equations ensure that the same is true for the source. As explained in chapter 2, if we consider 'the variation of the metric as we go away from the horizon' as 'dynamics', then the membrane equations play the role of 'constraint equations', whereas the equations we solved to determine the metric corrections are like the 'dynamical' ones. Now in any theory of gravity, it is enough to solve the 'dynamical equations' everywhere and the constraint equation only along one constant 'time slice' (in our case which would be a constant ψ slice); gauge invariance will ensure that the full set of equations are solved everywhere [68]. This theorem guarantees that the rest of the three independent components of $\mathcal{E}_{AB}$ must vanish provided we have solved the equations correctly. These components are

$$\begin{aligned}
&u^A O^B \mathcal{E}_{AB} = H^{(2)} + S^{(2)} \equiv \mathcal{E}^{(2)} \\
&\frac{1}{D} P^{AB} \mathcal{E}_{AB} = H^{(tr)} + S^{(tr)} \equiv \mathcal{E}^{(tr)} \\
&O^A P^C_B \mathcal{E}_{AC} = H^{(V_2)}_B + S^{(V_2)}_B \equiv \mathcal{E}^{(V_2)}_B
\end{aligned}$$

Therefore the fact that these components do vanish on our solution is an important consistency check of our whole procedure and the final answer. Computationally it turns out to be quite non-trivial. In fact we have to take help from Mathematica to prove them.

Vanishing of $\mathcal{E}^{(2)}$

From eq (3.12) it follows that

$$\begin{aligned}
H^{(2)} &= \frac{1}{2}\left(\frac{K}{D}\right)^2 \sum_{n=1}^{2} \mathfrak{s}_n \left(f''_n + f'_n - \frac{e^{-R}}{2} h'_n\right) + \left(\frac{K}{D}\right)\left(\frac{\nabla\cdot\mathfrak{v}}{2D}\right) v' \\
&= \frac{1}{2}\left(\frac{K}{D}\right)^2 \mathfrak{s}_1 (f''_1 + f'_1) + \frac{1}{2}\left(\frac{K}{D}\right) \mathfrak{s}_2 \left[N\left(f''_2 + f'_2\right) + v'\right] \\
&= e^{-R}\mathfrak{s}_1 + \frac{1}{2}\left(\frac{K}{D}\right)\mathfrak{s}_2 \left[-\frac{e^{-R}}{1-e^{-R}}\, v + \left(\frac{D}{K}\right)^3 \frac{R^2\, e^{-R}}{1-e^{-R}} + v'\right] \\
&= e^{-R}\mathfrak{s}_1 - \frac{1}{2}\left(\frac{D}{K}\right)^2 R^2 e^{-R}\, \mathfrak{s}_2
\end{aligned} \tag{3.25}$$

Here we have used the fact that metric correction in the trace sector (i.e., $h_n(R)$) vanishes. Also we have used equation (3.24) for the divergence of $\mathfrak{v}_C$ and the last three equations from (3.6) for the expressions of $f_n(R)$ and $v(R)$.

From equation (3.14) we could see that $H^{(2)}$ is exactly the minus of $S^{(2)}$ as required.

Vanishing of $\mathcal{E}^{(tr)}$

This follows trivially from (3.14) and (3.12), as both $S^{(tr)}$ and $H^{(tr)}$ vanish at this order.

Vanishing of $\mathcal{E}_B^{(V_2)}$

From equation (3.14) we see that $S_C^{(V_2)} = 0$, therefore $H_C^{(V_2)}$ should also vanish on our solution. The equation below checks that this is true.

$$\begin{aligned} H_C^{(V_2)} &\equiv \frac{1}{2}\left(\frac{K}{D}\right)^2 (v'' + v')\,\mathfrak{v}_C + \frac{1}{2}\left(\frac{K}{D}\right) t' \frac{\left(\nabla_D \mathfrak{t}_C^D\right)}{D} \\ &= \frac{1}{2}\left(\frac{K}{D}\right)\left[\left(\frac{K}{D}\right)(v'' + v') + t'\right]\mathfrak{v}_C \\ &= 0 \end{aligned} \tag{3.26}$$

In the second line we have used the identity (see Appendix B.2.1 for the derivation),

$$\nabla_D \left(\mathfrak{t}_C^D\right) = D\,\mathfrak{v}_C \tag{3.27}$$

In the last line we have used the first and the second equation of (3.6) for the expressions of $v(R)$ and $t(R)$.

3.4 Discussions

In this chapter, we have found new dynamical 'black hole' type solutions of the Einstein's equation in presence of cosmological constant in an expansion in the inverse powers of dimension. We have done the calculation up to second subleading order. The spacetime, determined here, will necessarily possess an event horizon. The dynamics of the horizon could be mapped to the dynamics of a velocity field on a dynamical membrane, embedded in the asymptotic background. We have determined the equation for this dual dynamics of the membrane and the velocity field also in an expansion in $\left(\frac{1}{D}\right)$.

As a check we have matched the spectrum of the Quasi-Normal modes. This matching is not included in this thesis, see [64] for details. Another important check would be to match the metric with the large dimension limit of known black hole solutions. Apart from just a check on our results, this exercise could also give hints to some exact but non-trivial solutions of our membrane equations. This might lead to some techniques to solve the membrane equation analytically.

As we have mentioned in the introduction, one of our key motivation for this second subleading calculation is to have some insight in entropy production, which is expected to take place only at this order. Calculation of this entropy production could be one interesting project.

Chapter 4

Stress tensor for large-D membrane at sub-leading order

This chapter is based on [71].

4.1 Introduction

As mentioned in the introduction 1.2, here our main goal is to compute membrane stress tensor up to second subleading order in $\frac{1}{D}$ expansion. In our case, the membrane, which is a codimension-one hypersurface, is embedded in AdS/ dS space. More precisely, the metric of the embedding space satisfies the following equation

$$R_{AB} - \left(\frac{R - \Lambda}{2}\right) G_{AB} = 0$$

Where, dimension(D) dependence of Λ is parametrized as follows

$$\Lambda = [(D-1)(D-2)]\lambda, \quad \lambda \sim \mathcal{O}(1)$$

The membrane is characterized by its shape (encoded in its extrinsic curvature $\mathcal{K}_{\mu\nu}$) and a velocity field (u_μ), unit normalized with respect to the induced metric of the membrane. Before going into any details of the computation, we will first give the final answer. The membrane stress tensor, that we report below, is a symmetric two-indexed tensor, constructed out of this velocity field, extrinsic curvature and its derivatives.

4.1.1 Final Result

In this subsection, we shall write our final result - the expression of the membrane stress tensor up to order $\mathcal{O}\left(\frac{1}{D}\right)$. Conservation of this stress tensor would give the membrane equation derived in Chapter 3. For convenience, we shall decompose the stress tensor in the following way

$$\boxed{8\pi T_{\mu\nu} = \mathcal{S}_1\; u_\mu u_\nu + \mathcal{S}_2\; g^{(ind)}_{\mu\nu} + \mathcal{V}_\mu\; u_\nu + \mathcal{V}_\nu\; u_\mu + \mathcal{W}_{\mu\nu}} \tag{4.1}$$

Where,

$$\begin{split}\mathcal{S}_1 =& \frac{\mathcal{K}}{2} + \frac{1}{2}\bigg(\frac{\bar{\nabla}^2\mathcal{K}}{\mathcal{K}^2} - \lambda\,\frac{D-1}{\mathcal{K}} - \frac{1}{K}\mathcal{K}_{\alpha\beta}\mathcal{K}^{\alpha\beta}\bigg) + \frac{1}{\mathcal{K}}\bigg[- u\cdot\mathcal{K}\cdot\mathcal{K}\cdot u - 13\bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\bigg)^2\\
&+ 2\,u^\alpha\mathcal{K}_{\alpha\beta}\bigg(\frac{\bar{\nabla}^\beta\mathcal{K}}{\mathcal{K}}\bigg) + 14\,\bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\bigg)(u\cdot\mathcal{K}\cdot u) - \frac{\mathcal{K}}{D}\bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\bigg) + \frac{\mathcal{K}}{D}(u\cdot\mathcal{K}\cdot u) + \frac{1}{\mathcal{K}^3}\bar{\nabla}^2\left(\bar{\nabla}^2\mathcal{K}\right)\\
&- 4\,(u\cdot\mathcal{K}\cdot u)^2 - 8\,\lambda\frac{D}{\mathcal{K}}\bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\bigg) + 4\,\lambda\frac{D}{\mathcal{K}}\,(u\cdot\mathcal{K}\cdot u) - 2\,\bigg(\frac{\bar{\nabla}_\alpha\mathcal{K}}{\mathcal{K}}\bigg)\bigg(\frac{\bar{\nabla}^\alpha\mathcal{K}}{\mathcal{K}}\bigg) + \lambda - \lambda^2\,\frac{D^2}{\mathcal{K}^2}\bigg]\\
&+ \frac{1}{\mathcal{K}}\,(2\,\text{Zeta}[3]-1)\,\bigg[-\frac{\mathcal{K}}{D}\bigg(\frac{(u\cdot\nabla)\mathcal{K}}{\mathcal{K}} - u\cdot\mathcal{K}\cdot u\bigg) - \lambda - u\cdot\mathcal{K}\cdot\mathcal{K}\cdot u + 2\,\bigg(\frac{\nabla_\alpha\mathcal{K}}{\mathcal{K}}\bigg)\,u^\beta\mathcal{K}^\alpha_\beta\\
&- \bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\bigg)^2 + 2\,\bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\bigg)\,(u\cdot\mathcal{K}\cdot u) - \bigg(\frac{\bar{\nabla}^\alpha\mathcal{K}}{\mathcal{K}}\bigg)\bigg(\frac{\bar{\nabla}_\alpha\mathcal{K}}{\mathcal{K}}\bigg) - (u\cdot\mathcal{K}\cdot u)^2\bigg]\\
\mathcal{S}_2 =& -\frac{1}{2}\,(u\cdot\mathcal{K}\cdot u) - \frac{1}{2\mathcal{K}}\,\mathcal{K}^{\alpha\beta}\mathcal{K}_{\alpha\beta} - \frac{1}{\mathcal{K}}\bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} - \frac{1}{2}\,(u\cdot\mathcal{K}\cdot u) - \frac{\mathcal{K}}{2\,D}\bigg)\,(u\cdot\mathcal{K}\cdot u) + \frac{\lambda}{K}\\
&+ \frac{1}{\mathcal{K}}\,\mathcal{K}^{\alpha\beta}(\nabla_\alpha u_\beta) - \frac{2}{\mathcal{K}}\,u_\alpha\mathcal{K}^{\alpha\beta}\bigg(\frac{1}{2}\frac{\bar{\nabla}_\beta\mathcal{K}}{\mathcal{K}} - \frac{\bar{\nabla}^2 u_\beta}{\mathcal{K}}\bigg)\end{split} \tag{4.2}$$

$$\begin{split}\mathcal{V}_\mu =& \frac{1}{2}\bigg(\frac{\bar{\nabla}_\mu\mathcal{K}}{\mathcal{K}}\bigg) - \bigg(\frac{\bar{\nabla}^2 u_\mu}{\mathcal{K}}\bigg) + \frac{1}{\mathcal{K}}\mathcal{K}^\alpha_\mu\mathcal{K}_{\alpha\beta}u^\beta - \frac{1}{\mathcal{K}^3}\bar{\nabla}^2\left(\bar{\nabla}^2 u_\mu\right) + \frac{1}{\mathcal{K}}\bar{\nabla}_\mu\bigg(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\bigg)\\
&+ \frac{1}{\mathcal{K}}\bigg(\frac{\bar{\nabla}^2 u_\mu}{\mathcal{K}}\bigg)\bigg(-2\,(u\cdot\mathcal{K}\cdot u) + 4\,\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} + 2\,\lambda\frac{D}{\mathcal{K}} - \frac{\mathcal{K}}{D}\bigg) + \frac{1}{2\,\mathcal{K}}\bigg(\frac{\bar{\nabla}_\mu\mathcal{K}}{\mathcal{K}}\bigg)(u\cdot\mathcal{K}\cdot u)\end{split} \tag{4.3}$$

$$\begin{split}\mathcal{W}_{\mu\nu} =& \frac{1}{2}\mathcal{K}_{\mu\nu} - \frac{1}{2}\left(\bar{\nabla}_\mu u_\nu + \bar{\nabla}_\nu u_\mu\right) - \frac{1}{\mathcal{K}}\,\mathcal{K}_{\mu\nu}\,(u\cdot\mathcal{K}\cdot u) + \frac{1}{2\,\mathcal{K}}\left(\bar{\nabla}_\mu u_\nu + \bar{\nabla}_\nu u_\mu\right)(u\cdot\mathcal{K}\cdot u)\\
&+ \frac{1}{2\,\mathcal{K}}\bigg[\bar{\nabla}_\mu\bigg(\frac{\bar{\nabla}^2 u_\nu}{\mathcal{K}}\bigg) + \bar{\nabla}_\nu\bigg(\frac{\bar{\nabla}^2 u_\mu}{\mathcal{K}}\bigg) + \bar{\nabla}_\mu\left(u^\alpha\mathcal{K}_{\alpha\nu}\right) + \bar{\nabla}_\nu\left(u^\alpha\mathcal{K}_{\alpha\mu}\right) - 2\,\bar{\nabla}_\mu\bigg(\frac{\bar{\nabla}_\nu\mathcal{K}}{\mathcal{K}}\bigg)\bigg]\\
&- \frac{1}{\mathcal{K}}\left(\bar{\nabla}^\alpha u_\mu\right)\left(\bar{\nabla}_\alpha u_\nu\right) - \frac{1}{\mathcal{K}}\bigg(\frac{\bar{\nabla}^2 u_\mu}{\mathcal{K}}\bigg)\bigg(\frac{\bar{\nabla}^2 u_\nu}{\mathcal{K}}\bigg)\end{split} \tag{4.4}$$

Here, $g^{(ind)}_{\mu\nu}$ is the induced metric on the membrane, $\bar{\nabla}_\mu$ is the covariant derivative with respect to $g^{(ind)}_{\mu\nu}$. Membrane velocity u_μ can also be viewed as a vector field u_A in the full background spacetime. u_μ is related to u_A through the following equation

$$u_\mu = \left(\frac{\partial X^A}{\partial y^\mu}\right) u_A \tag{4.5}$$

Where, X^A are the coordinates in the full spacetime and y^μ are the coordinates on the membrane world volume.

The extrinsic curvature of the membrane $\mathcal{K}_{\mu\nu}$ is defined as follows

$$\mathcal{K}_{\mu\nu} = \left(\frac{\partial X^A}{\partial y^\mu}\right)\left(\frac{\partial X^B}{\partial y^\nu}\right) K_{AB}, \quad \text{Where,} \;\; K_{AB} = \Pi_A^C \nabla_C n_B \tag{4.6}$$

Here, n_A is the normal to the membrane and Π_{AB} is the projector orthogonal to the membrane defined as $\Pi_{AB} = g_{AB} - n_A n_B$.

4.1.2 Strategy

The two key principles that fix this stress tensor are the following

- Conservation of the stress tensor should reproduce the membrane equation up to the relevant order.
- This stress tensor should be the source of the gravitational radiation, generated from the massive fluctuating membrane.

In fact, it is the second principle that finally determines the algorithm to be used to derive the stress tensor. The algorithm is such that the first principle is automatically ensured and we have used it in the end as a consistency check for our long calculation.

Below, we shall just write down the steps to be used so that the final construction is consistent with the second principle. However, we shall not write the justification for any of these steps as they are explained in detail in [4] and explanation is completely independent of the order in terms of $\frac{1}{D}$ expansion.

- Step-1: Codimension-one membrane is given by a single scalar equation $\psi = 1$. Define $\psi > 1$ region as ‘ outside of the membrane’ and $\psi < 1$ as ‘ inside of the membrane’. ‘Outside region’ is the one that extends towards asymptotic infinity and contains the gravitational radiation.

- Step-2: Next, we would like to write a spacetime metric for both outside and inside region, with the following properties.

  1. The metric would solve Einstein’s equation (in presence of cosmological constant) linearized around pure AdS/dS metric.
  2. The metric would fall off as ψ^{-D} in the outside region and would be regular in the inside region.
  3. The metric should be continuous across the membrane though its first normal derivative need not be.

  It turns out that in $\frac{1}{D}$ expansion, the above two conditions uniquely fix the metric on both sides, in terms of the induced metric on the membrane, which we read off from the large-D metric determined in Chapter 3.

- Step-3: Once we have determined the metric on both sides, the discontinuity of its normal derivative across the membrane is also fixed unambiguously. The conserved stress tensor associated with the membrane is computed from this discontinuity. More precisely, it is the difference between the two Brown York stress tensors on the membrane evaluated with respect to the inside and outside metric.

$$T_{AB} = T^{(in)}_{AB} - T^{(out)}_{AB} \tag{4.7}$$

  Here,

$$8\pi T^{(in)}_{AB} = K^{(in)}_{AB} - K^{(in)}\mathfrak{p}^{(in)}_{AB} \text{ and, } 8\pi T^{(out)}_{AB} = K^{(out)}_{AB} - K^{(out)}\mathfrak{p}^{(in)}_{AB} \tag{4.8}$$

are respectively the Brown York stress tensors of internal solution and external solution evaluated on the membrane. $K^{(in)}_{AB}$ and $\mathfrak{p}^{(in)}_{AB}$ are respectively extrinsic curvature and projector on to the membrane viewed as a submanifold of the background spacetime perturbed by the internal solution. Similarly, $K^{(out)}_{AB}$ and $\mathfrak{p}^{(out)}_{AB}$ are respectively extrinsic curvature and projector on to the membrane viewed as a submanifold of the background spacetime perturbed by the external solution. $T^{(out)}_{AB}$ and $T^{(in)}_{AB}$ both satisfies $n^A T^{(out/in)}_{AB} = 0$. So, T_{AB} can equally well be regarded as a tensor $T_{\mu\nu}$ that lives on the membrane world volume.

Calculationally, this is very lengthy. In the main text, we have just written the final results, most of the lengthy derivations are in the appendices. The organization of this chapter is as follows: In section 4.2 we have linearized the Large -D solution known up to subleading order and have changed the gauge and subsidiary condition (as discussed just below eq.(4.9)). In section 4.3 we have constructed a linearized solution of Einstein's equation in the inside region of the membrane. In section 4.4 we have calculated the membrane stress tensor and in the section 4.5 we have shown that the subleading order membrane equation follows from the conservation of this stress tensor.

4.2 Linearized Solution : Outside($\psi > 1$)

In this section, we shall work out the metric in the outside region. However, what we are finally interested in is just the difference between Brown York stress tensor across the membrane. To compute it, we need to know the metric only very near the membrane. The large D solution as described in Chapter 3, already determined the metric in this near membrane region even at non-linear order. For our purpose, we shall simply read off the 'outside metric' from Chapter 3. In fact, we have to pick out only the part that is enough to solve the linearized equations. In other words, we need only that part of the metric which could be recast as

$$G^{(out)}_{AB} = g_{AB} + \psi^{-D}\mathfrak{h}_{AB} = g_{AB} + \psi^{-D}\sum_{m=0}^{\infty}(\psi-1)^m h^{(m)}_{AB} \tag{4.9}$$

In the first subsection, we have described the large-D solution and read off the piece needed.

The main calculation of this section involves a change of gauge and 'subsidiary conditions' (conventions that fix how the basic fields would evolve away from the membrane, see Chapter 2 for more details). In the next two subsections, we have described the new set of conventions, that are more useful for our purpose and performed the required changes on the metric, read off in the first subsection. Needless to say, all steps are worked out in an expansion in $\frac{1}{D}$.

4.2.1 Large-D Metric up to sub-subleading order : Linearized

In this subsection, we will just quote the solution of Einstein's equation up to second subleading order in $\frac{1}{D}$ expansion as derived in chapter 3 and we will linearize the solution in ψ^{-D}. The solution is given by

$$G_{AB} = g_{AB} + \psi^{-D}O_A O_B + \left(\frac{1}{D}\right)^2 G^{(2)}_{AB} + \cdots \tag{4.10}$$

Here, g_{AB} is the background metric and $O_A = n_A - u_A$.

$$\begin{split} G^{(2)}_{AB} = &\Big[O_A O_B \left(f_1(R)\,\mathfrak{s}_1 + f_2(R)\,\mathfrak{s}_2\right) + t(R)\,\mathfrak{t}_{AB} + v(R)\left(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\right)\Big] \\ &\text{where}\ \ R \equiv D(\psi-1),\ \ P_{AB} = g_{AB} - n_A n_B + u_A u_B \\ &\text{and,}\ \ n^A\,\mathfrak{v}_A = u^A\,\mathfrak{v}_A = 0,\ \ n^A\,\mathfrak{t}_{AB} = u^A\,\mathfrak{t}_{AB} = 0,\ \ g^{AB}\,\mathfrak{t}_{AB} = 0 \end{split} \tag{4.11}$$

Where,

$$
\begin{aligned}
t(R) &= -\,2\left(\frac{D}{K}\right)^2\int_R^\infty \frac{y\,dy}{e^y-1}\\
v(R) &= 2\left(\frac{D}{K}\right)^3\left[\int_R^\infty e^{-x}dx\int_0^x \frac{y\,e^y}{e^y-1}dy\;-\;e^{-R}\int_0^\infty e^{-x}dx\int_0^x \frac{y\,e^y}{e^y-1}dy\right]\\
f_1(R) &= -2\left(\frac{D}{K}\right)^2\int_R^\infty x\,e^{-x}dx+2\,e^{-R}\left(\frac{D}{K}\right)^2\int_0^\infty x\,e^{-x}dx\\
f_2(R) &= \left(\frac{D}{K}\right)\left[\int_R^\infty e^{-x}dx\int_0^x \frac{v(y)}{1-e^{-y}}dy-e^{-R}\int_0^\infty e^{-x}dx\int_0^x \frac{v(y)}{1-e^{-y}}dy\right]\\
&\quad-\left(\frac{D}{K}\right)^4\left[\int_R^\infty e^{-x}dx\int_0^x \frac{y^2\,e^{-y}}{1-e^{-y}}dy-e^{-R}\int_0^\infty e^{-x}dx\int_0^x \frac{y^2\,e^{-y}}{1-e^{-y}}dy\right]
\end{aligned}
\tag{4.12}
$$

And,

$$
\begin{aligned}
\mathfrak{t}_{AB} &= P_A^C P_B^D\Bigg[\bar{R}_{FCDE}O^E O^F+\frac{K}{D}\Bigg(K_{CD}-\frac{\hat{\nabla}_C u_D+\hat{\nabla}_D u_C}{2}\Bigg)\\
&\quad-P^{EF}(K_{EC}-\hat{\nabla}_E u_C)(K_{FD}-\hat{\nabla}_F u_D)\Bigg]\\
\mathfrak{v}_A &= P_A^B\Bigg[\frac{K}{D}\left(n^D u^E O^F \bar{R}_{FBDE}\right)+\frac{K^2}{2D^2}\left(\frac{\hat{\nabla}_B K}{K}+(u\cdot\hat{\nabla})u_B-2\,u^D K_{DB}\right)\\
&\quad-P^{FD}\left(\frac{\hat{\nabla}_F K}{D}-\frac{K}{D}(u^E K_{EF})\right)\left(K_{DB}-\hat{\nabla}_D u_B\right)\Bigg]\\
\mathfrak{s}_1 &= u^E u^F n^D n^C \bar{R}_{CEFD}+\left(\frac{u\cdot\hat{\nabla}K}{K}\right)^2+\frac{\hat{\nabla}_A K}{K}\left[4\,u^B K^A_B-2\left[(u\cdot\hat{\nabla})u^A\right]-\frac{\hat{\nabla}^A K}{K}\right]\\
&\quad-(\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B)-(u\cdot K\cdot u)^2-\left[(u\cdot\hat{\nabla})u_A\right]\left[(u\cdot\hat{\nabla})u^A\right]+2\left[(u\cdot\hat{\nabla})u^A\right](u^B K_{BA})\\
&\quad-3\,(u\cdot K\cdot K\cdot u)-\frac{K}{D}\left(\frac{u\cdot\hat{\nabla}K}{K}-u\cdot K\cdot u\right)\\
\mathfrak{s}_2 &= \frac{K^2}{D^2}\Bigg[-\frac{K}{D}\left(\frac{u\cdot\hat{\nabla}K}{K}-u\cdot K\cdot u\right)-2\,\lambda-(u\cdot K\cdot K\cdot u)+2\left(\frac{\hat{\nabla}_A K}{K}\right)u^B K^A_B-\left(\frac{u\cdot\hat{\nabla}K}{K}\right)^2\\
&\quad+2\left(\frac{u\cdot\hat{\nabla}K}{K}\right)(u\cdot K\cdot u)-\left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right)-(u\cdot K\cdot u)^2+n^B n^D u^E u^F \bar{R}_{FBDE}\Bigg]
\end{aligned}
\tag{4.13}
$$

Here, $\bar{R}_{ABCD}$ is the Riemann tensor of the background metric g_{AB} and $\hat{\nabla}$ is defined through the following equation - for a generic n-index tensor $W_{A_1A_2\cdots A_n}$

$$\hat{\nabla}_A W_{A_1A_2\cdots A_n} = \Pi^C_A \Pi^{C_1}_{A_1} \Pi^{C_2}_{A_2} \cdots \Pi^{C_n}_{A_n} \nabla_C W_{C_1C_2\cdots C_n} \tag{4.14}$$

We want the sub-subleading order metric in linearized order in ψ^{-D}. So, we need to calculate the above integration (4.12) in linearized order in ψ^{-D}. The answers are the following. See C.1 for details.

$$\begin{aligned}
t(R) &= -2\left(\frac{D}{K}\right)^2 e^{-R}\,[R+1] + \mathcal{O}\left(e^{-2R}\right)\\
v(R) &= 2\left(\frac{D}{K}\right)^3 \left(1+R+\frac{R^2}{2}\right) e^{-R} + \mathcal{O}\left(e^{-2R}\right)\\
f_1(R) &= -2\left(\frac{D}{K}\right)^2 R\,e^{-R} + \mathcal{O}\left(e^{-2R}\right)\\
f_2(R) &= 2\left(\frac{D}{K}\right)^4 e^{-R}\,(2\,\text{Zeta}[3]-1) + \mathcal{O}(e^{-2R})
\end{aligned} \tag{4.15}$$

Using (4.15), we can write the full metric G_{AB} as

$$\begin{aligned}
G_{AB} &= g_{AB} + \psi^{-D} O_A O_B + \psi^{-D}\frac{1}{D^2}\Bigg[-2\left(\frac{D}{K}\right)^2 (R+1)\,\mathfrak{t}_{AB} - 2\left(\frac{D}{K}\right)^2 R\,\mathfrak{s}_1\,O_A O_B\\
&\quad + 2\left(\frac{D}{K}\right)^4 (2\,\text{Zeta}[3]-1)\,\mathfrak{s}_2\,O_A O_B + 2\left(\frac{D}{K}\right)^3\left(1+R+\frac{R^2}{2}\right)\left(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\right)\Bigg]\\
&= g_{AB} + \psi^{-D}\left[O_A O_B + \frac{1}{K^2}\left\{2\frac{D^2}{K^2}(2\,\text{Zeta}[3]-1)\,\mathfrak{s}_2 O_A O_B - 2\,\mathfrak{t}_{AB} + 2\frac{D}{K}\left(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\right)\right\}\right]\\
&\quad + R\,\psi^{-D}\frac{1}{K^2}\left[-2\,\mathfrak{t}_{AB} - 2\,\mathfrak{s}_1\,O_A O_B + 2\left(\frac{D}{K}\right)\left(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\right)\right]\\
&\quad + R^2\,\psi^{-D}\frac{1}{K^2}\left(\frac{D}{K}\right)\left(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\right) + \mathcal{O}\left(\frac{1}{D^3}, \psi^{-2D}\right)
\end{aligned} \tag{4.16}$$

Now, if we write G_{AB} as

$$G_{AB} = g_{AB} + \psi^{-D} M_{AB} = g_{AB} + \psi^{-D}\sum_{n=0}^{\infty}(\psi-1)^n M^{(n)}_{AB} \tag{4.17}$$

We will get

$$
\begin{aligned}
M^{(0)}_{AB} &= O_A O_B + \frac{2}{K^2}\Big[-\mathfrak{t}_{AB} + \Big(\frac{D}{K}\Big)^2 (2\,\text{Zeta}[3]-1)\,\mathfrak{s}_2 O_A O_B + \frac{D}{K}\big(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\big)\Big] + \mathcal{O}\Big(\frac{1}{D}\Big)^3 \\
M^{(1)}_{AB} &= -\frac{2D}{K^2}\Big[\mathfrak{t}_{AB} + \mathfrak{s}_1\, O_A O_B - \frac{D}{K}\big(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\big)\Big] + \mathcal{O}\Big(\frac{1}{D}\Big)^2 \\
M^{(2)}_{AB} &= \Big(\frac{D}{K}\Big)^3\big(\mathfrak{v}_A O_B + \mathfrak{v}_B O_A\big) + \mathcal{O}\Big(\frac{1}{D}\Big)
\end{aligned}
\tag{4.18}
$$

4.2.2 Change of Gauge Condition

Large-D solution (Chapter 2 and Chapter 3) has been derived in the gauge condition $O^A h_{AB} = 0$. But it turns out that, for the calculation of the stress tensor, it is more convenient to use the gauge condition $n^A h_{AB} = 0$. In this subsection, we will implement this gauge transformation.

$$G_{AB} = g_{AB} + \psi^{-D} M_{AB} \tag{4.19}$$

We do the following infinitesimal coordinate transformation

$$x^A \to x'^A = x^A - \psi^{-D}\xi^A(x^A) \tag{4.20}$$

Under the above coordinate transformation, metric transforms as follows

$$G'_{AB}(x') = G_{AB}(x') + \nabla'_A\left[\psi^{-D}\xi_B(x')\right] + \nabla'_B\left[\psi^{-D}\xi_A(x')\right] \tag{4.21}$$

Now, using (4.17), we get

$$\boxed{M'_{AB} = M_{AB} + \psi^D\nabla_A\left[\psi^{-D}\xi_B\right] + \psi^D\nabla_B\left[\psi^{-D}\xi_A\right]} \tag{4.22}$$

We choose the coordinate transformation in a way such that $n^A M'_{AB} = 0$. Now using the expansion $\xi_A = \sum_{n=0}^{\infty}(\psi-1)^n \xi^{(n)}_A$ we get

$$-n^A\sum_{m=0}^{\infty}(\psi-1)^m M^{(m)}_{AB} = \psi^D(n\cdot\nabla)\left[\psi^{-D}\sum_{m=0}^{\infty}(\psi-1)^m\xi^{(m)}_B\right] + \psi^D n^A\nabla_B\left[\psi^{-D}\sum_{m=0}^{\infty}(\psi-1)^m\xi^{(m)}_A\right] \tag{4.23}$$

Now, using the following decomposition

$$\begin{aligned}
\xi_B^{(0)} &= \xi_B^{(0,0)} + \frac{1}{D}\xi_B^{(0,1)} + \frac{1}{D^2}\xi_B^{(0,2)} + \frac{1}{D^3}\xi_B^{(0,3)} + \mathcal{O}\left(\frac{1}{D}\right)^4 \\
\xi_B^{(1)} &= \xi_B^{(1,0)} + \frac{1}{D}\xi_B^{(1,1)} + \frac{1}{D^2}\xi_B^{(1,2)} + \mathcal{O}\left(\frac{1}{D}\right)^3 \\
\xi_B^{(2)} &= \xi_B^{(2,0)} + \frac{1}{D}\xi_B^{(2,1)} + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned} \tag{4.24}$$

from (4.23), we can determine $\xi_A^{(m,n)}$ order by order in $\frac{1}{D}$ expansion in terms of $M_{AB}^{(n)}$. See Appendix C.2 for details. Different components of $\xi_B^{(2)}$ become

$$\begin{aligned}
\xi_B^{(2,0)} &= 0 \\
\xi_B^{(2,1)} &= \frac{1}{N}\left[n^A M_{AB}^{(2)} + n^A M_{AB}^{(1)} - \frac{n_B}{2}\left(n\cdot M^{(2)}\cdot n + n\cdot M^{(1)}\cdot n\right)\right]
\end{aligned} \tag{4.25}$$

Different components of $\xi_B^{(1)}$ become

$$\begin{aligned}
\xi_B^{(1,0)} &= 0 \\
\xi_B^{(1,1)} &= \frac{1}{N}\left[n^A M_{AB}^{(1)} + n^A M_{AB}^{(0)} - \frac{n_B}{2}\left(n\cdot M^{(1)}\cdot n + n\cdot M^{(0)}\cdot n\right)\right] \\
\xi_B^{(1,2)} &= \frac{1}{N}\left[(n\cdot\nabla)\xi_B^{(1,1)} + (n\cdot\nabla)\xi_B^{(0,1)}\right] + \frac{1}{N}\left[n^A\nabla_B\xi_A^{(1,1)} + n^A\nabla_B\xi_A^{(0,1)}\right] \\
&\quad + 2\,\xi_B^{(2,1)} + \xi_B^{(1,1)} - \frac{n_B}{N}\left[n^A(n\cdot\nabla)\xi_A^{(1,1)} + n^A(n\cdot\nabla)\xi_A^{(0,1)}\right]
\end{aligned} \tag{4.26}$$

Different components of $\xi_B^{(0)}$ become

$$\begin{aligned}
\xi_B^{(0,0)} &= 0 \\
\xi_B^{(0,1)} &= \frac{1}{N}\left[n^A M_{AB}^{(0)} - \frac{n_B}{2}\left(n\cdot M^{(0)}\cdot n\right)\right] \\
\xi_B^{(0,2)} &= \frac{1}{N}\left[(n\cdot\nabla)\xi_B^{(0,1)} + n^A\nabla_B\xi_A^{(0,1)}\right] + \xi_B^{(1,1)} - \frac{n_B}{N}\left[n^A(n\cdot\nabla)\xi_A^{(0,1)}\right] \\
\xi_B^{(0,3)} &= \frac{1}{N}\left[(n\cdot\nabla)\xi_B^{(0,2)} + n^A\nabla_B\xi_A^{(0,2)}\right] + \xi_B^{(1,2)} - \frac{n_B}{N}\left[n^A(n\cdot\nabla)\xi_A^{(0,2)}\right]
\end{aligned} \tag{4.27}$$

Using (4.25), (4.26) and (4.27) we can calculate M'_{AB} from (4.22). We expect the final answer to be fully projected and that is what we get. See Appendix C.2 for details.

$$\begin{aligned}
M'_{AB} &= \Pi_A^C\Pi_B^{C'}\left[M_{CC'}^{(0)} + (\psi-1)M_{CC'}^{(1)} + (\psi-1)^2 M_{CC'}^{(2)}\right] + \hat{\nabla}_A\xi_B^{(0)} + \hat{\nabla}_B\xi_A^{(0)} \\
&\quad + (\psi-1)\left(\hat{\nabla}_A\xi_B^{(1)} + \hat{\nabla}_B\xi_A^{(1)}\right) + \mathcal{O}\left(\frac{1}{D}\right)^3
\end{aligned} \tag{4.28}$$

Using (4.18), (C.53) and (C.54) we can finally write M'_{AB} as

$$\boxed{M'_{AB} = (\psi - 1)^m M'^{(m)}_{AB}} \tag{4.29}$$

Where,

$$\begin{aligned}
M'^{(0)}_{AB} = {} & u_A u_B + \frac{1}{\psi K}\Big[u_A \frac{\hat{\nabla}_B K}{K} + u_B \frac{\hat{\nabla}_A K}{K} + K_{AB} - \hat{\nabla}_B u_A - \hat{\nabla}_A u_B\Big] \\
& + \frac{2}{K^2}\Big[- \mathfrak{t}_{AB} + \frac{D^2}{K^2}\left(2\,\text{Zeta}[3] - 1\right)\mathfrak{s}_2\, u_A u_B - \frac{D}{K}\big(\mathfrak{v}_A u_B + \mathfrak{v}_B u_A\big)\Big] \\
& + \frac{1}{K^2}\Big[- \frac{(n\cdot\nabla)K}{K}\Big(4\,u_A \frac{\hat{\nabla}_B K}{K} + K_{AB} - 2\,\hat{\nabla}_B u_A\Big) + 2\,u_A \hat{\nabla}_B\Big(\frac{n\cdot\nabla K}{K}\Big) \\
& + \hat{\nabla}_B\left\{u^E K_{AE} - (n\cdot\nabla)u_A\right\} - 2\frac{\hat{\nabla}_B K}{K}\left\{u^E K_{AE} - \Pi^C_A (n\cdot\nabla)u_C\right\}\Big] \\
& + \frac{1}{K^2}\Big[- \frac{n\cdot\nabla K}{K}\Big(4\,u_B \frac{\hat{\nabla}_A K}{K} + K_{AB} - 2\,\hat{\nabla}_A u_B\Big) + 2\,u_B \hat{\nabla}_A\Big(\frac{n\cdot\nabla K}{K}\Big) \\
& + \hat{\nabla}_A\left\{u^E K_{BE} - (n\cdot\nabla)u_B\right\} - 2\frac{\hat{\nabla}_A K}{K}\left\{u^E K_{BE} - \Pi^C_B (n\cdot\nabla)u_C\right\}\Big] + \mathcal{O}\Big(\frac{1}{D}\Big)^3
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
M'^{(1)}_{AB} = {} & -\frac{2D}{K^2}\Big[\mathfrak{t}_{AB} + \mathfrak{s}_1\, u_A u_B + \frac{D}{K}\big(\,\mathfrak{v}_A u_B + \mathfrak{v}_B u_A\big)\Big] \\
& + \frac{1}{K}\Big[u_A \frac{\hat{\nabla}_B K}{K} + u_B \frac{\hat{\nabla}_A K}{K} + K_{AB} - \hat{\nabla}_B u_A - \hat{\nabla}_A u_B\Big] + \mathcal{O}\Big(\frac{1}{D}\Big)^2
\end{aligned} \tag{4.31}$$

4.2.3 Change of Subsidiary Condition

$M'^{(m)}_{AB}$ can not yet be identified with $h^{(m)}_{AB}$ - we have used in the calculation of the stress tensor. Because, we have imposed the condition $\Pi^C_A \Pi^{C'}_B (n.\nabla) h^{(m)}_{CC'} = 0$ on $h^{(m)}_{CC'}$. We will expand $M'^{(m)}_{AB}$ in a power series expansion in $(\psi - 1)$ and will determine different coefficients by satisfying $\Pi^C_A \Pi^{C'}_B (n.\nabla) h^{(m)}_{CC'} = 0$.

We define $h^{(0)}_{AB}$ in the following way such that $\Pi^C_A \Pi^{C'}_B (n.\nabla) h^{(0)}_{AB} = 0$

$$h^{(0)}_{AB} = M'^{(0)}_{AB} - (\psi - 1) C^{(0)}_{AB} - (\psi - 1)^2 E^{(0)}_{AB} + \mathcal{O}(\psi - 1)^3 \tag{4.32}$$

Acting on the above equation by $\Pi^C_A \Pi^{C'}_B (n.\nabla)$ and them equating the coefficient of $(\psi - 1)^0$ we get

$$C^{(0)}_{AB} = \frac{1}{N} \Pi^C_A \Pi^{C'}_B (n.\nabla) M'^{(0)}_{CC'} \tag{4.33}$$

Equating the coefficient of $(\psi - 1)$ we get

$$E^{(0)}_{CC'} = -\frac{1}{2N}\Pi^A_C \Pi^B_{C'}(n.\nabla)C^{(0)}_{AB} \tag{4.34}$$

The final form form of $h^{(0)}_{AB}$ on $\psi = 1$ takes the following form. See Appendix C.2.1 for details

$$\boxed{h^{(0)}_{AB} = \mathcal{S}^{(0)}\, u_A u_B + u_A \mathcal{H}^{(0)}_B + u_B \mathcal{H}^{(0)}_A + \mathcal{W}^{(0)}_{AB}} \tag{4.35}$$

Where,

$$\begin{aligned}
\mathcal{S}^{(0)} = 1 &- \frac{2}{K^2}\bigg[u\cdot K\cdot K\cdot u - 3\left(\frac{(u\cdot\nabla)K}{K}\right)^2 - 2\,u^B K_{BD}\left(\frac{\hat{\nabla}^D K}{K}\right) + 2\,u\cdot K\cdot u\,\left(\frac{(u\cdot\nabla)K}{K}\right)\\
&+ \frac{K}{D}\left(\frac{(u\cdot\nabla)K}{K}\right) - \frac{K}{D}(u\cdot K\cdot u)\bigg]\\
&+ \frac{2}{K^2}\,(2\,\text{Zeta}[3]-1)\,\bigg[-\frac{K}{D}\left(\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right) - \lambda - u\cdot K\cdot K\cdot u + 2\left(\frac{\nabla_A K}{K}\right)u^B K^A_B\\
&- \left(\frac{(u\cdot\nabla)K}{K}\right)^2 + 2\,\frac{(u\cdot\nabla)K}{K}(u\cdot K\cdot u) - \left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right) - (u\cdot K\cdot u)^2\bigg]
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
\mathcal{H}^{(0)}_A = &\frac{1}{K}\left(\frac{\hat{\nabla}_A K}{K}\right) + \frac{2}{K^2}\hat{\nabla}_A\left(\frac{\hat{\nabla}^2 K}{K^2}\right) + \frac{2}{K^2}K^F_A\left(\frac{\hat{\nabla}_F K}{K}\right) - \frac{2}{K^2}\left(\hat{\nabla}^F u_A\right)\left(\frac{\hat{\nabla}_F K}{K}\right)\\
&+ \frac{2}{K^2}\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left[u\cdot K\cdot u - 2\,\frac{(u\cdot\nabla)K}{K}\right] + \frac{2}{K^2}\left(\frac{\hat{\nabla}_A K}{K}\right)\left[u\cdot K\cdot u - 2\,\frac{(u\cdot\nabla)K}{K} + \lambda\,\frac{D}{K} + \frac{K}{2D}\right]
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
\mathcal{W}^{(0)}_{AB} = &\frac{1}{K}\left[K_{AB} - \hat{\nabla}_A u_B - \hat{\nabla}_B u_A\right]\\
&- \frac{2}{K^2}\,K_{AB}\left[\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right] - \frac{2}{K^2}\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right)\left[\frac{K}{2D} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u\right]\\
&+ \frac{2}{K^2}K^F_A K_{FB} - \frac{2}{K^2}\left(K^F_A\hat{\nabla}_F u_B + K^F_B\hat{\nabla}_F u_A\right) + \frac{2}{K^2}(\hat{\nabla}^F u_A)(\hat{\nabla}_F u_B) + \frac{2}{K^2}\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left(\frac{\hat{\nabla}^2 u_B}{K}\right)\\
&+ \frac{2}{K^2}\left(\frac{\hat{\nabla}_A K}{K}\right)\left(\frac{\hat{\nabla}_B K}{K}\right) - \frac{2}{K^2}\left[\left(\frac{\hat{\nabla}_A K}{K}\right)u^E K_{EB} + \left(\frac{\hat{\nabla}_B K}{K}\right)u^E K_{EA}\right]\\
&+ \frac{1}{K^2}\left[\hat{\nabla}_A\left(u^E K_{EB}\right) + \hat{\nabla}_B\left(u^E K_{EA}\right)\right] - \frac{1}{K^2}\left[\hat{\nabla}_A\left(\frac{\hat{\nabla}^2 u_B}{K}\right) + \hat{\nabla}_B\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\right]
\end{aligned} \tag{4.38}$$

Now, $h^{(1)}_{AB}$ on the surface $\psi = 1$ becomes

$$h^{(1)}_{AB} = M'^{(1)}_{AB} + C^{(0)}_{AB} \tag{4.39}$$

The final form form of $h^{(1)}_{AB}$ on $\psi = 1$ takes the following form. See Appendix C.2.1 for details

$$\boxed{h^{(1)}_{AB} = \mathcal{S}^{(1)}\, u_A u_B + u_A \mathcal{H}^{(1)}_B + u_B \mathcal{H}^{(1)}_A + \mathcal{W}^{(1)}_{AB}} \tag{4.40}$$

Where

$$\mathcal{S}^{(1)} = -2\,\lambda \left(\frac{D}{K^2}\right) \tag{4.41}$$

$$\begin{aligned}
\mathcal{H}^{(1)}_A =&\ \frac{D}{K}\left(\frac{\hat{\nabla}^2 u_A}{K}\right) + \frac{D}{K^2}\left(\frac{\hat{\nabla}_A K}{K}\right)\left[-5\,\frac{(u\cdot\nabla)K}{K} + 2\,u\cdot K\cdot u - \lambda\,\frac{D}{K}\right] \\
&+ \frac{D}{K^2}\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left[-12\,\frac{(u\cdot\nabla)K}{K} + 6\,u\cdot K\cdot u - 2\,\lambda\,\frac{D}{K} + 2\,\frac{K}{D}\right] \\
&+ \frac{D}{K^2}\left[-u^B K_{BD} K^D_A + \frac{1}{K^2}\hat{\nabla}^2\left(\hat{\nabla}^2 u_A\right) - 3\left(\frac{\hat{\nabla}^B K}{K}\right)\hat{\nabla}_B u_A + \frac{1}{K^2}\hat{\nabla}_A\left(\hat{\nabla}^2 K\right) + K^D_A\left(\frac{\hat{\nabla}_D K}{K}\right)\right]
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
\mathcal{W}^{(1)}_{AB} =&\ \frac{D}{K^2}\left[u\cdot K\cdot u - \frac{K}{D}\right] K_{AB} + \frac{D}{K^2}\left[\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D}{K}\right]\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right) + \frac{D}{K^2}\,K^F_A K_{FB} \\
&- \frac{D}{K^2}\,\lambda\,\Pi_{AB} - \frac{D}{K^2}\left(K^F_A\,\hat{\nabla}_F u_B + K^F_B\,\hat{\nabla}_F u_A\right) + 2\,\frac{D}{K^2}\left(\hat{\nabla}^F u_A\right)\left(\hat{\nabla}_F u_B\right) \\
&+ 2\,\frac{D}{K^2}\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left(\frac{\hat{\nabla}^2 u_B}{K}\right) + \frac{D}{K^2}\frac{1}{K}\hat{\nabla}_A\left(\hat{\nabla}_B K\right) - \frac{D}{K^2}\left[\left(\frac{\hat{\nabla}_A K}{K}\right)u^E K_{EB} + \left(\frac{\hat{\nabla}_B K}{K}\right)u^E K_{EA}\right] \\
&- \frac{D}{K^2}\frac{1}{K}\left[\hat{\nabla}_A\left(\hat{\nabla}^2 u_B\right) + \hat{\nabla}_B\left(\hat{\nabla}^2 u_A\right)\right] + \frac{D}{K^2}\left[\left(\frac{\hat{\nabla}_A K}{K}\right)\left(\frac{\hat{\nabla}^2 u_B}{K}\right) + \left(\frac{\hat{\nabla}_B K}{K}\right)\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\right]
\end{aligned} \tag{4.43}$$

So, finally, we have brought the large-D solution in the following form

$$G^{(out)}_{AB} = g_{AB} + \psi^{-D}\mathfrak{h}_{AB} = g_{AB} + \psi^{-D}\sum_{m=0}^{\infty}(\psi - 1)^m h^{(m)}_{AB} \tag{4.44}$$

Where, $h^{(m)}_{AB}$ satisfies $n^A h^{(m)}_{AB} = 0$ and $\Pi^A_C \Pi^B_{C'}(n\cdot\nabla)h^{(m)}_{AB} = 0$

4.3 Linearized Solution : Inside($\psi < 1$)

In this section, we shall construct the 'inside solution' i.e, the metric for region $\psi < 1$. As we have mentioned before, we want this metric to be regular throughout the 'inside region' in order to make sure that the membrane is the sole source of the gravitational radiation in this system.

Note that the solution presented in Chapter 3 continued to be a solution even when $\psi < 1$. However, this solution diverges at the location of the black hole, the point where ψ approaches zero and also it does not have any discontinuity across the event horizon - the location of the membrane. Therefore, unlike the 'outside solution' we have to construct the inside solution from scratch maintaining the regularity and the fact that on the membrane it reduces to the same induced metric as the one read off from the 'outside solution'.

We shall write the inside metric in the following form

$$G^{(in)}_{AB} = g_{AB} + \tilde{\mathfrak{h}}_{AB} = g_{AB} + \sum_{m=0}^{\infty}(\psi - 1)^m \tilde{h}^{(m)}_{AB} \tag{4.45}$$

Where, g_{AB} is background metric. $\tilde{h}^{(m)}_{AB}$ satisfies the gauge condition $n^A \tilde{h}^{(m)}_{AB} = 0$. At linearized order, Christoffel symbol for (4.45) is given by

$$\Gamma^A_{BC} = \bar{\Gamma}^A_{BC} + \underbrace{\frac{1}{2} g^{AC'} \left[\nabla_C \tilde{\mathfrak{h}}_{BC'} + \nabla_B \tilde{\mathfrak{h}}_{CC'} - \nabla_{C'} \tilde{\mathfrak{h}}_{BC}\right]}_{\delta\Gamma^A_{BC}} + \mathcal{O}\left(\tilde{\mathfrak{h}}\right)^2 \tag{4.46}$$

Where, $\bar{\Gamma}^A_{BC}$ is Christoffel symbol of g_{AB} and ∇_A is covariant derivative with respect to g_{AB}. Now, Ricci tensor is given by

$$R^{(in)}_{AB} = \bar{R}_{AB} + \nabla_D \left[\delta\Gamma^D_{AB}\right] - \nabla_B \left[\delta\Gamma^D_{AD}\right] \tag{4.47}$$

Where, $\bar{R}_{AB}$ is Ricci tensor for g_{AB}.

$$\delta\Gamma^A_{BA} = \frac{1}{2} g^{AC'} \left[\nabla_A \tilde{\mathfrak{h}}_{BC'} + \nabla_B \tilde{\mathfrak{h}}_{AC'} - \nabla_A \tilde{\mathfrak{h}}_{BC'}\right] = \frac{1}{2} \nabla_B \tilde{\mathfrak{h}} \tag{4.48}$$

Where, $\tilde{\mathfrak{h}} = g^{AC'}\tilde{\mathfrak{h}}_{AC'}$. So, Ricci tensor for inside region($\psi < 1$)

$$R^{(in)}_{AB} = \bar{R}_{AB} + \frac{1}{2}\nabla_D\nabla_A\tilde{\mathfrak{h}}^D_B + \frac{1}{2}\nabla_D\nabla_B\tilde{\mathfrak{h}}^D_A - \frac{1}{2}\nabla^2\tilde{\mathfrak{h}}_{AB} - \frac{1}{2}\nabla_B\nabla_A\tilde{\mathfrak{h}} \tag{4.49}$$

Einstein's equation in the inside region

$$\begin{aligned} & R^{(\text{in})}_{AB} - (D-1)\lambda\, G^{(\text{in})}_{AB} = 0 \\ \Rightarrow & \frac{1}{2}\nabla_D\nabla_A\tilde{\mathfrak{h}}^D_B + \frac{1}{2}\nabla_D\nabla_B\tilde{\mathfrak{h}}^D_A - \frac{1}{2}\nabla^2\tilde{\mathfrak{h}}_{AB} - \frac{1}{2}\nabla_B\nabla_A\tilde{\mathfrak{h}} - (D-1)\lambda\tilde{\mathfrak{h}}_{AB} = 0 \end{aligned} \tag{4.50}$$

Projecting the above equation perpendicular to n_A and n_B we get

$$\begin{aligned} \Pi^A_C\Pi^B_{C'}\Big[& \nabla_A\nabla_E\tilde{\mathfrak{h}}^E_B + \nabla_B\nabla_E\tilde{\mathfrak{h}}^E_A - \nabla^2\tilde{\mathfrak{h}}_{AB} - \nabla_B\nabla_A\tilde{\mathfrak{h}} + 2\bar{R}_{EABC}\tilde{\mathfrak{h}}^{EC} \\ & + \bar{R}_{AC}\tilde{\mathfrak{h}}^C_B + \bar{R}_{BC}\tilde{\mathfrak{h}}^C_A - 2(D-1)\lambda\tilde{\mathfrak{h}}_{AB}\Big] = 0 \end{aligned} \tag{4.51}$$

Using the following decomposition for $\tilde{h}^{(1)}_{AB}$

$$\tilde{h}^{(1)}_{AB} = \tilde{h}^{(1,1)}_{AB} + \frac{1}{D}\tilde{h}^{(1,2)}_{AB} \tag{4.52}$$

We can solve for $\tilde{h}^{(1,1)}_{AB}, \tilde{h}^{(1,2)}_{AB}, \tilde{h}^{(2)}_{AB}$ by solving (4.51) order by order in $\frac{1}{D}$ expansion. The final form form of $\tilde{h}^{(1,1)}_{AB}$ on $\psi = 1$ takes the following form. See Appendix C.2.2 for details

$$\boxed{\tilde{h}^{(1,1)}_{CC'} = \tilde{\mathcal{S}}^{(1,1)}\, u_C u_{C'} + u_C\tilde{\mathcal{H}}^{(1,1)}_{C'} + u_{C'}\tilde{\mathcal{H}}^{(1,1)}_C + \tilde{\mathcal{W}}^{(1,1)}_{CC'}} \tag{4.53}$$

where,

$$\tilde{\mathcal{S}}^{(1,1)} = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{4.54}$$

$$\begin{aligned} \tilde{\mathcal{H}}^{(1,1)}_C = & -\frac{D}{K}\left(\frac{\hat{\nabla}^2 u_C}{K}\right) + \frac{D}{K^2}\left[\frac{\hat{\nabla}^2 K}{K^2} - \lambda\frac{D}{K} - \frac{K}{D}\right]\left(\frac{\hat{\nabla}^2 u_C}{K}\right) - \frac{D}{K^4}\hat{\nabla}_C\left(\hat{\nabla}^2 K\right) - \frac{D}{K^2}K^D_C\left(\frac{\hat{\nabla}_D K}{K}\right) \\ & + \frac{D}{K^2}K^F_C K_{FD}u^D + \frac{D}{K^2}\left(\frac{\hat{\nabla}^F K}{K}\right)\left(\hat{\nabla}_F u_C\right) + \frac{D}{K^2}\left(\frac{\hat{\nabla}_C K}{K}\right)\left[2\,\frac{\hat{\nabla}^2 K}{K^2} + \frac{u\cdot\nabla K}{K} - \lambda\frac{D}{K}\right] \end{aligned} \tag{4.55}$$

$$
\begin{aligned}
\tilde{\mathcal{W}}^{(1,1)}_{CC'} =& -2\frac{D}{K^2}\left(\hat{\nabla}^D u_C\right)\left(\hat{\nabla}_D u_{C'}\right) - 2\frac{D}{K^2}(u\cdot K\cdot u)K_{CC'} + \lambda\frac{D}{K^2}\Pi_{CC'} \\
&- \frac{D}{K^2}\left[\frac{\hat{\nabla}^2 K}{K^2} - \lambda\frac{D}{K}\right]\left(\hat{\nabla}_C u_{C'} + \hat{\nabla}_{C'} u_C\right) - \frac{D}{K^2}\left[\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}_{C'} K}{K}\right) + \left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right)\left(\frac{\hat{\nabla}_C K}{K}\right)\right] \\
&+ \frac{D}{K^2}\left[\left(\frac{\hat{\nabla}_C K}{K}\right)u^F K_{FC'} + \left(\frac{\hat{\nabla}_{C'} K}{K}\right)u^F K_{FC}\right] - \frac{D}{K^2}K^E_C K_{EC'} - \frac{D}{K^2}\frac{1}{K}\hat{\nabla}_C\left(\hat{\nabla}_{C'} K\right) \\
&+ \frac{D}{K^2}\left[K^D_C\left(\hat{\nabla}_D u_{C'}\right) + K^D_{C'}\left(\hat{\nabla}_D u_C\right)\right] + \frac{D}{K^2}\frac{1}{K}\left[\hat{\nabla}_C\left(\hat{\nabla}^2 u_{C'}\right) + \hat{\nabla}_{C'}\left(\hat{\nabla}^2 u_C\right)\right]
\end{aligned} \tag{4.56}
$$

The final form form of $\tilde{h}^{(1,2)}_{AB}$ on $\psi = 1$ takes the following form. See Appendix C.2.2 for details

$$
\boxed{\tilde{h}^{(1,2)}_{CC'} = \tilde{\mathcal{S}}^{(1,2)}\, u_C u_{C'} + u_C \tilde{\mathcal{H}}^{(1,2)}_{C'} + u_{C'}\tilde{\mathcal{H}}^{(1,2)}_C + \tilde{\mathcal{W}}^{(1,2)}_{CC'}} \tag{4.57}
$$

Where,

$$
\tilde{\mathcal{S}}^{(1,2)} = 2\,\lambda\left(\frac{D}{K}\right)^2 \tag{4.58}
$$

$$
\begin{aligned}
\tilde{\mathcal{H}}^{(1,2)}_C =& \frac{D}{K}\left[-1 + \frac{D}{K}\left(\frac{\hat{\nabla}^2 K}{K^2}\right) + \lambda\,\frac{D^2}{K^2}\right]\left(\frac{\hat{\nabla}^2 u_C}{K}\right) \\
&+ \frac{D^2}{K^2}\left[-\left(\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2}\right) - 2\left(\frac{\hat{\nabla}^E K}{K}\right)\left(\hat{\nabla}_E u_C\right) + 2\hat{\nabla}_C\left(\frac{(u\cdot\nabla)K}{K}\right)\right]
\end{aligned} \tag{4.59}
$$

$$
\tilde{\mathcal{W}}^{(1,2)}_{CC'} = -2\,\frac{D^2}{K^2}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) + 2\,\frac{D^2}{K^2}\left(\hat{\nabla}_C u_{C'} + \hat{\nabla}_{C'} u_C\right)\frac{(u\cdot\nabla)K}{K} \tag{4.60}
$$

The final form form of $\tilde{h}^{(2)}_{AB}$ on $\psi = 1$ takes the following form. See Appendix C.2.2 for details

$$
\boxed{\tilde{h}^{(2)}_{CC'} = \tilde{\mathcal{S}}^{(2)}\, u_C u_{C'} + u_C \tilde{\mathcal{H}}^{(2)}_{C'} + u_{C'}\tilde{\mathcal{H}}^{(2)}_C + \tilde{\mathcal{W}}^{(2)}_{CC'}} \tag{4.61}
$$

Where,

$$
\tilde{\mathcal{S}}^{(2)} = \mathcal{O}\left(\frac{1}{D}\right) \tag{4.62}
$$

$$
\tilde{\mathcal{H}}^{(2)}_C = \frac{D}{K}\left[-\frac{1}{2} - 2\,\frac{D}{K}\left(\frac{\hat{\nabla}^2 K}{K^2}\right) + \lambda\,\frac{D^2}{K^2}\right]\left(\frac{\hat{\nabla}^2 u_C}{K}\right) + \frac{D^2}{2K^2}\left[\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2} - 2\left(\frac{\hat{\nabla}^E K}{K}\right)\left(\hat{\nabla}_E u_C\right)\right] \tag{4.63}
$$

$$
\tilde{\mathcal{W}}^{(2)}_{CC'} = \frac{D^2}{K^2}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) \tag{4.64}
$$

Adding (4.53) and (4.57) we get

$$\boxed{\tilde{h}^{(1)}_{CC'} = \tilde{\mathcal{S}}^{(1)}\, u_C u_{C'} + u_C \tilde{\mathcal{H}}^{(1)}_{C'} + u_{C'} \tilde{\mathcal{H}}^{(1)}_{C} + \tilde{\mathcal{W}}^{(1)}_{CC'}} \tag{4.65}$$

Where,

$$\tilde{\mathcal{S}}^{(1)} = 2\,\lambda\,\frac{D}{K^2} \tag{4.66}$$

$$\begin{aligned}
\tilde{\mathcal{H}}^{(1)}_{C} = & -\frac{D}{K}\left(\frac{\hat{\nabla}^2 u_C}{K}\right) + \frac{D}{K^2}\left[2\,\frac{\hat{\nabla}^2 K}{K^2} - 2\,\frac{K}{D}\right]\left(\frac{\hat{\nabla}^2 u_C}{K}\right) - \frac{D}{K^4}\hat{\nabla}_C\left(\hat{\nabla}^2 K\right) - \frac{D}{K^2}K^D_C\left(\frac{\hat{\nabla}_D K}{K}\right) \\
& + \frac{D}{K^2}K^F_C K_{FD}u^D - \frac{D}{K^2}\left(\frac{\hat{\nabla}^F K}{K}\right)\left(\hat{\nabla}_F u_C\right) + \frac{D}{K^2}\left(\frac{\hat{\nabla}_C K}{K}\right)\left[2\,\frac{\hat{\nabla}^2 K}{K^2} + \frac{u\cdot\nabla K}{K} - \lambda\frac{D}{K}\right] \\
& - \frac{D}{K^2}\left(\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2}\right) + 2\,\frac{D}{K^2}\hat{\nabla}_C\left(\frac{(u\cdot\nabla)K}{K}\right)
\end{aligned} \tag{4.67}$$

$$\begin{aligned}
\tilde{\mathcal{W}}^{(1)}_{CC'} = & -2\frac{D}{K^2}\left(\hat{\nabla}^D u_C\right)\left(\hat{\nabla}_D u_{C'}\right) - 2\frac{D}{K^2}(u\cdot K\cdot u)K_{CC'} + \lambda\frac{D}{K^2}\Pi_{CC'} \\
& - \frac{D}{K^2}\left[\frac{\hat{\nabla}^2 K}{K^2} - \lambda\frac{D}{K} - 2\,\frac{(u\cdot\nabla)K}{K}\right]\left(\hat{\nabla}_C u_{C'} + \hat{\nabla}_{C'} u_C\right) \\
& - \frac{D}{K^2}\left[2\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) + \left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}_{C'} K}{K}\right) + \left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right)\left(\frac{\hat{\nabla}_C K}{K}\right)\right] \\
& + \frac{D}{K^2}\left[\left(\frac{\hat{\nabla}_C K}{K}\right)u^F K_{FC'} + \left(\frac{\hat{\nabla}_{C'} K}{K}\right)u^F K_{FC}\right] - \frac{D}{K^2}K^E_C K_{EC'} - \frac{D}{K^2}\frac{1}{K}\hat{\nabla}_C\left(\hat{\nabla}_{C'} K\right) \\
& + \frac{D}{K^2}\left[K^D_C\left(\hat{\nabla}_D u_{C'}\right) + K^D_{C'}\left(\hat{\nabla}_D u_C\right)\right] + \frac{D}{K^2}\frac{1}{K}\left[\hat{\nabla}_C\left(\hat{\nabla}^2 u_{C'}\right) + \hat{\nabla}_{C'}\left(\hat{\nabla}^2 u_C\right)\right]
\end{aligned} \tag{4.68}$$

4.4 Stress Tensor

In this section, we will derive the expression for membrane stress tensor. The membrane stress tensor is given by the discontinuity of the Brown-York stress tensor across the membrane.[1]

[1]See subsection 3.3 of [4] for detailed discussion on this

4.4.1 Outside($\psi > 1$) Stress Tensor

The outside stress tensor is given by

$$8\pi T^{(out)}_{AB} = K^{(out)}_{AB} - K^{(out)}\mathfrak{p}^{(out)}_{AB}\Big|_{\psi=1} \tag{4.69}$$

Where, $\mathfrak{p}^{(out)}_{AB} = G^{(out)}_{AB} - n^{(out)}_A n^{(out)}_B$; $G^{(out)}_{AB} = g_{AB} + \psi^{-D}\mathfrak{h}_{AB}$; $n^{(out)}_A = \dfrac{\partial_A\psi}{\sqrt{G^{AB}_{(out)}\partial_A\psi\,\partial_B\psi}}$

$$K^{(out)}_{AB} = \left[\mathfrak{p}^{(out)}\right]^C_A \left[\mathfrak{p}^{(out)}\right]^{C'}_B \left(\tilde{\nabla}_C n^{(out)}_{C'}\right)_{\psi=1} \tag{4.70}$$

Where, $\mathfrak{p}^{(out)}_{AB} = G^{(out)}_{AB} - n^{(out)}_A n^{(out)}_B$ and, $\tilde{\nabla}$ is covariant derivative with respect to $G^{(out)}_{AB}$ (4.71)

The final expression for $K^{(out)}_{AB}$ and $K^{(out)}$ are the followings. See Appendix C.3 for details.

$$\begin{aligned} K^{(out)}_{AB} &= K_{AB} - \frac{ND}{2}h^{(0)}_{AB} + \frac{N}{2}h^{(1)}_{AB} + \frac{1}{2}\left(h^{(0)}_{BD}K^D_A + h^{(0)}_{AD}K^D_B\right) \\ K^{(out)} &= K - \frac{ND}{2}h^{(0)} + \frac{N}{2}h^{(1)} \end{aligned} \tag{4.72}$$

Putting the expression for $K^{(\mathrm{out})}_{AB}$ and $K^{(\mathrm{out})}$ from (4.72) in (4.69) we get the final expression of $T^{(out)}_{AB}$.

$$\begin{aligned} 8\pi T^{(out)}_{AB} = K_{AB} &- \frac{ND}{2}h^{(0)}_{AB} + \frac{N}{2}h^{(1)}_{AB} + \frac{1}{2}\left(h^{(0)}_{BD}K^D_A + h^{(0)}_{AD}K^D_B\right) \\ &- \left(\Pi_{AB} + h^{(0)}_{AB}\right)\left(K - \frac{ND}{2}h^{(0)} + \frac{N}{2}h^{(1)}\right) \end{aligned} \tag{4.73}$$

4.4.2 Inside($\psi < 1$) Stress Tensor

The inside stress tensor is given by

$$8\pi T^{(in)}_{AB} = K^{(in)}_{AB} - K^{(in)}\mathfrak{p}^{(in)}_{AB}\Big|_{\psi=1} \tag{4.74}$$

$$\text{Where, } \mathfrak{p}^{(in)}_{AB} = G^{(in)}_{AB} - n^{(in)}_A n^{(in)}_B;\; G^{(in)}_{AB} = g_{AB} + \tilde{\mathfrak{h}}_{AB};\; n^{(in)}_A = \frac{\partial_A\psi}{\sqrt{G^{AB}_{(in)}\partial_A\psi\,\partial_B\psi}} \tag{4.75}$$

Now,

$$K_{AB}^{(in)} = \left[\mathfrak{p}^{(in)}\right]_A^C \left[\mathfrak{p}^{(in)}\right]_B^{C'} \left(\breve{\nabla}_C n_{C'}^{(in)}\right)_{\psi=1} \tag{4.76}$$

Where, $\mathfrak{p}_{AB}^{(in)} = G_{AB}^{(in)} - n_A^{(in)} n_B^{(in)}$ and, $\breve{\nabla}$ is covariant derivative with respect to $G_{AB}^{(in)}$ (4.77)

The final expression for $K_{AB}^{(in)}$ and $K^{(in)}$ are the followings. See Appendix C.3 for details.

$$\begin{aligned} K_{AB}^{(\text{in})} &= K_{AB} + \frac{1}{2}\left(\tilde{h}_{BF}^{(0)} K_A^F + \tilde{h}_{AF}^{(0)} K_B^F + N\tilde{h}_{AB}^{(1)}\right) \\ K^{(\text{in})} &= K + \frac{N}{2}\tilde{h}^{(1)} \end{aligned} \tag{4.78}$$

Putting the expression for $K_{AB}^{(in)}$ and $K^{(in)}$ from (4.78) in (4.74) and using the fact that $\tilde{h}_{AB}^{(0)} = h_{AB}^{(0)}$ we get the final expression of $T_{AB}^{(in)}$.

$$8\pi T_{AB}^{(\text{in})} = K_{AB} + \frac{1}{2}\left(h_{BF}^{(0)} K_A^F + h_{AF}^{(0)} K_B^F + N\tilde{h}_{AB}^{(1)}\right) - \left(\Pi_{AB} + h_{AB}^{(0)}\right)\left(K + \frac{N}{2}\tilde{h}^{(1)}\right) \tag{4.79}$$

4.4.3 Membrane Stress Tensor

Membrane stress tensor is given by

$$\begin{aligned} 8\pi T_{AB} &= 8\pi\left[T_{AB}^{(\text{in})} - T_{AB}^{(\text{out})}\right] \\ &= \frac{ND}{2}\left[h_{AB}^{(0)} - \Pi_{AB} h^{(0)}\right] - \frac{N}{2}\left[h_{AB}^{(1)} - \tilde{h}_{AB}^{(1)} - \Pi_{AB}\left(h^{(1)} - \tilde{h}^{(1)}\right)\right] + \mathcal{O}\left(h\right)^2 \end{aligned} \tag{4.80}$$

We can simplify the calculation of stress tensor by using a trick. We define

$$8\pi T_{AB}^{(NT)} = \frac{ND}{2} h_{AB}^{(0)} - \frac{N}{2}\left[h_{AB}^{(1)} - \tilde{h}_{AB}^{(1)}\right] \tag{4.81}$$

Then from (4.80) we can very easily see that $T_{AB} - T_{AB}^{(NT)} \propto \Pi_{AB}$. Let's call this proportionality factor Δ. With this notation membrane stress tensor becomes

$$8\pi T_{AB} = 8\pi\left[T_{AB}^{(NT)} + \Delta\Pi_{AB}\right] \tag{4.82}$$

Now, from the condition $K^{AB}T_{AB} = 0$ we get

$$8\pi\, \Delta = -\frac{1}{K}\, 8\pi \left(K^{AB} T^{(NT)}_{AB}\right) \tag{4.83}$$

Using (4.35), (4.40), (4.65) and identity (C.161) in (4.81) and after some simplification we get the final form of $T^{(NT)}_{AB}$ as

$$8\pi T^{(NT)}_{AB} = \mathcal{S}_1\, u_A u_B + \mathcal{V}_A\, u_B + \mathcal{V}_B\, u_A + \widetilde{\mathcal{W}}_{AB} \tag{4.84}$$

Where,

$$\begin{aligned}
\mathcal{S}_1 =& \frac{K}{2} + \frac{1}{2}\bigg(\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D-1}{K} - \frac{1}{K}K_{AB}K^{AB}\bigg) \\
&+ \frac{1}{K}\bigg[- u\cdot K\cdot K\cdot u - 13\bigg(\frac{u\cdot\nabla K}{K}\bigg)^2 + 2\, u^B K_{BD}\bigg(\frac{\hat{\nabla}^D K}{K}\bigg) + 14\,\bigg(\frac{u\cdot\nabla K}{K}\bigg)(u\cdot K\cdot u) \\
&- \frac{K}{D}\bigg(\frac{u\cdot\nabla K}{K}\bigg) + \frac{K}{D}(u\cdot K\cdot u) + \frac{1}{K^3}\hat{\nabla}^2\left(\hat{\nabla}^2 K\right) - 4\,(u\cdot K\cdot u)^2 - 8\,\lambda\frac{D}{K}\,\bigg(\frac{u\cdot\nabla K}{K}\bigg) \\
&+ 4\,\lambda\frac{D}{K}\,(u\cdot K\cdot u) - 2\,\bigg(\frac{\hat{\nabla}_B K}{K}\bigg)\bigg(\frac{\hat{\nabla}^B K}{K}\bigg) + \lambda - \lambda^2\,\frac{D^2}{K^2}\bigg] \\
&+ \frac{1}{K}\,(2\,\text{Zeta}[3]-1)\,\bigg[-\frac{K}{D}\bigg(\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\bigg) - \lambda - u\cdot K\cdot K\cdot u + 2\,\bigg(\frac{\nabla_A K}{K}\bigg)\, u^B K^A_B \\
&- \bigg(\frac{u\cdot\nabla K}{K}\bigg)^2 + 2\,\bigg(\frac{u\cdot\nabla K}{K}\bigg)\,(u\cdot K\cdot u) - \bigg(\frac{\hat{\nabla}^D K}{K}\bigg)\bigg(\frac{\hat{\nabla}_D K}{K}\bigg) - (u\cdot K\cdot u)^2\bigg]
\end{aligned} \tag{4.85}$$

$$\begin{aligned}
\mathcal{V}_A =& \frac{1}{2}\bigg(\frac{\hat{\nabla}_A K}{K}\bigg) - \bigg(\frac{\hat{\nabla}^2 u_A}{K}\bigg) + \frac{1}{K}K^F_A K_{FD}u^D - \frac{1}{K^3}\hat{\nabla}^2\left(\hat{\nabla}^2 u_A\right) + \frac{1}{K}\hat{\nabla}_A\bigg(\frac{u\cdot\nabla K}{K}\bigg) \\
&+ \frac{1}{K}\bigg(\frac{\hat{\nabla}^2 u_A}{K}\bigg)\bigg(-2\,(u\cdot K\cdot u) + 4\,\frac{u\cdot\nabla K}{K} + 2\,\lambda\frac{D}{K} - \frac{K}{D}\bigg) + \frac{1}{2\,K}\bigg(\frac{\hat{\nabla}_A K}{K}\bigg)(u\cdot K\cdot u)
\end{aligned} \tag{4.86}$$

And,

$$\begin{aligned}
\widetilde{\mathcal{W}}_{AB} =& \frac{1}{2}K_{AB} - \frac{1}{2}\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right) - \frac{1}{K}\,K_{AB}\,(u\cdot K\cdot u) + \frac{1}{2\,K}\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right)(u\cdot K\cdot u) \\
&- \frac{1}{K}\left(\hat{\nabla}^F u_A\right)\left(\hat{\nabla}_F u_B\right) - \frac{1}{K}\bigg(\frac{\hat{\nabla}^2 u_A}{K}\bigg)\bigg(\frac{\hat{\nabla}^2 u_B}{K}\bigg) + \frac{\lambda}{K}\,\Pi_{AB} \\
&+ \frac{1}{2\,K}\bigg[\hat{\nabla}_A\bigg(\frac{\hat{\nabla}^2 u_B}{K}\bigg) + \hat{\nabla}_B\bigg(\frac{\hat{\nabla}^2 u_A}{K}\bigg) + \hat{\nabla}_A\left(u^E K_{EB}\right) + \hat{\nabla}_B\left(u^E K_{EA}\right) - 2\,\hat{\nabla}_A\bigg(\frac{\hat{\nabla}_B K}{K}\bigg)\bigg]
\end{aligned} \tag{4.87}$$

Now, we can calculate Δ

$$
\begin{aligned}
8\pi\Delta &= -\frac{1}{K}\,8\pi\left(K^{AB}T^{(NT)}_{AB}\right)\\
&= -\frac{1}{2}\,(u\cdot K\cdot u) - \frac{1}{2K}\,K^{AB}K_{AB} - \frac{1}{2K}\left(\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D}{K} - \frac{K}{D}\right)(u\cdot K\cdot u)\\
&\quad - \frac{2}{K}\,u_A K^{AB}\left(\frac{1}{2}\frac{\hat{\nabla}_B K}{K} - \frac{\hat{\nabla}^2 u_B}{K}\right) + \frac{1}{K}\,K^{AB}(\nabla_A u_B)
\end{aligned}
\tag{4.88}
$$

So, the full stress tensor becomes

$$
\boxed{8\pi T_{AB} = \mathcal{S}_1\,u_A u_B + \mathcal{V}_A\,u_B + \mathcal{V}_B\,u_A + \widetilde{\mathcal{W}}_{AB} + \tilde{\mathcal{S}}_2\,\Pi_{AB}}
\tag{4.89}
$$

Where, $\mathcal{S}_1$, $\mathcal{V}_A$, $\widetilde{\mathcal{W}}_{AB}$ are given respectively by (4.85), (4.86), (4.87) and $\widetilde{\mathcal{S}}_2$ is given by

$$
\begin{aligned}
\widetilde{\mathcal{S}}_2 &= -\frac{1}{2}\,(u\cdot K\cdot u) - \frac{1}{2K}\,K^{AB}K_{AB} - \frac{1}{2K}\left(\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D}{K} - \frac{K}{D}\right)(u\cdot K\cdot u)\\
&\quad - \frac{2}{K}\,u_A K^{AB}\left(\frac{1}{2}\frac{\hat{\nabla}_B K}{K} - \frac{\hat{\nabla}^2 u_B}{K}\right) + \frac{1}{K}\,K^{AB}(\nabla_A u_B)
\end{aligned}
\tag{4.90}
$$

4.5 Conservation of the Membrane Stress Tensor

The final expression of membrane stress tensor (4.1) is very large. It would be quite difficult to calculate the divergence of stress tensor by hand. We have written a $Mathematica$ code to calculate the divergence of the stress tensor, and verified that the divergence of the membrane stress tensor indeed gives the membrane equation. Specifically, we have checked the followings

- $u^A\hat{\nabla}^B T_{AB}$ gives scalar membrane equation (eq.(3.8) of Chapter 3)
- $P^A_C\hat{\nabla}^B T_{AB}$ gives vector membrane equation (eq.(3.8) of Chapter 3)

Here, we want to make some comments about how we have done the large-D calculation in *Mathematica*. We choose the following background metric

$$
ds^2 = -e^{2r}dt^2 + dr^2 + e^{2r}dx_a dx^a + e^{2r}dx_i dx^i
\tag{4.91}
$$

which is pure AdS metric written in a slightly different coordinates than usual Poincare patch coordinates($r \to \log r$ will give usual Poincare patch metric). Here, 'a' runs over some finite p dimension and i runs over large $D - p - 2$ dimension. ψ and u_A are only functions of (t, r, x_a) and does not depend on x_i. We can effectively do our calculation in finite $p + 2$ dimension. We will calculate the contribution that will come from the large $D - p - 2$ dimension by hand and will accordingly take into account. For example, if we want to calculate $\hat{\nabla}^B \hat{\nabla}_B u_A$ (where A, B runs over full D dimension), the first thing to note is that it has non zero component only along 'a' direction and it is given by

$$\hat{\nabla}^B \hat{\nabla}_B u_a = \hat{\nabla}^b \hat{\nabla}_b u_a + \frac{1}{2} \frac{D-p-2}{e^{2r}} \left(\hat{\nabla}^b e^{2r}\right) \left(\hat{\nabla}_b u_a\right) - \frac{D-p-2}{4\, e^{4r}} \left(\hat{\nabla}_a e^{2r}\right) \left[(u \cdot \partial) e^{2r}\right] \tag{4.92}$$

Where $\hat{\nabla}_b$ is projected covariant derivative with respect to finite $p+2$ dimensional metric. Similarly, we can calculate all the quantities appearing in the expression of the stress tensor.

4.6 Discussions

In this Chapter, we have calculated the membrane stress tensor up to order $\mathcal{O}\left(\frac{1}{D}\right)$ and showed that the conservation of this stress tensor gives the subleading order membrane equation.

Very briefly, our procedure is as follows : given the large-D solution outside the membrane - linearize the solution - search for a regular solution inside the membrane region with the condition that the induced metric is continuous on both sides of the membrane - construct the Brown York stress tensor for inside and outside region - the difference of the Brown York stress tensor across the membrane is the membrane stress tensor.

As it turns out, the computation leading to the stress tensor at subsubleading orders is extremely tedious, though the final result is relatively compact and simple (presented in Section 4.1.1). Still one might wonder what is the point of taking up such a calculation.

The key motivation we have already mentioned in the introduction 1.2. It is about the finite D completion of membrane stress tensor [56].

Though this second order membrane stress tensor is just a small step towards this final goal. We think, the following would be the next few steps, which might help to construct a finite D completion of the membrane stress tensor (if it exists), by generating more data

- A detailed matching with the hydrodynamic stress tensor dual to the same gravity system in the regime of overlap for these two perturbation techniques (namely $\frac{1}{D}$ expansion and derivative expansion (see [66,72])). Now after computing the membrane stress tensor, we could extend this matching to include the effect of the gravitational radiation as well.

- Recasting known rotating black hole solutions in arbitrary D, in the language of large D expansion, capturing few terms that could contribute in a stationary situation, to all orders.

- Finally, evaluating the second order membrane stress tensor on the rotating black holes, hoping some novel pattern or truncation would emerge out of this exercise, that will tell us in general how stationarity is encoded in this large-D expansion technique.

We find all of the above projects are interesting, themselves. They will teach us a lot about how perturbation works in gravity and how they could be used to have analytic control over the otherwise difficult to handle dynamics of gravitating systems. We leave all these for future work.

Chapter 5

Comparison between ‘Fluid-Gravity’ and ‘Membrane-Gravity’ dualities

This chapter is based on [72].

As discussed in the introduction 1.3, here we will describe a comparison between ‘Fluid-Gravity’ and ‘Membrane-Gravity’ dualities up to first subleading order on both sides. The organization of this chapter is as follows.

In section 5.1 we first discussed the overlap regime of these two perturbation schemes. Next in the section 5.2 we discussed the map between the bulk of the ‘black-hole’ spacetime and the pure AdS mentioned above, and described an algorithm to construct the map, whenever it exists. In section 5.3 we compared the two metrics and the two sets of dual equations (controlling the fluid-dynamics and the membrane dynamics respectively) within the overlap regime, up to the first subleading order on both sides. This section contains the main calculation of this chapter. We worked out the map between these two sets of dual variables, leading to a map between large D relativistic hydrodynamics and the membrane dynamics. Finally, in the section 5.5 we concluded and discussed the future directions.

5.1 The overlap regime

In this section, we shall discuss whether we could apply both ‘derivative expansion’ and $\left(\frac{1}{D}\right)$ expansion simultaneously. We shall first define the perturbation parameters for both these two techniques in a precise way and also fix the range of their validity. We shall see that these two parameters are completely independent of each other and therefore their ratio could be tuned to any value, large or small.

Next, we shall compare the forms of the two metrics, determined using these two techniques, assuming the ratio (between the two perturbation parameters) to have any arbitrary value.

5.1.1 Perturbation parameter in 'derivative expansion'

Here we shall very briefly describe the method of 'derivative expansion'. See [63] for a more elaborate discussion.

The technique of 'derivative expansion' could be applied to construct a certain class of solutions to Einstein's equation in the presence of negative cosmological constant in arbitrary dimension D.

$$\begin{gathered}\textit{The key gravity equation:}\\ \mathcal{E}_{AB} \equiv R_{AB} + (D-1)\lambda^2 g_{AB} = 0\end{gathered} \tag{5.1}$$

λ is the inverse of AdS radius. From now on, we shall choose units such that λ is set to one. These gravity solutions are of 'black hole' type, meaning they would necessarily have a singularity shielded by some horizon [59]. They are in one-to-one correspondence with the solutions of relativistic Navier-Stokes equations in $(D-1)$ dimensional flat spacetime (without any restriction on the value of D). In fact, we could use the hydrodynamic variables themselves to label the different gravity solutions, constructed using this technique of 'derivative expansion'. The labeling hydrodynamic variables are

1. Unit normalized velocity: $u^\mu(x)$
2. Local temperature: $T(x) = \left(\frac{D-1}{4\pi}\right) r_H(x)$

At the moment r_H is just some arbitrary length scale, which would eventually be related to the horizon scale of the dual black brane metric.

$\{x^\mu\}, \quad \mu = \{0, 1, \cdots, D-2\}$ are the coordinates on the flat spacetime whose metric is simply given by the Minkowski metric, $\eta_{\mu\nu} = Diag\{-1, 1, 1, 1\cdots\}$.

'Derivative expansion' enters right into the definition of the hydrodynamic limit. The velocity and the temperature of fluid are functions of spacetime but the functional dependence must be slow with respect to the length scale $r_H(x)$. For a generic fluid flow at a generic point, it implies the following.

Choose an arbitrary point x_0^μ; scale the coordinates (or set the units) such that in the transformed coordinate $r_H(x_0) = 1$. Now the technique of derivative expansion would be applicable provided in this scaled coordinate system

$$\begin{aligned} &|\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_n} r_H|_{x_0} << |\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_{n-1}} r_H|_{x_0} << \cdots << |\bar\partial_{\alpha_1} r_H|_{x_0} << 1 \quad \forall\, n, \alpha_i, x_0 \\ &|\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_n} u^\mu|_{x_0} << |\bar\partial_{\alpha_1}\bar\partial_{\alpha_2}\cdots\bar\partial_{\alpha_{n-1}} u^\mu|_{x_0} << \cdots << |\bar\partial_{\alpha_1} u^\mu|_{x_0} << |u^\mu| \quad \forall\, n, \alpha_i, x_0 \end{aligned} \tag{5.2}$$

In other words, the number of ∂_α derivatives in a given term determines how suppressed the term is[1]. In terms of original x^μ coordinates, each derivative ∂_μ corresponds to $r_H\bar\partial_\mu$. Therefore if we work in x^μ (which, unlike $\bar x^\mu$, are not defined around any given point) coordinates, the parameter that controls the perturbation is schematically $\sim r_H^{-1}\partial_\mu$. [2].

The starting point of this perturbation is a boosted black brane in asymptotically AdS space. The metric has the following form

(in coordinates denoted as $\{r, x^\mu\}, \quad \mu = \{0, 1, \cdots, D-2\}$. Units are chosen so that

[1] The conditions as described in (5.2) are for a generic situation. For a particular fluid profile, it could happen that at a given point in spacetime some nth order term is comparable to or even smaller than some $(n+1)$th order term. One might have to rearrange the fluid expansion around such anomalous points if they exist, but they do not imply a 'breakdown' of hydrodynamic approximation. As long as all derivatives in appropriate dimensionless coordinates are suppressed compared to one, 'derivative expansion' could be applied.

[2] For a conformal fluid in finite dimension, there is only one length scale, set by the local temperature which also sets the scale of derivative expansion. But if we take $D \to \infty$, $T(x)$ and $r_H \sim \frac{T(x)}{D}$ are two parametrically separated scales and it becomes important to know which one among these two scales controls the derivative expansion. In the condition (5.2) we have chosen r_H to be the relevant scale and set it to order $\mathcal{O}(1)$. Indeed the results in [61] seem to indicate that terms of different derivative orders in hydrodynamic stress tensor, dual to gravity are weighted by factors of $r_H \sim \frac{T(x)}{D}$, and not T alone.
Note that here the temperature of the fluid would scale as D, which is different from the D scaling of the temperature, imposed in [46].

dimensionful constant, λ, appearing in equation (5.1) is set to one)[3].

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2\, f\left(r/r_H\right)\; u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ \text{where}\quad f(z) &= 1 - \frac{1}{z^{D-1}},\;\; P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu \end{aligned} \tag{5.3}$$

Equation (5.3) is an exact solution to equation (5.1) provided u_μ and r_H are constants.

Now the algorithm for 'derivative expansion' runs as follows. Suppose, u^μ and r_H are not constants but are functions of $\{x^\mu\}$. Equation (5.3) will no longer be a solution. If we evaluate the gravity equation $\mathcal{E}_{AB}$ on (5.3), the RHS will certainly be proportional to the derivatives of u_μ and r_H. But u_μ and r_H being the hydrodynamic variables, their derivatives are 'small' at every point in the sense described in (5.2). Therefore a 'small' correction in the leading ansatz could solve the equation.

The r dependence of these 'small corrections ' could be determined exactly while the $\{x^\mu\}$ dependence would be treated in perturbation in terms of the labeling data $u^\mu(x)$ and $r_H(x)$ and their derivatives. $u^\mu(x)$ and $r_H(x)$ themselves would be constrained to satisfy the hydrodynamic equation, order by order in derivative expansion. While dealing with the full set of gravity equations (5.1), these equations on the hydrodynamic variables or the labeling data would emerge as the 'constraint equations' of the theory of classical gravity.

5.1.2 Perturbation parameter in $\left(\frac{1}{D}\right)$ expansion

This is a perturbation technique, which is applicable only in a large number of spacetime dimension (denoted as D), as a series expansion in powers of $\left(\frac{1}{D}\right)$. Clearly $\left(\frac{1}{D}\right)$ is the perturbation parameter (a dimensionless number to begin with) here, which must satisfy

$$\left(\frac{1}{D}\right) << 1$$

Unlike the derivative expansion, the $\left(\frac{1}{D}\right)$ expansion does not necessarily need the presence of cosmological constant, but we could also apply it if the cosmological constant is present

[3] Note that the scaling of λ with D is up to us. At finite D it is of no relevance, but it matters while taking the large D limit. Here λ would be fixed to one as we would take D to ∞.

provided we keep λ, the AdS radius (see equation (5.1) in subsection - 5.1.1) fixed as we take D large. Note that the choice $\lambda = 1$, as we have done in previous subsection, is consistent with this 'D- scaling'.

The starting point here is the following metric.

$$dS^2 \equiv G_{AB}\, dX^A dX^B = g_{AB}\, dX^A dX^B + \psi^{-D}(O_A\, dX^A)^2 \tag{5.4}$$

where, g_{AB}, ψ and O_A are defined as follows.

1. g_{AB} is a smooth metric of pure AdS geometry which we shall refer to as 'background'. We could choose any coordinate as long as the metric is smooth and all components of the Riemann curvature tensors are of order $\mathcal{O}(1)$ or smaller in terms of large D - order counting.

2. $\left(\psi^{-D}\right)$ is a harmonic function with respect to the metric g_{AB}.

3. O_A is the one-form dual to the tangent vector to a null geodesic in the background satisfying $O_A n_B\, g^{AB} = 1$. Where, n_A is the unit normal on the constant ψ hypersurfaces (viewed as hypersurfaces embedded in the background).

The metric (5.4) would solve the Einstein's equation (5.1) at leading order (which turns out to be of order $\mathcal{O}(D^2)$) provided the divergence of the $\mathcal{O}(1)$ vector field, $U^A \equiv n^A - O^A$ with respect to the background metric is also of order $\mathcal{O}(1)$.

$$\begin{gathered}\nabla \cdot U \equiv \left(\nabla \cdot n - \nabla \cdot O\right)_{\psi=1} = \mathcal{O}\left(1\right) \\ \text{where } \nabla \equiv \text{covariant derivative w.r.t. } g_{AB}\end{gathered} \tag{5.5}$$

Naively equation (5.5) does not seem to constrain the vector field U^A since each of its components along with their derivatives in every direction are of order $\mathcal{O}(1)$ (this is what we mean by an 'order $\mathcal{O}(1)$ vector field'). However, it is indeed a constraint within the validity-regime of $\left(\frac{1}{D}\right)$ expansion. We could apply large D techniques provided for a generic $\mathcal{O}(1)$

vector field $V^A\partial_A$, its divergence is of order $\mathcal{O}(D)$[4].

One easy way to ensure such scaling would be to assume that the dynamics is confined within a finite number of dimensions and the rest of the geometry is protected by some large symmetry [65].

From now on, we shall assume such symmetry to be present in all the dynamics we discuss, including the dual hydrodynamics, labeling the different geometries constructed in 'derivative expansion'. For example, we shall assume that the divergence of the fluid velocity u^μ, which we shall denote by $\Theta(\equiv \partial_\mu u^\mu)$, is always of order $\mathcal{O}(D)$, whereas the velocity vector itself is of order $\mathcal{O}(1)$.

Now we shall briefly describe some general features of this leading geometry in (5.9). See [65] for a detailed discussion.

Firstly note that with the above conditions, the hypersurface $\psi = 1$ becomes null and we could identify this surface with the event horizon of the full spacetime.

Also, if one is finitely away from the $\psi = 1$ hypersurface, the factor ψ^{-D} vanishes for large D and the metric reduces to its asymptotic form g_{AB}.

Next, consider the region of thickness of the order of $\mathcal{O}\left(\frac{1}{D}\right)$ around $\psi = 1$ hypersurface. This is the region[5], where $\left(\frac{1}{D}\right)$ expansion would lead to a nontrivial correction to the leading

[4]This requirement certainly restricts the allowed dynamics that could be handled using this method. But it is not as restrictive as it might seem to begin with. To see it explicitly, let us choose a coordinate system $\{z, y^\mu\}$ for the background.

$$
\begin{aligned}
g_{zz} &= \frac{1}{z^2}, \quad g_{\mu\nu} = z^2\eta_{\mu\nu} \quad \text{Det}[g] = -z^{(D-2)} \\
\nabla\cdot V &= z^{-(D-2)}\partial_z\left[z^{(D-2)}V^z\right] + \partial_\mu V^\mu \\
&= \partial_z V^z + \partial_\mu V^\mu + (D-2)\left(\frac{V_z}{z}\right)
\end{aligned}
\tag{5.6}
$$

Here clearly the first term is of order $\mathcal{O}(1)$. The second term could potentially be of order $\mathcal{O}(D)$ since a large number of indices are summed over. Still to precisely cancel against the last term, which certainly is of order $\mathcal{O}(D)$ as long as $\left(\frac{V_z}{z}\right)$ is not very small, it requires some fine tuning. Equation (5.5) says that $U^A\partial_A$ is such a fine-tuned vector field.

[5]Following [65] , we shall refer to this region as 'membrane region'

geometry. To see why, let us do the following coordinate transformation.

$$X^A = X_0^A + \frac{\tilde{x}^A}{D} \quad \partial_A = D\,\tilde{\partial}_A$$

where $\{X_0^A\}$ is an arbitrary point on the $\psi = 1$ hypersurface. In these new coordinates

$$dS^2 = D^2 \mathcal{G}_{AB}\, d\tilde{x}^A d\tilde{x}^B, \quad \text{where} \quad \mathcal{G}_{AB} = G_{AB}\left(X_0 + \frac{\tilde{x}}{D}\right) \tag{5.7}$$

Now, if $\tilde{x}^A$ is not as large as D, it is possible to expand ψ^{-D}, O_A and g_{AB} around X_0^A.

$$\begin{aligned}
&\psi^{-D}(X^A) = e^{-\tilde{x}^A N_A} + \mathcal{O}\left(\frac{1}{D}\right), \quad \text{where} \quad N_A = [\partial_A \psi]_{X_0^A} \\
&O^A(X) = O^A|_{X_0^A} + +\mathcal{O}\left(\frac{1}{D}\right), \quad g_{AB}(X) = g_{AB}|_{X_0^A} + \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{5.8}$$

Note that from the second condition (see the discussion below equation (5.4)) it follows that

$$\text{Extrinsic curvature of } (\psi = 1) \text{ surface} = K|_{\psi=1} = D\sqrt{N_A N_B \bar{G}^{AB}} + \mathcal{O}(1)$$

Substituting equation (5.8) in equation (5.7) we find

$$\begin{aligned}
\mathcal{G}_{AB} = &\, O_A(X_0)\, n_B(X_0) + O_B(X_0)\, n_A(X_0) + P_{AB}(X_0) \\
&- \left(1 - e^{-\tilde{x}^A N_A}\right) O_A(X_0)\, O_B(X_0) + \mathcal{O}\left(\frac{1}{D}\right) \\
&\text{where} \quad P_{AB}(X^0) \equiv \text{projector perpendicular to } n_A(X_0) \text{ and } O_A(X_0) \\
&n_A = \frac{\partial_A \psi}{\sqrt{(\partial_A \psi)(\partial_B \psi) g^{AB}}}
\end{aligned} \tag{5.9}$$

Clearly, at the very leading order, the metric will have non-trivial variation only along the direction of N_A - the normal to the $\psi = 1$ hypersurface at point X_0^A. Variations along all other directions are suppressed by factors of $\left(\frac{1}{D}\right)$. This is very similar to the metric in equation (5.3) where at leading order the non-trivial variation is only along a single direction - r. Therefore, within this 'membrane region', $\left(\frac{1}{D}\right)$ expansion would *almost* reduce to derivative expansion along directions other than N_A provided the metric (5.9) solves equation(5.1) at very leading order. The conditions, listed below equation (5.4) along with equation (5.5) ensure that this is the case.

Once the leading solution is found, the same algorithm, described in the previous subsection, would work and we could find the subleading corrections handling the variations of N_A and O_A along the constant ψ hypersurface. All such variations would be suppressed as long as none of the components of N_A, O_A and their derivatives (in the unscaled X^A coordinates) are as large as D. In other words, we should be able choose a coordinate system, along the horizon (or the hypersurface $\psi = 1$) such that

$$\left[g^{AB}\left(\partial_A\,\psi^{-D}\right)\left(\partial_B\,\psi^{-D}\right)\right]^{-\frac{1}{2}}\partial_A\,|_{\text{horizon}} << 1 \tag{5.10}$$

It is enough to impose this inequality only on the $\psi = 1$ hypersurface; the conditions listed below equation (5.4) will ensure that they are true on all constant ψ surfaces.

These conditions also specify the defining data (analogue of fluid-velocity and temperature in case of 'derivative expansion') for the class of metrics, generated by $\left(\frac{1}{D}\right)$ expansion. Here, the gravity solutions are expressed in terms of the auxiliary function ψ and the one-form $O_A\,dX^A$. These two auxiliary fields satisfy the second and the third conditions, listed below equation (5.4). However, the above-mentioned conditions, being differential equations, could not fix the fields completely unless some boundary conditions are specified along any fixed surface. The most natural choice for this hypersurface is the surface given by $\psi = 1$, which, by construction, is the horizon of the full spacetime geometry. Different metric solutions are classified by the shape of this surface and the components of O_A projected along the surface. Just as in 'derivative expansion', we could solve for the metric correction only if these defining data (the projected O_A field and the shape of the surface, encoded in its extrinsic curvature) satisfy the constraint equation, which we shall refer to as the 'membrane equation'.

5.1.3 Comparison between two perturbation schemes

In subsection-(5.1.2), we have seen that within the membrane region, $\mathcal{O}\left(\frac{1}{D}\right)$ expansion is *almost* like 'derivative expansion' as described in subsection-(5.1.1). Still, it is also clear that they are not quite the same. The leading ansatz itself looks quite different for the two schemes, and there is no question of overlap if these two techniques compute perturbations around two entirely different geometries. So, to find an 'overlap regime', the first step would be to see where in the parameter-space and in what sense, equation (5.3) and (5.7) describe the same leading geometry.

Note that though the leading geometries look different algebraically, they both have similar geometric properties - namely the existence of a curvature singularity. In metric (5.3) it is located at $r = 0$ and the metric (5.7) is singular at $\psi = 0$. Also, the singularity is shielded by some event-horizon[6].

To see the similarities more explicitly, let us first choose a coordinate system $X^A \equiv \{\rho, X^\mu\}$, such that the background metric- g_{AB} in equation (5.8) takes the form

$$g_{AB}\, dX^A\, dX^B = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu} dX^\mu dX^\nu, \tag{5.11}$$

In this coordinate system, the following metric is an exact solution of equation (5.1)

$$ds^2 = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu} dX^\mu dX^\nu + \left(\frac{\rho}{r_H}\right)^{-(D-1)} \left(\frac{d\rho}{\rho} - \rho\, dt\right)^2 \tag{5.12}$$

This is simply the Schwarzschild black brane solution, written in Kerr-Schild form. Now let us note the following features of this metric [65].

- The function $\left(\frac{\rho}{r_H}\right)^{-(D-1)}$ is harmonic with respect to the background up to correction of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

$$\nabla^2 \left(\frac{\rho}{r_H}\right)^{-(D-1)} = \mathcal{O}\left(\frac{1}{D}\right)^2$$

[6]So far, the way both the techniques of 'large-D expansion' and 'derivative expansion' are developed, the existence of a horizon is a must. It would be interesting to know whether we could depart from this condition and still apply either of these two techniques to construct 'horizon free' or non-singular smooth geometries.

Hence the function $\left(\frac{\rho}{r_H}\right)^{-(D-1)}$ could be identified with ψ^{-D} appearing in the metric (5.4) up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

- The one form $\left(\frac{d\rho}{\rho} - \rho\, dt\right)$ is null and satisfies the geodesic equation. Further, contraction of this one-form with the unit normal to constant ρ hypersurfaces is one. Hence this one form could be identified with the null one form $O_A dX^A$

Hence it follows that the metric in (5.12), which is an exact solution of (5.1), could be cast in the form of our leading ansatz up to corrections subleading in $\left(\frac{1}{D}\right)$ expansion. We could also expand the metric in equation (5.12) around a given point on the horizon $\rho = r_H$, the same way we have done (see equation (5.9)) in the previous subsection with the following set of identifications.

$$\begin{aligned} &N_A\, dX^A|_{\rho=1} = \frac{d\rho}{r_H}, \;\; O_A\, dX^A|_{\rho=1} = \frac{d\rho}{r_H} - r_H\, dt \\ &n_A\, dX^A = \frac{N_A\, dX^A}{\sqrt{N_A N^A}} = \frac{d\rho}{r_H} \end{aligned} \tag{5.13}$$

The very leading term in this expansion, once written in terms of N_A and O_A would have exactly the same form as that of the metric in equation (5.7). The main difference between our leading ansatz, equation (5.4) and equation (5.12) is that in the later N_A and O_A satisfy equation (5.13) everywhere along the horizon, in the same $\{\rho, y^\mu\}$ coordinates. For our leading ansatz (5.4) also, it is true that we could always choose a local $\{\rho, t\}$ coordinates by reversing the equations in (5.13). But for a generic ψ and O_A, this could not be done globally and this is the reason why our leading ansatz is not an exact solution of (5.1). However, the deviation from the exact solution would clearly be proportional to the derivatives of N_A and O_A and therefore subleading. So finally we conclude that locally around a point on the horizon, the leading ansatz for $\left(\frac{1}{D}\right)$ expansion looks like a Schwarzschild black brane written in a Kerr-Schild form with the local ρ and t coordinates, respectively oriented along the direction of the normal N_A and the direction O_A projected along the membrane $\psi = 1$.

Now let us come to the leading ansatz for the metric in derivative expansion. As it is explained in detail in [58], the leading ansatz in derivative expansion, equation (5.3) , reduces to Schwarzschild black brane in Eddington-Finkelstein coordinates if we choose $r_H = constant$ and $u^\mu = \{1, 0, 0, \cdots\}$. Also locally at any point $\{x_0^\mu\}$, we could always choose a coordinate system such that $u^\mu(x_0) = \{1, 0, 0, \cdots\}$, or in other words by appropriate choice of coordinates locally the metric described in (5.3) could always be made to look like a Schwarzschild black brane, though in a different gauge than in equation (5.4). Clearly, the starting point of these different expansions are 'locally 'same and it is possible to have an overlap regime.

But the difference lies in the concept of 'locality' and also in the space of defining data. In case of 'large-D' expansion, the classifying data of the metric is specified on the horizon whereas for 'derivative expansion' it is defined on the boundary of AdS space.
The range of validity for 'large-D' expansion is given in equation (5.10). If we replace $\partial_A \psi^{-D}|_{\text{horizon}}$ by $(-DN_A)$ the condition (5.10) reduces to the existence of coordinate system such that

$$\partial_A \,|_{\text{horizon}} << D \tag{5.14}$$

which looks very similar to the validity regime for 'derivative expansion' , as already mentioned in subsection (5.1.1)

$$r_H^{-1} \partial_\mu << 1 \tag{5.15}$$

If we could somehow map each point on the boundary to a point on the horizon (viewed as a hypersurface embedded in the background), the same $\{x^\mu\}$ coordinates could be used as coordinates along the horizon. In that case, whenever r_H is of order $\mathcal{O}(1)$ in terms of 'large-D' order counting, the inequality (5.15) would imply equation (5.14). In other words, as $D \to \infty$, all solutions of 'derivative expansion' could be legitimately expanded further in $\left(\frac{1}{D}\right)$, though the reverse may not be true.

Now we know that ∂_A and ∂_μ are simply related (without any extra factor of D) for the case of exact Schwarzschild black brane solutions. This is just the well-known coordinate transformation one should use to go from Kerr-Schild to Eddington-Finkelstein form of the black brane metric. This transformation also gives the required map from the horizon to boundary coordinates. Once perturbations are introduced on both sides, we expect the relation between these two sets of coordinate systems would get corrected, but in a controlled and perturbative manner, thus maintaining the above argument for the existence of overlap.

So in summary, there does exist a region of overlap between these two perturbative techniques. In this chapter, our goal is to match them in the regime of overlap. As it is clear from the above discussion, the key step involves determining the map between ∂_A and ∂_μ, which we are going to elaborate in the next section.

5.2 Transforming to 'large-D' gauge

From the discussion of section - (5.1) it follows that if the spacetime dimension D is very large, we could always apply '$\left(\frac{1}{D}\right)$ expansion' whenever 'derivative expansion' is applicable. Therefore a metric, corrected in derivative expansion in arbitrary dimension, when further expanded in $\left(\frac{1}{D}\right)$, should reproduce the metric generated independently using the method of '$\left(\frac{1}{D}\right)$ expansion'. More precisely if we take the metric of equation (4.1) from [61] and expand it in $\left(\frac{1}{D}\right)$, it should match with the metric given in equation (8.1) of [65] after appropriate transformation.

In this section, our goal is to understand what these 'appropriate transformations' are.

Let us explain it in little more detail.

As we have mentioned before, both of these two perturbative techniques generate black brane geometries, in terms of a set of 'dynamical data', confined to a codimension one hypersurface. In the first case, it is the boundary of the Asymptotic AdS space and in the second case, it is the event horizon viewed as a hypersurface embedded in pure AdS. So both the techniques require a map from the full spacetime geometry to a codimension one membrane.

The details of this map are quite clear for the case of 'derivative expansion'.

The data-set that distinguishes between different dynamical geometries, here is the profile of a relativistic conformal fluid (its velocity and temperature). In other words, given a unit normalized velocity field and temperature, defined on a $(D-1)$ dimensional flat spacetime and satisfying the relativistic Navier-Stokes equation, we should be able to uniquely construct a D dimensional spacetime with a dynamical event horizon such that its metric is a solution to (5.1). The $(D-1)$ dimensional space is identified with the conformal boundary of this D dimensional black brane geometry, which we shall refer to as bulk. This construction [61] uses a very specific coordinate system, that encodes how a point in the bulk could be associated with a point in the boundary. In [73], the authors have also explained how to reverse the construction of [58], [61]. They have given an algorithm to read off the dual fluid variables starting from any black brane geometry that admits derivative expansion but written in arbitrary coordinates. This explicitly proves the claim of one-to-one correspondence between the dynamical black brane geometry, admitting derivative expansion and the fluid profile, satisfying relativistic Navier-Stokes equation. This algorithm has been heavily used to cast the rotating black-holes in the 'hydrodynamic form' [61].

Similarly, according to [65], there exists a one-to-one correspondence between dynamical black brane geometries in $\left(\frac{1}{D}\right)$ expansion and a codimension-one 'membrane dynamics' in pure AdS space, though [65] shows the correspondence in only one direction. It starts from valid membrane data and integrates it outward towards infinity to construct the corresponding black brane geometry. But to explicitly show this correspondence, we also need to know the reverse. In other words, we should know how to associate a point on the membrane to a point on the bulk and how to read off the membrane data, starting from a dynamical black brane geometry that admits an expansion in $\left(\frac{1}{D}\right)$, but written in some arbitrary coordinates.

In the next subsection, we shall formulate an algorithm to determine this 'membrane-bulk map', analogous to the discussion of [73] in the context of transforming the rotating black holes to the hydrodynamic form.

5.2.1 Bulk-Membrane map

[7]

The 'large-D expansion' technique, as developed in [65], would always generate the dynamical black brane metric G_{AB} in a 'split' form. This 'split' is specified in terms of an auxiliary function ψ and an auxiliary vector field $O^A\partial_A$. In terms of equation,

$$G_{AB} = g_{AB} + G^{(\text{rest})}_{AB} \tag{5.16}$$

where g_{AB} is the background and $G^{(\text{rest})}_{AB}$ is such that there exists a null geodesic vector field $O^A\partial_A$ in the background, satisfying

$$O^A\, G_{AB} = O^A\, g_{AB} \;\;\Rightarrow\;\; O^A\, G^{(\text{rest})}_{AB} = 0 \tag{5.17}$$

The normalization of this null geodesic vector is determined in terms of the function ψ, defined as follows.

1. $\left(\psi^{-D}\right)$ is a harmonic function with respect to the metric g_{AB}.

2. $\psi = 1$ hypersurface, when viewed as an embedded surface in full spacetime, becomes the dynamical event horizon. This is how the boundary condition on ψ is specified.

After fixing ψ, the normalization of O^A is fixed through the following condition.

$$O^A n_A = 1.$$

[7] This subsection has been worked out by Shiraz Minwalla in a different context. We sincerely thank him for explaining it in detail to us. This 'bulk-membrane' map is the key concept needed for the required 'matching' of the two perturbative gravity solutions.

where n_A is the unit normal on the constant ψ hypersurfaces (viewed as hypersurfaces embedded in the background).

The equations (5.16) and (5.17) together specify a map between two entirely different geometries, with metric g_{AB} and G_{AB} respectively, both satisfying equation (5.1). So if we want to recast an arbitrary dynamical black brane metric, which admits $\left(\frac{1}{D}\right)$ expansion, in the form as described in (5.16), the first step would be to figure out this map or the 'split' of the spacetime between 'background' and the 'rest', so that the equation (5.17) is obeyed.

Now from the discussion of the previous subsection, we see that this 'map' is crucially dependent on the vector field $O^A\partial_A$ and the function ψ. But both of them are defined using the 'background' geometry and we immediately face a problem, since given an arbitrary black brane metric, it is the 'background' that we are after.
For example, given a black brane metric, we could always determine the location of the event horizon, but we would never know its embedding in the background, unless we know the 'split' and therefore we would not be able to construct the ψ function, by exploiting the harmonicity condition on ψ^{-D}. If we do not know ψ we would not be able to orient or normalize O^A, as required.

So, we must have some equivalent formulation of this 'split' just in terms of the full spacetime metric. The following observation allows us to do it. We could show that if G_{AB} admits a split between g_{AB} and $G^{(\text{rest})}_{AB}$ satisfying $O^A G^{(\text{rest})}_{AB} = 0$, then the vector - $O^A\partial_A$, which is a null geodesic with respect to g_{AB}, is also a null geodesic with respect to G_{AB}.

Proof:

We know that

$$(O\cdot\nabla)O^A = \kappa\, O^A$$

where ∇ denotes the covariant derivative with respect to g_{AB} and κ is the proportionality factor. We would like to show that

$$(O\cdot\check{\nabla})O^A \propto O^A, \quad \text{where } \check{\nabla} \text{ is covariant derivative w.r.t. } G_{AB}$$

Suppose $\check{\Gamma}^A_{BC}$ denotes the Christoffel symbol corresponding to $\check{\nabla}_A$ and Γ^A_{BC} denotes the Christoffel symbol corresponding to ∇_A. These two would be related as follows [65].

$$\check{\Gamma}^A_{BC} = \Gamma^A_{BC} + \underbrace{\frac{1}{2}\left(\nabla_B\left[G^{(\text{rest})}\right]^A_C + \nabla_C\left[G^{(\text{rest})}\right]^A_B - \nabla^A\left[G^{(\text{rest})}\right]_{BC}\right)}_{\delta\Gamma^A_{BC}} \tag{5.18}$$

Here all raising and lowering of indices have been done using g_{AB}. Note that

$$\begin{aligned}
O^B O^C\,\delta\Gamma^A_{BC} &= O^B(O\cdot\nabla)\left[G^{(\text{rest})}\right]^A_B - \frac{1}{2}O^B O^C\nabla^A\left[G^{(\text{rest})}\right]_{BC}\\
&= -\left[G^{(\text{rest})}\right]^A_B\left[(O\cdot\nabla)O^B\right] + \frac{1}{2}\left(\nabla^A O^C\right)\left[G^{(\text{rest})}\right]_{BC}O^B\\
&= \kappa\left(O^C\left[G^{(\text{rest})}\right]^A_C\right) = 0
\end{aligned} \tag{5.19}$$

What we want to show simply follows from equation (5.19)

$$(O\cdot\check{\nabla})O^A = (O\cdot\nabla)O^A = \kappa\, O^A \tag{5.20}$$

So we could determine O^A by solving the null geodesic equation with respect to the full spacetime metric G_{AB}. But to determine it fully, we also need to know κ, fixed by the normalization of O^A. As mentioned before, the normalization used previously in the application of 'large-D' technique is not suitable for our purpose, since it requires the knowledge of the 'background' beforehand. But luckily the form of the 'split', which is defined by the condition $\left[O^A G^{(\text{rest})}_{AB} = 0\right]$ is independent of the normalization of O^A.

So we shall first determine another null geodesic field (let us denote it by $\bar{O}_A$ to remind ourselves of the difference in normalization) which is affinely parametrized and whose inner-product with the normal to event horizon (which, up to normalization, could again be determined without any knowledge of the 'split') is one.

Now we are at a stage to define the map between the 'background' and the full spacetime geometry.

Suppose $\{Y^A\}$ denote the coordinates in the background geometry (in our case pure AdS, the metric is denoted by g_{AB}) and $\{X^A\}$ are the coordinates of the full spacetime (the dynamical black brane, the metric is denoted by G_{AB}). Let us denote the invertible functions that give a one to one correspondence between these two spaces as $\{f^A\}$.

$$Y^A = f^A(\{X\}) \tag{5.21}$$

The equations that will determine f^A s are the following

$$\bar{O}^A\, G_{AB}|_{\{X\}} = \bar{O}^A \left(\frac{\partial f^C}{\partial X^A}\right)\left(\frac{\partial f^{C'}}{\partial X^B}\right) g_{CC'}|_{\{X\}} \tag{5.22}$$

[8] Here $\bar{O}^A$ are affinely parametrized the null geodesics in the full spacetime geometries i.e.,

$$\bar{O}\cdot\check{\nabla}\bar{O}^A = 0 \tag{5.23}$$

Equation (5.23) would fix $\bar{O}_A$ completely once we specify the angles it would make with the tangents of the horizons, which is effectively a set of $(D-1)$ numbers. Now what we are actually interested in is not $\bar{O}_A$ but O_A which is related to $\bar{O}_A$ with a normalization. Therefore we are free to choose the normalization of $\bar{O}_A$, since anyway, we have to re-normalize it again. This will fix one of the $(D-1)$ initial conditions. Rest we shall keep arbitrary.

We shall assume

$$\begin{aligned}
\bar{O}^A N_A|_{\text{horizon}} &= 1\\
\bar{O}^A l_A^{(i)}|_{\text{horizon}} &= \text{ some arbitrary functions of horizon cordinates}
\end{aligned} \tag{5.24}$$

where N_A is the null normal to the event horizon (with some arbitrary normalization) and $l^A_{(i)}\partial_A$ s are the unit normalized space-like tangent vectors to the horizon.

It turns out that the hydrodynamic metric could be split for a very specific choice of these

[8]The subscript $\{X\}$ in equation (5.22) denotes that both LHS and RHS of equation (5.22) have to be evaluated in terms $\{X^A\}$ coordinates.

spatial initial conditions and we shall fix them order by order in derivative expansion by matching the hydrodynamic and the 'large-D' metric. Once $\bar{O}^A$ is fixed (in terms of these arbitrary angles), we could determine f^A s up to some integration constants by solving equation (5.22).

Equation (5.22) further says that if we apply the map (5.21) as a coordinate transformation on the 'background', then in the new $\{X^A\}$ coordinates the map would just be an 'identity' map and the full spacetime metric G_{AB} would admit the split as given in equation (5.16) satisfying (5.17) [9].

Once we have figured out how to split the full spacetime metric into 'background' and the 'rest', we know how to view the event horizon as a surface embedded in the 'background' and therefore the auxiliary function ψ (by solving the harmonicity of ψ^{-D} w.r.t the background) everywhere. Now we can normalize $\bar{O}^A$ as it has been done in [65]. Using these ψ and O^A (appropriately normalized) one should be able to recast any arbitrary metric, that admits large-D expansion, exactly in the form of [65].

5.3 Bulk-Membrane map in metric dual to Hydrodynamics

In this subsection, we shall implement the above algorithm, described in the previous subsection, for the metric dual to hydrodynamics. For convenience, we are summarizing the steps again.

- Determine the equation for the event horizon.
- Determine the null normal to the horizon.

[9] We would also like to emphasize that what we are describing here is not just a gauge or coordinate transformation. The 'split' mentioned in equation (5.16) is a genuine point-wise map between two entirely different geometries. Once we have figured out the 'map', we are free to transform the coordinates further; both G_{AB} and g_{AB} would change, but the 'map' will still be there.

- Solve equation (5.23) to determine $\bar{O}^A$ everywhere. We need the normal, derived in the previous step, to impose the boundary condition.

- Choose any arbitrary coordinate system $\{Y^A\}$, where the 'background' has a smooth metric g_{AB}.

- Now solve the equation (5.22) to determine the mapping functions f^A 's.

For a generic dynamical metric, it is not easy to implement all these steps. But in this case what would help us is the 'derivative expansion' and the fact that f^A 's are exactly known at zero derivative order; it is simply the coordinate transformation between Eddington-Finkelstein and Kerr-Schild form of a static black brane metric.

Though the zeroth order transformation is already known, as a 'warm-up' exercise we shall re-derive it using the above algorithm. The condition of 'staticity' and translational symmetry of the metric allow us to solve relevant equations exactly in this case.

5.3.1 Zeroth order in 'derivative expansion':

At zeroth order in derivative expansion, the metric dual to hydrodynamics has the following form

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 f\left(r/r_H\right) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ \text{where} \quad & P_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu, \quad f(z) \equiv \left[1 - z^{-(D-1)}\right], \quad u_\mu u_\nu \eta^{\mu\nu} = -1 \end{aligned} \tag{5.25}$$

We could read off the components of the metric and its inverse.

$$\begin{aligned} &G_{rr} = 0, \quad G_{\mu r} = -u_\mu, \quad G_{\mu\nu} = -r^2 f\left(r/r_H\right) u_\mu u_\nu + r^2 P_{\mu\nu} \\ &G^{rr} = r^2 f\left(r/r_H\right), \quad G^{\mu r} = u^\mu, \quad G^{\mu\nu} = \frac{1}{r^2} P^{\mu\nu} \end{aligned} \tag{5.26}$$

At zero derivative order, both r_H and u^μ could be treated as constants, The event horizon and the null normal to it are given by

$$\text{Event Horizon} : \mathcal{S} = r - r_H = 0, \quad N_A\, dX^A = dX^A \partial_A \mathcal{S} = dr \tag{5.27}$$

Now we shall figure out the 'map' that will lead to the desired 'split' between 'background' and 'rest'.

We have already determined the event horizon. Next, we have to solve for $\bar{O}^A$, satisfying the conditions

$$\bar{O}^B \check{\nabla}_B \bar{O}^A = 0, \;\; \bar{O}^A \bar{O}^B G_{AB} = 0, \;\; \bar{O}^A N_A|_{r=r_H} = \bar{O}^r|_{r=r_H} = 1$$

At zero derivative order, G_{AB} has translational symmetry in all the x^μ. The conditions on $\bar{O}^A$ does not break this symmetry. Hence $\bar{O}^A$ must have the form

$$\bar{O}^A \partial_A = h_1(r)\, \partial_r + h_2(r)\, u^\mu \partial_\mu \tag{5.28}$$

Now we shall process the condition that O^A is a null vector field.

$$\begin{aligned} &\bar{O}^A \bar{O}^B G_{AB} = 0 \\ \Rightarrow\; &2h_2(r) h_1(r) G_{\mu r} u^\mu + h_2(r)^2 u^\mu u^\nu G_{\mu\nu} = 0 \\ \Rightarrow\; &h_2(r)\left[2h_1(r) - r^2 f\left(r/r_H\right) h_2(r)\right] = 0 \\ \Rightarrow\; &h_2(r) = 0 \end{aligned} \tag{5.29}$$

So finally $\bar{O}^A \partial_A = h_1(r) \partial_r$[10].

Substituting this form of $\bar{O}^A$ in the geodesic equation we could see that $h_1(r)$ has to be a constant and then boundary condition simply says that $h_1(r) = 1$

$$\bar{O}^A \partial_A = \bar{O}^r \partial_r = \partial_r \tag{5.30}$$

Now let us choose a coordinate system $Y^A = \{\rho, y^\mu\}$ for the 'background' where the metric takes the following form

$$ds^2_{background} = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu}\, dy^\mu\, dy^\nu \tag{5.31}$$

[10] Actually, there is two solutions to (5.29). If we assume $h_2(r) \neq 0$ and finite everywhere, then

$$h_1(r) = \frac{r^2}{2} f\left(r/r_H\right) h_2(r)$$

This implies that $h_1(r)$ will vanish at the horizon $r = r_H$ (which is a zero of the function $f\left(r/r_H\right)$), contradicting the boundary condition on $\bar{O}^r$.

Again the symmetries motivate us to take the following form for the mapping, which gives the one to one correspondence between the background coordinates $\{Y^A\} = \{\rho, y^\mu\}$ and black brane coordinates $\{X^A\} = \{r, x^\mu\}$

$$y^\mu = x^\mu + g(r)u^\mu, \quad \rho = h(r) \tag{5.32}$$

Let us apply the map (5.32) as a coordinate transformation on the background. In the new coordinates (where the map is just an 'identity') the background metric takes the following form

$$g_{rr} = \left(\frac{h'}{h}\right)^2 - (g'h)^2, \;\; g_{\mu r} = g'h^2 u_\mu, \;\; g_{\mu\nu} = h^2\, \eta_{\mu\nu} \tag{5.33}$$

Here we have suppressed the r dependence and derivative w.r.t r is denoted by prime ($'$). In this coordinates equation (5.22) takes the form

$$\left(\frac{h'}{h}\right)^2 - (g'h)^2 = 0, \;\; g'h^2 = -1 \tag{5.34}$$

These two equation could be solved very simply. The general solution

$$h(r) = \pm(r + c_1), \quad g(r) = \frac{1}{r + c_1} + c_2 \tag{5.35}$$

where c_1 and c_2 are two arbitrary constants.

We shall choose the plus sign in $h(r)$ to make sure that whenever r increases, ρ also increases.

Now we have to fix the integration constants. Note that once we know the map, we know the form of $G^{\text{(rest)}}_{AB}$, satisfying equation (5.17) by construction.

$$\begin{aligned} &G^{\text{(rest)}}_{rr} = G^{\text{(rest)}}_{r\mu} = 0 \\ &G^{\text{(rest)}}_{\mu\nu} = \left[(r + c_1)^2 - r^2 f(r/r_H)\right] u_\mu u_\nu + \left[r^2 - (r + c_1)^2\right] P_{\mu\nu} \end{aligned} \tag{5.36}$$

Now we further want that if $D \to \infty$, the metric should reduce to its asymptotic form at any finite distance from the event horizon or in other words, $G^{\text{(rest)}}_{\mu\nu}$ must vanish outside the ‘membrane region’ (a region with ‘thickness’ of the order of $\mathcal{O}\left(\frac{1}{D}\right)$ around the ‘membrane’, see section (5.1.2)). This condition will force us to set $c_1 = 0$. The other constant c_2 is not appearing in the final form of the metric at all, so this ambiguity will remain here at this order and it is simply a consequence of the translational symmetry in x^μ and y^μ directions. For simplicity, we shall also choose $c_2 = 0$. So the final form of the map at zeroth order

$$\rho = r, \qquad y^\mu = x^\mu + \frac{u^\mu}{r} \tag{5.37}$$

5.3.2 First order in derivative expansion

In this subsection, we shall extend the computation of the previous subsection up to the first order in derivative expansion. Here u^μ and r_H depends on x^μ but any term that has more than one derivatives of u^μ and r_H has been neglected. All calculations presented in this subsection generically will have corrections at order $\mathcal{O}(\partial^2)$.

At first order in derivative expansion the metric dual to hydrodynamics has the following form [61]

$$\begin{aligned} ds^2 = & -2u_\mu dx^\mu dr - r^2 f\left(r/r_H\right) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ & + r\left[-\left(u_\mu a_\nu + u_\nu a_\mu\right) + \frac{2\Theta}{D-2} u_\mu u_\nu + 2F\left(r/r_H\right)\sigma_{\mu\nu}\right] dx^\mu dx^\nu \end{aligned} \tag{5.38}$$

Where,

$$F(r) = r\int_r^\infty dx \frac{x^{D-2}-1}{x(x^{D-1}-1)}$$

And

$$a_\mu = (\eta^{\alpha\beta} u_\alpha \partial_\beta) u_\mu\ , \quad \Theta = \eta^{\alpha\beta}\partial_\alpha u_\beta\ , \quad \sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2}\right) - \left(\frac{\Theta}{D-2}\right) P^{\mu\nu} \tag{5.39}$$

We shall often refer to this metric, described in equation (5.38), as ‘hydrodynamic metric’. Here both r_H and u_μ are functions of x^μ; but they are not completely arbitrary. the hydrodynamic metric will solve the Einstein’s equation (up to corrections of order $\mathcal{O}(\partial^2)$) provided the derivatives of r_H and u_μ satisfies the following equations[11].

$$\frac{(\eta^{\alpha\beta}u_\alpha\partial_\beta)r_H}{r_H} + \frac{\Theta}{D-2} = 0, \quad P^{\mu\nu}\left(\frac{\partial_\mu r_H}{r_H}\right) + a^\nu = 0 \tag{5.40}$$

We read off the components of the metric and its inverse

$$\begin{aligned} G_{\mu r} =& -u_\mu, \quad G_{rr} = 0 \\ G_{\mu\nu} =& -r^2 f\left(r/r_H\right) u_\mu u_\nu + r^2 P_{\mu\nu} \\ &+ r\left[-\left(u_\mu a_\nu + u_\nu a_\mu\right) + \left(\frac{2\Theta}{D-2}\right) u_\mu u_\nu + 2F\left(r/r_H\right)\sigma_{\mu\nu}\right] \end{aligned} \tag{5.41}$$

$$\begin{aligned} G^{rr} &= r^2 f(r/r_H) - r\left(\frac{2\Theta}{D-2}\right), \quad G^{\mu r} = u^\mu - \frac{a^\mu}{r} \\ G^{\mu\nu} &= \frac{P^{\mu\nu}}{r^2} - \frac{2F(r/r_H)}{r^3}\,\sigma^{\mu\nu} \end{aligned} \tag{5.42}$$

The horizon is still given by the surface (no correction at first order in derivative, though the normal gets corrected since $\partial_\mu r_H$ is not negligible now.)

$$\text{Event Horizon} : \mathcal{S} = r - r_H = 0, \quad N_A\, dX^A = dX^A\partial_A\mathcal{S} = dr - dx^\mu\,\partial_\mu r_H \tag{5.43}$$

We need the Christoffel symbols to compute the geodesic equation.

$$\begin{aligned} &\check\Gamma^r_{rr} = 0, \qquad \check\Gamma^\mu_{rr} = 0 \\ &\check\Gamma^r_{\alpha r} = \left[r f(r/r_H) + \frac{r^2}{2r_H} f'(r/r_H) - \frac{\Theta}{D-2}\right] u_\alpha \\ &\check\Gamma^\mu_{r\delta} = \frac{1}{2r^2}\left[2rP^\mu_\delta - \partial_\delta u^\mu - u_\delta a^\mu + \partial^\mu u_\delta + u^\mu a_\delta - 2F(r/r_H)\sigma^\mu_\delta + 2\left(r/r_H\right)F'(r/r_H)\sigma^\mu_\delta\right] \end{aligned} \tag{5.44}$$

At first order in derivative expansion, the most general correction that could be added to $\bar{O}^A$, maintaining it as a null vector with respect to the first order corrected metric:

$$\bar{O}^A\partial_A = \partial_r + w_1(r)\,\Theta\,\partial_r + w_2(r)\,a^\mu\partial_\mu \tag{5.45}$$

[11]These two equations are just the stress tensor conservation equation for a $(D-1)$ dimensional ideal conformal fluid.

We shall fix $w_1(r)$ and $w_2(r)$ using the geodesic equation.

The r component of the geodesic equation gives the following.

$$
\begin{aligned}
&(\bar{O}\cdot\check{\nabla})\bar{O}^r = 0\\
\Rightarrow&\bar{O}^r\check{\nabla}_r\bar{O}^r + \bar{O}^\mu\check{\nabla}_\mu O^r = 0\\
\Rightarrow&\bar{O}^r\partial_r\bar{O}^r + \check{\Gamma}^r_{rr}\bar{O}^r\bar{O}^r + 2\bar{O}^r\bar{O}^\alpha\check{\Gamma}^r_{\alpha r} = 0\\
\Rightarrow&(1+w_1(r)\Theta)w_1'(r)\Theta + 2(1+w_1(r)\Theta)(w_2(r)a^\alpha)\check{\Gamma}^r_{\alpha r} = 0\\
\Rightarrow&w_1'(r) = 0\\
\Rightarrow&w_1(r) = A_1, \quad \text{where } A_1 \text{ is a constant}
\end{aligned}
$$

From the μ component of the geodesic equation we find

$$
\begin{aligned}
&(\bar{O}\cdot\check{\nabla})\bar{O}^\mu = 0\\
\Rightarrow\ &\bar{O}^r\check{\nabla}_r\bar{O}^\mu + \bar{O}^\lambda\check{\nabla}_\lambda\bar{O}^\mu = 0\\
\Rightarrow\ &\bar{O}^r\partial_r\bar{O}^\mu + \bar{O}^r\bar{O}^r\check{\Gamma}^\mu_{rr} + 2\bar{O}^r\bar{O}^\delta\check{\Gamma}^\mu_{r\delta} = 0\\
\Rightarrow\ &\left[w_2'(r) + \frac{2w_2(r)}{r}\right]a^\mu = 0\\
\Rightarrow\ &w_2(r) = \left(\frac{A_2}{r^2}\right), \quad \text{where } A_2 \text{ is another integration constant}
\end{aligned}
$$

At this stage

$$
\bar{O}^A\partial_A = \partial_r + A_1\Theta\,\partial_r + \left(\frac{A_2}{r^2}\right)\,a^\mu\partial_\mu \tag{5.46}
$$

We could partially fix the integration constants using the boundary conditions.

At horizon

$$
\begin{aligned}
&\bar{O}^A N_A|_{r=r_H} = 1 \;\Rightarrow\; (1+A_1\Theta) = 1 \Rightarrow\; A_1 = 0\\
&\bar{O}^\mu\partial_\mu r_H = \mathcal{O}\left(\partial^2\right) \;\Rightarrow\; \text{No constraint on } A_2
\end{aligned} \tag{5.47}
$$

Hence it follows that .

$$\begin{aligned} \bar{O}^A\partial_A &= \partial_r + \left(\frac{A_2}{r^2}\right)\ a^\mu\partial_\mu + \text{terms 2nd order in derivative expansion} \\ \Rightarrow \bar{O}_A\ dX^A &= -u_\mu\ dx^\mu + A_2\ a_\mu\ dx^\mu + \text{terms 2nd order in derivative expansion} \end{aligned} \tag{5.48}$$

Next, we have to solve for the 'mapping functions'. Let us choose the same coordinates $\{Y^A\}$, as in the previous subsection so that the background takes the form of equation (5.31). We expect that the mapping functions (5.37) will get corrected by first order terms in derivative expansion.

$$y^\mu = x^\mu + \frac{u^\mu(x)}{r} + f_1(r)\Theta\ u^\mu(x) + f_2(r)\ a^\mu(x), \quad \rho = r + f_3(r)\ \Theta \tag{5.49}$$

As before, we shall apply the map (5.49) as a coordinate transformation on the background. In the new coordinates (where the map is just an 'identity') the background metric takes the following form

$$\begin{aligned} g_{rr} &= 2\left(f_1'(r) + \frac{f_3'(r)}{r^2} - \frac{2f_3(r)}{r^3}\right)\ \Theta \\ g_{\mu r} &= -\left[1 - \left(r^2 f_1'(r) - \frac{2f_3(r)}{r}\right)\Theta\right] u_\mu + r^2 f_2'(r)\ a_\mu \\ g_{\mu\nu} &= r^2\left(1 + \frac{2f_3(r)}{r}\ \Theta\right)\ \eta_{\mu\nu} + r\left(\partial_\nu u_\mu + \partial_\mu u_\nu\right) \end{aligned} \tag{5.50}$$

Substituting equation (5.50) in equation (5.22) we find

$$\begin{aligned} & g_{\mu r} + \left(\frac{A_2}{r^2}\right)\ a^\nu g_{\nu\mu} = -u_\mu + A_2\ a_\mu + \mathcal{O}\left(\partial^2\right), \quad g_{rr} = 0 \\ \Rightarrow\ & r^2 f_1'(r) - \frac{2f_3(r)}{r} = 0, \quad f_2'(r) = 0, \quad f_1'(r) + \frac{f_3'(r)}{r^2} - \frac{f_3(r)}{r^3} = 0 \end{aligned} \tag{5.51}$$

The general solution for equation (5.51):

$$\begin{aligned} & f_3(r) = C_3, \quad f_2(r) = C_2, \quad f_1(r) = C_1 - \frac{C_3}{r^2} \\ & \text{where } C_1, C_2 \text{ and } C_3 \text{ are arbitray constants} \end{aligned} \tag{5.52}$$

In the new $X^A = \{r, x^\mu\}$ coordinates the metric of the background takes the following form

$$
\begin{aligned}
ds^2_{\text{background}} &= g_{AB}dX^A dX^B \\
&= -2u_\mu dx^\mu\, dr + r^2\eta_{\mu\nu}dx^\mu\, dx^\nu \\
&\quad + r\left[2C_3\Theta\, \eta_{\mu\nu} + (\partial_\mu u_\nu + \partial_\nu u_\mu)\right] dx^\mu dx^\nu \\
&= -2u_\mu dx^\mu\, dr + r^2\eta_{\mu\nu}dx^\mu\, dx^\nu \\
&\quad + 2r\left[-C_3\Theta\, u_\mu u_\nu + \left(C_3 + \frac{1}{D-2}\right)\Theta\, P_{\mu\nu} - \left(\frac{a_\mu u_\nu + a_\nu u_\mu}{2}\right) + \sigma_{\mu\nu}\right] dx^\mu dx^\nu
\end{aligned}
\tag{5.53}
$$

In the last step we have rewritten $(\partial_\mu u_\nu + \partial_\nu u_\mu)$ using the following identity

$$
\partial_\mu u_\nu + \partial_\nu u_\mu = 2\sigma_{\mu\nu} + \left(\frac{2\Theta}{D-2}\right) P_{\mu\nu} - (a_\mu u_\nu + a_\nu u_\mu) \tag{5.54}
$$

Once we know the background, we could determine $G^{(\text{rest})}_{AB}$.

$$
\begin{aligned}
&G^{(\text{rest})}_{rr} = 0, \quad G^{(\text{rest})}_{\mu r} = 0 \\
&G^{(\text{rest})}_{\mu\nu} = r^2\left(\frac{r_H}{r}\right)^{D-1} u_\mu u_\nu - 2r\,\tilde{C}_3\,\Theta\,\eta_{\mu\nu} + 2r\left[F(r/r_H) - 1\right]\sigma_{\mu\nu} \\
&\text{where}\quad \tilde{C}_3 \equiv C_3 + \frac{1}{D-2}
\end{aligned}
\tag{5.55}
$$

5.4 Hydrodynamic metric in $\left(\frac{1}{D}\right)$ expansion

In this section, we would like to expand the 'hydrodynamic metric' (already split into 'background' and 'rest' in the previous section) in an expansion in $\left(\frac{1}{D}\right)$ and compare it against the metric described in [65].

This comparison involves two steps. The first one is, of course, an exact match of the two metrics up to the required order. The second step involves the mapping of the evolution of the data. Let us explain it in a little more detail.

As we have mentioned before, both 'hydrodynamic metric' and 'large - D' metric is determined in terms of data, defined on a codimension one hypersurfaces - in the first case it is the velocity and temperature of a relativistic fluid living on the boundary of asymptotic AdS and in the second case it is the horizon viewed as a membrane embedded in the background with fluctuating shape and velocity. However, we cannot choose the data arbitrarily. The hydrodynamic metric or the large D metric will solve Einstein's equation only if the corresponding data satisfy certain evolution equation. For matching of these two metrics, the evolution of the data also should match. More precisely, we should be able to re-express the membrane velocity and shape in terms of fluid velocity and temperature and further, we have to show that once hydrodynamic equations are satisfied, the membrane equation is also true up to the required order.

Below we shall first compare the two metrics and in the next subsection, we shall prove the equivalence of the evolution of these two sets of defining data.

5.4.1 Comparison between the two metrics

If the hydrodynamic metric has to match with the final metric described in [65], the first requirement is that $G^{(\text{rest})}_{\mu\nu}$ must vanish as one goes finitely away from the horizon. This is possible provided $\tilde{C}_3$ is zero and also the function $[F(r/r_H) - 1]$ has a certain type of fall-off behavior at large r. Now $\tilde{C}_3$ being an integration constant we could easily set it to zero. In appendix (D.1) we have analyzed the integral (5.39) and therefore the function $[F(r/r_H) - 1]$. It turns out that at large D this integral could be approximated as follows.

$$F(z) = F\left(1 + \frac{Z}{D}\right) = 1 - \left(\frac{1}{D}\right)^2 \sum_{m=1} \left(\frac{1 + mZ}{m^2}\right) e^{-mZ} + \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{5.56}$$

Hence $[F(r/r_H) - 1]$ vanishes[12] up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

After substituting equation (5.56) and the value for the integration constant $\tilde{C}_3$, the black

[12] Also, note that the vanishing has appropriate fall-off behavior (exponential decay in the scaled Z variable) as required by large D corrections

brane metric dual to hydrodynamics takes the following form

$$dS^2 = dS^2_{\text{background}} + r^2 \left(\frac{r_H}{r}\right)^{D-1} (u_\mu \, dx^\mu)^2 + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{5.57}$$

where $dS^2_{\text{background}}$ is given by equation (5.53)

As we have mentioned before, the metric in [65] is described in terms of one auxiliary function ψ and one auxiliary null one-form $O_A dX^A$. For convenience we are quoting the metric here again.

$$dS^2 = dS^2_{\text{background}} + \psi^{-D} \left(O_A \, dX^A\right)^2 + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{5.58}$$

Here ψ^{-D} is harmonic with respect to the background with $\psi = 1$ being the event horizon of the full spacetime and O_A is simply proportional to $\bar{O}_A$ determined in the previous sub-section. The proportionality factor (let us denote it by the scalar function $\Phi(X)$) is fixed using the condition that the component of O_A along the unit normal of $\psi =$ constant hyper-surfaces is one everywhere. In terms of equations, the above conditions could be expressed as

$$\bar{O}^A = \Phi(X) \, O^A, \quad \Phi(X) = \frac{\bar{O}^A \, \partial_A \psi}{\sqrt{(\partial_A \psi)(\partial^A \psi)}} \quad \text{where } \partial^A \psi \equiv g^{AB} \, \partial_B \psi \tag{5.59}$$

Rewriting (5.58) in terms of $\bar{O}_A$,

$$\begin{aligned} dS^2 &= dS^2_{\text{background}} + \left(\frac{\psi^{-D}}{\Phi^2}\right) \left(\bar{O}_A \, dX^A\right)^2 + \mathcal{O}\left(\frac{1}{D}\right)^2 \\ &= dS^2_{\text{background}} + \left(\frac{\psi^{-D}}{\Phi^2}\right) (u_\mu - A_2 \, a_\mu) \, (u_\nu - A_2 \, a_\nu) \; dx^\mu dx^\nu + \mathcal{O}\left(\frac{1}{D}\right)^2 \end{aligned} \tag{5.60}$$

The metric in (5.60) will match exactly with the metric in (5.57) provided we set A_2 to zero and identify $\left[\Phi^2 r^2 \left(\frac{r_H}{r}\right)^{D-1}\right]$ with the harmonic function ψ^{-D} up to corrections of order $\left(\frac{1}{D}\right)^2$. Hence in terms of equation, what we finally have to verify is the following

$$\psi^{-D} - \Phi^2 r^2 \left(\frac{r_H}{r}\right)^{D-1} = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{5.61}$$

where ψ satisfies

$$\nabla^2 \psi^{-D} = 0 \tag{5.62}$$

with the boundary condition that $\psi = 1$ should reduce to the horizon, i.e., the hypersurface given by $r = r_H$, in an expansion in $\left(\frac{1}{D}\right)$.

Now we shall first determine ψ and then Φ. Note that both ψ and the norm of $\partial_A \psi$ are scalar functions and it is much easier to compute them in a coordinate system where the background metric has a simple form. Therefore we shall solve the equation in the $\{\rho, y^\mu\}$ coordinate system and then transform the answer to the $\{r, x^\mu\}$ coordinates for final matching. First, we need to know the position of the horizon in $\{Y^A\}$ coordinates since that will provide the required boundary condition for ψ. We know that in $\{X^A\} = \{r, x^\mu\}$ coordinates the horizon is at $r = r_H(x) + \mathcal{O}(\partial^2)$. Now $\{X^A\}$ and $\{Y^A\}$ coordinates are related as follows.

$$\begin{aligned}
\rho &= r - \frac{\Theta(x)}{D-2} + \mathcal{O}(\partial^2), \\
y^\mu &= x^\mu + \frac{u^\mu(x)}{r} + \left(\frac{\Theta(x)}{D-2}\right)\left(\frac{u^\mu(x)}{r^2}\right) + C_1\, \Theta(x)\, u^\mu(x) + C_2\, a^\mu(x) + \mathcal{O}(\partial^2)
\end{aligned} \tag{5.63}$$

The inverse transformation:

$$\begin{aligned}
r &= \rho + \frac{\Theta(y)}{D-2} + \mathcal{O}(\partial^2) \\
x^\mu &= y^\mu - \frac{u^\mu(x)}{\rho} - C_1\, \Theta(x)\, u^\mu(x) - C_2\, a^\mu(x) + \mathcal{O}(\partial^2) \\
&= y^\mu - \frac{u^\mu(y)}{\rho} + \frac{a^\mu(y)}{\rho^2} - C_1\, \Theta(y)\, u^\mu(y) - C_2\, a^\mu(y) + \mathcal{O}(\partial^2)
\end{aligned} \tag{5.64}$$

Therefore in terms of $\{Y^A\}$ coordinates the horizon is at

$$\begin{aligned}
\rho &= r_H\left(x^\mu\right) - \left(\frac{\Theta}{D-2}\right) + \mathcal{O}\left(\partial^2\right) \\
&= r_H(y^\mu) - \frac{(u\cdot\partial)\, r_H}{r_H} - \left(\frac{\Theta}{D-2}\right) + \mathcal{O}\left(\partial^2\right) = r_H(y^\mu) + \mathcal{O}\left(\partial^2\right)
\end{aligned} \tag{5.65}$$

Here, for any term that is of first order in derivative to begin with, this coordinate transformation will generate change of order $\mathcal{O}(\partial^2)$ and therefore negligible in our computation.

In the last line, we have used equation (5.40).

Once we know the position of the horizon, we could solve for ψ. In $\{\rho, y^\mu\}$ coordinates the expressions for ψ and its norm are as follows (see appendix (D.2 for derivation).

$$\begin{aligned}
\psi(\rho, y^\mu) &= 1 + \left(1 - \frac{1}{D}\right)\left(\frac{\rho}{r_H(y)} - 1\right) + \mathcal{O}\left(\frac{1}{D}\right)^3 \\
\Rightarrow\ dY^A\,\partial_A\psi &= \left(1 - \frac{1}{D}\right)\left(\frac{d\rho}{r_H(y)}\right) - \rho\left(1 - \frac{1}{D}\right)\left(\frac{\partial_\mu r_H(y)}{r_H^2(y)}\right) dy^\mu \\
\Rightarrow\ \partial^A\psi\,\partial_A\psi &= \left(\frac{\rho}{r_H(y)}\right)^2\left(1 - \frac{1}{D}\right)^2 + \mathcal{O}(\partial)^2
\end{aligned} \tag{5.66}$$

Clearly this solution satisfies the boundary condition that $\psi = 1 \Rightarrow \rho = r_H(y) + \mathcal{O}(\partial^2)$.

Now we have to transform these quantities in $\{X^A\}$ coordinates. We shall first transform the quantity $\left[\frac{\rho}{r_H(y)}\right]$.

$$\begin{aligned}
\frac{\rho}{r_H(y)} &= \frac{r - \frac{\Theta}{D-2}}{r_H(x) + \frac{(\eta^{\alpha\beta}u_\alpha\partial_\beta)r_H}{r}} + \mathcal{O}(\partial^2) \\
&= \left(\frac{1}{r_H(x)}\right)\left(r - \frac{\Theta}{D-2}\right)\left(1 - \frac{(\eta^{\alpha\beta}u_\alpha\partial_\beta)r_H}{r\,r_H}\right) + \mathcal{O}(\partial^2) \\
&= \left(\frac{1}{r_H(x)}\right)\left(r - \frac{\Theta}{D-2} - \frac{(\eta^{\alpha\beta}u_\alpha\partial_\beta)r_H}{r_H}\right) + \mathcal{O}(\partial^2) = \frac{r}{r_H(x)} + \mathcal{O}(\partial^2)
\end{aligned} \tag{5.67}$$

From equation (5.67) it follows that

$$\begin{aligned}
\psi(r, x^\mu) &= 1 + \left(1 - \frac{1}{D}\right)\left(\frac{r}{r_H(x)} - 1\right) + \mathcal{O}\left(\frac{1}{D^3},\, \partial^2\right) \\
\Rightarrow\ dX^A\,\partial_A\psi &= \left(1 - \frac{1}{D}\right)\left(\frac{dr}{r_H}\right) - r\left(1 - \frac{1}{D}\right)\left(\frac{\partial_\mu r_H}{r_H^2}\right) dx^\mu + \mathcal{O}\left(\frac{1}{D^2},\, \partial^2\right) \\
\Rightarrow\ \partial^A\psi\,\partial_A\psi &= \left(\frac{r}{r_H}\right)^2\left(1 - \frac{1}{D}\right)^2 + \mathcal{O}\left(\frac{1}{D^2},\, \partial^2\right)
\end{aligned} \tag{5.68}$$

Substituting this solution in equation (5.59) we find $\Phi(X) = \frac{1}{r}$.

Now we have all the ingredients to verify equation (5.61). Let us introduce a new $\mathcal{O}(1)$ variable R such that

$$\frac{r}{r_H} = 1 + \frac{R}{D}$$

In terms of R we find

$$\begin{aligned}\psi^{-D}-\Phi^2r^2\left(\frac{r_H}{r}\right)^{D-1} &= \psi^{-D}-\left(\frac{r}{r_H}\right)^{-(D-1)}\\ &= \left[1+\left(1-\frac{1}{D}\right)\left(\frac{R}{D}\right)\right]^{-D}-\left(1+\frac{R}{D}\right)^{-(D-1)}\\ &= -\frac{1}{2}\left(\frac{R}{D}\right)^2 e^{-R}+\mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{5.69}$$

This is exactly what is required to have a match between the ‘hydrodynamic metric’ and the ‘large-D’ metric up to the expected order.

5.4.2 Comparison between the evolution of two sets of data

As mentioned before, the ‘hydrodynamic metric’ is defined in terms of the velocity and the temperature [13] of the relativistic conformal fluid moving in a flat Minkowski spacetime of dimension $(D-1)$. In case of large - D expansion, the metric is given in terms of a $(D-1)$ dimensional time-like fluctuating membrane embedded in pure AdS spacetime with a dynamical velocity field on it. Both of these two sets of data are controlled by separate equations. For ‘derivative expansion’ , the governing equation of data is given in (5.40). In ‘large-D’ technique, the relevant equation is the following [65]

$$\bar\nabla\cdot U=0,\qquad \left[\frac{\bar\nabla^2 U_\alpha}{\mathcal{K}}-\frac{\bar\nabla_\alpha\mathcal{K}}{\mathcal{K}}+U^\beta\mathcal{K}_{\beta\alpha}-U\cdot\bar\nabla U_\alpha\right]\mathcal{P}^\alpha_\gamma=0 \tag{5.70}$$

Here the equation is written as an intrinsic equation on the membrane world-volume. All raising, lowering and contraction of the indices are done with respect to the induced metric on the dynamical membrane. U_α is the velocity of the membrane, expressed in terms of

[13] The temperature and the horizon radius are related by the following relation

$$r_H=\frac{4\pi\,T}{(D-1)}$$

In our choice of units

$$r_H\sim\mathcal{O}(1)\ \Rightarrow\ T\sim\mathcal{O}(D)$$

its intrinsic coordinates. $\mathcal{K}_{\beta\alpha}$ is the extrinsic curvature of the membrane, expressed as a symmetric tensor on the membrane world-volume. $\mathcal{K}$ denotes its trace. $\mathcal{P}^{\alpha}_{\gamma}$ is the projector perpendicular to U^{α}.

In this subsection, our goal is to show that equation (5.40) implies equation (5.70) up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

Our first job would be to express the U^{α} and $\mathcal{K}_{\alpha\beta}$ in terms of velocity u^{μ} and temperature (or r_H) of the relativistic fluid. Remember that though both u^{μ} and U^{α} are unit normalized velocity vector, they are defined on completely different spaces, one being a flat Minkowski metric and the other is the curved (both intrinsic and extrinsic curvature, being nonzero) membrane world volume.

For convenience, we shall work in $\{Y^A\} = \{\rho, y^{\mu}\}$ coordinates where the background metric is simple. We shall first compute the unit normal to the membrane and different components of its extrinsic curvature, to begin with in terms of background coordinates and then we shall re-express it as an intrinsic symmetric tensor on the membrane.

The unit normal to the membrane is given by

$$
\begin{aligned}
n_A\, dY^A|_{\text{membrane}} &\equiv dY^A \left[\frac{\partial_A \psi}{\sqrt{\partial^A\psi\,\partial_A\psi}}\right]_{\text{membrane}} \\
&= \frac{d\rho - dy^{\mu}\,\partial_{\mu} r_H(y)}{r_H(y)}
\end{aligned}
\tag{5.71}
$$

The extrinsic curvature is defined as follows.

$$
\begin{aligned}
&K_{AB} = \Pi^C_A\, \nabla_C n_B = \Pi^C_A \left(\partial_C n_B - \Gamma^D_{CB} n_D\right) \\
&\text{where } \Pi^B_A = \delta^B_A - n_A\, n^B \ \text{ and } \nabla \text{ is the covariant derivative w.r.t background}
\end{aligned}
\tag{5.72}
$$

Now let us choose $\{y^{\mu}\}$ as the intrinsic coordinate on the membrane world volume. In this choice of coordinates, the extrinsic curvature $\mathcal{K}_{\alpha\beta}$ will have the following structure.

$$
\mathcal{K}_{\alpha\beta} = K_{\rho\rho}\,(\partial_\alpha r_H)\,(\partial_\beta r_H) + \left[K_{\rho\alpha}\,(\partial_\beta r_H) + K_{\rho\beta}\,(\partial_\alpha r_H)\right] + K_{\alpha\beta} \tag{5.73}
$$

Note that the first term in the RHS of equation (5.73) does not contribute at first order derivative expansion.

After using equation (5.72) and (5.73), at this order the final expression for $\mathcal{K}_{\mu\nu}$ turns out to be very simple (see appendix (D.3) for the details of the computation).

$$\mathcal{K}_{\alpha\beta} = r_H^2\ \eta_{\alpha\beta} + \mathcal{O}(\partial^2), \quad \mathcal{K} = (D-1) \tag{5.74}$$

The induced metric on the membrane is given by

$$g_{\alpha\beta} = r_H^2\ \eta_{\alpha\beta} + \mathcal{O}(\partial^2) \tag{5.75}$$

Now we shall determine the velocity U^α. The velocity is defined as the projection of O^A on the membrane which, by construction, would be unit normalized with respect to the induced metric of the membrane. In $\{Y^A\}$ coordinates, $O_A\ dY^A$ takes the following form

$$\begin{aligned}
&O_A\ dX^A|_{\text{membrane}} = -\left[r\ u_\mu(x)\ dx^\mu\right]_{\text{membrane}}\\
&= -\left(r_H(y) + \frac{\Theta}{D-2}\right)\left[u_\mu(y) - \frac{a_\mu(y)}{r_H}\right]\left[\left(\frac{\partial x^\mu}{\partial\rho}\right)d\rho + \left(\frac{\partial x^\mu}{\partial y^\nu}\right)dy^\nu\right]_{\rho=r_H(y)}\\
&= -\left(r_H(y) + \frac{\Theta}{D-2}\right)\left[u_\mu(y) - \frac{a_\mu(y)}{r_H}\right]\left[\left(\frac{u^\mu(y)}{r_H^2(y)} - \frac{2a^\mu(y)}{r_H^3(y)}\right)d\rho + \left(\delta^\mu_\nu - \frac{\partial_\nu u^\mu}{r_H}\right)dy^\nu\right]\\
&= \left(\frac{1}{r_H(y)} + \frac{\Theta}{(D-2)r_H^2}\right)d\rho + \left[-r_H(y)\ u_\mu(y) - \left(\frac{\Theta}{D-2}\right)u_\mu + a_\mu(y)\right]\ dy^\mu\\
&= \left(\frac{1}{r_H(y)} + \frac{\Theta}{(D-2)r_H^2}\right)d\rho + \left[-r_H(y)\ u_\mu(y) - \left(\frac{\partial_\mu r_H}{r_H}\right)\right]\ dy^\mu
\end{aligned} \tag{5.76}$$

In the last line, we have used equation (5.40), which is the governing equation for the data in the hydrodynamic side of the duality.

From equations (5.76) and (5.71) it follows that

$$U_A\ dY^A \equiv -\ dY^A\left[O_A - n_A\right]_{\text{membrane}} = -\left(\frac{1}{r_H^2}\right)\left(\frac{\Theta}{D-2}\right)d\rho + r_H\ u_\mu\ dy^\mu \tag{5.77}$$

Now U_α is just rewriting of U_A in terms of the intrinsic coordinates of the membrane. Following the same method as in equation (5.73) we find

$$U_\alpha\ dy^\alpha \equiv \left[r_H\ u_\alpha + \mathcal{O}(\partial^2)\right]\ dy^\alpha \tag{5.78}$$

Once we know $\mathcal{K}_{\alpha\beta}$, U^α and the induced metric on the membrane, we could compute each term in the equation (5.70).

$$
\begin{aligned}
&\bar\nabla\cdot U = \left(\frac{D-2}{r_H}\right)\left[\frac{\Theta}{D-2}+\frac{(\eta^{\alpha\beta}u_\alpha\partial_\beta)r_H}{r_H}\right]+\mathcal{O}\left(\partial^2\right)=\mathcal{O}\left(\partial^2\right)\\
&\bar\nabla^2 U_\alpha = \mathcal{O}\left(\partial^2\right)\\
&(U\cdot\bar\nabla)U_\beta = a_\beta + \frac{P^\alpha_\beta\partial_\alpha\, r_H}{r_H}+\mathcal{O}\left(\partial^2\right)=\mathcal{O}\left(\partial^2\right)\\
&U^\alpha\,\mathcal{K}_{\alpha\beta}\,\mathcal{P}^\beta_\gamma=\mathcal{O}(\partial^2)\\
&\bar\nabla_\alpha\mathcal{K}=\mathcal{O}(\partial^2)
\end{aligned}
\tag{5.79}
$$

As it is clear from the notation, in the LHS of each equation the relevant metric is the induced metric on the membrane whereas in RHS it is the flat Minkowski metric $\eta_{\alpha\beta}$. Substituting equations (5.79) in equation (5.70) we could easily show that membrane equation follows as a consequence of fluid equation.

In this context let us mention the work in [56]. Here the authors have computed the boundary stress tensor dual to a slowly varying membrane embedded in AdS. They have found the dual fluid velocity in terms of the membrane velocity. It could be easily checked that equation (5.78) is indeed the inverse of what they have found up to correction of order $\mathcal{O}(\partial^2)$.

5.5 Discussions

In this chapter, we have compared dynamical black brane solutions of Einstein's equation (in presence of negative cosmological constant) generated by two different perturbative schemes, namely 'derivative expansion' and Large-dimension expansion. In both the cases, the spacetime necessarily has an event horizon. We have shown that in a large number of dimensions whenever 'derivative expansion' is applicable, we can expand the metric further in $\left(\frac{1}{D}\right)$, (though the reverse may not be true always). We have found a perfect match in this overlap regime of these two perturbative techniques up to first subleading order on both sides.

This calculation has been extended to the next order on both sides in [66]. This has also been extended to Einstein-Maxwell system in presence of negative cosmological constant in [67].

In some sense, our analysis serves as a consistency test for these two methods. But this comparison could teach us something more. This is about the dual systems of these two gravity solutions.

The dynamical black brane metric generated by 'derivative expansion' in D dimension is dual to the relativistic conformal hydrodynamics living in $(D-1)$ dimensional flat space-time. The variables of hydrodynamics are fluid velocity and temperature, which are the data that label different black brane solutions in derivative expansion.

On the other hand, the metric generated in 'large D expansion' is dual to a co-dimension one dynamical membrane embedded in pure AdS and coupled with a velocity field. Here also the labeling data of the metric live on a $(D-1)$ dimensional hypersurface and they consist of a scalar function - the shape of the membrane and a unit normalized velocity field. This is very similar to hydrodynamics in terms of counting, though the governing equations and the physical significance of the variables are entirely different.

However, we have already seen that these two systems of equations are approximately equivalent after an appropriate field redefinition. In this chapter, we have verified it at the very leading order and we expect that the project of comparing the two metrics up to second subleading order would extend this equivalence to the next order on both sides.

In fact, it is expected that this equivalence is valid to all orders [56]. In other words, in the overlap regime, these two equations must be exactly equivalent to each other if we consider all orders on both sides [56], though to see this equivalence we need to re-express the variables of one side in terms of the other [32, 46, 56].

This equivalence actually involves some interesting resummation of one series into the other. Even the leading term in derivative expansion can encode many terms of $\left(\frac{1}{D}\right)$ expansion and on the other hand, the leading membrane equation might have information about many higher order transport coefficients. At linearized level, this has been nicely captured in the analysis in [29]. The frequencies of Quasi-normal modes do exhibit such resummation. In [56], the authors have proposed a resummed stress tensor that could exactly reproduce the fluid stress tensor exactly up to the first order in derivative expansion. It would be very interesting to understand this structure in full detail, at a non-linear level. This might lead to a fluid-membrane duality in large number of dimensions where gravity does not have any role to play.

Appendix A

Appendices for Chapter 2

A.1 Calculation of the homogeneous part - H_{AB}

In this section we shall give details of the computation for (2.35), (2.36), (2.37), (2.38) and their decoupled form as described in equations (2.40), (2.41), (2.42) and (2.43). As mentioned before, we can determine the metric up to $\mathcal{O}\left(\frac{1}{D}\right)$ by solving the gravity equation (2.18) up to order $\mathcal{O}(D)$. At this order $G^{(1)}_{AB}$ contributes simply as a linear fluctuation over the zeroth order metric $G^{[0]}_{AB} = g_{AB} + G^{(0)}_{AB}$. So here we shall first compute the form of the gravity equation (2.18), linearized about $G^{[0]}_{AB}$.

Let us denote the perturbed metric as

$$\mathfrak{g}_{AB} = G^{[0]}_{AB} + \frac{1}{D}G^{(1)}_{AB} = g_{AB} + \psi^{-D}O_AO_B + \frac{1}{D}G^{(1)}_{AB}$$

Also, as it is clear from our discussion, in this linearized calculation we need to compute only the leading D piece.

The linearized variation of the Christoffel symbols and the Ricci Tensor take the form

$$\begin{aligned}\delta\Gamma^A_{BC} &= \frac{1}{D}\left(\frac{g^{AM} - \psi^{-D}O^AO^M}{2}\right)\left(D_B\, G^{(1)}_{MC} + D_C\, G^{(1)}_{MB} - D_M\, G^{(1)}_{BC}\right)\\ \delta R_{AB} &= \left(D_C\, \delta\Gamma^C_{AB} - D_B\, \delta\Gamma^C_{AC}\right)\end{aligned} \tag{A.1}$$

In equation (A.1), D_A denotes the covariant derivative w.r.t $G^{[0]}_{AB}$. Now we can easily convert D_A to ∇_A (i.e. the covariant derivative w.r.t g_{AB}) by introducing some new terms to account for the correction to Christoffel symbols generated from the extra piece $(\psi^{-D}O_AO_B)$.

$$\delta R_{AB} = \underbrace{\nabla_C \,\delta\Gamma^C_{AB}}_{\text{Term1}} - \underbrace{\nabla_B \,\delta\Gamma^C_{AC}}_{\text{Term2}} - \underbrace{\left(\tilde{\Gamma}^M_{CA}\delta\Gamma^C_{MB} + \tilde{\Gamma}^M_{CB}\delta\Gamma^C_{MA}\right)}_{\text{Term3}} + \underbrace{\tilde{\Gamma}^M_{AB}\delta\Gamma^C_{MC}}_{\text{Term4}}$$

where

$$\begin{aligned}\delta\Gamma^A_{BC} &= \frac{1}{D}\left(\frac{g^{AM} - \psi^{-D}O^A O^M}{2}\right)\Big(\nabla_B \, G^{(1)}_{MC} + \nabla_C \, G^{(1)}_{MB} \\ &\qquad - \nabla_M \, G^{(1)}_{BC} - 2\tilde{\Gamma}^{M'}_{BC} \, G^{(1)}_{MM'}\Big) \\ \tilde{\Gamma}^A_{BC} &= -\psi^{-D}\left(\frac{DN}{2\psi}\right)\left[O^A\left(n_B O_C + n_C O_B\right) - n^A O_B O_C + \psi^{-D} O^A O_B O_C\right]\end{aligned} \tag{A.2}$$

A.1.1 Scalar sector

In this subsection we shall compute H^{scalar}_{AB}. The relevant part of δG_{AB} has the following form.

$$\begin{aligned} G^{(1)}_{AB}|_{scalar} &= O_A O_B \sum_n f_n(R)\, \mathfrak{s}_n \\ R &= D(\psi - 1)\end{aligned} \tag{A.3}$$

To compute H^{scalar}_{AB} we have to substitute equation (A.2) in (A.3) and compute only the leading D piece.

$$\begin{aligned}\delta\Gamma^C_{AB} &= \left(\frac{N}{2}\right)\sum_n \Big[\left(O^C n_A O_B + O^C n_B O_A - n^C O_A O_B\right) f'_n \\ &\qquad + \left(\psi^{-D} O^C O_A O_B\right)\left(f'_n - f_n\right)\Big]\mathfrak{s}_n + \text{Subleading terms} \\ \delta\Gamma^C_{AC} &= 0\end{aligned} \tag{A.4}$$

Now we shall substitute equation (A.4) in each of the four terms in equation (A.2).

$$\begin{aligned}
\text{Term1} &= \nabla_C \, \delta\Gamma^C_{AB} \\
&= \left(\frac{DN^2}{2}\right) \sum_n \mathfrak{s}_n \Big[(f''_n + f'_n)\,(n_B O_A + n_A O_B - O_B O_A) \\
&\qquad\qquad + \psi^{-D}\,(f''_n - f'_n)\, O_B O_A \Big] + \text{Subleading terms} \\
\text{Term2} &= \nabla_B \, \delta\Gamma^C_{AC} = 0
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\text{Term3} &= \left(\tilde\Gamma^M_{CA}\delta\Gamma^C_{MB} + \tilde\Gamma^M_{CB}\delta\Gamma^C_{MA}\right) \\
&= -\left(DN^2\right)\psi^{-D} \sum_n \Big[f'_n \, O_B O_A \Big] \mathfrak{s}_n + \text{Subleading terms} \\
\text{Term4} &= \tilde\Gamma^M_{AB}\delta\Gamma^C_{MC} = 0
\end{aligned}$$

So finally

$$\begin{aligned}
H^{scalar}_{AB} &= \text{Term1} - \text{Term3} \\
&= \left(\frac{DN^2}{2}\right) \sum_n \mathfrak{s}_n \,(f''_n + f'_n) \Big[n_B O_A + n_A O_B - \left(1 - \psi^{-D}\right) O_B O_A \Big]
\end{aligned} \tag{A.6}$$

A.1.2 Vector sector

In this subsection we shall compute H^{vector}_{AB}. The relevant part of δG_{AB} has the following form.

$$G^{(1)}_{AB}|_{vector} = \sum_n v_n(R)\Big([\mathfrak{v}_n]_A O_B + [\mathfrak{v}_n]_B O_A\Big), \quad R \equiv D(\psi - 1) \tag{A.7}$$

Now we shall substitute equation (A.8) in each of the four terms in equation (A.2).

$$\begin{aligned}
\delta\Gamma^A_{BC} = \left(\frac{N}{2}\right) \sum_n \Big\{ & O^A\,(n_B[\mathfrak{v}_n]_C + n_C[\mathfrak{v}_n]_B)\, v'_n \\
& - \left(n^A - \psi^{-D} O^A\right)(O_B[\mathfrak{v}_n]_C + O_C[\mathfrak{v}_n]_B)\, v'_n \\
& + \left[v'_n(n_B O_C + n_C O_B) - v_n\left(\psi^{-D} O_B O_C\right)\right][\mathfrak{v}_n]^A \Big\} \\
& + \text{Subleading terms}
\end{aligned} \tag{A.8}$$

$$\delta\Gamma^C_{AC} = 0$$

$$\begin{aligned}
\text{Term1} &= \nabla_C \, \delta\Gamma^C_{AB} \\
&= \left(\frac{DN^2}{2}\right) \sum_n \left(u_B \left[\mathfrak{v}_n\right]_A + u_A \left[\mathfrak{v}_n\right]_B\right) \left(v''_n + v'_n\right) \\
&\quad + \left(\frac{DN^2}{2}\right) \sum_n \psi^{-D} \left(O_B \left[\mathfrak{v}_n\right]_A + O_A \left[\mathfrak{v}_n\right]_B\right) v''_n \\
&\quad + \left(\frac{N}{2}\right) \sum_n \left(\nabla \cdot \mathfrak{v}_n\right) \left[v'_n \left(n_A O_B + n_B O_A\right) - \psi^{-D} v_n O_B O_A\right] \\
&\quad + \text{Subleading terms}
\end{aligned} \tag{A.9}$$

$$\text{Term2} = \nabla_B \, \delta\Gamma^C_{AC} = 0$$

$$\begin{aligned}
\text{Term3} &= \left(\tilde{\Gamma}^M_{CA} \delta\Gamma^C_{MB} + \tilde{\Gamma}^M_{CB} \delta\Gamma^C_{MA}\right) \\
&= -\left(\frac{DN^2}{2}\right) \psi^{-D} \sum_n v'_n \left(O_B \left[\mathfrak{v}_n\right]_A + O_A \left[\mathfrak{v}_n\right]_B\right) + \text{Subleading terms}
\end{aligned}$$

$$\text{Term4} = \tilde{\Gamma}^M_{AB} \delta\Gamma^C_{MC} = 0$$

So finally

$$\begin{aligned}
H^{vector}_{AB} &= \text{Term1} - \text{Term3} \\
&= \left(\frac{N}{2}\right) \sum_n \left(\nabla \cdot \mathfrak{v}_n\right) \left[v'_n \left(n_A O_B + n_B O_A\right) - \psi^{-D} v_n O_B O_A\right] \\
&\quad + \left(\frac{DN^2}{2}\right) \sum_n \left(v''_n + v'_n\right) \left\{ \left(u_B \left[\mathfrak{v}_n\right]_A + u_A \left[\mathfrak{v}_n\right]_B\right) \right. \\
&\qquad\qquad \left. + \psi^{-D} \left(O_B \left[\mathfrak{v}_n\right]_A + O_A \left[\mathfrak{v}_n\right]_B\right) \right\}
\end{aligned} \tag{A.10}$$

A.1.3 Tensor sector

In this subsection we shall compute H^{tensor}_{AB}. The relevant part of δG_{AB} has the following form.

$$G^{(1)}_{AB}|_{tensor} = \sum_n t_n(R)\, [\mathsf{t}_n]_{AB}, \quad R \equiv D(\psi - 1) \tag{A.11}$$

$$\begin{aligned}\delta\Gamma^A_{BC} = {} & \left(\frac{N}{2}\right)\sum_n t'_n \left\{ [\mathsf{t}_n]^A_C\, n_B + [\mathsf{t}_n]^A_B\, n_C - \left(n^A - \psi^{-D} O^A\right)[\mathsf{t}_n]_{BC} \right\} \\ & + \text{Subleading terms} \end{aligned} \tag{A.12}$$

$$\delta\Gamma^C_{AC} = 0$$

Now we shall substitute equation (A.12) in each of the four terms in equation (A.2).

$$\begin{aligned}\text{Term1} &= \nabla_C\, \delta\Gamma^C_{AB} \\ &= -\left(\frac{DN^2}{2}\right)\sum_n \left[t''_n(1-\psi^{-D}) + t'_n\right][\mathsf{t}_n]_{AB} \\ &\quad + \left(\frac{N}{2}\right)\sum_n t'_n \Big(n_B\left(\nabla_C\,[\mathsf{t}_n]^C_A\right) + n_A\left(\nabla_C\,[\mathsf{t}_n]^C_B\right)\Big) \\ &\quad + \text{Subleading terms}\end{aligned} \tag{A.13}$$

$$\begin{aligned}\text{Term2} &= \nabla_B\, \delta\Gamma^C_{AC} = 0 \\ \text{Term3} &= \left(\tilde\Gamma^M_{CA}\delta\Gamma^C_{MB} + \tilde\Gamma^M_{CB}\delta\Gamma^C_{MA}\right) = \text{Subleading terms} \\ \text{Term4} &= \tilde\Gamma^M_{AB}\delta\Gamma^C_{MC} = 0\end{aligned}$$

So finally

$$\begin{aligned}H^{tensor}_{AB} &= \text{Term1} \\ &= -\left(\frac{DN^2}{2}\right)\sum_n \left[t''_n(1-\psi^{-D}) + t'_n\right][\mathsf{t}_n]_{AB} \\ &\quad + \left(\frac{N}{2}\right)\sum_n t'_n \Big(n_B\left(\nabla_C\,[\mathsf{t}_n]^C_A\right) + n_A\left(\nabla_C\,[\mathsf{t}_n]^C_B\right)\Big)\end{aligned} \tag{A.14}$$

A.1.4 Trace sector

In this subsection we shall compute H^{trace}_{AB}. The relevant part of δG_{AB} has the following form.

$$G^{(1)}_{AB}|_{trace} = \left(\frac{1}{D}\right) P_{AB} \sum_n h_n(R)\mathfrak{s}_n, \quad R \equiv D(\psi - 1) \tag{A.15}$$

As explained in section (2.5), we have an extra factor of $\left(\frac{1}{D}\right)$ compared to the expressions of δG_{AB} in tensor, vector and the scalar sector.

$$\begin{aligned}\delta\Gamma^C_{AB} &= \left(\frac{N}{2D}\right) \sum_n h'_n \Big[\left(n_A P^C_B + n_B P^C_A - n^C P_{AB}\right) + \left(\psi^{-D} O^C P_{AB}\right) \Big] \mathfrak{s}_n \\ &\quad + \text{Subleading terms} \\ \delta\Gamma^C_{AC} &= \left(\frac{N}{2}\right) n_A \sum_n h'_n\, \mathfrak{s}_n + \text{Subleading terms}\end{aligned} \tag{A.16}$$

Now we shall substitute equation (A.16) in each of the four terms in equation (A.2).

$$\begin{aligned}\text{Term1} &= \nabla_C\, \delta\Gamma^C_{AB} \\ &= -\left(\frac{N^2}{2}\right) \sum_n \mathfrak{s}_n \Big\{ \left[\left(1 - \psi^{-D}\right) h''_n + h'_n\right] P_{AB} + 2h'_n\, n_A n_B \Big\} \\ &\quad + \text{Subleading terms} \\ \\ \text{Term2} &= \nabla_B\, \delta\Gamma^C_{AC} \\ &= \left(\frac{DN^2}{2}\right) \sum_n \mathfrak{s}_n \Big[h''_n\, n_A n_B \Big] + \text{Subleading terms} \\ \\ \text{Term3} &= \left(\tilde{\Gamma}^M_{CA} \delta\Gamma^C_{MB} + \tilde{\Gamma}^M_{CB} \delta\Gamma^C_{MA}\right) = \text{Subleading terms} \\ \\ \text{Term4} &= \tilde{\Gamma}^M_{AB} \delta\Gamma^C_{MC} \\ &= -\left(\frac{DN^2}{4}\right) \psi^{-D} \sum_n \mathfrak{s}_n h'_n \Big[n_B O_A + n_A O_B - \left(1 - \psi^{-D}\right) O_B O_A \Big] \\ &\quad + \text{Subleading terms}\end{aligned} \tag{A.17}$$

So finally

$$
\begin{aligned}
H^{trace}_{AB} &= \text{Term1} - \text{Term2} + \text{Term4} \\
&= -\left(\frac{DN^2}{4}\right)\sum_n \mathfrak{s}_n \Big\{ 2h''_n\, n_A n_B + h'_n \left[n_B O_A + n_A O_B - \left(1-\psi^{-D}\right) O_B O_A\right]\psi^{-D} \Big\} \\
&\quad -\left(\frac{N^2}{2}\right)\sum_n \mathfrak{s}_n \Big\{ 2h'_n\, n_A n_B + \left[h'_n + \left(1-\psi^{-D}\right) h''_n\right] P_{AB} \Big\} + \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned}
\tag{A.18}
$$

Note that in the above equation, the second line is of order $\mathcal{O}(1)$. Since in our calculation we are only interested up to order $\mathcal{O}(D)$, we could ignore the second line. For our purpose

$$
\begin{aligned}
H^{trace}_{AB} &= \text{Term1} - \text{Term2} + \text{Term4} \\
&= -\left(\frac{DN^2}{4}\right)\sum_n \mathfrak{s}_n \Big\{ 2h''_n\, n_A n_B + h'_n \left[n_B O_A + n_A O_B - \left(1-\psi^{-D}\right) O_B O_A\right]\psi^{-D} \Big\}
\end{aligned}
\tag{A.19}
$$

A.2 Calculation of the sources - S_{AB}

In this section we shall give details of calculation of S_{AB}. As mentioned in subsection (2.5.6) we have to evaluate $\mathcal{E}_{AB}$ on $G^{[0]}_{AB}$.

$$
\begin{aligned}
\mathcal{E}_{AB} &= R_{AB}\big|_{G^{[0]}_{AB}} - (D-1)\lambda G^{[0]}_{AB} \\
&= \bar{R}_{AB} + \delta R_{AB} - (D-1)\lambda G^{[0]}_{AB} \\
&= (D-1)\lambda g_{AB} + \delta R_{AB} - (D-1)\lambda\left(g_{AB} + G^{(0)}_{AB}\right) \\
&= \delta R_{AB} - D\lambda\, G^{(0)}_{AB} + \text{Subleading Terms}
\end{aligned}
\tag{A.20}
$$

Where $\bar{R}_{AB}$ is the Ricci tensor evaluated on the background metric g_{AB} and δR_{AB} is simply the difference between the Ricci tensor evaluated on $G^{[0]}_{AB}$ and Ricci tensor evaluated on g_{AB}.

Using this notation

$$
S_{AB} = \delta R_{AB} - D\lambda\, G^{(0)}_{AB} \tag{A.21}
$$

Now for our case,

$$
G^{[0]}_{AB} = g_{AB} + \psi^{-D} O_A O_B, \quad G^{(0)}_{AB} = \psi^{-D} O_A O_B \tag{A.22}
$$

As the one form field 'O' is null, the inverse of the above metric (A.22) becomes very simple.

$$G^{[0]AB} = g^{AB} - \psi^{-D} O^A O^B \tag{A.23}$$

Substituting lead ansatz in equation (2.25) we find

$$\begin{aligned}\delta\Gamma^A_{BC} = \left[\frac{g^{AM} - \psi^{-D} O^A O^M}{2}\right] \Big[&\nabla_B \left(\psi^{-D} O_C O_M\right) + \nabla_C \left(\psi^{-D} O_B O_M\right) \\ &- \nabla_M \left(\psi^{-D} O_C O_B\right) \Big]\end{aligned} \tag{A.24}$$

Here ∇ is covariant derivative with respect to the background metric g_{AB}.

For the convenience of computation we shall decompose $\delta\Gamma^A_{BC}$ in two parts

$$\delta\Gamma^A_{BC} = \delta\Gamma^A_{BC}|_{\text{lin.}} + \delta\Gamma^A_{BC}|_{\text{non-lin.}} \tag{A.25}$$

where

$$\begin{aligned}\delta\Gamma^A_{BC}|_{\text{lin.}} &= \frac{1}{2}\left\{\nabla_B(\psi^{-D} O_C O^A) + \nabla_C(\psi^{-D} O_B O^A) - \nabla^A(\psi^{-D} O_B O_C)\right\} \\ \delta\Gamma^A_{BC}|_{\text{non-linear}} &= \frac{1}{2}\psi^{-D} O^A (O \cdot \nabla)(\psi^{-D} O_B O_C)\end{aligned} \tag{A.26}$$

From (2.24) we know that Ricci tensor can be written as

$$R_{AB} = \bar{R}_{AB} + \underbrace{\nabla_C \left[\delta\Gamma^C_{AB}\right] - \nabla_B \left[\delta\Gamma^C_{CA}\right] + \left[\delta\Gamma^C_{CE}\right]\left[\delta\Gamma^E_{AB}\right] - \left[\delta\Gamma^C_{BE}\right]\left[\delta\Gamma^E_{AC}\right]}_{\delta R_{AB}} \tag{A.27}$$

From equation (A.24) it follows

$$\begin{aligned}\delta\Gamma^C_{CA} = &\frac{1}{2}\left\{\nabla_C(\psi^{-D} O_A O^C) + \nabla_A(\psi^{-D} O_C O^C) - \nabla^C(\psi^{-D} O_C O_A)\right\} \\ &+ \frac{1}{2}\psi^{-D} O^C (O \cdot \nabla)(\psi^{-D} O_C O_A) \\ = &\ 0\end{aligned} \tag{A.28}$$

The expression for δR_{AB} simplifies once we substitute equation (A.24)

$$\begin{aligned}\delta R_{AB} &= \nabla_C \left[\delta\Gamma^C_{AB}\right] - \left[\delta\Gamma^C_{BE}\right]\left[\delta\Gamma^E_{AC}\right] \\ &= \underbrace{\nabla_C \left[\delta\Gamma^C_{AB}|_{\text{linear}}\right]}_{\delta R_{AB}|_{\text{linear}}} + \underbrace{\nabla_C \left[\delta\Gamma^C_{AB}|_{\text{non-linear}}\right]}_{\delta R^{(1)}_{AB}|_{\text{non-linear}}} - \underbrace{\left[\delta\Gamma^C_{BE}\right]\left[\delta\Gamma^E_{AC}\right]}_{\delta R^{(2)}_{AB}|_{\text{non-linear}}}\end{aligned} \tag{A.29}$$

At first we present the calculation of $\delta R^{(2)}_{AB}|_{\text{non-linear}}$

$$\begin{aligned}\delta R^{(2)}_{AB}|_{\text{non-lin.}} = & \underbrace{-\left[\delta\Gamma^{C}_{BE}|_{\text{lin.}}\right]\left[\delta\Gamma^{E}_{AC}|_{\text{lin.}}\right]}_{\text{Term-1}} \underbrace{-\left[\delta\Gamma^{C}_{BE}|_{\text{lin.}}\right]\left[\delta\Gamma^{E}_{AC}|_{\text{non-lin.}}\right]}_{\text{Term-2}} \\ & \underbrace{-\left[\delta\Gamma^{C}_{BE}|_{\text{non-lin.}}\right]\left[\delta\Gamma^{E}_{AC}|_{\text{lin.}}\right]}_{\text{Term-3}} \underbrace{-\left[\delta\Gamma^{C}_{BE}|_{\text{non-lin.}}\right]\left[\delta\Gamma^{E}_{AC}|_{\text{non-lin.}}\right]}_{\text{Term-4}}\end{aligned} \tag{A.30}$$

$$\begin{aligned}\text{Term-4} \equiv & -\left[\delta\Gamma^{C}_{BE}|_{\text{non-lin.}}\right]\left[\delta\Gamma^{E}_{AC}|_{\text{non-lin.}}\right] \\ = & -\left\{\frac{1}{2}\psi^{-D}O^{C}(O\cdot\nabla)(\psi^{-D}O_BO_E)\right\}\left\{\frac{1}{2}\psi^{-D}O^{E}(O\cdot\nabla)(\psi^{-D}O_CO_A)\right\} \\ = & \, 0\end{aligned} \tag{A.31}$$

$$\begin{aligned}\text{Term-3} \equiv & -\left[\delta\Gamma^{C}_{BE}|_{\text{non-lin.}}\right]\left[\delta\Gamma^{E}_{AC}|_{\text{lin.}}\right] \\ = & -\left\{\frac{1}{2}\psi^{-D}O^{C}(O\cdot\nabla)(\psi^{-D}O_BO_E)\right\} \\ & \qquad \frac{1}{2}\left\{\nabla_C(\psi^{-D}O_AO^E)+\nabla_A(\psi^{-D}O_CO^E)-\nabla^E(\psi^{-D}O_CO_A)\right\} \\ = & -\frac{1}{4}\psi^{-D}\left\{(O\cdot\nabla)(\psi^{-D}O_BO_E)\right\}\left\{(O\cdot\nabla)(\psi^{-D}O_AO^E)\right\} \\ = & -\frac{1}{4}\psi^{-3D}\{(O\cdot\nabla)O_E\}\{(O\cdot\nabla)O^E\}O_BO_A \\ = & \, 0\end{aligned} \tag{A.32}$$

In the last step we have used (2.29).

Similarly,

$$\text{Term-2} = 0 \tag{A.33}$$

Now we shall compute Term-1, which is non-zero and a bit complicated.

$$
\begin{aligned}
\delta R^{(2)}_{AB}|_{\text{non-lin.}} =& -\left[\delta\Gamma^C_{BE}|_{\text{lin.}}\right]\left[\delta\Gamma^E_{AC}|_{\text{lin.}}\right] \\
=& -\frac{1}{2}\Big\{\nabla_B(\psi^{-D}O_E O^C) + \nabla_E(\psi^{-D}O_B O^C) - \nabla^C(\psi^{-D}O_B O_E)\Big\} \\
& \qquad \frac{1}{2}\Big\{\nabla_C(\psi^{-D}O_A O^E) + \nabla_A(\psi^{-D}O_C O^E) - \nabla^E(\psi^{-D}O_C O_A)\Big\} \\
=& -\frac{1}{4}\Big\{\psi^{-2D}(\nabla_B O_E)(\nabla_C O^E)O^C O_A - \psi^{-2D}O_E(\nabla_B O^C)O_A(\nabla^E O_C) \\
& + \nabla_E(\psi^{-D}O_B O^C)\nabla_C(\psi^{-D}O_A O^E) + \psi^{-2D}O_B(\nabla_E O^C)O^E(\nabla_A O_C) \\
& - \psi^{-2D}O_B(\nabla_E O^C)(\nabla^E O_C)O_A - \psi^{-2D}O_B(\nabla^C O_E)O_A(\nabla_C O^E) \\
& - \psi^{-2D}O_B(\nabla^C O_E)O_C(\nabla_A O^E) + \nabla^C(\psi^{-D}O_B O_E)\nabla^E(\psi^{-D}O_C O_A)\Big\} \\
=& \frac{1}{2}\psi^{-2D}(\nabla_E O^C)(\nabla^E O_C)O_B O_A - \frac{1}{2}\nabla_E(\psi^{-D}O_B O^C)\nabla_C(\psi^{-D}O_A O^E) \\
=& -\frac{1}{2}\left[(O\cdot\nabla)\left(\psi^{-D}O_B\right)\right]\left[(O\cdot\nabla)\left(\psi^{-D}O_A\right)\right] \\
& + \psi^{-2D}\left(\frac{DN}{2\psi}\right)2\left[n^E(O\cdot\nabla)O_E\right]O_B O_A \\
& + \left(\frac{\psi^{-2D}}{2}\right)\left[(\nabla_E O_C)\left(\nabla^E O^C - \nabla^C O^E\right)\right]O_B O_A + \mathcal{O}(1)
\end{aligned}
\tag{A.34}
$$

Now using the fact that

$$
(\nabla_E O_C)\left(\nabla^E O^C\right) = (\nabla_E O_C)\left(\nabla^C O^E\right) = \frac{K^2}{D} + \mathcal{O}(1)
$$

we finally find

$$
\begin{aligned}
&\delta R^{(2)}_{AB}|_{\text{non-lin.}} \\
&= -\frac{1}{2}(O\cdot\nabla)\left[\psi^{-D}O_B\right]\ (O\cdot\nabla)\left[\psi^{-D}O_A\right] + \psi^{-2D}\left(\frac{DN}{\psi}\right)\left[n^E(O\cdot\nabla)O_E\right]O_B O_A \\
&= -\frac{1}{2}(O\cdot\nabla)\left[\psi^{-D}O_B\right]\ (O\cdot\nabla)\left[\psi^{-D}O_A\right] + \psi^{-2D}K\left[u^C(O\cdot\nabla)n_C\right]O_B O_A + \mathcal{O}(1)
\end{aligned}
\tag{A.35}
$$

In the last line we have used the fact that

$$
\left(\frac{DN}{\psi}\right) = K + \mathcal{O}(1)
$$

Now we proceed to the calculation of $\delta R^{(1)}_{AB}|_{\text{non-lin.}}$

$$\begin{aligned}
\delta R^{(1)}_{AB}|_{\text{non-lin.}} &= \nabla_C \left[\delta\Gamma^C_{AB}|_{\text{non-linear}}\right] = \nabla_C \left\{\frac{1}{2}\psi^{-D} O^C (O\cdot\nabla)(\psi^{-D} O_B O_A)\right\} \\
&= \left(\frac{\psi^{-D}}{2}\right)\left[(\nabla\cdot O)\,(O\cdot\nabla)\left(\psi^{-D}O_B O_A\right) + O_A (O\cdot\nabla)\left[(O\cdot\nabla)(\psi^{-D}O_B)\right]\right] \\
&+ \frac{1}{2}\left[(O\cdot\nabla)\left(\psi^{-D}O_A\right)\right]\,\left[(O\cdot\nabla)\left(\psi^{-D}O_B\right)\right] + \frac{1}{2}(O\cdot\nabla)\left[\psi^{-2D}O_B (O\cdot\nabla) O_A\right] \\
&= \left(\frac{\psi^{-2D}}{2}\right)\left(\frac{DN}{\psi} - \nabla\cdot O\right)\left[\left(\frac{DN}{\psi}\right) O_B O_A - (O\cdot\nabla)(O_B O_A)\right] \\
&\quad + \frac{1}{2}\left[(O\cdot\nabla)\left(\psi^{-D}O_A\right)\right]\,\left[(O\cdot\nabla)\left(\psi^{-D}O_B\right)\right] \\
&\quad - \left(\frac{\psi^{-2D}}{2}\right)(O\cdot\nabla)\left[\frac{DN}{\psi} O_A O_B\right] + \mathcal{O}(1)
\end{aligned} \tag{A.36}$$

We can use identity(A.75) to simplify (A.36)

$$\begin{aligned}
&\delta R^{(1)}_{AB}|_{\text{non-lin.}} \\
&= \left(\frac{\psi^{-2D}}{2}\right)\left[(n\cdot\nabla)K + K(\nabla\cdot u)\right] O_B O_A - \left(\frac{\psi^{-2D}}{2}\right)(O\cdot\nabla)\left[K\, O_A O_B\right] \\
&+ \frac{1}{2}\left[(O\cdot\nabla)\left(\psi^{-D}O_A\right)\right]\,\left[(O\cdot\nabla)\left(\psi^{-D}O_B\right)\right] + \mathcal{O}(1) \\
&= \left(\frac{\psi^{-2D}}{2}\right)\left[(u\cdot\nabla)K + K(\nabla\cdot u)\right] O_B O_A - \psi^{-2D}\, K\left[u^C (O\cdot\nabla) n_C\right] O_A O_B \\
&+ \frac{1}{2}\left[(O\cdot\nabla)\left(\psi^{-D}O_A\right)\right]\,\left[(O\cdot\nabla)\left(\psi^{-D}O_B\right)\right] + \mathcal{O}(1)
\end{aligned} \tag{A.37}$$

In the last step we have used the subsidiary condition on O_A.

$$(O\cdot\nabla)O_A = \left[u^C (O\cdot\nabla) n_C\right] O_A \tag{A.38}$$

Adding (A.35) and (A.37) we get the desired expression for $\delta R_{AB}|_{\text{non-lin.}}$

$$\delta R_{AB}|_{\text{non-lin.}} = \left(\frac{\psi^{-2D}}{2}\right)\left[(u\cdot\nabla)K + K(\nabla\cdot u)\right] O_B O_A + \mathcal{O}(1) \tag{A.39}$$

Finally, $\delta R_{AB}|_{\text{non-lin.}}$ becomes

$$\begin{aligned}
&\delta R_{AB}|_{\text{non-lin.}} \\
&= \left(\frac{\psi^{-2D}}{2}\right)\left[\frac{DN}{\psi}(\nabla\cdot u) + (u\cdot\nabla)\left(\frac{DN}{\psi}\right)\right] O_A O_B \\
&= \left(\frac{\psi^{-2D}}{2}\right)\left(\frac{DN}{\psi}(\hat{\nabla}\cdot u)\right)\, O_A O_B
\end{aligned} \tag{A.40}$$

Where, $\hat{\nabla}$ is defined as follows, for any general tensor with n indices $W_{A_1A_2\cdots A_n}$

$$\hat{\nabla}_A W_{A_1A_2\cdots A_n} = \Pi_A^C\, \Pi_{A_1}^{C_1}\Pi_{A_2}^{C_2}\cdots\Pi_{A_n}^{C_n}\left(\nabla_C W_{C_1C_2\cdots C_n}\right) \tag{A.41}$$

Now, we shall calculate the linear terms in Ricci tensor

$$\begin{aligned}
\delta R_{AB}|_{\text{lin.}} &= \nabla_C\left[\delta\Gamma^C_{BA}|_{\text{lin.}}\right] \\
&= \underbrace{\frac{1}{2}\nabla_C\left\{\nabla_B\left(\psi^{-D}O_AO^C\right)\right\}}_{T_1} + \underbrace{\frac{1}{2}\nabla_C\left\{\nabla_A\left(\psi^{-D}O_BO^C\right)\right\}}_{T_2} \underbrace{-\frac{1}{2}\nabla_C\left\{\nabla^C\left(\psi^{-D}O_AO_B\right)\right\}}_{T_3}
\end{aligned} \tag{A.42}$$

$$\begin{aligned}
T_1 &= \frac{1}{2}\nabla_C\left\{\nabla_B\left(\psi^{-D}O_AO^C\right)\right\} \\
&= \frac{1}{2}[\nabla_C,\nabla_B]\left(\psi^{-D}O_AO^C\right) + \frac{1}{2}\nabla_B\nabla_C\left(\psi^{-D}O_AO^C\right) \\
&= \left(\frac{\psi^{-D}}{2}\right)\bar{R}_{EB}O^EO_A - \frac{1}{2}\nabla_B\left[\psi^{-D}\left\{\left(\frac{DN}{\psi} - \nabla\cdot O\right)O_A - (O\cdot\nabla)\,O_A\right\}\right] + \mathcal{O}(1) \\
&= \left(\frac{D\lambda}{2}\right)\psi^{-D}O_AO_B + \left(\frac{DN}{2\psi}\right)\psi^{-D}\left[\frac{DN}{\psi} - \nabla\cdot O - u^C(O\cdot\nabla)n_C\right]n_BO_A \\
&\qquad - \frac{\psi^{-D}}{2}O_A\nabla_B\left(\hat{\nabla}\cdot u\right) + \mathcal{O}(1)
\end{aligned} \tag{A.43}$$

In the last step we have used subsidiary condition on O and also the fact that

$$\left(\frac{DN}{\psi} - \nabla\cdot O\right) = \left(\frac{DN}{\psi} - K\right) + \mathcal{O}(1) \sim \mathcal{O}(1)$$

Similarly,

$$\begin{aligned}
T_2 = \left(\frac{D\lambda}{2}\right)\psi^{-D}O_AO_B + \left(\frac{DN}{2\psi}\right)\psi^{-D}\left[\frac{DN}{\psi} - \nabla\cdot O - u^C(O\cdot\nabla)n_C\right]n_AO_B \\
- \frac{\psi^{-D}}{2}O_B\nabla_A\left(\hat{\nabla}\cdot u\right) + \mathcal{O}(1)
\end{aligned} \tag{A.44}$$

$$\begin{aligned} T_3 &= -\frac{1}{2}\nabla_C\nabla^C(\psi^{-D}O_BO_A) \\ &= -\frac{1}{2}\left(\nabla^2\psi^D\right)O_AO_B - \left(\nabla_C\psi^{-D}\right)\left(\nabla^C O_AO_B\right) - \frac{\psi^{-D}}{2}\nabla^2(O_AO_B) \\ &= \psi^{-D}\left[\left(\frac{DN}{\psi}\right)(n\cdot\nabla)(O_AO_B) - \frac{1}{2}\nabla^2(O_AO_B)\right] \end{aligned} \tag{A.45}$$

Adding (A.43),(A.44),(A.45) we get the expression for $[R_L]_{AB}$

$$\begin{aligned} &\delta R_{AB}|_{\text{lin.}} \\ &= \psi^{-D}\Bigg[\left(\frac{DN}{\psi}\right)(n\cdot\nabla)(O_AO_B) + D\lambda\, O_AO_B - \frac{1}{2}(O_A\nabla_B + O_B\nabla_A)\left(\hat{\nabla}\cdot u\right) \\ &\qquad - \frac{1}{2}\nabla^2(O_AO_B) + \left(\frac{DN}{2\psi}\right)\left(\frac{DN}{\psi} - \nabla\cdot O - u^C(O\cdot\nabla)n_C\right)(n_BO_A + n_AO_B)\Bigg] + \mathcal{O}(1) \\ &= \psi^{-D}\Bigg[K\,(n\cdot\nabla)(O_AO_B) - \frac{1}{2}\nabla^2(O_AO_B) + D\lambda\, O_AO_B - \frac{1}{2}(O_A\nabla_B + O_B\nabla_A)\left(\hat{\nabla}\cdot u\right) \\ &\qquad + \frac{K}{2}\left(\frac{(n\cdot\nabla)K}{K} + \nabla\cdot u - u^C(O\cdot\nabla)n_C\right)(n_BO_A + n_AO_B)\Bigg] + \mathcal{O}(1) \end{aligned} \tag{A.46}$$

Using, the following identities

$$(n\cdot\nabla)(O_AO_B) = 2[u^C(n\cdot\nabla)n_C]O_AO_B + \left(O_AP^C_B + O_BP^C_A\right)[(u\cdot\nabla)O_C] \tag{A.47}$$

$$\begin{aligned} O_B\nabla^2O_A + O_A\nabla^2O_B = 2\left[K[u^D(n\cdot\nabla)n_D] + (u\cdot\nabla)K\right]O_AO_B + (O_BP^C_A + O_AP^C_B)\nabla^2O_C \\ - [\left(\nabla^C O_D\right)(\nabla_C O^D)][n_AO_B + n_BO_A] \end{aligned} \tag{A.48}$$

We have used the identity (A.76) for the derivation of the above equation.

$$\begin{aligned} (O_A\nabla_B + O_B\nabla_A)\left(\hat{\nabla}\cdot u\right) &= \left(P^E_AO_B + P^E_BO_A\right)\hat{\nabla}_E\left(\hat{\nabla}\cdot u\right) + 2\,O_AO_B(u\cdot\nabla)(\hat{\nabla}\cdot u) \\ &\quad + (n_AO_B + n_BO_A)(O\cdot\nabla)(\hat{\nabla}\cdot u) \\ &= (n_AO_B + n_BO_A)(n\cdot\nabla)(\hat{\nabla}\cdot u) + \mathcal{O}(1) \end{aligned} \tag{A.49}$$

The expression of $\delta R_{AB}|_{\text{lin.}}$ becomes

$$\begin{aligned}
&\delta R_{AB}|_{\text{lin.}}\\
&= \psi^{-D}\Bigg[D\,\lambda\,(O_A O_B) + \left(O_B P_A^C + O_A P_B^C\right)\left(K(u\cdot\nabla)O_C - \frac{1}{2}\nabla^2 O_C\right)\\
&+ (n_A O_B + N_B O_A)\left\{\frac{K}{2}\left(\frac{n\cdot\nabla K}{K} + \nabla\cdot u - u^C(O\cdot\nabla)n_C\right) + \frac{1}{2}K_{CD}K^{CD} - \frac{1}{2}(n\cdot\nabla)\left(\hat{\nabla}\cdot u\right)\right\}\Bigg]
\end{aligned} \tag{A.50}$$

Substituting (A.50) and (A.40) in (A.21) we get the source term $S_{AB}^{(-1)}$

$$\begin{aligned}
&S_{AB} = \delta R_{AB} - D\lambda G_{AB}^{(0)} = \delta R_{AB}|_{\text{lin.}} + \delta R_{AB}|_{\text{non-lin.}} - D\lambda\,\psi^{-D}O_A O_B\\
&= \psi^{-D}\Bigg[\psi^{-D}\left(\frac{K}{2}\right)\left(\hat{\nabla}\cdot u\right)O_B O_A + \left(O_B P_A^C + O_A P_B^C\right)\left(K(u\cdot\nabla)O_C - \frac{1}{2}\nabla^2 O_C\right)\\
&\quad + (n_A O_B + N_B O_A)\left\{\frac{K}{2}\left(\frac{n\cdot\nabla K}{K} + \nabla\cdot u - u^C(O\cdot\nabla)n_C\right) + \frac{1}{2}K_{CD}K^{CD} - \frac{1}{2}(n\cdot\nabla)(\hat{\nabla}\cdot u)\right\}\Bigg]\\
&= \psi^{-D}\left(\frac{K}{2}\right)\Bigg\{\psi^{-D}\left(\hat{\nabla}\cdot u\right)O_B O_A + \left(O_B P_A^C + O_A P_B^C\right)\left[\frac{\hat{\nabla}^2 u_C}{K} - \frac{\nabla_C K}{K} + u^D K_{DC} - (u\cdot\nabla)u_C\right]\\
&+ (n_A O_B + n_B O_A)\left[\frac{1}{K}K_{CD}K^{CD} - \frac{1}{K}(n\cdot\nabla)(\hat{\nabla}\cdot u) + \frac{n\cdot\nabla K}{K} + \hat{\nabla}\cdot u - 2\,\frac{u\cdot\nabla K}{K} + u\cdot K\cdot u\right]\Bigg\}
\end{aligned} \tag{A.51}$$

In the last line we have used the following identity (see appendix A.6 for derivation)

$$P_B^C\nabla^2 O_C = P_B^C\left[\nabla_C K - \hat{\nabla}^2 u_C + K\left(u^D K_{DC} - (u\cdot\nabla)u_C\right)\right] + \mathcal{O}(1) \tag{A.52}$$

Now,

$$S_{AB} = \psi^{-D}\left(\frac{K}{2}\right)\Bigg\{\psi^{-D}(\hat{\nabla}\cdot u)O_B O_A + \left(O_B P^C_A + O_A P^C_B\right)\left[\frac{\hat{\nabla}^2 u_C}{K} - \frac{\nabla_C K}{K} + u^D K_{DC} - u\cdot\nabla u_C\right]$$
$$+ (n_A O_B + n_B O_A)\Bigg[\frac{1}{K}K_{CD}K^{CD} - \frac{\hat{\nabla}^2 K}{K^2} + 2\,\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u - \frac{1}{K}u^D\bar{R}_{DE}u^E + \frac{(n\cdot\nabla)K}{K}$$
$$+ \hat{\nabla}\cdot u - 2\,\frac{(u\cdot\nabla)K}{K} + (u\cdot K\cdot u)\Bigg]\Bigg\} \tag{A.53}$$

In the last line we have used the following identity

$$(n\cdot\nabla)\left(\hat{\nabla}\cdot u\right) = \left(\frac{\hat{\nabla}^2 K}{K} - 2\,(u\cdot\nabla)K + K\,(u\cdot K\cdot u) + u^D\bar{R}_{DE}u^E\right) \tag{A.54}$$

Where $\left(\hat{\nabla}\cdot u\right)$ is given in appendix A.4.

We will use the following two identity to further simplify S_{AB}

$$u^C\bar{R}_{DC}\,u^D = -n^C\bar{R}_{DC}\,n^D \tag{A.55}$$

$$\text{and,}\quad (n\cdot\nabla K) = -n^A\bar{R}_{AD}\,n^D + \frac{\hat{\nabla}^2 K}{K} - K_{AB}K^{AB} \tag{A.56}$$

The first one (A.55) of the above two identities follows from the fact that $\bar{R}_{DC}$ (Ricci tensor evaluated on the background) is proportional to the background metric g_{DC} and both u and n are normalized time-like and space-like vectors respectively. For the derivation of the second one (A.56) see A.6.4.

Using (A.55) and (A.56), we get S_{AB}

$$S_{AB} = \psi^{-D}\left(\frac{K}{2}\right)\Bigg[\psi^{-D}\left(\hat{\nabla}\cdot u\right)O_B O_A + (n_A O_B + n_B O_A)\left(\hat{\nabla}\cdot u\right)$$
$$+ (O_B P^C_A + O_A P^C_B)\left(\frac{\hat{\nabla}^2 u_C}{K} - \frac{\nabla_C K}{K} + u^D K_{DC} - (u\cdot\nabla)u_C\right)\Bigg] \tag{A.57}$$

Let us note the presence of '$K(\hat{\nabla}\cdot u)$' term in S_{AB}. From the leading order calculation it follows that it is of order $\mathcal{O}(D)$ on $\psi = 1$ hypersurface(see eq (2.22)). This is sort of 'anomalous', since naive order counting suggests that this term should be of order $\mathcal{O}(D^2)$ and this may not be the case once we are away from the membrane.

Now for any generic term, which is of order $\mathcal{O}(1)$ when evaluated on $(\psi = 1)$ hypersurface, will have corrections of order $\mathcal{O}\left(\frac{1}{D}\right)$ (or further suppressed) as one goes away from $\psi = 1$. But, for 'anomalous' term like $K(\hat{\nabla} \cdot u)$ that is not the case. Below, we shall examine this term in more detail. We can expand $(\hat{\nabla} \cdot u)$ in $\left[\psi - 1 = \frac{R}{D}\right]$ as follows

$$
\begin{aligned}
\hat{\nabla} \cdot u &= \left(\hat{\nabla} \cdot u\right)_{\psi=1} + \frac{\psi - 1}{N}(n \cdot \nabla)\left(\hat{\nabla} \cdot u\right)_{\psi=1} \\
&= \left(\hat{\nabla} \cdot u\right)_{R=0} + \left(\frac{R}{K}\right)\left(\frac{\hat{\nabla}^2 K}{K} - 2\,(u \cdot \nabla)K + K\,(u \cdot K \cdot u) + u^D \bar{R}_{DE} u^E\right)_{R=0} \\
&= \left(\hat{\nabla} \cdot u\right)_{R=0} - \frac{R}{K}\left(\hat{\nabla} \cdot E\right)_{R=0}
\end{aligned}
\tag{A.58}
$$

We don't need to expand any other term since $\hat{\nabla} \cdot u$ is the only 'anomalous' term in this order. Substituting (A.58) in (A.57) we get the final expression for S_{AB}

$$
\begin{aligned}
S_{AB} = e^{-R}\left(\frac{K}{2}\right)\Bigg[& e^{-R}\left(\left(\hat{\nabla} \cdot u\right)_{R=0} - \frac{R}{K}\left(\hat{\nabla} \cdot E\right)_{R=0}\right) O_B O_A \\
&+ (n_A O_B + n_B O_A)\left(\left(\hat{\nabla} \cdot u\right)_{R=0} - \frac{R}{K}\left(\hat{\nabla} \cdot E\right)\right)_{R=0} \\
&+ (O_B P_A^C + O_A P_B^C)\left(\frac{\hat{\nabla}^2 u_C}{K} - \frac{\nabla_C K}{K} + u^D K_{DC} - (u \cdot \nabla) u_C\right)_{R=0}\Bigg]
\end{aligned}
\tag{A.59}
$$

A.3 Intermediate steps for matching with AdS Black Hole

Since we know that the horizon is not at $r = 1$, this implies $\psi(r = 1) \neq 1$.

We shall assume the following expansion of ψ around $r = 1$.

$$
\psi(r) = 1 + \frac{X_1}{D} + \frac{X_2}{D^2} + \left(a_{10} + \frac{a_{11}}{D}\right)(r - 1) + a_{20}(r - 1)^2 + \mathcal{O}\left(\frac{1}{D}\right)^3
$$
$$
\text{where } X_1,\ X_2,\ a_{10},\ a_{11},\ a_{20} \text{ are constants and } (r - 1) \sim \mathcal{O}\left(\frac{1}{D}\right)
\tag{A.60}
$$

Substituting equation (A.60) in equation (2.86) and solving it order by order in $\left(\frac{1}{D}\right)$ we find the following solutions for the coefficients.

$$a_{10} = 1, \quad a_{11} = X_1 - 2, \quad a_{20} = 0 \tag{A.61}$$

To fix X_1 and X_2 we have to use the fact that $\psi = 1$ correspond to horizon. We can determine the horizon of Schwarzschild-AdS black hole r_0 order by order in $\left(\frac{1}{D}\right)$.

$$r_0 = 1 - \frac{\log 2}{D} + \left(\frac{1}{D}\right)^2 \left[-2\log 2 + \frac{(\log 2)^2}{2}\right] + \mathcal{O}\left(\frac{1}{D}\right)^3 \tag{A.62}$$

Noe setting $\psi(r_0) = 1$ we find

$$X_1 = \log 2, \quad X_2 = \frac{(\log 2)^2}{2}$$

So finally we found

$$\begin{aligned}\psi(r) = 1 + \frac{\log 2}{D} + \left(\frac{1}{D}\right)^2 \left[\frac{(\log 2)^2}{2}\right] \\ + \left[1 + \left(\frac{\log 2 - 2}{D}\right)\right](r-1) + \mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{A.63}$$

A.4 The derivation of $(\nabla \cdot u)$

Note that to compute the full spacetime divergence of u^A we also need to know the normal derivative of u^A away from the membrane.

$$\begin{aligned}\nabla \cdot u &= P^{AB}\nabla_A u_B + n^B (n\cdot\nabla) u_B \\ &= P^{AB}\nabla_A u_B - u^B (n\cdot\nabla) n_B \\ &= P^{AB}\nabla_A u_B - \frac{(u\cdot\nabla)K}{K}\end{aligned} \tag{A.64}$$

In the last line we have used the identity $\left[(n\cdot\nabla)n_A = \frac{\Pi_A^C \nabla_C K}{K}\right]$.

We know that the first term in equation (A.64) is of order $\mathcal{O}(1)$ on the membrane. It follows from the equation of motion at zeroth order. However, to determine the source term we need to know this expression even away from the $(\psi = 1)$ hypersurface. Below we shall determine this term in an expansion in $(\psi - 1)$ and we shall see that the coefficient of the linear term is also of order $\mathcal{O}(1)$.

Consider the expansion of u^A from $(\psi = 1)$ hypersurface.

$$u^A = u^A|_{\psi=1} + \frac{\psi - 1}{N}[(n \cdot \nabla)u^A]|_{\psi=1} + \cdots \tag{A.65}$$

Substituting this expansion in first term of the equation (A.64) we find

$$\begin{aligned}
&(P^{AB}\nabla_A u_B)\\
&= P^{AB}\nabla_A\left(u_B|_{\psi=1} + \frac{\psi - 1}{N}[(n \cdot \nabla)u_B]|_{\psi=1} + \cdots\right)\\
&= P^{AB}\nabla_A u_B|_{\psi=1} + P^{AB}\nabla_A\left(\frac{\psi - 1}{N}[(n \cdot \nabla)u_B]|_{\psi=1} + \cdots\right)\\
&= P^{AB}\nabla_A u_B|_{\psi=1} + \left(\frac{\psi - 1}{N}\right)P^{AB}\nabla_A[(n \cdot \nabla)u_B]|_{\psi=1} - P^{AB}\left(\frac{\psi - 1}{N^2}\right)(\nabla_A N)[(n \cdot \nabla)u_B]|_{\psi=1}\\
&= P^{AB}\nabla_A u_B|_{\psi=1} + \left(\frac{\psi - 1}{N}\right)P^{AB}\nabla_A[(n \cdot \nabla)u_B]|_{\psi=1} + \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{A.66}$$

Now we shall process the coefficient of $(\psi - 1)$.

$$\begin{aligned}
P^{AB}\nabla_A[(n \cdot \nabla)u_B] &= P^{AB}\nabla_A\left[-n_B\frac{(u \cdot \nabla)K}{K} + P^D_B[(n \cdot \nabla)n_D - (u \cdot \nabla)O_D]\right]\\
&= -(u \cdot \nabla)K + K[n^D(u \cdot \nabla)O_D] + P^{AD}\nabla_A[(n \cdot \nabla)n_D - (u \cdot \nabla)O_D]\\
&= -u^D\bar{R}_{DE}n^E + K\,(u \cdot K \cdot u) - 2\,(u \cdot \nabla)K + \frac{\hat{\nabla}^2 K}{K} + u^D\bar{R}_{DE}u^E\\
&= \frac{\hat{\nabla}^2 K}{K} - 2\,(u \cdot \nabla)K + K\,(u \cdot K \cdot u) + u^D\bar{R}_{DE}u^E
\end{aligned} \tag{A.67}$$

Note that $(\psi - 1)$ is also of order $\mathcal{O}\left(\frac{1}{D}\right)$. Therefore combining equations (A.66) and (A.67) we find

$$\nabla \cdot u = \left(\hat{\nabla} \cdot u\right)\Big|_{\psi=1} - \frac{(u \cdot \nabla)K}{K} + \frac{\psi - 1}{N}\left[\frac{\hat{\nabla}^2 K}{K} - 2\,(u \cdot \nabla)K + K\,(u \cdot K \cdot u) + u^D\bar{R}_{DE}u^E\right] + \mathcal{O}\left(\frac{1}{D}\right)$$

A.5 The divergence of the vector constraint equation at 1st order

The membrane equation at 1st order is given in equation (2.60). For convenience we are quoting the equation here again.

$$P^A_B\left[\frac{\hat{\nabla}^2 u_A}{K} - \frac{\nabla_A K}{K} + u_C K^C_A - (u\cdot\nabla)u_A\right] = \mathcal{O}\left(\frac{1}{D}\right) \tag{A.68}$$

We could compute the divergence of each of the term separately.

$$\text{Divergence} \equiv \underbrace{\nabla^B\left(P^A_B\frac{\hat{\nabla}^2 u_A}{K}\right)}_{Term-1} - \underbrace{\nabla^B\left(P^A_B\frac{\nabla_A K}{K}\right)}_{Term-2} + \underbrace{\nabla^B\left(P^A_B\, u_C K^C_A\right)}_{Term-3} - \underbrace{\nabla^B\left(P^A_B\,(u\cdot\nabla)u_A\right)}_{Term-4} \tag{A.69}$$

$$\begin{aligned}
\text{Term-1} &\equiv \nabla^B\left(P^A_B\frac{\hat{\nabla}^2 u_A}{K}\right)\\
&= -n^A[\nabla^2 u_A - K(n\cdot\nabla)u_A] + \frac{1}{K}P^A_B\nabla^B\left[\nabla^2 u_A - K(n\cdot\nabla)u_A\right]\\
&= (u\cdot\nabla)K + \frac{1}{K}P^{AB}\left[-\bar{R}_{BD}(\nabla^D u_A) + \bar{R}_{BEAD}(\nabla^E u^D) + \nabla^E(\bar{R}_{BEAD}\,u^D) + \nabla^2(\nabla_B u_A)\right]\\
&\qquad - P^{AB}\nabla_B[(n\cdot\nabla)u_A]\\
&= (u\cdot\nabla)K + \frac{1}{K}\nabla^2(\nabla\cdot u) - P^{AB}\nabla_B[(n\cdot\nabla)u_A]\\
&= (u\cdot\nabla)K
\end{aligned} \tag{A.70}$$

In the last line we have used (A.66) for the expression of $(\nabla\cdot u)$

$$\begin{aligned}
\text{Term-2} &\equiv \nabla^B\left(P^A_B\frac{\nabla_A K}{K}\right)\\
&= \frac{\nabla^2 K}{K} - (n\cdot\nabla)K\\
&= \frac{\hat{\nabla}^2 K}{K}
\end{aligned} \tag{A.71}$$

$$
\begin{aligned}
\text{Term-3} &\equiv \nabla^B \left(P^A_B \, u_C K^C_A\right) \\
&= P^A_B \, u^C \nabla^B (\Pi^D_A \nabla_D n_C) \\
&= -K u^C (n\cdot\nabla) n_C + u^C \nabla^2 n_C \\
&= (u\cdot\nabla) K
\end{aligned}
\tag{A.72}
$$

$$
\begin{aligned}
\text{Term-4} &\equiv \nabla^B \left(P^A_B \, (u\cdot\nabla) u_A\right) \\
&= -K \, n^A (u\cdot\nabla) u_A + P^{AB} (\nabla_B u^E)(\nabla_E u_A) + P^{AB} u^E (\nabla_B \nabla_E u_A) \\
&= K \, (u\cdot K\cdot u) + P^{AB} u^E \bar{R}_{BEAD} u^D \\
&= K \, (u\cdot K\cdot u) + u^E \bar{R}_{ED} u^D
\end{aligned}
\tag{A.73}
$$

Adding equations (A.70), (A.71), (A.72) and (A.73) we get the divergence of the vector constraint equation as

$$
\text{Divergence} \equiv -u^C \bar{R}_{DC} u^D - \frac{\hat{\nabla}^2 K}{K} + 2(u\cdot\nabla)K - K(u^A K_{AB} u^B) = 0 \tag{A.74}
$$

A.6 Identities

In this appendix we shall prove some identities that we have used for our computation.

A.6.1 Proof of (2.21) from (2.26)

$$
\begin{aligned}
&\nabla^2(\psi^{-D}) = 0 \\
\Rightarrow \; &\frac{DN}{\psi} - K = \frac{(n\cdot\nabla)N}{N} - \frac{N}{\psi} \\
&\qquad\qquad = \frac{(n\cdot\nabla)(\psi\, K)}{\psi\, K} - \frac{N}{\psi} + \mathcal{O}\left(\frac{1}{D}\right) \\
&\qquad\qquad = \frac{(n\cdot\nabla)K}{K} + \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned}
\tag{A.75}
$$

A.6.2 Proof of equations (A.48)

We have used the following identity for derivation of (A.48)

$$\begin{aligned} u^A\nabla^2 n_A &= u^A\nabla_C\left[n^C(n\cdot\nabla)n_A + K_A^C\right] \\ &= K\left[u^A(n\cdot\nabla)n_A\right] + u^A\nabla_C K_A^C + \mathcal{O}(1) \\ &= K\left[u^A(n\cdot\nabla)n_A\right] + u^A\nabla_C K_A^C + \mathcal{O}(1) \\ &= K\left[u^A(n\cdot\nabla)n_A\right] + (u\cdot\nabla)K + \mathcal{O}(1) \\ &= 2(u\cdot\nabla)K + \mathcal{O}(1) \end{aligned} \tag{A.76}$$

In deriving equation (A.76) we have used the following identity

$$(n\cdot\nabla)n_A = \Pi_A^C\left[\frac{\nabla_C K}{K}\right] + \mathcal{O}\left(\frac{1}{D}\right) \tag{A.77}$$

Proof of (A.77)

$$\begin{aligned} \nabla_A N^2 &= \nabla_A[(\nabla_B\psi)(\nabla^B\psi)] \\ \Rightarrow 2N\nabla_A N &= 2(\nabla^B\psi)(\nabla_A\nabla_B\psi) \\ \Rightarrow\ N\nabla_A N &= (\nabla^B\psi)(\nabla_B\nabla_A\psi) \\ \Rightarrow\ N\nabla_A N &= Nn^B\nabla_B(Nn_A) \\ \Rightarrow\quad \nabla_A N &= (n\cdot\nabla)(Nn_A) \\ \Rightarrow (n\cdot\nabla)n_A &= \Pi_A^C\left(\frac{\nabla_C N}{N}\right) \\ &= \Pi_A^C\left[\frac{\nabla_C K}{K}\right] + \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{A.78}$$

A.6.3 Proof of (A.52)

$$P_B^C\nabla^2 O_C = P_B^C(\nabla^2 n_C - \nabla^2 u_C) \tag{A.79}$$

$$
\begin{aligned}
P_B^C\nabla^2 n_C &= P_B^C\ \nabla^D\nabla_D n_C \\
&= P_B^C\ \nabla^D\nabla_D\left(\frac{\nabla_C\psi}{N}\right) \\
&= P_B^C\ \nabla^D\left(\frac{\nabla_D\nabla_C\psi}{N} - \frac{1}{N^2}(\nabla_D N)(\nabla_C\psi)\right) \\
&= P_B^C\ \left(\frac{\nabla^D\nabla_C\nabla_D\psi}{N} - \frac{2}{N^2}(\nabla_D N)(\nabla^D\nabla_C\psi)\right) \\
&= \frac{1}{N}P_B^C\left([\nabla_D,\nabla_C]\nabla^D\psi + \nabla_C\nabla_D\nabla^D\psi\right) + \mathcal{O}(1) \\
&= \frac{1}{N}P_B^C\left(\bar{R}_{DCE}{}^{D}\ \nabla^E\psi + \nabla_C\nabla_D(Nn^D)\right) + \mathcal{O}(1) \\
&= \frac{1}{N}P_B^C\nabla_C\left(n_D\nabla^D N + N\nabla^D n_D\right) + \mathcal{O}(1) \\
&= P_B^C\left(\frac{N\nabla_C\nabla^D n_D}{N} + \frac{(\nabla_C N)(\nabla^D n_D)}{N}\right) + \mathcal{O}(1) \\
&= P_B^C\left(\nabla_C K + \frac{\nabla_C K}{K}K\right) + \mathcal{O}(1) \\
&= 2P_B^C\nabla_C K + \mathcal{O}(1)
\end{aligned}
\tag{A.80}
$$

$$
\begin{aligned}
P_B^C(\nabla^2 u_C) &= P_B^C\ \nabla^D\nabla_D u_C \\
&= P_B^C\nabla^D\left(\Pi_C^E\ \nabla_D u_E + n_C n^E\nabla_D u_E\right) \\
&= P_B^C\nabla^D\left(\Pi_C^E\ \nabla_D u_E\right) + \mathcal{O}(1) \\
&= P_B^C\nabla^D\left(\Pi_C^E\ \Pi_D^F\ \nabla_F u_E + \Pi_C^E\ n_D(n\cdot\nabla)u_E\right) + \mathcal{O}(1) \\
&= P_B^C\nabla^D\left(\Pi_C^E\ \Pi_D^F\ \nabla_F u_E\right) + P_B^C K(n\cdot\nabla)u_C + \mathcal{O}(1) \\
&= P_B^C\Pi_N^D\nabla^N\left(\Pi_C^E\ \Pi_D^F\ \nabla_F u_E\right) + P_B^C n^D(n\cdot\nabla)\left(\Pi_C^E\ \Pi_D^F\ \nabla_F u_E\right) \\
&\qquad + P_B^C K(n\cdot\nabla)u_C + \mathcal{O}(1) \\
&= P_B^C\Pi_N^D\nabla^N\left(\Pi_C^E\ \Pi_D^F\ \nabla_F u_E\right) + P_B^C K(n\cdot\nabla)u_C + \mathcal{O}(1)
\end{aligned}
\tag{A.81}
$$

Adding(A.80) and (A.81) we get the expression for $P_B^C\nabla^2 O_C$

$$P_B^C \nabla^2 O_C = P_B^C \left[2\nabla_C K - \Pi_N^D \nabla^N \left(\Pi_C^E \, \Pi_D^F \, \nabla_F u_E\right) - K(n\cdot\nabla)u_C\right] + \mathcal{O}(1) \tag{A.82}$$

Now from our subsidiary condition,

$$\begin{aligned} &P_B^C (O\cdot\nabla)O_C = 0 \\ \Rightarrow &P_B^C (n\cdot\nabla)u_C = P_B^C \left[(n\cdot\nabla)n_C - (u\cdot\nabla)n_C + (u\cdot\nabla)u_C\right] \end{aligned} \tag{A.83}$$

Substituting (A.83) in (A.82) we get,

$$\begin{aligned} P_B^C \nabla^2 O_C = P_B^C &\Big\{ 2\nabla_C K - \Pi_N^D \nabla^N \left(\Pi_C^E \, \Pi_D^F \, \nabla_F u_E\right) \\ &- K\left[(n\cdot\nabla)n_C - (u\cdot\nabla)n_C + (u\cdot\nabla)u_C\right] \Big\} + \mathcal{O}(1) \\ = P_B^C &\left\{ \nabla_C K - \Pi_N^D \nabla^N \left(\Pi_C^E \, \Pi_D^F \, \nabla_F u_E\right) + K\left[u^D K_{DC} - (u\cdot\nabla)u_C\right] \right\} + \mathcal{O}(1) \\ = P_B^C &\left\{ \nabla_C K - \hat{\nabla}^2 u_C + K\left[u^D K_{DC} - (u\cdot\nabla)u_C\right] \right\} + \mathcal{O}(1) \end{aligned} \tag{A.84}$$

Where $\hat{\nabla}$ is defined in eq (A.41).

A.6.4 Proof of (A.56)

$$\begin{aligned} (n\cdot\nabla)\mathcal{K} &= (n^A\nabla_A)(\nabla_B n^B) \\ &= n^A\left[\nabla_A, \nabla_B\right]n^B + n^A\nabla_B(\nabla_A n^B) \\ &= -n^A \bar{R}_{ABD}{}^{B}\, n^D + \nabla_B\left[(n\cdot\nabla)n^B\right] - K_{AB}K^{AB} \\ &= -n^A \bar{R}_{AD} n^D + \nabla_B\left[\frac{\Pi^{BA}\nabla_A K}{K}\right] - K_{AB}K^{AB} \\ &= -n^A \bar{R}_{AD}\, n^D + \frac{\hat{\nabla}^2 K}{K} - K_{AB}K^{AB} \end{aligned} \tag{A.85}$$

Appendix B

Appendices for Chapter 3

B.1 Calculation of the sources - S_{AB}

In this section we shall give the details of the calculation of S_{AB}. As mentioned before, the source will be given by E_{AB} calculated on $G^{[0]}_{AB} = g_{AB} + \psi^{-D} O_A O_B$.

We shall follow Appendix A.2 for computation. The first step would be to decompose the source in the following way.

$$
\begin{aligned}
S_{AB} &\equiv E_{AB}|_{G^{[0]}_{AB}} \\
&= R_{AB}|_{G^{[0]}_{AB}} - (D-1)\lambda G^{[0]}_{AB} \\
&= -(D-1)\lambda\, \psi^{-D} O_A O_B + \underbrace{\nabla_C \left[\delta\Gamma^C_{AB}|_{\text{lin}}\right]}_{\delta R_{AB}|_{\text{lin}}} + \underbrace{\nabla_C \left[\delta\Gamma^C_{AB}|_{\text{non-lin}}\right]}_{\delta R^{(1)}_{AB}|\text{non-lin}} \underbrace{- \left[\delta\Gamma^C_{BE}\right]\left[\delta\Gamma^E_{AC}\right]}_{\delta R^{(2)}_{AB}|\text{non-lin}}
\end{aligned}
\tag{B.1}
$$

where

$$
\begin{aligned}
\delta\Gamma^A_{BC}|_{\text{lin.}} &= \frac{1}{2}\left\{\nabla_B(\psi^{-D} O_C O^A) + \nabla_C(\psi^{-D} O_B O^A) - \nabla^A(\psi^{-D} O_B O_C)\right\} \\
\delta\Gamma^A_{BC}|_{\text{non-lin}} &= \frac{1}{2}\psi^{-D} O^A (O\cdot\nabla)(\psi^{-D} O_B O_C) \\
\delta\Gamma^A_{BC} &= \delta\Gamma^A_{BC}|_{\text{lin.}} + \delta\Gamma^A_{BC}|_{\text{non-lin.}}
\end{aligned}
\tag{B.2}
$$

At first we present the calculation of $\delta R^{(2)}_{AB}|_{\text{non-linear}}$

$$
\begin{aligned}
\delta R^{(2)}_{AB}|_{\text{non-lin.}} = &\underbrace{-\left[\delta\Gamma^C_{BE}|_{\text{lin.}}\right]\left[\delta\Gamma^E_{AC}|_{\text{lin.}}\right]}_{\text{Term-1}} \underbrace{-\left[\delta\Gamma^C_{BE}|_{\text{lin.}}\right]\left[\delta\Gamma^E_{AC}|_{\text{non-lin.}}\right]}_{\text{Term-2}} \\
&\underbrace{-\left[\delta\Gamma^C_{BE}|_{\text{non-lin.}}\right]\left[\delta\Gamma^E_{AC}|_{\text{lin.}}\right]}_{\text{Term-3}} \underbrace{-\left[\delta\Gamma^C_{BE}|_{\text{non-lin.}}\right]\left[\delta\Gamma^E_{AC}|_{\text{non-lin.}}\right]}_{\text{Term-4}}
\end{aligned}
\tag{B.3}
$$

As previously, in this case also, Term-2=Term-3=Term-4=0;

Now we need to calculate Term-1.

$$
\begin{aligned}
\delta R^{(2)}_{AB}|_{\text{non-lin.}} &= -\left[\delta\Gamma^{C}_{BE}|_{\text{lin.}}\right]\left[\delta\Gamma^{E}_{AC}|_{\text{lin.}}\right] \\
&= \frac{1}{2}\psi^{-2D}(\nabla_E O^C)(\nabla^E O_C)O_B O_A - \frac{1}{2}\nabla_E(\psi^{-D}O_B O^C)\nabla_C(\psi^{-D}O_A O^E) \\
&= -\frac{1}{2}[(O\cdot\nabla)(\psi^{-D}O_B)][(O\cdot\nabla)(\psi^{-D}O_A)] + \psi^{-2D}\left(\frac{DN}{\psi}\right)\, Q\, O_A O_B \\
&+ \frac{\psi^{-2D}}{2}(\nabla_E O_C)(\nabla^E O^C - \nabla^C O^E)O_B O_A - \psi^{-2D}\, Q^2\, O_B O_A
\end{aligned}
\tag{B.4}
$$

Where, $Q \equiv u^E(O\cdot\nabla)n_E$

$$
\begin{aligned}
\delta R^{(2)}_{AB}|_{\text{non-lin.}} &= -\frac{1}{2}[(O\cdot\nabla)(\psi^{-D}O_B)][(O\cdot\nabla)(\psi^{-D}O_A)] + \psi^{-2D}K\, Q\, O_A O_B \\
&+ \frac{\psi^{-2D}}{2}\left[(\nabla_E O_C)(\nabla^E O^C - \nabla^C O^E) - 2\, Q^2 + 2\, Q\frac{(n\cdot\nabla)K}{K}\right] O_B O_A
\end{aligned}
\tag{B.5}
$$

In deriving (B.5) we have used,

$$
\frac{DN}{\psi} = K + \frac{(n\cdot\nabla)K}{K} \tag{B.6}
$$

Now we proceed to the calculation of $\delta R^{(1)}_{AB}|_{\text{non-lin.}}$

$$
\begin{aligned}
&\delta R^{(1)}_{AB}|_{\text{non-lin.}} \\
&= \nabla_C \left[\frac{1}{2}\psi^{-D} O^C (O\cdot\nabla)\left(\psi^{-D} O_A O_B\right)\right] \\
&= \left(\frac{\psi^{-D}}{2}\right)\left[(\nabla\cdot O)\,(O\cdot\nabla)\left(\psi^{-D} O_B O_A\right) + O_A (O\cdot\nabla)\left[(O\cdot\nabla)(\psi^{-D} O_B)\right]\right] \\
&\quad + \frac{1}{2}\left[(O\cdot\nabla)\left(\psi^{-D} O_A\right)\right]\ \left[(O\cdot\nabla)\left(\psi^{-D} O_B\right)\right] + \frac{1}{2}(O\cdot\nabla)\left[\psi^{-2D} O_B (O\cdot\nabla) O_A\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}[(O\cdot\nabla)(\psi^{-D}O_A)][(O\cdot\nabla)(\psi^{-D}O_B)] - \frac{\psi^{-2D}}{2}(O\cdot\nabla)[K\, O_A O_B] \\
&+ \frac{\psi^{-2D}}{2}\left(\frac{DN}{\psi} - \nabla\cdot O\right)\left(\frac{DN}{\psi} - 2\,Q\right) O_A O_B \\
&+ \frac{\psi^{-2D}}{2}\left[3\,Q^2 + 2\,(O\cdot\nabla)Q - (O\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right) - \frac{(n\cdot\nabla)K}{K} 2\,Q\right] O_A O_B
\end{aligned}
\tag{B.7}
$$

Now,

$$
\begin{aligned}
&\left(\frac{DN}{\psi} - \nabla\cdot O\right)\left(\frac{D\,N}{\psi} - 2\,Q\right) \\
&= \left[\frac{(n\cdot\nabla)K}{K} + \frac{(n\cdot\nabla)^2 K}{K^2} - 2\,\frac{[(n\cdot\nabla)K]^2}{K^3} + \hat{\nabla}\cdot u - \frac{1}{K}(u\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right)\right. \\
&\qquad \left. - \frac{(u\cdot\nabla)K}{K} + \frac{1}{K}\frac{(n\cdot\nabla)K}{K}\frac{(u\cdot\nabla)K}{K}\right]\left[K + \frac{(n\cdot\nabla)K}{K} - 2\,Q\right] \\
&= K\left(\hat{\nabla}\cdot u\right) + (O\cdot\nabla)K + \frac{(n\cdot\nabla)^2 K}{K} - 2\left[\frac{(n\cdot\nabla)K}{K}\right]^2 + \frac{(O\cdot\nabla)K}{K}\frac{(n\cdot\nabla)K}{K} \\
&- 2Q\frac{(O\cdot\nabla)K}{K} - (u\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right) + \frac{(n\cdot\nabla)K}{K}\frac{(u\cdot\nabla)K}{K} + (\hat{\nabla}\cdot u)\frac{(n\cdot\nabla)K}{K} - 2Q(\hat{\nabla}\cdot u)
\end{aligned}
\tag{B.8}
$$

Where, $\hat{\nabla}$ is defined in (2.58)

In deriving (B.8) we have used (see B.2.3 for derivation),

$$
\nabla\cdot u = \hat{\nabla}\cdot u - \frac{(u\cdot\nabla)K}{K} - \frac{1}{K}(u\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right) + \frac{1}{K}\frac{(n\cdot\nabla)K}{K}\frac{(u\cdot\nabla)K}{K}
\tag{B.9}
$$

Using, (B.8) we get the final expression of $\delta R^{(1)}_{AB}|_{\text{non-lin.}}$,

$$
\begin{aligned}
&\delta R^{(1)}_{AB}|_{\text{non-lin.}} \\
&= \frac{\psi^{-2D}}{2} K(\hat{\nabla}\cdot u)\, O_A O_B - \psi^{-2D} K\, Q\, O_A O_B + \frac{1}{2}\left[(O\cdot\nabla)\left(\psi^{-D} O_A\right)\right]\left[(O\cdot\nabla)(\psi^{-D} O_B)\right] \\
&+ \frac{\psi^{-2D}}{2}\left[3\,Q^2 + 2(O\cdot\nabla)Q - 2\,Q\left(\frac{(n\cdot\nabla)K}{K} + \frac{(O\cdot\nabla)K}{K}\right) + (\tilde{\nabla}\cdot u)\left(\frac{(n\cdot\nabla)K}{K} - 2Q\right)\right] O_A O_B
\end{aligned}
\tag{B.10}
$$

Adding (B.5) and (B.10) we get

$$
\begin{aligned}
&\delta R_{AB}|_{\text{non-lin.}} \\
&\equiv \delta R^{(1)}_{AB}|_{\text{non-lin.}} + \delta R^{(2)}_{AB}|_{\text{non-lin.}} \\
&= \frac{1}{2}\psi^{-2D}\, K(\hat{\nabla}\cdot u)\, O_A O_B + \frac{1}{2}\psi^{-2D}\Bigg[(\nabla_E O^C)(\nabla^E O_C - \nabla_C O^E) + Q^2 + 2(O\cdot\nabla)Q \\
&\qquad - 2Q\frac{(O\cdot\nabla)K}{K} + (\hat{\nabla}\cdot u)\left(\frac{(n\cdot\nabla)K}{K} - 2Q\right)\Bigg] O_A O_B
\end{aligned}
\tag{B.11}
$$

Let us note the presence of '$K(\hat{\nabla}\cdot u)$ ' term in $\delta R_{AB}|_{\text{non-lin.}}$. From the membrane equation at first subleading order, it follows that this term is of order $\mathcal{O}(1)$ on $\psi = 1$ hypersurface. This is sort of 'anomalous', since naive order counting suggests that this term should be or order $\mathcal{O}(D^2)$ and this may not be the case once we are away from the membrane.

Now for any generic term, which is of order $\mathcal{O}(1)$ when evaluated on $(\psi = 1)$ hypersurface, will have corrections of order $\mathcal{O}\left(\frac{1}{D}\right)$ (or further suppressed) as one goes away from $\psi = 1$. While integrating the ODEs, this is the reason we could ignore all the implicit ψ dependence in the source. However from the above discussion we could see that such reasoning does not work for '$K(\hat{\nabla}\cdot u)$ ' (or in fact any such 'anomalous' term). Below we shall examine this term in more detail.

We can expand $(\hat{\nabla}\cdot u)$ in $\left[\psi-1=\frac{R}{D}\right]$ as follows

$$
\begin{aligned}
\hat{\nabla}\cdot u = {} & (\hat{\nabla}\cdot u)\Big|_{\psi=1} + \frac{\psi-1}{N}(n\cdot\nabla)(\hat{\nabla}\cdot u)\Big|_{\psi=1} + \frac{(\psi-1)^2}{2N^2}\left[\frac{(n\cdot\nabla)N}{N}\right]\Bigg|_{\psi=1}\left[(n\cdot\nabla)(\hat{\nabla}\cdot u)\right]\Bigg|_{\psi=1} \\
& + \frac{(\psi-1)^2}{2N^2}\left[(n\cdot\nabla)(n\cdot\nabla)\left(\hat{\nabla}\cdot u\right)\right]\Bigg|_{\psi=1} + \mathcal{O}(\psi-1)^3 \\
= {} & \left(\hat{\nabla}\cdot u\right)\Bigg|_{R=0} - R\left[\frac{\hat{\nabla}\cdot E}{K}\right]_{R=0} - \frac{R^2}{2}\left[\left(\frac{(n\cdot\nabla)K}{K^3}\right)\left(\hat{\nabla}\cdot E\right)\right]_{R=0} \\
& + R^2\left[\left(\frac{D^2}{K^3}\right)\,\mathfrak{s}_2\right]_{R=0} + \mathcal{O}\left(\frac{1}{D}\right)^2 \\
= {} & \left(\hat{\nabla}\cdot u\right)\Bigg|_{R=0} - R\left[\frac{\hat{\nabla}\cdot E}{K}\right]_{R=0} + R^2\left[\left(\frac{D^2}{K^3}\right)\,\mathfrak{s}_2\right]_{R=0} + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned}
\tag{B.12}
$$

Where E_A is given in equation (3.16).

In the second line we have used the following two identities (to prove them we have used *Mathematica Version-11*),

$$
\begin{aligned}
(n\cdot\nabla)(\hat{\nabla}\cdot u)\Big|_{R=0} &= -(\hat{\nabla}\cdot E)\Big|_{R=0} + \mathcal{O}\left(\frac{1}{D}\right) \\
(n\cdot\nabla)(n\cdot\nabla)(\hat{\nabla}\cdot u)\Big|_{R=0} &= 2\,D^2\left(\frac{\mathfrak{s}_2}{K}\right)\Bigg|_{R=0} + \mathcal{O}(1)
\end{aligned}
\tag{B.13}
$$

Clearly the second and the third term in the last line of equation (B.12) (which encode the value of $(\hat{\nabla}\cdot u)$ off the membrane) could contribute in $\delta R_{AB}|_{\text{non-lin.}}$ at order $\mathcal{O}(1)$.

Substituting (B.12) in equation (B.11) we find

$$
\begin{aligned}
&\delta R_{AB}\big|_{\text{non-lin.}} \\
&= \psi^{-2D}\left(\frac{K}{2}\right)\left[\left(\hat{\nabla}\cdot u\right)_{\psi=1} - R\left(\frac{\hat{\nabla}\cdot E}{K}\right)_{\psi=1} - \frac{1}{2K}\left[\nabla_{(E}u_{F)}\nabla_{(C}u_{D)}P^{FC}P^{ED}\right]\right]O_A O_B \\
&\quad + \frac{\psi^{-2D}}{2}R^2\left(\frac{D^2}{K^2}\right)(\mathfrak{s}_2)\,O_A O_B \\
&- \psi^{-2D}\Bigg[2u^A K^C_A \frac{\nabla_C K}{K} - (\nabla_C u_A)(\nabla^C u^A) - (u\cdot K\cdot K\cdot u) + 3\left(\frac{(u\cdot\nabla)K}{K}\right)^2 \\
&- \frac{K}{D}\left(\frac{(u\cdot\nabla)K}{K}\right) + \frac{K}{D}(u\cdot K\cdot u) - 2\,\frac{(u\cdot\nabla)K}{K}(u\cdot K\cdot u) - u^E u^F \bar{R}_{EDFC}O^C O^D\Bigg]O_A O_B \\
&= \psi^{-2D}\left(\frac{K}{2}\right)\left[\left(\hat{\nabla}\cdot u\right)_{\psi=1} - R\left(\frac{\hat{\nabla}\cdot E}{K}\right)_{\psi=1} - \frac{1}{2K}\left[\nabla_{(E}u_{F)}\nabla_{(C}u_{D)}P^{FC}P^{ED}\right]\right]O_A O_B \\
&+ \frac{\psi^{-2D}}{2}R^2\left(\frac{D^2}{K^2}\right)(\mathfrak{s}_2)\,O_A O_B \;-\psi^{-2D}\Bigg[\left(\frac{u\cdot\nabla K}{K}\right)^2 + 4\,u^A K^B_A\frac{\nabla_B K}{K} - (\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B) \\
&- (u\cdot K\cdot u)^2 - 2\,\frac{\hat{\nabla}^A K}{K}\left[(u\cdot\nabla)u_A\right] - \left[(u\cdot\hat{\nabla})u_A\right]\left[(u\cdot\hat{\nabla})u^A\right] + 2\left[(u\cdot\nabla)u^A\right](u^B K_{BA}) \\
&- 3(u\cdot K\cdot K\cdot u) - \frac{\hat{\nabla}_A K}{K}\frac{\hat{\nabla}^A K}{K} - \frac{K}{D}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) + u^E u^F n^D n^C \bar{R}_{CEFD}\Bigg]O_A O_B \\
\\
&= e^{-2R}\left(\frac{K}{2}\right)\left[\left(\hat{\nabla}\cdot u\right)_{R=0} - \frac{1}{2K}\left(\nabla_E u_F + \nabla_F u_E\right)\left(\nabla_C u_D + \nabla_D u_C\right)P^{FC}P^{ED}\right]O_A\,O_B \\
&\quad + \left(\frac{e^{-2R}}{2}\right)\left[-R\left(\hat{\nabla}\cdot E\right)_{R=0} + R^2\left(\frac{D^2}{K^2}\,\mathfrak{s}_2\right)_{R=0}\right]O_A O_B - e^{-2R}(\mathfrak{s}_1)\,O_A O_B
\end{aligned}
\tag{B.14}
$$

where

$$\mathfrak{s}_1 = u^E u^F n^D n^C \bar{R}_{CEFD} + \left(\frac{u\cdot\nabla K}{K}\right)^2 + \frac{\hat{\nabla}_A K}{K}\left[4\,u^B K^A_B - 2\left[(u\cdot\nabla)u_A\right] - \frac{\hat{\nabla}^A K}{K}\right]$$
$$-(\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B) - (u\cdot K\cdot u)^2 - \left[(u\cdot\hat{\nabla})u_A\right]\left[(u\cdot\hat{\nabla})u^A\right] + 2\left[(u\cdot\nabla)u^A\right](u^B K_{BA})$$
$$-3(u\cdot K\cdot K\cdot u) - \frac{K}{D}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)$$

$$\mathfrak{s}_2 = \frac{K^2}{D^2}\Bigg[-\frac{K}{D}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) - 2\,\lambda - (u\cdot K\cdot K\cdot u) + 2\left(\frac{\nabla_A K}{K}\right)u^B K^A_B - \left(\frac{u\cdot\nabla K}{K}\right)^2$$
$$+2\left(\frac{u\cdot\nabla K}{K}\right)(u\cdot K\cdot u) - \left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right) - (u\cdot K\cdot u)^2 + n^B n^D u^E u^F \bar{R}_{FBDE}\Bigg] \tag{B.15}$$

Now we shall calculate those terms in Ricci tensor that are linear in ψ^{-D}

$$\delta R_{AB}|_{\text{lin.}} = \nabla_C\left[\delta\Gamma^C_{BA}|_{\text{lin.}}\right]$$
$$= \underbrace{\frac{1}{2}\nabla_C\left\{\nabla_B\left(\psi^{-D}O_A O^C\right)\right\}}_{T_1} + \underbrace{\frac{1}{2}\nabla_C\left\{\nabla_A\left(\psi^{-D}O_B O^C\right)\right\}}_{T_2} \underbrace{-\frac{1}{2}\nabla_C\left\{\nabla^C\left(\psi^{-D}O_A O_B\right)\right\}}_{T_3} \tag{B.16}$$

$$T_1 = \frac{1}{2}\nabla_C\left\{\nabla_B\left(\psi^{-D}O_A O^C\right)\right\}$$
$$= \frac{1}{2}[\nabla_C,\nabla_B]\left(\psi^{-D}O_A O^C\right) + \frac{1}{2}\nabla_B\nabla_C\left(\psi^{-D}O_A O^C\right)$$
$$= \frac{\psi^{-D}}{2}\left(\bar{R}_{BD}O^D O_A + \bar{R}_{CBAD}O^D O^C\right) - \frac{1}{2}\nabla_B\left[\psi^{-D}\left\{\left(\frac{DN}{\psi} - \nabla\cdot O\right)O_A - Q\,O_A\right\}\right]$$
$$= \frac{\psi^{-D}}{2}\left(\bar{R}_{BD}O^D O_A + \bar{R}_{CBAD}O^D O^C\right) + \left(\frac{DN}{2\psi}\right)\psi^{-D}\left[\frac{DN}{\psi} - \nabla\cdot O - Q\right]n_B O_A$$
$$-\frac{1}{2}\psi^{-D}\nabla_B\left\{\left(\frac{DN}{\psi} - \nabla\cdot O - Q\right)O_A\right\}$$
$$= \frac{\psi^{-D}}{2}\left(\bar{R}_{BD}O^D O_A + \bar{R}_{CBAD}O^D O^C\right) + \frac{\psi^{-D}}{2}\Big[(n\cdot\nabla)K + K(\nabla\cdot u - Q)\Big]n_B O_A$$
$$+\frac{\psi^{-D}}{2}\left[\frac{(n\cdot\nabla)^2 K}{K} - 2\left(\frac{(n\cdot\nabla)K}{K}\right)^2 - \frac{K}{D}\left(\frac{(n\cdot\nabla)K}{K}\right)\right]n_B O_A + \frac{\psi^{-D}}{2}\left(\frac{K}{D}\right)(\nabla_B O_A)$$
$$-\frac{\psi^{-D}}{2}O_A\nabla_B\left[\frac{(n\cdot\nabla)K}{K} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u + \hat{\nabla}\cdot u\right] \tag{B.17}$$

Similarly, we will get T_2 by interchanging A and B indices

$$\begin{aligned} T_3 &= -\frac{1}{2}\nabla_C\nabla^C(\psi^{-D}O_BO_A) \\ &= -\frac{1}{2}\left(\nabla^2\psi^{D}\right)O_AO_B - \left(\nabla_C\psi^{-D}\right)\left(\nabla^C O_AO_B\right) - \frac{\psi^{-D}}{2}\nabla^2(O_AO_B) \\ &= \psi^{-D}\left[\left(\frac{DN}{\psi}\right)(n\cdot\nabla)\left(O_AO_B\right) - \frac{1}{2}\nabla^2(O_AO_B)\right] \end{aligned} \tag{B.18}$$

Adding T_1, T_2, T_3 we get the expression for $\delta R_{AB}|_{\text{lin.}}$

$$\begin{aligned} &\delta R_{AB}|_{\text{lin}} \\ &= \psi^{-D}\,(D-1)\,\lambda\,O_AO_B + \psi^{-D}\bar{R}_{CABD}O^DO^C + \psi^{-D}K\,(n\cdot\nabla)(O_AO_B) \\ &+ \frac{\psi^{-D}}{2}(n_BO_A + n_AO_B)[(n\cdot\nabla)K + K(\nabla\cdot u - Q)] - \frac{\psi^{-D}}{2}\left(O_A\nabla^2O_B + O_B\nabla^2O_A\right) \\ &+ \frac{\psi^{-D}}{2}\left\{\frac{(n\cdot\nabla)^2K}{K} - 2\left[\frac{(n\cdot\nabla)K}{K}\right]^2 - \frac{K}{D}\,\frac{(n\cdot\nabla)K}{K}\right\}(n_BO_A + O_Bn_A) \\ &+ \psi^{-D}\left\{\left[\frac{(n\cdot\nabla)K}{K}\right](n\cdot\nabla)(O_AO_B) - (\nabla_CO_A)(\nabla^CO_B)\right\} + \frac{\psi^{-D}}{2}\frac{K}{D}\left[\nabla_BO_A + \nabla_AO_B\right] \\ &- \frac{\psi^{-D}}{2}(O_A\nabla_B + O_B\nabla_A)\left[\frac{(n\cdot\nabla)K}{K} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u + \hat{\nabla}\cdot u\right] \end{aligned} \tag{B.19}$$

Now, we shall decompose the source in the way as mentioned in (3.13). Note that the decomposition of a general 2-index symmetric tensor (C_{AB}) is the following

$$\begin{aligned} C_{AB} = {} & P_A^DP_B^EC_{DE} + (P_A^EO_B + P_B^EO_A)C_{ED}u^D + (P_A^En_B + P_B^En_A)C_{ED}O^D \\ & + (n_AO_B + n_BO_A)(O^EC_{ED}u^D) + O_AO_B(u^EC_{ED}u^D) + n_An_B(O^EC_{ED}O^D) \end{aligned} \tag{B.20}$$

Using (B.20) we shall first decompose each of the tensor structure appearing in (B.19)

$$\begin{aligned} (n\cdot\nabla)(O_AO_B) &= 2\left[u^C(n\cdot\nabla)n_C\right]O_AO_B + (O_AP_B^C + O_BP_A^C)(n\cdot\nabla)O_C \\ &= 2\left[u^C(n\cdot\nabla)n_C\right]O_AO_B + (O_AP_B^C + O_BP_A^C)(u\cdot\nabla)O_C \end{aligned} \tag{B.21}$$

$$O_B \nabla^2 O_A + O_A \nabla^2 O_B$$
$$= 2\left[K[u^D(n\cdot\nabla)n_D] + (u\cdot\nabla)K - u^D K^C_D\left(\frac{\nabla_C K}{K}\right) + u^D(n\cdot\nabla)^2 n_D + (\nabla_C u_D)(\nabla^C u^D)\right] O_A O_B$$
$$- [\left(\nabla^C O_D\right)\left(\nabla_C O^D\right)][n_A O_B + n_B O_A] + (O_B P^C_A + O_A P^C_B)\nabla^2 O_C \tag{B.22}$$

$$(\nabla_C O_A)(\nabla^C O_B) = (u^D \nabla_C n_D)(u^E \nabla^C n_E) O_A O_B + (\nabla_D O_C)(\nabla^D O_{C'}) P^C_A P^{C'}_B$$
$$+ (O_B P^C_A + O_A P^C_B)[(\nabla_F O_C)(u^D \nabla^F n_D)] \tag{B.23}$$

$$\nabla_B O_A + \nabla_A O_B = 2\,(u\cdot K\cdot u) O_A O_B + \, Q\,(n_A O_B + n_B O_A) + P^C_A P^{C'}_B(\nabla_C O_{C'} + \nabla_{C'} O_C)$$
$$+ (O_B P^C_A + O_A P^C_B)[(u\cdot\nabla)O_C + u^D K_{CD}] \tag{B.24}$$

$$(O_A\nabla_B + O_B\nabla_A)\left[\frac{(n\cdot\nabla)K}{K} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u + \hat{\nabla}\cdot u\right]$$
$$= -2\,\frac{(u\cdot\nabla)K}{D}\,O_A O_B - \left(O_A P^C_B + O_B P^C_A\right)\frac{\nabla_C K}{D}$$
$$+ (O_A n_B + O_B n_A)(O\cdot\nabla)\left[\frac{(n\cdot\nabla)K}{K} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u + \hat{\nabla}\cdot u\right] \tag{B.25}$$

$$\bar{R}_{CABD} O^D O^C = P^E_A P^F_B\,\bar{R}_{CEFD} O^D O^C + O_A O_B\, u^E u^F \bar{R}_{CEFD} O^D O^C$$
$$+ (P^E_A O_B + P^E_B O_A)\bar{R}_{CEFD} O^D O^C u^F \tag{B.26}$$

Using (B.21), (B.22), (B.23), (B.24), (B.25) we can decompose $\delta R_{AB}|_{\text{lin}}$ in the following way

$$\delta R_{AB}|_{\text{lin}} = \delta R^{(S_1)}_{\text{lin}} O_A O_B + \delta R^{(S_2)}_{\text{lin}}(n_A O_B + n_B O_A) + \delta R^{(S_3)}_{\text{lin}} n_A n_B + \delta R^{(tr)}_{\text{lin}} P_{AB}$$
$$+ (O_A P^C_B + O_B P^C_A)\left[\delta R^{(V_1)}_{\text{lin}}\right]_C + (n_A P^C_B + n_B P^C_A)\left[\delta R^{(V_2)}_{\text{lin}}\right]_C + \left[\delta R^{(T)}_{\text{lin}}\right]_{AB} \tag{B.27}$$

Where

$$\delta R^{(S_1)} = \psi^{-D}(D-1)\,\lambda + \psi^{-D}\Big[u^E u^F \bar{R}_{CEFD} n^D n^C - (u\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right) + u^A K_A^C \frac{\nabla_C K}{K}$$
$$- \frac{(n\cdot\nabla)K}{K}\frac{(u\cdot\nabla)K}{K} + 2\frac{(n\cdot\nabla)K}{K}[u^C(n\cdot\nabla)n_C] - (u^D\nabla_C n_D)(u^E\nabla^C n_E)$$
$$- u^A(n\cdot\nabla)^2 n_A - (\nabla_C u_A)(\nabla^C u^A) + \frac{K}{D}(u\cdot K\cdot u) + \frac{K}{D}\frac{(u\cdot\nabla)K}{K}\Big]$$
$$= \psi^{-D}(D-1)\,\lambda + \psi^{-D}\Big[2u^A K_A^C \frac{\nabla_C K}{K} - (\nabla_C u_A)(\nabla^C u^A) - (u\cdot K\cdot K\cdot u) - \frac{K}{D}\frac{(u\cdot\nabla)K}{K}$$
$$+ 3\left(\frac{(u\cdot\nabla)K}{K}\right)^2 + \frac{K}{D}(u\cdot K\cdot u) - 2\,\frac{(u\cdot\nabla)K}{K}(u\cdot K\cdot u) + u^E u^F \bar{R}_{CEFD} n^D n^C\Big]$$
$$= \psi^{-D}(D-1)\,\lambda + \psi^{-D}\,\mathfrak{s}_1 \tag{B.28}$$

Where,

$$\mathfrak{s}_1 = \left(\frac{u\cdot\nabla K}{K}\right)^2 + \frac{\hat{\nabla}_A K}{K}\left[4\,u^B K_B^A - 2\,[(u\cdot\nabla)u_A] - \frac{\hat{\nabla}^A K}{K}\right] - (\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B)$$
$$- (u\cdot K\cdot u)^2 - \left[(u\cdot\hat{\nabla})u_A\right][(u\cdot\hat{\nabla})u^A] + 2\left[(u\cdot\nabla)u^A\right](u^B K_{BA}) - 3\,(u\cdot K\cdot K\cdot u)$$
$$- \frac{K}{D}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) + u^E u^F \bar{R}_{CEFD}\, n^D n^C \tag{B.29}$$

$$\delta R^{(S_2)} = \frac{\psi^{-D}}{2}\Bigg[K\left\{\hat{\nabla}\cdot u - \frac{(u\cdot\nabla)K}{K} - \frac{1}{K}(u\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right) + \frac{1}{K}\frac{(n\cdot\nabla)K}{K}\frac{(u\cdot\nabla)K}{K}\right\}$$
$$+ (n\cdot\nabla)K - K\,Q + \frac{(n\cdot\nabla)^2 K}{K} - 2\,\left(\frac{(n\cdot\nabla)K}{K}\right)^2 - \frac{K}{D}\frac{(n\cdot\nabla)K}{K} + \frac{K}{D}\,Q$$
$$+ (\nabla^C O_A)(\nabla_C O^A) - (O\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u + \hat{\nabla}\cdot u\right)\Bigg] \tag{B.30}$$

We shall massage the above expression for $\delta R^{(S_2)}$ a little more.

Let us note the presence of '$K(\hat{\nabla}\cdot u)$' term in $\delta R^{(S_2)}$. From the discussion just below the equation (B.11) it is clear that we need to take the expansion of $\hat{\nabla}\cdot u$ in $\psi - 1$. The $\psi - 1$

expansion of $(\hat{\nabla}\cdot u)$ is given by (B.12)

$$\hat{\nabla}\cdot u = \left(\hat{\nabla}\cdot u\right)_{R=0} - R\left[\frac{\hat{\nabla}\cdot E}{K}\right]_{R=0} + R^2\left[\left(\frac{D^2}{K^3}\right)\mathfrak{s}_2\right]_{R=0} + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{B.31}$$

Substituting equation (B.31) in equation (B.30) we find

$$\begin{aligned}
\delta R^{(S_2)} =& \frac{\psi^{-D}}{2}\left[K\left(\hat{\nabla}\cdot u\right)_{R=0} - R\left(\hat{\nabla}\cdot E\right)_{R=0} + R^2\left[\left(\frac{D^2}{K^2}\right)\mathfrak{s}_2\right]_{R=0}\right] \\
&+ \frac{\psi^{-D}}{2}\Bigg[-K\left\{\frac{(u\cdot\nabla)K}{K} + \frac{1}{K}(u\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right) - \frac{1}{K}\frac{(n\cdot\nabla)K}{K}\frac{(u\cdot\nabla)K}{K}\right\} \\
&+ (n\cdot\nabla)K - K\,Q + \frac{(n\cdot\nabla)^2K}{K} - 2\left(\frac{(n\cdot\nabla)K}{K}\right)^2 - \frac{K}{D}\frac{(n\cdot\nabla)K}{K} + \frac{K}{D}Q \\
&+ (\nabla^C O_A)(\nabla_C O^A) - (O\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u + \hat{\nabla}\cdot u\right)\Bigg]
\end{aligned} \tag{B.32}$$

Now it turns out that it is possible to rewrite the last three lines of equation (B.32) in terms of the already defined scalar structures $\mathfrak{s}_1$ plus few extra terms which could be expressed as functions of membrane equation.

We have used *Mathematica Version 11* for this purpose[1]

$$\begin{aligned}
\delta R^{(S_2)} =& e^{-R}\left[-\mathfrak{s}_1 + \frac{K}{2}\left((\hat{\nabla}\cdot u) - \frac{1}{2K}\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}P^{AC}P^{BD}\right)\right]\Bigg|_{R=0} \\
&+ \frac{e^{-R}}{2}\left[-R\left(\hat{\nabla}\cdot E\right)_{R=0} + R^2\left[\left(\frac{D^2}{K^2}\right)\mathfrak{s}_2\right]_{R=0}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned} \tag{B.34}$$

[1]More precisely *Mathematica* has been used to rearrange $\delta R^{(S_2)}$ on $R = 0$ hypersurface . Away from the membrane the calculation is relatively less tedious and could be done by hand. On $\psi = 1$ i.e., on R=0, $\delta R^{(S_2)}$ becomes

$$\delta R^{(S_2)}\Big|_{R=0} = e^{-R}\left[-\mathfrak{s}_1 + \frac{K}{2}\left((\hat{\nabla}\cdot u) - \frac{1}{2K}\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}P^{AC}P^{BD}\right)\right]\Bigg|_{R=0} \tag{B.33}$$

$$\text{Where,}\qquad \nabla_{(A}u_{B)} = \nabla_A u_B + \nabla_B u_A$$

For *Mathematica* computation we do have to choose a specific background and coordinate system. Since we have an independent proof that the final answer is 'background-covariant', such a choice does not imply any loss of generality. However, we need to do an appropriate 'geometrization' of the answer that we get from *Mathematica*, so that we could write it in a 'background covariant form' as desired. See [14], [3] for details of this procedure.

This type of rewriting helps to see the consistency of the set of coupled ODEs manifestly (see section - 3.3.1).

Let us continue with derivation for the rest of the components of the source.

$$\delta R^{(S_3)} = 0 \tag{B.35}$$

$$\begin{aligned}
\delta R^{(tr)} &= \frac{\psi^{-D}}{2}\frac{P^{CC'}}{D-2}\left[-2(\nabla_D O_C)(\nabla^D O_{C'}) + \frac{K}{D}(\nabla_C O_{C'} + \nabla_{C'} O_C)\right] \\
&= \frac{\psi^{-D}}{2}\frac{1}{D-2}\left[-2\,P^{CC'}(\nabla_D n_C)(\nabla^D n_{C'}) + \frac{K}{D}P^{CC'}(\nabla_C n_{C'} + \nabla_{C'} n_C)\right] + \mathcal{O}\left(\frac{1}{D}\right) \\
&= \frac{\psi^{-D}}{2}\frac{1}{D-2}\left(-2\frac{K^2}{D} + 2\frac{K^2}{D}\right) + \mathcal{O}\left(\frac{1}{D}\right) \\
&= 0
\end{aligned} \tag{B.36}$$

$$\begin{aligned}
&\left[\delta R^{(V_1)}_{\text{lin}}\right]_A \\
&= \frac{\psi^{-D}}{2}P^C_A\left[2\,K(u\cdot\nabla)O_C - \nabla^2 O_C\right] + \frac{\psi^{-D}}{2}P^C_A\Bigg[2\,\bar{R}_{ECFD}\,O^D O^E u^F \\
&+ 2\frac{(n\cdot\nabla)K}{K}[(u\cdot\nabla)O_C] + \frac{\nabla_C K}{D}2(\nabla_F O_C)(u^D\nabla^F n_D) + \frac{K}{D}(u\cdot\nabla)O_C + \frac{K}{D}(u^D K_{CD})\Bigg] \\
&= \frac{e^{-R}}{2}P^C_A\left[2\,K(u\cdot\nabla)O_C - \nabla^2 O_C\right]\Big|_{\psi=1} + \frac{e^{-R}}{2}\left(\frac{\psi-1}{N}\right)(n\cdot\nabla)\left[P^C_A\left(2K(u\cdot\nabla)O_C - \nabla^2 O_C\right)\right]\Big|_{\psi=1} \\
&\quad + \frac{e^{-R}}{2}P^C_A\Bigg[2\,\bar{R}_{ECFD}\,O^D O^E u^F + 2\frac{(n\cdot\nabla)K}{K}[(u\cdot\nabla)O_C] + \frac{\nabla_C K}{D} \\
&\qquad - 2(\nabla_F O_C)(u^D\nabla^F n_D) + \frac{K}{D}(u\cdot\nabla)O_C + \frac{K}{D}u^D K_{CD}\Bigg]_{\psi=1} \\
&= \left(\frac{e^{-R}}{2}\right)\left[K\,E^{\text{vector}}_A - 2R\left(\frac{D}{K}\right)\mathfrak{v}_A\right]
\end{aligned} \tag{B.37}$$

In the last line we have used the following two identities (see appendix B.2.4 and B.2.5 for derivation)

$$(n\cdot\nabla)\left[P^C_A\left(2\,K(u\cdot\nabla)O_C-\nabla^2 O_C\right)\right]_{R=0}=-2D\,\mathfrak{v}_A \tag{B.38}$$

$$\begin{split}P^C_A\Bigg[2\,K(u\cdot\nabla)O_C-\nabla^2 O_C+2\,\bar{R}_{ECFD}\,O^D O^E u^F+2\frac{(n\cdot\nabla)K}{K}[(u\cdot\nabla)O_C]+\frac{\nabla_C K}{D}\\ -2(\nabla_F O_C)(u^D\nabla^F n_D)+\frac{K}{D}(u\cdot\nabla)O_C+\frac{K}{D}u^D K_{CD}\Bigg]_{\psi=1}=K\,E^{\text{vector}}_A\end{split} \tag{B.39}$$

Where E^{vector}_A is the subleading (see equation (3.16)) membrane equation, and $\mathfrak{v}_A$ is given by

$$\begin{split}\mathfrak{v}_A=P^B_A\Bigg[\frac{K}{D}\left(n^D u^E O^F\bar{R}_{FBDE}\right)+\frac{K^2}{2D^2}\left(\frac{\nabla_B K}{K}+(u\cdot\nabla)u_B-2\,u^D K_{DB}\right)\\ -P^{FD}\left(\frac{\nabla_F K}{D}-\frac{K}{D}(u^E K_{EF})\right)(K_{DB}-\nabla_D u_B)\Bigg]\end{split} \tag{B.40}$$

Note that the simplification of $\left[\delta R^{(V_1)}_{\text{lin}}\right]$ involves the same issues as in $\delta R^{(S_2)}$. The first line of the RHS of equation (B.37) is of order $\mathcal{O}(D)$ by naive order counting. However, because of the membrane equation at first subleading order, this is of $\mathcal{O}(1)$ on $\psi=1$ hypersurface. Away from the hypersurface this may not be the case and we have to expand the first line around $\psi=1$ and take into account at least the first term in the expansion. This is what has been done in the second line of equation (B.37). In the final step we have re-written $\left[\delta R^{(V_1)}_{\text{lin}}\right]$ in terms of already-defined vector structure $\mathfrak{v}_A$ plus terms proportional to membrane equation.

The rest of the components of S_{AB} are easy to compute without any further subtlety.

$$\left[\delta R^{(V_2)}_{\text{lin}}\right]_C=0 \tag{B.41}$$

$$
\begin{aligned}
&\left[\delta R^{(T)}_{\text{lin}}\right]_{AB} \\
&= \frac{\psi^{-D}}{2} P^C_A P^{C'}_B \left[2\,\bar{R}_{FCC'D} O^D O^F - 2(\nabla_D O_C)(\nabla^D O_{C'}) + \frac{K}{D}(\nabla_C O_{C'} + \nabla_{C'} O_C)\right] \\
&\quad - \frac{\psi^{-D}}{2}\frac{P_{AB}}{D-2} P^{CC'} \left[-2(\nabla_D O_C)(\nabla^D O_{C'}) + \frac{K}{D}(\nabla_C O_{C'} + \nabla_{C'} O_C)\right] \\
&= \frac{\psi^{-D}}{2} P^C_A P^{C'}_B \left[2\,\bar{R}_{FCC'D} O^D O^F - 2(\nabla_D O_C)(\nabla^D O_{C'}) + \frac{K}{D}(\nabla_C O_{C'} + \nabla_{C'} O_C)\right] \\
&= \psi^{-D} P^C_A P^{C'}_B \left[\frac{K}{D}\left(K_{CC'} - \frac{\nabla_C u_{C'} + \nabla_{C'} u_C}{2}\right) - P^E_F (K_{EC} - \nabla_E u_C)(K^F{}_{C'} - \nabla^F u_{C'})\right] \\
&\qquad + \psi^{-D} P^C_A P^{C'}_B\, \bar{R}_{FCC'D} O^D O^F \\
&= \psi^{-D}\, \mathfrak{t}_{AB}
\end{aligned}
\tag{B.42}
$$

Where,

$$
\begin{aligned}
\mathfrak{t}_{AB} = P^C_A P^D_B \Bigg[&+ \bar{R}_{FCDE} O^E O^F + \frac{K}{D}\left(K_{CD} - \frac{\nabla_C u_D + \nabla_D u_C}{2}\right) \\
&- P^{EF}(K_{EC} - \nabla_E u_C)(K_{FD} - \nabla_F u_D)\Bigg]
\end{aligned}
\tag{B.43}
$$

In deriving (B.42) we have used the following identity

$$
P^C_A(\nabla_D O_C) = P^E_D P^C_A (\nabla_E O_C) - O_D[P^C_A (u\cdot\nabla) O_C] \tag{B.44}
$$

Which follows from the subsidiary condition.

B.2 Some identities

In this appendix we shall prove some of the identities that we have used to compute the metric correction.

B.2.1 The derivation of the Identity (3.27)

$$
[\mathfrak{t}_1]_{CC'} = P^A_C P^B_{C'} \left[\frac{K}{D}\left(K_{AB} - \frac{\nabla_A u_B + \nabla_B u_A}{2}\right) - P^D_E (K_{DA} - \nabla_D u_A)(K^E{}_B - \nabla^E u_B)\right] \tag{B.45}
$$

$$
\begin{aligned}
&\nabla^C[\mathfrak{t}_1]_{CC'} \\
&= \underbrace{\frac{K}{D}\nabla^C\left(P^A_C P^B_{C'} K_{AB}\right)}_{\text{Term-1}} - \underbrace{\left[\nabla^C\left\{P^A_C P^D_F (K_{DA} - \nabla_D u_A)\right\}\right]\left[P^B_{C'} P^{EF}(K_{EB} - \nabla_E u_B)\right]}_{\text{Term-2}} \\
&- \underbrace{\frac{K}{D}\nabla^C\left(P^A_C P^B_{C'}\frac{\nabla_A u_B + \nabla_B u_A}{2}\right)}_{Term-3} - \underbrace{\left[P^A_C P^D_F (K_{DA} - \nabla_D u_A)\right]\left[\nabla^C\left\{P^B_{C'} P^{EF}(K_{EB} - \nabla_E u_B)\right\}\right]}_{Term-4}
\end{aligned}
\tag{B.46}
$$

After a bit of straight forward calculation the each of the above terms become

$$
\text{Term-1} \equiv \frac{K}{D} P^E_{C'} \nabla_E K \tag{B.47}
$$

$$
\text{Term-2} \equiv P^{EA} P^B_{C'}[\nabla_E K - K(u^D K_{DE})](K_{AB} - \nabla_A u_B) \tag{B.48}
$$

$$
\text{Term-3} \equiv \frac{K}{2D} P^E_{C'}[\nabla_E K + K(u\cdot\nabla)u_E] \tag{B.49}
$$

$$
\text{Term-4} \equiv \frac{K}{D} P^F_{C'}[K\, u^D K_{DF} - K(u\cdot\nabla)u_F] \tag{B.50}
$$

Adding (B.47), (B.48), (B.49) and (B.50) we get

$$
\begin{aligned}
\nabla^C[\mathfrak{t}_1]_{CC'} = &\frac{K}{2D} P^B_C\left[\nabla_B K + K(u\cdot\nabla)u_B - 2K(u^A K_{AB})\right] \\
&- P^{BD} P^A_C\left(\nabla_B K - K(u^E K_{EB})\right)\left[K_{DA} - \nabla_D u_A\right]
\end{aligned}
\tag{B.51}
$$

B.2.2 The derivation of scalar structure $\mathfrak{s}_2$ (3.24)

The scalar structure $\mathfrak{s}_2$ is defined as

$$
\mathfrak{s}_2 = \frac{\nabla\cdot\mathfrak{v}}{D} \tag{B.52}
$$

$$
\begin{aligned}
\mathfrak{v}_A = P^B_A\Bigg[&\frac{K}{D}\left(n^D u^E O^F \bar{R}_{FBDE}\right) + \frac{K^2}{2D^2}\left(\frac{\nabla_B K}{K} + (u\cdot\nabla)u_B - 2\,u^D K_{DB}\right) \\
&- P^{FD}\left(\frac{\nabla_F K}{D} - \frac{K}{D}(u^E K_{EF})\right)(K_{DB} - \nabla_D u_B)\Bigg]
\end{aligned}
\tag{B.53}
$$

Now,

$$\nabla^A \mathfrak{v}_A = -K\Bigg[\frac{K^2}{2D^2}\left(\frac{(n\cdot\nabla)K}{K} + n^B(u\cdot\nabla)u_B\right) - P^{FD}\left(\frac{\nabla_F K}{D} - \frac{K}{D}u^E K_{EF}\right)\left(-n^B\nabla_D u_B\right)$$
$$+\frac{K}{D}n^D u^E O^F n^B \bar{R}_{FBDE}\Bigg] + P^B_A\Bigg[\frac{K^2}{2D^2}\left(\frac{\nabla^A\nabla_B K}{K} + \nabla^A[(u\cdot\nabla)u_B] - 2\,u^D\nabla^A K_{DB}\right)$$
$$-\left(\nabla^A P^{FD}\right)\left(\frac{\nabla_F K}{D}\right)K_{DB} - P^{FD}\left(\frac{\nabla^A\nabla_F K}{D} - \frac{K}{D}\,u^E\nabla^A(K_{EF})\right)K_{DB}$$
$$-P^{FD}\left(\frac{\nabla_F K}{D} - \frac{K}{D}\left(u^E K_{EF}\right)\right)\left(\nabla^A K_{DB} - \nabla^A\nabla_D u_B\right) + \frac{K}{D}\left(K^{AD}\right)u^E O^F \bar{R}_{FBDE}\Bigg]$$

$$= \frac{K^2}{D}\Bigg[-\frac{K}{2D}\left(\frac{(n\cdot\nabla)K}{K} - u\cdot K\cdot u\right) + P^{FD}\left(\frac{\nabla_F K}{K} - u^E K_{EF}\right)\left(u^B\nabla_D n_B\right)$$
$$+ n^D u^E u^F n^B \bar{R}_{FBDE} + \frac{1}{2D}\left(\frac{\nabla^2 K}{K}\right) - \frac{\lambda}{2} - \frac{1}{D}\left(u^D\nabla^A K_{DA}\right) + \frac{K}{D}\left(\frac{(n\cdot\nabla)K}{K}\right)$$
$$-P^F_A\frac{1}{D}\left(\frac{\nabla^A\nabla_F K}{K} - u^E\nabla^A K_{EF}\right) - P^{FD}\left(\frac{\nabla_F K}{K} - u^E K_{EF}\right)\left(\frac{\nabla^A K_{DA}}{K}\right) - \lambda\Bigg] \tag{B.54}$$

Now using

$$\frac{\nabla^2 K}{K^2} = \frac{\hat{\nabla}^2 K}{K^2} + \frac{(n\cdot\nabla)K}{K} + \mathcal{O}\left(\frac{1}{D}\right)$$
$$\text{and,}\quad \frac{\hat{\nabla}^2 K}{K^2} = 2\left(\frac{u\cdot\nabla K}{K}\right) - u\cdot K\cdot u + \frac{\lambda(D-1)}{K} \tag{B.55}$$

We get the final expression

$$\nabla^A \mathfrak{v}_A = \frac{K^2}{D}\Bigg[n^B n^D u^E u^F \bar{R}_{FBDE} - \frac{K}{D}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) - 2\,\lambda$$
$$-\left(u\cdot K\cdot K\cdot u\right) + 2\left(\frac{\nabla_A K}{K}\right)u^B K^A_B - \left(\frac{u\cdot\nabla K}{K}\right)^2$$
$$+2\left(\frac{u\cdot\nabla K}{K}\right)(u\cdot K\cdot u) - \left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right) - (u\cdot K\cdot u)^2\Bigg]$$
$$= D\,\mathfrak{s}_2 \tag{B.56}$$

B.2.3 The derivation of the Identity (B.9)

$$\nabla\cdot u=\hat{\nabla}\cdot u-\frac{(u\cdot\nabla)K}{K}-\frac{1}{K}(u\cdot\nabla)\left(\frac{(n\cdot\nabla)K}{K}\right)+\frac{1}{K}\frac{(n\cdot\nabla)K}{K}\frac{(u\cdot\nabla)K}{K}\tag{B.57}$$

$$\begin{aligned}\nabla\cdot u&=\hat{\nabla}\cdot u+n_B(n\cdot\nabla)u^B\\&=\hat{\nabla}\cdot u-u^B\left[\psi K+\psi\frac{(n\cdot\nabla)N}{N}-N\right]^{-1}\hat{\nabla}_B\left[\psi K+\psi\frac{(n\cdot\nabla)N}{N}-N\right]\end{aligned}\tag{B.58}$$

In the last line we have used the following relation

$$ND=\psi K+\psi\frac{(n\cdot\nabla)N}{N}-N\tag{B.59}$$

$$\begin{aligned}\nabla\cdot u&=\hat{\nabla}\cdot u-u^B\left[\psi K+\psi\frac{(n\cdot\nabla)N}{N}-N\right]^{-1}\hat{\nabla}_B\left[\psi K+\psi\frac{(n\cdot\nabla)N}{N}-N\right]\\&=\hat{\nabla}\cdot u-\left[1-\frac{(n\cdot\nabla)N}{NK}+\frac{N}{\psi K}\right]\left[\frac{(u\cdot\nabla)K}{K}+\frac{1}{K}(u\cdot\nabla)\left\{\frac{(n\cdot\nabla)N}{N}-\frac{N}{\psi}\right\}\right]\\&=\hat{\nabla}\cdot u-\frac{(u\cdot\nabla)K}{K}-\frac{1}{K}(u\cdot\nabla)\left\{\frac{(n\cdot\nabla)N}{N}-\frac{N}{\psi}\right\}+\left[\frac{(n\cdot\nabla)N}{NK}-\frac{N}{\psi K}\right]\frac{(u\cdot\nabla)K}{K}\\&=\hat{\nabla}\cdot u-\frac{(u\cdot\nabla)K}{K}-\frac{1}{K}(u\cdot\nabla)\left[\frac{(n\cdot\nabla)K}{K}\right]+\frac{1}{K}\left(\frac{(n\cdot\nabla)K}{K}\right)\left(\frac{(u\cdot\nabla)K}{K}\right)\end{aligned}\tag{B.60}$$

In the last line we have used

$$\frac{(n\cdot\nabla)N}{N}=\frac{(n\cdot\nabla)K}{K}+\frac{K}{D}\tag{B.61}$$

B.2.4 The derivation of the identity (B.38)

$$\begin{aligned}&(n\cdot\nabla)\left[P^C_D\left\{2\,K(u\cdot\nabla)O_C-\nabla^2O_C\right\}\right]\\&=(n\cdot\nabla)\left[P^C_D\left\{-2\,K(n\cdot\nabla)u_C+\nabla^2u_C\right\}\right]\\&=\underbrace{\left[(n\cdot\nabla)P^C_D\right]\left[-2\,K(n\cdot\nabla)u_C+\nabla^2u_C\right]}_{\text{1 st Term}}+\underbrace{P^C_D(n\cdot\nabla)\left[-2\,K(n\cdot\nabla)u_C+\nabla^2u_C\right]}_{\text{2 nd Term}}\end{aligned}\tag{B.62}$$

$$
\begin{aligned}
&\textbf{1 st Term} \equiv \left[(n\cdot\nabla)P^C_D\right]\left[-2\,K(n\cdot\nabla)u_C+\nabla^2 u_C\right]\\
&= -n_D[(n\cdot\nabla)n^C]\left[-2\,K(n\cdot\nabla)u_C+\nabla^2 u_C\right]+u_D[(n\cdot\nabla)u^C]\left[-2\,K(n\cdot\nabla)u_C+\nabla^2 u_C\right]\\
&\quad-[(n\cdot\nabla)n_D]\left[-2\,Kn^C(n\cdot\nabla)u_C+n^C\nabla^2 u_C\right]\\
&=0
\end{aligned}
\tag{B.63}
$$

Where, we have used

$$
\begin{aligned}
&(n\cdot\nabla)n_D = -u_D\left[u^B(n\cdot\nabla)n_B\right]+P^B_D(n\cdot\nabla)n_B\\
&(n\cdot\nabla)u_D = n_D\left[n^B(n\cdot\nabla)u_B\right]+P^B_D(n\cdot\nabla)u_B\\
&\text{And, }-2\,K(n\cdot\nabla)u_C+\nabla^2 u_C = n_C\left[2\,Ku_D(n\cdot\nabla)n^D-u_D\nabla^2 n^D\right]
\end{aligned}
\tag{B.64}
$$

The third one follows from the fact that,

$$
\begin{aligned}
&P^C_B\left[-2\,K(n\cdot\nabla)u_C+\nabla^2 u_C\right]\\
&=P^C_B\left[\hat{\nabla}^2 u_C-\ K(n\cdot\nabla)u_C\right]\\
&=P^C_B\left[\hat{\nabla}^2 u_C-\hat{\nabla}_C K-K(u\cdot\nabla)u_C+Ku^D K_{DC}\right]\\
&=0
\end{aligned}
\tag{B.65}
$$

Where, $[E_1]^{\text{vector}}_B$ is the leading order membrane equation.

$$
\begin{aligned}
&\textbf{2 nd Term} \equiv P^C_D(n\cdot\nabla)\left[-2\,K(n\cdot\nabla)u_C+\nabla^2 u_C\right]\\
&\qquad= P^C_D\left\{-2[(n\cdot\nabla)K][(n\cdot\nabla)u_C]-2\,K\,(n\cdot\nabla)[(n\cdot\nabla)u_C]+(n\cdot\nabla)(\nabla^2 u_C)\right\}
\end{aligned}
\tag{B.66}
$$

Now,

$$
\begin{aligned}
&P^C_D(n\cdot\nabla)(\nabla^2 u_C)\\
&= P^C_D\, n^E\nabla_E\nabla_F\nabla^F u_C\\
&= P^C_D\, n^E[\nabla_E,\nabla_F]\nabla^F u_C + P^C_D\, n^E\nabla_F\nabla_E\nabla^F u_C\\
&= P^C_D\left[-\lambda\,(D-1)(n\cdot\nabla)u_C + n^E\bar{R}_{EFCB}\left(\nabla^F u^B\right) + n^E\nabla^F[\nabla_E,\nabla_F]u_C + n^E\nabla^F\nabla_F\nabla_E u_C\right]\\
&= P^C_D\Bigg[-\lambda\,(D-1)(n\cdot\nabla)u_C + n^E\bar{R}_{EFCB}\left(\nabla^F u^B\right) + n^E u^B\left(\nabla^F\bar{R}_{EFCB}\right) + n^E\bar{R}_{EFCB}\left(\nabla^F u^B\right)\\
&\quad + \hat{\nabla}^2[(n\cdot\nabla)u_C] - (\nabla^2 n^E)(\nabla_E u_C) - 2\,(\nabla_F n^E)(\nabla^F\nabla_E u_C) + K\,(n\cdot\nabla)[(n\cdot\nabla)u_C]\Bigg]\\
&= P^C_D\Bigg[\hat{\nabla}^2[(n\cdot\nabla)u_C] - (\nabla^2 n^E)(\nabla_E u_C) - 2\,(\nabla_F n^E)(\nabla^F\nabla_E u_C) + K\,(n\cdot\nabla)[(n\cdot\nabla)u_C]\\
&\quad -\;\lambda\,(D-1)(n\cdot\nabla)u_C\Bigg]\\
&= P^C_D\Bigg[\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K} - \frac{1}{K^2}\left(\hat{\nabla}^2 K\right)\hat{\nabla}^2 u_C - \left(\hat{\nabla}^2 n_C\right)\frac{u\cdot\nabla K}{K} - (\nabla^2 n^E)(\nabla_E u_C)\\
&\quad - 2\,(\nabla_F n^E)(\nabla^F\nabla_E u_C) + K\,(n\cdot\nabla)[(n\cdot\nabla)u_C] - \lambda\,(D-1)(n\cdot\nabla)u_C\Bigg]
\end{aligned}
\tag{B.67}
$$

In the last line we have used,

$$
\begin{aligned}
P^C_D\hat{\nabla}^2[(n\cdot\nabla)u_C] &= P^C_D\hat{\nabla}^2\left[P^E_C\frac{\hat{\nabla}^2 u_E}{K} - n_C\frac{u\cdot\nabla K}{K}\right]\\
&= P^C_D\left[\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K} - \frac{1}{K^2}\left(\hat{\nabla}^2 K\right)\hat{\nabla}^2 u_C - \left(\hat{\nabla}^2 n_C\right)\frac{u\cdot\nabla K}{K}\right]
\end{aligned}
\tag{B.68}
$$

Using (B.67) in (B.66) we get,

2-nd Term

$$
\begin{aligned}
&= -P^C_D\lambda(D-1)(n\cdot\nabla)u_C + P^C_D\Bigg[-2[(n\cdot\nabla)K][(n\cdot\nabla)u_C] - K\,(n\cdot\nabla)[(n\cdot\nabla)u_C] + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K}\\
&\quad - \frac{1}{K^2}\left(\hat{\nabla}^2 K\right)\hat{\nabla}^2 u_C - \left(\hat{\nabla}^2 n_C\right)\frac{u\cdot\nabla K}{K} - (\nabla^2 n^E)(\nabla_E u_C) - 2\,(\nabla_D n^E)(\nabla^D\nabla_E u_C)\Bigg]
\end{aligned}
\tag{B.69}
$$

Using the following identity whose derivation is a bit lengthy, and we are skipping the derivation

$$
\begin{aligned}
&P_B^C(n\cdot\nabla)[(n\cdot\nabla)u_C] \\
&= P_B^C\Bigg[-4\,\frac{u\cdot\nabla K}{K}\,[(u\cdot\nabla)u_C] + [(u\cdot\nabla)u_C]\,(u\cdot K\cdot u) - 7\,\frac{u\cdot\nabla K}{K}\frac{\nabla_C K}{K} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2} \\
&+ 3\,(u\cdot K\cdot u)\frac{\nabla_C K}{K} - \frac{K}{D}u^D K_{DC} + 4\left(u^D K_{DC}\right)\frac{u\cdot\nabla K}{K} - u^D K_{DC}(u\cdot K\cdot u) - 2K_C^D\frac{\nabla_D K}{K} \\
&- 2(u_E K^{ED})(\nabla_D u_C) + 2\,K^{AF}K_{AC}u_F - 2\frac{\lambda(D-1)}{K}\frac{\hat{\nabla}^2 u_C}{K} - 2\,u^F n^E O^A \bar{R}_{EFCA}\Bigg]
\end{aligned}
\tag{B.70}
$$

Now,

2-nd Term

$$
\begin{aligned}
&= P_B^C\left[-\frac{K^2}{D}\left((u\cdot\nabla)u_C - u^D K_{DC} + \frac{\nabla_C K}{K}\right)\right] + P_B^C K\Bigg[2\,u^F n^E O^A \bar{R}_{EFCA} \\
&+ 2\,K_C^D\frac{\nabla_D K}{K} + 2(u_E K^{ED})(\nabla_D u_C) - 2\,K^{AF}K_{AC}u_F - 2\,\frac{\hat{\nabla}^E K}{K}(\nabla_E u_C) \\
&- 2\frac{u\cdot\nabla K}{K}(u\cdot\nabla)u_C + 2\frac{u\cdot\nabla K}{K}u^D K_{DC} + 2(u\cdot K\cdot u)[(u\cdot\nabla)u_C] - 2(u\cdot K\cdot u)(u^D K_{DC})\Bigg] \\
&= -2\,D\,\mathfrak{v}_B
\end{aligned}
\tag{B.71}
$$

Finally, we get

$$
(n\cdot\nabla)\left[P_D^C\left\{2\,K(u\cdot\nabla)O_C - \nabla^2 O_C\right\}\right] = -2D\,\mathfrak{v}_D \tag{B.72}
$$

B.2.5 The derivation of the identity (B.39)

We can divide the L.H.S. of (B.39) as follows

$$
\begin{aligned}
&P_B^C\Bigg[2\,K(u\cdot\nabla)O_C - \nabla^2 O_C + 2\,n^D O^E u^F \bar{R}_{ECFD} + 2\frac{(n\cdot\nabla)K}{K}[(u\cdot\nabla)O_C] + \frac{\nabla_C K}{D} \\
&\quad - 2(\nabla_F O_C)(u^D\nabla^F n_D) + \frac{K}{D}(u\cdot\nabla)O_C + \frac{K}{D}u^D K_{CD}\Bigg] \equiv P_A^C\,\nabla^2 u_C - P_A^C\nabla^2 n_C + W
\end{aligned}
\tag{B.73}
$$

where W is what we get by subtracting off $P_A^C\,\nabla^2 u_C - P_A^C\nabla^2 n_C$ from the LHS of equation (B.73).

First we shall simplify W

$$\begin{aligned}W = P_B^C\Big[&2\,K(u\cdot\nabla)O_C + 2\,n^D O^E u^F \bar{R}_{ECFD}\; + 2\frac{(n\cdot\nabla)K}{K}[(u\cdot\nabla)O_C] + \frac{\nabla_C K}{D}\\ &- 2(\nabla_F O_C)(u^D\nabla^F n_D) + \frac{K}{D}(u\cdot\nabla)O_C + \frac{K}{D}u^D K_{CD}\Big]\\ = P_B^C\Big[&2K\left(u^D K_{DC}\right) - 2K(u\cdot\nabla)u_C + 2\,u^D K_{DC}\left(\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right)\\ &- 2[(u\cdot\nabla)u_C]\left(\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right) + \frac{\nabla_C K}{D} - 2\,u_D K_{FC}K^{FD}\\ &+ 2(\nabla_F u_C)\left(u_D K^{FD}\right) + \frac{K}{D}[(u\cdot\nabla)u_C] + 2\,n^D O^E u^F \bar{R}_{ECFD}\Big]\end{aligned} \tag{B.74}$$

Now, we shall simplify $P_A^C\,\nabla^2 n_C$

$$\begin{aligned}P_B^C\nabla^2 n_C &= P_B^C\nabla^D\left(\nabla_D n_C\right)\\ &= P_B^C\nabla^D\left[K_{DC} + n_D(n\cdot\nabla)n_C\right]\\ &= \underbrace{P_B^C\nabla^D K_{DC}}_{T_1} + \underbrace{P_B^C K(n\cdot\nabla)n_C}_{T_2} + \underbrace{P_B^C(n\cdot\nabla)\left[(n\cdot\nabla)n_C\right]}_{T_3}\end{aligned} \tag{B.75}$$

$$\begin{aligned}T_1 &\equiv P_B^C\nabla^D K_{DC}\\ &= P_B^C\nabla^D K_{CD}\\ &= P_B^C\nabla^D\left(\Pi_C^E\nabla_E n_D\right)\\ &= P_B^C\left[(\nabla^D\Pi_C^E)(\nabla_E n_D) + \Pi_C^E\left(\nabla^D\nabla_E n_D\right)\right]\\ &= P_B^C\left\{-(\nabla^D n_C)[(n\cdot\nabla)n_D] + \Pi_C^E\nabla_E\nabla^D n_D\right\} + P_B^E[\nabla_D,\nabla_E]n^D\\ &= -P_B^C K_C^D\left(\frac{\nabla_D K}{K}\right) + P_B^C\nabla_C K - P_B^E\bar{R}_{DEC}{}^D n^C\\ &= -P_B^C K_C^D\left(\frac{\nabla_D K}{K}\right) + P_B^C\,\nabla_C K\end{aligned} \tag{B.76}$$

$$
\begin{aligned}
T_2 &\equiv P_B^C K[(n\cdot\nabla)n_C] \\
&= P_B^C \; K \; \frac{\nabla_C(ND)}{ND} \\
&= P_B^C K \frac{1}{\psi K + \psi\frac{(n\cdot\nabla)N}{N} - N} \nabla_C \left(\psi K + \psi \frac{(n\cdot\nabla)N}{N} - N\right) \\
&= P_B^C \left(1 - \frac{(n\cdot\nabla)N}{NK} + \frac{N}{\psi K}\right) \nabla_C \left(K + \frac{(n\cdot\nabla)N}{N} - \frac{N}{\psi}\right) \\
&= P_B^C \; \nabla_C \left(K + \frac{(n\cdot\nabla)N}{N} - \frac{N}{\psi}\right) + P_B^C \left(-\frac{(n\cdot\nabla)N}{NK} + \frac{N}{\psi K}\right) \nabla_C K \\
&= P_B^C \nabla_C K + P_B^C \nabla_C \left(\frac{(n\cdot\nabla)K}{K}\right) - P_B^C \left(\frac{(n\cdot\nabla)K}{K}\right)\left(\frac{\nabla_C K}{K}\right)
\end{aligned}
\tag{B.77}
$$

In the first line we have used

$$
ND = \psi K + \psi \frac{(n\cdot\nabla)N}{N} - N \tag{B.78}
$$

And, in the last line we have used

$$
\frac{(n\cdot\nabla)N}{N} = \frac{(n\cdot\nabla)K}{K} + \frac{K}{D} \tag{B.79}
$$

$$
\begin{aligned}
T_3 &\equiv P^C_B(n\cdot\nabla)[(n\cdot\nabla)n_C] \\
&= P^C_B\left[(n\cdot\nabla)\Pi^D_C\right]\left(\frac{\nabla_D N}{N}\right) + P^C_B(n\cdot\nabla)\left(\frac{\nabla_C N}{N}\right) \\
&= -P^C_B\left[(n\cdot\nabla)n_C\right]\left(\frac{(n\cdot\nabla)N}{N}\right) - P^C_B\frac{1}{N^2}[(n\cdot\nabla)N](\nabla_C N) + P^C_B\frac{1}{N}[(n\cdot\nabla)(\nabla_C N)] \\
&= -P^C_B\left(\frac{\nabla_C K}{K}\right)\left(\frac{(n\cdot\nabla)N}{N}\right) - P^C_B\left(\frac{(n\cdot\nabla)N}{N}\right)\left(\frac{\nabla_C K}{K}\right) + P^C_B\frac{1}{N}n^D\nabla_C\nabla_D N \\
&= -2P^C_B\left(\frac{\nabla_C K}{K}\right)\left(\frac{(n\cdot\nabla)N}{N}\right) + P^C_B\frac{1}{N}\nabla_C[(n\cdot\nabla)N] - P^C_B\frac{1}{N}\left(\nabla_C n^D\right)\left(\nabla_D N\right) \\
&= -2P^C_B\left(\frac{\nabla_C K}{K}\right)\left(\frac{(n\cdot\nabla)N}{N}\right) + P^C_B\nabla_C\left(\frac{(n\cdot\nabla)N}{N}\right) + \frac{1}{N^2}P^C_B(\nabla_C N)[(n\cdot\nabla)N] \\
&\qquad\qquad - P^C_B\frac{1}{N}\left(\nabla_C n^D\right)\left(\nabla_D N\right) \\
&= -2P^C_B\left(\frac{\nabla_C K}{K}\right)\left(\frac{(n\cdot\nabla)K}{K} + \frac{K}{D}\right) + P^C_B\nabla_C\left(\frac{(n\cdot\nabla)K}{K} + \frac{K}{D}\right) \\
&\quad + P^C_B\left(\frac{\nabla_C K}{K}\right)\left(\frac{(n\cdot\nabla)K}{K} + \frac{K}{D}\right) - P^C_B K^D_C\left(\frac{\nabla_D K}{K}\right) \\
&= -2P^C_B\left(\frac{\nabla_C K}{K}\right)\left(2\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right) + P^C_B\nabla_C\left(\frac{\hat{\nabla}^2 K}{K^2}\right) + P^C_B\frac{\nabla_C K}{K}\frac{\lambda(D-1)}{K} \\
&\quad + P^C_B\left(\frac{\nabla_C K}{K}\right)\left(2\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right) - P^C_B K^D_C\frac{\nabla_D K}{K}
\end{aligned}
\tag{B.80}
$$

In the last line we have used

$$
\frac{(n\cdot\nabla)K}{K} = \frac{\hat{\nabla}^2 K}{K^2} - \frac{(D-1)\lambda}{K} - \frac{K}{D}
\tag{B.81}
$$

And, divergence of leading order vector membrane equation

$$
\frac{\hat{\nabla}^2 K}{K^2} = 2\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u + \frac{\lambda(D-1)}{K}
\tag{B.82}
$$

Adding (B.76) (B.77) and (B.80) we get

$$
\begin{aligned}
P^C_B\nabla^2 n_C = P^C_B\Bigg[&2\nabla_C K - 2K^D_C\left(\frac{\nabla_D K}{K}\right) + \frac{2}{K^2}\nabla_C\left(\hat{\nabla}^2 K\right) - 2\,\frac{\nabla_C K}{K}\frac{\lambda(D-1)}{K} \\
&- 6\left(\frac{\nabla_C K}{K}\right)\left(2\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right)\Bigg]
\end{aligned}
\tag{B.83}
$$

Now, we shall simplify $P_B^C \nabla^2 u_C$

$$
\begin{aligned}
&P_B^C \hat{\nabla}^2 u_C \\
&= P_B^C \hat{\nabla}^E \left(\Pi_E^F \Pi_C^D \nabla_F u_D\right) \\
&= P_B^C \Pi_M^E \nabla^M \left(\Pi_E^F \Pi_C^D \nabla_F u_D\right) \\
&= P_B^C \Pi_M^N \left(\nabla^M \Pi_N^F\right) (\nabla_F u_C) + P_B^C \Pi_M^F \left(\nabla^M \Pi_C^D\right) (\nabla_F u_D) + P_B^C \Pi_M^F \nabla^M \nabla_F u_C \\
&= P_B^C \bigg[- \Pi_M^N \, n^F \left(\nabla^M n_N\right) (\nabla_F u_C) - \Pi_M^F n^D \left(\nabla^M n_C\right) (\nabla_F u_D) + \nabla^2 u_C - n^F n_M \nabla^M \nabla_F u_C \bigg] \\
&= P_B^C \bigg[- n^F \, K(\nabla_F u_C) - n^D \left(\nabla^M n_C\right) (\nabla_M u_D) + n^D [(n \cdot \nabla) n_C][(n \cdot \nabla) u_D] \\
&\qquad + \nabla^2 u_C - n_M \nabla^M \left(n^F \nabla_F u_C\right) + n_M \left(\nabla^M n^F\right) (\nabla_F u_C) \bigg] \\
&= P_B^C \bigg[- K[(n \cdot \nabla) u_C] - \left(\nabla^M n_C\right) (n^D \nabla_M u_D) + [(n \cdot \nabla) n_C][n^D (n \cdot \nabla) u_D] \\
&\qquad + \nabla^2 u_C - (n \cdot \nabla)[(n \cdot \nabla) u_C] + \left[(n \cdot \nabla) n^F\right] (\nabla_F u_C) \bigg] \\
&\Rightarrow P_B^C \nabla^2 u_C = P_B^C \bigg[\hat{\nabla}^2 u_C + K[(n \cdot \nabla) u_C] + \left(\nabla^M n_C\right) (n^D \nabla_M u_D) \\
&\qquad - [(n \cdot \nabla) n_C][n^D (n \cdot \nabla) u_D] + (n \cdot \nabla)[(n \cdot \nabla) u_C] - \left[(n \cdot \nabla) n^F\right] (\nabla_F u_C) \bigg]
\end{aligned}
\tag{B.84}
$$

$$
\begin{aligned}
\text{Now,} \quad & P_B^C \left(\nabla^M n_C\right) \left(n^D \nabla_M u_D\right) \\
&= -P_B^C \left[K_C^M + n^M (n \cdot \nabla) n_C\right] \left[u_D K_M^D + u_D n_M (n \cdot \nabla) n^D\right] \\
&= -P_B^C K_C^M K_M^D u_D - P_B^C [(n \cdot \nabla) n_C] \left[u^D (n \cdot \nabla) n_D\right] \\
&= -P_B^C K_C^M K_M^D u_D - P_B^C \frac{\nabla_C K}{K} \left(\frac{u \cdot \hat{\nabla} K}{K}\right)
\end{aligned}
\tag{B.85}
$$

Putting (B.85) in (B.84) we get

$$\begin{aligned}
P^C_B\nabla^2 u_C &= P^C_B\hat{\nabla}^2 u_C + P^C_B\ K[(n\cdot\nabla)u_C] - P^C_B K^M_C K^D_M u_D - P^C_B\frac{\nabla_C K}{K}\left(\frac{u\cdot\nabla K}{K}\right)\\
&+ P^C_B\frac{\nabla_C K}{K}\left(\frac{u\cdot\nabla K}{K}\right) + P^C_B\ (n\cdot\nabla)[(n\cdot\nabla)u_C] - P^C_B\frac{\hat{\nabla}^F K}{K}(\nabla_F u_C)\\
\Rightarrow P^C_B\nabla^2 u_C &= P^C_B\hat{\nabla}^2 u_C + P^C_B\ K[(n\cdot\nabla)u_C] - P^C_B K^M_C K^D_M u_D + P^C_B\ (n\cdot\nabla)[(n\cdot\nabla)u_C]\\
&- P^C_B\frac{\hat{\nabla}^F K}{K}(\nabla_F u_C)
\end{aligned} \tag{B.86}$$

As we have mentioned before derivation of $P^C_B(n\cdot\nabla)[(n\cdot\nabla)u_C]$ is lengthy, we shall use the result mentioned in eq(B.70)

Using (B.70) for $P^C_B(n\cdot\nabla)[(n\cdot\nabla)u_C]$ we get the final expression for $P^C_B\nabla^2 u_C$

$$\begin{aligned}
P^C_B\nabla^2 u_C = P^C_B\Bigg[&\hat{\nabla}^2 u_C + K[(n\cdot\nabla)u_C] - 4\frac{(u\cdot\nabla)K}{K}\,[(u\cdot\nabla)u_C] + [(u\cdot\nabla)u_C]\,(u\cdot K\cdot u)\\
&- 7\left(\frac{u\cdot\nabla K}{K}\right)\frac{\nabla_C K}{K} - \frac{\hat{\nabla}_D K}{K}\left(\nabla^D u_C\right) + 3\,(u\cdot K\cdot u)\frac{\nabla_C K}{K} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2} - \frac{K}{D}u^D K_{DC}\\
&+ 4\left(u^D K_{DC}\right)\frac{u\cdot\nabla K}{K} - u^D K_{DC}(u\cdot K\cdot u) - 2\,K^D_C\frac{\nabla_D K}{K} - 2(u_E K^{ED})(\nabla_D u_C)\\
&+ K^{AF}K_{AC}u_F - 2\frac{(D-1)\lambda}{K}\left(\frac{\nabla_C K}{K} - u^E K_{EC} + (u\cdot\nabla)u_C\right) - 2\,n^E u^F O^A\bar{R}_{EFCA}\Bigg]
\end{aligned} \tag{B.87}$$

Adding (B.74) (B.83) and (B.87) we get the final expression

$$\begin{aligned}
&\frac{1}{K}\left(P^C_B\ \nabla^2 u_C - P^C_B\nabla^2 n_C + W\right)\\
&= \left[\frac{\hat{\nabla}^2 u_C}{K} - \frac{\hat{\nabla}_C K}{K} + u^E K_{EC} - u\cdot\hat{\nabla}u_C\right]P^C_B + \left[\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^3} - \frac{u^E K_{ED}K^D_C}{K} - \frac{(\hat{\nabla}_C K)(u\cdot\hat{\nabla}K)}{K^3}\right.\\
&- \frac{(\hat{\nabla}_E K)(\hat{\nabla}^E u_C)}{K^2} - \frac{2K^{DE}\hat{\nabla}_D\hat{\nabla}_E u_C}{K^2} - \frac{\hat{\nabla}_C\hat{\nabla}^2 K}{K^3} + \frac{\hat{\nabla}_C(K_{ED}K^{ED}K)}{K^3} + 3\frac{(u\cdot K\cdot u)(u\cdot\hat{\nabla}u_C)}{K}\\
&- 3\frac{(u\cdot K\cdot u)(u^E K_{EC})}{K} - 6\frac{(u\cdot\hat{\nabla}K)(u\cdot\hat{\nabla}u_C)}{K^2} + 6\frac{(u\cdot\hat{\nabla}K)(u^E K_{EC})}{K^2} + 3\frac{u\cdot\hat{\nabla}u_C}{D-3}\\
&\left.- 3\frac{u^E K_{EC}}{D-3} - \frac{(D-1)\lambda}{K^2}\left(\frac{\hat{\nabla}_C K}{K} - 2u^D K_{DC} + 2(u\cdot\hat{\nabla})u_C\right)\right]P^C_B\\
&\equiv E^{\text{vector}}_B
\end{aligned} \tag{B.88}$$

Where, in the last step we have used the following identity

$$
\begin{aligned}
P_B^C(n\cdot\nabla)u_C =& P_B^C\left[\frac{\nabla_C K}{K} + \frac{1}{K}\nabla_C\left(\frac{\hat{\nabla}^2 K}{K^2} - \frac{(D-1)\lambda}{K} - \frac{K}{D}\right) - u^D K_{DC} + (u\cdot\nabla)u_C\right. \\
&\left. - \frac{1}{K}\left(\frac{\nabla_C K}{K}\right)\left(2\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u - \frac{K}{D}\right)\right]
\end{aligned}
\tag{B.89}
$$

Appendix C

Appendices for Chapter 4

C.1 Calculation of integrals (4.12) at linear order

$$
\begin{aligned}
t(R) &= -2\left(\frac{D}{K}\right)^2 \int_R^\infty \frac{y\,dy}{e^y-1} \\
&= -2\left(\frac{D}{K}\right)^2 \Big[-R\,\text{Log}\left[1-e^{-R}\right] + \text{PolyLog}\left[2, e^{-R}\right]\Big]
\end{aligned} \tag{C.1}
$$

Where PolyLog$[n, z]$ is defined as

$$\text{PolyLog}[n, z] \equiv \text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

We just want e^{-R} term of the integration. Expand in e^{-R} we get.

$$
\begin{aligned}
t(R) &= -2\left(\frac{D}{K}\right)^2 \left[R\,e^{-R} + e^{-R}\right] + \mathcal{O}\left(e^{-2R}\right) \\
&= -2\left(\frac{D}{K}\right)^2 e^{-R}\left[R+1\right]
\end{aligned} \tag{C.2}
$$

$$\boxed{t(R) = -2\left(\frac{D}{K}\right)^2 e^{-R}\left[R+1\right] + \mathcal{O}\left(e^{-2R}\right)} \tag{C.3}$$

$$v(R) = 2\left(\frac{D}{K}\right)^3 \left[\int_R^\infty e^{-x}dx \int_0^x \frac{y\,e^y}{e^y-1}dy - e^{-R}\int_0^\infty e^{-x}dx \int_0^x \frac{y\,e^y}{e^y-1}dy\right] \tag{C.4}$$

Now,

$$\int_0^x \frac{y\,e^y}{e^y-1}dy = \frac{\pi^2}{6} + \frac{x^2}{2} + x\,\text{Log}\left[1-e^{-x}\right] - \text{PolyLog}\left[2, e^{-x}\right] \tag{C.5}$$

$$
\begin{aligned}
&\Rightarrow \int_R^\infty e^{-x}dx \int_0^x \frac{y\,e^y}{e^y-1}dy \\
&= \int_R^\infty e^{-x}\left(\frac{\pi^2}{6} + \frac{x^2}{2} + x\,\text{Log}\left[1-e^{-x}\right] - \text{PolyLog}\left[2, e^{-x}\right]\right)dx \\
&= e^{-R}\left(\frac{\pi^2}{6}\right) + e^{-R}\left(\frac{R^2}{2}\right) - (1-e^{-R})R\,\text{Log}\left[1-e^{-R}\right] + (1-e^{-R})\,\text{PolyLog}\left[2, e^{-R}\right]
\end{aligned} \tag{C.6}
$$

$$\Rightarrow \int_0^\infty e^{-x}dx \int_0^x \frac{y\, e^y}{e^y - 1}dy = \frac{\pi^2}{6} \tag{C.7}$$

Substituting (C.6) and (C.7) in (C.4) we get the final expression

$$v(R) = 2\left(\frac{D}{K}\right)^3 \left[e^{-R}\left(\frac{R^2}{2}\right) - (1-e^{-R})R\,\text{Log}\left[1-e^{-R}\right] + (1-e^{-R})\,\text{PolyLog}\left[2, e^{-R}\right]\right] \tag{C.8}$$

Expanding as before in e^{-R} we get

$$\boxed{v(R) = 2\left(\frac{D}{K}\right)^3 \left(1 + R + \frac{R^2}{2}\right) e^{-R} + \mathcal{O}\left(e^{-2R}\right)} \tag{C.9}$$

The $f_1(R)$ integration is very straightforward

$$\begin{aligned} f_1(R) &= 2\left(\frac{D}{K}\right)^2 \left[-\int_R^\infty x\, e^{-x}dx + e^{-R}\int_0^\infty x\, e^{-x}dx\right] \\ &= -2\left(\frac{D}{K}\right)^2 R\, e^{-R} \end{aligned} \tag{C.10}$$

$$\boxed{f_1(R) = -2\left(\frac{D}{K}\right)^2 R\, e^{-R} + \mathcal{O}\left(e^{-2R}\right)} \tag{C.11}$$

Calculation of $f_2(R)$ is a bit complicated

$$\begin{aligned} f_2(R) = \left(\frac{D}{K}\right) &\left[\int_R^\infty e^{-x}dx \int_0^x \frac{v(y)}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx \int_0^x \frac{v(y)}{1-e^{-y}}dy\right] \\ - \left(\frac{D}{K}\right)^4 &\left[\int_R^\infty e^{-x}dx \int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx \int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}}dy\right] \end{aligned}$$

First we will calculate the second line of $f_2(R)$

$$\int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}}dy = x^2\,\text{Log}[1-e^{-x}] - 2\,x\,\text{PolyLog}[2, e^{-x}] - 2\,\text{PolyLog}[3, e^{-x}] + 2\,\text{Zeta}[3] \tag{C.12}$$

Where Zeta$[n]$ is the 'Riemann Zeta function' given by

$$\text{Zeta}[n] \equiv \zeta[n] = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

Now, we need to do the following integration

$$\begin{aligned}
&\int_0^{\infty} e^{-x} dx \int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}} dy \\
&= \int_0^{\infty} e^{-x} \Big[x^2 \text{ Log}[1-e^{-x}] - 2\, x \text{ PolyLog}[2, e^{-x}] - 2 \text{ PolyLog}[3, e^{-x}] + 2 \text{ Zeta}[3] \Big] dx \\
&= 2\,(-1 + \text{Zeta}[3])
\end{aligned}$$

Now, we want to calculate the following integration

$$\begin{aligned}
&\int_R^{\infty} e^{-x} dx \int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}} dy \\
&= \int_R^{\infty} e^{-x} \Big[x^2 \text{ Log}[1-e^{-x}] - 2\, x \text{ PolyLog}[2, e^{-x}] - 2 \text{ PolyLog}[3, e^{-x}] + 2 \text{ Zeta}[3] \Big] dx
\end{aligned}$$

We can expand the integrand in e^{-x} and then can do the integration term by term. Doing the integration term by term, we get

$$\int_R^{\infty} e^{-x} dx \int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}} dy = 2\, e^{-R} \text{ Zeta}[3] + \mathcal{O}(e^{-2R}) \tag{C.13}$$

So, finally the second line of $f_2(R)$ becomes

$$\begin{aligned}
&-\left(\frac{D}{K}\right)^4 \left[\int_R^{\infty} e^{-x} dx \int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}} dy - e^{-R} \int_0^{\infty} e^{-x} dx \int_0^x \frac{y^2\, e^{-y}}{1-e^{-y}} dy \right] \\
&= -2 \left(\frac{D}{K}\right)^4 e^{-R}
\end{aligned} \tag{C.14}$$

Now we will calculate the first line of $f_2(R)$

$$\left(\frac{D}{K}\right) \left[\int_R^{\infty} e^{-x} dx \int_0^x \frac{v(y)}{1-e^{-y}} dy - e^{-R} \int_0^{\infty} e^{-x} dx \int_0^x \frac{v(y)}{1-e^{-y}} dy \right] \tag{C.15}$$

Using eq (C.8) we get

$$\begin{aligned}
\int_0^x \frac{v(y)}{1-e^{-y}} dy &= 2 \left(\frac{D}{K}\right)^3 \int_0^x dy \left[\frac{y^2\, e^{-y}}{2(1-e^{-y})} - y \text{ Log}\left[1-e^{-y}\right] + \text{PolyLog}\left[2, e^{-y}\right] \right] \\
&= 2 \left(\frac{D}{K}\right)^3 \left[\frac{x^2}{2} \text{ Log}[1-e^{-x}] - 2x\text{PolyLog}[2, e^{-x}] - 3 \text{ PolyLog}[3, e^{-x}] + 3 \text{ Zeta}[3] \right]
\end{aligned} \tag{C.16}$$

Now we need to do the following integration

$$
\begin{aligned}
&\int_0^\infty e^{-x}dx \int_0^x \frac{v(y)}{1-e^{-y}}dy \\
&= 2\left(\frac{D}{K}\right)^3 \int_0^\infty e^{-x}dx \left[\frac{x^2}{2}\,\text{Log}[1-e^{-x}] - 2\,x\,\text{PolyLog}[2,e^{-x}] - 3\,\text{PolyLog}[3,e^{-x}] + 3\,\text{Zeta}[3]\right] \\
&= 2\left(\frac{D}{K}\right)^3 \text{Zeta}[3]
\end{aligned}
$$

Now, we will calculate the following integration. Expanding the integrand in e^{-x} and doing the integration term by term we get

$$
\int_R^\infty e^{-x}dx \int_0^x \frac{v(y)}{1-e^{-y}}dy = 2\left(\frac{D}{K}\right)^3 3\,e^{-R}\,\text{Zeta}[3] + \mathcal{O}(e^{-2R}) \tag{C.17}
$$

So, finally the first line of $f_2(R)$ becomes

$$
\begin{aligned}
&\left(\frac{D}{K}\right)\left[\int_R^\infty e^{-x}dx \int_0^x \frac{v(y)}{1-e^{-y}}dy - e^{-R}\int_0^\infty e^{-x}dx \int_0^x \frac{v(y)}{1-e^{-y}}dy\right] \\
&= 4\left(\frac{D}{K}\right)^4 e^{-R}\,\text{Zeta}[3] + \mathcal{O}(e^{-2R})
\end{aligned} \tag{C.18}
$$

$f_2(R)$ becomes

$$
\boxed{f_2(R) = 2\left(\frac{D}{K}\right)^4 e^{-R}\,(2\,\text{Zeta}[3] - 1) + \mathcal{O}(e^{-2R})} \tag{C.19}
$$

C.2 Some Details of Linearized Calculation

C.2.1 Outside $(\psi > 1)$

From (4.23)

$$
\begin{aligned}
& - n^A \sum_{m=0}^{\infty} (\psi - 1)^m M_{AB}^{(m)} \\
&= \sum_{m=0}^{\infty} \Bigg[- \frac{ND}{\psi} (\psi - 1)^m \xi_B^{(m)} + m(\psi - 1)^{m-1} N \xi_B^{(m)} + (\psi - 1)^m (n \cdot \nabla) \xi_B^{(m)} \\
&\quad - \frac{ND}{\psi} (\psi - 1)^m \, n_B \left(n \cdot \xi^{(m)}\right) + N \, m(\psi - 1)^{m-1} \, n_B \left(n \cdot \xi^{(m)}\right) + (\psi - 1)^m \, n^A \nabla_B \xi_A^{(m)} \Bigg] \\
&= \sum_{m=0}^{\infty} \Bigg[- ND(\psi - 1)^m \xi_B^{(m)} [1 + (\psi - 1)]^{-1} + m(\psi - 1)^{m-1} N \xi_B^{(m)} + (\psi - 1)^m (n \cdot \nabla) \xi_B^{(m)} \\
& - ND(\psi - 1)^m n_B (n \cdot \xi^{(m)}) [1 + (\psi - 1)]^{-1} + Nm(\psi - 1)^{m-1} \, n_B (n \cdot \xi^{(m)}) + (\psi - 1)^m n^A \nabla_B \xi_A^{(m)} \Bigg]
\end{aligned}
$$

Comparing coefficient of $(\psi - 1)^0$ we get

$$
n^A M_{AB}^{(0)} = ND \left[\xi_B^{(0)} + n_B \left(n \cdot \xi^{(0)}\right) \right] - \left[(n \cdot \nabla) \xi_B^{(0)} + n^A \nabla_B \xi_A^{(0)} \right] - N \left[\xi_B^{(1)} + n_B \left(n \cdot \xi^{(1)}\right) \right] \tag{C.20}
$$

Comparing coefficient of $(\psi - 1)^1$ we get

$$
\begin{aligned}
n^A M_{AB}^{(1)} = {} & ND \left[\xi_B^{(1)} - \xi_B^{(0)} \right] + ND \left[n_B \left(n \cdot \xi^{(1)}\right) - n_B \left(n \cdot \xi^{(0)}\right) \right] - \left[(n \cdot \nabla) \xi_B^{(1)} + n^A \nabla_B \xi_A^{(1)} \right] \\
& - 2N \left[\xi_B^{(2)} + n_B \left(n \cdot \xi^{(2)}\right) \right]
\end{aligned} \tag{C.21}
$$

Comparing coefficient of $(\psi - 1)^2$ we get

$$
\begin{aligned}
n^A M_{AB}^{(2)} = {} & ND \left[\xi_B^{(2)} - \xi_B^{(1)} + \xi_B^{(0)} + n_B \left(n \cdot \xi^{(2)}\right) - n_B \left(n \cdot \xi^{(1)}\right) + n_B \left(n \cdot \xi^{(0)}\right) \right] \\
& - \left[(n \cdot \nabla) \xi_B^{(2)} + n^A \nabla_B \xi_A^{(2)} \right] - 3N \left[\xi_B^{(3)} + n_B \left(n \cdot \xi^{(3)}\right) \right]
\end{aligned} \tag{C.22}
$$

M_{AB} is correct up to order $\mathcal{O}\left(\frac{1}{D}\right)^2$. So, we want ξ_A to be correct up to order $\mathcal{O}\left(\frac{1}{D}\right)^3$. This implies we want $\xi_A^{(0)}$ to be correct up to order $\mathcal{O}\left(\frac{1}{D}\right)^3$, $\xi_A^{(1)}$ to be correct up to order $\mathcal{O}\left(\frac{1}{D}\right)^2$

and $\xi_A^{(2)}$ to be correct up to order $\mathcal{O}\left(\frac{1}{D}\right)$. Now, using the following expansion

$$\begin{aligned} \xi_B^{(0)} &= \xi_B^{(0,0)} + \frac{1}{D}\xi_B^{(0,1)} + \frac{1}{D^2}\xi_B^{(0,2)} + \frac{1}{D^3}\xi_B^{(0,3)} + \mathcal{O}\left(\frac{1}{D}\right)^4 \\ \xi_B^{(1)} &= \xi_B^{(1,0)} + \frac{1}{D}\xi_B^{(1,1)} + \frac{1}{D^2}\xi_B^{(1,2)} + \mathcal{O}\left(\frac{1}{D}\right)^3 \\ \xi_B^{(2)} &= \xi_B^{(2,0)} + \frac{1}{D}\xi_B^{(2,1)} + \mathcal{O}\left(\frac{1}{D}\right)^2 \end{aligned} \tag{C.23}$$

From (C.20) we get

$$\begin{aligned} & ND\left[\xi_B^{(0,0)} + n_B\left(n\cdot\xi^{(0,0)}\right)\right] = 0 \\ \Rightarrow\ & ND\left[\left(n\cdot\xi^{(0,0)}\right) + \left(n\cdot\xi^{(0,0)}\right)\right] = 0 \\ \Rightarrow\ & \left(n\cdot\xi^{(0,0)}\right) = 0 \\ \Rightarrow\ & \xi_B^{(0,0)} = 0 \end{aligned} \tag{C.24}$$

From (C.21), at leading order

$$\begin{aligned} & ND\left[\xi_B^{(1,0)} - \xi_B^{(0,0)} + n_B\left(n\cdot\xi^{(1,0)}\right) - n_B\left(n\cdot\xi^{(0,0)}\right)\right] = 0 \\ \Rightarrow\ & ND\left[\xi_B^{(1,0)} + n_B\left(n\cdot\xi^{(1,0)}\right)\right] = 0 \\ \Rightarrow\ & \xi_B^{(1,0)} = 0 \end{aligned} \tag{C.25}$$

Similarly, from (C.22)

$$\xi_B^{(2,0)} = 0 \tag{C.26}$$

Now, we will calculate $\xi_B^{(2,1)}$. From (C.22) at $\mathcal{O}(1)$

$$n^A M_{AB}^{(2)} = N\left[\xi_B^{(2,1)} - \xi_B^{(1,1)} + \xi_B^{(0,1)} + n_B\left(n\cdot\xi^{(2,1)}\right) - n_B\left(n\cdot\xi^{(1,1)}\right) + n_B\left(n\cdot\xi^{(0,1)}\right)\right] \tag{C.27}$$

From (C.21) at $\mathcal{O}(1)$

$$n^A M_{AB}^{(1)} = N\left[\xi_B^{(1,1)} - \xi_B^{(0,1)} + n_B\left(n\cdot\xi^{(1,1)}\right) - n_B\left(n\cdot\xi^{(0,1)}\right)\right] \tag{C.28}$$

Adding (C.28) and (C.27) we get

$$
\begin{aligned}
& n^A M^{(2)}_{AB} + n^A M^{(1)}_{AB} = N\left[\xi^{(2,1)}_B + n_B\left(n\cdot\xi^{(2,1)}\right)\right] \\
\Rightarrow\ & n\cdot M^{(2)}\cdot n + n\cdot M^{(1)}\cdot n = 2N\left(n\cdot\xi^{(2,1)}\right) \\
\Rightarrow\ & n\cdot\xi^{(2,1)} = \frac{1}{2N}\left(n\cdot M^{(2)}\cdot n + n\cdot M^{(1)}\cdot n\right)
\end{aligned}
\tag{C.29}
$$

Finally we get,

$$
\xi^{(2,1)}_B = \frac{1}{N}\left[n^A M^{(2)}_{AB} + n^A M^{(1)}_{AB} - \frac{n_B}{2}\left(n\cdot M^{(2)}\cdot n + n\cdot M^{(1)}\cdot n\right)\right] \tag{C.30}
$$

Adding (C.20) and (C.21) we get,

$$
\begin{aligned}
& n^A M^{(1)}_{AB} + n^A M^{(0)}_{AB} \\
&= ND\left[\xi^{(1)}_B + n_B\left(n\cdot\xi^{(1)}\right)\right] - \left[(n\cdot\nabla)\xi^{(1)}_B + (n\cdot\nabla)\xi^{(0)}_B\right] - \left[n^A\nabla_B\xi^{(1)}_A + n^A\nabla_B\xi^{(0)}_A\right] \\
&- N\left[\xi^{(1)}_B + n_B\left(n\cdot\xi^{(1)}\right)\right] - 2N\left[\xi^{(2)}_B + n_B\left(n\cdot\xi^{(2)}\right)\right]
\end{aligned}
\tag{C.31}
$$

From (C.31), at order $\mathcal{O}(1)$ we get

$$
\begin{aligned}
& n^A M^{(1)}_{AB} + n^A M^{(0)}_{AB} = N\left[\xi^{(1,1)}_B + n_B\left(n\cdot\xi^{(1,1)}\right)\right] \\
\Rightarrow\ & n\cdot M^{(1)}\cdot n + n\cdot M^{(0)}\cdot n = 2N\left(n\cdot\xi^{(1,1)}\right) \\
\Rightarrow\ & n\cdot\xi^{(1,1)} = \frac{1}{2N}\left(n\cdot M^{(1)}\cdot n + n\cdot M^{(0)}\cdot n\right) \\
\Rightarrow\ & \xi^{(1,1)}_B = \frac{1}{N}\left[n^A M^{(1)}_{AB} + n^A M^{(0)}_{AB} - \frac{n_B}{2}\left(n\cdot M^{(1)}\cdot n + n\cdot M^{(0)}\cdot n\right)\right]
\end{aligned}
\tag{C.32}
$$

From (C.31) at order $\mathcal{O}\left(\frac{1}{D}\right)$,

$$
\begin{aligned}
& N\left[\xi^{(1,2)}_B + n_B\left(n\cdot\xi^{(1,2)}\right)\right] - \left[(n\cdot\nabla)\xi^{(1,1)}_B + (n\cdot\nabla)\xi^{(0,1)}_B\right] - \left[n^A\nabla_B\xi^{(1,1)}_A + n^A\nabla_B\xi^{(0,1)}_A\right] \\
& - N\left[\xi^{(1,1)}_B + n_B\left(n\cdot\xi^{(1,1)}\right)\right] - 2N\left[\xi^{(2,1)}_B + n_B\left(n\cdot\xi^{(2,1)}\right)\right] = 0 \\
\Rightarrow\ & \left(n\cdot\xi^{(1,2)}\right) = \frac{1}{N}\left[n^B(n\cdot\nabla)\xi^{(1,1)}_B + n^B(n\cdot\nabla)\xi^{(0,1)}_B\right] + 2\left(n\cdot\xi^{(2,1)}\right) + \left(n\cdot\xi^{(1,1)}\right) \\
\Rightarrow\ & \xi^{(1,2)}_B = \frac{1}{N}\left[(n\cdot\nabla)\xi^{(1,1)}_B + (n\cdot\nabla)\xi^{(0,1)}_B\right] + \frac{1}{N}\left[n^A\nabla_B\xi^{(1,1)}_A + n^A\nabla_B\xi^{(0,1)}_A\right] \\
& \quad + 2\,\xi^{(2,1)}_B + \xi^{(1,1)}_B - \frac{n_B}{N}\left[n^A(n\cdot\nabla)\xi^{(1,1)}_A + n^A(n\cdot\nabla)\xi^{(0,1)}_A\right]
\end{aligned}
\tag{C.33}
$$

Now, we will calculate $\xi_A^{(0)}$. From (C.20), at order $\mathcal{O}(1)$

$$\begin{aligned} n^A M_{AB}^{(0)} &= N \left[\xi_B^{(0,1)} + n_B \left(n \cdot \xi^{(0,1)} \right) \right] \\ \Rightarrow n \cdot M \cdot n &= 2N \left(n \cdot \xi^{(0,1)} \right) \\ \Rightarrow \xi_B^{(0,1)} &= \frac{1}{N} \left[n^A M_{AB}^{(0)} - \frac{n_B}{2} \left(n \cdot M^{(0)} \cdot n \right) \right] \end{aligned} \tag{C.34}$$

From (C.20) at order $\mathcal{O}\left(\frac{1}{D}\right)$

$$\begin{aligned} & N \left[\xi_B^{(0,2)} + n_B \left(n \cdot \xi^{(0,2)} \right) \right] - \left[(n \cdot \nabla) \xi_B^{(0,1)} + n^A \nabla_B \xi_A^{(0,1)} \right] - N \left[\xi_B^{(1,1)} + n_B \left(n \cdot \xi^{(1,1)} \right) \right] = 0 \\ \Rightarrow & 2N \left(n \cdot \xi^{(0,2)} \right) = 2\, n^B (n \cdot \nabla) \xi_B^{(0,1)} + 2N \left(n \cdot \xi^{(1,1)} \right) \\ \Rightarrow & \xi_B^{(0,2)} = \frac{1}{N} \left[(n \cdot \nabla) \xi_B^{(0,1)} + n^A \nabla_B \xi_A^{(0,1)} \right] + \xi_B^{(1,1)} - \frac{n_B}{N} \left[n^A (n \cdot \nabla) \xi_A^{(0,1)} \right] \end{aligned} \tag{C.35}$$

From (C.20) at order $\mathcal{O}\left(\frac{1}{D}\right)^2$

$$\begin{aligned} & N \left[\xi_B^{(0,3)} + n_B \left(n \cdot \xi^{(0,3)} \right) \right] - \left[(n \cdot \nabla) \xi_B^{(0,2)} + n^A \nabla_B \xi_A^{(0,2)} \right] - N \left[\xi_B^{(1,2)} + n_B \left(n \cdot \xi^{(1,2)} \right) \right] = 0 \\ \Rightarrow & n \cdot \xi^{(0,3)} = \frac{1}{N} \left[n^B (n \cdot \nabla) \xi_B^{(0,2)} \right] + n \cdot \xi^{(1,2)} \\ \Rightarrow & \xi_B^{(0,3)} = \frac{1}{N} \left[(n \cdot \nabla) \xi_B^{(0,2)} + n^A \nabla_B \xi_A^{(0,2)} \right] + \xi_B^{(1,2)} - \frac{n_B}{N} \left[n^A (n \cdot \nabla) \xi_A^{(0,2)} \right] \end{aligned} \tag{C.36}$$

Using, (C.30) (C.32), (C.33), (C.34), (C.35) and (C.36) in (4.22) we get

$$
\begin{aligned}
M'_{AB} &= M_{AB} + \psi^D \nabla_A \left[\psi^{-D}\xi_B\right] + \psi^D \nabla_B \left[\psi^{-D}\xi_A\right] \\
&= M^{(0)}_{AB} + (\psi-1)M^{(1)}_{AB} + (\psi-1)^2 M^{(2)}_{AB} + \underbrace{\nabla_A \xi_B - \left(\frac{ND}{\psi}\right) n_A \xi_B}_{\mathcal{L}_{AB}} + \mathcal{L}_{BA} + \mathcal{O}\left(\frac{1}{D}\right)^3 \\
&= M^{(0)}_{AB} + (\psi-1)M^{(1)}_{AB} + (\psi-1)^2 M^{(2)}_{AB} + \nabla_A \left[\xi^{(0)}_B + (\psi-1)\xi^{(1)}_B + (\psi-1)^2 \xi^{(2)}_B\right] \\
&- ND[1+(\psi-1)]^{-1} n_A \left[\xi^{(0)}_B + (\psi-1)\xi^{(1)}_B + (\psi-1)^2\xi^{(2)}_B\right] + \mathcal{L}_{BA} \\
&= M^{(0)}_{AB} + (\psi-1)M^{(1)}_{AB} + (\psi-1)^2 M^{(2)}_{AB} + \nabla_A \xi^{(0)}_B + N n_A \xi^{(1)}_B + (\psi-1)\nabla_A \xi^{(1)}_B \\
&+ (\psi-1)^2 \nabla_A \xi^{(2)}_B + 2N(\psi-1) n_A \xi^{(2)}_B - ND\, n_A \xi^{(0)}_B - ND(\psi-1) n_A \xi^{(1)}_B \\
&- ND(\psi-1)^2 n_A \xi^{(2)}_B + ND(\psi-1) n_A \xi^{(0)}_B + ND(\psi-1)^2 n_A \xi^{(1)}_B - ND(\psi-1)^2 n_A \xi^{(0)}_B + \mathcal{L}_{BA} \\
&= M^{(0)}_{AB} + \nabla_A \xi^{(0)}_B + N n_A \xi^{(1)}_B - ND\, n_A \xi^{(0)}_B \\
&+ (\psi-1)\left[M^{(1)}_{AB} + \nabla_A \xi^{(1)}_B + 2N n_A \xi^{(2)}_B - ND\, n_A \xi^{(1)}_B + ND\, n_A \xi^{(0)}_B\right] \\
&+ (\psi-1)^2 \left[M^{(2)}_{AB} + \nabla_A \xi^{(2)}_B - ND\, n_A \xi^{(2)}_B + ND\, n_A \xi^{(1)}_B - ND\, n_A \xi^{(0)}_B\right] + \mathcal{L}_{BA}
\end{aligned}
\tag{C.37}
$$

Now writing the expression for $\mathcal{L}_{BA}$ we finally get

$$
\begin{aligned}
M'_{AB} = &\left[M^{(0)}_{AB} + \nabla_A \xi^{(0)}_B + N n_A \xi^{(1)}_B - ND\, n_A \xi^{(0)}_B + \nabla_B \xi^{(0)}_A + N n_B \xi^{(1)}_A - ND\, n_B \xi^{(0)}_A\right] \\
&+ (\psi-1)\Big[M^{(1)}_{AB} + \nabla_A \xi^{(1)}_B + 2N n_A \xi^{(2)}_B - ND\, n_A \xi^{(1)}_B + ND\, n_A \xi^{(0)}_B \\
&+ \nabla_B \xi^{(1)}_A + 2N n_B \xi^{(2)}_A - ND\, n_B \xi^{(1)}_A + ND\, n_B \xi^{(0)}_A\Big] \\
&+ (\psi-1)^2 \Big[M^{(2)}_{AB} + \nabla_A \xi^{(2)}_B - ND\, n_A \xi^{(2)}_B + ND\, n_A \xi^{(1)}_B - ND\, n_A \xi^{(0)}_B \\
&+ \nabla_B \xi^{(2)}_A - ND\, n_B \xi^{(2)}_A + ND\, n_B \xi^{(1)}_A - ND\, n_B \xi^{(0)}_A\Big]
\end{aligned}
\tag{C.38}
$$

Now, we will simplify (C.38). First, we will simplify the first square bracketed terms.

$$
\begin{aligned}
&\nabla_A\xi^{(0)}_B - ND\, n_A\xi^{(0)}_B + Nn_A\xi^{(1)}_B \\
&= \nabla_A\xi_B^{(0)} - Nn_A\left[\xi_B^{(0,1)} + \frac{1}{D}\xi_B^{(0,2)} + \frac{1}{D^2}\xi_B^{(0,3)}\right] + n_A\frac{N}{D}\left[\xi_B^{(1,1)}+\frac{1}{D}\xi_B^{(1,2)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^3 \\
&= \nabla_A\xi_B^{(0)} - n_A\left[n^D M^{(0)}_{DB} - \frac{n_B}{2}\left(n\cdot M^{(0)}\cdot n\right)\right] - n_A\frac{1}{D}\left[(n\cdot\nabla)\xi_B^{(0,1)} + n^D\nabla_B\xi_D^{(0,1)}\right] \\
&- \frac{N}{D}n_A\xi_B^{(1,1)} + n_An_B\frac{1}{D}\left[n^D(n\cdot\nabla)\xi_D^{(0,1)}\right] - \frac{n_A}{D^2}\left[(n\cdot\nabla)\xi_B^{(0,2)} + n^D\nabla_B\xi_D^{(0,2)}\right] - \frac{N}{D^2}n_A\,\xi_B^{(1,2)} \\
&+ \frac{1}{D^2}n_An_B\left[n^D(n\cdot\nabla)\xi_D^{(0,2)}\right] + \frac{N}{D}n_A\xi_B^{(1,1)} + \frac{N}{D^2}n_A\xi_B^{(1,2)}
\end{aligned}
\tag{C.39}
$$

Using, (C.39) and it's symmetric part the first square bracketed terms become

$$
\begin{aligned}
&M^{(0)}_{AB} + \nabla_A\xi^{(0)}_B + Nn_A\xi^{(1)}_B - ND\, n_A\xi^{(0)}_B + \nabla_B\xi^{(0)}_A + Nn_B\xi^{(1)}_A - ND\, n_B\xi^{(0)}_A \\
&= \Pi^C_A\Pi^{C'}_B\left[M^{(0)}_{CC'} + \nabla_C\left(\frac{1}{D}\xi^{(0,1)}_{C'} + \frac{1}{D^2}\xi^{(0,2)}_{C'}\right) + \nabla_{C'}\left(\frac{1}{D}\xi^{(0,1)}_{C} + \frac{1}{D^2}\xi^{(0,2)}_{C}\right)\right]
\end{aligned}
\tag{C.40}
$$

Now, we will simplify the second square bracketed term of (C.38)

$$
\begin{aligned}
&\nabla_A\xi^{(1)}_B + 2Nn_A\xi^{(2)}_B - ND\, n_A\xi^{(1)}_B + ND\, n_A\xi^{(0)}_B \\
&= \nabla_A\xi^{(1)}_B + 2Nn_A\xi^{(2)}_B - N\,n_A\left[\xi_B^{(1,1)} + \frac{1}{D}\xi_B^{(1,2)}\right] + N\,n_A\left[\xi_B^{(0,1)} + \frac{1}{D}\xi_B^{(0,2)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2 \\
&= \nabla_A\xi^{(1)}_B + \cancel{2Nn_A\xi^{(2)}_B} - n_A\left[n^D M^{(1)}_{DB} + \cancel{n^D M^{(0)}_{DB}} - \frac{n_B}{2}\left(n\cdot M^{(1)}\cdot n + \cancel{n\cdot M^{(0)}\cdot n}\right)\right] \\
&- \frac{n_A}{D}\left[(n\cdot\nabla)\xi_B^{(1,1)} + \cancel{(n\cdot\nabla)\xi_B^{(0,1)}}\right] - \frac{n_A}{D}\left[n^D\nabla_B\xi_D^{(1,1)} + \cancel{n^D\nabla_B\xi_D^{(0,1)}}\right] \\
&- \cancel{\frac{2}{D}Nn_A\xi^{(2)}_B} - \cancel{\frac{N}{D}n_A\,\xi_B^{(1,1)}} + n_An_B\frac{1}{D}\left[n^D(n\cdot\nabla)\xi_D^{(1,1)} + \cancel{n^D(n\cdot\nabla)\xi_D^{(0,1)}}\right] \\
&+ \frac{n_A}{D}\left[\cancel{n^D M^{(0)}_{DB} - \frac{n_B}{2}\left(n\cdot M^{(0)}\cdot n\right)}\right] + \frac{n_A}{D}\left[\cancel{(n\cdot\nabla)\xi_B^{(0,1)} + n^D\nabla_B\xi_D^{(0,1)}}\right] + \cancel{\frac{N}{D}n_A\xi_B^{(1,1)}} \\
&- \cancel{n_An_B\frac{1}{D}\left[n^D(n\cdot\nabla)\xi_D^{(0,1)}\right]} \\
&= \nabla_A\xi^{(1)}_B - n_A\left[n^D M^{(1)}_{DB} - \frac{n_B}{2}\left(n\cdot M^{(1)}\cdot n\right)\right] - \frac{n_A}{D}\left[(n\cdot\nabla)\xi_B^{(1,1)}\right] \\
&- \frac{n_A}{D}\left[n^D\nabla_B\xi_D^{(1,1)}\right] + n_An_B\frac{1}{D}\left[n^D(n\cdot\nabla)\xi_D^{(1,1)}\right]
\end{aligned}
\tag{C.41}
$$

Adding $M^{(1)}_{AB}$, (C.41) and it's symmetric part we get

$$\begin{aligned}
&M^{(1)}_{AB} + \nabla_A \xi^{(1)}_B + 2N n_A \xi^{(2)}_B - ND\, n_A \xi^{(1)}_B + ND\, n_A \xi^{(0)}_B \\
&+ \nabla_B \xi^{(1)}_A + 2N n_B \xi^{(2)}_A - ND\, n_B \xi^{(1)}_A + ND\, n_B \xi^{(0)}_A + \mathcal{O}\left(\frac{1}{D}\right)^2 \\
&= \Pi^C_A \Pi^{C'}_B \left[M^{(1)}_{CC'} + \frac{1}{D}\left(\nabla_C \xi^{(1,1)}_{C'} + \nabla_{C'} \xi^{(1,1)}_C \right)\right] + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned} \tag{C.42}$$

Finally, we will try to simplify the third square bracketed term of (4.22)

$$\begin{aligned}
&\nabla_A \xi^{(2)}_B - ND\, n_A \xi^{(2)}_B + ND\, n_A \xi^{(1)}_B - ND\, n_A \xi^{(0)}_B \\
&= -N\, n_A \xi^{(2,1)}_B + N\, n_A \xi^{(1,1)}_B - N\, n_A \xi^{(0,1)}_B + \mathcal{O}\left(\frac{1}{D}\right) \\
&= -n_A \left[n^D M^{(2)}_{DB} + \cancel{n^D M^{(1)}_{DB}} - \frac{n_B}{2}\left(n \cdot M^{(2)} \cdot n + \cancel{n \cdot M^{(1)} \cdot n} \right)\right] \\
&+ n_A \left[\cancel{n^D M^{(1)}_{DB}} + \cancel{n^D M^{(0)}_{DB}} - \frac{n_B}{2}\left(\cancel{n \cdot M^{(1)} \cdot n} + \cancel{n \cdot M^{(0)} \cdot n} \right)\right] \\
&- n_A \left[\cancel{n^D M^{(0)}_{DB}} - \cancel{\frac{n_B}{2}\left(n \cdot M^{(0)} \cdot n \right)} \right] \\
&= -n_A \left[n^D M^{(2)}_{DB} - \frac{n_B}{2}\left(n \cdot M^{(2)} \cdot n \right)\right]
\end{aligned} \tag{C.43}$$

Using, (C.43) and it's symmetric part the third square bracketed terms become

$$\begin{aligned}
&M^{(2)}_{AB} + \nabla_A \xi^{(2)}_B - ND\, n_A \xi^{(2)}_B + ND\, n_A \xi^{(1)}_B - ND\, n_A \xi^{(0)}_B \\
&+ \nabla_B \xi^{(2)}_A - ND\, n_B \xi^{(2)}_A + ND\, n_B \xi^{(1)}_A - ND\, n_B \xi^{(0)}_A + \mathcal{O}\left(\frac{1}{D}\right) \\
&= \Pi^C_A \Pi^{C'}_B M^{(2)}_{CC'} + \mathcal{O}\left(\frac{1}{D}\right)
\end{aligned} \tag{C.44}$$

Finally, adding (C.40), (C.42) and (C.44) we get the final expression of M'_{AB}(4.28)

$$\begin{aligned}
M'_{AB} &= \Pi^C_A \Pi^{C'}_B \left[M^{(0)}_{CC'} + (\psi - 1) M^{(1)}_{CC'} + (\psi - 1)^2 M^{(2)}_{CC'} \right] + \hat{\nabla}_A \xi^{(0)}_B + \hat{\nabla}_B \xi^{(0)}_A \\
&+ (\psi - 1)\left(\hat{\nabla}_A \xi^{(1)}_B + \hat{\nabla}_B \xi^{(1)}_A \right) + \mathcal{O}\left(\frac{1}{D}\right)^3
\end{aligned} \tag{C.45}$$

Now we will calculate different terms in (4.28). First we will calculate $\xi_A^{(0,1)}$

$$\begin{aligned}\xi_A^{(0,1)} &= \frac{1}{N}\left[n^B M_{BA}^{(0)} - \frac{n_A}{2}\left(n\cdot M^{(0)}\cdot n\right)\right]\\ &= \frac{1}{N}\Bigg[O_A + \frac{2}{K^2}\left\{\left(\frac{D}{K}\right)^2 (2\,\text{Zeta}[3]-1)\,\mathfrak{s}_2 O_A + \frac{D}{K}\mathfrak{v}_A\right\}\\ &\quad - \frac{n_A}{2}\left\{1+\frac{2}{K^2}\left(\frac{D}{K}\right)^2 (2\,\text{Zeta}[3]-1)\,\mathfrak{s}_2\right\}\Bigg]\\ &= \frac{1}{N}\left[\frac{n_A}{2} - u_A\right] + \mathcal{O}\left(\frac{1}{D}\right)^2\end{aligned} \tag{C.46}$$

Next, we will calculate $\xi_A^{(0,2)}$

$$\begin{aligned}\xi_A^{(0,2)} &= \frac{1}{N}\left[(n\cdot\nabla)\xi_A^{(0,1)} + n^B\nabla_A\xi_B^{(0,1)}\right] - \frac{n_A}{N}\left[n^B(n\cdot\nabla)\xi_B^{(0,1)}\right]\\ &+ \frac{1}{N}\left[n^B M_{BA}^{(1)} + n^B M_{BA}^{(0)} - \frac{n_A}{2}\left(n\cdot M^{(1)}\cdot n + n\cdot M^{(0)}\cdot n\right)\right]\end{aligned} \tag{C.47}$$

Now, we need to calculate different terms of (C.47)

$$\begin{aligned}&\frac{1}{N}\left[n^B M_{BA}^{(1)} + n^B M_{BA}^{(0)} - \frac{n_A}{2}\left(n\cdot M^{(1)}\cdot n + n\cdot M^{(0)}\cdot n\right)\right] = \frac{1}{N}\left[\frac{n_A}{2} - u_A\right] + \mathcal{O}\left(\frac{1}{D}\right)^2\\ &\nabla_A\xi_B^{(0,1)} = \frac{1}{N}\left[\frac{\nabla_A n_B}{2} - \nabla_A u_B\right] - \frac{\nabla_A N}{N^2 D}\left[\frac{n_B}{2} - u_B\right]\\ &(n\cdot\nabla)\xi_B^{(0,1)} = \frac{1}{N}\left[\frac{(n\cdot\nabla)n_B}{2} - (n\cdot\nabla)u_B\right] - \frac{(n\cdot\nabla)N}{N^2}\left[\frac{n_B}{2} - u_B\right]\\ &n^B\nabla_A\xi_B^{(0,1)} = \frac{1}{N}\left[u^B\nabla_A n_B\right] - \frac{1}{2ND}\left(\frac{\nabla_A N}{N}\right)\\ &n^B(n\cdot\nabla)\xi_B^{(0,1)} = \frac{1}{N}\left[u^B(n\cdot\nabla)n_B\right] - \frac{1}{2ND}\left(\frac{(n\cdot\nabla)N}{N}\right)\end{aligned} \tag{C.48}$$

Using (C.48) in (C.47) we get

$$\begin{aligned}\xi_A^{(0,2)} &= \frac{1}{N}\left[\frac{n_A}{2} - u_A\right] + \frac{1}{N^2}\Bigg[\frac{(n\cdot\nabla)n_A}{2} - (n\cdot\nabla)u_A - \frac{(n\cdot\nabla)N}{N}\left(\frac{n_A}{2} - u_A\right)\\ &\quad + u^B\nabla_A n_B - \frac{1}{2}\left(\frac{\nabla_A N}{N}\right) - n_A\left(u^B(n\cdot\nabla)n_B - \frac{1}{2}\frac{(n\cdot\nabla)N}{N}\right)\Bigg]\\ &= \frac{1}{N}\left[\frac{n_A}{2} - u_A\right] + \frac{1}{N^2}\left[\frac{(n\cdot\nabla)n_A}{2} - (n\cdot\nabla)u_A + u_A\frac{(n\cdot\nabla)N}{N} + u^B\mathcal{K}_{AB} - \frac{1}{2}\frac{\nabla_A N}{N}\right]\\ &= \frac{1}{N}\left[\frac{n_A}{2} - u_A\right] - \frac{1}{N^2}\left[\frac{1}{2}n_A\frac{(n\cdot\nabla)N}{N} + (n\cdot\nabla)u_A - u_A\frac{(n\cdot\nabla)N}{N} - u^B\mathcal{K}_{AB}\right] + \mathcal{O}\left(\frac{1}{D}\right)\end{aligned} \tag{C.49}$$

Adding (C.46) and (C.49) we get the expression of $\xi_A^{(0)}$

$$\xi_A^{(0)} = \frac{1}{ND}\left[\frac{n_A}{2} - u_A\right] + \frac{1}{ND^2}\left[\frac{n_A}{2} - u_A\right] - \frac{1}{N^2D^2}\left[\frac{n_A}{2}\left(\frac{n\cdot\nabla N}{N}\right) + (n\cdot\nabla)u_A - u_A\left(\frac{n\cdot\nabla N}{N}\right) - u^B K_{AB}\right] + \mathcal{O}\left(\frac{1}{D}\right)^3$$

Next, we will calculate $\xi_A^{(1,1)}$

$$\begin{aligned}\xi_A^{(1,1)} &= \frac{1}{N}\left[n^B M_{BA}^{(1)} + n^B M_{BA}^{(0)} - \frac{n_A}{2}\left(n\cdot M^{(1)}\cdot n + n\cdot M^{(0)}\cdot n\right)\right] + \mathcal{O}\left(\frac{1}{D}\right)^2 \\ &= \frac{1}{N}\left[\frac{n_A}{2} - u_A\right] + \mathcal{O}\left(\frac{1}{D}\right)^2\end{aligned} \tag{C.50}$$

So, expression of $\xi_A^{(1)}$ we get

$$\xi_A^{(1)} = \frac{1}{ND}\left[\frac{n_A}{2} - u_A\right] + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.51}$$

Now, we will calculate $\Pi_C^A \Pi_{C'}^B \left(\nabla_B \xi_A^{(0)}\right)$

$$\begin{aligned}\Pi_C^A \Pi_{C'}^B \left(\nabla_B \xi_A^{(0)}\right) &= \frac{1}{ND}\Pi_C^A \Pi_{C'}^B u_A\left(\frac{\nabla_B N}{N}\right) + \frac{1}{ND}\Pi_C^A \Pi_{C'}^B\left[\frac{\nabla_B n_A}{2} - \nabla_B u_A\right] \\ &+ \Pi_C^A \Pi_{C'}^B\left[(\nabla_B n_A)\left(\frac{1}{2ND^2} - \frac{1}{2N^2D^2}\frac{(n\cdot\nabla)N}{N}\right) - \nabla_B\left(\frac{u_A}{ND^2}\right)\right. \\ &\left. + \nabla_B\left\{\frac{1}{N^2D^2}\left(-(n\cdot\nabla)u_A + u_A\frac{(n\cdot\nabla)N}{N} + u^E K_{AE}\right)\right\}\right] + \mathcal{O}\left(\frac{1}{D}\right)^3\end{aligned} \tag{C.52}$$

Using the identity (C.159) and (C.160) we can write the above equation as

$$\begin{aligned}&\Pi_C^A \Pi_{C'}^B \left(\nabla_B \xi_A^{(0)}\right) \\ &= \frac{1}{\psi K}\left(1 - \frac{(n\cdot\nabla)N}{NK} + \frac{N}{\psi K}\right)\Pi_C^A \Pi_{C'}^B\left[u_A\left(\frac{\nabla_B K}{K}\right) + \frac{1}{2}\nabla_B n_A - \nabla_B u_A\right] \\ &+ \frac{1}{\psi K^2}\Pi_C^A \Pi_{C'}^B\left[u_A \nabla_B\left(\frac{(n\cdot\nabla)K}{K}\right) - u_A\left(\frac{\nabla_B K}{K}\right)\frac{(n\cdot\nabla)K}{K}\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{D^2}\Pi^A_C\Pi^B_{C'}\Bigg[\frac{1}{2N}K_{BA}\left(1-\frac{1}{N}\frac{(n\cdot\nabla)N}{N}\right)+\frac{1}{N^2}u_A(\nabla_B N)-\frac{1}{N}(\nabla_B u_A)\\
&-\frac{2}{N^3}(\nabla_B N)\left(u^E K_{AE}-(n\cdot\nabla)u_A+u_A\frac{(n\cdot\nabla)N}{N}\right)\\
&+\frac{1}{N^2}\nabla_B\left(u^E K_{AE}-(n\cdot\nabla)u_A+u_A\frac{(n\cdot\nabla)N}{N}\right)\Bigg]+\mathcal{O}\left(\frac{1}{D}\right)^3\\
&=\frac{1}{\psi K}\left[u_C\left(\frac{\hat\nabla_{C'}K}{K}\right)+\frac{1}{2}K_{CC'}-\hat\nabla_{C'}u_C\right]-\frac{1}{K^2}\left(\frac{n\cdot\nabla K}{K}\right)\left[u_C\left(\frac{\hat\nabla_{C'}K}{K}\right)+\frac{1}{2}K_{CC'}-\hat\nabla_{C'}u_C\right]\\
&+\frac{1}{K^2}\left[u_C\hat\nabla_{C'}\left(\frac{(n\cdot\nabla)K}{K}\right)-u_C\left(\frac{\hat\nabla_{C'}K}{K}\right)\frac{(n\cdot\nabla)K}{K}\right]\\
&+\frac{1}{K^2}\Bigg[-\frac{1}{2}K_{CC'}\left(\frac{(n\cdot\nabla)K}{K}\right)-2\,\frac{\hat\nabla_{C'}K}{K}\left(u^E K_{CE}-\Pi^E_C(n\cdot\nabla)u_E+u_C\frac{(n\cdot\nabla)K}{K}\right)\\
&+\hat\nabla_{C'}\left(u^E K_{CE}-(n\cdot\nabla)u_C+u_C\frac{(n\cdot\nabla)K}{K}\right)\Bigg]+\mathcal{O}\left(\frac{1}{D}\right)^3
\end{aligned}
\tag{C.53}
$$

Now,

$$
\Pi^A_C\Pi^B_{C'}\left(\nabla_B\xi^{(1)}_A\right)=\frac{1}{K}\left[u_C\left(\frac{\hat\nabla_{C'}K}{K}\right)+\frac{1}{2}K_{CC'}-\hat\nabla_{C'}u_C\right]+\mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.54}
$$

Calculation of $h^{(0)}_{CC'}$

From (4.32) we get

$$
h^{(0)}_{AB}\big|_{\psi=1}=M'^{(0)}_{AB}\big|_{\psi=1} \tag{C.55}
$$

First we will write $\mathfrak{t}_{AB}$ and $\mathfrak{v}_A$ in a convenient way. From (4.13), $\mathfrak{t}_{AB}$ can be written as

$$
\mathfrak{t}_{AB}=\mathcal{Y}_{AB}+u_A\mathcal{X}_B+u_B\mathcal{X}_A+\mathcal{Z}u_Au_B \tag{C.56}
$$

Where,

$$\begin{aligned}
\mathcal{Y}_{AB} &= \frac{K}{D}K_{AB} - \frac{K}{2D}\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right) - K^F_A K_{FB} + K^F_A\, \hat{\nabla}_F u_B + K^F_B\, \hat{\nabla}_F u_A - (\hat{\nabla}^F u_A)(\hat{\nabla}_F u_B)\\
&\quad - \left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left(\frac{\hat{\nabla}^2 u_B}{K}\right) + \left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left(\frac{\hat{\nabla}_B K}{K}\right) + \left(\frac{\hat{\nabla}^2 u_B}{K}\right)\left(\frac{\hat{\nabla}_A K}{K}\right) - \left(\frac{\hat{\nabla}_A K}{K}\right)\left(\frac{\hat{\nabla}_B K}{K}\right)\\
\mathcal{X}_A &= \frac{K}{D}\left[u^C K_{CA} - \frac{1}{2}(u\cdot\hat{\nabla})u_A\right] - u^C K_{CE} K^E_A + u^C K_{EC}\left(\hat{\nabla}^E u_A\right) + \frac{(u\cdot\nabla)K}{K}\left[\frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K}\right]\\
\mathcal{Z} &= \frac{K}{D}\, u\cdot K\cdot u - u^C K^F_C K_{FD} u^D - \left(\frac{u\cdot\nabla K}{K}\right)^2
\end{aligned} \tag{C.57}$$

From (4.13), $\mathfrak{v}_A$ can be written as

$$\mathfrak{v}_A = \mathcal{N}_A + \mathcal{J} u_A \tag{C.58}$$

Where,

$$\begin{aligned}
\mathcal{N}_A &= \frac{K^2}{2D^2}\left[\frac{\hat{\nabla}^2 u_A}{K} - u^D K_{DA}\right] - \left[\frac{\hat{\nabla}_F K}{D} - \frac{K}{D}\, u^E K_{EF}\right] K^F_A + \left[\frac{\hat{\nabla}_F K}{D} - \frac{K}{D}\, u^E K_{EF}\right]\left(\hat{\nabla}^F u_A\right)\\
&\quad + \left[\frac{(u\cdot\nabla)K}{D} - \frac{K}{D}\, u\cdot K\cdot u\right]\frac{\hat{\nabla}^2 u_A}{K} - \left[\frac{(u\cdot\nabla)K}{D} - \frac{K}{D}\, u\cdot K\cdot u\right]\frac{\hat{\nabla}_A K}{K}\\
\mathcal{J} &= -\frac{K^2}{2D^2}\, u\cdot K\cdot u - u^B K_{BD}\left(\frac{\hat{\nabla}^D K}{D} - \frac{K}{D}\, u^E K^D_E\right) - \frac{u\cdot\nabla K}{K}\left(\frac{u\cdot\nabla K}{D} - \frac{K}{D}\, u\cdot K\cdot u\right)
\end{aligned} \tag{C.59}$$

Using (C.56) and (C.58) we can write $h^{(0)}_{AB}\big|_{\psi=1}$ as

$$h^{(0)}_{AB} = \widetilde{\mathcal{S}}^{(0)}\, u_A u_B + u_A \widetilde{\mathcal{H}}^{(0)}_B + u_B \widetilde{\mathcal{H}}^{(0)}_A + \mathcal{W}^{(0)}_{AB} \tag{C.60}$$

Where,

$$
\begin{aligned}
\widetilde{\mathcal{S}}^{(0)} &= 1 - \frac{2}{K^2}\left[\cancel{\frac{K}{D}\, u\cdot K}\cdot u - u\cdot K\cdot K\cdot u - \left(\frac{u\cdot\nabla K}{K}\right)^2\right] \\
&+ \frac{2}{K^2}\left(2\,\text{Zeta}[3]-1\right)\Bigg[-\frac{K}{D}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) - 2\,\lambda - (u\cdot K\cdot K\cdot u) + 2\left(\frac{\nabla_A K}{K}\right)u^B K^A_B \\
&- \left(\frac{u\cdot\nabla K}{K}\right)^2 + 2\left(\frac{u\cdot\nabla K}{K}\right)(u\cdot K\cdot u) - \left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right) - (u\cdot K\cdot u)^2 + \lambda\Bigg] \\
&- \frac{2}{K^2}\left[\cancel{-\frac{K}{D}\, u\cdot K\cdot u} - 2\,u^B K_{BD}\left(\frac{\hat{\nabla}^D K}{K} - u^E K^D_E\right) - 2\,\frac{u\cdot\nabla K}{K}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)\right] \\
&= 1 - \frac{2}{K^2}\left[u\cdot K\cdot K\cdot u - 3\left(\frac{(u\cdot\nabla)K}{K}\right)^2 - 2\,u^B K_{BD}\left(\frac{\hat{\nabla}^D K}{K}\right) + 2\,u\cdot K\cdot u\left(\frac{u\cdot\nabla K}{K}\right)\right] \\
&+ \frac{2}{K^2}\left(2\,\text{Zeta}[3]-1\right)\Bigg[-\frac{K}{D}\left(\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right) - \lambda - u\cdot K\cdot K\cdot u + 2\left(\frac{\nabla_A K}{K}\right)u^B K^A_B \\
&- \left(\frac{u\cdot\nabla K}{K}\right)^2 + 2\,\frac{(u\cdot\nabla)K}{K}(u\cdot K\cdot u) - \left(\frac{\hat{\nabla}^D K}{K}\right)\left(\frac{\hat{\nabla}_D K}{K}\right) - (u\cdot K\cdot u)^2\Bigg]
\end{aligned}
\tag{C.61}
$$

$$
\begin{aligned}
\widetilde{\mathcal{H}}^{(0)}_A &= \frac{1}{K}\frac{\hat{\nabla}_A K}{K} - \frac{2}{K^2}\Bigg[\frac{K}{D}\left(u^C K_{CA} - \frac{1}{2}(u\cdot\hat{\nabla})u_A\right) - u^C K_{CE}K^E_A + u^C K_{EC}\left(\hat{\nabla}^E u_A\right) \\
&+ \frac{u\cdot\nabla K}{K}\left(\frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K}\right)\Bigg] - \frac{2}{K^2}\frac{D}{K}\Bigg[\frac{K^2}{2D^2}\left(\frac{\hat{\nabla}^2 u_A}{K} - u^D K_{DA}\right) - \left(\frac{\hat{\nabla}_F K}{D} - \frac{K}{D}u^E K_{EF}\right)K^F_A \\
&+ \left(\frac{\hat{\nabla}_F K}{D} - \frac{K}{D}\,u^E K_{EF}\right)\hat{\nabla}^F u_A + \left(\frac{u\cdot\nabla K}{D} - \frac{K}{D}\,u\cdot K\cdot u\right)\frac{\hat{\nabla}^2 u_A}{K} \\
&- \left(\frac{u\cdot\nabla K}{D} - \frac{K}{D}\,u\cdot K\cdot u\right)\frac{\hat{\nabla}_A K}{K}\Bigg] - \frac{4}{K^2}\left(\frac{n\cdot\nabla K}{K}\right)\left(\frac{\hat{\nabla}_A K}{K}\right) + \frac{2}{K^2}\hat{\nabla}_A\left(\frac{n\cdot\nabla K}{K}\right)
\end{aligned}
\tag{C.62}
$$

Using the following two identity

$$
\begin{aligned}
\hat{\nabla}_A\left(\frac{n\cdot\nabla K}{K}\right) &= \hat{\nabla}_A\left(\frac{\hat{\nabla}^2 K}{K^2}\right) + \lambda\frac{D}{K}\left(\frac{\hat{\nabla}_A K}{K}\right) - \frac{\hat{\nabla}_A K}{D} \\
\left(u\cdot\hat{\nabla}\right)u_A &= \frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K} + u^D K_{DA} + u_A\left(-\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u\right)
\end{aligned}
\tag{C.63}
$$

we get,

$$
\begin{aligned}
\widetilde{\mathcal{H}}_A^{(0)} &= \frac{1}{K}\frac{\hat{\nabla}_A K}{K} + u^C K_{CA}\left[\cancel{\frac{2}{K^2}\frac{K}{2D}} - \cancel{\frac{2}{K^2}\frac{K}{D}} + \cancel{\frac{2}{K^2}\frac{K}{2D}}\right] + \frac{2}{K^2}\hat{\nabla}_A\left(\frac{\hat{\nabla}^2 K}{K^2}\right) \\
&+ u^C K_{EC}\left(\hat{\nabla}^E u_A\right)\left[\cancel{-\frac{2}{K^2}} + \cancel{\frac{2}{K^2}}\right] + \frac{2}{K^2}\frac{D}{K}K_A^F\left(\frac{\hat{\nabla}_F K}{D}\right) - \frac{2}{K^2}\frac{D}{K}\left(\hat{\nabla}^F u_A\right)\left(\frac{\hat{\nabla}_F K}{D}\right) \\
&+ \frac{\hat{\nabla}^2 u_A}{K}\left[\cancel{\frac{2}{K^2}\frac{K}{2D}} - \frac{2}{K^2}\left(\frac{u\cdot\nabla K}{K}\right) - \cancel{\frac{2}{K^2}\frac{K}{2D}} - \frac{2}{K^2}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)\right] \\
&+ \frac{\hat{\nabla}_A K}{K}\Bigg[-\frac{2}{K^2}\frac{K}{2D} + \frac{2}{K^2}\frac{(u\cdot\nabla)K}{K} + \frac{2}{K^2}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) + \lambda\frac{2}{K^2}\frac{D}{K} - \frac{2}{K^2}\frac{K}{D} \\
&- \frac{4}{K^2}\left(2\,\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u - \frac{K}{D}\right)\Bigg] + \frac{2}{K^2}\frac{K}{2D}u_A\left(-\frac{u\cdot\nabla K}{K} + u\cdot K\cdot u\right) \\
&= \frac{1}{K}\left(\frac{\hat{\nabla}_A K}{K}\right) + \frac{2}{K^2}\hat{\nabla}_A\left(\frac{\hat{\nabla}^2 K}{K^2}\right) + \frac{2}{K^2}K_A^F\left(\frac{\hat{\nabla}_F K}{K}\right) - \frac{2}{K^2}\left(\hat{\nabla}^F u_A\right)\left(\frac{\hat{\nabla}_F K}{K}\right) \\
&+ \frac{2}{K^2}\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left[u\cdot K\cdot u - 2\,\frac{(u\cdot\nabla)K}{K}\right] + \frac{2}{K^2}\left(\frac{\hat{\nabla}_A K}{K}\right)\left[u\cdot K\cdot u - 2\,\frac{(u\cdot\nabla)K}{K} + \lambda\,\frac{D}{K} + \frac{K}{2D}\right] \\
&+ \frac{2}{K^2}\frac{K}{2D}u_A\left(-\frac{u\cdot\nabla K}{K} + u\cdot K\cdot u\right)
\end{aligned}
\tag{C.64}
$$

$$
\begin{aligned}
\mathcal{W}_{AB}^{(0)} &= \frac{1}{K}\left[K_{AB} - \hat{\nabla}_A u_B - \hat{\nabla}_B u_A\right] - \frac{2}{K^2}\Bigg[\frac{K}{D}K_{AB} - \frac{K}{2D}\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right) - K_A^F K_{FB} + K_A^F\hat{\nabla}_F u_B \\
&+ K_B^F\hat{\nabla}_F u_A - \left(\hat{\nabla}^F u_A\right)\left(\hat{\nabla}_F u_B\right) - \frac{\hat{\nabla}^2 u_A}{K}\frac{\hat{\nabla}^2 u_B}{K} + \cancel{\frac{\hat{\nabla}^2 u_A}{K}\frac{\hat{\nabla}_B K}{K}} + \cancel{\frac{\hat{\nabla}^2 u_B}{K}\frac{\hat{\nabla}_A K}{K}} - \frac{\hat{\nabla}_A K}{K}\frac{\hat{\nabla}_B K}{K}\Bigg] \\
&- \frac{2}{K^2}\left[\frac{n\cdot\nabla K}{K}\left(K_{AB} - \hat{\nabla}_A u_B - \hat{\nabla}_B u_A\right)\right] \\
&+ \frac{1}{K^2}\left[\hat{\nabla}_A\left(u^E K_{BE} - \frac{\hat{\nabla}^2 u_B}{K}\right) + \hat{\nabla}_B\left(u^E K_{AE} - \frac{\hat{\nabla}^2 u_A}{K}\right) + 2K_{AB}\frac{u\cdot\nabla K}{K}\right] \\
&- \frac{2}{K^2}\left[\left(\frac{\hat{\nabla}_B K}{K}\right)u^E K_{AE} - \cancel{\frac{\hat{\nabla}_B K}{K}\frac{\hat{\nabla}^2 u_A}{K}} + \left(\frac{\hat{\nabla}_A K}{K}\right)u^E K_{BE} - \cancel{\frac{\hat{\nabla}_A K}{K}\frac{\hat{\nabla}^2 u_B}{K}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{K}\left[K_{AB} - \hat{\nabla}_A u_B - \hat{\nabla}_B u_A\right] \\
&- \frac{2}{K^2}\, K_{AB}\left[\frac{(u\cdot\nabla)K}{K} - u\cdot K\cdot u\right] - \frac{2}{K^2}\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right)\left[\frac{K}{2D} - 2\,\frac{(u\cdot\nabla)K}{K} + u\cdot K\cdot u\right] \\
&+ \frac{2}{K^2}K_A^F K_{FB} - \frac{2}{K^2}\left(K_A^F\hat{\nabla}_F u_B + K_B^F\hat{\nabla}_F u_A\right) + \frac{2}{K^2}\left(\hat{\nabla}^F u_A\right)\left(\hat{\nabla}_F u_B\right) + \frac{2}{K^2}\frac{\hat{\nabla}^2 u_A}{K}\frac{\hat{\nabla}^2 u_B}{K} \\
&+ \frac{2}{K^2}\left(\frac{\hat{\nabla}_A K}{K}\right)\left(\frac{\hat{\nabla}_B K}{K}\right) - \frac{2}{K^2}\left[\left(\frac{\hat{\nabla}_A K}{K}\right)u^E K_{EB} + \left(\frac{\hat{\nabla}_B K}{K}\right)u^E K_{EA}\right] \\
&+ \frac{1}{K^2}\left[\hat{\nabla}_A\left(u^E K_{EB}\right) + \hat{\nabla}_B\left(u^E K_{EA}\right)\right] - \frac{1}{K^2}\left[\hat{\nabla}_A\left(\frac{\hat{\nabla}^2 u_B}{K}\right) + \hat{\nabla}_B\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\right]
\end{aligned}
\tag{C.65}
$$

Calculation of $h^{(1)}_{CC'}$

From (4.39) $h^{(1)}_{AB}$ on $\psi = 1$ is given by

$$
\begin{aligned}
h^{(1)}_{AB} &= M'^{(1)}_{AB} + C^{(0)}_{AB} \\
&= C^{(0)}_{AB} - \frac{2D}{K^2}\left[\mathfrak{t}_{AB} + \mathfrak{s}_1\, u_A u_B + \frac{D}{K}\left(\mathfrak{v}_A u_B + \mathfrak{v}_B u_A\right)\right] \\
&+ \frac{1}{K}\left[u_A\frac{\hat{\nabla}_B K}{K} + u_B\frac{\hat{\nabla}_A K}{K} + K_{AB} - \hat{\nabla}_B u_A - \hat{\nabla}_A u_B\right] + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned}
\tag{C.66}
$$

From (4.33)

$$
\begin{aligned}
C^{(0)}_{CC'} &= \frac{1}{N}\Pi^A_C\Pi^B_{C'}(n.\nabla)M'^{(0)}_{AB} \\
&= \frac{1}{N}\left[u_C\Pi^E_{C'}(n\cdot\nabla)u_E + u_{C'}\Pi^E_C(n\cdot\nabla)u_E\right] \\
&- \frac{1}{NK}\frac{(n\cdot\nabla)N}{N}\left[u_C\frac{\hat{\nabla}_{C'}K}{K} + u_{C'}\frac{\hat{\nabla}_C K}{K} + K_{CC'} - \left(\hat{\nabla}_C u_{C'} + \hat{\nabla}_{C'}u_C\right)\right] \\
&+ \frac{1}{NK}\Pi^E_C\Pi^F_{C'}(n\cdot\nabla)\left[u_E\Pi^B_F\frac{\nabla_B K}{K} + u_F\Pi^B_E\frac{\nabla_B K}{K} + K_{EF} - \Pi^A_E\Pi^B_F\left(\nabla_A u_B + \nabla_B u_A\right)\right] + \mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned}
\tag{C.67}
$$

To simplify the above expression we will use the following identity. We will not give the derivations of these identities. The derivations are quite straightforward

$$\Pi_C^E \Pi_{C'}^F (n\cdot\nabla) K_{EF} = -\frac{\hat{\nabla}_C K}{K}\frac{\hat{\nabla}_{C'} K}{K} - \lambda\,\Pi_{CC'} + \hat{\nabla}_C\left(\frac{\hat{\nabla}_{C'} K}{K}\right) - K_C^E K_{EC'}$$

$$\begin{aligned}\Pi_C^E \Pi_{C'}^F (n\cdot\nabla)\left[u_E \Pi_F^B \frac{\nabla_B K}{K}\right] &= \frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}_{C'} K}{K} - u_C\left(\frac{n\cdot\nabla K}{K}\right)\frac{\hat{\nabla}_{C'} K}{K} + u_C\Bigg[\frac{1}{K^2}\hat{\nabla}_{C'}\left(\hat{\nabla}^2 K\right)\\ &- \frac{\hat{\nabla}_{C'} K}{K}\left(2\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D}{K} - \frac{K}{D}\right) - 2\,\frac{K}{D}\left(\frac{\hat{\nabla}_{C'} K}{K}\right) - K_{C'}^D\left(\frac{\nabla_D K}{K}\right)\Bigg]\end{aligned} \tag{C.68}$$

$$\begin{aligned}\Pi_C^E \Pi_{C'}^F (n\cdot\nabla)\left[\Pi_E^A \Pi_F^B \nabla_A u_B\right] &= -\frac{\hat{\nabla}_C K}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} + \frac{\hat{\nabla}_{C'} K}{K} u^B K_{BC} - K_{CC'}\frac{u\cdot\nabla K}{K}\\ &+ \hat{\nabla}_C\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) - K_C^D\left(\hat{\nabla}_D u_{C'}\right)\end{aligned} \tag{C.69}$$

Using (C.68) and (C.69) we can write $C^{(0)}_{CC'}$ as

$$C^{(0)}_{CC'} = u_C\,\tau_{C'} + u_{C'}\,\tau_C + \Xi_{CC'} \tag{C.70}$$

Where,

$$\begin{aligned}\tau_C &= \frac{1}{N}\Pi_C^E (n\cdot\nabla) u_E + \frac{1}{NK}\Bigg[-\frac{\hat{\nabla}_C K}{K}\left(2\,\frac{(n\cdot\nabla)K}{K} + 2\,\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D}{K}\right) + \frac{1}{K^2}\hat{\nabla}_C\left(\hat{\nabla}^2 K\right)\\ &- 2\,\frac{K}{D}\left(\frac{\hat{\nabla}_C K}{K}\right) - K_C^D\left(\frac{\nabla_D K}{K}\right)\Bigg]\\ \Xi_{CC'} &= -\frac{1}{NK}\frac{(n\cdot\nabla)N}{N}\left[K_{CC'} - \hat{\nabla}_C u_{C'} - \hat{\nabla}_{C'} u_C\right] + \frac{1}{NK}\Bigg[2\,\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}_{C'} K}{K} + 2\,\frac{\hat{\nabla}^2 u_{C'}}{K}\frac{\hat{\nabla}_C K}{K}\\ &- \frac{\hat{\nabla}_C K}{K}\frac{\hat{\nabla}_{C'} K}{K} - \lambda\,\Pi_{CC'} + \hat{\nabla}_C\left(\frac{\hat{\nabla}_{C'} K}{K}\right) - K_C^E K_{EC'} - \frac{\hat{\nabla}_{C'} K}{K} u^B K_{BC} - \frac{\hat{\nabla}_C K}{K} u^B K_{BC'}\\ &+ 2\,K_{CC'}\frac{(u\cdot\nabla)K}{K} + K_C^D\left(\hat{\nabla}_D u_{C'}\right) + K_{C'}^D\left(\hat{\nabla}_D u_C\right) - \hat{\nabla}_C\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) - \hat{\nabla}_{C'}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\Bigg]\end{aligned} \tag{C.71}$$

Using (C.56), (C.58) and (C.70) we can write $h^{(1)}_{AB}$ as

$$h^{(1)}_{AB} = \Phi\; u_A u_B + u_A\,\Omega_B + u_B\,\Omega_A + \mathcal{W}^{(1)}_{AB} \tag{C.72}$$

Where,

$$\Phi = -2\,\frac{D}{K^2}\left[\frac{K}{D}\,u\cdot K\cdot u - u^C K_C^F K_{FD} u^D - \left(\frac{u\cdot\nabla K}{K}\right)^2\right] - 2\,\frac{D}{K^2}\Bigg[\lambda + \left(\frac{u\cdot\nabla K}{K}\right)^2$$
$$+ \frac{\hat{\nabla}_A K}{K}\left(4\,u^B K_B^A - 2\left[(u\cdot\hat{\nabla})u^A\right] - \frac{\hat{\nabla}^A K}{K}\right) - (\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B) - [(u\cdot\hat{\nabla})u_A][(u\cdot\hat{\nabla})u^A]$$
$$- (u\cdot K\cdot u)^2 + 2\left[(u\cdot\hat{\nabla})u^A\right](u^B K_{BA}) - 3\,(u\cdot K\cdot K\cdot u) - \frac{K}{D}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)\Bigg]$$
$$- 4\,\frac{D^2}{K^3}\left[-\frac{K^2}{2D^2}\,u\cdot K\cdot u - u^B K_{BD}\left(\frac{\hat{\nabla}^D K}{D} - \frac{K}{D}\,u^E K_E^D\right) - \frac{u\cdot\nabla K}{K}\left(\frac{u\cdot\nabla K}{D} - \frac{K}{D}u\cdot K\cdot u\right)\right]$$

$$= -2\,\frac{D}{K^2}\Bigg[-2\,u\cdot K\cdot K\cdot u + \lambda + \frac{\hat{\nabla}_A K}{K}\left(2\,u^B K_B^A - \frac{\hat{\nabla}^A K}{K}\right) - (\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B)$$
$$- (u\cdot K\cdot u)^2 - [(u\cdot\hat{\nabla})u_A]\left((u\cdot\hat{\nabla})u^A + 2\,\frac{\hat{\nabla}^A K}{K} - 2\,u_B K^{BA}\right) + \frac{K}{D}\,u\cdot K\cdot u$$
$$- \frac{u\cdot\nabla K}{K}\left(\frac{K}{D} + 2\,\frac{u\cdot\nabla K}{K} - 2\,u\cdot K\cdot u\right)\Bigg] \tag{C.73}$$

Using 2nd identity of (C.63) we can write the above equation as

$$\Phi = -2\,\frac{D}{K^2}\Bigg[-2\,u\cdot K\cdot K\cdot u + \lambda + \frac{\hat{\nabla}_A K}{K}\left(2\,u^B K_B^A - \frac{\hat{\nabla}^A K}{K}\right) - (\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B)$$
$$- (u\cdot K\cdot u)^2 - \left(\frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K} + u^E K_{EA}\right)\left(\frac{\hat{\nabla}^2 u^A}{K} + \frac{\hat{\nabla}^A K}{K} - u_D K^{DA}\right) + \frac{K}{D}\,u\cdot K\cdot u$$
$$+ \left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) - \frac{u\cdot\nabla K}{K}\left(\frac{K}{D} + 2\,\frac{u\cdot\nabla K}{K} - 2\,u\cdot K\cdot u\right)\Bigg]$$
$$= -2\,\frac{D}{K^2}\Bigg[\lambda - u\cdot K\cdot K\cdot u - (\hat{\nabla}_A u_B)(\hat{\nabla}^A u^B) - \frac{\hat{\nabla}^2 u_A}{K}\frac{\hat{\nabla}^2 u^A}{K} - \left(\frac{u\cdot\nabla K}{K}\right)^2$$
$$- \frac{K}{D}\,\frac{u\cdot\nabla K}{K} + \frac{K}{D}\,u\cdot K\cdot u\Bigg] \tag{C.74}$$

$$\Omega_A = -2\,\frac{D}{K^2}\Bigg[\frac{K}{D}\left(u^C K_{CA} - \frac{1}{2}(u\cdot\hat{\nabla})u_A\right) - u^C K_{CE}K^E_A + u^C K_{EC}\left(\hat{\nabla}^E u_A\right)$$
$$+\frac{(u\cdot\nabla)K}{K}\left(\frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K}\right)\Bigg] - 2\,\frac{D}{K^2}\Bigg[\frac{K}{2D}\left(\frac{\hat{\nabla}^2 u_A}{K} - u^D K_{DA}\right) - \left(\frac{\hat{\nabla}_F K}{K} - u^E K_{EF}\right)K^F_A$$
$$+\left(\frac{\hat{\nabla}_F K}{K} - u^E K_{EF}\right)\left(\hat{\nabla}^F u_A\right) + \left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)\frac{\hat{\nabla}^2 u_A}{K} - \left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)\frac{\hat{\nabla}_A K}{K}\Bigg]$$
$$+\frac{1}{K}\frac{\hat{\nabla}_A K}{K} + \frac{1}{N}\Pi^E_A(n\cdot\nabla)u_E + \frac{D}{K^2}\Bigg[-\frac{\hat{\nabla}_A K}{K}\left(2\,\frac{(n\cdot\nabla)K}{K} + 2\,\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D}{K}\right)$$
$$+\frac{1}{K^2}\hat{\nabla}_A\left(\hat{\nabla}^2 K\right) - 2\,\frac{K}{D}\left(\frac{\hat{\nabla}_A K}{K}\right) - K^D_A\left(\frac{\nabla_D K}{K}\right)\Bigg] \tag{C.75}$$

To simplify the above expression we will use the following identity

$$\frac{1}{N}\Pi^E_A(n\cdot\nabla)u_E = \frac{D}{K}\left(\frac{\hat{\nabla}^2 u_A}{K}\right) + \frac{D}{K^2}\Bigg[-u^B K_{BD}K^D_A + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_A}{K^2} - \frac{\hat{\nabla}^B K}{K}\,\hat{\nabla}_B u_A - \frac{u\cdot\nabla K}{K}\frac{\hat{\nabla}_A K}{K}$$
$$+\frac{\hat{\nabla}^2 u_A}{K}\left(-8\,\frac{u\cdot\nabla K}{K} + 4\,u\cdot K\cdot u - 2\,\lambda\,\frac{D}{K} + 2\,\frac{K}{D}\right)\Bigg]$$
$$+u_A\,\frac{D}{K^2}\Bigg[-(\hat{\nabla}_D u_E)(\hat{\nabla}^D u^E) - u\cdot K\cdot K\cdot u - \frac{\hat{\nabla}^2 u_E}{K}\,\frac{\hat{\nabla}^2 u^E}{K} - \left(\frac{u\cdot\nabla K}{K}\right)^2\Bigg] \tag{C.76}$$

To prove the above identity we have used subsidiary condition $P^A_B(O\cdot\nabla)O_A=0$ and the second order membrane equation(2.17 in [64]). Using (C.76) we get

$$
\begin{split}
\Omega_A =& \frac{D}{K}\,\frac{\hat{\nabla}^2 u_A}{K} + \frac{D}{K^2}\frac{K}{D}\left(\frac{\hat{\nabla}_A K}{K}\right) + \frac{D}{K^2}\Bigg[-u^B K_{BD}K^D_A + \frac{1}{K^2}\hat{\nabla}^2\hat{\nabla}^2 u_A - \frac{\hat{\nabla}^B K}{K}\,\hat{\nabla}_B u_A \\
& - \frac{u\cdot\nabla K}{K}\frac{\hat{\nabla}_A K}{K} + \frac{\hat{\nabla}^2 u_A}{K}\left(-8\,\frac{u\cdot\nabla K}{K} + 4\,u\cdot K\cdot u - 2\,\lambda\,\frac{D}{K} + 2\,\frac{K}{D}\right)\Bigg] \\
& + \frac{D}{K^2}\frac{\hat{\nabla}_A K}{K}\left(-2\,\frac{n\cdot\nabla K}{K} - 2\,\frac{\hat{\nabla}^2 K}{K^2} + \lambda\,\frac{D}{K}\right) + \frac{D}{K^2}\frac{1}{K^2}\hat{\nabla}_A\left(\hat{\nabla}^2 K\right) - 2\,\frac{D}{K^2}\,\frac{K}{D}\left(\frac{\hat{\nabla}_A K}{K}\right) \\
& - \frac{D}{K^2}K^D_A\left(\frac{\hat{\nabla}_D K}{K}\right) + \frac{D}{K^2}\Bigg[-2\,\cancel{\frac{K}{D}u^C K_{CA}} + \frac{K}{D}\left(\frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K} + \cancel{u^D K_{DA}}\right) + 2\,u^C K_{CE}K^E_A \\
& - 2\,\cancel{u^C K_{EC}\left(\hat{\nabla}^E u_A\right)} - 2\,\frac{(u\cdot\nabla)K}{K}\left(\frac{\hat{\nabla}^2 u_A}{K} - \frac{\hat{\nabla}_A K}{K}\right) - \frac{K}{D}\left(\frac{\hat{\nabla}^2 u_A}{K} - \cancel{u^D K_{DA}}\right) \\
& + 2\,K^F_A\left(\frac{\hat{\nabla}_F K}{K} - u^E K_{EF}\right) - 2\,(\hat{\nabla}^F u_A)\left(\frac{\hat{\nabla}_F K}{K} - \cancel{u^E K_{EF}}\right) - 2\,\frac{\hat{\nabla}^2 u_A}{K}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right) \\
& + 2\,\frac{\hat{\nabla}_A K}{K}\left(\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u\right)\Bigg] + u_A\,\frac{D}{K^2}\Bigg[-(\hat{\nabla}_D u_E)(\hat{\nabla}^D u^E) - u\cdot K\cdot K\cdot u \\
& - \frac{\hat{\nabla}^2 u_E}{K}\,\frac{\hat{\nabla}^2 u^E}{K} - \left(\frac{u\cdot\nabla K}{K}\right)^2 + \frac{K}{D}\left(-\frac{u\cdot\nabla K}{K} + u\cdot K\cdot u\right)\Bigg] \\
=& \frac{D}{K}\left(\frac{\hat{\nabla}^2 u_A}{K}\right) + \frac{D}{K^2}\left(\frac{\hat{\nabla}_A K}{K}\right)\left[-5\,\frac{(u\cdot\nabla)K}{K} + 2\,u\cdot K\cdot u - \lambda\,\frac{D}{K}\right] \\
& + \frac{D}{K^2}\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left[-12\,\frac{(u\cdot\nabla)K}{K} + 6\,u\cdot K\cdot u - 2\,\lambda\,\frac{D}{K} + 2\,\frac{K}{D}\right] \\
& + \frac{D}{K^2}\left[-u^B K_{BD}K^D_A + \frac{1}{K^2}\hat{\nabla}^2\left(\hat{\nabla}^2 u_A\right) - 3\left(\frac{\hat{\nabla}^B K}{K}\right)\hat{\nabla}_B u_A + \frac{1}{K^2}\hat{\nabla}_A\left(\hat{\nabla}^2 K\right) + K^D_A\left(\frac{\hat{\nabla}_D K}{K}\right)\right] \\
& + u_A\,\frac{D}{K^2}\Bigg[-(\hat{\nabla}_D u_E)(\hat{\nabla}^D u^E) - u\cdot K\cdot K\cdot u - \frac{\hat{\nabla}^2 u_E}{K}\,\frac{\hat{\nabla}^2 u^E}{K} - \left(\frac{u\cdot\nabla K}{K}\right)^2 \\
& + \frac{K}{D}\left(-\frac{u\cdot\nabla K}{K} + u\cdot K\cdot u\right)\Bigg]
\end{split}
\tag{C.77}
$$

$$
\begin{aligned}
\mathcal{W}^{(1)}_{AB} = &-2\frac{D}{K^2}\Bigg[\frac{K}{D}K_{AB} - \frac{K}{2D}(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A) - K_A^F K_{FB} + K_A^F\,\hat{\nabla}_F u_B + K_B^F\,\hat{\nabla}_F u_A - \hat{\nabla}^F u_A \hat{\nabla}_F u_B \\
&- \frac{\hat{\nabla}^2 u_A}{K}\frac{\hat{\nabla}^2 u_B}{K} + \cancel{\frac{\hat{\nabla}^2 u_A}{K}\frac{\hat{\nabla}_B K}{K}} + \cancel{\frac{\hat{\nabla}^2 u_B}{K}\frac{\hat{\nabla}_A K}{K}} - \frac{\hat{\nabla}_A K}{K}\frac{\hat{\nabla}_B K}{K}\Bigg] + \cancel{\frac{1}{K}\Big[K_{AB} - \hat{\nabla}_B u_A - \hat{\nabla}_A u_B\Big]} \\
&- \frac{1}{NK}\left[\cancel{N} + \frac{(n\cdot\nabla)K}{K}\right]\Big[K_{AB} - \hat{\nabla}_A u_B - \hat{\nabla}_B u_A\Big] + \frac{D}{K^2}\Bigg[\cancel{2\,\frac{\hat{\nabla}^2 u_A}{K}\frac{\hat{\nabla}_B K}{K}} + \cancel{2\,\frac{\hat{\nabla}^2 u_B}{K}\frac{\hat{\nabla}_A K}{K}} \\
&- \frac{\hat{\nabla}_A K}{K}\frac{\hat{\nabla}_B K}{K} - \lambda\,\Pi_{AB} + \hat{\nabla}_A\left(\frac{\hat{\nabla}_B K}{K}\right) - K_A^E K_{EB} - \frac{\hat{\nabla}_B K}{K}u^E K_{EA} - \frac{\hat{\nabla}_A K}{K}u^E K_{EB} \\
&+ 2\,K_{AB}\frac{(u\cdot\nabla)K}{K} + K_A^D\left(\hat{\nabla}_D u_B\right) + K_B^D\left(\hat{\nabla}_D u_A\right) - \hat{\nabla}_A\left(\frac{\hat{\nabla}^2 u_B}{K}\right) - \hat{\nabla}_B\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\Bigg] \\
= &\frac{D}{K^2}\left[u\cdot K\cdot u - \frac{K}{D}\right]K_{AB} + \frac{D}{K^2}\left[\frac{\hat{\nabla}^2 K}{K^2} - \lambda\,\frac{D}{K}\right]\left(\hat{\nabla}_A u_B + \hat{\nabla}_B u_A\right) + \frac{D}{K^2}\,K_A^F K_{FB} \\
&- \frac{D}{K^2}\,\lambda\,\Pi_{AB} - \frac{D}{K^2}\left(K_A^F\,\hat{\nabla}_F u_B + K_B^F\,\hat{\nabla}_F u_A\right) + 2\,\frac{D}{K^2}\left(\hat{\nabla}^F u_A\right)\left(\hat{\nabla}_F u_B\right) \\
&+ 2\,\frac{D}{K^2}\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\left(\frac{\hat{\nabla}^2 u_B}{K}\right) + \frac{D}{K^2}\frac{1}{K}\hat{\nabla}_A\left(\hat{\nabla}_B K\right) - \frac{D}{K^2}\left[\left(\frac{\hat{\nabla}_A K}{K}\right)u^E K_{EB} + \left(\frac{\hat{\nabla}_B K}{K}\right)u^E K_{EA}\right] \\
&- \frac{D}{K^2}\frac{1}{K}\left[\hat{\nabla}_A\left(\hat{\nabla}^2 u_B\right) + \hat{\nabla}_B\left(\hat{\nabla}^2 u_A\right)\right] + \frac{D}{K^2}\left[\left(\frac{\hat{\nabla}_A K}{K}\right)\left(\frac{\hat{\nabla}^2 u_B}{K}\right) + \left(\frac{\hat{\nabla}_B K}{K}\right)\left(\frac{\hat{\nabla}^2 u_A}{K}\right)\right]
\end{aligned}
\tag{C.78}
$$

C.2.2 Inside$(\psi < 1)$

From (4.51)

$$
\begin{aligned}
\Rightarrow\ \Pi_C^A \Pi_{C'}^B \Bigg[&\underbrace{\nabla_A\nabla_E\tilde{\mathfrak{h}}_B^E + \nabla_B\nabla_E\tilde{\mathfrak{h}}_A^E}_{\text{Part-1}}\underbrace{-\nabla^2\tilde{\mathfrak{h}}_{AB}}_{\text{Part-2}}\underbrace{-\nabla_B\nabla_A\tilde{\mathfrak{h}}}_{\text{Part-3}} \\
&\underbrace{+2\,\bar{R}_{EABC}\tilde{\mathfrak{h}}^{EC} + \bar{R}_{AC}\tilde{\mathfrak{h}}_B^C + \bar{R}_{BC}\tilde{\mathfrak{h}}_A^C - 2(D-1)\lambda\tilde{\mathfrak{h}}_{AB}}_{\text{Part-4}}\Bigg] = 0
\end{aligned}
\tag{C.79}
$$

Now, we will simplify the above equation

$$
\begin{aligned}
\text{Part-1} &= \Pi^A_C \Pi^B_{C'} \left[\nabla_A \nabla_E \tilde{\mathfrak{h}}^E_B + \nabla_B \nabla_E \tilde{\mathfrak{h}}^E_A\right] \\
&= \Pi^A_C \Pi^B_{C'} \sum_{m=0}^{\infty} \left[\nabla_A \left((\psi-1)^m \nabla_E [\tilde{h}^{(m)}]^E_B\right) + \nabla_B \left((\psi-1)^m \nabla_E [\tilde{h}^{(m)}]^E_A\right)\right] \\
&= \Pi^A_C \Pi^B_{C'} \sum_{m=0}^{\infty} (\psi-1)^m \left[\nabla_A \nabla_E [\tilde{h}^{(m)}]^E_B + \nabla_B \nabla_E [\tilde{h}^{(m)}]^E_A\right]
\end{aligned}
\tag{C.80}
$$

$$
\begin{aligned}
\text{Part-2} &= -\Pi^A_C \Pi^B_{C'} \nabla^2 \tilde{\mathfrak{h}}_{AB} \\
&= -\Pi^A_C \Pi^B_{C'} \nabla^D \sum_{m=0}^{\infty} \left[m(\psi-1)^{m-1} N n_D \tilde{h}^{(m)}_{AB} + (\psi-1)^m \nabla_D \tilde{h}^{(m)}_{AB}\right] \\
&= -\Pi^A_C \Pi^B_{C'} \sum_{m=0}^{\infty} \Big[m(m-1)(\psi-1)^{m-2} N^2 \tilde{h}^{(m)}_{AB} + m(\psi-1)^{m-1} [(n\cdot\nabla)N] \tilde{h}^{(m)}_{AB} \\
&\quad + m(\psi-1)^{m-1} N K \tilde{h}^{(m)}_{AB} + 2m(\psi-1)^{m-1} N (n\cdot\nabla) \tilde{h}^{(m)}_{AB} + (\psi-1)^m \nabla^2 \tilde{h}^{(m)}_{AB}\Big]
\end{aligned}
\tag{C.81}
$$

$$
\begin{aligned}
\text{Part-3} &= -\Pi^A_C \Pi^B_{C'} \left[\nabla_B \nabla_A \tilde{\mathfrak{h}}\right] \\
&= -\Pi^A_C \Pi^B_{C'} \nabla_B \sum_{m=0}^{\infty} \left[m(\psi-1)^{m-1} N n_A \tilde{h}^{(m)} + (\psi-1)^m \nabla_A \tilde{h}^{(m)}\right] \\
&= -\Pi^A_C \Pi^B_{C'} \sum_{m=0}^{\infty} \left[m(\psi-1)^{m-1} N (\nabla_B n_A) \tilde{h}^{(m)} + (\psi-1)^m \nabla_B \nabla_A \tilde{h}^{(m)}\right]
\end{aligned}
\tag{C.82}
$$

$$
\begin{aligned}
\text{Part-4} &= \Pi^A_C \Pi^B_{C'} \left[2\, \bar{R}_{EABC} \tilde{\mathfrak{h}}^{EC} + \bar{R}_{AC} \tilde{\mathfrak{h}}^C_B + \bar{R}_{BC} \tilde{\mathfrak{h}}^C_A - 2(D-1)\lambda \tilde{\mathfrak{h}}_{AB}\right] \\
&= \Pi^A_C \Pi^B_{C'} \left[2\lambda \left(g_{EB} g_{AC} - g_{EC} g_{AB}\right) \tilde{\mathfrak{h}}^{EC} + 2\lambda(D-1) \tilde{\mathfrak{h}}_{AB} - 2\lambda(D-1)\lambda \tilde{\mathfrak{h}}_{AB}\right] \\
&= \Pi^A_C \Pi^B_{C'}\, 2\lambda \sum_{m=0}^{\infty} (\psi-1)^m \left[\tilde{h}^{(m)}_{AB} - \tilde{h}^{(m)} g_{AB}\right]
\end{aligned}
\tag{C.83}
$$

Collecting the coefficient of $(\psi-1)^0$ of (C.79)

$$
\begin{aligned}
&\Pi^A_C \Pi^B_{C'} \Bigg[\nabla_A \nabla_E \left[\tilde{h}^{(0)}\right]^E_B + \nabla_B \nabla_E \left[\tilde{h}^{(0)}\right]^E_A - 2N^2 \tilde{h}^{(2)}_{AB} - [(n\cdot\nabla)N] \tilde{h}^{(1)}_{AB} - NK \tilde{h}^{(1)}_{AB} \\
&- 2N(n\cdot\nabla) \tilde{h}^{(1)}_{AB} - \nabla^2 \tilde{h}^{(0)}_{AB} - N\, K_{AB}\, \tilde{h}^{(1)} - \nabla_B \nabla_A \tilde{h}^{(0)} + 2\,\lambda\, \tilde{h}^{(0)}_{AB} - 2\,\lambda\, \tilde{h}^{(0)}\, g_{AB}\Bigg] = 0
\end{aligned}
\tag{C.84}
$$

Using (4.52), the leading order($\mathcal{O}(D)$) terms of (C.84)

$$\Pi^A_C \Pi^B_{C'} \left[\nabla_A \nabla_E \big[\tilde{h}^{(0)}\big]^E_B + \nabla_B \nabla_E \big[\tilde{h}^{(0)}\big]^E_A - NK \tilde{h}^{(1,1)}_{CC'} - \nabla^2 \tilde{h}^{(0)}_{AB} - NK_{AB} \tilde{h}^{(1,1)} \right] = 0 \tag{C.85}$$

In the last equation, we have used the fact that $\tilde{h}^{(0)}$ can nowhere be $\mathcal{O}(D)$. Taking trace of (C.85)

$$\begin{aligned} &\Pi^{AB} \left[2\, \nabla_A \nabla_E \big[\tilde{h}^{(0)}\big]^E_B - \nabla^2 \tilde{h}^{(0)}_{AB} \right] - 2NK \tilde{h}^{(1,1)} = 0 \\ \Rightarrow\ & \tilde{h}^{(1,1)} = \frac{1}{2NK} \Pi^{AB} \left[2\, \nabla_A \nabla_E \big[\tilde{h}^{(0)}\big]^E_B - \nabla^2 \tilde{h}^{(0)}_{AB} \right] \end{aligned} \tag{C.86}$$

Now, from (C.85)

$$\tilde{h}^{(1,1)}_{CC'} = \Pi^A_C \Pi^B_{C'} \frac{1}{NK} \left[\nabla_A \nabla_E \big[\tilde{h}^{(0)}\big]^E_B + \nabla_B \nabla_E \big[\tilde{h}^{(0)}\big]^E_A - \nabla^2 \tilde{h}^{(0)}_{AB} \right] - \frac{1}{K} K_{CC'} \tilde{h}^{(1,1)} \tag{C.87}$$

From, subleading order($\mathcal{O}(1)$) of (C.84)

$$\begin{aligned} &\Pi^A_C \Pi^B_{C'} \Bigg[-2N^2 \tilde{h}^{(2)}_{AB} - [(n\cdot\nabla)N] \tilde{h}^{(1,1)}_{AB} - \frac{NK}{D} \tilde{h}^{(1,2)}_{AB} - 2N(n\cdot\nabla) \tilde{h}^{(1,1)}_{AB} - \frac{N}{D} K_{AB} \tilde{h}^{(1,2)} \\ &- \nabla_B \nabla_A \tilde{h}^{(0)} + 2\,\lambda \tilde{h}^{(0)}_{AB} - 2\,\lambda \tilde{h}^{(0)} g_{AB} \Bigg] = \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{C.88}$$

Taking trace,

$$\tilde{h}^{(1,2)} = \frac{D}{2NK} \Big[-2N^2 \tilde{h}^{(2)} - [(n\cdot\nabla)N] \tilde{h}^{(1,1)} - 2\lambda \tilde{h}^{(0)} (D-2) - \Pi^{AB} \Big\{ 2N(n\cdot\nabla) \tilde{h}^{(1,1)}_{AB} + \nabla_B \nabla_A \tilde{h}^{(0)} \Big\} \Big] + \mathcal{O}(1) \tag{C.89}$$

Now, from (C.88)

$$\begin{aligned} \tilde{h}^{(1,2)}_{CC'} = \frac{D}{NK} \Pi^A_C \Pi^B_{C'} \Bigg[&-2N^2 \tilde{h}^{(2)}_{AB} - [(n\cdot\nabla)N] \tilde{h}^{(1,1)}_{AB} - 2N(n\cdot\nabla) \tilde{h}^{(1,1)}_{AB} - \frac{N}{D} K_{AB} \tilde{h}^{(1,2)} \\ &- \nabla_B \nabla_A \tilde{h}^{(0)} + 2\,\lambda \tilde{h}^{(0)}_{AB} - 2\,\lambda \tilde{h}^{(0)} g_{AB} \Bigg] + \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{C.90}$$

Collecting coefficients of $(\psi - 1)$ of (C.79) at order($\mathcal{O}(D)$)

$$\Pi^A_C \Pi^B_{C'} \Big[\nabla_A \nabla_E \big[\tilde{h}^{(1,1)}\big]^E_B + \nabla_B \nabla_E \big[\tilde{h}^{(1,1)}\big]^E_A - 2NK\tilde{h}^{(2)}_{AB} - \nabla^2 \tilde{h}^{(1,1)}_{AB} - 2NK_{AB}\tilde{h}^{(2)} - \nabla_B \nabla_A \tilde{h}^{(1,1)} - 2\lambda \tilde{h}^{(1,1)} g_{AB} \Big] = \mathcal{O}(1) \tag{C.91}$$

Taking trace,

$$\tilde{h}^{(2)} = \Pi^{AB} \frac{1}{4NK} \Big[2\, \nabla_A \nabla_E \big[\tilde{h}^{(1,1)}\big]^E_B - \nabla^2 \tilde{h}^{(1,1)}_{AB} - \nabla_B \nabla_A \tilde{h}^{(1,1)} - 2\lambda \tilde{h}^{(1,1)} g_{AB} \Big] + \mathcal{O}(1) \tag{C.92}$$

From (C.91)

$$\tilde{h}^{(2)}_{CC'} = \Pi^A_C \Pi^B_{C'} \frac{1}{2NK} \Big[\nabla_A \nabla_E \big[\tilde{h}^{(1,1)}\big]^E_B + \nabla_B \nabla_E \big[\tilde{h}^{(1,1)}\big]^E_A - \nabla^2 \tilde{h}^{(1,1)}_{AB} - 2NK_{AB}\tilde{h}^{(2)} - \nabla_B \nabla_A \tilde{h}^{(1,1)} - 2\lambda \tilde{h}^{(1,1)} g_{AB} \Big] + \mathcal{O}\left(\frac{1}{D}\right) \tag{C.93}$$

Calculation of $\tilde{h}^{(1,1)}_{CC'}$

From, (C.87)

$$\tilde{h}^{(1,1)}_{CC'} = \underbrace{\Pi^A_C \Pi^B_{C'} \frac{1}{NK} \Big[\nabla_A \nabla_E \big[\tilde{h}^{(0)}\big]^E_B + \nabla_B \nabla_E \big[\tilde{h}^{(0)}\big]^E_A \Big]}_{\tilde{h}^{(1,1)}_{CC'}|_{\text{part-1}}} \underbrace{- \Pi^A_C \Pi^B_{C'} \frac{1}{NK} \nabla^2 \tilde{h}^{(0)}_{AB}}_{\tilde{h}^{(1,1)}_{CC'}|_{\text{part-2}}} \underbrace{- \frac{1}{K} K_{CC'} \tilde{h}^{(1,1)}}_{\tilde{h}^{(1,1)}_{CC'}|_{\text{part-3}}} \tag{C.94}$$

$$\tilde{h}^{(1,1)}_{CC'}|_{\text{part-1}} = \Pi^A_C \Pi^B_{C'} \frac{1}{NK} \Big[\nabla_A \nabla_E \big[h^{(0)}\big]^E_B + \nabla_B \nabla_E \big[h^{(0)}\big]^E_A \Big] + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.95}$$

We want to calculate the above expression on $\psi = 1$. But to calculate $\tilde{h}^{(1,1)}_{CC'}|_{\text{part-1}}$ on $\psi = 1$ we need the $(\psi - 1)$ dependent terms of $h^{(0)}_{AB}$. From (4.32)

$$\begin{aligned} \big[h^{(0)}\big]^E_B &= \big[M'^{(0)}\big]^E_B - (\psi - 1)\big[C^{(0)}\big]^E_B + \mathcal{O}(\psi - 1)^2 \\ \Rightarrow\ \nabla_E \big[h^{(0)}\big]^E_B &= \nabla_E \big[M'^{(0)}\big]^E_B + \mathcal{O}(\psi - 1) \end{aligned} \tag{C.96}$$

Now,

$$\big[M'^{(0)}\big]^E_B = u^E u_B + \frac{1}{K} \Big[u^E \Pi^C_B \Big(\frac{\nabla_C K}{K}\Big) + u_B \Pi^{CE} \Big(\frac{\nabla_C K}{K}\Big) + K^E_B - \Pi^{CE} \Pi^{C'}_B (\nabla_C u_{C'} + \nabla_{C'} u_C) \Big] + \mathcal{O}\Big(\frac{1}{D}\Big)^2 \tag{C.97}$$

After a bit of simplification divergence of the above equation becomes

$$\nabla_E\big[M'^{(0)}\big]^E_B = u_B\,(\nabla\cdot u) + (u\cdot\nabla)u_B + \frac{1}{K}\left[u_B\frac{\hat{\nabla}^2 K}{K} + \hat{\nabla}_B K - n_B\,K^{AC}K_{AC} - \hat{\nabla}^2 u_B - K u^C K_{CB} - \lambda\,D u_B\right] + \mathcal{O}\left(\frac{1}{D}\right) \tag{C.98}$$

In the derivation of the above equation we have used the following identities

$$\begin{aligned}\nabla^2 K &= \hat{\nabla}^2 K + K(n\cdot\nabla)K + \mathcal{O}(D)\\ \hat{\nabla}^2 u_A &= \Pi^D_A\left[\nabla^2 u_D - K(n\cdot\nabla)u_D\right] + \mathcal{O}(1)\\ \nabla^A K_{AB} &= \hat{\nabla}_B K - n_B K^{AC}K_{AC} + \mathcal{O}(1)\end{aligned} \tag{C.99}$$

Now,

$$\begin{aligned}\nabla_E\big[h^{(0)}\big]^E_B &= u_B\,(\nabla\cdot u) + u_B\frac{\hat{\nabla}^2 K}{K^2} - n_B\,\frac{K}{D} - \lambda\,\frac{D}{K}u_B - n_B(u\cdot K\cdot u)\\ &\quad + \left[-\frac{\hat{\nabla}^2 u_B}{K} + \frac{\hat{\nabla}_B K}{K} - u^E K_{EB} + (u\cdot\hat{\nabla})u_B\right] + \mathcal{O}\left(\frac{1}{D}\right)\\ &= -2\,u_B\frac{u\cdot\nabla K}{K} + u_B\frac{\hat{\nabla}^2 K}{K^2} - n_B\,\frac{K}{D} - \lambda\,\frac{D}{K}u_B - n_B(u\cdot K\cdot u) + u_B(u\cdot K\cdot u)\\ &= -n_B\,\frac{K}{D} - n_B(u\cdot K\cdot u)\end{aligned} \tag{C.100}$$

In the last line we have used the divergence of leading order membrane equation

$$\frac{\hat{\nabla}^2 K}{K^2} = 2\,\frac{u\cdot\nabla K}{K} - u\cdot K\cdot u + \lambda\,\frac{D}{K} + \mathcal{O}\left(\frac{1}{D}\right) \tag{C.101}$$

From (C.100)

$$\Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla_A\nabla_E\big[h^{(0)}\big]^E_B = -\frac{1}{NK}\left[\frac{K}{D} + u\cdot K\cdot u\right]K_{CC'} \tag{C.102}$$

So, finally we get

$$\tilde{h}^{(1,1)}_{CC'}|_{\text{part-1}} = -\frac{2}{NK}\left[\frac{K}{D} + u\cdot K\cdot u\right]K_{CC'} + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.103}$$

Now, we will calculate

$$\tilde{h}^{(1,1)}_{CC'}|_{\text{part-2}} = -\Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla^2 h^{(0)}_{AB} + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.104}$$

We want to calculate the above expression on $\psi = 1$. But to calculate $\tilde{h}^{(1,1)}_{CC'}|_{\text{part-2}}$ on $\psi = 1$ we need the $(\psi - 1)$ and $(\psi - 1)^2$ dependent terms of $h^{(0)}_{AB}$.

$$\tilde{h}^{(1,1)}_{CC'}|_{\text{part-2}} = \underbrace{-\Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla^2 M'^{(0)}_{AB}}_{\text{Term-1}} + \underbrace{\Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla^2\left[(\psi-1)C^{(0)}_{AB}\right]}_{\text{Term-2}} + \underbrace{\Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla^2\left[(\psi-1)^2 E^{(0)}_{AB}\right]}_{\text{Term-3}} \tag{C.105}$$

$$\begin{aligned}\text{Term-3} &= \Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla^2\left[(\psi-1)^2 E^{(0)}_{AB}\right]\\ &= \frac{2N}{K}E^{(0)}_{CC'}\end{aligned} \tag{C.106}$$

From (4.34)

$$\begin{aligned}E^{(0)}_{CC'} &= -\frac{1}{2N}\Pi^A_C\Pi^B_{C'}(n.\nabla)C^{(0)}_{AB}\\ &= \frac{1}{2N}\frac{(n\cdot\nabla)N}{N^2}\left[u_C\frac{\hat{\nabla}^2 u_{C'}}{K} + u_{C'}\frac{\hat{\nabla}^2 u_C}{K}\right] - \frac{1}{2N^2}\left[2\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} + u_C\left(\frac{\hat{\nabla}_{C'}K}{K}\right)\frac{(u\cdot\nabla)K}{K}\right.\\ &\left. + u_{C'}\left(\frac{\hat{\nabla}_C K}{K}\right)\frac{(u\cdot\nabla)K}{K} + u_C\Pi^B_{C'}(n\cdot\nabla)(n\cdot\nabla)u_B + u_{C'}\Pi^B_C(n\cdot\nabla)(n\cdot\nabla)u_B\right] + \mathcal{O}\left(\frac{1}{D}\right)\end{aligned} \tag{C.107}$$

Using (C.107) we can write Term-3 as

$$\text{Term-3} = \mathcal{A}^{(3)}_{CC'} + u_C\mathcal{B}^{(3)}_{C'} + u_{C'}\mathcal{B}^{(3)}_C \tag{C.108}$$

Where,

$$\begin{aligned}\mathcal{A}^{(3)}_{CC'} &= -2\,\frac{D}{K^2}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right)\\ \mathcal{B}^{(3)}_C &= \frac{D}{K^2}\left[\frac{\hat{\nabla}^2 K}{K^2} - \lambda\frac{D}{K}\right]\frac{\hat{\nabla}^2 u_C}{K} - \frac{D}{K^2}\left[\frac{\hat{\nabla}_C K}{K}\frac{(u\cdot\nabla)K}{K} + \Pi^B_C(n\cdot\nabla)(n\cdot\nabla)u_B\right]\end{aligned} \tag{C.109}$$

Now,

$$\begin{aligned}\text{Term-2} &= \Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla^2\left[(\psi-1)C^{(0)}_{AB}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2\\ &= C^{(0)}_{CC'} + \frac{1}{K}\frac{(n\cdot\nabla)N}{N}C^{(0)}_{CC'} + \frac{2}{K}\Pi^A_C\Pi^B_{C'}(n\cdot\nabla)C^{(0)}_{AB}\end{aligned} \tag{C.110}$$

Using (C.70) and (C.107) we can write Term-2 as

$$\text{Term-2} = \mathcal{A}^{(2)}_{CC'} + u_C\mathcal{B}^{(2)}_{C'} + u_{C'}\mathcal{B}^{(2)}_C \tag{C.111}$$

Where,

$$\mathcal{B}^{(2)}_C = \frac{1}{N}\Pi^E_C(n\cdot\nabla)u_E + \frac{1}{NK}\Bigg[-\frac{\hat\nabla_C K}{K}\left(2\,\frac{(n\cdot\nabla)K}{K} + 2\,\frac{\hat\nabla^2 K}{K^2} - \lambda\,\frac{D}{K}\right) + \frac{1}{K^2}\hat\nabla_C\left(\hat\nabla^2 K\right)$$
$$-2\,\frac{K}{D}\left(\frac{\hat\nabla_C K}{K}\right) - K^D_C\left(\frac{\nabla_D K}{K}\right)\Bigg] + \frac{1}{NK}\frac{(n\cdot\nabla)N}{N}\left[\Pi^E_C(n\cdot\nabla)u_E\right]$$
$$+\frac{2}{K}\Bigg[-\frac{1}{N}\left(\frac{n\cdot\nabla N}{N}\right)\frac{\hat\nabla^2 u_C}{K} + \frac{1}{N}\left(\frac{\hat\nabla_C K}{K}\right)\frac{u\cdot\nabla K}{K} + \frac{1}{N}\Pi^B_C(n\cdot\nabla)(n\cdot\nabla)u_B\Bigg]$$
$$\mathcal{A}^{(2)}_{CC'} = -\frac{1}{NK}\frac{(n\cdot\nabla)N}{N}\left[K_{CC'} - \hat\nabla_C u_{C'} - \hat\nabla_{C'} u_C\right] + \frac{1}{NK}\Bigg[2\,\frac{\hat\nabla^2 u_C}{K}\frac{\hat\nabla_{C'}K}{K} + 2\,\frac{\hat\nabla^2 u_{C'}}{K}\frac{\hat\nabla_C K}{K}$$
$$-\frac{\hat\nabla_C K}{K}\frac{\hat\nabla_{C'}K}{K} - \lambda\,\Pi_{CC'} + \hat\nabla_C\left(\frac{\hat\nabla_{C'}K}{K}\right) - K^E_C K_{EC'} - \frac{\hat\nabla_{C'}K}{K}u^B K_{BC} - \frac{\hat\nabla_C K}{K}u^B K_{BC'}$$
$$+2\,K_{CC'}\frac{(u\cdot\nabla)K}{K} + K^D_C\left(\hat\nabla_D u_{C'}\right) + K^D_{C'}\left(\hat\nabla_D u_C\right) - \hat\nabla_C\left(\frac{\hat\nabla^2 u_{C'}}{K}\right) - \hat\nabla_{C'}\left(\frac{\hat\nabla^2 u_C}{K}\right)\Bigg]$$
$$+\frac{2}{NK}\left[2\,\frac{\hat\nabla^2 u_C}{K}\frac{\hat\nabla^2 u_{C'}}{K}\right] \tag{C.112}$$

$$\text{Term-1} = -\Pi^A_C\Pi^B_{C'}\frac{1}{NK}\nabla^2 M'^{(0)}_{AB} + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.113}$$

Here,

$$M'^{(0)}_{AB} = u_A u_B + \frac{1}{ND}\left[u_A\Pi^E_B\left(\frac{\nabla_E K}{K}\right) + u_B\Pi^E_A\left(\frac{\nabla_E K}{K}\right) + K_{AB} - \Pi^E_A\Pi^F_B(\nabla_E u_F + \nabla_F u_E)\right] + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.114}$$

$$\Pi^A_C\Pi^B_{C'}\nabla^2 M'^{(0)}_{AB}$$
$$= u_C\Pi^B_{C'}\nabla^2 u_B + u_{C'}\Pi^B_C\nabla^2 u_B + 2\,\Pi^A_C\Pi^B_{C'}(\nabla^D u_A)(\nabla_D u_B)$$
$$-\frac{1}{ND}\left(\frac{\nabla^2 N}{N}\right)\left[u_C\,\Pi^E_{C'}\frac{\nabla_E K}{K} + u_{C'}\,\Pi^E_C\frac{\nabla_E K}{K} + K_{CC'} - \Pi^E_C\Pi^F_{C'}(\nabla_E u_F + \nabla_F u_E)\right]$$
$$+\frac{1}{ND}\Bigg[\Pi^A_C(\nabla^2 u_A)\Pi^E_{C'}\frac{\nabla_E K}{K} + u_C\Pi^B_{C'}\nabla^2\left(\Pi^E_B\frac{\nabla_E K}{K}\right) + \Pi^A_{C'}(\nabla^2 u_A)\Pi^E_C\frac{\nabla_E K}{K}$$
$$+u_{C'}\Pi^B_C\nabla^2\left(\Pi^E_B\frac{\nabla_E K}{K}\right) + \Pi^A_C\Pi^B_{C'}\nabla^2 K_{AB} - \Pi^A_C\Pi^B_{C'}\nabla^2\left\{\Pi^E_A\Pi^F_B(\nabla_E u_F + \nabla_F u_E)\right\}\Bigg] + \mathcal{O}\left(\frac{1}{D}\right) \tag{C.115}$$

We will use the following identities to simplify (C.115). we are just stating the identities without proof, proofs are quite straightforward.

$$
\begin{aligned}
\frac{1}{N} &= \frac{D}{K}\left[1-\frac{1}{K}\left(\frac{\hat{\nabla}^2 K}{K^2}-\lambda\,\frac{D}{K}-\frac{K}{D}\right)\right]+\mathcal{O}\left(\frac{1}{D}\right)^2\\
\Pi^D_B\nabla^2 u_D &= \hat{\nabla}^2 u_B + K\Pi^D_B(n\cdot\nabla)u_D - K^F_B u^D K_{FD} + \Pi^D_B(n\cdot\nabla)(n\cdot\nabla)u_D\\
&\quad - \frac{\hat{\nabla}^F K}{K}(\hat{\nabla}_F u_B)+\mathcal{O}\left(\frac{1}{D}\right)\\
\Pi^A_C\Pi^B_{C'}(\nabla^2 K_{AB}) &= -2\,(\hat{\nabla}_C K)\left(\frac{\hat{\nabla}_{C'}K}{K}\right) - 2\,\lambda K\Pi_{CC'} + \lambda(D-1)K_{CC'}\\
&\quad + 2\hat{\nabla}_C(\hat{\nabla}_{C'}K) - K_{CC'}\frac{K^2}{D}\\
\Pi^B_{C'}\nabla^2\left(\Pi^E_B\frac{\nabla_E K}{K}\right) &= -(\hat{\nabla}_{C'}K)\left[4\,\frac{\hat{\nabla}^2 K}{K^2}-3\,\lambda\frac{D}{K}\right]+\frac{2}{K}\hat{\nabla}_{C'}(\hat{\nabla}^2 K)\\
\Pi^A_C\Pi^B_{C'}\nabla^2\left[\Pi^E_A\Pi^F_B\nabla_E u_F\right] &= -2\,(\hat{\nabla}_C K)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) + 2\,(\hat{\nabla}_{C'}K)u^F K_{FC} + \lambda(D-1)(\hat{\nabla}_C u_{C'})\\
&\quad + 2\,\hat{\nabla}_C(\hat{\nabla}^2 u_{C'}) - 2\,K_{CC'}(u\cdot\nabla)K\\
\Pi^A_C\Pi^B_{C'}(\nabla^D u_A)(\nabla_D u_A) &= (\hat{\nabla}^D u_C)(\hat{\nabla}_D u_{C'}) + \left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right)
\end{aligned}
\tag{C.116}
$$

Using (C.116), we can write Term-1 as

$$
\text{Term-1} = \mathcal{A}^{(1)}_{CC'} + u_C\mathcal{B}^{(1)}_{C'} + u_{C'}\mathcal{B}^{(1)}_C \tag{C.117}
$$

Where,

$$
\begin{aligned}
\mathcal{B}^{(1)}_C &= -\frac{D}{K^2}\left[1-\frac{1}{K}\left(\frac{\hat{\nabla}^2 K}{K^2}-\lambda\,\frac{D}{K}-\frac{K}{D}\right)\right]\left[\hat{\nabla}^2 u_C + K\Pi^D_C(n\cdot\nabla)u_D\right]\\
&\quad -\frac{D}{K^2}\left[-K^F_C K_{FD}u^D + \Pi^D_C(n\cdot\nabla)(n\cdot\nabla)u_D - \frac{\hat{\nabla}^F K}{K}(\hat{\nabla}_F u_C)\right] - 2\,\frac{D}{K^4}\hat{\nabla}_C(\hat{\nabla}^2 K)\\
&\quad +\frac{1}{K}\left[1+\frac{D}{K}\left(2\,\frac{\hat{\nabla}^2 K}{K^2}-\lambda\,\frac{D}{K}-\frac{K}{D}\right)\right]\frac{\hat{\nabla}_C K}{K} + \frac{D}{K^2}\left[4\,\frac{\hat{\nabla}^2 K}{K^2}-3\,\lambda\frac{D}{K}\right]\frac{\hat{\nabla}_C K}{K}
\end{aligned}
\tag{C.118}
$$

$$\mathcal{A}^{(1)}_{CC'} = -2\,\frac{D}{K^2}\big(\hat{\nabla}^D u_C\big)\big(\hat{\nabla}_D u_{C'}\big) - 2\,\frac{D}{K^2}\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K}$$
$$+\frac{1}{K}\left[1+\frac{D}{K}\left(2\,\frac{\hat{\nabla}^2 K}{K^2}-\lambda\frac{D}{K}-\frac{K}{D}\right)\right]\left[K_{CC'}-\hat{\nabla}_C u_{C'}-\hat{\nabla}_{C'}u_C\right]$$
$$-\frac{D}{K^2}\Bigg[4\,\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}_{C'}K}{K}+4\,\frac{\hat{\nabla}^2 u_{C'}}{K}\frac{\hat{\nabla}_C K}{K}-2\,\frac{\hat{\nabla}_C K}{K}\frac{\hat{\nabla}_{C'}K}{K}-2\,\lambda\Pi_{CC'}+\lambda\,\frac{D}{K}K_{CC'}$$
$$+\frac{2}{K}\hat{\nabla}_C\big(\hat{\nabla}_{C'}K\big)-\frac{K}{D}K_{CC'}-2\,\frac{\hat{\nabla}_{C'}K}{K}u^F K_{FC}-2\,\frac{\hat{\nabla}_C K}{K}u^F K_{FC'}-\lambda\,\frac{D}{K}\left(\hat{\nabla}_C u_{C'}+\hat{\nabla}_{C'}u_C\right)$$
$$-\frac{2}{K}\hat{\nabla}_C\big(\hat{\nabla}^2 u_{C'}\big)-\frac{2}{K}\hat{\nabla}_{C'}\big(\hat{\nabla}^2 u_C\big)+4\,K_{CC'}\frac{u\cdot\nabla K}{K}\Bigg] \tag{C.119}$$

Adding, (C.117), (C.111) and (C.108) we get final expression of $\tilde{h}^{(1,1)}_{CC'}|_{\text{part-2}}$

$$\tilde{h}^{(1,1)}_{CC'}|_{\text{part-2}} = \left(\mathcal{A}^{(1)}_{CC'}+\mathcal{A}^{(2)}_{CC'}+\mathcal{A}^{(3)}_{CC'}\right)+u_C\left(\mathcal{B}^{(1)}_{C'}+\mathcal{B}^{(2)}_{C'}+\mathcal{B}^{(3)}_{C'}\right)+u_{C'}\left(\mathcal{B}^{(1)}_{C}+\mathcal{B}^{(2)}_{C}+\mathcal{B}^{(3)}_{C}\right) \tag{C.120}$$

$$\begin{aligned}
\tilde{h}^{(1,1)}_{CC'}|_{\text{part-3}} &= -\frac{1}{K}K_{CC'}\tilde{h}^{(1,1)}+\mathcal{O}\left(\frac{1}{D}\right)^2\\
&= -\frac{1}{K}K_{CC'}\frac{1}{2}\Pi^{AB}\left[\tilde{h}^{(1,1)}_{AB}|_{\text{part-1}}+\tilde{h}^{(1,1)}_{AB}|_{\text{part-2}}\right]+\mathcal{O}\left(\frac{1}{D}\right)^2\\
&= -\frac{1}{2K}K_{CC'}\left[-\frac{2}{N}\left(\frac{K}{D}+u\cdot K\cdot u\right)\right]-\frac{1}{2K}K_{CC'}\left[\Pi^{AB}\tilde{h}^{(1,1)}_{AB}|_{\text{part-2}}\right]\\
&= \frac{D}{K^2}K_{CC'}\left(\frac{K}{D}+u\cdot K\cdot u\right)-\frac{1}{2K}K_{CC'}\left[\Pi^{AB}\left(\mathcal{A}^{(1)}_{AB}+\mathcal{A}^{(2)}_{AB}\right)\right]\\
&= \frac{D}{K^2}K_{CC'}\left(\frac{K}{D}+u\cdot K\cdot u\right)+\mathcal{O}\left(\frac{1}{D}\right)^2
\end{aligned} \tag{C.121}$$

In the derivation of (C.121) we have used the following identity

$$\frac{1}{K}\Pi^{AB}\hat{\nabla}_A\left(\frac{\hat{\nabla}^2 u_B}{K}\right)=\frac{u\cdot\nabla K}{K}+\mathcal{O}\left(\frac{1}{D}\right) \tag{C.122}$$

Adding (C.103),(C.120) and (C.121) we get the final expression of $\tilde{h}^{(1,1)}_{CC'}$ as given in (4.53).

Calculation of $\tilde{h}^{(2)}_{CC'}$

From (C.93), the non-vanishing terms of $\tilde{h}^{(2)}_{CC'}$ are the following

$$\tilde{h}^{(2)}_{CC'} = \underbrace{\Pi^A_C \Pi^B_{C'} \frac{1}{2NK}\left[\nabla_A \nabla_E \left[\tilde{h}^{(1,1)}\right]^E_B + \nabla_B \nabla_E \left[\tilde{h}^{(1,1)}\right]^E_A\right]}_{\tilde{h}^{(2)}_{CC'}\big|_{\text{Part-1}}} \underbrace{-\Pi^A_C \Pi^B_{C'} \frac{1}{2NK}\left[\nabla^2 \tilde{h}^{(1,1)}_{AB}\right]}_{\tilde{h}^{(2)}_{CC'}\big|_{\text{Part-2}}} + \mathcal{O}\left(\frac{1}{D}\right) \tag{C.123}$$

For the calculation of $\tilde{h}^{(2)}_{CC'}$ we need $(\psi - 1)$ dependent terms of $\tilde{h}^{(1,1)}_{CC'}$. The expression of $\tilde{h}^{(1,1)}_{CC'}$ up to the relevant order is given by

$$\begin{aligned}
\tilde{h}^{(1,1)}_{CC'} = &-\frac{1}{NK}\left[u_C \Pi^B_{C'}\left(\nabla^2 u_B\right) + u_{C'} \Pi^B_C \left(\nabla^2 u_B\right)\right] + \frac{1}{N}\left[u_C \Pi^B_{C'} (n\cdot\nabla) u_B + u_{C'} \Pi^B_C (n\cdot\nabla) u_B\right] \\
&- (\psi - 1)\frac{1}{N}\left[1 + \frac{D}{K}\left(\frac{\nabla^2 K}{K^2}\right) - \frac{(n\cdot\nabla)N}{N^2}\right]\left[u_C\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) + u_{C'}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\right] \\
&+ (\psi - 1)\frac{1}{N^2}\left[2\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) + u_C\left(\frac{\hat{\nabla}^2 \hat{\nabla}^2 u_{C'}}{K^2}\right) + u_{C'}\left(\frac{\hat{\nabla}^2 \hat{\nabla}^2 u_C}{K^2}\right)\right. \\
&\left. - u_C\left(\frac{\hat{\nabla}^2 K}{K^2}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) - u_{C'}\left(\frac{\hat{\nabla}^2 K}{K^2}\right)\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\right] \\
&- (\psi - 1)\frac{D}{NK}\left[\left(\hat{\nabla}_C u_{C'} + \hat{\nabla}_{C'} u_C\right)\left(\frac{u\cdot\nabla K}{K}\right) + u_C \hat{\nabla}_{C'}\left(\frac{u\cdot\nabla K}{K}\right) + u_{C'} \hat{\nabla}_C\left(\frac{u\cdot\nabla K}{K}\right)\right]
\end{aligned} \tag{C.124}$$

$$\begin{aligned}
\nabla_E \left[\tilde{h}^{(1,1)}\right]^E_B &= -\frac{1}{N}\left[-u_B\, K n^C \left(\frac{\nabla^2 u_C}{K}\right)\right] + \frac{1}{N}\left[-K u_B\, n^C (n\cdot\nabla) u_C\right] \\
&= \frac{K}{N} u_B\, \frac{n^C \nabla^2 u_C}{K} - \frac{K}{N} u_B \left[n^C (n\cdot\nabla) u_C\right] \\
&= -\frac{K}{N} u_B \left(\frac{u\cdot\nabla K}{K}\right)
\end{aligned} \tag{C.125}$$

From, (C.123)

$$\tilde{h}^{(2)}_{CC'}\big|_{\text{Part-1}} = -\frac{D}{2NK}\left[\left(\hat{\nabla}_C u_{C'} + \hat{\nabla}_{C'} u_C\right)\frac{u\cdot\nabla K}{K} + u_C \hat{\nabla}_{C'}\left(\frac{u\cdot\nabla K}{K}\right) + u_{C'} \hat{\nabla}_C\left(\frac{u\cdot\nabla K}{K}\right)\right] \tag{C.126}$$

From, (C.123)

$$
\begin{aligned}
\tilde{h}^{(2)}_{CC'}\Big|_{\text{Part-2}} &= -\frac{1}{2NK}\Pi^A_C\Pi^B_{C'}\left[\nabla^2\tilde{h}^{(1,1)}_{AB}\right] \\
&= \underbrace{-\frac{1}{2NK}\Pi^A_C\Pi^B_{C'}\nabla^2\left[-\frac{1}{NK}\left(u_A\Pi^E_B\nabla^2u_E+u_B\Pi^E_A\nabla^2u_E\right)\right]}_{\text{Term-1}} \\
&\quad\underbrace{-\frac{1}{2NK}\Pi^A_C\Pi^B_{C'}\nabla^2\left[\frac{1}{N}\left\{u_A\Pi^E_B(n\cdot\nabla)u_E+u_B\Pi^E_A(n\cdot\nabla)u_E\right\}\right]}_{\text{Term-2}} \\
&\quad+\frac{1}{2N}\left[1+\frac{D}{K}\frac{\nabla^2K}{K^2}-\frac{(n\cdot\nabla)N}{N^2}\right]\left[u_C\frac{\hat{\nabla}^2u_{C'}}{K}+u_{C'}\frac{\hat{\nabla}^2u_C}{K}\right] \\
&\quad-\frac{1}{2N^2}\left[2\frac{\hat{\nabla}^2u_C}{K}\frac{\hat{\nabla}^2u_{C'}}{K}+u_C\frac{\hat{\nabla}^2\hat{\nabla}^2u_{C'}}{K^2}+u_{C'}\frac{\hat{\nabla}^2\hat{\nabla}^2u_C}{K^2}-u_C\frac{\hat{\nabla}^2K}{K^2}\frac{\hat{\nabla}^2u_{C'}}{K}-u_{C'}\frac{\hat{\nabla}^2K}{K^2}\frac{\hat{\nabla}^2u_C}{K}\right] \\
&\quad+\frac{D}{2NK}\left[\left(\hat{\nabla}_Cu_{C'}+\hat{\nabla}_{C'}u_C\right)\left(\frac{u\cdot\nabla K}{K}\right)+u_C\hat{\nabla}_{C'}\left(\frac{u\cdot\nabla K}{K}\right)+u_{C'}\hat{\nabla}_C\left(\frac{u\cdot\nabla K}{K}\right)\right]
\end{aligned}
\tag{C.127}
$$

$$
\begin{aligned}
\text{Term-1} &= -\frac{1}{2NK}\Pi^A_C\Pi^B_{C'}\nabla^2\left[-\frac{1}{NK}\left(u_A\Pi^E_B\nabla^2u_E+u_B\Pi^E_A\nabla^2u_E\right)\right] \\
&= -\frac{1}{2NK}\left[1+\frac{2D}{K}\frac{\nabla^2K}{K^2}\right]\left[u_C\Pi^E_{C'}\nabla^2u_E+u_{C'}\Pi^E_C\nabla^2u_E\right] \\
&\quad+\frac{1}{2NK}\frac{D}{K^2}\left[2\,\Pi^E_C\left(\nabla^2u_E\right)\Pi^F_{C'}\left(\nabla^2u_F\right)+u_C\Pi^E_{C'}\nabla^2\left(\Pi^F_E\nabla^2u_F\right)+u_{C'}\Pi^E_C\nabla^2\left(\Pi^F_E\nabla^2u_F\right)\right]
\end{aligned}
\tag{C.128}
$$

Using the identity,

$$
\begin{aligned}
\Pi^B_F\nabla^2\left(\Pi^C_B\nabla^2u_C\right) &= 2\,\hat{\nabla}^2\left(\hat{\nabla}^2u_F\right)+K^2\left(\frac{\hat{\nabla}_FK}{K}\right)\left(\frac{u\cdot\nabla K}{K}\right)-\lambda(D-1)K\left(\frac{\hat{\nabla}^2u_F}{K}\right) \\
&\quad+K\hat{\nabla}^2\left(\frac{\hat{\nabla}^2u_F}{K}\right)+K^2\Pi^E_F(n\cdot\nabla)(n\cdot\nabla)u_E-2K\left(\hat{\nabla}^EK\right)\hat{\nabla}_Eu_F-3\frac{K^3}{D}\left(\frac{\hat{\nabla}^2u_F}{K}\right)
\end{aligned}
\tag{C.129}
$$

we get,

$$\begin{aligned}
\text{Term-1} &= \frac{4D}{NK}\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} - \frac{1}{2N}\left[1+\frac{2D}{K}\left(\frac{\hat{\nabla}^2 K}{K^2}+\frac{n\cdot\nabla K}{K}\right)\right]\left[2u_C\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right)+2u_{C'}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\right]\\
&+\frac{D}{2NK}u_C\Bigg[-\lambda\frac{D}{K}\frac{\hat{\nabla}^2 u_{C'}}{K}+3\,\frac{\hat{\nabla}^2\hat{\nabla}^2 u_{C'}}{K^2}-\frac{\hat{\nabla}^2 K}{K^2}\frac{\hat{\nabla}^2 u_{C'}}{K}+\frac{\hat{\nabla}_{C'}K}{K}\frac{u\cdot\nabla K}{K}\\
&+\Pi^E_{C'}(n\cdot\nabla)(n\cdot\nabla)u_E-2\,\frac{\hat{\nabla}^E K}{K}(\hat{\nabla}_E u_{C'})-3\,\frac{K}{D}\frac{\hat{\nabla}^2 u_{C'}}{K}\Bigg]\\
&+\frac{D}{2NK}u_{C'}\Bigg[-\lambda\frac{D}{K}\frac{\hat{\nabla}^2 u_C}{K}+3\,\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2}-\frac{\hat{\nabla}^2 K}{K^2}\frac{\hat{\nabla}^2 u_C}{K}+\frac{\hat{\nabla}_C K}{K}\frac{u\cdot\nabla K}{K}\\
&+\Pi^E_C(n\cdot\nabla)(n\cdot\nabla)u_E-2\,\frac{\hat{\nabla}^E K}{K}(\hat{\nabla}_E u_C)-3\,\frac{K}{D}\frac{\hat{\nabla}^2 u_C}{K}\Bigg]
\end{aligned} \tag{C.130}$$

$$\begin{aligned}
\text{Term-2} &= -\frac{1}{2NK}\Pi^A_C\Pi^B_{C'}\nabla^2\left[\frac{1}{N}\left\{u_A\Pi^E_B(n\cdot\nabla)u_E+u_B\Pi^E_A(n\cdot\nabla)u_E\right\}\right]\\
&=\frac{1}{2NK}\frac{\nabla^2 N}{N^2}\left[u_C\Pi^E_{C'}(n\cdot\nabla)u_E+u_{C'}\Pi^E_C(n\cdot\nabla)u_E\right]\\
&-\frac{1}{2N^2K}\Big[\Pi^A_C(\nabla^2 u_A)\Pi^E_{C'}(n\cdot\nabla)u_E+\Pi^A_{C'}(\nabla^2 u_A)\Pi^E_C(n\cdot\nabla)u_E\\
&+u_C\Pi^B_{C'}\nabla^2\{\Pi^E_B(n\cdot\nabla)u_E\}+u_{C'}\Pi^B_C\nabla^2\{\Pi^E_B(n\cdot\nabla)u_E\}\Big]\\
&=\frac{1}{2N}\left[1+\frac{D}{K}\left(\frac{\nabla^2 K}{K^2}\right)\right]\left[u_C\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right)+u_{C'}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\right]-\frac{2}{N^2}\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K}\\
&-\frac{1}{2N^2}u_C\left[\frac{\hat{\nabla}^2\hat{\nabla}^2 u_{C'}}{K^2}-\frac{\hat{\nabla}^2 K}{K^2}\frac{\hat{\nabla}^2 u_{C'}}{K}+\frac{\hat{\nabla}_{C'}K}{K}\frac{u\cdot\nabla K}{K}+\Pi^E_{C'}(n\cdot\nabla)(n\cdot\nabla)u_E\right]\\
&-\frac{1}{2N^2}u_{C'}\left[\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2}-\frac{\hat{\nabla}^2 K}{K^2}\frac{\hat{\nabla}^2 u_C}{K}+\frac{\hat{\nabla}_C K}{K}\frac{u\cdot\nabla K}{K}+\Pi^E_C(n\cdot\nabla)(n\cdot\nabla)u_E\right]
\end{aligned} \tag{C.131}$$

Adding (C.126) and (C.127) we get the final expression of $\tilde{h}^{(2)}_{CC'}$ as given in (4.61) after using (C.130) and (C.131)

Calculation of $\tilde{h}^{(1,2)}_{CC'}$

From (C.90), the non-vanishing terms of $\tilde{h}^{(1,2)}_{CC'}$ are the followings

$$\tilde{h}^{(1,2)}_{CC'} = \underbrace{-2\,\frac{D}{K}\Pi^A_C\Pi^B_{C'}(n\cdot\nabla)\tilde{h}^{(1,1)}_{AB}}_{\tilde{h}^{(1,2)}_{CC'}\big|_{\text{Part-1}}} \underbrace{-\frac{D}{NK}\Pi^A_C\Pi^B_{C'}\left[2N^2\tilde{h}^{(2)}_{AB} + \{(n\cdot\nabla)N\}\tilde{h}^{(1,1)}_{AB} - 2\lambda\tilde{h}^{(0)}_{AB}\right]}_{\tilde{h}^{(1,2)}_{CC'}\big|_{\text{Part-2}}} + \mathcal{O}\left(\frac{1}{D}\right) \tag{C.132}$$

$$\begin{aligned}
\tilde{h}^{(1,2)}_{CC'}\big|_{\text{Part-1}} &= -2\,\frac{D}{K}\Pi^A_C\Pi^B_{C'}(n\cdot\nabla)\tilde{h}^{(1,1)}_{AB} \\
&= \underbrace{2\,\frac{D}{K}\Pi^A_C\Pi^B_{C'}(n\cdot\nabla)\left[\frac{1}{NK}\{u_A\Pi^E_B\left(\nabla^2 u_E\right) + u_B\Pi^E_A\left(\nabla^2 u_E\right)\}\right]}_{\text{term-1}} \\
&\quad \underbrace{-2\,\frac{D}{K}\Pi^A_C\Pi^B_{C'}(n\cdot\nabla)\left[\frac{1}{N}\{u_A\Pi^E_B(n\cdot\nabla)u_E + u_B\Pi^E_A(n\cdot\nabla)u_E\}\right]}_{\text{term-2}} \\
&\quad + 2\,\frac{D}{K}\left[1 + \frac{D}{K}\left(\frac{\nabla^2 K}{K^2}\right) - \frac{(n\cdot\nabla)N}{N^2}\right]\left[u_C\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) + u_{C'}\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\right] \\
&\quad - 2\,\frac{D}{NK}\Bigg[2\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) + u_C\left(\frac{\hat{\nabla}^2\hat{\nabla}^2 u_{C'}}{K^2}\right) + u_{C'}\left(\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2}\right) \\
&\quad - u_C\left(\frac{\hat{\nabla}^2 K}{K^2}\right)\left(\frac{\hat{\nabla}^2 u_{C'}}{K}\right) - u_{C'}\left(\frac{\hat{\nabla}^2 K}{K^2}\right)\left(\frac{\hat{\nabla}^2 u_C}{K}\right)\Bigg] \\
&\quad + 2\,\frac{D^2}{K^2}\left[\left(\hat{\nabla}_C u_{C'} + \hat{\nabla}_{C'} u_C\right)\left(\frac{u\cdot\nabla K}{K}\right) + u_C\hat{\nabla}_{C'}\left(\frac{u\cdot\nabla K}{K}\right) + u_{C'}\hat{\nabla}_C\left(\frac{u\cdot\nabla K}{K}\right)\right]
\end{aligned} \tag{C.133}$$

$$\begin{aligned}
\text{term-1} &= \frac{2D}{K}\Pi^A_C\Pi^B_{C'}(n\cdot\nabla)\left[\frac{1}{NK}\{u_A\Pi^E_B\left(\nabla^2 u_E\right) + u_B\Pi^E_A\left(\nabla^2 u_E\right)\}\right] \\
&= -\frac{2D}{NK^2}\left[\frac{(n\cdot\nabla)N}{N} + \frac{(n\cdot\nabla)K}{K}\right]\left[u_C\Pi^E_{C'}\nabla^2 u_E + u_{C'}\Pi^E_C\nabla^2 u_E\right] \\
&+ \frac{2D}{NK^2}\Big[\{\Pi^A_C(n\cdot\nabla)u_A\}\Pi^E_{C'}\nabla^2 u_E + \{\Pi^A_{C'}(n\cdot\nabla)u_A\}\Pi^E_C\nabla^2 u_E + u_C\Pi^B_{C'}(n\cdot\nabla)\left(\Pi^E_B\nabla^2 u_E\right) \\
&+ u_{C'}\Pi^B_C(n\cdot\nabla)\left(\Pi^E_B\nabla^2 u_E\right)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2D}{NK}\left[N + 2\,\frac{(n\cdot\nabla)K}{K}\right]\left[2\,u_C\frac{\hat{\nabla}^2 u_{C'}}{K} + 2\,u_{C'}\frac{\hat{\nabla}^2 u_C}{K}\right] \\
&+ \frac{2D}{NK}\bigg[4\,\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} - \frac{1}{K}u_C\Pi^B_{C'}\big\{(n\cdot\nabla)n_B\big\}n^E\nabla^2 u_E + \frac{1}{K}u_C\Pi^E_{C'}(n\cdot\nabla)\big(\nabla^2 u_E\big) \\
&- \frac{1}{K}u_{C'}\Pi^B_C\big\{(n\cdot\nabla)n_B\big\}n^E\nabla^2 u_E + \frac{1}{K}u_{C'}\Pi^E_C(n\cdot\nabla)\big(\nabla^2 u_E\big)\bigg]
\end{aligned}
\tag{C.134}
$$

Using the identity

$$
\begin{aligned}
\Pi^C_F(n\cdot\nabla)\big(\nabla^2 u_C\big) &= -\lambda\, D\frac{\hat{\nabla}^2 u_F}{K} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_F}{K} - K\frac{\hat{\nabla}^2 K}{K^2}\frac{\hat{\nabla}^2 u_F}{K} - K\frac{\hat{\nabla}_F K}{K}\frac{(u\cdot\nabla)K}{K} \\
&\quad + K\Pi^E_F(n\cdot\nabla)(n\cdot\nabla)u_E - 2\big(\hat{\nabla}^E K\big)\big(\hat{\nabla}_E u_F\big) - 3\,\frac{K^2}{D}\frac{\hat{\nabla}^2 u_F}{K}
\end{aligned}
\tag{C.135}
$$

we get

$$
\begin{aligned}
\text{term-1} &= -\frac{2D}{NK}\left[N + 2\,\frac{(n\cdot\nabla)K}{K}\right]\left[2\,u_C\frac{\hat{\nabla}^2 u_{C'}}{K} + 2\,u_{C'}\frac{\hat{\nabla}^2 u_C}{K}\right] \\
&+ \frac{2D}{NK}\left[4\,\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} + 2\,u_C\frac{\hat{\nabla}_{C'}K}{K}\frac{(u\cdot\nabla)K}{K} + 2\,u_{C'}\frac{\hat{\nabla}_C K}{K}\frac{(u\cdot\nabla)K}{K}\right] \\
&+ \frac{2D}{NK}u_C\bigg[-\lambda\,\frac{D}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_{C'}}{K^2} - \frac{\hat{\nabla}^2 K}{K^2}\frac{\hat{\nabla}^2 u_{C'}}{K} - \frac{\hat{\nabla}_{C'}K}{K}\frac{(u\cdot\nabla)K}{K} \\
&+ \Pi^E_{C'}(n\cdot\nabla)(n\cdot\nabla)u_E - 2\bigg(\frac{\hat{\nabla}^E K}{K}\bigg)\big(\hat{\nabla}_E u_{C'}\big) - 3\,\frac{K}{D}\frac{\hat{\nabla}^2 u_{C'}}{K}\bigg] \\
&+ \frac{2D}{NK}u_{C'}\bigg[-\lambda\,\frac{D}{K}\frac{\hat{\nabla}^2 u_C}{K} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2} - \frac{\hat{\nabla}^2 K}{K^2}\frac{\hat{\nabla}^2 u_C}{K} - \frac{\hat{\nabla}_C K}{K}\frac{(u\cdot\nabla)K}{K} \\
&+ \Pi^E_C(n\cdot\nabla)(n\cdot\nabla)u_E - 2\bigg(\frac{\hat{\nabla}^E K}{K}\bigg)\big(\hat{\nabla}_E u_C\big) - 3\,\frac{K}{D}\frac{\hat{\nabla}^2 u_C}{K}\bigg]
\end{aligned}
\tag{C.136}
$$

$$
\begin{aligned}
\text{term-2} &= -\frac{2D}{K}\Pi^A_C\Pi^B_{C'}(n\cdot\nabla)\left[\frac{1}{N}\big\{u_A\Pi^E_B(n\cdot\nabla)u_E + u_B\Pi^E_A(n\cdot\nabla)u_E\big\}\right] \\
&= -\frac{2D}{K}\Pi^A_C\Pi^B_{C'}\left(-\frac{n\cdot\nabla N}{N^2}\right)\big[u_A\Pi^E_B(n\cdot\nabla)u_E + u_B\Pi^E_A(n\cdot\nabla)u_E\big] \\
&- \frac{2D}{NK}\Pi^A_C\Pi^B_{C'}\bigg[\big\{(n\cdot\nabla)u_A\big\}\Pi^E_B(n\cdot\nabla)u_E - u_A\big\{(n\cdot\nabla)n_B\big\}n^E(n\cdot\nabla)u_E \\
&+ u_A\,\Pi^E_B(n\cdot\nabla)(n\cdot\nabla)u_E + \big\{(n\cdot\nabla)u_B\big\}\Pi^E_A(n\cdot\nabla)u_E - u_B\big\{(n\cdot\nabla)n_A\big\}n^E(n\cdot\nabla)u_E \\
&+ u_B\,\Pi^E_A(n\cdot\nabla)(n\cdot\nabla)u_E\bigg]
\end{aligned}
$$

$$= \frac{2D}{NK}\left[N + \frac{n\cdot\nabla K}{K}\right]\left[u_C\frac{\hat{\nabla}^2 u_{C'}}{K} + u_{C'}\frac{\hat{\nabla}^2 u_C}{K}\right] - \frac{2D}{NK}\Bigg[2\,\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} + u_C\frac{\hat{\nabla}_{C'}K}{K}\frac{(u\cdot\nabla)K}{K}$$
$$+ u_{C'}\frac{\hat{\nabla}_C K}{K}\frac{(u\cdot\nabla)K}{K} + u_C\Pi^E_{C'}(n\cdot\nabla)(n\cdot\nabla)u_E + u_{C'}\Pi^E_C(n\cdot\nabla)(n\cdot\nabla)u_E\Bigg] \tag{C.137}$$

$$\begin{aligned}\tilde{h}^{(1,2)}_{CC'}\big|_{\text{Part-2}} &= -\frac{D}{NK}\Pi^A_C\Pi^B_{C'}\left[2N^2\tilde{h}^{(2)}_{AB} + \{(n\cdot\nabla)N\}\tilde{h}^{(1,1)}_{AB} - 2\lambda\tilde{h}^{(0)}_{AB}\right]\\
&= -2\,\tilde{h}^{(2)}_{CC'} - \frac{D}{K}\left[\frac{n\cdot\nabla K}{K} + N\right]\tilde{h}^{(1,1)}_{AB} + 2\,\lambda\left(\frac{D}{K}\right)^2 u_C u_{C'}\\
&= 2\,\lambda\frac{D^2}{K^2}u_C u_{C'} - 2\,\frac{D^2}{K^2}\frac{\hat{\nabla}^2 u_C}{K}\frac{\hat{\nabla}^2 u_{C'}}{K} + \frac{D^2}{K^2}\left[\frac{\hat{\nabla}^2 K}{K^2} - \lambda\frac{D}{K}\right]\left[u_C\frac{\hat{\nabla}^2 u_{C'}}{K} + u_{C'}\frac{\hat{\nabla}^2 u_C}{K}\right]\\
&\quad - 2u_C\left[\frac{D}{K}\left\{-\frac{1}{2} - 2\,\frac{D}{K}\frac{\hat{\nabla}^2 K}{K^2} + \lambda\,\frac{D^2}{K^2}\right\}\frac{\hat{\nabla}^2 u_{C'}}{K} + \frac{D^2}{2K^2}\left\{\frac{\hat{\nabla}^2\hat{\nabla}^2 u_{C'}}{K^2} - 2\frac{\hat{\nabla}^E K}{K}(\hat{\nabla}_E u_{C'})\right\}\right]\\
&\quad - 2u_{C'}\left[\frac{D}{K}\left\{-\frac{1}{2} - 2\,\frac{D}{K}\frac{\hat{\nabla}^2 K}{K^2} + \lambda\,\frac{D^2}{K^2}\right\}\frac{\hat{\nabla}^2 u_C}{K} + \frac{D^2}{2K^2}\left\{\frac{\hat{\nabla}^2\hat{\nabla}^2 u_C}{K^2} - 2\frac{\hat{\nabla}^E K}{K}(\hat{\nabla}_E u_C)\right\}\right]\end{aligned} \tag{C.138}$$

Adding (C.133) and (C.138) we get the final expression of $\tilde{h}^{(1,2)}_{CC'}$ as given in (4.57) after using (C.134) and (C.137)

C.3 Some Details of Stress Tensor Calculation

Outside($\psi > 1$)

$$G^{(\text{out})}_{AB} = g_{AB} + \psi^{-D}\mathfrak{h}_{AB} \tag{C.139}$$

Inverse of (C.139) at linear order is

$$G^{AB}_{(\text{out})} = g^{AB} - \psi^{-D}\mathfrak{h}^{AB} + \mathcal{O}(h)^2 \qquad \text{here, } \mathfrak{h}^{AB} = g^{AC}g^{BD}\mathfrak{h}_{CD} \tag{C.140}$$

Using, the gauge condition $n^A\mathfrak{h}_{AB} = 0$, we get

$$n^{(out)}_A = n_A \tag{C.141}$$

Now,

$$\begin{aligned} \mathfrak{p}^{(\text{out})}_{AB} &= G^{(\text{out})}_{AB} - n^{(\text{out})}_A n^{(\text{out})}_B \\ &= g_{AB} + \psi^{-D}\mathfrak{h}_{AB} - n_A n_B \\ &= \Pi_{AB} + \psi^{-D}\mathfrak{h}_{AB} \end{aligned} \tag{C.142}$$

$$\left[\mathfrak{p}^{(\text{out})}\right]^A_B = \delta^A_B - n^A n_B = \Pi^A_B \tag{C.143}$$

Now, from (4.70)

$$\begin{aligned} K^{(\text{out})}_{AB} &= \left[\mathfrak{p}^{(\text{out})}\right]^C_A \left[\mathfrak{p}^{(\text{out})}\right]^{C'}_B \left(\tilde{\nabla}_C n_{C'}\right)_{\psi=1} \\ &= \Pi^C_A \Pi^{C'}_B \left(\partial_C n_{C'} - \tilde{\Gamma}^E_{CC'} n_E\right)\Big|_{\psi=1} \end{aligned} \tag{C.144}$$

Where,

$$\tilde{\Gamma}^E_{CC'} = \Gamma^E_{CC'} + \delta\tilde{\Gamma}^E_{CC'} \tag{C.145}$$

Here, $\Gamma^E_{CC'}$ is Christoffel symbol with respect to g_{AB} and $\delta\tilde{\Gamma}^E_{CC'}$ is defined as

$$\delta\tilde{\Gamma}^E_{CC'} = \frac{1}{2}[G^{(\text{out})}]^{EF}\left[\nabla_C(\psi^{-D}\mathfrak{h}_{C'F}) + \nabla_{C'}(\psi^{-D}\mathfrak{h}_{CF}) - \nabla_F(\psi^{-D}\mathfrak{h}_{CC'})\right] \tag{C.146}$$

Here, ∇_C is covariant derivative with respect to g_{AB}

$$K^{(\text{out})}_{AB} = K_{AB} - \Pi^C_A \Pi^{C'}_B n_E \delta\tilde{\Gamma}^E_{CC'}\Big|_{\psi=1} \tag{C.147}$$

Now,

$$\begin{aligned} &-\Pi^C_A \Pi^{C'}_B n_E \delta\tilde{\Gamma}^E_{CC'}\Big|_{\psi=1} \\ &= -\frac{1}{2}\Pi^C_A \Pi^{C'}_B n^F \left[\nabla_C(\psi^{-D}\mathfrak{h}_{C'F}) + \nabla_{C'}(\psi^{-D}\mathfrak{h}_{CF}) - \nabla_F(\psi^{-D}\mathfrak{h}_{CC'})\right]\Big|_{\psi=1} \\ &= -\frac{1}{2}\Pi^C_A \Pi^{C'}_B n^F \left[\psi^{-D}\nabla_C\mathfrak{h}_{C'F} + \psi^{-D}\nabla_{C'}\mathfrak{h}_{CF} + ND\psi^{-D-1}n_F\mathfrak{h}_{CC'} - \psi^{-D}\nabla_F\mathfrak{h}_{CC'}\right] \\ &= -\frac{1}{2}\Pi^C_A \Pi^{C'}_B \left[-\mathfrak{h}_{C'F}(\nabla_C n^F) - \mathfrak{h}_{CF}(\nabla_{C'} n^F) + ND\mathfrak{h}_{CC'} - (n\cdot\nabla)\mathfrak{h}_{CC'}\right] \\ &= -\frac{1}{2}\Pi^C_A \Pi^{C'}_B \left[-h^{(0)}_{C'F}\left(\nabla_C n^F\right) - h^{(0)}_{CF}\left(\nabla_{C'} n^F\right) + NDh^{(0)}_{CC'} - Nh^{(1)}_{CC'} - (n\cdot\nabla)h^{(0)}_{CC'}\right]\Big|_{\psi=1} \\ &= -\frac{1}{2}\Pi^C_A \Pi^{C'}_B \left[-h^{(0)}_{C'F}K^F_C - h^{(0)}_{CF}K^F_{C'} + NDh^{(0)}_{CC'} - Nh^{(1)}_{CC'}\right] \end{aligned} \tag{C.148}$$

Finally, we get

$$K^{\text{(out)}}_{AB} = K_{AB} - \frac{ND}{2} h^{(0)}_{AB} + \frac{N}{2} h^{(1)}_{AB} + \frac{1}{2}\left(h^{(0)}_{BD} K^{D}_{A} + h^{(0)}_{AD} K^{D}_{B}\right) \tag{C.149}$$

Trace of $K^{\text{(out)}}_{AB}$

$$\begin{aligned} K^{\text{(out)}} &= \left(g^{AB} - \psi^{-D}\mathfrak{h}^{AB}\right) K^{\text{(out)}}_{AB}\Big|_{\psi=1} \\ &= K - \frac{ND}{2} h^{(0)} + \frac{N}{2} h^{(1)} + \frac{1}{2} g^{AB}\left(h^{(0)}_{BD} K^{D}_{A} + h^{(0)}_{AD} K^{D}_{B}\right) - h^{(0)}_{AB} K^{AB} \\ &= K - \frac{ND}{2} h^{(0)} + \frac{N}{2} h^{(1)} \end{aligned} \tag{C.150}$$

Inside($\psi < 1$)

As, in the previous subsection

$$n^{\text{(in)}}_{A} = n_{A}, \quad \mathfrak{p}^{\text{(in)}}_{AB} = \Pi_{AB} + \tilde{\mathfrak{h}}_{AB} \text{ and, } \quad \left[\mathfrak{p}^{\text{(in)}}\right]^{A}_{B} = \Pi^{A}_{B} \tag{C.151}$$

Now, from (4.76)

$$\begin{aligned} K^{(in)}_{AB} &= \left[\mathfrak{p}^{(in)}\right]^{C}_{A} \left[\mathfrak{p}^{(in)}\right]^{C'}_{B} \left(\breve{\nabla}_{C} n^{(in)}_{C'}\right)_{\psi=1} \\ &= \Pi^{C}_{A}\Pi^{C'}_{B}\left(\partial_{C} n_{C'} - \hat{\Gamma}^{E}_{CC'} n_{E}\right)\Big|_{\psi=1} \end{aligned} \tag{C.152}$$

Where,

$$\hat{\Gamma}^{E}_{CC'} = \Gamma^{E}_{CC'} + \delta\hat{\Gamma}^{E}_{CC'} \tag{C.153}$$

Here, $\Gamma^{E}_{CC'}$ is Christoffel symbol with respect to g_{AB} and $\delta\hat{\Gamma}^{E}_{CC'}$ is defined as

$$\delta\hat{\Gamma}^{E}_{CC'} = \frac{1}{2}[G^{\text{(in)}}]^{EF}\left(\nabla_{C}\tilde{\mathfrak{h}}_{C'F} + \nabla_{C'}\tilde{\mathfrak{h}}_{CF} - \nabla_{F}\tilde{\mathfrak{h}}_{CC'}\right) \tag{C.154}$$

Here, ∇_{C} is covariant derivative with respect to g_{AB} Now,

$$K^{\text{(in)}}_{AB} = K_{AB} - \Pi^{C}_{A}\Pi^{C'}_{B} n_{E}\delta\hat{\Gamma}^{E}_{CC'}\Big|_{\psi=1} \tag{C.155}$$

Now,

$$
\begin{aligned}
-\Pi_A^C \Pi_B^{C'} n_E \delta\hat{\Gamma}^E_{CC'}\Big|_{\psi=1} &= -\frac{1}{2}\Pi_A^C \Pi_B^{C'} n^F \left(\nabla_C \tilde{\mathfrak{h}}_{C'F} + \nabla_{C'} \tilde{\mathfrak{h}}_{CF} - \nabla_F \tilde{\mathfrak{h}}_{CC'}\right) \\
&= \frac{1}{2}\Pi_A^C \Pi_B^{C'} \left[\tilde{\mathfrak{h}}_{C'F}(\nabla_C n^F) + \tilde{\mathfrak{h}}_{CF}(\nabla_{C'} n^F) + (n\cdot\nabla)\sum_{m=0}^{\infty}(\psi-1)^m \tilde{h}^{(m)}_{CC'}\right]_{\psi=1} \\
&= \frac{1}{2}\Pi_A^C \Pi_B^{C'} \left[\tilde{h}^{(0)}_{C'F} K^F_C + \tilde{h}^{(0)}_{CF} K^F_{C'} + N\tilde{h}^{(1)}_{CC'}\right]_{\psi=1} \\
&= \frac{1}{2}\tilde{h}^{(0)}_{BF} K^F_A + \frac{1}{2}\tilde{h}^{(0)}_{AF} K^F_B + \frac{1}{2}N\tilde{h}^{(1)}_{AB}
\end{aligned}
\tag{C.156}
$$

So, we get

$$
K^{(\text{in})}_{AB} = K_{AB} + \frac{1}{2}\left(\tilde{h}^{(0)}_{BF} K^F_A + \tilde{h}^{(0)}_{AF} K^F_B + N\tilde{h}^{(1)}_{AB}\right) \tag{C.157}
$$

Stress of extrinsic curvature is given by

$$
\begin{aligned}
K^{(\text{in})} &= \left(g^{AB} - \tilde{\mathfrak{h}}^{AB}\right) K^{(\text{in})}_{AB}\Big|_{\psi=1} \\
&= \left(g^{AB} - [\tilde{h}^{(0)}]^{AB}\right) K^{(\text{in})}_{AB} \\
&= K + \frac{1}{2}\left(\tilde{h}^{(0)}_{AF} K^{FA} + \tilde{h}^{(0)}_{AF} K^{FA} + N\tilde{h}^{(1)}\right) - [\tilde{h}^{(0)}]^{AB} K_{AB} \\
&= K + \frac{N}{2}\tilde{h}^{(1)}
\end{aligned}
\tag{C.158}
$$

C.4 Important Identities

In this appendix we will mention the identities we have used in chapter 4. The identities have been calculated on $\psi = 1$ hypersurface. We are not giving the derivations simply due to the fact that the derivations are very lengthy but nevertheless the derivations are quite straightforward.

Identity-1:

$$
\frac{\hat{\nabla}_B N}{N} = \frac{\hat{\nabla}_B K}{K} + \frac{1}{K}\hat{\nabla}_B\left(\frac{n\cdot\nabla K}{K}\right) - \frac{1}{K}\left(\frac{\hat{\nabla}_B K}{K}\right)\left(\frac{n\cdot\nabla K}{K}\right) + \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.159}
$$

Identity-2:

$$\frac{(n\cdot\nabla)N}{N}=\frac{K}{D}+\frac{(n\cdot\nabla)K}{K}+\frac{1}{D}\frac{(n\cdot\nabla)K}{K}+\frac{(n\cdot\nabla)(n\cdot\nabla)K}{K^2}-\frac{2}{K}\left(\frac{n\cdot\nabla K}{K}\right)^2+\mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.160}$$

Identity-3:

$$ND=K+\frac{(n\cdot\nabla)K}{K}+\frac{(n\cdot\nabla)(n\cdot\nabla)K}{K^2}-\frac{2}{K}\left(\frac{n\cdot\nabla K}{K}\right)^2+\mathcal{O}\left(\frac{1}{D}\right)^2 \tag{C.161}$$

Identity-4:

$$\begin{aligned}\frac{(n\cdot\nabla)K}{K}&=\frac{\hat{\nabla}^2K}{K^2}-\frac{1}{K}K_{AB}K^{AB}-\frac{\lambda(D-1)}{K}+\frac{1}{K^4}\hat{\nabla}^2\big(\hat{\nabla}^2K\big)-\frac{2}{K}\left(\frac{\hat{\nabla}^2K}{K^2}\right)\left(\frac{\hat{\nabla}^2K}{K^2}\right)\\&+\lambda\,\frac{D}{K^2}\left(\frac{\hat{\nabla}^2K}{K^2}\right)-\frac{1}{D}\left(\frac{\hat{\nabla}^2K}{K^2}\right)-\frac{1}{K}\left(\frac{\hat{\nabla}^2K}{K^2}-\lambda\,\frac{D}{K}-\frac{K}{D}\right)\left(\frac{\hat{\nabla}^2K}{K^2}\right)\\&-\frac{2}{K}\left(\frac{\hat{\nabla}^EK}{K}\right)\left(\frac{\hat{\nabla}_EK}{K}\right)+\mathcal{O}\left(\frac{1}{D}\right)^2\end{aligned} \tag{C.162}$$

Identity-5:

$$\begin{aligned}\frac{(n\cdot\nabla)(n\cdot\nabla)K}{2K^2}=\frac{1}{K}\Bigg[&-\frac{3}{2}\left(\frac{\hat{\nabla}^2K}{K^2}\right)\left(\frac{\hat{\nabla}^2K}{K^2}\right)+\lambda\,\frac{D}{K}\left(\frac{\hat{\nabla}^2K}{K^2}\right)+\frac{1}{2K^3}\hat{\nabla}^2\big(\hat{\nabla}^2K\big)\\&-\left(\frac{\hat{\nabla}^EK}{K}\right)\left(\frac{\hat{\nabla}_EK}{K}\right)-2\,\frac{K}{D}\left(\frac{\hat{\nabla}^2K}{K^2}\right)+\lambda+\frac{K^2}{D^2}\Bigg]+\mathcal{O}\left(\frac{1}{D}\right)^2\end{aligned} \tag{C.163}$$

Appendix D

Appendices for Chapter 5

D.1 Analysis of $F(r/r_H)$

In this section, we shall evaluate the integral (5.39) in large D limit. For convenience we are quoting the equation here.

$$F(y) = y \int_y^\infty dx \frac{x^{D-2} - 1}{x(x^{D-1} - 1)} \tag{D.1}$$

We would like to evaluate this integral systematically for large D. Let us first evaluate the integral for $y \geq 2$. In this case, since D is very large, $x^D >> 1$ throughout the range of integration. So we shall expand the integrand in the following way.

$$\begin{aligned}
\frac{x^{D-2} - 1}{x(x^{D-1} - 1)} &= \left(\frac{1}{x^2}\right)\left(1 - x^{-(D-2)}\right)\left(1 - x^{-(D-1)}\right)^{-1} \\
&= \left(\frac{1}{x^2}\right)\left(1 - x^{-(D-2)}\right)\left(1 + \sum_{m=1} x^{-m(D-1)}\right) \\
&= \left(\frac{1}{x^2}\right)\left(1 + \sum_{m=1}\left[x^{-m(D-1)} - x^{-m(D-1)+1}\right]\right)
\end{aligned} \tag{D.2}$$

Integrating (D.2) we find

$$y \int_{y\geq 2}^\infty dx \frac{x^{D-2} - 1}{x(x^{D-1} - 1)} = 1 + \sum_{m=1}\left[\left(\frac{1}{(D-1)m+1}\right) y^{-(D-1)m} - \left(\frac{1}{(D-1)m}\right) y^{-(D-1)m+1}\right] \tag{D.3}$$

Clearly, the sums in the RHS of (D.3) are convergent for $y \geq 2$. Let us denote the RHS as $k(y)$. However, the expansion in (D.2) is not valid inside the 'membrane region', i.e., when $y - 1 \sim \mathcal{O}\left(\frac{1}{D}\right)$ and naively $k(y)$ is not the answer for the integral.

But consider the function $\tilde{k}(y) = F(y) - k(y)$. This function vanishes for all $y \geq 2$ and also by construction, it is a smooth function at $y = 2$ (none of the derivatives diverge). Hence $\tilde{k}(y)$ must vanish for every y. So we conclude, for every allowed y (i.e., $y \geq 1$)

$$F(y) = 1 + \sum_{m=1} \left[\left(\frac{1}{(D-1)m+1} \right) y^{-(D-1)m} - \left(\frac{1}{(D-1)m} \right) y^{-(D-1)m+1} \right] \tag{D.4}$$

Note that $F(y)$ reduces to 1 as $y \to \infty$ as required in section (5.3.2).

Now we would like to expand $F(y)$ in a series in $\left(\frac{1}{D}\right)$, where y is in the membrane regime.

$$y = 1 + \frac{Y}{D}, \quad Y \sim \mathcal{O}(1)$$

In this regime $F(y)$ takes the following form

$$F(y) = F\left(1 + \frac{Y}{D}\right) = 1 - \left(\frac{1}{D}\right)^2 \sum_{m=1} \left(\frac{1+mY}{m^2}\right) e^{-mY} + \mathcal{O}\left(\frac{1}{D^3}\right) \tag{D.5}$$

In chapter 5, we consider only the first subleading correction in $\left(\frac{1}{D}\right)$ expansion. Therefore $F(y)$ could be set to 1 for our purpose.

D.2 Derivation of ψ in $\{Y^A\} = \{\rho, y^\mu\}$ coordinates

In this section, we shall give the derivation of ψ as mentioned in eq (5.62). We want to solve ψ such that $\nabla^2 \psi^{-D} = 0$. Where ∇ is the covariant derivative with respect to the background metric

$$ds^2_{background} = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu}\, dy^\mu\, dy^\nu \tag{D.6}$$

we can expand ψ as follows

$$\psi = 1 + \left(A_{10} + \epsilon\, B_{10} + \frac{A_{11} + \epsilon\, B_{11}}{D}\right)(\rho - r_H) + (A_{20} + \epsilon\, B_{20})(\rho - r_H)^2 + \mathcal{O}\left(\frac{1}{D^3}\right) \tag{D.7}$$

Here ϵ denotes that B_{ij}'s are $\mathcal{O}(\partial)$ terms.

$$
\begin{aligned}
&\nabla^2\left(\psi^{-D}\right) = 0\\
\Rightarrow\ &\psi\left(\nabla^2\psi\right) - (D+1)(\nabla^A\psi)(\nabla_A\psi) = 0\\
\Rightarrow\ &\psi\,\rho^2\left[\partial_\rho\partial_\rho\psi - \Gamma^\rho_{\rho\rho}(\partial_\rho\psi) - \Gamma^\mu_{\rho\rho}(\partial_\mu\psi)\right] + \frac{\psi}{\rho^2}\,\eta^{\mu\nu}\left[-\Gamma^\rho_{\mu\nu}(\partial_\rho\psi) - \Gamma^\alpha_{\mu\nu}\partial_\alpha\psi\right]\\
&\qquad -(D+1)\,\rho^2\,(\partial_\rho\psi)^2 + \mathcal{O}(\partial)^2 = 0
\end{aligned}
\tag{D.8}
$$

The required Christoffel symbols are

$$
\Gamma^\rho_{\rho\rho} = -\frac{1}{\rho};\qquad \Gamma^\mu_{\rho\rho} = 0;\qquad \Gamma^\rho_{\mu\nu} = -\rho^3\eta_{\mu\nu};\qquad \Gamma^\alpha_{\mu\nu} = 0;
\tag{D.9}
$$

Using the above Christoffel symbol we get

$$
\psi\left[\rho^2\,\partial^2_\rho\psi + D\rho\,\partial_\rho\psi\right] - (D+1)\,\rho^2\,(\partial_\rho\psi)^2 = 0
\tag{D.10}
$$

Now,

$$
\begin{aligned}
\partial_\rho\psi &= \left(A_{10} + \epsilon\,B_{10} + \frac{A_{11} + \epsilon\,B_{11}}{D}\right) + 2\,(A_{20} + \epsilon\,B_{20})(\rho - r_H)\\
\partial^2_\rho\psi &= 2\,(A_{20} + \epsilon\,B_{20})
\end{aligned}
\tag{D.11}
$$

Solving, (D.10) order by order in derivative expansion we get the following solution

$$
\psi(\rho, y^\mu) = 1 + \left(1 - \frac{1}{D}\right)\left(\frac{\rho}{r_H(y^\mu)} - 1\right) + \mathcal{O}\left(\frac{1}{D}\right)^3
\tag{D.12}
$$

D.3 Computing different terms in membrane equation

In this section we shall give the details of calculations of different terms that appear in the membrane equation. The different components of the projector defined in (5.72) are given by

$$
\Pi^\rho_\rho = 0;\qquad \Pi^\rho_\mu = \partial_\mu r_H;\qquad \Pi^\mu_\rho = \frac{1}{r_H^4}(\partial^\mu r_H);\qquad \Pi^\mu_\nu = \delta^\mu_\nu
\tag{D.13}
$$

The different components of the Christoffel symbol of the background metric in $Y^A = \{\rho, y^\mu\}$ co-ordinates are given by

$$\Gamma^{\rho}_{\rho\rho} = -\frac{1}{\rho}; \quad \Gamma^{\rho}_{\mu\rho} = 0; \quad \Gamma^{\rho}_{\mu\nu} = -\rho^3\eta_{\mu\nu}; \quad \Gamma^{\nu}_{\mu\rho} = \frac{1}{\rho}\delta^{\nu}_{\mu}; \quad \Gamma^{\alpha}_{\mu\nu} = 0; \quad \Gamma^{\mu}_{\rho\rho} = 0; \tag{D.14}$$

From (5.73) it is clear that we need only $K_{\rho\alpha}$ and $K_{\alpha\beta}$ component of extrinsic curvature

$$\begin{aligned} K_{\rho\mu} &= \Pi^{C}_{\rho}\Big(\partial_C n_\mu - \Gamma^{D}_{C\mu} n_D\Big) \\ &= \Pi^{\nu}_{\rho}\Big(\partial_\nu n_\mu - \Gamma^{\rho}_{\nu\mu} n_\rho\Big) \\ &= \frac{\partial_\mu r_H}{r_H^2} \\ K_{\mu\nu} &= \Pi^{C}_{\mu}\Big(\partial_C n_\nu - \Gamma^{D}_{C\nu} n_D\Big) \\ &= \Pi^{\rho}_{\mu}\Big(\partial_\rho n_\nu - \Gamma^{\rho}_{\rho\nu} n_\rho\Big) + \Pi^{\alpha}_{\mu}\Big(\partial_\alpha n_\nu - \Gamma^{\rho}_{\alpha\nu} n_\rho\Big) \\ &= -\delta^{\alpha}_{\mu}\,\Gamma^{\rho}_{\alpha\nu} n_\rho \\ &= \rho^2\,\eta_{\mu\nu} \end{aligned} \tag{D.15}$$

Now, as mentioned in (5.73) in terms of the intrinsic coordinates on the membrane the extrinsic curvature will have the structure

$$\begin{aligned} \mathcal{K}_{\alpha\beta} &= K_{\rho\rho}\,(\partial_\alpha r_H)\,(\partial_\beta r_H) + [K_{\rho\alpha}\,(\partial_\beta r_H) + K_{\rho\beta}\,(\partial_\alpha r_H)] + K_{\alpha\beta} \\ &= r_H^2\,\eta_{\alpha\beta} + \mathcal{O}(\partial)^2 \end{aligned} \tag{D.16}$$

The trace of the extrinsic curvature

$$\mathcal{K} = (D-1) + \mathcal{O}(\partial^2) \tag{D.17}$$

For the calculation of the extrinsic curvature we need background metric, where for the rest of the calculation we require induced metric on the horizon. The induced metric on the horizon is given by

$$g^{(ind)}_{\alpha\beta} = r_H^2\,\eta_{\alpha\beta} + \mathcal{O}(\partial^2) \tag{D.18}$$

The Christoffel symbol of the induced metric

$$\bar{\Gamma}^{\delta}_{\beta\alpha} = \left(\delta^{\delta}_{\beta}\frac{\partial_{\alpha} r_H}{r_H} + \delta^{\delta}_{\alpha}\frac{\partial_{\beta} r_H}{r_H} - \eta_{\alpha\beta}\frac{\partial^{\delta} r_H}{r_H}\right) \tag{D.19}$$

Now we shall calculate all the terms mentioned in (5.79). First, we shall calculate

$$\begin{aligned}
\bar{\nabla}\cdot U &= g^{\alpha\beta}_{(ind)}\,\bar{\nabla}_{\alpha}U_{\beta}\\
&= \frac{\eta^{\alpha\beta}}{r_H^2}\left[\partial_{\alpha}U_{\beta} - \bar{\Gamma}^{\delta}_{\alpha\beta}U_{\delta}\right] + \mathcal{O}(\partial)^2\\
&= \frac{\eta^{\alpha\beta}}{r_H^2}\left[\partial_{\alpha}\left(r_H\, u_{\beta}\right) - \left(r_H\, u_{\delta}\right)\left(\delta^{\delta}_{\beta}\frac{\partial_{\alpha} r_H}{r_H} + \delta^{\delta}_{\alpha}\frac{\partial_{\beta} r_H}{r_H} - \eta_{\alpha\beta}\frac{\partial^{\delta} r_H}{r_H}\right)\right] + \mathcal{O}(\partial)^2\\
&= (D-2)\left(\frac{(\eta^{\alpha\beta}u_{\alpha}\partial_{\beta})r_H}{r_H^2}\right) + \frac{\partial\cdot u}{r_H} + \mathcal{O}(\partial)^2
\end{aligned} \tag{D.20}$$

Now we shall calculate $\bar{\nabla}^2 U_{\mu}$ and $\left(U\cdot\bar{\nabla}\right)U_{\alpha}$

$$\begin{aligned}
\bar{\nabla}^2 U_{\mu} &= g^{\alpha\beta}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}U_{\mu}\\
&= g^{\alpha\beta}\left[\partial_{\alpha}(\bar{\nabla}_{\beta}U_{\mu}) - \bar{\Gamma}^{\delta}_{\alpha\beta}(\bar{\nabla}_{\delta}U_{\mu}) - \bar{\Gamma}^{\delta}_{\alpha\mu}(\bar{\nabla}_{\beta}U_{\delta})\right]\\
&= \mathcal{O}(\partial)^2
\end{aligned} \tag{D.21}$$

$$\begin{aligned}
\left(U\cdot\bar{\nabla}\right)U_{\alpha} &= U^{\beta}(\partial_{\beta}U_{\alpha}) - U^{\beta}\,\bar{\Gamma}^{\delta}_{\beta\alpha}U_{\delta}\\
&= \frac{u^{\beta}}{r_H}\left(r_H(\partial_{\beta}u_{\alpha}) + u_{\alpha}(\partial_{\beta}r_H)\right) - \frac{u^{\beta}}{r_H}(r_H\, u_{\delta})\left(\delta^{\delta}_{\beta}\frac{\partial_{\alpha} r_H}{r_H} + \delta^{\delta}_{\alpha}\frac{\partial_{\beta} r_H}{r_H} - \eta_{\alpha\beta}\frac{\partial^{\delta} r_H}{r_H}\right) + \mathcal{O}(\partial^2)\\
&= (\eta^{\mu\nu}u_{\mu}\partial_{\nu})u_{\alpha} + u_{\alpha}\left(\frac{(\eta^{\mu\nu}u_{\mu}\partial_{\nu})r_H}{r_H}\right) + \frac{\partial_{\alpha}r_H}{r_H} + \mathcal{O}(\partial^2)
\end{aligned} \tag{D.22}$$

Now,

$$\begin{aligned}
U^{\alpha}\,\mathcal{K}_{\alpha\beta}\,\mathcal{P}^{\beta}_{\gamma} &= (\delta^{\beta}_{\gamma} + U^{\beta}\,U_{\gamma})(U^{\alpha}\,r_H^2\,\eta_{\alpha\beta}) + \mathcal{O}(\partial^2)\\
&= (\delta^{\beta}_{\gamma} + U^{\beta}\,U_{\gamma})U_{\beta} + \mathcal{O}(\partial^2)\\
&= \mathcal{O}(\partial^2)
\end{aligned} \tag{D.23}$$

Appendix E

Notations

In this appendix, we shall summarize the notations we have used in this thesis.

Table E.1: Notations

Background spacetime indices	Capital Latin (A, B, C, D)
Indices on the membrane	Small Greek (α, β, μ, ν)
Induced metric on the membrane as embedded in g_{AB}	$g^{(ind)}_{\mu\nu}$
Full non-linear metric outside the membrane	G_{AB}
Linearized metric outside the membrane	$G^{(out)}_{AB} = g_{AB} + \psi^{-D}\mathfrak{h}_{AB}$
Linearized metric inside the membrane	$G^{(in)}_{AB} = g_{AB} + \tilde{\mathfrak{h}}_{AB}$
Projector on the membrane as embedded in g_{AB}	$\Pi_{AB} = g_{AB} - n_A\, n_B$
Projector perpendicular to both the normal of the membrane as embedded in g_{AB} and the velocity	$P_{AB} = g_{AB} - n_A\, n_B + u_A u_B$
Projector on the membrane as embedded in $G^{(out)}_{AB}$	$\mathfrak{p}^{(out)}_{AB} = G^{(out)}_{AB} - n^{(out)}_A n^{(out)}_B$
Projector on the membrane as embedded in $G^{(in)}_{AB}$	$\mathfrak{p}^{(in)}_{AB} = G^{(in)}_{AB} - n^{(in)}_A n^{(in)}_B$
Covariant derivative w.r.t. g_{AB}	∇_A
Covariant derivative w.r.t. $g^{(ind)}_{\mu\nu}$	$\bar{\nabla}_\mu$
Covariant derivative w.r.t. G_{AB}	$\check{\nabla}_A$
Covariant derivative w.r.t. $G^{(out)}_{AB}$	$\tilde{\nabla}_A$
Covariant derivative w.r.t. $G^{(in)}_{AB}$	$\breve{\nabla}_A$

Covariant derivative w.r.t. g_{AB} projected along the membrane	$\hat{\nabla}_A$ see (2.58) for definition
Extrinsic curvature of the membrane when embedded in $G^{(out)}_{AB}$	$K^{(out)}_{AB}$
Extrinsic curvature of the membrane when embedded in $G^{(in)}_{AB}$	$K^{(in)}_{AB}$
Extrinsic curvature of the membrane when embedded in g_{AB}	K_{AB}
Pull back of K_{AB} on $\psi = 1$ hypersurface	$\mathcal{K}_{\mu\nu}$ see (2.62) for definition

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