HARDY TYPE INEQUALITIES FOR DUNKL OPERATORS

By

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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Dedicated to My Brother

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Contents

Summary

Hardy inequalities are of fundamental importance in many areas of mathematics and theoretical physics. Since their discovery a rich theory has been developed on Hardy inequalities and it became a broad area of mathematical research. The original inequality on positive real numbers extended to N-dimensional Euclidean space and more general domains. Several versions and improvements of classical Hardy inequality are available in the literature. Many of these Hardy type inequalities have been studied in the Dunkl setting recently. S. Thangavelu, Y. Xu, Ó. Ciaurri, F. Soltani, B. Amri, D. V. Gorbachev and A. Velicu are a few of the authors studied about such Hardy type inequalities.

The Dunkl operators was first introduced by C. F. Dunkl in 1989. The Theory of Dunkl operators in the study of special functions with reflection symmetries is very young. In recent years this operator have found considerable attention in various branches of mathematics and Physics. Dunkl transform is an extension of Fourier transform which defines an isometry on the weighted space $L^2(\mathbb{R}^d, d\mu_k(x))$. It enjoys many similar properties of classical Fourier transform. The Dunkl Laplacian associated to a multiplicity function k on fixed root system R and reflection group G is defined by $\Delta_k = \sum_{j=1}^{N} T_j^2$ where T_j's are the Dunkl operators. Also the Dunkl gradient is defined as $\nabla_k = (T_1, T_2, ..., T_N)$. If the multiplicity function k is identically equal to zero the operators T_j, Δ_k and ∇_k reduce to ∂_j, Δ and to ∇ respectively. We denote $\gamma_k = \sum_{\alpha \in R_+} k(\alpha)$, $d_k = N+2\gamma_k$ and $d\mu_k(x)$ as the Dunkl weighted measure.

The main theme of this thesis is to study different type of Hardy inequalities associated with the Dunkl operators. We will establish Hardy inequalities, trace Hardy inequalities and Stein-Weiss inequalities associated to Dunkl operators for the Euclidean space, half-space and cone. We begin with Hardy type inequalities for the L^2 space. We prove classical Hardy inequality for G-invariant function. Using this result we proved certain classical Hardy inequalities for half-space and cone.

Let R be a root system on $\mathbb{R}^N$ and k be a multiplicity function on R. Define a root system R_1 on $\mathbb{R}_+^{N+1}$ as $R_1 = R \times \{0\}$. Also define the multiplicity function k_1

on R_1 as $k_1(x,0) := k(x)$. Now with this root system on $\mathbb{R}^{N+1}_+$ the Dunkl gradient is given as $\tilde{\nabla}_k = (\nabla_k, \partial_{x_{N+1}})$. To establish a Hardy inequality on the half-space we always consider this Euclidean extension of the root system. Similarly to establish a Hardy inequality on $\mathbb{R}^N_{l_+}$, we fix a root system on $\mathbb{R}^N_{l_+}$ which is actually an extension of a root system on $\mathbb{R}^{N-l}$. That is, if R is a root system on R^{N-l} then extend the root system R to a root system R' of $\mathbb{R}^N_{l_+}$ by defining $R' := \{(x,0) \in \mathbb{R}^N : x \in R\}$. Also the multiplicity function k on R can be extended to k' on R' by $k'(x,0) = k(x)$. Now if ∇_k is the Dunkl gradient on $\mathbb{R}^{N-l}$, with this root system R' we can write the Dunkl gradient on $\mathbb{R}^N_{l_+}$ as $\tilde{\nabla}_k = (\nabla_k, \frac{\partial}{\partial x_{N-l+1}}, ..., \frac{\partial}{\partial x_N})$.

For $0 < s < 1$, the fractional Dunkl Laplacian $(-\Delta_k)^s$ is defined using Dunkl transform, that is $\mathcal{F}_k((-\Delta_k)^s f)(\xi) := |\xi|^s \mathcal{F}_k(\xi)$. Caffarelli and Silvestre[2006] developed an idea of Drichlet to Neumann map to study about the fractional Laplacians. We extended this idea to the Dunkl setting and proved some trace Hardy type inequalities by identifying proper extension problems. As a corollary to this we also obtained fractional Hardy inequalities for the Dunkl Laplacian. Also we found an independant method to prove fractional Hardy inequality for Dunkl fractional Laplacian.

Using ground state substitution technique Frank and Seiringer [2008] proved fractional Hardy inequalities for L^p space. Also he proved an improved Hardy inequality for $p \geqslant 2$. The symmetry of the kernel $|x-y|^{-(N+ps)}$ plays a vital role in proving these Hardy inequalities in Euclidean setting. Note that this kernel is nothing but the translation of the function $|x|^{-(N+ps)}$. To work with Dunkl case it is essential to consider the kernel which is Dunkl translation of $|x|^{-(d_k+ps)}$. Motivated by an article by Gorbachev[2019], we define the kernel $\Phi_\delta(x,y)$, which is actually Dunkl translation of $|x|^{-(d_k+\delta)}$ with $\delta \neq -d_k$. Extending Frank's idea to the Dunkl setting we first establish a Dunkl fractional Hardy inequality on $\mathbb{R}^N$. Also we proved these results for upper half space and cone.

Since this Hardy inequality is strict for all non zero functions in $C_0^\infty(\mathbb{R}^N)$ the natural question to ask is whether the inequality can be improved on the completion of $C_0^\infty(\mathbb{R}^N)$, that is, whether some positive term can be added to improve the inequality. For $p \geqslant 2$, the answer is affirmative and there are many articles investigated on different types of improved Hardy inequalities. We also obtain an improved Dunkl type Hardy inequality in the case $p \geqslant 2$. Abdellaoui et al.[2014, 2016, 2017] proved a version of improved fractional Hardy inequalities where the reminder term again is a norm of fractional gradient. We also proved such improved fractional Hardy inequalities for fractional Dunkl Laplacian. We proved certain types of Stein-Weiss inequalities and fractional Stein-Weiss inequalities by extending the same technique.

Chapter 1

Introduction and Preliminaries of Dunkl Theory

1.1 Introduction

The main theme of this thesis is to study different type of Hardy inequalities associated with the Dunkl operators. We will establish Hardy inequalities, trace Hardy inequalities and Stein-Weiss inequalities associated to Dunkl operators for the Euclidean space, half-space and cone. Hardy inequalities are of fundamental importance in many areas of mathematics and theoretical physics. Since their discovery a rich theory has been developed on Hardy inequalities and it became a broad area of mathematical research. The original inequality on positive real numbers extended to N-dimensional Euclidean space and more general domains.

The original Hardy inequality is first discussed by G. H. Hardy in [20] and it is of the form

$$\int_0^\infty |u'(x)|^p dx \geqslant C \int_0^\infty \frac{|u(x)|^p}{|x|^p} dx, \tag{1.1.1}$$

where $1 < p < \infty$ and $u \in C_0^\infty(0, \infty)$. The inequality is strict for any non zero function u and $C = (p-1)/p$ is the best constant. Later it is generalized to higher dimensions. For $1 < p < \infty$ the higher dimension analogue of (1.1.1) can be stated as

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p \geqslant \lambda(N,p) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx, \tag{1.1.2}$$

where $u \in C_0^\infty(\mathbb{R}^N)$. Let $\dot{H}^{1,p}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with the norm $\|u\|_{\dot{H}^{1,p}} := \|\nabla u\|_p$. The inequality in (1.1.2) can be extended to $\dot{H}^{1,p}(\mathbb{R}^N)$. The best constant $\lambda(N,p)$ of (1.1.2) is obtained by

$$\lambda(N,p) = \inf_{\substack{u \in \dot{H}^{1,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^p dx}{\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx}.$$

When $N \geqslant 2$ and $p = N$ the Hardy inequality does not hold. That is we can find functions f in $C_0^\infty(\mathbb{R}^N)$ such that the integral $\int_{\mathbb{R}^N} \frac{|f(x)|^N}{|x|^N} dx$ diverges. For instance, choose f such that $0 \leqslant f(x) \leqslant 1$ for all $x \in \mathbb{R}^N$ and $f(x) = 1$ inside the ball $B(0, r/2)$ and $f(x) = 0$ outside the ball $B(0,r)$. Now a simple calculation will give us $\int_{\mathbb{R}^N} \frac{|f(x)|^N}{|x|^N} dx = \infty$. For $p = 2$ the Hardy inequality (1.1.2) is also known as uncertainty principle. One can also understand the Hardy inequality as a continuous embedding of $\dot{H}^{1,p}(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ with respect to a weight $|x|^{-p}$. This embedding is known as Hardy-Sobolev embedding. For $N > p$ the best constant $\lambda(N,p)$ is never achieved in the space $\dot{H}^{1,p}(\mathbb{R}^N)$, that is, there does not exist a non zero function u in $\dot{H}^{1,p}(\mathbb{R}^N)$ such that the equality in (1.1.2) holds. But still we can find a minimizing sequence in $\dot{H}^{1,p}(\mathbb{R}^N)$ for the best constant. For $\epsilon > 0$, consider the functions $u_\epsilon := |x|^{-\frac{N-p}{p}+\epsilon}$ and we can see that

$$\lambda(N,p) = \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}^N} |\nabla u_\epsilon(x)|^p dx}{\int_{\mathbb{R}^N} \frac{|u_\epsilon(x)|^p}{|x|^p} dx} = \left(\frac{N-p}{p}\right)^p.$$

All u_ϵ's are elements of $\dot{H}^{1,p}(\mathbb{R}^N)$, but the limiting function $u(x) = |x|^{-\frac{N-p}{p}}$ is not in $\dot{H}^{1,p}(\mathbb{R}^N)$. It is interesting to see that $u(x) = |x|^{-\frac{N-p}{p}}$ is a solution of the following equation

$$-div(|\nabla u|^{p-2}\nabla u) = \left(\frac{N-p}{p}\right)^p |u|^{p-1}$$

in the sense of distributions. The expression on the left hand side is called $p-$Laplace operator and denoted by $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$. When $p = 2$, Δ_p reduces to the classical Euclidean Laplacian Δ. Since the inequality (1.1.2) is strict for all non zero functions in $\dot{H}^{1,p}(\mathbb{R}^N)$ the natural question to ask is whether the inequality can be improved, that is whether some positive term can be added on the right hand side of (1.1.2). For $p \geqslant 2$, the answer is affirmative and there are many articles in which the authors investigated on different types of improved Hardy inequalities. We would like to refer the article by Frank and Seiringer(see [16]) .

The theory of Dunkl operators was first introduced in [11]. Dunkl Fourier transform is an extension of the classical Fourier transform which defines an isometry on the weighted space $L^2(\mathbb{R}^d, d\mu_k(x))$. It enjoys many similar properties of classical Fourier transform. The Dunkl Laplacian associated to a multiplicity function k on a reflection group G is defined by $\Delta_k = \sum_{j=1}^N T_j^2$, where T_j's are the Dunkl operators. Also the Dunkl gradient is defined as $\nabla_k = (T_1, T_2, ..., T_N)$. If the multiplicity function k is identically equal to zero the operators T_j, Δ_k and ∇_k reduce to ∂_j, Δ and to ∇ respectively. From the physical science point of view, the Dunkl transform has applications in quantum many body problems. We discuss all the necessary preliminaries for this thesis later in this chapter.

In Chapter 2 we will discuss certain Hardy inequalities in Dunkl setting with $p = 2$. We need to fix a root system R and a multiplicity function k on R. Also associated to the root system R we define $\gamma_k = \sum_{\alpha\in R_+} k(\alpha)$, $d_k = N + 2\gamma_k$ and $\lambda_k = (d_k - 2)/2$. An L^2 analogue of (1.1.2) for the Dunkl gradient ∇_k is stated as follows.

Theorem 1.1.1. *Let $d_k \geqslant 3$. Let u be a G-invariant function such that $u \in C_0^\infty(\mathbb{R}^N)$. We have the following inequality*

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \geqslant \lambda_k^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} d\mu_k(x), \tag{1.1.3}$$

where λ_k^2 is the optimal constant.

For $1 < p < \infty$, $1 \leqslant l \leqslant N$ and $\alpha + l > 0$, a more generalized Hardy inequality of the form

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p |y|^{\alpha+p} dx \geqslant C \int_{\mathbb{R}^N} |u(x)|^p |y|^\alpha dx, \tag{1.1.4}$$

where $x = (y, z) \in \mathbb{R}^l \times \mathbb{R}^{N-l}$, with the optimal constant $C = \frac{(\alpha+l)^p}{p^p}$, was given by Simone Secchi et al. in [28]. Now for fixed root systems R_1 and R_2 with multiplicity functions k_1 and k_2 corresponding to the spaces $\mathbb{R}^l$ and $\mathbb{R}^{N-l}$ respectively. We define a root system R on $\mathbb{R}^N$ and a multiplicity function k on R. If $1 \leqslant l \leqslant N$ and if $x \in \mathbb{R}^N$ we can write $x = (y, z)$ where $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^{N-l}$. Now the following theorem is a generalization of the Theorem 1.1.1.

Theorem 1.1.2. *Let $l + 2\gamma_{k_1} - 2 > 0$, then for each G-invariant $u \in C_0^\infty(\mathbb{R}^N)$, we have the following inequality:*

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^2 d\mu_k(x) \geqslant \left(\frac{l + 2\gamma_{k_1} - 2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x).$$

Moreover the constant appearing above is optimal.

The half-space is defined as the set $\mathbb{R}^N_+ = \{(x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\}$. A Hardy inequality on the upper half-space can be written as

$$\int_{\mathbb{R}^N_+} |\nabla u(x)|^2 \, dx \geqslant \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u(x)|^2}{x_N^2} dx.$$

Later in [39], J. Tidblom proved that, for all $u \in C_0^\infty(\mathbb{R}^N_+)$ the following inequality holds:

$$\int_{\mathbb{R}^N_+} |\nabla u(x)|^2 dx \geqslant \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u(x)|^2}{x_N^2} dx + \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u(x)|^2}{x_{N-1}^2 + x_N^2} dx.$$

Using the above Hardy inequality (1.1.4), Jing-Wen Luan et al. proved the following Hardy inequality for half-space in [23].

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geqslant \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u|^2}{x_N^2} dx + \frac{(l-1)^2}{4} \int_{\mathbb{R}^N_+} \frac{|u|^2}{x_{N-l+1}^2 + ... + x_N^2} dx, \tag{1.1.5}$$

where $\frac{(l-1)^2}{4}$ is the best constant.

Now we prove a generalized version of Hardy inequality on the half-space in Dunkl setting. Let R be a root system on $\mathbb{R}^N$ and k be a multiplicity function on R. Define a root system R_1 on $\mathbb{R}^{N+1}_+$ as $R_1 = R \times \{0\}$. Also define the multiplicity function k_1 on R_1 as $k_1(x, 0) := k(x)$. Now with this root system on $\mathbb{R}^{N+1}_+$ the Dunkl gradient is given as $\tilde{\nabla}_k = (\nabla_k, \partial_{x_{N+1}})$.

Theorem 1.1.3. *Let $\tilde{\nabla}_k$ be the gradient on $\mathbb{R}^{N+1}$ as mentioned above. For $l \in \{1/2, 1, 3/2, 2, ..., N/2, ..\}$ and for any G-invariant $u \in C_0^\infty(\mathbb{R}^{N+1}_+)$,*

$$\int_{\mathbb{R}^{N+1}_+} |\tilde{\nabla}_k u|^2 d\mu_k(x) dx_{N+1} + l(l-1) \int_{\mathbb{R}^{N+1}_+} \frac{|u(x)|^2}{x_{N+1}^2} d\mu_k(x) dx_{N+1}$$
$$\geqslant \frac{(N + 2\gamma_k + 2l - 1)^2}{4} \int_{\mathbb{R}^{N+1}_+} \frac{|u(x)|^2}{|x|^2} d\mu_k(x) dx_{N+1},$$

where $\frac{(N+2\gamma_k+2l-1)^2}{4}$ is optimal.

If we put $l = \frac{1}{2}$ and $k = 0$ in the above theorem we get the classical Hardy inequality for half-space.

A cone is a subset of $\mathbb{R}^N$ and is a generalization of upper half-space. It is denoted by $\mathbb{R}^N_{l_+}$ for $1 \leqslant l \leqslant N$ and defined by $\mathbb{R}^N_{l_+} = \{(x_1, ..x_N) \in \mathbb{R}^N : x_{N-l+1} > 0, x_{N-l+2} > 0.., x_N > 0\}$. In 2012, in [34], Dan Su et al. found the sharp Hardy inequality for the cone. Their result states that, for $N \geqslant 3$ and $u \in C_0^\infty(\mathbb{R}^N_{l_+})$,

$$\int_{\mathbb{R}^N_{l_+}} |\nabla u|^2 dx \geqslant \frac{(N-2+2l)^2}{4} \int_{\mathbb{R}^N_{l_+}} \frac{|u|^2}{|x|^2} dx, \tag{1.1.6}$$

and the constant $\frac{(N-2+2l)^2}{4}$ is sharp. Using a similar method used in the proof of Theorem 1.1.3 we can extend the result to cone. To establish a Hardy inequality on $\mathbb{R}^N_{l_+}$, we fix a root system on $\mathbb{R}^N_{l_+}$ which is actually an extension of a root system on $\mathbb{R}^{N-l}$. That is, if R is a root system on R^{N-l} then extend the root system R to a root system R' of $\mathbb{R}^N_{l_+}$ by defining $R' := \{(x, 0) \in \mathbb{R}^N : x \in R\}$. Also the multiplicity function k on R can be extended to k' on R' by $k'(x, 0) = k(x)$. Now if ∇_k is the Dunkl gradient on $\mathbb{R}^{N-l}$, with this root system R' we can write the Dunkl gradient on $\mathbb{R}^N_{l_+}$ as $\tilde{\nabla}_k = (\nabla_k, \frac{\partial}{\partial x_{N-l+1}}, ..., \frac{\partial}{\partial x_N})$.

Theorem 1.1.4. *Let $N + 2\gamma_k \geqslant 3$. Let $u \in C_0^\infty(\mathbb{R}^N_{l_+})$ and $G-$invariant. Then the following inequality holds:*

$$\int_{\mathbb{R}^N_{l_+}} |\tilde{\nabla}_k u|^2 d\mu_k(x) dx_{N-l+1}...dx_N \geqslant \frac{(N+2l+2\gamma_k-2)^2}{4} \int_{\mathbb{R}^N_{l_+}} \frac{|u|^2}{|x|^2} d\mu_k(x) dx_{N-l+1}...dx_N,$$

where the constant $\frac{(N+2l+2\gamma_k-2)^2}{4}$ is sharp.

Fractional powers of linear operators appear in many areas of mathematics. In particular fractional powers of Laplacian are nowadays classical objects. In recent years fractional powers of non local equations of fractional order, in particular, fractional Laplacian, gained a lot of attention from partial differential equations and harmonic analysis. For $0 < s < 1$ the fractional power of Laplacian $(-\Delta)^s$ is defined as $\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s}\hat{f}(\xi)$. There are many more equivalent definitions for fractional Laplacian. One of the references to understand the different equivalent definitions is [24].

Using the classical Laplacian Δ, the Hardy inequality in the equation (1.1.2) can also be written as

$$\langle(-\Delta)u, u\rangle \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}dx. \tag{1.1.7}$$

Analogous to (1.1.7) a Hardy inequality for the operator $(-\Delta)^s$ is stated as follows. For the functions u such that $u, (-\Delta)^s u \in L^2(\mathbb{R}^N)$,

$$\langle(-\Delta)^s u, u\rangle \geqslant 4^s \frac{\Gamma(\frac{N+s}{4})^2}{\Gamma(\frac{N-s}{4})^2} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}dx. \tag{1.1.8}$$

The constant $4^s \frac{\Gamma(\frac{N+s}{4})^2}{\Gamma(\frac{N-s}{4})^2}$ is sharp and never achieved. The left-hand side of the equation (1.1.8) can be written as

$$\langle(-\Delta)^s u, u\rangle = \frac{4^s\Gamma(N/2+s)}{2|\Gamma(-s)|\pi^{N/2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy. \tag{1.1.9}$$

In view of this we can see that the fractional Hardy inequality in equation (1.1.8) is equivalent to the inequality,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \geqslant 2\pi^{N/2}\frac{\Gamma(\frac{N+s}{4})^2}{\Gamma(\frac{N-s}{4})^2}\frac{|\Gamma(-s)|}{\Gamma(N/2+s)} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}dx. \tag{1.1.10}$$

Another version of this Hardy inequality, in which the homogeneous weight $|x|^{-2s}$ is replaced by $(\delta^2+|x|^2)^{-s}$, is of the form

$$\langle(-\Delta)^s u, u\rangle \geqslant \frac{\Gamma(\frac{N+s}{2})}{\Gamma(\frac{N-s}{2})}\delta^{2s}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{(\delta^2+|x|^2)^s}dx, \qquad \delta>0, \tag{1.1.11}$$

where the constant is sharp since it is achieved for the functions $(\delta^2+|x|^2)^{-\frac{N-s}{2}}$.

Similar to the case of fractional Laplacian, the fractional powers of Dunkl Laplacian can also be defined through Dunkl Fourier transform. For $0<s<1$, the fractional Dunkl Laplacian is defined as $\mathcal{F}_k((-\Delta_k)^s f)(\xi) = |\xi|^{2s}\mathcal{F}_k(\xi)$ for suitable function f.

Theorem 1.1.5. *F Let $N\geqslant 1$ and $0<s<1$ be such that $d_k/2>s$. Then for $f\in C_0^\infty(\mathbb{R}^N)$ we have*

$$\langle(-\Delta_k)^s f, f\rangle \geqslant 4^s\left(\frac{\Gamma(\frac{d_k+s}{2})}{\Gamma(\frac{d_k-s}{2})}\right)^2\int_{\mathbb{R}^N}\frac{|f(x)|^2}{|x|^{2s}}d\mu_k(x).$$

The fractional Hardy inequality in the half-space has been investigated by many authors. For suitable functions, a fractional Hardy inequality for the half-space $\mathbb{R}^N_+$ is stated as

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \geqslant D_{N,2,s}\int_{\mathbb{R}^N_+}\frac{|u(x)|^2}{x_N^{2s}}dx, \tag{1.1.12}$$

where the constant $D_{N,2,s}$, given by

$$D_{N,2,s} = 2\pi^{(N-1)/2}\frac{\Gamma((1+2s)/2)}{\Gamma((N+2s)/2)}\int_0^1\left|1-r^{(2s-1)/2}\right|^2\frac{dr}{(1-r)^{1+2s}},$$

is optimal. For further reading and improvements of fractional Hardy inequalities we refer to [6, 8, 12, 15, 14, 16, 40]. To obtain a fractional Hardy inequality on

$\mathbb{R}^N_+$, as we discussed above, we fix the root system $R_2 = R \times \{0\}$ on $\mathbb{R}^N_+$ which is an extension of the root system R on $\mathbb{R}^{N-1}$. With this root system the Dunkl gradient $\tilde{\Delta}_{k_2}$ on $\mathbb{R}^N_+$ looks like $\tilde{\Delta}_{k_2} = \Delta_k + \frac{\partial^2}{\partial x_N^2}$.

Theorem 1.1.6. *Let $u \in C_0^\infty(\mathbb{R}^N_+)$, $0 < s < 1$ and $N/2 + \gamma_k > s$. We have*

$$\langle (-\tilde{\Delta}_{k_2})^{s/2} u, u \rangle_{\mathbb{R}^N_+} \geqslant \frac{\Gamma(\frac{N+2+s}{2} + \gamma_k)}{\Gamma(\frac{N+2-s}{2} + \gamma_k)} \int_{\mathbb{R}^N_+} \frac{u(x, x_N)^2}{(1 + |x|^2 + x_N^2)^s} d\mu_k(x) dx_N.$$

Let $\tilde{\Delta}_{k_2} = \Delta_k + \sum_{j=N-l+1}^{N} \frac{\partial^2}{\partial x_j^2}$ be the Dunkl Laplacian on $\mathbb{R}^N_{l_+}$ with the root system $R_2 = R \times \{(0)_l\}$ defined on $\mathbb{R}^N$ which is an extension of the root system R on $\mathbb{R}^{N-l}$.

Theorem 1.1.7. *Let $0 < s < 1$ and $N/2 + \gamma_k > s$. For $u \in C_0^\infty(\mathbb{R}^N_{l_+})$ the following inequality holds*

$$\langle (-\tilde{\Delta}_{k_2})^{s/2} u, u \rangle_{\mathbb{R}^N_{l_+}} \geqslant C_{\gamma_k, s} \int_{\mathbb{R}^N_{l_+}} \frac{u^2}{(1 + |x|^2 + x_{N-l+1}^2 + ... + x_N^2)^s} d\mu_k(x) dx_{N-l+1} ... dx_N,$$

where $C_{\gamma_k,s} = \frac{\Gamma(\frac{N+2l+s}{2} + \gamma_k)}{\Gamma(\frac{N+2l-s}{2} + \gamma_k)}$.

We have already defined fractional Laplacian earlier. Since it is a non local operator there are a lot of technical difficulties to deal with. In a recent paper [9] in 2007, Caffarelli and Silvestre studied about fractional Laplacian through the Dirichlet to Neumann map. Their idea is to relate fractional Laplacian to a local operator by adding a new variable 'ρ'. For any function f, the extension problem can be stated as follows.

$$div(\rho^{1-2s}\nabla_{(x,\rho)}u(x,\rho)) = 0, \qquad (x,\rho) \in \mathbb{R}^{N+1}_+$$
$$u(x,0) = f(x),$$

the energy of which is given by

$$J[u] = \int_0^\infty \int_{\mathbb{R}^N} \rho^{1-2s}|\nabla_{(x,\rho)}u(x,\rho)|^2 dx d\rho.$$

Then the authors of [9] established that

$$\lim_{\rho\to 0} \rho^{1-2s}\frac{\partial u}{\partial \rho}(x,\rho) = -2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}(-\Delta)^s f(x). \tag{1.1.13}$$

Using this idea many authors studied trace Hardy type inequalities on different domains by identifying proper extension problems, for instance [13, 33, 36]. For $0 < s < 1$ and suitable functions on $\mathbb{R}^N \times \mathbb{R}_+$, the trace Hardy inequality states that

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla u(x,\rho)|^2 \rho^{1-s} dx d\rho \geqslant 2\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\left(\frac{\Gamma((N+s)/4)}{\Gamma((N-s)4)}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{|x|^s} dx. \tag{1.1.14}$$

Also the trace Hardy inequalities with non-homogeneous weight $(\delta^2+|x|^2)^s$, which are of the form

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla u(x,\rho)|^2 \rho^{1-s} dx d\rho \geqslant 2\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\frac{\Gamma((N+s)/2)}{\Gamma((N-s)2)}\delta^{2s} \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{(\delta^2+|x|^2)^s} dx \tag{1.1.15}$$

with $\delta > 0$, have also been studied.

We also use the idea of Dirichlet to Neumann map for the fractional Dunkl

Laplacian after identifying a proper extension problem.

Theorem 1.1.8. *Let $0 < s < 1$ and $u \in C_0^\infty(\mathbb{R}_+^{N+1})$, then*

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u(x,\rho)|^2 \rho^{1-s} d\mu_k(x) d\rho \geqslant C(d_k, s, \delta) \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{\delta(x)} d\mu_k(x).$$

If we choose the function $\delta(x) = (1+|x|^2)^s$ we get the trace Hardy inequality with non-homogeneous weight and the optimal constant $C(d_k, s, \delta) = 2\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)}$. Similarly, when $\delta(x) = |x|^s$ we get a trace Hardy inequality with homogeneous weight with a constant $C(d_k, s, \delta) = 2c_h^{-1}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\left(\frac{\Gamma(\frac{N+2\gamma_k+s}{4})}{\Gamma(\frac{N+2\gamma_k-s}{4})}\right)^2$. By a suitable choice of f, we can prove the following Hardy inequality for $(-\Delta_k)^{s/2}$. Analogous to (1.1.13) we establish a relation between the extension problem and fractional Dunkl Laplacian and obtain a Hardy inequality for fractional power of Dunkl Laplacian.

Corollary 1.1.9. *For $f \in L^2(\mathbb{R}^N, h_k^2(x))$ for which $\Delta_k^{s/2} f \in L^2(\mathbb{R}^N, d\mu_k(x))$,*

$$\langle (-\Delta_k)^{s/2} f, f\rangle \geqslant 2^s \frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{(1+|x|^2)^s} d\mu_k(x).$$

Further the fractional Hardy inequality for Dunkl Laplacian for upper half space and the cone with both homogeneous and non-homogeneous weights are also proved in Chapter 2.

We will discuss in the next chapter certain Hardy inequality for Dunkl Laplacian for $L^2(\Omega)$, where Ω is $\mathbb{R}^N$, $\mathbb{R}_+^N$ or $\mathbb{R}_{l+}^N$. In chapter 3 we are interested to prove these Hardy inequalities for the space $L^p(\Omega)$ where $1 < p < \infty$. First we give a proof of the following classical L^p Hardy inequality in the Dunkl case.

Theorem 1.1.10. *Let $1 \leqslant p < \infty$. Let u be a real valued $G-$ invariant function. For $u \in C_0^\infty(\mathbb{R}^N)$ when $d_k > p$ or $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ when $d_k < p$, the following* *inequality holds*

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^p d\mu_k(x) \geqslant \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} d\mu_k(x). \tag{1.1.16}$$

The constant $\left|\frac{d_k-p}{p}\right|^p$ *given in the inequality is optimal.*

We also obtain an improved Hardy inequality in the case $p \geqslant 2$

Theorem 1.1.11. *Let* $2 \leqslant p < \infty$*. Let* u *be a real valued* $G-$*invariant function. For* $u \in C_0^\infty(\mathbb{R}^N)$ *when* $d_k > p$ *or* $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ *when* $d_k < p$*, the following inequality holds*

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) - \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x) \geqslant c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k - p}} d\mu_k(x), \tag{1.1.17}$$

where c_p *is given by*

$$c_p = \min_{0<\tau<1/2} \big((1-\tau)^p - \tau^p + p\tau^{p-1}\big). \tag{1.1.18}$$

When $p = 2$ *the equality holds and with* $c_2 = 1$*.*

Further we are also interested to prove a fractional weighted L^p Hardy inequality for the Dunkl Laplacian.

The equality in (1.1.9) allows us to write the L^2 fractional Hardy inequality as in (1.1.10). But when $p \neq 2$ one cannot have the equivalence of $\|(-\Delta^{s/2})u\|_p^p$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy$. There are many studies in the literature regarding the fractional Hardy inequality of the form

$$\big\|(-\Delta^{s/2})u\big\|_p^p \geqslant C(N,s,p) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx$$

for instance Herbst, in [21] calculated the sharp constant in the above inequality. In case of fractional L^p Hardy inequality, instead of our Euclidean Laplacian Δ, we are interested in a more general Laplace operator called $p-$Laplace operator. The $p-$Laplace operator is denoted as Δ_p and is defined as $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$. For $1 < p < \infty$ and u is smooth enough, the fractional power of $p-$Laplacian,$(-\Delta)^s_p$, is defined as

$$(-\Delta)^s_p u(x) := \lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy.$$

In this thesis we are interested in the fractional Hardy inequalities of the form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \geqslant C'(N,s,p)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx$$

in the Dunkl setting. The basic study of fractional power of Dunkl Laplacian can be done in a similar fashion to the Euclidean case. We adopt the ideas of Frank et al. used in [16] in proving fractional Hardy inequality. The authors of [16] used the technique of ground state substitution to establish the inequality. In general, the idea is to find a Hardy inequality for the functional $E[u]$, which is given by

$$E[u] := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p k(x,y)dxdy$$

where $k(x,y)$ is a non-negative measurable function on $\mathbb{R}^N\times\mathbb{R}^N$ which is symmetric in x and y. The Euler-Lagrange equation of the functional $E[u]$ is given by

$$2\int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))k(x,y)dy = V(x)w(x)^{p-1}. \qquad (1.1.19)$$

for some real valued function V on $\mathbb{R}^N$. A positive function w satisfying (1.1.19) is known as 'virtual ground state' corresponding to the energy functional $E[u] - \int_{\mathbb{R}^N} V|u|^p dx$. In our case we are interested in the kernel $k(x,y)$ of the form $k(x,y) = |x-y|^{-(N+ps)}$. Since it is singular on the diagonal $x = y$, to overcome the divergence of integral we have to use some regularization of principal value of integrals.

The symmetry of the kernel $|x-y|^{-(N+ps)}$ play a vital role in proving these Hardy inequalities. Note that this kernel is nothing but the translation of the function $|x|^{-(N+ps)}$. To work with Dunkl case it is essential to consider the kernel which is Dunkl translation of $|x|^{-(d_k+ps)}$. Motivated from Gorbachev et al. (see [17, Lemma 2.3]) we define the kernel $\Phi_{ps}(x,y)$, which is actually Dunkl translation of $|x|^{-(d_k+\delta)}$, as

$$\Phi_\delta(x,y) := \frac{1}{\Gamma((d_k+\delta)/2)} \int_0^\infty s^{\frac{d_k+\delta}{2}-1} \tau_y^k \left(e^{-s|.|^2}\right)(x) ds \quad \delta \neq -d_k. \quad (1.1.20)$$

The fractional Hardy inequality in Dunkl setting is stated as follows:

Theorem 1.1.12. *Let $d_k \geqslant 1$ and $0 < s < 1$. For $u \in \dot{W}_p^s(\mathbb{R}^N)$ when $2 \leqslant p < d_k/s$ or $u \in \dot{W}_p^s(\mathbb{R}^N \setminus \{0\})$ when $p > d_k/s$, the following inequality holds;*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) \geqslant C_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x),$$

where $\Phi_{ps}(x,y)$ is given in (1.1.20) and

$$C_{d_k,s,p} := 2 \int_0^1 r^{ps-1} |1 - r^{(d_k-ps)/p}|^p \Phi_{N,s,p}(r) dr,$$

with

$$\Phi_{N,s,p}(r) := \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})}\int_0^\pi \frac{sin^{d_k-2}\theta}{(1-2r\ cos\ \theta + r^2)^{\frac{d_k+ps}{2}}}d\theta,\ N \geqslant 2,$$
$$\Phi_{1,s,p}(r) := \left(\tau_r^k\big(|.|^{d_k+ps}\big) + \tau_{-r}^k\big(|.|^{d_k+ps}\big)\right)(1),\ N = 1. \quad (1.1.21)$$

The constant $C_{d_k,s,p}$ *is sharp. If* $p = 1$*, equality holds iff* u *is proportional to a symmetric decreasing function. If* $p > 1$*, the inequality is strict for any function* $0 \not\equiv u \in \dot{W}_p^s(\mathbb{R}^N)$ *or* $\dot{W}_p^s(\mathbb{R}^N \setminus \{0\})$*, respectively. Further for* $p \geqslant 2$ *the following inequality holds.*

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)\\ &\geqslant C_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)\\ &\qquad + c_p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(x)-v(y)|^p\Phi_{ps}(x,y)\frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}}\frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}},\end{aligned} \quad (1.1.22)$$

where $v := |x|^{(d_k-ps)/p}u$ *and* c_p *is given in (1.1.18).* $c_2 = 1$ *and the equality holds in* $p = 2$ *case.*

As in the case of classical Hardy inequality we will consider extended root system to establish fractional Hardy inequalities on half-space and cone.

Set $G_s(u)$ as

$$G_s(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - C(N,p,s)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx,$$

where the constant $C_{N,p,s}$ is the sharp constant in the fractional Hardy inequality obtained by Frank et al in [16]. For $p \geqslant 2$, $0 < s < 1$ and c_p R.L Frank and R.

Seiringer have proved the sharp Hardy inequality with a remainder term in [16].

$$G_s(u) \geqslant c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{(N-ps)/2}} \frac{dy}{|y|^{(N-ps)/2}},$$

where $v := |x|^{(N-ps)/2}u$. The result is true for all $u \in C_0^\infty(\mathbb{R}^N)$ if $ps < N$ and for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $ps > N$ and the inequality turns out to be an equality if $p = 2$. Combining three different articles [3, 1, 2] due to B. Abdellaoui et al. we can get an improved fractional Hardy inequality for $1 < p < \infty$. The combined statement is as follows: For $0 < s < 1$, $ps < N$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain we have:

$$G_s(u) \geqslant C(N,q,s,\Omega) \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}} dxdy. \tag{1.1.23}$$

The result is true for all $1 < q < p < \infty$ and for all functions $u \in C_0^\infty(\Omega)$. Note that this inequality is true for all $1 < p < \infty$ and the remainder term here is a $p-$norm of a fractional gradient. The inequality in (1.1.22) gives an improved fractional Hardy inequality for $p \geqslant 2$ in the Dunkl setting. We look for an improved term which is $p-$norm of a fractional Dunkl gradient for all $1 < p < \infty$.

Theorem 1.1.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $1 \leqslant q < p < \infty$. Then for all $u \in C_0^\infty(\Omega)$*

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x)$$

$$\geqslant C \iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y), \tag{1.1.24}$$

where

$$\Lambda_{d_k,s,p} = 2\int_0^1 r^{ps-1}|1-r^{(d_k-ps)/p}|^p\Phi(r)dr, \tag{1.1.25}$$

with

$$\Phi(r) = \begin{cases} \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1)}{2})}\int_0^\pi \frac{\sin^{d_k-2}\theta}{(1-2r\cos\theta+r^2)^{\frac{d_k+ps}{2}}} & \text{for } N \geqslant 2 \\ \Big(\tau_r^k(|.|^{-d_k-ps}) + \tau_{-r}^k(|.|^{-d_k-ps})\Big)(1) & \text{for } N = 1 \end{cases}$$

and C is a positive constant depending on Ω, d_k, q and s.

Stein-Weiss inequality is one of the most important inequality in mathematics. It states that for every $0 < \beta < N$ and for every $\varphi \in L^2(\mathbb{R}^N)$ there exists a positive constant such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta}{2}}|x-y|^{N-\beta}|y|^{\frac{\beta}{2}}}dxdy \leqslant C\|\varphi\|_2^2. \tag{1.1.26}$$

Moreover the authors of [21] have found the optimal constant $C = \frac{1}{2^\beta}\left(\frac{\Gamma(\frac{N-\beta}{4})}{\Gamma(\frac{N+\beta}{4})}\right)^2$. We prove a more generalized version this inequality in the Dunkl setting.

Theorem 1.1.14. *Let $0 < \beta < d_k$. Then for every $\varphi \in L^2(\mathbb{R}^N, d\mu_k(x))$ the Stein-Weiss inequality is given by*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}}\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y) \leqslant \frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2\int_{\mathbb{R}^N}|\varphi|^2 d\mu_k(x) \tag{1.1.27}$$

where the constant appearing on the right-hand side is optimal.

Let $\dot{H}^s(\mathbb{R}^N)$ denotes the fractional homogeneous Sobolev space equipped with the norm

$$\|\varphi\|_{\dot{H}^s}^2 := \frac{s\Gamma(\frac{N+2s}{2})}{2^{2(1-s)}\pi^{N/2}\Gamma(1-s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}dxdy. \tag{1.1.28}$$

In [25] V. Moroz and J. V. Schaftingen proved a fractional Stein-Weiss inequality on $\dot{H}^s(\mathbb{R}^N)$. They adopted the ground state substitution techniques developed by Frank et al. in [16]. the following theorem is the Stein Weiss potential for the fractional Dunkl gradient estimate.

Theorem 1.1.15. *Let $s \in (0,1)$, $s < d_k/2$ and $\beta < d_k$. The for all $\varphi \in W^{s,2}(\mathbb{R}^N)$ the following inequality holds*

$$\frac{1}{2^{\beta+s}}\left(\frac{\Gamma(\frac{d_k-2s}{4})\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+2s}{4})\Gamma(\frac{d_k+\beta}{4})}\right)^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(y)|^2 \Phi_{2s} d\mu_k(x) d\mu_k(y)$$
$$\geqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}} \Phi_{-\beta}(x,y) d\mu_k(x) d\mu_k(y) \tag{1.1.29}$$

and the constant is optimal.

1.2 Preliminaries of Dunkl Theory

In this section we give some basics on Dunkl theory which we will be using in this thesis. We suggest readers [11, 27, 35, 38] to get more details of Fourier analysis related to Dunkl operators. For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote σ_α as the reflection in the hyper plane $\langle\alpha\rangle^\perp$ orthogonal to α, that is

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha,$$

where $|\alpha| := \sqrt{\langle \alpha, \alpha\rangle}$.

Definition 1.2.1. Let $R \subset \mathbb{R}^N\setminus\{0\}$ be a finite set. Then R is called a root system, if

(1) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$

(2) $\sigma_\alpha(R) = R$ for all $\alpha \in R$.

A root system can be written as the disjoint union of $R_+ \cup (-R_+)$ and R_+ and $(-R_+)$ are separated by a hyper plane passing through the origin. R_+ is called as the set of positive roots of the root system. The subgroup $G = G(R) \subseteq O(N, \mathbb{R})$ which is generated by reflections $\{\sigma_\alpha : \alpha \in R\}$ is called reflection group (or Coxeter-group) associated with R.

For any root system R in $\mathbb{R}^N$, the reflection group $G = G(R)$ is finite and the set of reflections contained in $G(R)$ is exactly $\{\sigma_\alpha, \alpha \in R\}$. For the convenience of the calculations we assume that R is normalized, that is $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$. A function $k : R \to \mathbb{C}$ is said be a multiplicity function if it is invariant under the natural action of G on R. In this thesis we consider only the multiplicity functions from R to $(0, \infty)$.

Fix a root system R and a multiplicity function k on R. Then for $\xi \in \mathbb{R}^N$, the Dunkl operators T_ξ is defined by

$$T_\xi f(x) = \partial_\xi f(x) + E_\xi f(x), \quad f \in C^1(\mathbb{R}^N)$$

where

$$E_\xi f(x) = \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. Since k is G-invariant T_ξ does not depend on the choice of R_+. For the standard basis vector we use the abbreviation $T_i = T_{e_i}$. For each i, T_i has the following properties:

i) $T_i T_j = T_j T_i$.

ii) For $f, g \in C^1(\mathbb{R}^N)$, $T_\xi(fg) = T_\xi(f).g + f.T_\xi(g)$, provided at least one of them is G-invariant.

The Dunkl Laplacian Δ_k is defined by $\Delta_k = \sum_{j=1}^{N} T_j^2$ which can also be expressed as

$$\Delta_k f(x) = \Delta_0 f(x) + 2\sum_{\alpha\in R_+} k(\alpha)\left(\frac{\langle \nabla_0 f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\right), \quad (1.2.1)$$

where Δ_0 and ∇_0 are the usual Euclidean Laplacian and gradient operators on $\mathbb{R}^N$ respectively.

For a fixed reflection group G and the multiplicity function k, the weight function h_k is defined by

$$h_k(x) = \prod_{\alpha\in R_+} |\langle x, \alpha\rangle|^{k(\alpha)},\ x \in \mathbb{R}^N.$$

This is a positive homogeneous function of degree $\gamma_k := \sum_{\alpha\in R_+} k(\alpha)$ and is invariant under the reflection group G. Throughout this paper we assume that $k(\alpha) \geqslant 0$ we denote the weighted measure $h_k^2(x)dx$ by $d\mu_k(x)$. By the $G-$invariance of k we have $k(\alpha) = k(-\alpha)$ for all $\alpha \in R$ and hence $h_k(x)$ does not depend on the choice of R_+.

For $f \in C_b^1$, the space of bounded functions of class C^1, and $g \in \mathcal{S}(\mathbb{R}^N)$, the space of Schwartz class functions,

$$\int\limits_{\mathbb{R}^N} T_i f(x) g(x) d\mu_k(x) = -\int\limits_{\mathbb{R}^N} f(x) T_i g(x) d\mu_k(x).$$

We use the notations $d_k = N + 2\gamma_k$ and $\lambda_k = \frac{N-2}{2} + \gamma_k$ whenever required. Using the spherical polar coordinates $x = rx'$, where $x' \in \mathbb{S}^{N-1}$, we can write

$$\int\limits_{\mathbb{R}^N} f(x) d\mu_k(x) = \int\limits_0^\infty \int\limits_{\mathbb{S}^{N-1}} f(rx') d\mu_k(x') d\sigma(x') r^{(2\lambda_k+1)} dr$$

and deduce that

$$c_k^{-1} = \int\limits_{\mathbb{R}^N} e^{-|x|^2/2} d\mu_k(x) = 2^{\lambda_k}\Gamma(\lambda_k + 1)a_k^{-1}, \text{ where } a_k^{-1} = \int\limits_{\mathbb{S}^{N-1}} h_k^2(x')(x')d\sigma(x').$$

It is known that for any $y \in \mathbb{R}^N$, there exists a unique real analytic solution $f = E_k(\cdot, y)$ of system $T_i f = y_i f$ $1 \leqslant i \leqslant N$ satisfying $f(0) = 1$. $E_k(x, y)$ is called the Dunkl kernel and it is a generalization of the exponential function $e^{<x,y>}$. We list some of the important properties of $E_k(x, y)$ below:

i) $E_k(x, y) = E_k(y, x)$.

ii) $E_k(\lambda x, y) = E_k(x, \lambda y)$ where $\lambda \in \mathbb{C}$.

iii) $E_k(\sigma x, \sigma y) = E_k(x, y)$ for all $\sigma \in G$.

iv) $|E_k(x, y)| \leqslant e^{|x||y|}$.

v) $|E_k(ix, y)| \leqslant 1$ for all $x, y \in \mathbb{R}^N$.

Dunkl Fourier transform is a generalization of classical Fourier transform and it is defined in terms of the Dunkl kernel. For $f \in L^1(\mathbb{R}^N, d\mu_k(x))$, it's Dunkl Fourier transform is defined by

$$\mathcal{F}_k f(\xi) = c_k^{-1} \int\limits_{\mathbb{R}^N} f(x) E_k(-i\xi, x) d\mu_k(x). \tag{1.2.2}$$

It possess many analogous properties of Fourier transform.

i) Dunkl Fourier transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$.

ii) (Plancheral formula) Dunkl Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R}^N, d\mu_k(x))$.

iii) (Inversion formula) If $\mathcal{F}_k f \in L^1(\mathbb{R}^N, d\mu_k(x))$ then $f(x) = \mathcal{F}_k(\mathcal{F}_k f)(-x)$.

Dunkl translation operator $\tau_y f$ is defined by $\mathcal{F}_k(\tau_y f)(\xi) = E_k(iy, \xi)\mathcal{F}_k f(\xi)$ and it makes sense for all $f \in L^2(\mathbb{R}^N, d\mu_k(x))$ as $E_k(iy, \xi)$ is a bounded function. But for$f \in \mathcal{S}(\mathbb{R}^N)$ the above equation makes sense pointwise. The property $\tau_y f(x) = \tau_{-x} f(-y)$ of the translation operator will be used later. Dunkl translation operator is bounded on $L^2(\mathbb{R}^n, d\mu_k(x))$. However, L^p boundedness of the Dunkl translation operator is not known. We define Dunkl convolution of f, g in Schwartz space by

$$f *_k g(x) = \int_{\mathbb{R}^N} \tau_x^k f(-y) g(y) d\mu_k(y). \tag{1.2.3}$$

Convolution operator is associative and commutative and it satisfies the following properties:

i) For $f, g \in \mathcal{S}(\mathbb{R}^N)$, $f *_k g \in \mathcal{S}(\mathbb{R}^N)$ and $\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g)$.

ii) Let $1 \leqslant p, q, r \leqslant \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and $f \in L^p(\mathbb{R}^N, d\mu_k(x)), g \in L^q(\mathbb{R}^N, d\mu_k(x))$ is radial then $f *_k g \in L^r(\mathbb{R}^N, d\mu_k(x))$ and moreover it satisfies

$$\|f *_k g\|_r \leqslant \|f\|_p \|g\|_q. \tag{1.2.4}$$

The Riesz potential in the Dunkl setting is defined by S. Thangavelu and Y. Xu in [38]. Let α be a real number such that $0 < \alpha < d_k$, then for every $u \in \mathcal{S}$ the weighted Riesz potential $I_\alpha^k u$ is defined as

$$I_\alpha^k u(x) := (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^N} \tau_y^k u(x) |y|^{\alpha - d_k} d\mu_k(y), \tag{1.2.5}$$

where $\gamma_\alpha^k = 2^{\alpha - d_k/2}\Gamma(\alpha/2)/\Gamma((d_k - \alpha)/2)$.

Chapter 2

Hardy and Trace Hardy Inequalities for $L^2(\mathbb{R}^N, d\mu_k(x))$

In this chapter we will study Hardy inequality, trace Hardy inequality, fractional Hardy inequality for Dunkl operators. We start with the optimal classical Hardy inequality for Dunkl gradient in the space $L^2(\mathbb{R}^N, d\mu_k(x))$. Using this result we prove the optimal Hardy inequalities for the half-space and cone. Later we will establish a trace Hardy inequality and fractional Hardy inequality using the technique of extension problem developed by Caffarelli and Silvestre in a well celebrated paper [9].

2.1 Introduction

For $N \geqslant 3$ and $u \in C_0^\infty(\mathbb{R}^N)$, the classical Hardy inequality states that

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx, \tag{2.1.1}$$

where ∇ is the classical gradient on $\mathbb{R}^N$ and the constant $(\frac{N-2}{2})^2$ is sharp. For $1 < p < \infty$, $1 \leqslant l \leqslant N$ and $\alpha + l > 0$, a more generalized Hardy inequality of the form

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p |y|^{\alpha+p} dx \geqslant C \int_{\mathbb{R}^N} |u(x)|^p |y|^{\alpha} dx, \tag{2.1.2}$$

where $x = (y, z) \in \mathbb{R}^l \times \mathbb{R}^{N-l}$, with the optimal constant $C = \frac{(\alpha+l)^p}{p^p}$, was given by Simone Secchi et al. in [28].

Let $\mathbb{R}^N_+ = \{(x_1, ..., x_N) \in \mathbb{R}^N | x_N > 0\}$ be the half-space. A Hardy inequality on the upper half-space can be written as

$$\int_{\mathbb{R}^N_+} |\nabla u(x)|^2 \; dx \geqslant \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u(x)|^2}{x_N^2} dx.$$

Later in [39], J. Tidblom proved that, for all $u \in C_0^\infty(\mathbb{R}^N_+)$ the following inequality holds:

$$\int_{\mathbb{R}^N_+} |\nabla u(x)|^2 dx \geqslant \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u(x)|^2}{x_N^2} dx + \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u(x)|^2}{x_{N-1}^2 + x_N^2} dx.$$

Using the above Hardy inequality (2.1.2), Jing-Wen Luan et al. have proven the following Hardy inequality for half-space in [23].

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geqslant \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u|^2}{x_N^2} dx + \frac{(l-1)^2}{4} \int_{\mathbb{R}^N_+} \frac{|u|^2}{x_{N-l+1}^2 + ... + x_N^2} dx, \tag{2.1.3}$$

where $\frac{(l-1)^2}{4}$ is the best constant. Later in 2012, in [34], Dan Su et al. found the sharp Hardy inequality for the cone $\mathbb{R}^N_{l_+} = \{(x_1, ..., x_N) : x_{N-l+1} > 0, ..., x_N > 0\}$, $1 \leqslant l \leqslant N$. Their result states that, for $N \geqslant 3$ and $u \in C_0^\infty(\mathbb{R}^N_{l_+})$,

$$\int_{\mathbb{R}^N_{l_+}} |\nabla u|^2 dx \geqslant \frac{(N-2+2l)^2}{4} \int_{\mathbb{R}^N_{l_+}} \frac{|u|^2}{|x|^2} dx, \tag{2.1.4}$$

and the constant $\frac{(N-2+2l)^2}{4}$ is sharp.

For the classical Laplacian $\Delta = -\sum_{j=1}^{N} \partial_j^2$ the Hardy inequality in the equation (2.1.1) can be also written as

$$\langle \Delta u, u \rangle \geqslant \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx. \tag{2.1.5}$$

There are many results regarding the Hardy inequalities of fractional powers of Laplacian. For $0 < s < 1$, a Hardy inequality for Δ^s, the fractional power of Laplacian, is stated as

$$\langle \Delta^s u, u \rangle \geqslant 4^s \frac{\Gamma(\frac{N+s}{4})^2}{\Gamma(\frac{N-s}{4})^2} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx, \tag{2.1.6}$$

for the functions u such that $u, \Delta^s u \in L^2(\mathbb{R}^N)$. The constant $4^s \frac{\Gamma(\frac{N+s}{4})^2}{\Gamma(\frac{N-s}{4})^2}$ is sharp and never achieved. Another version of this Hardy inequality, in which the homogeneous weight $|x|^{-2s}$ is replaced by $(\delta^2 + |x|^2)^{-s}$, is of the form

$$\langle \Delta^s u, u \rangle \geqslant \frac{\Gamma(\frac{N+s}{2})}{\Gamma(\frac{N-s}{2})} \delta^{2s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{(\delta^2 + |x|^2)^s} dx, \qquad \delta > 0, \tag{2.1.7}$$

where the constant is sharp since it is achieved for the functions $(\delta^2 + |x|^2)^{-\frac{N-s}{2}}$. The left-hand side of the equation (2.1.6) can be written as

$$\langle \Delta^s u, u \rangle = \frac{4^s \Gamma(N/2 + s)}{2|\Gamma(-s)|\pi^{N/2}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy. \tag{2.1.8}$$

In view of this we can see that the fractional Hardy inequality in equation (2.1.6) is equivalent to the inequality

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \geqslant 2\pi^{N/2}\frac{\Gamma(\frac{N+s}{4})^2}{\Gamma(\frac{N-s}{4})^2}\frac{|\Gamma(-s)|}{\Gamma(N/2+s)}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}dx. \tag{2.1.9}$$

The fractional Hardy inequality in the half-space has been investigated by many authors. For suitable functions, a fractional Hardy inequality for the half-space $\mathbb{R}^N_+$ is stated as

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \geqslant D_{N,2,s}\int_{\mathbb{R}^N_+}\frac{|u(x)|^2}{x_N^{2s}}dx, \tag{2.1.10}$$

where the constant $D_{N,2,s}$ given by,

$$D_{N,2,s} = 2\pi^{(N-1)/2}\frac{\Gamma((1+2s)/2)}{\Gamma((N+2s)/2)}\int_0^1 \left|1-r^{(2s-1)/2}\right|^2\frac{dr}{(1-r)^{1+2s}},$$

is optimal. For further reading and improvements of fractional Hardy inequalities we refer to [6, 8, 12, 15, 14, 16, 40]. For $0 < s < 1$ and suitable functions on $\mathbb{R}^N \times \mathbb{R}_+$, the trace Hardy inequality states that

$$\int_0^\infty\int_{\mathbb{R}^N}|\nabla u(x,\rho)|^2\rho^{1-s}dxd\rho \geqslant 2\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\left(\frac{\Gamma((N+s)/4)}{\Gamma((N-s)4)}\right)^2\int_{\mathbb{R}^N}\frac{|u(x,0)|^2}{|x|^s}dx. \tag{2.1.11}$$

Also the trace Hardy inequalities with non-homogeneous weight $(\delta^2+|x|^2)^s$, which are of the form

$$\int_0^\infty\int_{\mathbb{R}^N}|\nabla u(x,\rho)|^2\rho^{1-s}dxd\rho \geqslant 2\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\frac{\Gamma((N+s)/2)}{\Gamma((N-s)2)}\delta^{2s}\int_{\mathbb{R}^N}\frac{|u(x,0)|^2}{(\delta^2+|x|^2)^s}dx \tag{2.1.12}$$

with $\delta > 0$, have also been studied. The inequalities in (2.1.11) and (2.1.12) are obtained by means of the solution of an initial value problem, for $0 < s < 1$

$$(-\Delta + \partial_\rho^2 + \frac{1-s}{\rho}\partial)v(x,\rho) = 0, \qquad x \in \mathbb{R}^N, \rho > 0; \qquad v(x,0) = f(x). \tag{2.1.13}$$

The initial value problem given above is known as the extension problem for the Laplacian. Caferalli and Silvestre studied about the solution of the extension problem in [9] and they established the relationship

$$\lim_{\rho \to 0} \rho^{1-s}\partial_\rho v(x,\rho) = 2^{1-s}\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\Delta^{s/2}f(x). \tag{2.1.14}$$

Their techniques have been used in many papers to study the different extension problems and certain types of trace Hardy type inequalities; we refer for instance [13, 33, 36].

This chapter is organized as follows. In Section 2.2 we will prove the Hardy inequality and a few uncertainty principles and in Section 2.3 we will prove Hardy inequalities for half-space and cone. In Section 2.4 our idea is to prove the trace Hardy inequalities and Hardy inequalities for the Dunkl fractional Laplacian. To obtain the above we solve the extension problem related to Dunkl fractional Laplacian and establish the connection between the solution of the extension problem and fractional Dunkl Laplacian. In Sections 2.5 and 2.6 we prove the fractional Hardy inequalities on half-space and cone.

2.2 Hardy Inequality and Uncertainty Principles

In this section we will first prove a general theorem which offers Hardy inequality, uncertainty principle and a few other theorems as corollary.

Theorem 2.2.1. *Let w be a positive radial function and let V be a function satisfying $-\Delta_k w + Vw \geqslant 0$ in $\mathbb{R}^N$. Then for all G-invariant $u \in C_0^1(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} (|\nabla_k u|^2 + V|u|^2) d\mu_k(x) \geqslant \int_{\mathbb{R}^N} |\nabla_k (w^{-1}u)|^2 w^2 d\mu_k(x). \tag{2.2.1}$$

Proof. Let $u = wv$ and w is a radial function. Then we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \\ =& \int_{\mathbb{R}^N} |w\nabla_k v + v\nabla_k w|^2 d\mu_k(x) \\ =& \int_{\mathbb{R}^N} \left(|\nabla_k v|^2 w^2 + v^2 |\nabla_k w|^2 + 2wv \sum_j T_j v T_j w \right) d\mu_k(x). \end{aligned} \tag{2.2.2}$$

We will consider each of the integral separately and finally substitute it in the original equation and get the inequality. First we consider the following integral

$$\begin{aligned} &\int_{\mathbb{R}^N} v^2 T_j w T_j w d\mu_k(x) \\ &= -\int_{\mathbb{R}^N} w T_j (v^2 T_j w) d\mu_k(x) \\ &= -\int_{\mathbb{R}^N} w(\partial_j + E_j)(v^2 \partial_j w) d\mu_k(x) \\ &= -\int_{\mathbb{R}^N} wv^2 \partial_j^2 w d\mu_k(x) - \int_{\mathbb{R}^N} wE_j(v^2 \partial_j w) d\mu_k(x) - \int_{\mathbb{R}^N} 2wv\partial_j v \partial_j w d\mu_k(x) \\ &= -\int_{\mathbb{R}^N} wv^2 \partial_j^2 w d\mu_k(x) - \int_{\mathbb{R}^N} wE_j(v^2 \frac{w'(r)}{r} x_j) d\mu_k(x) - \int_{\mathbb{R}^N} 2wv\partial_j v \partial_j w d\mu_k(x) \end{aligned}$$

$$= -\int_{\mathbb{R}^N} wv^2\partial_j^2 w d\mu_k(x) - \int_{\mathbb{R}^N} \frac{ww'(r)}{r} E_j(v^2 x_j) d\mu_k(x) - \int_{\mathbb{R}^N} 2wv\partial_j v\partial_j w d\mu_k(x)$$
$$= -\int_{\mathbb{R}^N} wv^2\partial_j^2 w d\mu_k(x) - 2\int_{\mathbb{R}^N} wv\partial_j v\partial_j w d\mu_k(x) - \int_{\mathbb{R}^N} \frac{ww'(r)}{r}\times$$
$$\left\{v^2(\sigma_\alpha x)\sum_{\alpha\in R_+} k(\alpha)\frac{\alpha_j^2}{|\alpha|^2}\right\} d\mu_k(x).$$

Taking the summation over j,

$$\int_{\mathbb{R}^N}\sum_j v^2 T_j w T_j w d\mu_k(x) = -\int_{\mathbb{R}^N} wv^2\Delta_0^2 w d\mu_k(x) - 2\int_{\mathbb{R}^N} vw\nabla_o v.\nabla_0 vw d\mu_k(x)$$
$$-2\gamma_k\int_{\mathbb{R}^N}\frac{ww'(r)}{r} v^2(x) d\mu_k(x). \tag{2.2.3}$$

Since w is radial and v is G-invariant, we have

$$\sum_j T_j v T_j w = \sum_j \partial_j v\partial_j w. \tag{2.2.4}$$

Substituting (2.2.3) and (2.2.4) into (2.2.2) we get

$$\int_{\mathbb{R}^N} |w\nabla_k v + v\nabla_k w|^2 d\mu_k(x)$$
$$= \int_{\mathbb{R}^N} |\nabla_k v|^2 w^2 d\mu_k(x) - \int_{\mathbb{R}^N} wv^2\Delta_0 w d\mu_k(x) - 2\int_{\mathbb{R}^N} vw\nabla_0 v.\nabla_0 w d\mu_k(x)$$
$$- 2\gamma_k\int_{\mathbb{R}^N}\frac{ww'(r)}{r} v^2(x) d\mu_k(x) + 2\int_{\mathbb{R}^N} wv\nabla_0 v.\nabla_0 w d\mu_k(x)$$
$$= \int_{\mathbb{R}^N} |\nabla_k v|^2 w^2 d\mu_k(x) - \int_{\mathbb{R}^N} wv^2\Delta_0 w d\mu_k(x) - 2\gamma_k\int_{\mathbb{R}^N}\frac{ww'(r)}{r} v^2(x) d\mu_k(x).$$

Simplifying, we get

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \;=\; \int_{\mathbb{R}^N} |\nabla_k v|^2 w^2 d\mu_k(x) - \int_{\mathbb{R}^N} wv^2\Delta_k w d\mu_k(x). \tag{2.2.5}$$

By substitution $-\Delta_k w + Vw \geqslant 0$, we get the desired inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \geqslant \int_{\mathbb{R}^N} |\nabla_k v|^2 w^2 d\mu_k(x) - \int_{\mathbb{R}^N} V u^2 d\mu_k(x).$$

□

2.2.1 Applications of the Theorem

In this section we will assume that the function u is $G-$invariant and $u \in C_0^\infty(\mathbb{R}^N)$. By using the above theorem we can prove some important theorems by simply choosing the appropriate functions w and V.

Hardy Inequality

Assume that $\lambda_k = \frac{N-2}{2} + \gamma_k > 0$. Choose $w(x) = |x|^{-\lambda_k}$. Since it is a radial function we can directly calculate the Dunkl Laplacian of the function $w(x)$. The Dunkl Laplacian for the radial function is given by

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{2\lambda_k + 1}{r}\frac{\partial}{\partial r}.$$

So,

$$\begin{aligned} \Delta_k w &= \lambda_k(\lambda_k+1)|x|^{-(\lambda_k+2)} - (2\lambda_k+1)(\lambda_k)|x|^{-(\lambda_k+2)} \\ &= -\lambda_k^2 |x|^{-2}|x|^{-\lambda_k}. \end{aligned}$$

Now choose the function V as $V(x) = -\lambda_k^2|x|^{-2}$ so that the equality $-\Delta_k w + Vw = 0$ holds. Substituting in the Theorem 2.2.1, we obtain

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \geqslant \lambda_k^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} d\mu_k(x). \tag{2.2.6}$$

Remark 2.2.2. The optimality of the Hardy inequality in the Euclidean case for $N > 1$ has been done by I. Peral and J.L. Vazquez in [26]. We can adapt the similar technique in the case of Dunkl setting too. Use the following sequence of radial functions, $U_{\epsilon,k}$,

$$U_{\epsilon,k}(x) = \begin{cases} A_{N,\epsilon} & \text{if } 0 \leqslant |x| \leqslant 1 \\ A_{N,\epsilon}|x|^{-\frac{N-2}{2}-\gamma_k-\epsilon} & \text{if } |x| > 1, \end{cases}$$

where $A_{N,\epsilon} = 2/(N - 2 + 2\gamma_k + 2\epsilon)$, and proceed as in the proof Lemma 4.1 of [26].

Heisenberg Uncertainty Principle

Now let

$$w(x) = e^{-\frac{\alpha|x|^2}{2}}.$$

Using the Dunkl Laplacian for radial functions and get

$$\begin{aligned} \Delta_k w &= \alpha^2 r^2 e^{-\alpha\frac{|x|^2}{2}} - \alpha e^{-\alpha\frac{|x|^2}{2}} - (2\lambda_k + 1)\alpha e^{-\alpha\frac{|x|^2}{2}} \\ &= e^{-\alpha\frac{|x|^2}{2}}(\alpha^2|x|^2 - (2\lambda_k + 2)\alpha) = (\alpha^2|x|^2 - (2\lambda_k + 2)\alpha)w(x). \end{aligned}$$

So we choose $V(x) = \alpha^2|x|^2 - (2\lambda_k + 2)\alpha$. Using the Theorem 2.2.1

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \geqslant - \int_{\mathbb{R}^N} \big(\alpha^2|x|^2 - \alpha(2\lambda_k + 2)\big)|u|^2 d\mu_k(x).$$

Now optimizing for α we get

$$\left(\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x)\right)^{1/2}\left(\int_{\mathbb{R}^N} |x|^2|u|^2 d\mu_k(x)\right)^{1/2} \geqslant \left(\frac{2\lambda_k + 2}{2}\right)\int_{\mathbb{R}^N} |u|^2 d\mu_k(x).$$

Hydrogen Uncertainty Principle

In this case let us choose

$$w(x) = e^{-\alpha|x|},$$

then the Dunkl Laplacian of w is given by

$$\begin{aligned}\Delta_k w(x) &= \alpha^2 e^{-\alpha|x|} - (2\lambda_k + 1)\frac{\alpha}{|x|}e^{-\alpha|x|} \\ &= e^{-\alpha|x|}\left(\alpha^2 - (2\lambda_k + 1)\frac{\alpha}{|x|}\right)w(x).\end{aligned}$$

So we choose

$$V(x) = \alpha^2 - \frac{\alpha}{|x|}(2\lambda_k + 1)$$

and using the Theorem 2.2.1 to obtain the inequality,

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \geqslant -\int_{\mathbb{R}^N} \big(\alpha^2 - \frac{\alpha}{|x|}(2\lambda_k + 1)\big)|u|^2 d\mu_k(x).$$

Now optimize for α we get

$$\left(\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x)\right)^{1/2}\left(\int_{\mathbb{R}^N} |u|^2 d\mu_k(x)\right)^{1/2} \geqslant \frac{2\lambda_k + 1}{2}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} d\mu_k(x).$$

Linear Sobolev Inequality

Let $N + 2\gamma_k \geqslant 2$ and let

$$w(x) = (1 + |x|^2)^{-\frac{N-2}{2}-\gamma_k} = (1 + |x|^2)^{-t}.$$

$$\Delta_k w(x) = (1+|x|^2)^{-t-2}\Bigg(4t(t+1)|x|^2 - (1+|x|^2)(2t+2(2\lambda_k+1)t)\Bigg)$$

identify that λ_k is nothing but t. Then by replacing λ_k by t we get

$$\begin{aligned}\Delta_k w(x) &= (1+|x|^2)^{-t-2}\Bigg(4t(t+1)|x|^2 - 4t(t+1)(1+|x|^2)\Bigg)\\ &= -4t(t+1)(1+|x|^2)^{-t-2}\\ &= (1+|x|^2)^{-t}\Bigg((1+|x|^2)^{-2}(4t(t+1)\Bigg).\end{aligned}$$

Substituting the value of t and simplifying we get

$$\Delta_k w(x) = (1+|x|^2)^{-\frac{N-2}{2}-\gamma_k}\Bigg(-(N+2\gamma_k)(N-2+2\gamma_k)(1+|x|^2)^{-2}\Bigg)$$

and now choose

$$V(x) = -(N+2\gamma_k)(N-2+2\gamma_k)(1+|x|^2)^{-2},$$

to satisfy the equation $\Delta_k w(x) = V(x)w(x)$. By substituting in the Theorem 2.2.1 we obtain

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) \geqslant (N+2\gamma_k)(N-2+2\gamma_k)\int_{\mathbb{R}^N} \frac{|u|^2}{(1+|x|^2)^2} d\mu_k(x).$$

2.3 Hardy Inequality on the Half-Space and Cone

Let R_1 and R_2 be root systems and k_1, k_2 be multiplicity function corresponding to the space $\mathbb{R}^l$ and $\mathbb{R}^{N-l}$ respectively. Then $R = (R_1 \times (0)_{N-l}) \cup ((0)_l \times R_2)$ will be a root system on $\mathbb{R}^N$. Let us define the multiplicity function k on R by natural extension of k_1 and k_2. Let $x = (y, z) \in \mathbb{R}^l \times \mathbb{R}^{N-l}$, then it is easy to observe that $h_k^2(x) = h_{k_1}^2(y) h_{k_2}^2(z)$.

If $1 \leqslant l \leqslant N$ and if $x \in \mathbb{R}^N$ we can write $x = (y, z)$ where $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^{N-l}$.

Theorem 2.3.1. *Let $l + 2\gamma_{k_1} - 2 > 0$, then for each G-invariant $u \in C_0^\infty(\mathbb{R}^N)$, we have the following inequality:*

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^2 d\mu_k(x) \geqslant \left(\frac{l + 2\gamma_{k_1} - 2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x).$$

Moreover the constant appearing above is optimal.

Proof. For $u \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x) = \int_{\mathbb{R}^{N-l}} h_{k_2}^2(z) dz \int_{\mathbb{R}^l} \frac{|u(x)|^2}{|y|^2} h_{k_1}^2(y) dy.$$

Using the Hardy inequality we get

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x) &\leqslant \frac{4}{(l + 2\gamma_{k_1} - 2)^2} \int_{\mathbb{R}^{N-l}} h_{k_2}^2(z) dz \int_{\mathbb{R}^l} |\nabla_{k_1, y} u(y)|^2 h_{k_1}^2(y) dy \\ &\leqslant \frac{4}{(l + 2\gamma_{k_1} - 2)^2} \int_{\mathbb{R}^N} |\nabla_k u(x)|^2 d\mu_k(x). \end{aligned}$$

The last inequality hold since $|\nabla_k u(x)| \geqslant |\nabla_{k_1, y} u(y)|$.

To prove the constant $(\frac{l+2\gamma_{k_1}-2}{2})^2$ is optimal, consider $u(y, z) = v(y) w(z)$, where $v \in C_0^\infty(\mathbb{R}^l)$ and $w \in C_0^\infty(\mathbb{R}^{N-l})$. It is clear that

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^2 d\mu_k(x) = \int_{\mathbb{R}^N} \left[|\nabla_{k_1} v(y)|^2 |w(z)|^2 + |\nabla_{k_2} w(z)|^2 |v(y)|^2 \right] d\mu_k(x).$$

Consider the convex function from $[0, \infty) \times [0, \infty)$ to $[0, \infty)$ defined as, $(s, t) \mapsto$

(s^2+t^2). Now by the convexity we have

$$(s^2+t^2) \leqslant (1-\lambda)^{-1}s^2 + \lambda^{-1}t^2,$$

for all $s, t \geqslant 0$ and $0 < \lambda < 1$. By using this relation we obtain,

$$\begin{aligned}
& \frac{\int_{\mathbb{R}^N} |\nabla_k u(x)|^2 d\mu_k(x)}{\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x)} \\
= \;& \frac{\int_{\mathbb{R}^N} \left[|\nabla_{k_1} v(y)|^2 |w(z)|^2 + |\nabla_{k_2} w(z)|^2 |v(y)|^2\right] d\mu_k(x)}{\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x)} \\
\leqslant \;& (1-\lambda)^{-1} \frac{\int_{\mathbb{R}^N} |\nabla_{k_1} v(y)|^2 |w(z)|^2 d\mu_k(x)}{\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x)} + \lambda^{-1} \frac{\int_{\mathbb{R}^N} |\nabla_{k_2} w(z)|^2 |v(y)|^2 d\mu_k(x)}{\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x)} \\
= \;& (1-\lambda)^{-1} \frac{\int_{\mathbb{R}^l} |\nabla_{k_1} v(y)|^2 h_{k_1}^2(y) dy}{\int_{\mathbb{R}^l} \frac{|v(y)|^2}{|y|^2} h_{k_1}^2(y) dy} + \lambda^{-1} \frac{\int_{\mathbb{R}^{N-l}} |\nabla_{k_2} w(z)|^2 h_{k_2}^2(z) dz \int_{\mathbb{R}^l} |v(y)|^2 h_{k_1}^2(y) dy}{\int_{\mathbb{R}^{N-l}} |w(z)|^2 h_{k_2}^2(z) dz \int_{\mathbb{R}^l} \frac{|v(y)|^2}{|y|^2} h_{k_1}^2(y) dy}.
\end{aligned}$$

Since w is radial we have

$$\inf_{\substack{w \in C_0^\infty(\mathbb{R}^{N-l}) \\ w \neq 0}} \frac{\int_{\mathbb{R}^{N-l}} |\nabla_k w(z)|^2 h_{k_2}^2(z) dz}{\int_{\mathbb{R}^{N-l}} |w(z)|^2 h_{k_2}^2(z) dz} = \inf_{\substack{w \in C_0^\infty(\mathbb{R}^{N-l}) \\ w \neq 0}} \frac{\int_{\mathbb{R}^{N-l}} |\nabla_0 w(z)|^2 dz}{\int_{\mathbb{R}^{N-l}} |w(z)|^2) dz} = 0.$$

Hence for $0 < \lambda < 1$, by the optimality of the Hardy inequality on $\mathbb{R}^l$ with the root system R_1, we get

$$\begin{aligned}
\inf_{\substack{u \in C_0^\infty(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla_k u(x)|^2 d\mu_k(x)}{\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|y|^2} d\mu_k(x)} &\leqslant (1-\lambda)^{-1} \inf_{\substack{v \in C_0^\infty(\mathbb{R}^l) \\ v \neq 0}} \frac{\int_{\mathbb{R}^l} |\nabla_k v(y)|^2 h_{k_1}^2(y) dy}{\int_{\mathbb{R}^l} \frac{|v(y)|^2}{|y|^2} h_{k_1}^2(y) dy} \\
&\leqslant (1-\lambda)^{-1} \left(\frac{l-2}{2} + \gamma_{k_1}\right)^2.
\end{aligned}$$

Letting $\lambda \to 0$ and we get the constant $\left(\frac{l-2}{2} + \gamma_{k_1}\right)^2$ in Theorem 2.3.1 is optimal.

□

2.3.1 Hardy Inequality on the Half-Space $\mathbb{R}_+^{N+1}$

A straight forward calculation gives the following relations which will be used to prove the next theorem.

$$(\sqrt{x_{N+1}})^{-1}\big(-\Delta_k - \frac{\partial^2}{\partial x_{N+1}^2} - \frac{1}{4x_{N+1}^2}\big)(\sqrt{x_{N+1}}g(x))$$
$$= -\Delta_k g(x) - \big(\frac{\partial^2}{\partial x_{N+1}^2} + \frac{1}{x_{N+1}}\frac{\partial}{\partial x_{N+1}}\big)g(x). \quad (2.3.1)$$

$$-x_{N+1}^{-l}\left(\sum_{j=1}^{N} T_j^2 + \frac{\partial^2}{\partial x_{N+1}^2} - \frac{l(l-1)}{x_{N+1}^2}\right)x_{N+1}^l g(x)$$
$$= -\left(\sum_{j=1}^{N} T_j^2 + \frac{\partial^2}{\partial x_{N+1}^2} + \frac{2l}{x_{N+1}}\frac{\partial}{\partial x_{N+1}}\right)g(x). \quad (2.3.2)$$

$$-\prod_{i=N-l+1}^{N} x_i^{-1}\left(\sum_{j=1}^{N-l} T_j^2 + \sum_{j=N-l+1}^{N}\frac{\partial^2}{\partial x_j^2}\right)\prod_{i=N-l+1}^{N} x_i g(x)$$
$$= -\sum_{j=1}^{N-l} T_j^2 g(x) - \sum_{j=N-l+1}^{N}\left(\frac{\partial^2}{\partial x_j^2} + \frac{2}{x_j}\frac{\partial}{\partial x_j}\right)g(x). \quad (2.3.3)$$

Extend the root system R of $\mathbb{R}^N$ to $\mathbb{R}^{N+1}$ by $R\times\{0\}$ and extend the corresponding multiplicity function to $\mathbb{R}^{N+1}$ by $k(x,0) = k(x)$, where $x \in \mathbb{R}^N$. Let $\tilde{\nabla}_k = (\nabla_k, \frac{\partial}{\partial x_{N+1}})$ be the gradient on $\mathbb{R}^{N+1}$, where ∇_k is the Dunkl gradient on $\mathbb{R}^N$. With this notation we have the following theorem which can be considered as a Hardy inequality in the upper half space.

Theorem 2.3.2. *write* $\tilde{x} = (x, x_{N+1})$. *Let* $u \in C_0^\infty(\mathbb{R}_+^{N+1})$ *such that* u *is* G-

invariant. Then

$$\int_{\mathbb{R}_+^{N+1}} |\tilde{\nabla}_k u(\tilde{x})|^2 \; d\mu_k(x) dx_{N+1}$$
$$\geqslant \frac{1}{4} \int_{\mathbb{R}_+^{N+1}} \frac{|u(\tilde{x})|^2}{x_{N+1}^2} d\mu_k(x) dx_{N+1}$$
$$+ \frac{(N+2\gamma_k)^2}{4} \int_{\mathbb{R}_+^{N+1}} \frac{|u(\tilde{x})|^2}{x_1^2 + ... + x_{N+1}^2} h_k^2(x) dx dx_{N+1}.$$

Proof. Let $\tilde{\nabla} = (\nabla_k, \nabla_0)$ be the gradient on $\mathbb{R}_x^N \times \mathbb{R}_y^2$ where ∇_k be the Dunkl gradient on $\mathbb{R}^N$ and ∇_0 be the Euclidean gradient on $\mathbb{R}_y^2$. Using the optimal Hardy inequality given in the equation (2.2.6) for $v \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}_2)$:

$$\int_{\mathbb{R}^N \times \mathbb{R}^2} |\tilde{\nabla} v(x,y)|^2 d\mu_k(x) dy \geqslant \frac{(N+2\gamma_k)^2}{4} \int_{\mathbb{R}^N \times \mathbb{R}^2} \frac{|v(x,y)|^2}{x_1^2 + ... + x_N^2 + y_1^2 + y_2^2} d\mu_k(x) dy.$$

Let $v(x,y) = v(x,|y|)$ and using the above relations we get

$$\int_{\mathbb{R}_x^N \times \mathbb{R}_y^2} |\tilde{\nabla} v(x,y)|^2 d\mu_k(x) dy$$
$$= - \int_{\mathbb{R}_x^N \times \mathbb{R}_y^2} \left(\Delta_k + \frac{\partial^2}{\partial x_{N+1}^2} + \frac{1}{x_{N+1}} \frac{\partial}{\partial x_{N+1}} \right) v(x,y).v(x,y) d\mu_k(x) dy$$
$$= - \int_{\mathbb{R}_x^N \times \mathbb{R}_y^2} (\sqrt{x_{N+1}})^{-1} \left(- \Delta_k + \frac{\partial^2}{\partial x_{N+1}^2} - \frac{1}{4x_{n+1}^2} \right) \sqrt{x_{N+1}} v(x,|y|).v(x,|y|) d\mu_k(x) dy$$
$$= - \int_{\mathbb{R}_+^{N+1}} (\sqrt{x_{N+1}})^{-1} \left(- \Delta_k + \frac{\partial^2}{\partial x_{N+1}^2} - \frac{1}{4x_{n+1}^2} \right) \sqrt{x_{N+1}} v(x,|y|) v(x,|y|) x_{N+1} \int_0^{2\pi} d\theta.$$

Using the above relation (2.3.1) and substituting $u = \sqrt{x_{N+1}}v$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^2} |\nabla_k v(x,y)|^2 d\mu_k(x)dy \\
&\quad = \left(\int_{\mathbb{R}_+^{N+1}} |\tilde{\nabla}_k u(x,x_{N+1})|^2 - \frac{1}{4}\int_{\mathbb{R}_+^{N+1}} \frac{|u(x,x_{N+1})|^2}{x_{N+1}^2} \right) \int_0^{2\pi} d\theta \\
&\quad \geqslant \frac{(N+2\gamma_k)^2}{4} \int_{\mathbb{R}_x^N\times\mathbb{R}_y^2} \frac{|v(x,y)|^2}{x_1^2+...+x_N^2+y_1^2+y_2^2} d\mu_k(x)dy \\
&\quad = \frac{(N+2\gamma_k)^2}{4} \int_{\mathbb{R}_+^{N+1}} \frac{|u(x,x_{N+1})|^2}{x_1^2+...+x_{N+1}^2} d\mu_k(x)dx_{N+1} \int_0^{2\pi} d\theta.
\end{aligned}$$

So we have the inequality

$$\begin{aligned}
&\int_{\mathbb{R}_+^{N+1}} |\tilde{\nabla}_k u(x,x_{N+1})|^2 - \frac{1}{4}\int_{\mathbb{R}_+^{N+1}} \frac{|u(x,x_{N+1})|^2}{x_{N+1}^2} d\mu_k(x)dx_{N+1} \\
&\qquad \geqslant \frac{(N+2\gamma_k)^2}{4} \int_{\mathbb{R}_+^{N+1}} \frac{|u(x,x_{N+1})|^2}{x_1^2+...+x_{N+1}^2} d\mu_k(x)dx_{N+1}.
\end{aligned}$$

□

Now we can prove a slightly generalized version of above theorem. If we put $l = \frac{1}{2}$ in the following lemma we will get the above result and the $l = 1$ case will be used to prove the Hardy inequality for the cone.

Lemma 2.3.3. *Let $\tilde{\nabla}_k$ be the gradient on $\mathbb{R}^{N+1}$ as mentioned earlier. For $l \in \{1/2, 1, 3/2, 2, ..., N/2, ..\}$ and for all G-invariant $u \in C_0^\infty(\mathbb{R}_+^{N+1})$,*

$$\begin{aligned}
&\int_{\mathbb{R}_+^{N+1}} |\tilde{\nabla}_k u|^2 d\mu_k(x)dx_{N+1} + l(l-1)\int_{\mathbb{R}_+^{N+1}} \frac{|u(x)|^2}{x_{N+1}^2} d\mu_k(x)dx_{N+1} \\
&\qquad \geqslant \frac{(N+2\gamma_k+2l-1)^2}{4} \int_{\mathbb{R}_+^{N+1}} \frac{|u(x)|^2}{|x|^2} d\mu_k(x)dx_{N+1},
\end{aligned}$$

where $\frac{(N+2\gamma_k+2l-1)^2}{4}$ is sharp.

Proof. Let $\tilde{\nabla} = (\nabla_k, \nabla_0)$ be the gradient on $\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}$ where ∇_k is the Dunkl gradient over $\mathbb{R}_x^N$ and ∇_0 is the Euclidean gradient over $\mathbb{R}_y^{2l+1}$. From the Hardy inequality (2.2.6) it follows that, for $v \in C_0^\infty(\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1})$,

$$\int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} |\tilde{\nabla} v(x,y)|^2 d\mu_k(x) dy \geqslant \frac{(N+2\gamma_k+2l-1)^2}{4} \int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} \frac{|v(x,y)|^2}{x_1^2+...+x_N^2+|y|^2} d\mu_k(x) dy. \tag{2.3.4}$$

Consider the functions with $v(x,y) = v(x,|y|)$ and write $x_{N+1} = |y|$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} |\tilde{\nabla} v(x,y)|^2 d\mu_k(x) dy \\
&\quad = -\int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} \left(\Delta_k + \sum_{m=1}^{2l+1} \frac{\partial^2}{\partial y_m^2}\right) v(x,y).v(x,y) d\mu_k(x) dy \\
&\quad = -\int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} \left(\Delta_k + \frac{\partial^2}{\partial x_{N+1}^2} + \frac{2l}{x_{N+1}} \frac{\partial}{\partial x_{N+1}}\right) v(x,|y|).v(x,|y|) d\mu_k(x) dy.
\end{aligned}$$

Substituting $u = x_{N+1}^l v$ in the above equation and using the relation (2.3.2) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} |\tilde{\nabla} v|^2 d\mu_k(x) dy \\
&= \int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} x_{N+1}^{-l} \bigg(-\Delta_k - \frac{\partial^2}{\partial x_{N+1}^2} + \frac{l(l-1)}{x_{N+1}^2}\bigg) x_{N+1}^l v(x,|y|).v(x,|y|) d\mu_k(x) dy \\
&= \|\mathbb{S}^{2l+1}\| \int_{\mathbb{R}_+^{N+1}} \bigg(-\Delta_k - \frac{\partial^2}{\partial x_{N+1}^2} + \frac{l(l-1)}{x_{N+1}^2}\bigg) x_{N+1}^l v(x,x_{N+1}).x_{N+1}^l v(x,x_{N+1}) d\mu_k(x) dx_{N+1} \\
&= \|\mathbb{S}^{2l+1}\| \bigg(\int_{\mathbb{R}_+^{N+1}} |\tilde{\nabla}_k u(x,x_{N+1})|^2 + l(l-1) \int_{\mathbb{R}_+^{N+1}} \frac{|u|^2}{x_{N+1}^2}\bigg) d\mu_k(x) dx_{N+1} \\
&\geqslant \frac{(N+2\gamma_k+2l-1)^2}{4} \int_{\mathbb{R}_x^N \times \mathbb{R}_y^{2l+1}} \frac{|v(x,y)|^2}{x_1^2+...+x_N^2+|y|^2} d\mu_k(x) dy \\
&= \||\mathbb{S}^{2l+1}\| \frac{(N+2\gamma_k+2l-1)^2}{4} \int_{\mathbb{R}_+^{N+1}} \frac{|u(x,x_{N+1})|^2}{|x|^2+x_{N+1}^2} d\mu_k(x) dx_{N+1}.
\end{aligned}$$

Hence the required inequality

$$\int_{\mathbb{R}^{N+1}_+} |\tilde{\nabla}_k u(x, x_{N+1})|^2 + l(l-1) \int_{\mathbb{R}^{N+1}_+} \frac{|u|^2}{x^2_{N+1}} d\mu_k(x) dx_{N+1}$$
$$\geqslant \frac{(N + 2\gamma_k + 2l - 1)^2}{4} \int_{\mathbb{R}^{N+1}_+} \frac{|u(x, x_{N+1})|^2}{|x|^2 + x^2_{N+1}} d\mu_k(x) dx_{N+1}. \qquad (2.3.5)$$

Since the the Hardy inequality in (2.3.4) is sharp, the constant in (2.3.5) is sharp.

□

2.3.2 Hardy Inequality on the Cone $\mathbb{R}^N_{l_+}$

Let $\tilde{\nabla}_k = (\nabla_k, \frac{\partial}{\partial x_{N-l+1}}, ..., \frac{\partial}{\partial x_N})$ be the gradient on $\mathbb{R}^N$. Now we are going to prove Hardy inequality for the cone using the Lemma 2.3.3.

Theorem 2.3.4. *Let $N + 2\gamma_k \geqslant 3$. Let u be a G-invariant function and $u \in C_0^\infty(\mathbb{R}^N_{l_+})$. Then the following inequality holds:*

$$\int_{\mathbb{R}^N_{l_+}} |\tilde{\nabla}_k u|^2 d\mu_k(x) dx_{N-l+1} ... dx_N$$
$$\geqslant \frac{(N + 2l + 2\gamma_k - 2)^2}{4} \int_{\mathbb{R}^N_{l_+}} \frac{|u|^2}{|x|^2} d\mu_k(x) dx_{N-l+1} ... dx_N,$$

where the constant $\frac{(N+2l+2\gamma_k-2)^2}{4}$ is sharp.

Proof. As in the previous cases, without abusing the notation, we use $\tilde{\nabla} = (\nabla_k, \nabla_0)$ where ∇_k is the Dunkl gradient on $\mathbb{R}^{N-l}_x$ and ∇_0 is Euclidean gradient on $\mathbb{R}^{3l}_y$. The sharp Hardy inequality for $v \in C_0^\infty(\mathbb{R}^{N-l}_x \times \mathbb{R}^{3l}_y)$ is given by

$$\int_{\mathbb{R}^{N-l}_x \times \mathbb{R}^{3l}_y} (|\nabla_k v|^2 + |\nabla_0 v|^2) h_k^2(x) h_k^2(y) dx dy$$
$$\geqslant \frac{(N + 2l + 2\gamma_k - 2)^2}{4} \int_{\mathbb{R}^{N-l}_x \times \mathbb{R}^{3l}_y} \frac{|v|^2}{|x|^2 + |y|^2} h_k^2(x) h_k^2(y) dx dy,$$

where $x \in \mathbb{R}_x^{N-l}$ and $y \in \mathbb{R}_y^{3l}$. Set

$$x_{N-l+1} = \sqrt{y_1^2 + y_2^2 + y_3^2}, \quad x_{N-l+2} = \sqrt{y_4^2 + y_5^2 + y_6^2}, \cdots, x_N = \sqrt{y_{3l-2}^2 + y_{3l-1}^2 + y_{3l}^2}$$

and let $v(x, y) = v(x_1, ..x_N)$. Consider the integral

$$\begin{aligned}
&\int_{\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l}} |\tilde{\nabla} v|^2 d\mu_k(x) dy \\
&= -\int_{\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l}} \left(\Delta_k + \sum_{j=1}^{3l} \frac{\partial^2}{\partial y_j^2} \right) v(x, y).v(x, y) d\mu_k(x) dy \\
&= -\int_{\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l}} \left(\Delta_k + \sum_{j=N-l+1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{2}{x_j} \frac{\partial}{\partial x_j} \right) v(x, y).v(x, y) d\mu_k(x) dy.
\end{aligned}$$

For convenience we will denote the operator $\Delta_k + \sum_{j=N-l+1}^{N} \frac{\partial^2}{\partial x_j^2}$ by M_k and the surface area measure of $\mathbb{S}^3$ by $\|\mathbb{S}^3\|$. Now using the relation (2.3.3) and putting $u = \left(\prod_{j=N-l+1}^{N} x_j \right) v$ we have

$$\begin{aligned}
&\int_{\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l}} |\tilde{\nabla} v|^2 d\mu_k(x) dy \\
&= -\int_{\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l}} \prod_{j=N-l+1}^{N} x_j^{-1} M_k \prod_{j=N-l+1}^{N} x_j v(x_1, \cdots, x_N).v(x_1, \cdots, x_N) d\mu_k(x) dy \\
&= -\|\mathbb{S}^3\|^l \int_{\mathbb{R}_+^{N+1}} \prod_{j=N-l+1}^{N} x_j M_k \\
&\qquad \prod_{j=N-l+1}^{N} x_j v(x_1, \cdots, x_N).v(x_1, \cdots, x_N) d\mu_k(x) dx_{N-l+1} \cdots dx_N \\
&= -\|\mathbb{S}^3\|^l \int_{\mathbb{R}_{l_+}^{N}} |\tilde{\nabla}_k u|^2 d\mu_k(x) dx_{N-l+1} \cdots dx_N \\
&\geqslant \frac{(N + 2l + 2\gamma_k - 2)^2}{4} \int_{\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l}} \frac{|v|^2}{|x|^2 + |y|^2} d\mu_k(x) dy
\end{aligned}$$

$$= \|\mathbb{S}^3\|^l \frac{(N+2l+2\gamma_k-2)^2}{4} \int_{\mathbb{R}^N_{l_+}} \frac{|u|^2}{x_1^2+...+x_N^2} d\mu_k(x) dx_{N-l+1}\cdots dx_N.$$

Hence the theorem. □

Remark 2.3.5. The proof of Theorem 2.3.2, Lemma 2.3.3 and Theorem 2.3.4 are mainly based on the Hardy inequality proved in the Subsection 2.2.1. The proof of all these theorems are given in such a way that the optimality follow from the optimality of the Hardy inequality.

2.4 Trace Hardy Inequality and Fractional Hardy Inequality

The Hardy inequality for the upper half space is stated as

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geqslant \frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u|^2}{x_N^2} dx$$

for $u \in C_0^\infty(\mathbb{R}^N)$ and $\mathbb{R}^N_+ = \{(x_1, x_2, \cdots, x_N) : x_N > 0\}$ and $\frac{1}{4}$ is the best possible constant. In the recent times a lot of attention is given to the analysis of fractional power of Laplacians. Fractional Laplacian, $(-\Delta)^s$, for $s \in (0,1)$ can be defined using Fourier transform as $\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s}\hat{f}(\xi)$. It also can be expressed as

$$(-\Delta)^s f(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{f(x)-f(\xi)}{|x-\xi|^{N+2s}} d\xi,$$

where

$$c_{N,s} = \frac{s2^s\Gamma(\frac{N+2s}{2})}{\pi^{N/2}\Gamma(1-s)}.$$

Similarly in the case of Dunkl setting we can define the fractional power of Dunkl Laplacian in a number of ways. For $0 < s < 1$, the fractional power of Dunkl Laplacian is defined as $\mathcal{F}_k((-\Delta_k)^s f)(\xi) = |\xi|^{2s}\mathcal{F}_k(f)(\xi)$. One of the other equivalent definitions that we will use is

$$(-\Delta_k)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\Delta_k} f(x) - f(x))\frac{dt}{t^{1+s}}.$$

Our aim is to prove the trace Hardy inequality in Dunkl case. In order to do this we first calculate

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u - \frac{u}{v}\nabla_{k,\rho} v|^2 \rho^a d\mu_k(x) d\rho,$$

where $\nabla_{k,\rho} := (\nabla_k, \partial_\rho)$ and we assume that u and v are real valued and u is G-invariant. Now consider the integral

$$\int_{\mathbb{R}^N} (T_j u - \frac{u}{v} T_j v)^2 d\mu_k(x) = \int_{\mathbb{R}^N} (T_j u)^2 + \frac{u^2}{v^2}((T_j v)^2 - 2\frac{u}{v} T_j u T_j v) d\mu_k(x).$$

Let us consider terms of the right hand side of the equation separately. Assume that v is radial and use the integration by parts formula for Dunkl operator given in Section 1.2,

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{u^2}{v^2}(T_jv)^2d\mu_k(x) &= \int_{\mathbb{R}^N}\frac{u^2}{v^2}\partial_jv(T_jv)d\mu_k(x) = -\int_{\mathbb{R}^N}\partial_j(\frac{1}{v})u^2T_jvd\mu_k(x)\\ &= -\int_{\mathbb{R}^N}T_j(\frac{1}{v})u^2T_jvd\mu_k(x) = \int_{\mathbb{R}^N}\frac{1}{v}T_j(u^2T_jv)d\mu_k(x)\\ &= \int_{\mathbb{R}^N}\frac{1}{v}T_j(u^2\partial_jv)d\mu_k(x)\\ &= \int_{\mathbb{R}^N}\frac{1}{v}(\partial_j+E_j)(u^2\partial_jv)d\mu_k(x)\\ &= \int_{\mathbb{R}^N}\frac{1}{v}(2u\partial_ju\partial_jv+u^2\partial_j^2v+\frac{v'(r)}{r}E_j(x_ju^2))d\mu_k(x)\end{aligned}$$

$$= \int_{\mathbb{R}^N}\left(2\frac{u}{v}\partial_ju\partial_jv+\frac{u^2}{v}\partial_j^2v+\frac{v'(r)}{rv}E_j(x_ju^2)\right)d\mu_k(x).$$

By applying the definition of E_j and summing over j we get

$$\sum_{j=1}^{N}E_j(x_ju^2) = \sum_{\alpha\in R_+}k(\alpha)(u^2(x)+u^2(\sigma_\alpha x)).$$

Using this, we can arrive at

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{u^2}{v^2}|\nabla_kv|^2d\mu_k(x)\\ &\quad= \int_{\mathbb{R}^N}2\frac{u}{v}\nabla_0u.\nabla_0vd\mu_k(x)+\int_{\mathbb{R}^N}\frac{u^2}{v}\Delta_0vd\mu_k(x)\\ &\qquad\qquad+\int_{\mathbb{R}^N}\frac{v'(r)}{rv}\sum_{\alpha\in R_+}k(\alpha)(u^2(x)+u^2(\sigma_\alpha x))d\mu_k(x)\\ &\quad= \int_{\mathbb{R}^N}2\frac{u}{v}\nabla_0u.\nabla_0vd\mu_k(x)+\int_{\mathbb{R}^N}\frac{u^2}{v}\Delta_0vd\mu_k(x)+2\gamma_k\int_{\mathbb{R}^N}\frac{v'(r)}{rv}u^2(x)d\mu_k(x).\end{aligned}$$

On the other hand

$$\begin{aligned}
&-2\int_{\mathbb{R}^N}\frac{u}{v}T_juT_jvd\mu_k(x)\\
&\quad=-2\int_{\mathbb{R}^N}\frac{u}{v}(\partial_j+E_j)(u)\partial_jvd\mu_k(x)\\
&\quad=-2\int_{\mathbb{R}^N}\frac{u}{v}\partial_ju\partial_jvd\mu_k(x)-2\int_{\mathbb{R}^N}\frac{u}{v}\partial_jv\sum_{\alpha\in R_+}k(\alpha)\frac{u(x)-u(\sigma_\alpha(x))}{\langle x,\alpha\rangle}\alpha_jd\mu_k(x).
\end{aligned}$$

Summing over j gives

$$\begin{aligned}
&-2\int_{\mathbb{R}^N}\frac{u}{v}\nabla_ku.\nabla_kvd\mu_k(x)\\
=\;&-2\int_{\mathbb{R}^N}\frac{u}{v}\nabla_0u.\nabla_0vd\mu_k(x)-2\int_{\mathbb{R}^N}\frac{u}{v}\frac{v'(r)}{r}\sum_{\alpha\in R_+}k(\alpha)(u(x)-u(\sigma_\alpha x))d\mu_k(x).
\end{aligned}$$

The above two simplified expression will lead to

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{u^2}{v^2}|\nabla_kv|^2d\mu_k(x)-2\int_{\mathbb{R}^N}\frac{u}{v}\nabla_ku.\nabla_kvd\mu_k(x)\\
&\qquad=2\int_{\mathbb{R}^N}\frac{u}{v}\frac{v'(r)}{r}\sum_{\alpha\in R_+}k(\alpha)u(\sigma_\alpha x)d\mu_k(x)+\int_{\mathbb{R}^N}\frac{u^2}{v}\Delta_0vh_k^2(x).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla_ku-\frac{u}{v}\nabla_kv|^2d\mu_k(x)\\
&=\int_{\mathbb{R}^N}|\nabla_ku|^2d\mu_k(x)+2\int_{\mathbb{R}^N}\frac{u}{v}\frac{v'(r)}{r}\sum_{\alpha\in R_+}k(\alpha)u(\sigma_\alpha x)d\mu_k(x)+\int_{\mathbb{R}^N}\frac{u^2}{v}\Delta_0vd\mu_k(x).
\end{aligned}$$

Using the G-invariance of u and the expression of Dunkl Laplacian for radial function, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_k u - \frac{u}{v}\nabla_k v|^2 d\mu_k(x) \\ &\leqslant \int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) + 2\gamma_k \int_{\mathbb{R}^N} \frac{u^2}{v}\frac{v'(r)}{r} d\mu_k(x) + \int_{\mathbb{R}^N} \frac{u^2}{v}\Delta_0 v d\mu_k(x) \\ &= \int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k(x) + \int_{\mathbb{R}^N} \frac{u^2}{v}\Delta_k v d\mu_k(x).\end{aligned}$$

On the other hand doing a similar calculation with ρ-derivative gives

$$\begin{aligned}\int_0^\infty \left(\frac{u^2}{v^2}(\frac{\partial v}{\partial \rho})^2 - 2\frac{u}{v}\frac{\partial u}{\partial \rho}\frac{\partial v}{\partial \rho}\right)\rho^a d\rho \\ &= \int_0^\infty \frac{u^2}{v}\frac{\partial}{\partial \rho}(\rho^a \frac{\partial v}{\partial \rho}) d\rho + \frac{|u(x,0)|^2}{v(x,0)} \lim_{\rho\to 0}(\rho^a \partial_\rho v)(x,\rho).\end{aligned}$$

We can write

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u - \frac{u}{v}\nabla_{k,\rho} v|^2 \rho^a d\mu_k(x) d\rho$$

$$= \int_0^\infty \int_{\mathbb{R}^N} \left[\sum_{j=1}^N (T_j u - \frac{u}{v}T_j v)^2 + (\frac{\partial u}{\partial \rho} - \frac{u}{v}\frac{\partial v}{\partial \rho})^2\right] \rho^a d\mu_k(x) d\rho.$$

Now substitute the above equations and adding up we get

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u - \frac{u}{v}\nabla_{k,\rho} v|^2 \rho^a d\mu_k(x) d\rho \\ &\leqslant \int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u|^2 \rho^a d\mu_k(x) d\rho + \int_0^\infty \int_{\mathbb{R}^N} \frac{u^2}{v}\rho^a L_{k_a} v h_k^2(x) dx d\rho \\ &\quad + \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{v(x,0)} \lim_{\rho\to 0}(\rho^a \frac{\partial v}{\partial \rho})(x,0) d\mu_k(x),\end{aligned}$$

where L_{k_a} is the differential operator

$$L_{k_a} = \Delta_k + \partial_\rho^2 + \frac{a}{\rho}\partial_\rho. \tag{2.4.1}$$

Therefore, if v satisfies the equation $L_{k_a}v = 0$ on $\mathbb{R}_+^{N+1}$, then the above inequality reduces to

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u(x,\rho)|^2 \rho^a h_k^2(x) dx d\rho \geqslant - \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{v(x,0)} \lim_{\rho\to 0}(\rho^a \frac{\partial v}{\partial \rho})(x,\rho) d\mu_k(x). \tag{2.4.2}$$

Now we are interested in solving $L_{k_a}v = 0$ with a given initial condition, say $v(x,0) = f(x)$. This is actually the extension problem for Dunkl Laplacian. We use the techniques developed by L. Caffarelli and L. Silvestre in [9] to obtain the solution of the extension problem for the Dunkl Laplacian and relation of the extension problem with the fractional power of Dunkl Laplacian.

When $a = M - 1$ is a positive integer, note that L_{k_a} is given by the action of $\Delta_{k,x} + \Delta_0$ on $\mathbb{R}^{N+M}$ on functions $v(x,y)$ which are radial in the y variable.

$$(\Delta_{k,x} + \Delta_0)v(x,y) = \Delta_{k,x}v + (\partial_\rho^2 + \frac{M-1}{\rho}\partial_\rho)v(x,y)$$

with $|y| = \rho$. Then the solution of $L_{k_a}v = 0$ can be obtained by considering the fundamental solution of $\Delta_k + \Delta_0$ on $\mathbb{R}^{N+M}$. That is, the function

$$v(x,\rho) = (\rho^2 + |x|^2)^{-\frac{N+M}{2} - \gamma_k + 1}$$

solves $L_{k_a}v = 0$ even if a is not a positive integer. The choice $a = 1 - s$ leads to the solution

$$v(x,\rho) = (\rho^2 + |x|^2)^{-\frac{N-s}{2}-\gamma_k} = \rho^{-(N-s)-2\gamma_k}(1 + \rho^{-2}|x|^2)^{-\frac{N-s}{2}-\gamma_k}.$$

Define ψ_α, for any $\alpha \geqslant 0$ by $\psi_\alpha(x) = (1 + |x|^2)^{-\alpha}$. Then we can see

$$v(x,\rho) = \rho^{-(N-s)-2\gamma_k}(1 + \rho^{-2}|x|^2)^{-\frac{N-s}{2}-\gamma_k} = \rho^{-(N-s)-2\gamma_k}\psi_{\frac{N+s}{2}+\gamma_k}(\rho^{-1}x).$$

By taking convolution we see that $f *_k \rho^{-(N+2\gamma_k-s)}\psi_{\frac{N-s}{2}+\gamma_k}(\rho^{-1}.)$ also satisfies the extended equation (2.4.1) with $a = 1-s$. The function $\rho^{-(N+2\gamma_k-s)}\psi_{\frac{N-s}{2}+\gamma_k}(\rho^{-1}x)$ does not give an approximate identity as $\psi_{\frac{N-s}{2}+\gamma_k}$ is not integrable. Hence the convolution $f *_k \rho^{-(N+2\gamma_k-s)}\psi_{\frac{N-s}{2}+\gamma_k}(\rho^{-1}.)$ does not converge to f as $\rho \to 0$. It can be observed that

$$\rho^s(\rho^2 + |x|^2)^{-\frac{N+s}{2}-\gamma_k} = \rho^{-N-2\gamma_k}\psi_{\frac{N+s}{2}+\gamma_k}(\rho^{-1}x). \tag{2.4.3}$$

and this function satisfies the equation (2.4.1) with $a = 1 - s$. It also defines an approximate identity. Therefore, we have

$$f *_k \rho^{-N}\psi_{\frac{N+s}{2}+\gamma_k}(\rho^{-1}.) \to f$$

as $\rho \to 0$. If $0 \leqslant s \leqslant N/2 + \gamma_k$, the function $\psi_{\frac{N-s}{2}+\gamma_k} \in L^2(\mathbb{R}^N, d\mu_k(x))$ and so we can talk about $\mathcal{F}_k(\psi_{\frac{N-s}{2}+\gamma_k})$.

Theorem 2.4.1. *For* $0 < s < N/2 + \gamma_k$

$$\Delta_k^{s/2}\psi_{\frac{N-s}{2}+\gamma_k}(x) = 2^s\frac{\Gamma(\frac{N+s}{2} + \gamma_k)}{\Gamma(\frac{N-s}{2} + \gamma_k)}\psi_{\frac{N+s}{2}+\gamma_k}.$$

Proof. This relation can be proved by calculating the Dunkl transforms of $\psi_{\frac{N+s}{2}+\gamma_k}$ and $\psi_{\frac{N-s}{2}+\gamma_k}$. The gamma integral

$$a^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-at} t^{\alpha-1} dt, \quad a > 0,\ \alpha > 0$$

gives

$$\psi_{\frac{N+s}{2}+\gamma_k}(x) = \frac{1}{\Gamma(\frac{N+s}{2}+\gamma_k)} \int_0^\infty e^{(1+|x|^2)t} t^{\frac{N+s}{2}+\gamma_k-1} dt.$$

We know that

$$e^{-t|\xi|^2} = {c_h}^{-1} \int_{\mathbb{R}^N} \frac{1}{(2t)^{\lambda_k+1}} e^{-\frac{|x|^2}{4t}} E_k(x,-i\xi) d\mu_k(x). \tag{2.4.4}$$

Using the definition of Dunkl Fourier transform

$$\mathcal{F}_k \psi_{\frac{N+s}{2}+\gamma}(\xi) = c_h^{-1} \frac{1}{\Gamma(\frac{N+s}{2}+\gamma_k)} \int_{\mathbb{R}^N} \int_0^\infty e^{-(1+|x|^2)t} t^{\frac{N+s}{2}+\gamma-1} E_k(x,-i\xi) d\mu_k(x) dt.$$

Using the equation (2.4.4) we calculate

$$\mathcal{F}_k \psi_{\frac{N+s}{2}+\gamma}(\xi) = \frac{2^{-(\lambda_k+1)}}{\Gamma(\frac{N+s}{2}+\gamma_k)} \int_0^\infty e^{-t} e^{-\frac{1}{4t}|\xi|^2} t^{\frac{s}{2}-1} dt.$$

Now apply the change of variable $t \to \frac{1}{4t}|\xi|^2$ to obtain

$$\mathcal{F}_k \psi_{\frac{N+s}{2}+\gamma}(\xi) = \frac{2^{-(\lambda_k+1)} 2^{-s} |\xi|^s}{\Gamma(\frac{N+s}{2}+\gamma_k)} \int_0^\infty e^{-\frac{1}{4t}|\xi|^2} e^{-t} t^{-s/2-1} dt.$$

Hence we conclude that

$$\mathcal{F}_k \psi_{\frac{N+s}{2}+\gamma_k}(\xi) = 2^{-s} \frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)} |\xi|^s \mathcal{F}_k \psi_{\frac{N-s}{2}+\gamma_k}(\xi).$$

□

Since $\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g)$ we obtain

$$f * \psi_{\frac{N+s}{2}+\gamma_k} = 2^{-s}\frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)}\Delta_k^{s/2} f *_k \psi_{\frac{N-s}{2}+\gamma_k}. \tag{2.4.5}$$

Now we let

$$v_{s,\rho}(x) = \rho^s(\rho^2+|x|^2)^{-\frac{N+s}{2}-\gamma_k} = \rho^{-N-2\gamma}\psi_{\frac{N+s}{2}+\gamma_k}(\rho^{-1}x)$$

and using the change of variable we get

$$\mathcal{F}_k(v_{s,\rho})(\xi) = \mathcal{F}_k(\psi_{\frac{N+s}{2}+\gamma_k})(\rho\xi) = 2^{-s}\frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)}\rho^s|\xi|^s\mathcal{F}_k(\psi_{\frac{N-s}{2}+\gamma_k})(\rho\xi).$$

Therefore, it follows that

$$f *_k v_{s,\rho}(x) = 2^{-s}\frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)}\Big((\Delta_k^{s/2} f) *_k \rho^{-N-2\gamma+s}\psi_{\frac{N-s}{2}+\gamma_k}(\rho^{-1}.\Big)(x).$$

Since $\rho^{-(N+2\gamma_k-s)}\psi_{\frac{N-s}{2}+\gamma_k}(\rho^{-1}x)$ satisfies the equation $L_{k_a}u=0$, $v_{s,\rho}$ and $f *_k v_{s,\rho}$ also satisfies the same equation. Since $v_{s,\rho}$ is an approximate identity we obtain

$$\lim_{\rho\to 0} f *_k v_{s,\rho} = a_N(s) f$$

in $L^p(\mathbb{R}^N, h_k^2(x))$, $1 \leqslant p \leqslant \infty$, where

$$a_N(s) = \int_{\mathbb{R}^N}(1+|x|^2)^{-\frac{N+s}{2}-\gamma_k}dx.$$

We are going to calculate $a_N(s)$ explicitly. In fact

$$a_N(s) = a_k^{-1}\int_0^\infty (1+r^2)^{-\frac{N+s}{2}-\gamma_k} r^{N-1+2\gamma_k}dr.$$

Changing the variable to r by substituting $r^2 = t$ we obtain

$$a_N(s) = \frac{a_k^{-1}}{2} \int_0^\infty (1+t)^{-\frac{N+s}{2}-\gamma_k} t^{\frac{N}{2}+\gamma_k-1} dr. \tag{2.4.6}$$

In view of the formula

$$\int_0^\infty (1+t)^{-b} t^{a-1} dt = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)},$$

$a_N(s)$ can be written as

$$a_N(s) = \frac{a_k^{-1}}{2} \frac{\Gamma(\frac{N+2\gamma_k}{2})\Gamma(\frac{s}{2})}{\Gamma(\frac{N+2\gamma_k+s}{2})}. \tag{2.4.7}$$

Therefore,

$$\lim_{\rho\to 0} f *_k v_{s,\rho}(x) = \frac{a_k^{-1}}{2} \frac{\Gamma(\frac{N}{2}+\gamma_k)\Gamma(\frac{s}{2})}{\Gamma(\frac{N+s}{2}+\gamma_k)} f(x).$$

Since

$$(\rho^2+|x|^2)^{-\frac{N-s}{2}-\gamma_k} = \rho^{-N-2\gamma_k+s} \psi_{\frac{N-s}{2}+\gamma_k}(\rho^{-1}x),$$

we have

$$\begin{aligned}
&\rho^{1-s}\partial_\rho (f *_k v_{s,\rho})(x) \\
&= \rho^{1-s}\partial_\rho \left(2^{-s} \frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)} (\Delta_k^{s/2} f) *_k \rho^{-N-2\gamma_k+s} \psi_{\frac{N-s}{2}+\gamma_k}(\rho^{-1}.) \right)(x) \\
&= -2^{1-s} \frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)} \left(\frac{N-s}{2} + \gamma_k \right) (\Delta_k^{s/2} f) *_k \rho^{2-s} (\rho^2+|x|^2)^{-\frac{N+(2-s)}{2}-\gamma_k} \\
&= -2^{1-s} \frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)} \left(\frac{N-s}{2} + \gamma_k \right) (\Delta_k^{s/2} f) *_k v_{2-s,\rho}(x).
\end{aligned}$$

Consequently it yields

$$\lim_{\rho\to 0}\rho^{1-s}\partial_\rho(f *_k v_{s,\rho}) = -2^{1-s}\frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)}\left(\frac{N-s}{2}+\gamma_k\right)a_N(2-s)(\Delta_k^{s/2}f).$$

Using the explicit values of $a_N(s)$ and $a_N(2-s)$ we get

$$\lim_{\rho\to 0} -\frac{\rho^{1-s}\partial_\rho(f *_k v_{s,\rho})}{f *_k v_{s,\rho}} = 2^{1-s}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\frac{(\Delta_k^{s/2}f)}{f}.$$

Now using these results we get the following inequality:

$$\int_0^\infty\int_{\mathbb{R}^N}|\nabla_{k,\rho}u(x,\rho)|^2\rho^{1-s}h_k^2(x)dxd\rho \geqslant 2^{1-s}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\int_{\mathbb{R}^N}\frac{|u(x,0)|^2}{f(x)}(\Delta_k^{s/2}f)(x)d\mu_k(x). \tag{2.4.8}$$

The choice of $f(x) = (1+|x|^2)^{-\frac{N-s}{2}-\gamma_k}$ leads to the following theorem

Theorem 2.4.2. *Let $0 < s < 1$ and $u \in C_0^\infty(\mathbb{R}_+^{N+1})$ and G-invariant, then*

$$\int_0^\infty\int_{\mathbb{R}^N}|\nabla_{k,\rho}u(x,\rho)|^2\rho^{1-s}d\mu_k(x)d\rho \geqslant 2\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)}\int_{\mathbb{R}^N}\frac{|u(x,0)|^2}{(1+|x|^2)^s}d\mu_k(x).$$

By a suitable choice of f we can prove the following Hardy inequality for $\Delta_k^{s/2}$.

Corollary 2.4.3. *For $f \in L^2(\mathbb{R}^N, h_k^2(x))$ for which $\Delta_k^{s/2}f \in L^2(\mathbb{R}^N, d\mu_k(x))$,*

$$\langle\Delta_k^{s/2}f, f\rangle \geqslant 2^s\frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)}\int_{\mathbb{R}^N}\frac{|f(x)|^2}{(1+|x|^2)^s}d\mu_k(x).$$

Proof. Since $u = f * v_{s,\rho}$ satisfies

$$(\Delta_k + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho)u = 0,$$

with $u(x,0) = a_N(s)f(x)$ integrating $(\Delta_k + \partial_\rho^2 + \frac{1-s}{\rho})u$ with ρ variable and then integrate with x variable we get

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u|^2 \rho^{1-s} d\mu_k(x) d\rho = -\int_{\mathbb{R}^N} u(x,0) \lim_{\rho\to 0}(\rho^{1-s}\partial_\rho u)(x,\rho) d\mu_k(x).$$

Now use the above theorem and simplify to get

$$\frac{\Gamma(\frac{N-s}{2}+\gamma_k)}{\Gamma(\frac{N+s}{2}+\gamma_k)}(\frac{N-s}{2}+\gamma_k)a_N(2-s)a_N(s)\int_{\mathbb{R}^N} \Delta_k^{s/2} f(x) f(x) d\mu_k(x)$$
$$\geqslant 2\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)} a_N(s)^2 \int_{\mathbb{R}^N} \frac{|f(x)|^2}{(1+|x|^2)^s} d\mu_k(x).$$

Therefore, we arrive at the inequality

$$\langle \Delta_k^{s/2} f, f\rangle \geqslant 2^s \frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{(1+|x|^2)^s} d\mu_k(x).$$

□

Theorem 2.4.4. *For $0 < s < 1$ and $u \in C_0^\infty(\mathbb{R}_+^{N+1})$ and G-invariant, then*

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u(x,\rho)|^2 \rho^{1-s} h_k^2(x) dx d\rho$$
$$\geqslant 2c_h^{-1} \frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})} \left(\frac{\Gamma(\frac{N+2\gamma_k+s}{4})}{\Gamma(\frac{N+2\gamma_k-s}{4})}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{|x|^s} d\mu_k(x).$$

Proof. From the above trace Hardy inequality for non-homogeneous weight we have,

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u(x,\rho)|^2 \rho^{1-s} h_k^2(x) dx d\rho \geqslant 2\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})} \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{f(x)} (\Delta_k^{s/2} f)(x) d\mu_k(x).$$

For any $\varphi \in C_0^\infty(\mathbb{R}^N)$ we take $f = \varphi *_k u_{-s,\delta}(x)$ where

$$u_{-s,\delta}(x) = (\delta^2 + |x|^2)^{-\frac{N-s}{2}-\gamma_k}, \quad -1 < s < 1.$$

Then

$$\Delta_k^{s/2} f(x) = \Delta_k^{s/2}(\varphi *_k u_{-s,\delta})(x) = 2^s \frac{\Gamma(\frac{N+s}{2} + \gamma_k)}{\Gamma(\frac{N-s}{2} + \gamma_k)} \delta^s (\varphi *_k u_{s,\delta})(x).$$

Hence we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^N} &|\nabla_{k,\rho} u(x,\rho)|^2 \rho^{1-s} h_k^2(x) dx d\rho \\ &\geqslant 2 \frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})} \frac{\Gamma(\frac{N+s}{2} + \gamma_k)}{\Gamma(\frac{N-s}{2} + \gamma_k)} \int_{\mathbb{R}^N} u^2(x,0) \frac{\delta^s \varphi *_k u_{s,\delta}(x)}{\varphi *_k u_{-s,\delta}(x)} d\mu_k(x).\end{aligned}$$

Now we take $\varphi(x) = \psi(x)|x|^{-r}$, $0 < r < N + 2\gamma_k$,where $\psi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$. Then as $\delta \to 0$

$$\delta^s \varphi *_k u_{s,\delta}(x) \to a_N(s)\varphi(x) = a_N(s)\psi(x)|x|^{-r},$$

where $a_N(s)$ is given by (2.4.7)

Let ψ has the further property that $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ on the support of $u(x,0)$. On the other hand

$$\begin{aligned}\lim_{\delta\to 0} &\varphi *_k u_{-s,\delta}(x) \\ &= \int_{\mathbb{R}^N} \psi(y)|y|^{-r} \tau_y(|.|^{-N-2\gamma_k+s})(-x) h_k^2(y) dy \\ &\leqslant \int_{\mathbb{R}^N} |y|^{-r} \tau_y(|.|^{-N-2\gamma_k+s})(-x) h_k^2(y) dy = |x|^{-r} *_k |x|^{-N-2\gamma_k+s}.\end{aligned}$$

Now using the Plancheral theorem and Lemma 4.1 of [38]

$$\begin{aligned}\lim_{\delta\to 0}\varphi *_k u_{-s,\delta}(x) &\leqslant \mathcal{F}_k^{-1}\Big(\mathcal{F}_k\Big(|.|^{-r} *_k |.|^{-N-2\gamma_k+s}\Big)\Big)(x)\\ &= \mathcal{F}_k^{-1}\left(\frac{\Gamma(\frac{N+2\gamma_k-r}{2})\Gamma(s/2)}{2^{r-s}\Gamma(\frac{N+2\gamma_k-s}{2}\Gamma(r/2))}|.|^{-N-2\gamma_k-(r-s)}\right)(x)\\ &= D_k(N,r,s)|x|^{-(r-s)},\end{aligned}$$

where

$$D_k(N,r,s) = \frac{2^{-\frac{N}{2}-\gamma_k}\Gamma(\frac{r-s}{2})\Gamma(\frac{N+2\gamma_k-r}{2})\Gamma(\frac{s}{2})}{\Gamma(\frac{N+2\gamma_k-(r-s)}{2})\Gamma(\frac{N+2\gamma_k-s}{2})\Gamma(\frac{r}{2})}. \tag{2.4.9}$$

Using this now we have

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^N}|\nabla_{k,\rho}u(x,\rho)|^2\rho^{1-s}h_k^2(x)dxd\rho \geqslant 2\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)}\frac{a_N(s)}{D_k(N,r,s)}\\ \int_{\mathbb{R}^N}\frac{|u(x,0)|^2\psi(x)|x|^{-r}}{|x|^{-(r-s)}}d\mu_k(x).\end{aligned}$$

Choose $r = \frac{N+2\gamma_k+s}{2}$ and use the fact that $\psi = 1$ on the support of $u(x,0)$, we get the desired inequality

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}|\nabla_{k,\rho}u(x,\rho)|^2\rho^{1-s}d\mu_k(x)d\rho\\ &\qquad\geqslant 2c_h^{-1}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})}\left(\frac{\Gamma(\frac{N+2\gamma_k+s}{4})}{\Gamma(\frac{N+2\gamma_k-s}{4})}\right)^2\int_{\mathbb{R}^N}\frac{|u(x,0)|^2}{|x|^s}d\mu_k(x).\end{aligned}$$

□

Remark 2.4.5. If we put $f = u_{-s,\delta}$ in equation (2.4.8) we will get,

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_{k,\rho} u(x,\rho)|^2 \rho^{1-s} h_k^2(x) dx d\rho$$
$$\geqslant 2\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s}{2})} \frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)} \delta^{2s} \int_{\mathbb{R}^N} |u(x,0)|^2 \frac{u_{s,\delta}(x)}{u_{-s,\delta}(x)} d\mu_k(x).$$

It can be verified that the function $f(x) = u_{-s,\delta}(x)$ will optimize inequality. Using the above inequality and the Theorem 2.4.2 we get the following type of Hardy type inequality with non-homogeneous weight,

$$\langle \Delta_k^{s/2} f, f \rangle \geqslant 2^s \frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)} (\delta)^{2s} \int_{\mathbb{R}^N} \frac{f(x)^2}{(\delta^2+|x|^2)^s} d\mu_k(x).$$

The constant is sharp since we obtain the equality for the functions $f = u_{-s,\delta}$.

2.5 Sharp Fractional Hardy Inequality for the Dunkl Laplacian with Homogeneous Weight

We have already proven the Hardy inequality for Dunkl fractional Laplacian with non-homogeneous weight as a corollary of trace Hardy inequality. In this section we will prove the fractional Hardy inequality for Dunkl Laplacian when the weight function is homogeneous. We adopt the techniques from [35].

Let $x \in \mathbb{R}^N$ and $t > 0$ let G_t^k denotes the Dunkl heat kernel on $\mathbb{R}^N$, that is,

$$G_t^k(x) = \frac{1}{(2t)^{\gamma_k + N/2}} e^{\frac{-|x|^2}{4t}}.$$

For a function good enough the heat semigroup $e^{-t\Delta_k}$ is defined as the convolution

$G_t^k *_k f(x)$. Now,

$$e^{-t\Delta_k} f(x) = \int_{\mathbb{R}^N} f(y)\tau_y G_t^k(-x)d\mu_k(y)$$

and also $e^{-t\Delta_k}1 = 1$.

For $0 < s < 1$. We define another kernel $\mathcal{G}_k^s$ by

$$\mathcal{G}_s^k(x) = \frac{1}{|\Gamma(-s)|}\int_0^\infty G_t^k(x)t^{-s-1}dt.$$

Let $0 < \alpha < N/2 + \gamma_k$ and we define

$$g_\alpha^k(x) = \frac{1}{\Gamma(\alpha)2^{\frac{N}{2}+\gamma_k}}\int_o^\infty e^{\frac{-|x|^2}{4t}} t^{\alpha-1-N/2-\gamma_k} dt$$

Lemma 2.5.1. *Let $N \geqslant 1$. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < N/2 + \gamma_k$. We write for any $x \in \mathbb{R}^N$*

$$g_\alpha^k(x) = \frac{\Gamma(N/2 + \gamma_k - \alpha)}{\Gamma(\alpha).2^{2\alpha-\frac{N}{2}-\gamma_k}}|x|^{2\alpha-2\gamma_k-N}.$$

Then $\mathcal{G}_s^k$ can be expressed as,

$$\mathcal{G}_s^k(x) = \frac{4^s\Gamma(N/2 + \gamma_k + s)}{\Gamma(-s)2^{-\frac{N}{2}-\gamma_k}}|x|^{-2s-2\gamma_k-N}.$$

Also by using Lemma 4.1 of [38] we get $\mathcal{F}_k(g_\alpha^k)(\xi) = |\xi|^{-2\alpha}$.

Proposition 2.5.2. *Let $N \geqslant 1$ and $0 < s < 1$. Then, for all $f \in \mathcal{S}$, we have the following point wise representation*

$$\Delta_k^s f(x) = P.V \int_{\mathbb{R}^N} (f(x) - f(y))\tau_y \mathcal{G}_s(-x)d\mu_k(y)dy.$$

Proof. Using the definition of $e^{-t\Delta_k}$ and the fact that $e^{-t\Delta_k}1 = 1$ we have

$$e^{-t\Delta_k}f(x) - f(x) = e^{-t\Delta_k}f(x) - f(x)e^{-t\Delta_k}1(x)$$

$$= \int_{\mathbb{R}^N} \tau_y G_t^k(x) f(y) d\mu_k(y) - f(x) \int_{\mathbb{R}^N} \tau_y G_t^k(-x) d\mu_k(y).$$

Recall the definition of fractional power of Laplacian motivated by the numerical identity

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}, \lambda > 0.$$

Now we have

$$\begin{aligned}
\Delta_k^s f(x) \quad &= \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\Delta_k} f(x) - f(x)) \frac{dt}{t^{1+s}} \\
&= \frac{1}{\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^N} \tau_y G_t^k(-y)(f(y) - f(x)) dy \frac{dt}{t^{1+s}} \\
&= \frac{1}{\Gamma(-s)} \int_{\mathbb{R}^N} (f(y) - f(x)) \left(\int_o^\infty \tau_y G_t^k(-y) \frac{dt}{t^{1+s}} \right) d\mu_k(y) \\
&= \int_{\mathbb{R}^N} (f(x) - f(y)) \tau_y \mathcal{G}_s(-x) d\mu_k(y).
\end{aligned}$$

□

Lemma 2.5.3. *Let $N > 1$ and $0 < s < 1$ be such that $N/2 + \gamma_k > s$. Then, for $f \in C_0^\infty(\mathbb{R}^N)$*

$$\langle \Delta_k^s f, f \rangle = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x) - f(y)|^2 \tau_y \mathcal{G}_s^k(x) d\mu_k(x) d\mu_k(y).$$

Proof. Since $\tau_y \mathcal{G}_s(x) = \tau_{-x}\mathcal{G}_s(-y)$, it follows that

$$\begin{aligned}
\langle \Delta_k^s f, f\rangle &= \int_{\mathbb{R}^N} \Delta_k^s f(x) f(x) d\mu_k(x) \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (f(x)-f(y))\tau_y\mathcal{G}_s(-x) f(x) d\mu_k(y) d\mu_k(x) \\
&= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (f(x)-f(y))\tau_y\mathcal{G}_s(-x) f(y) d\mu_k(x) d\mu_k(y) \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x)-f(y)|^2\tau_y\mathcal{G}_s(-x) d\mu_k(x) d\mu_k(y).
\end{aligned}$$

□

Let the corresponding ground level representation $H_s^k[f]$ for f is given by

$$H_s^k[f] = \langle \Delta_k^s f, f\rangle - E_{N,s}\int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^{2s}} d\mu_k(x),$$

where $E_{N,s}$ is given by

$$E_{N,s} = 4^s\left(\frac{\Gamma(\frac{N}{4}+\frac{\gamma_k}{2}+\frac{s}{2})}{\Gamma(\frac{N}{4}+\frac{\gamma_k}{2}-\frac{s}{2})}\right)^2.$$

Now, if we prove $H_s^k[f]$ is positive then it is done.

Theorem 2.5.4. *Let $0<s<1, s<N/2+\gamma_k$ and $\alpha > s$. If $u \in C_0^\infty(\mathbb{R}^N)$ and $v(x) = u(x)(g_\alpha^k(x))^{-1}$. Then*

$$H_s^k[u] = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)|^2 \tau_y\mathcal{G}_s^k(x) g_\alpha^k(x) g_\alpha^k(y) d\mu_k(x) d\mu_k(y).$$

Proof. Polarize the expression given in Lemma 2.5.3 and obtain for any $f, g \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \Delta_k^s f, g\rangle = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (f(x)-f(y))(g(x)-g(y))\tau_y\mathcal{G}_s^k(-x) d\mu_k(x) d\mu_k(y). \quad (2.5.1)$$

We take $g(x) = g^k_\alpha(x)$ and $f(x) = |u(x)|^2(g^k_\alpha(x))^{-1}$. By Lemma 4.1 of [38] and Plancheral theorem for the Dunkl transform, the left hand side of the above Equality (2.5.1) become,

$$\begin{aligned}\langle \Delta^s_k f, g\rangle &= \int_{\mathbb{R}^N} |\xi|^{2s}\mathcal{F}_k(f)(\xi)\mathcal{F}_k(g)(\xi)d\mu_k(\xi)\\ &= \int_{\mathbb{R}^N} \mathcal{F}_k(f)(\xi)|\xi|^{-2(\alpha-s)}d\mu_k(\xi)\\ &= \int_{\mathbb{R}^N} f(x)g^k_{\alpha-s}(x)d\mu_k(x)\\ &= \int_{\mathbb{R}^N} |u(x)|^2\frac{g^k_{\alpha-s}(x)}{g^k_\alpha(x)}d\mu_k(x). \end{aligned} \tag{2.5.2}$$

Substituting f and g in the right hand side of (2.5.1) and see that,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(f(x)-f(y))(g(x)-g(y))\tau_y\mathcal{G}^k_s(-x)d\mu_k(x)d\mu_k(y)\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(|u(x)-u(y)|^2 - \left|\frac{u(x)}{g^k_\alpha(x)} - \frac{u(y)}{g^k_\alpha(y)}\right| g^k_\alpha(x)g^k_\alpha(y)\right)\tau_y\mathcal{G}^k_s(-x)d\mu_k(x)d\mu_k(y).\end{aligned} \tag{2.5.3}$$

Now we use the Equations (2.5.1), (2.5.3), Lemma 2.5.1 and Lemma 2.5.3 to get the required result. □

Corollary 2.5.5. *Let $N \geqslant 1$ and $0 < s < 1$ be such that $N/2 + \gamma_k > s$. Then for $f \in C^\infty_0(\mathbb{R}^N)$ we have*

$$E_{N,s}\int_{\mathbb{R}^N}\frac{|f(x)|^2}{|x|^{2s}}d\mu_k(x) \leqslant \langle \Delta^s_k f, f\rangle,$$

where the constant $E_{N,s}$ is given above.

Proof. By Lemma 2.5.1 and Theorem 2.5.4 we can write, for $\alpha > s$,

$$\begin{aligned}\langle \Delta_k^s f, f\rangle &\geqslant \int_{\mathbb{R}^N} |f(x)|^2 \frac{g_{\alpha-s}^k(x)}{g_\alpha^k(x)} d\mu_k(x) \\ &= \frac{4^s\Gamma(N/2+\gamma_k-\alpha+s)\Gamma(\alpha)}{\Gamma(\alpha-s)\Gamma(N/2+\gamma_k-\alpha)} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^{2s}} d\mu_k(x).\end{aligned}$$

Now choose $\alpha = \frac{N}{4} + \frac{\gamma_k}{2} + \frac{s}{2}$ and obtain the Hardy inequality. □

Remark 2.5.6. It is easy to see from the ground state representation $H_s^k[f]$ that the constant $E_{N,s}$ is sharp. The sharpness in the classical Euclidean case is discussed in [14]. Considering the functions which are converging to $|x|^{-\frac{N-2s}{2}-\gamma_k}$ and applying the limit in Theorem 2.5.4, we obtain the optimality for the Dunkl case.

2.6 Fractional Hardy Inequality for Half-space and Cone

Let $(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}^3$ and R be a root system on $\mathbb{R}^{N-1}$. Now let R_1 and R_2 be two root systems on $\mathbb{R}^{N+2}$ and $\mathbb{R}^N$ respectively defined as $R_1 = \{(x, 0) \in \mathbb{R}^{N+2} : x \in R\}$ and $R_2 = \{(x, 0) \in \mathbb{R}^N : x \in R\}$. Let $\tilde{\Delta}_{k_1}$ be the Dunkl Laplacian on $\mathbb{R}_x^{N-1} \times \mathbb{R}_y^3$ according to the root system R_1 which given by $\tilde{\Delta}_{k_1} = \Delta_k + \sum_{j=1}^3 \frac{\partial^2}{\partial y_j^2}$, where Δ_k is the Dunkl Laplacian on $\mathbb{R}^{N-1}$. Similarly the Dunkl Laplacian on $\mathbb{R}^N$ with respect to the root system R_2 is given by $\tilde{\Delta}_{k_2} = \Delta_k + \frac{\partial^2}{\partial x_N^2}$.

Theorem 2.6.1. *Let $u \in C_0^\infty(\mathbb{R}_+^N)$ and G-invariant. Also let $0 < s < 1$ and $N/2 + \gamma_k > s$. We have*

$$\langle \tilde{\Delta}_{k_2}^{s/2} u, u\rangle_{\mathbb{R}_+^N} \geqslant \frac{\Gamma(\frac{N+2+s}{2}+\gamma_k)}{\Gamma(\frac{N+2-s}{2}+\gamma_k)} \int_{\mathbb{R}_+^N} \frac{u(x, x_N)^2}{(1+|x|^2+x_N^2)^s} d\mu_k(x) dx_N.$$

Proof. Let us begin with the following calculation

$$\begin{aligned}
&\int_{\mathbb{R}_x^{N-1}\times\mathbb{R}_y^3} \tilde{\Delta}_{k_1}^{s/2} v(x,y).v(x,y)d\mu_k(x)d\mu_k(y) \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-1}\times\mathbb{R}_y^3} \left(e^{-t\tilde{\Delta}_{k_1}} v(x,y) - v(x,y)\right) v(x,y)d\mu_k(x)dy\frac{dt}{t^{1+\frac{s}{2}}} \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-1}\times\mathbb{R}_y^3} \left(e^{-t(\Delta_k+\sum_{j=1}^3 \frac{\partial^2}{\partial y_j^2})} v(x,y) - v(x,y)\right) v(x,y)d\mu_k(x)dy\frac{dt}{t^{1+\frac{s}{2}}} \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-1}\times\mathbb{R}_y^3} \left(e^{-t(\Delta_k+\frac{\partial^2}{\partial x_N^2}+\frac{2}{x_N}\frac{\partial}{\partial x_N})} v(x,y) - v(x,y)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad v(x,y)d\mu_k(x)dy\frac{dt}{t^{1+\frac{s}{2}}}.
\end{aligned}$$

We can directly calculate that

$$\left(\Delta_k + \frac{\partial^2}{\partial x_N^2} + \frac{2}{x_N}\frac{\partial}{\partial x_N}\right)^m v(x,y) = x_N^{-1}\left(\Delta_k + \frac{\partial^2}{\partial x_N^2}\right)^m x_N v(x,y).$$

for every $m \in \mathbb{N}$. So

$$e^{-t(\Delta_k+\frac{\partial^2}{\partial x_N^2}+\frac{2}{x_N}\frac{\partial}{\partial x_N})} v(x,y) = x_N^{-1} e^{-t(\Delta_k+\frac{\partial^2}{\partial x_N^2})} x_N v(x,y).$$

Further, assign that $v(x,y) = v(x,|y|)$, $|y| = x_N$, $u = u(x_1,...,x_N) = x_N v(x,x_N)$ and use the fractional Hardy inequality given in the Corollary 2.4.3 to obtain the desired Hardy inequality for the half-space.

$$\begin{aligned}
&\int_{\mathbb{R}_x^{N-1}\times\mathbb{R}_y^3} \tilde{\Delta}_{k_1}^{s/2} v(x,y).v(x,y)d\mu_k(x)dy \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-1}\times\mathbb{R}_y^3} x_N^{-1}\Big((e^{-t(\Delta_k+\frac{\partial^2}{\partial x_N^2})} x_N v(x,y) - x_N v(x,y)\Big) v(x,y)d\mu_k(x)dy\frac{dt}{t^{1+\frac{s}{2}}} \\
&= \|\mathbb{S}^3\| \int_0^\infty \int_{\mathbb{R}_+^N} x_N\Big((e^{-t(\Delta_k+\frac{\partial^2}{\partial x_N^2})} x_N v(x_1,...,x_N) - x_N v(x_1,...,x_N)\Big) \\
&\qquad\qquad\qquad v(x_1,...,x_N)d\mu_k(x)dx_N\frac{dt}{t^{1+\frac{s}{2}}} \\
&= \|\mathbb{S}^3\| \int_{\mathbb{R}_+^N} \left(\Delta_k + \frac{\partial^2}{\partial x_N^2}\right)^{s/2} x_N v(x_1,...,x_N).x_N v(x_1,...,x_N)d\mu_k(x)dx_N \\
&= \|\mathbb{S}^3\| \int_{\mathbb{R}_+^N} \tilde{\Delta}_{k_2}^{s/2} u(x_1,...,x_N)u(x_1,...,x_N)d\mu_k(x)dx_N \\
&\geqslant \frac{\Gamma(\frac{N+2+s}{2}+\gamma_k)}{\Gamma(\frac{N+2-s}{2}+\gamma_k)} \int_{\mathbb{R}_x^{N-1}\times\mathbb{R}_y^3} \frac{v(x,y)^2}{(1+|x|^2+|y|^2)^s}d\mu_k(x)dy \\
&\geqslant \|\mathbb{S}^3\|\frac{\Gamma(\frac{N+2+s}{2}+\gamma_k)}{\Gamma(\frac{N+2-s}{2}+\gamma_k)} \int_{\mathbb{R}_+^N} \frac{v(x,y)^2 x_N^2}{(1+|x|^2+x_N^2)^s}d\mu_k(x)dx_N \\
&= \|\mathbb{S}^3\|\frac{\Gamma(\frac{N+2+s}{2}+\gamma_k)}{\Gamma(\frac{N+2-s}{2}+\gamma_k)} \int_{\mathbb{R}_+^N} \frac{u(x,x_N)^2}{(1+|x|^2+x_N^2)^s}d\mu_k(x)dx_N.
\end{aligned}$$

□

Let $(x,y)\in\mathbb{R}^{N-l}\times\mathbb{R}^{3l}$ and R be a root system on $\mathbb{R}^{N-l}$. Now let R_1 and R_2 be two root systems on $\mathbb{R}^{N+2l}$ and $\mathbb{R}^N$ respectively and defined as $R_1 = \{(x,0)\in\mathbb{R}^{N+2l} : x\in R\}$ and $R_2 = \{(x,0)\in\mathbb{R}^N : x\in R\}$. Let $\tilde{\Delta}_{k_1}$ be the Dunkl Laplacian on $\mathbb{R}_x^{N-l}\times\mathbb{R}_y^{3l}$ according to the root system R_1 which is given by $\tilde{\Delta}_{k_1} = \Delta_k + \sum_{j=1}^{3l}\frac{\partial^2}{\partial y_j^2}$, where Δ_k is the Dunkl Laplacian on $\mathbb{R}^{N-l}$. Similarly the Dunkl Laplacian on $\mathbb{R}^N$ with respect to the root system R_2 is given by $\tilde{\Delta}_{k_2} = \Delta_k + \sum_{j=N-l+1}^{N}\frac{\partial^2}{\partial x_j^2}$.

Theorem 2.6.2. *Let $0<s<1$ and $N/2+\gamma_k>s$. For any G-invariant function* *u such that* $u \in C_0^\infty(\mathbb{R}^N_{l_+})$*, the following inequality holds*

$$\begin{aligned}
&\langle \tilde{\Delta}_{k_2}^{s/2} u, u\rangle_{\mathbb{R}^N_{l_+}} \\
&\geqslant \frac{\Gamma(\frac{N+2l+s}{2}+\gamma_k)}{\Gamma(\frac{N+2l-s}{2}+\gamma_k)} \int_{\mathbb{R}^N_{l_+}} \frac{u^2}{(1+|x|^2+x_{N-l+1}^2+...+x_N^2)^s} d\mu_k(x) dx_{N-l+1}...dx_N.
\end{aligned}$$

Proof. Let $v \in C_0^\infty(\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l})$,

$$\begin{aligned}
&\int_{\mathbb{R}_x^{N-l}\times\mathbb{R}_y^{3l}} \tilde{\Delta}_{k_1}^{s/2} v(x,y).v(x,y) h_k^2(x) h_k^2(y) dx dy \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-l}\times\mathbb{R}_y^{3l}} \left(e^{-t\tilde{\Delta}_{k_1}} v(x,y) - v(x,y)\right) v(x,y) d\mu_k(x) dy \frac{dt}{t^{1+\frac{s}{2}}} \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-l}\times\mathbb{R}_y^{3l}} \left(e^{-t(\Delta_k + \sum_{j=1}^{3l} \frac{\partial^2}{\partial y_j^2})} v(x,y) - v(x,y)\right) v(x,y) d\mu_k(x) dy \frac{dt}{t^{1+\frac{s}{2}}} \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-l}\times\mathbb{R}_y^{3l}} \left(e^{-t(\Delta_k + \sum_{j=N-l+1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{2}{x_j}\frac{\partial}{\partial x_j})} v(x,y) - v(x,y)\right) \\
&\qquad\qquad v(x,y) d\mu_k(x) dy \frac{dt}{t^{1+\frac{s}{2}}}.
\end{aligned}$$

As in the previous theorem by taking the positive integer powers of $\Delta_k + \frac{\partial^2}{\partial x_N^2} + \frac{2l}{x_N}\frac{\partial}{\partial x_N}$ we can verify that

$$e^{-t(\Delta_k + \sum_{j=N-l+1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{2}{x_j}\frac{\partial}{\partial x_j})} v(x,y) = \prod_{i=N-l+1}^{N} x_i^{-1} e^{-t(\Delta_k + \sum_{j=N-l+1}^{N} \frac{\partial^2}{\partial x_j^2})} \tilde{v}(x,y),$$

where $\tilde{v}(x,y) = \left(\prod_{i=N-l+1}^N x_i v(x,y)\right)$. Assume that $v(x,y) = v(x, x_{N-l+1}, ..., x_N)$ with $x_{N-l+j} = \sqrt{y_{3j-2}^2 + y_{3j-1}^2 + y_{3j}^2}$ for $1 \leqslant j \leqslant l$.

Furthermore, put $u = u(x_1,...,x_N) = \prod_{i=N-l+1}^N x_i v(x_1,...,x_N)$ and use the Corollary 2.4.3 for the functions in $\mathbb{R}_x^{N-l} \times \mathbb{R}_y^{3l}$.

$$
\begin{aligned}
&\int_{\mathbb{R}_x^{N-l}\times\mathbb{R}_y^{3l}} \tilde{\Delta}_{k_1}^{s/2} v(x,y).v(x,y)d\mu_k(x)dy \\
&= \int_0^\infty \int_{\mathbb{R}_x^{N-l}\times\mathbb{R}_y^{3l}} \prod_{i=N-l+1}^{N} x_i^{-1}\Big(e^{-t(\tilde{\Delta}_{k_2})}\tilde{v}(x,y) - \tilde{v}(x,y)\Big)v(x,y)d\mu_k(x)dy\frac{dt}{t^{1+\frac{s}{2}}} \\
&= \|\mathbb{S}^3\|^l \int_0^\infty \int_{\mathbb{R}^N_{l_+}} \prod_{i=N-l+1}^{N} x_i\Big(e^{-t(\tilde{\Delta}_{k_2})}\tilde{v}(x,y) - \tilde{v}(x,y)\Big)v(x,y)d\mu_k(x)dy\frac{dt}{t^{1+\frac{s}{2}}} \\
&= \|\mathbb{S}^3\|^l \int_{\mathbb{R}^N_{l_+}} \tilde{\Delta}_{k_2}^{s/2}\Big(\prod_{i=n-l+1}^{N} x_i v(x_1,...,x_N)\Big).\prod_{i=N-l+1}^{N} x_i v(x_1,...,x_N)d\mu_k(x)dx_{N-l+1}...dx_N \\
&= \|\mathbb{S}^3\|^l \int_{\mathbb{R}^N_{l_+}} \tilde{\Delta}_{k_2}^{s/2} u(x_1,...,x_N)u(x_1,...,x_N)d\mu_k(x)dx_{N-l+1}...dx_N \\
&\geqslant \frac{\Gamma(\frac{N+2l+s}{2}+\gamma_k)}{\Gamma(\frac{N+2l-s}{2}+\gamma_k)} \int_{\mathbb{R}_x^{N-l}\times\mathbb{R}^{3l}} \frac{v(x,y)^2}{(1+|x|^2+|y|^2)^s}d\mu_k(x)dy \\
&\geqslant \|\mathbb{S}^3\|^l \frac{\Gamma(\frac{N+2l+s}{2}+\gamma_k)}{\Gamma(\frac{N+2k-s}{2}+\gamma_k)} \int_{\mathbb{R}^N_{l_+}} \frac{v(x,y)^2\prod_{i=N-l+1}^{N} x_i^2}{(1+|x|^2+x_{N-l+1}^2+...x_N^2)^s}d\mu_k(x)dx_{N-l+1}...dx_N.
\end{aligned}
$$

□

We have proven Hardy inequality for fractional Dunkl Laplacian on the Half space and cone in the non-homogeneous case. We can prove the Hardy inequality in the homogeneous case with exactly similar arguments by using the Hardy inequality for the fractional Dunkl Laplacian in the homogeneous case.

We will just state fractional Hardy inequality for Dunkl Laplacian with homogeneous weight on the half-space and cone without proof. The same arguments used for non-homogeneous case can be applied. Instead of using the Hardy inequality with non homogeneous weight given in the Corollary 2.4.3 use the homogeneous version given the Corollary 2.5.5 in the proof. Also since the G-invariance is not assumed in Corollary 2.5.5 we don't assume it here either.

Let $\tilde{\Delta}_{k_2}$ be the Dunkl Laplacian on $\mathbb{R}^N$ defined above in the beginning of the

Section 2.6.

Theorem 2.6.3. *Let* $u \in C_0^\infty(\mathbb{R}^N)$ *and* $N/2 + \gamma_k > s$. *Then for* $0 < s < 1$ *we have*

$$\langle \tilde{\Delta}_{k_2}^s u, u\rangle_{\mathbb{R}_+^N} \geqslant 4^s \left(\frac{\Gamma(\frac{N+2}{4} + \frac{\gamma_k}{2} + \frac{s}{2})}{\Gamma(\frac{N+2}{4} + \frac{\gamma_k}{2} - \frac{s}{2})} \right)^2 \int_{\mathbb{R}_+^N} \frac{|u(x, x_N)|^2}{|x|^{2s}} d\mu_k(x) dx_N.$$

As in the Theorem 2.6.2 we use the same notation $\tilde{\Delta}_{k_2}$ for the Laplacian on $\mathbb{R}_{l_+}^N$ with the corresponding root system R_2 explained there.

Theorem 2.6.4. *Let* $u \in C_0^\infty(\mathbb{R}^N)$ *and* $N/2 + \gamma_k > s$. *Then for* $0 < s < 1$ *we have*

$$\langle \tilde{\Delta}_{k_2}^s u, u\rangle_{\mathbb{R}_{l_+}^N} \geqslant 4^s \left(\frac{\Gamma(\frac{N+2l}{4} + \frac{\gamma_k}{2} + \frac{s}{2})}{\Gamma(\frac{N+2l}{4} + \frac{\gamma_k}{2} - \frac{s}{2})} \right)^2 \int_{\mathbb{R}_{l_+}^N} \frac{|u(x, x_N)|^2}{|x|^{2s}} d\mu_k(x) dx_N.$$

Remark 2.6.5. From the Remark 2.4.5 it is clear that the constants in the Theorem 2.4.2, Corollary 2.4.3 are optimal. By the construction of the proof, this optimality is carried to the constants of Theorem 2.6.1 and the Theorem 2.6.2. Also since the constant in the Corollary 2.5.5 is sharp, and so constants appearing in the Theorem 2.6.3 and the Theorem 2.6.4 are optimal.

Chapter 3

L^p Hardy Type Inequalities and Stein-Weiss Inequalities for Dunkl Operators

In this chapter we discuss L^p Hardy inequalities, fractional Hardy inequalities and Stein-Weiss inequalities for the Dunkl gradient. We will first prove a classical L^p Hardy inequality for G-invariant functions with weighted measure. We will adopt the techniques of R. frank and R. Seiringer used in the article [16] to prove fractional Hardy inequalities. As in [16] we also obtain an improved inequality for $p \geqslant 2$. We extend this result to half space and cone by choosing suitable root systems. Also we will prove some Stein-Weiss inequalities in this chapter by using some 'ground state substitution' techniques.

3.1 Introduction

Hardy inequality is of fundamental importance in many areas of mathematical analysis and mathematical physics. A general Hardy inequality is of the form

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geqslant \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx,$$

for $u \in C_0^\infty(\mathbb{R}^N)$ or $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ respectively with respect to $1 \leqslant p < N$ or $p > N$. It is known that the constant $\left(\frac{|N-p|}{p}\right)^p$ is sharp and never attained in the corresponding spaces $\dot{W}_p^1(\mathbb{R}^N)$ or $\dot{W}_p^1(\mathbb{R}^N \setminus \{0\})$ respectively. A lot of work concerning fractional Hardy inequality has been developed in the literature. A remarkable work on the same is done by R.L Frank and R. Seiringer in [16]. They have proven the sharp Hardy inequality with sharp constants as follows: for $p \geqslant 1$, $0 < s < 1$ and $u \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geqslant C_{N,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx,$$

where the constant $C_{N,s,p}$ is sharp. Also they proved the fractional Hardy inequality with remainder term. That is, for $p \geqslant 2$ and $u \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - C_{N,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \\ &\geqslant c_p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{(N-ps)/2}} \frac{dy}{|y|^{(N-ps)/2}}, \end{aligned}$$

where $v := |x|^{(N-ps)/2} u$ and c_p is as in (3.2.19).

The same authors of [16] have proven the fractional Hardy inequality in half-spaces $\mathbb{R}_+^N$ with and without remainder terms in [16], where $\mathbb{R}_+^N = \{x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N : x_N > 0\}$. They have proven that, for some sharp constant

$D_{N,p,s}$

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \geqslant D_{N,p,s}\int_{\mathbb{R}^N_+}\frac{|u(x)|^p}{x_N^{ps}}\,dx,$$

for all $u \in \dot{W}^s_p(\mathbb{R}^N)$ with $ps \neq 1$. Similar to the case of $\mathbb{R}^N$ they obtained an improved fractional Hardy inequality which states for $p \geqslant 2$

$$\begin{aligned}&\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - D_{N,p,s}\int_{\mathbb{R}^N_+}\frac{|u(x)|^p}{x_N^{ps}}\,dx\\ &\geqslant c_p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\frac{dx}{x_N^{(1-ps)/2}}\frac{dy}{y_N^{(1-ps)/2}},\end{aligned}$$

where $v := x_N^{(1-ps)/p}u$ and c_p is given in (3.2.19).

Our aim in this chapter is to prove both Hardy and fractional Hardy inequality in Dunkl setting. We cite few papers in which authors studied some of the related inequalities in Dunkl setting. Pitts inequality for fractional Dunkl operator is studied by D. V. Gorbachev et al. in [18]. F. Soltani et al. have proven certain inequalities, namely Stein-Weiss inequality, Hardy-Littlewood-Sobolev inequality, uncertainty principles and some Pitts inequalities in the Dunkl setting in the papers [29, 30, 31]. In [10] Óscar Ciaurri et al. studied the Hardy-type inequalities for Dunkl Hermite operator. We mainly adapt the techniques used in [14] to prove the Hardy and fractional Hardy inequalities.

The chapter is organized as follows. In Section 3.2 we prove a generalized version of the classical L^p Hardy inequality in the Dunkl setting. We use the 'ground state substitution' technique to achieve it. For $p \geqslant 2$ we obtain an improved version of Hardy inequality in (3.2.21). In Section 3.3 we obtained an optimal fractional Hardy inequality for the Dunkl Laplacian. As in the Section 3.2 we obtain a fractional Hardy inequality with a remainder term for $p \geqslant 2$. The

Section 3.4 and Section 3.5 deals with similar type of fractional Hardy inequalities on half-space and cone respectively.

3.2 L^p Hardy Inequality

In this section we prove optimal L^p Hardy inequality for $1 \leqslant p < \infty$ and an improved Hardy inequality for $p \geqslant 2$ for $G-$ invariant real valued smooth function having compact support. Also we will prove a generalized L^p Hardy inequality with optimal constant for the same function space. However we can relax the condition $G-$ invariant function for certain case. We define the $p-$Dunkl Laplacian $\Delta_{k,p}$ by $\Delta_{k,p} f = div_k(|\nabla_k f|^{p-2}\nabla_k f)$, where $div_k(f_1, f_2, \cdots, f_N) = \sum_{j=1}^{N} T_j f_j$. We will compute $\Delta_{k,p} w$ for a radial function w which is needed to prove Hardy inequality. For a radial function w

$$\begin{aligned}
& div_k(|\nabla_k w|^{p-2}\nabla_k w) \\
&= \sum_{j=1}^{N} T_j\big(|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\big) = \sum_{j=1}^{N}(\partial_j + E_j)\big(|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\big) \\
&= \sum_{j=1}^{N}\Bigg((p-1)|w'(r)|^{p-2}w''(r)\big(\frac{x_j}{r}\big)^2 \\
&\qquad\qquad + |w'(r)|^{p-2}w'(r)\big(\frac{1}{r} - \frac{1}{r^2}\frac{x_j^2}{r}\big)\Bigg) + \frac{|w'(r)|^{p-2}w'(r)}{r}\sum_{j=1}^{N} E_j(x_j) \\
&= (p-1)|w'(r)|^{p-2}w''(r) + \big(\frac{N-1}{r} + 2\gamma_k\big)|w'(r)|^{p-2}w'(r).
\end{aligned}$$

Hence for a radial function w we have

$$\Delta_{k,p} w = (p-1)|w'(r)|^{p-2}w''(r) + \left(\frac{d_k - 1}{r}\right)|w'(r)|^{p-2}w'(r). \tag{3.2.1}$$

Theorem 3.2.1. *Let $1 \leqslant p < \infty$. Let u be a real valued G-invariant function.*

If $u \in C_0^\infty(\mathbb{R}^N)$ if $d_k > p$ and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_k < p$ then the following inequality holds:

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^p d\mu_k(x) \geqslant \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} d\mu_k(x). \tag{3.2.2}$$

The constant $\left|\frac{d_k-p}{p}\right|^p$ given in the inequality is optimal.

Proof. Let w be a positive radial function and let v be a G-invariant real valued function with $u = vw$. Use the inequality for real numbers a and b and for $p \geqslant 1$, $|a+b|^p \geqslant |a|^p + p|a|^{p-2}a.b$, we obtain

$$\begin{aligned} |\nabla_k u|^p &= |\nabla_k (vw)|^p \\ &= |v\nabla_k w + w\nabla_k v|^p \\ &\geqslant |v|^p|\nabla_k w|^p + p|v|^{p-2}|\nabla_k w|^{p-2} vw \nabla_k v.\nabla_k w. \end{aligned} \tag{3.2.3}$$

Since w is radial we write $w(x) = w(r)$ with $r = |x|$ and denote the derivatives as $w'(r) = \frac{dw}{dr}$ and $w''(r) = \frac{d^2w}{dr^2}$. First we will prove an inequality of the form

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geqslant \int_{\mathbb{R}^N} V|u|^p d\mu_k(x) \tag{3.2.4}$$

for the given radial function w and a function V, where w is a weak solution of the following equation

$$div_k\left(|\nabla_k w|^{p-2}\nabla_k w\right) + Vw^{p-1} = 0. \tag{3.2.5}$$

After proving the inequality (3.2.4) for the functions which satisfy (3.2.5), we will look for some explicit V and w which provide us the Hardy inequality.

In order to estimate the integral $\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x)$ we estimate the integral of each term in the right hand side of (3.2.3).

We start with

$$\begin{aligned}\int_{\mathbb{R}^N} |v|^p |\nabla_k w|^p d\mu_k(x) &= \int_{\mathbb{R}^N} |v|^p |\nabla_k w|^{p-2} \Bigg(\sum_{j=1}^N T_j w T_j w \Bigg) d\mu_k(x) \quad (3.2.6)\\ &= \sum_{j=1}^N \int_{\mathbb{R}^N} |v|^p |\nabla_k w|^{p-2} T_j w T_j w d\mu_k(x)\\ &= -\sum_{j=1}^N \int_{\mathbb{R}^N} w T_j \Bigg(|v|^p |\nabla_k w|^{p-2} T_j w \Bigg) d\mu_k(x).\end{aligned}$$

Let ∇_0 be the Eucledian gradient. Calculating $T_j\big(|v|^p|\nabla_k w|^{p-2}T_j w\big)$ separately, we obtain

$$\begin{aligned}T_j\Bigg(|v|^p|\nabla_k w|^{p-2}T_j w\Bigg) &= (\partial_j + E_j)\Bigg(|v|^p|\nabla_0 w|^{p-2}\partial_j w\Bigg) \quad (3.2.7)\\ &= \Bigg(p|v|^{p-1}\partial_j v\Bigg)|\nabla_o w|^{p-2}\partial_j w + |v|^p\partial_j\Bigg(|\nabla_0 w|^{p-2}\partial_j w\Bigg)\\ &\quad + E_j\Bigg(|v|^p|w'(r)|^{p-2}\frac{w'(r)}{r}x_j\Bigg).\end{aligned}$$

Since $\frac{|w'(r)|^{p-2}w'(r)}{r}$ is radial we can write

$$E_j\Bigg(\frac{|w'(r)|^{p-2}w'(r)}{r}|v|^p x_j\Bigg) = \frac{|w'(r)|^{p-2}w'(r)}{r}E_j(|v|^p x_j). \quad (3.2.8)$$

Using the definition of E_j and reflection one can easily calculate

$$\sum_{j=1}^N E_j(|v|^p x_j) = \sum_{\alpha\in R_+} k(\alpha)\big[|v(x)|^p + |v(\sigma_\alpha(x))|^p\big]. \quad (3.2.9)$$

Substituting (3.2.7), (3.2.8) and (3.2.9) in (3.2.6) and denoting the Euclidean divergence as div_0,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |v|^p |\nabla_k w|^p d\mu_k(x) \qquad (3.2.10)\\
&= -p\int_{\mathbb{R}^N} w|v|^{p-1}|\nabla_0 w|^{p-2}\nabla_0 v.\nabla_0 w d\mu_k(x)\\
&\quad -\int_{\mathbb{R}^N} w|v|^p div_0(|\nabla_0 w|^{p-2}\nabla_0 w) d\mu_k(x)\\
&\quad -\sum_\alpha k(\alpha)\int_{\mathbb{R}^N} \frac{w(r)|w'(r)|^{p-2}w'(r)}{r}\big(|v(x)|^p + |v(\sigma_\alpha x)|^p\big) d\mu_k(x).
\end{aligned}$$

Since radial functions and the Dunkl measure are invariant under reflection, a change of variable in the third integral on the right-hand side gives us

$$\begin{aligned}
&\int_{\mathbb{R}^N} |v|^p |\nabla_k w|^p d\mu_k(x) \qquad (3.2.11)\\
= &\ -p\int_{\mathbb{R}^N} w|v|^{p-2}v|\nabla_0 w|^{p-2}\nabla_0 v.\nabla_0 w d\mu_k(x)\\
&\ -\int_{\mathbb{R}^N} w|v|^p div_0(|\nabla_0 w|^{p-2}\nabla_0 w) d\mu_k(x)\\
&\ -2\gamma_k\int_{\mathbb{R}^N} \frac{|w'(r)|^{p-2}w'(r)w(r)}{r}|v(x)|^p d\mu_k(x).
\end{aligned}$$

Since w is radial we can write from (3.2.1)

$$div_k\Big(|\nabla_k w|^{p-2}\nabla_k w\Big) = div_0\Big(|\nabla_0 w|^{p-2}\nabla_0 w\Big) + 2\gamma_k\frac{|w'(r)|^{p-2}w'(r)}{r}.$$

Now we can write the above equation (3.2.11) as

$$\begin{aligned}
&\int_{\mathbb{R}^N} |v|^p |\nabla_k w|^p d\mu_k(x)\\
&= -p\int_{\mathbb{R}^N} w|v|^{p-2}v\nabla_0 v.\nabla_0 w|\nabla_0 w|^{p-2} d\mu_k(x)\\
&\qquad -\int_{\mathbb{R}^N} w(x)|v(x)|^p div_k\big(|\nabla_k w|^{p-2}\nabla_k w\big) d\mu_k(x).
\end{aligned}$$

Consider the second term on the right-hand side of (3.2.3) and integrating

$$
\begin{aligned}
& p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v.\nabla_k w d\mu_k(x)\\
&= p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x)\\
&\qquad +p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_0 w|^{p-2} vw\frac{w'(r)}{r}\Big(\sum_{j=1}^{N} E_j(v)x_j\Big)d\mu_k(x).
\end{aligned}
$$

Using the definition of E_j we find that

$$
\sum_{j=1}^{N} E_j(v)x_j = \sum_{\alpha\in R_+} k(\alpha)\big(v(x)-v(\sigma_\alpha x)\big).
$$

Since v is $G-$invariant we can write

$$
\begin{aligned}
& p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v.\nabla_k w d\mu_k(x) \qquad (3.2.12)\\
&= p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_0 w|^{p-2} vw\nabla_0 v.\nabla_k w d\mu_k(x)\\
&+p\int_{\mathbb{R}^N} |v|^{p-2}||\nabla_0 w|^{p-2} vw\frac{w'(r)}{r}\sum_{\alpha\in R_+}(k(\alpha)(v(x)-v(\sigma_\alpha x))d\mu_k(x)\\
&= p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x).
\end{aligned}
$$

Substituting all the above calculated estimations and integrals to the inequality (3.2.3),

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla_k(vw)|^p d\mu_k(x) \;\geqslant\; & -p\int_{\mathbb{R}^N} w|v|^{p-2}v\nabla_0 w.\nabla_0 v|\nabla_0 w|^{p-2}d\mu_k(x)\\
& - \int_{\mathbb{R}^N} w(x)|v(x)|^p div_k\Big(|\nabla_k w|^{p-2}\nabla_k w\Big)d\mu_k(x)\\
& + p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x).
\end{aligned}
$$

That is, we end up with

$$\int_{\mathbb{R}^N} |\nabla_k(vw)|^p d\mu_k(x) \geqslant -\int_{\mathbb{R}^N} w(x)|v(x)|^p div_k\Big(|\nabla_k w|^{p-2}\nabla_k w\Big) d\mu_k(x). \tag{3.2.13}$$

Now if w is a weak solution of the equation

$$div_k\Big(|\nabla_k w|^{p-2}\nabla_k w\Big) + V w^{p-1} = 0$$

for some function V, the above inequality (3.2.13) becomes

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geqslant \int_{\mathbb{R}^N} V|u|^p d\mu_k(x).$$

Now we will choose a w and V explicitly to obtain the desired Hardy inequality.

Let us choose $w(x) = |x|^{-(d_k-p)/p}$, that is $w(r) = r^{-(d_k-p)/p}$. By a straightforward calculation we get $w'(r) = -\frac{(d_k-p)}{p} r^{-(d_k-p)/p-1}$ and $w''(r) = \big(\frac{(d_k-p)}{p}\big)\big(\frac{(d_k-p)}{p}+1\big) r^{-((d_k-p))/p)-2}$. Using the Dunkl p-Laplacian for radial functions given in (3.2.1) we find that for $r \neq 0$

$$\Delta_{k,p} w(r) = -\left|\frac{d_k-p}{p}\right|^p r^{-\big(\big(\frac{(d_k-p)}{p}\big)(p-1)+p\big)}.$$

Choose $V(x) = \left|\frac{d_k-p}{p}\right|^p |x|^{-p}$ then w is a weak solution of $\Delta_{k,p} w = -V w^{p-1}$. Substituting V and w in (3.2.4) and obtain the desired Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geqslant \left|\frac{d_k-p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x).$$

To prove the optimality consider the functions u_ϵ below and take the limit as

$\epsilon \to 0$;

$$u_\epsilon(x) = \begin{cases} 1, & \text{if} |x| \leqslant 1 \\ |x|^{-\frac{|d_k - p|}{p} - \epsilon}, & \text{if } |x| > 1. \end{cases}$$

□

Remark 3.2.2. 1. We assumed that the function u in the Theorem 3.2.1 is $G-$invariant. Assume that $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ and $u = vw$ with some v and a radial function w with $w'(r) \geqslant 0$. Now by using the Hölder's inequality we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} |v|^{p-2} v(x) v(\sigma_\alpha x) \frac{w(r)w'(r)}{r} |\nabla_0 w|^{p-2} d\mu_k(x) \\ &= \int_{\mathbb{R}^N} |v|^{p-2} v(x) w(r) \frac{w'(r)|w'(r)|^{p-2}}{r} v(\sigma_\alpha x) d\mu_k(x) \\ &= \int_{\mathbb{R}^N} \left(\frac{|w'(r)|^{p-2} w'(r) w(r)}{r} \right)^{\frac{p-1}{p}} v(x)|v|^{p-2} \\ &\qquad \left(\frac{|w'(r)|^{p-2} w'(r) w(r)}{r} \right)^{\frac{1}{p}} v(\sigma_\alpha x) d\mu_k(x) \\ &\leqslant \left(\int_{\mathbb{R}^N} \frac{|w'(r)|^{p-1}}{r} |v(x)|^p d\mu_k(x) \right)^{\frac{p-1}{p}} \\ &\qquad \left(\int_{\mathbb{R}^N} \frac{|w'(r)|^{p-1} w(r)}{r} |v(\sigma_\alpha x)|^p d\mu_k(x) \right)^{\frac{1}{p}}. \end{aligned} \tag{3.2.14}$$

Therefore we conclude that

$$\int_{\mathbb{R}^N} |v|^{p-2} v(x) v(\sigma_\alpha x) \frac{w(r)w'(r)}{r} |\nabla_0 w|^{p-2} d\mu_k(x) \leqslant \int_{\mathbb{R}^N} \frac{|w'(r)|^{p-1} w(r)}{r} |v(x)|^p d\mu_k(x).$$

Using this we can rewrite the equation (3.2.12) as

$$\begin{aligned} p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v\nabla_k w d\mu_k(x) \qquad (3.2.15)\\ \geqslant p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x)\\ + p\gamma_k\int_{\mathbb{R}^N} |v|^{p-2}v^2(x)w(x)\frac{w'(r)}{r}|\nabla_k w|^{p-2}d\mu_k(x)\\ - p\gamma_k\int_{\mathbb{R}^N} |v|^{p}\frac{|w'(r)|}{r}w(x)|\nabla_k w|^{p-2}d\mu_k(x)\\ = p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_0 v.\nabla_0 w d\mu_k(x). \end{aligned}$$

Now by repeating exactly same steps of the proof for Theorem 3.2.1 we get the generalized Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) \geqslant \int_{\mathbb{R}^N} V|u|^p d\mu_k(x)$$

with some function V and w satisfies (3.2.5).

2. Let $w(x) = |x|^{-\frac{d_k - p}{p}}$ with $d_k < p$. Then $w'(r) \geqslant 0$ and by using the Remark 3.2.2(1) we get the Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x) \geqslant \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} |u|^p d\mu_k(x).$$

The above inequality is optimal and it is true for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$.

3. If $w'(r) < 0$ the Equation (3.2.15) will be of the form

$$p\int_{\mathbb{R}^N} |v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v\nabla_k w d\mu_k(x) \qquad (3.2.16)$$

(3.2.17)

$$
\begin{aligned}
&\geqslant p\int_{\mathbb{R}^N}|v|^{p-2}|\nabla_k w|^{p-2}vw\nabla_0 v.\nabla_0 w d\mu_k(x)\\
&\qquad\qquad + p\gamma_k\int_{\mathbb{R}^N}|v|^{p-2}v^2(x)w(x)\frac{w'(r)}{r}|\nabla_k w|^{p-2}d\mu_k(x)\\
&\qquad\qquad - p\gamma_k\int_{\mathbb{R}^N}|v|^{p}\frac{|w'(r)|}{r}w(x)|\nabla_k w|^{p-2}d\mu_k(x)\\
&= p\int_{\mathbb{R}^N}|v|^{p-2}|\nabla_k w|^{p-2}vw\nabla_0 v.\nabla_0 w d\mu_k(x)\\
&\qquad\qquad + 2p\gamma_k\int_{\mathbb{R}^N}|v|^{p}(x)w(x)\frac{w'(r)}{r}|\nabla_k w|^{p-2}d\mu_k(x).
\end{aligned}
$$

Now using (3.2.11) and (3.2.16)

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla_k(vw)|^p d\mu_k(x)\\
&\quad\geqslant -\int_{\mathbb{R}^N}w|v|^p div_0(|\nabla_0 w|^{p-2}\nabla_0 w)d\mu_k(x)\\
&\qquad\qquad + 2\gamma_k(p-1)\int_{\mathbb{R}^N}|v|^{p}(x)w(x)\frac{w'(r)}{r}|\nabla_k w|^{p-2}d\mu_k(x)\\
&\quad= -\int_{\mathbb{R}^N}w|v|^p\Big(div_0(|\nabla_0 w|^{p-2}\nabla_0 w)-2\gamma_k(p-1)\frac{|w'(r)|^{p-2}w'(r)}{r}\Big)d\mu_k(x).
\end{aligned}
$$

If w is a weak solution of the equation $L_p w + Vw^{p-1} = 0$ where

$$
\begin{aligned}
L_p w &:= div_0(|\nabla_0 w|^{p-2}\nabla_0 w) - 2\gamma_k(p-1)\frac{|w'(r)|^{p-2}w'(r)}{r}\\
&= div_k(|\nabla_0 w|^{p-2}\nabla_0 w) - 2\gamma_k p\frac{|w'(r)|^{p-2}w'(r)}{r},
\end{aligned}
$$

we have the Hardy inequality

$$
\int_{\mathbb{R}^N}|\nabla_k(u)|^p d\mu_k(x) \geqslant \int_{\mathbb{R}^N}V|u|^p d\mu_k(x).
$$

4. Let $u \in C_0^\infty(\mathbb{R}^N)$. Let $w := |x|^{-\frac{d_k-p}{p}}$ with $d_k > p$ and $v = |x|^{\frac{d_k-p}{p}}u$. Now

using the calculation carried out in (3.2.1) we can write

$$\begin{aligned} div_0(|\nabla_0 w|^{p-2}\nabla_0 w) &= (p-1)|w'(r)|^{p-2}w''(r) + \frac{(N-1)}{r}|w'(r)|^{p-2}w'(r) \\ &= -\left(\frac{d_k-p}{p}\right)^{p-1}\left(\frac{d_k-p}{p} - 2\gamma_k\right) r^{-\left(\left(\frac{(d_k-p)}{p}\right)(p-1)+p\right)}. \end{aligned}$$

Using this and the expression for L_p we can write

$$L_p(w) = -\left(\frac{d_k-p}{p}\right)^{p-1}\left(\frac{d_k-p}{p} - 2\gamma_k(p-1)\right) r^{-\left(\left(\frac{(d_k-p)}{p}\right)(p-1)+p\right)}.$$

Now for $V(x) = -\left(\frac{d_k-p}{p}\right)^{p-1}\left(\frac{d_k-p}{p} - 2\gamma_k(p-1)\right)|x|^{-p}$ we have the Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x) \geqslant \left(\frac{d_k-p}{p}\right)^{p-1}\left(\frac{d_k-p}{p} - 2\gamma_k(p-1)\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x). \tag{3.2.18}$$

We don't know about the sharpness of the constant appearing in (3.2.18).

Recall the algebraic inequality given in [16, Equation 2.13]; for $p \geqslant 2$

$$|a+b|^p \geqslant |a|^p + p|a|^{p-2}a.b + c_p|b|^p,$$

where a and b are real numbers and constant c_p is given by

$$c_p := \min_{0<\tau<1/2}\left((1-\tau)^p - \tau^p + p\tau^{p-1}\right) \tag{3.2.19}$$

and is sharp for this inequality. Using this the inequality (3.2.3) can be written as

$$|\nabla_k u|^p = |\nabla_k (vw)|^p \geqslant |v|^p|\nabla_k w|^p + p|v|^{p-2}|\nabla_k w|^{p-2} vw\nabla_k v.\nabla_k w + c_p|w|^p|\nabla_k v|^p. \tag{3.2.20}$$

For radial function w and reflection invariant function v such that $u = vw \in C_0^\infty(\mathbb{R}^N)$ if we use the inequality (3.2.20) instead of (3.2.3), the inequality (3.2.13) turns out to be

$$\int_{\mathbb{R}^N} |\nabla_k (vw)|^p d\mu_k(x) \geqslant - \int_{\mathbb{R}^N} w(x)|v(x)|^p div\Big(|\nabla_k w|^{p-2}\nabla_k w\Big) d\mu_k(x) + c_p\int_{\mathbb{R}^N} |w|^p|\nabla_k v|^p d\mu_k(x).$$

This improves the following Hardy inequality with a remainder term for $p \geqslant 2$.

Corollary 3.2.3. *Let $2 \leqslant p < \infty$. Let u be a real valued $G-$ invariant function. If $u \in C_0^\infty(\mathbb{R}^N)$ if $d_k > p$ and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_k < p$ then the following inequality holds:*

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) - \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x) \geqslant c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k - p}} d\mu_k(x), \tag{3.2.21}$$

where c_p is given by (3.2.19). When $p = 2$ the equality holds and with $c_2 = 1$.

Remark 3.2.4. By observing the Remark 3.2.2 we can make another remark on the Corollary 3.2.3. If $w(x) = |x|^{-\frac{d_k - p}{p}}$ with $d_k < p$, we obtain the following improved Hardy inequality for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) - \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} |u|^p d\mu_k(x) \geqslant c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k - p}} d\mu_k(x).$$

Also if $u \in C_0^\infty(\mathbb{R}^N)$ and if $w := |x|^{-\frac{d_k - p}{p}}$ with $d_k > p$ and $v = |x|^{\frac{d_k - p}{p}} u$. Now again by the Remark 3.2.2, we obtain the following improved Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k(u)|^p d\mu_k(x) - \left(\frac{d_k-p}{p}\right)^{p-1}\left(\frac{d_k-p}{p} - 2\gamma_k(p-1)\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x)$$
$$\geqslant c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k-p}} d\mu_k(x).$$

Now we will prove a generalized Hardy inequality which generalize the Theorem 3.2.1. Fix $1 \leqslant l \leqslant N$, we write $x \in \mathbb{R}^N$ as $x = (y, z)$ with $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^{N-l}$. Let R_1 be a root system on $\mathbb{R}^l$, and k_1 be multiplicity function on R_1. The Dunkl weight function associated with R_1 and k_1 is given by $h_{k_1}^2(x) = \prod_{\alpha\in R_{1,+}} |\langle x, \alpha\rangle|^{2k_1(\alpha)}$. Since k_1 is $G-$invariant we have $k_1(\alpha) = k_1(-\alpha)$ and thus the choice of any arbitrary positive subsystem $\mathbb{R}_{1,+}$ does not make any impact on the weight function. Now similarly for a root system R_2 and a multiplicity function k_2 on $\mathbb{R}^{N-l}$, we have the weight function $h_{k_2}^2(x) = \prod_{\alpha\in R_{2,+}} |\langle x, \alpha\rangle|^{2k_2(\alpha)}$. Define a root system on $\mathbb{R}^N$ as $R := \big(R_1 \times (0)_{N-l}\big) \cup \big((0)_l \times R_2\big)$. Also define the multiplicity function k on R as, $k(y, 0) = k_1(y)$ and $k(0, z) = k_2(z)$, where y and z belongs to R_1 and R_2 respectively. It is straightforward to check that R is a root system on $\mathbb{R}^N$ and k is a multiplicity function from R to positive reals. Corresponding to this R and k one can see that the Dunkl weighted measure on $\mathbb{R}^N$, denoted by $d\mu_k(x)$, is nothing but the product of the Dunkl weighted measures on $\mathbb{R}^l$ and $\mathbb{R}^{N-l}$. That is,

$$d\mu_k(x) = d\mu_{k_1}(y) d\mu_{k_2}(z) = h_k^2(x)dx = h_{k_1}^2(y)h_{k_2}^2(z)dydz.$$

With this preparation we state the following theorem.

Theorem 3.2.5. *Let $1 \leqslant p < \infty$ and let $1 \leqslant l \leqslant l \leqslant N$. Let u be a real valued $G-$invariant function. Assume that $u \in C_0^\infty(\mathbb{R}^N)$ if $d_{k_1} > p$ and $u \in$*

$C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_{k_1} < p$. Then the following inequality holds

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^p d\mu_k(x) \geqslant \left|\frac{d_{k_1} - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|y|^p} d\mu_k(x). \tag{3.2.22}$$

The constant $\left|\frac{d_{k_1}-p}{p}\right|^p$ given in the inequality is optimal.

Proof. The root system R with which we started allows us to write

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|y|^p} d\mu_k(x) = \int_{\mathbb{R}^{N-l}} d\mu_{k_1}(z) \int_{\mathbb{R}^l} \frac{|u(x)|^p}{|y|^p} d\mu_{k_2}(y). \tag{3.2.23}$$

Let $\nabla_{k_1,y}$ and $\nabla_{k_2,z}$ be the Dunkl gradient on $\mathbb{R}^l$ and $\mathbb{R}^{N-l}$ respectively. It is easy to see that $|\nabla_{k_1,y} u(y,z)| \leqslant |\nabla_k u(x)|$. By applying Theorem 3.2.1 to (3.2) we obtain the inequality (3.2.5). Now by using Lemma 3.2.1 and following the arguments from [28] we can prove that $\left|\frac{d_{k_1}-p}{p}\right|^p$ is optimal. □

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Remark 3.2.6. Remark 3.2.2 can be extended to the Theorem 3.2.5 similarly.

3.3 Fractional Hardy Inequality for $L^p(\mathbb{R}^N, d\mu_k(x))$

We have already seen that

$$\Delta^s u(x) = C \ P.V. \int_{\mathbb{R}^N} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} dy,$$

for some constant C. Using the symmetricity of the kernel $|x-y|^{-(N+2s)}$ with a constant $\tilde{C}$

$$\|(-\Delta^{s/2})u\|_2^2 = \langle \Delta^s u, u \rangle = \tilde{C} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy, \tag{3.3.1}$$

and thus the fractional L^2 Hardy inequality takes the form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \geqslant C(N,s)\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}dx,$$

the constant depends on N and s. One of the references to see the explicit calculation of this L^2 fractional Hardy inequality is [35, Appendix A]. However when $p \neq 2$ one cannot have the equivalence of $\|(-\Delta^{s/2})u\|_p^p$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy$ which we stated for $p = 2$ in (3.3.1). There are many studies done in the literature regarding the fractional Hardy inequality of the form

$$\left\|(-\Delta^{s/2})u\right\|_p^p \geqslant C(N,s,p)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx;$$

for instance Herbst in [21] calculated the sharp constant in the above inequality. But in this paper we are interested in the fractional Hardy inequalities of the form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \geqslant C'(N,s,p)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx \tag{3.3.2}$$

in the Dunkl setting.

The basic study of fractional power of Dunkl Laplacian can be done in a similar fashion to the Euclidean case. The kernel $|x-y|^{-(N+ps)}$ in (3.3.2) is actually the translation of the function $|x|^{-(N+ps)}$. We are motivated to consider the kernel which is Dunkl translation of $|x|^{-(d_k+ps)}$. taking the idea from [17, Lemma 2.3] we define the kernel $\Phi_\delta(x,y)$ as

$$\Phi_\delta(x,y) := \frac{1}{\Gamma((d_k+\delta)/2)}\int_0^\infty s^{\frac{d_k+\delta}{2}-1}\tau_y^k\left(e^{-s|.|^2}\right)(x)ds \quad d_k \neq \delta. \tag{3.3.3}$$

Theorem 3.3.1. *Let $d_k \geqslant 1$ and $0 < s < 1$. If $u \in \dot{W}^s_p(\mathbb{R}^N)$ when $2 \leqslant p < d_k/s$ or $u \in \dot{W}^{p,s}_k(\mathbb{R}^{\mathbb{N}} \setminus \{0\})$ when $p > d_k/s$, the following inequality holds;*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) \geqslant C_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x), \tag{3.3.4}$$

where $\Phi_{ps}(x,y)$ is given in (3.3.3) and

$$C_{d_k,s,p} := 2\int_0^1 r^{ps-1} |1 - r^{(d_k-ps)/p}|^p \Phi_{N,s,p}(r) dr, \tag{3.3.5}$$

with

$$\Phi_{N,s,p}(r) := \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})} \int_0^\pi \frac{sin^{d_k-2}\theta}{(1-2r\ cos\ \theta + r^2)^{\frac{d_k+ps}{2}}} d\theta, \ N \geqslant 2,$$
$$\Phi_{1,s,p}(r) := \Big(\tau^k_r\big(|.|^{d_k+ps}\big) + \tau^k_{-r}\big(|.|^{d_k+ps}\big)\Big)(1), \ N = 1. \tag{3.3.6}$$

The constant $C_{d_k,s,p}$ is sharp. If $p = 1$, equality holds iff u is proportional to a symmetric decreasing function. If $p > 1$, the inequality is strict for any function $0 \not\equiv u \in \dot{W}^s_p(\mathbb{R}^N)$ or $\dot{W}^s_p(\mathbb{R}^N \setminus \{0\})$, respectively. Further for $p \geqslant 2$ the following inequality holds.

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &|u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y)\\ &\geqslant C_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x)\\ &\qquad + c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)|^p \Phi_{ps}(x,y) \frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}} \frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}},\end{aligned} \tag{3.3.7}$$

where $v := |x|^{(d_k-ps)/p} u$, $C_{d_k,s,p}$ is given by (3.3.5) and c_p is given in (3.2.19). $c_2 = 1$ and the equality holds in $p = 2$ case.

Remark 3.3.2. The case when we choose the multiplicity function $k \equiv 0$ the Dunkl case will reduce to the classical case. So in that case we get the main results in [16] as a corollary of above theorems. That is [16, Theorem 1.1] and [16, Theorem 1.2] are obtained as a corollaries to Theorem 3.3.1.

Here is an auxiliary lemma which is proven in [16].

Lemma 3.3.3 (R. Frank, R. Seiringer). *Let $p \geqslant 1$. Then for all $0 \leqslant t \leqslant 1$ and $a \in \mathbb{C}$ one has*

$$|a - t|^p \geqslant (1 - t)^{p-1}(|a|^p - 1). \tag{3.3.8}$$

For $p > 1$ this inequality is strict unless $a = 1$ or $t = 0$. Moreover, if $p \geqslant 2$ then for all $0 \leqslant t \leqslant 1$ and all $a \in \mathbb{C}$ one has

$$|a - t|^p \geqslant (1 - t)^{p-1}\big(|a|^p - t\big) + c_p t^{p/2}|a - 1|^p, \tag{3.3.9}$$

with $0 < c_p \leqslant 1$ and c_p is given in (3.2.19). For $p = 2$, (3.3.9) is an equality with $c_2 = 1$. For $p > 2$, (3.3.9) is a strict equality unless $a = 1$ or $t = 0$.

For $N, p \geqslant 1$, let $\Phi_\epsilon(x, y)$ be symmetric positive real-valued functions defined on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\Phi_\epsilon \to \Phi_{ps}$ as $\epsilon \to 0$ with $\Phi_\epsilon \leqslant \Phi_{ps}$. Let us define the energy functional $E[u]$ as

$$E[u] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^p \Phi_{ps}(x, y) d\mu_k(x) d\mu_k(y),$$

where $\Phi_{ps}(x, y)$ is the kernel given in (3.3.3). Let us define the functions V_ϵ and

V as

$$V_\epsilon(x) := 2w(x)^{-p+1}\int_{\mathbb{R}^N}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_\epsilon(x,y)d\mu_k(y) \quad (3.3.10)$$

and $\int_{\mathbb{R}^N} Vf d\mu_k(x) := \lim_{\epsilon\to 0}\int_{\mathbb{R}^N} V_\epsilon f d\mu_k(x)$ for every $f \in C_0^\infty(\mathbb{R}^N)$. Following a similar argument as in the proof of [16, Proposition 2.2, Proposition 2.3] gives us the following two lemmas.

Lemma 3.3.4. *Let $u \in C_0^\infty(\mathbb{R}^N)$. If $E[u]$ and $\int V|u|^p$ are finite we have*

$$E[u] \geqslant \int_{\mathbb{R}^N} V(x)|u(x)|^p d\mu_k(x). \quad (3.3.11)$$

Lemma 3.3.5. *Let $p \geqslant 2$ and $u \in C_0^\infty(\mathbb{R}^N)$. If $E[u]$, $\int V|u|^p$ are finite and*

$$\int_{\mathbb{R}^N} |v(x)-v(y)|^p w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) < \infty, \quad (3.3.12)$$

then we have

$$\begin{aligned} &E[u] - \int_{\mathbb{R}^N} V(x)|u(x)|^p d\mu_k(x) \\ &\geqslant c_p \int_{\mathbb{R}^N} |v(x)-v(y)|^p w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y), \end{aligned} \quad (3.3.13)$$

where c_p is as in (3.2.19). If $p = 2$, (3.3.11) becomes an equality with $c_2 = 1$.

We will prove the following lemma which states that $w(x) = |x|^{-\frac{d_k-ps}{p}}$ solves the Euler-Lagrange equation related to the equation (3.3.4). For convenience in calculations we write $\alpha := (d_k - ps)/p$. Let $\Phi_\epsilon := \Phi_{ps}\chi_{||x|-|y||>\epsilon}$, then Φ_ϵ's are positive symmetric real valued functions which converges to Φ_{ps}, with $0 < \Phi_\epsilon \leqslant \Phi_{ps}$.

Lemma 3.3.6. *Let* $w(x) = |x|^{-\frac{d_k - ps}{p}}$*. The following limit converges uniformly for any compact subsets of* $\mathbb{R}^N$*;*

$$2\lim_{\epsilon\to 0}\int_{||x|-|y||>\epsilon}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_\epsilon(x,y)d\mu_k(y) = \frac{C_{d_k,s,p}}{|x|^{ps}}w(x)^{p-1}. \tag{3.3.14}$$

Proof. Let $|x| = r$ and $|y| = \rho$ and write $x = rx'$ and $y = \rho y'$. Using polar coordinates we obtain;

$$\begin{aligned}&\int_{||x|-|y||>\epsilon}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y) \\ &= \int_{|\rho-r|>\epsilon}\int_{\mathbb{S}^{N-1}}(r^{-\alpha}-\rho^{-\alpha})|r^{-\alpha}-\rho^{-\alpha}|^{p-2}\Phi_{ps}(rx',\rho y')\rho^{2\lambda_k+1}d\rho d\sigma_k(y')\end{aligned} \tag{3.3.15}$$

where $d\sigma_k(y') = h_k^2(y')d\sigma(y')$ with $d\sigma(y')$ is the (Euclidean) surface measure on the sphere $\mathbb{S}^{N-1}$. If $\rho < r$ we use the fact from [17, Lemma 2.3] that $\Phi_{ps}(rx',\rho y') = r^{-d_k-ps}\Phi_{ps}(x',\frac{\rho}{r}y')$ we get

$$\begin{aligned}&\int_{||x|-|y||>\epsilon}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y) \\ &= \int_{|\rho-r|>\epsilon}\int_{\mathbb{S}^{N-1}}\frac{sgn(\rho^\alpha-r^\alpha)|\rho^{-\alpha}-r^{-\alpha}|^{p-1}}{r^{d_k+ps}}\Phi_{ps}(x',\frac{\rho}{r}y')\rho^{2\lambda_k+1}d\sigma_k(y')d\rho,.\end{aligned} \tag{3.3.16}$$

Similarly, if $r < \rho$ from [17, Lemma 2.3] it follows that

$$\begin{aligned}&\int_{||x|-|y||>\epsilon}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y) \\ &= \int_{|\rho-r|>\epsilon}\int_{\mathbb{S}^{N-1}}\frac{sgn(\rho^\alpha-r^\alpha)|\rho^{-\alpha}-r^{-\alpha}|^{p-1}}{\rho^{1+ps}}\Phi_{ps}(\frac{r}{\rho}x',y')d\sigma_k(y')d\rho,\end{aligned} \tag{3.3.17}$$

It follows from [17, Lemma 2.3] that

$$\int_{\mathbb{S}^{N-1}} \Phi_{ps}(rx', \rho y')d\sigma_k(y') = \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})} \int_0^{\pi} \frac{sin^{d_k-2}\theta}{(r^2 - 2r\rho\ cos\ \theta + \rho^2)^{\frac{d_k+ps}{2}}} d\theta. \tag{3.3.18}$$

Using (3.3.16), (3.3.17) and (3.3.18) we can write (3.3.15) as

$$\begin{aligned} &\int_{||x|-|y||>\epsilon} (w(x) - w(y))|w(x) - w(y)|^{p-2}\Phi_{ps}(x, y)d\mu_k(y) \\ &= \frac{1}{r^{d_k-1}} \int_{|\rho-r|>\epsilon} \frac{sgn(\rho^\alpha - r^\alpha)}{|\rho - r|^{2-p(1-s)}} \varphi(\rho, r)d\rho, \end{aligned} \tag{3.3.19}$$

where $\varphi(\rho, r)$ is given by

$$\varphi(\rho, r) = |\frac{\rho^{-\alpha} - r^{-\alpha}}{r - \rho}|^{p-1}. \begin{cases} \rho^{d_k-1}(1 - \frac{\rho}{r})^{1+ps}\Phi_{N,s,p}(\frac{\rho}{r}), & \text{if } \rho < r, \\ r^{d_k-1}(1 - \frac{r}{\rho})^{1+ps}\Phi_{N,s,p}(\frac{r}{\rho}) & \text{if } \rho > r, \end{cases} \tag{3.3.20}$$

with $\Phi_{N,s,p}$ is given in (3.3.6).

We need to show the convergence of the integral

$$\int_{|\rho-r|>\epsilon} \frac{sgn(\rho^\alpha - r^\alpha)}{|\rho - r|^{2-p(1-s)}} \varphi(\rho, r)d\rho. \tag{3.3.21}$$

It is enough to show that the function $\phi(\rho, r)$ is Lipschitz continuous as a function of ρ at $\rho = r$. Writing $t = \rho/r$ it is sufficient to show the function $(1 - t)^{1+ps}\Phi_{N,s,p}(t)$ and its t-derivative is bounded at $t \to 1-$. As $N = 1$ it is trivial we do it for $N \geqslant 2$. The identity in [19, 3.665] states that

$$\int_{\mathbb{R}^N} \frac{sin^{2\mu-1}xdx}{(1 + 2a\ cos\ x + a^2)^\nu} = B\Big(\mu, \frac{1}{2}\Big)F\Big(\nu, \nu - \mu + \frac{1}{2}, \mu + \frac{1}{2}; a^2\Big), \tag{3.3.22}$$

where F is a hypergeometric function with $Re\ \mu > 0$ and $|a| < 1$. Using (3.3.22) we can write

$$\Phi_{N,s,p}(t) = \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})} B\left(\frac{d_k-1}{2}, \frac{1}{2}\right) F\left(\frac{d_k+ps}{2}, \frac{ps+2}{2}; \frac{d_k}{2}; t^2\right). \qquad (3.3.23)$$

Using the property that both $(1-z)^{a+b-c}F(a,b,c;z)$ and its derivative has a limit at $z \to 1-$ if $a+b-c>1$ we conclude $(1-t)^{1+ps}\Phi_{N,s,p}(t)$ and its t-derivative is bounded at $t \to 1-$.

Continuing the same argument from [16] we get (3.3.14) with

$$C_{d_k,s,p} = 2\lim_{\epsilon\to 0}\int_{|\rho-1|>\epsilon} \frac{sgn(\rho^\alpha - 1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho.$$

Now we will prove that this constant coincides with the constant given in (3.3.5).

$$\begin{aligned}
&2\lim_{\epsilon\to 0}\int_{|\rho-1|>\epsilon} \frac{sgn(\rho^\alpha - 1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho \\
&= 2\lim_{\epsilon\to 0}\left[\int_0^{1-\epsilon} \frac{sgn(\rho^\alpha - 1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho + \int_{1+\epsilon}^{\infty} \frac{sgn(\rho^\alpha - 1)}{|\rho-1|^{2-p(1-s)}}\varphi(\rho,1)d\rho\right] \\
&= 2\left[\int_0^1 \frac{sgn(\rho^\alpha - 1)}{(1-\rho)^{2-p(1-s)}}\varphi(\rho,1)d\rho + \int_0^1 \frac{sgn(1-\rho^\alpha)\rho^{-p(1-s)}}{(1-\rho)^{2-p(1-s)}}\varphi(\rho^{-1},1)d\rho\right] \\
&= 2sgn(\alpha)\int_0^1 \frac{\left(\rho^{-p(1-s)}\varphi(\rho^{-1},1) - \varphi(\rho,1)\right)}{(1-\rho)^{2-p(1-s)}}d\rho.
\end{aligned}$$

A straightforward calculation gives

$$\left(\rho^{-p(1-s)}\varphi(\rho^{-1},1) - \varphi(\rho,1)\right) = |1-\rho^\alpha|^{p-1}(1-\rho^\alpha)\Phi_{N,s,p}(\rho)(1-\rho)^{2-p(1-s)}$$

and it follows that

$$C_{d_k,s,p} = 2\int_0^1 \rho^{ps-1}|1-\rho^\alpha|^p\Phi_{N,s,p}(\rho)d\rho.$$

□

3.3.1 Proof of the Theorem 3.3.1

Now the Hardy inequalities (3.3.4) and (3.3.7) will follow by repeating the arguments of [16]. In case of the strictness, $p \geqslant 2$ due to the positive remainder term in (3.3.7), it is immediate that the inequality in (3.3.4) is strict. With similar arguments used to obtain [16, (2.18)], in our case we obtain

$$E[u] = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi_u(x,y)\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) + \int_{\mathbb{R}^N} V|u|^p d\mu_k(x), \qquad (3.3.24)$$

for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ with

$$\begin{aligned}\phi_u(x,y) =& |w(x)v(x) - w(y)v(y)|^p \\ & - (w(x)|v(x)|^p - w(y)|v(y)|^p)(w(x)-w(y))|w(x)-w(y)|^{p-2}.\end{aligned}$$

It can be proven easily that $\phi_u \geqslant 0$ (see [16]). This can be extended to $\dot{W}_k^{p,s}(\mathbb{R}^N \setminus \{0\})$ when $d_k < ps$ and to $\dot{W}_k^{p,s}(\mathbb{R}^N)$ when $d_k > ps$ by approximation.

Suppose $E[u] = \int_{\mathbb{R}^N} V|u|^p d\mu_k(x)$ for some $u \in \dot{W}_k^{p,s}(\mathbb{R}^N \setminus \{0\})$. Then it is true for $|u|$. Observing that $\Phi_{|u|} \geqslant 0$ and $\Phi_{ps}(x,y)$ is positive in (3.3.24) we can see that $\Phi_{|u|} \equiv 0$. From the Lemma 3.3.3 we obtain that v is a constant function and since $v = w^{-1}u$ we conclude that $u \equiv 0$. This gives that for any non-zero $u \in \dot{W}_k^{p,s}(\mathbb{R}^N \setminus \{0\})$ in case $d_k < ps$ or $u \in \dot{W}_k^{p,s}(\mathbb{R}^N)$ in case $d_k > ps$ the inequality (3.3.11) is strict.

Now for $p = 1$, we shall prove that the equality of (3.3.4) holds if and only if u is proportional to a symmetric decreasing function. Let χ_t be the characteristic function of a ball centered at origin with radius $R(t)$. Define $u = \int_0^\infty \chi_t dt$. Then for $p = 1$, we can write right hand side of the inequality (3.3.4) as

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{|x|^s} = \frac{\|\mathbb{S}^{N-1}\|_k}{d_k - s} \int_0^\infty R(t)^{d_k - s} dt,$$

where $\|\mathbb{S}^{N-1}\|_k$ is the surface measure of $\mathbb{S}^{N-1}$ with Dunkl weighted measure; one can calculate $\|\mathbb{S}^{N-1}\|_k = c_k^{-1}/(2^{(\frac{d_k}{2}-1)}\Gamma(d_k/2))$. Now the left-hand side of the same inequality (3.3.4) can be written as

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x) - u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) \\
&= 2 \iint\limits_{\{|x|<|y|\}} \left| \int (\chi_t(x) - \chi_t(y)) dt \right| \Phi_{ps} d\mu_k(x) d\mu_k(y) \\
&= 2 \iiint\limits_{\{|x|<R(t)<|y|\}} \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) dt \\
&= 2 \iint\limits_{\{|x|<1<|y|\}} \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) \int_0^\infty R(t)^{d_k - s} dt.
\end{aligned}$$

It gives the equality of (3.3.4) for the function u and $p = 1$.

The sharpness of the constant $C_{d_k,s,p}$ can be proved by the same arguments in [16]. But for the completion we give the proof here. To prove this, we will use the trial functions u_n and will show that, as $n \to \infty$,

$$\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^p k(x,y) d\mu_k(x) d\mu_k(y)}{\int_{\mathbb{R}^N} |u_n(x)|^p |x|^{-ps} d\mu_k(x)} \leqslant C_{d_k,s,p}(1 + \mathcal{O}(1)).$$

Let us define the functions u_n for $d_k > ps$ first. Let

$$\begin{aligned}
I \quad &:= \{x \in \mathbb{R}^N : 0 \leqslant |x| < 1\} \\
M_n \quad &:= \{x \in \mathbb{R}^N : 1 \leqslant |x| \leqslant n\} \\
O_n \quad &:= \{x \in \mathbb{R}^N : |x| \geqslant n\}.
\end{aligned}$$

Define

$$u_n(x) := \begin{cases} 1 - n^{-\alpha}, & \text{if } x \in I, \\ |x|^{-\alpha} - n^{-\alpha} & \text{if } x \in M_n, \\ 0 & \text{if } x \in O_n, \end{cases} \tag{3.3.25}$$

where $\alpha = \frac{(d_k - ps)}{p}$ Multiply the integrand of (3.3.14) with $u_n(x)$ and integrate with respect to x. Using the symmetricity of $\Phi_{ps}(x,y)$ we obtain as $\epsilon \to 0$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u_n(x)-u_n(y))(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) \\ = \; & C_{d_k,s,p}\int_{\mathbb{R}^N}\frac{u_n(x)w(x)^{p-1}}{|x|^{ps}}d\mu_k(x). \end{aligned} \tag{3.3.26}$$

Write

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u_n(x)-u_n(y))(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) \\ &\qquad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u_n(x)-u_n(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) + 2\mathcal{R}_0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_0 := &\int_{x\in I}\int_{y\in M_n}(1-w(y))((w(x)-w(y)^{p-1})-(1-w(y))^{p-1}) \\ &\qquad \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) \\ &+\int_{x\in M_n}\int_{y\in O_n}(w(x)-n^{-\alpha})((w(x)-w(y))^{p-1}-(w(x)-n^{-\alpha})^{p-1}) \\ &\qquad \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) \\ &+\int_{x\in I}\int_{y\in O_n}(1-n^{-\alpha})((w(x)-w(y))^{p-1}-(1-N^{-\alpha})^{p-1}) \\ &\qquad \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y). \end{aligned}$$

Since all the terms within all the three integral are non-negative, we have $\mathcal{R} \geqslant 0$. Divide the right-hand side of (3.3.26) by $C_{d_k,s,p}$ and add and subtract $\frac{u_n^p}{|x|^{ps}}$ to the integrand we obtain

$$\int_{\mathbb{R}^N} \frac{u_n^p}{|x|^{ps}} d\mu_k(x) + \mathcal{R}_1 + \mathcal{R}_2, \tag{3.3.27}$$

where

$$\mathcal{R}_1 := \int_I (1 - n^{-\alpha})(w(x)^{p-1} - (1 - n^{-\alpha})^{p-1} \frac{d\mu_k(x)}{|x|^{ps}} \tag{3.3.28}$$

$$\mathcal{R}_2 := \int_{M_n} (w(x) - n^{-\alpha})(w(x)^{p-1} - (w(x) - n^{-\alpha})^{p-1}) \frac{d\mu_k(x)}{|x|^{ps}}. \tag{3.3.29}$$

Observe that the integrands on both of the integrals are non-negative and we will show that $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{O}(1)$ as $n \to \infty$.

$$\begin{aligned} &\frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y)}{\int_{\mathbb{R}^N} |u_n(x)|^p |x|^{-ps} d\mu_k(x)} \qquad (3.3.30) \\ = \;& C_{d_k,s,p}\left(1 + \frac{\mathcal{R}_1 \mathcal{R}_2}{\int_{\mathbb{R}^N} |u_n(x)|^p |x|^{-ps} d\mu_k(x)}\right) - \frac{2\mathcal{R}_0}{\int_{\mathbb{R}^N} |u_n(x)|^p |x|^{-ps} d\mu_k(x)} \\ & \leqslant C_{d_k,s,p}(1 + o(1)). \end{aligned}$$

Now we need to prove that $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{O}(1)$ as $n \to \infty$. See that the integrand of $\mathcal{R}_1$ is bounded by $|x|^{\alpha - d_k}$ and it allows us to write $\mathcal{R}_1 \leqslant \int_{|x|<1} |x|^{\alpha - d_k} d\mu_k(x) < \infty$. Observe that $1 - (1-t)^{p-1} \leqslant t$ for $1 \leqslant p \leqslant 2$ and $1 - (1-t)^{p-1} \leqslant (p-1)t$ for $p > 2$, where $0 \leqslant t \leqslant 1$. Using this we can write

$$(w(x) - n^{-\alpha})(w(x)^{p-1} - (w(x) - n^{-\alpha})^{p-1}) \leqslant C_p n^{-\alpha} w(x)^{p-1}, \tag{3.3.31}$$

where $C_p = 1$ for $1 \leqslant p \leqslant 2$ and $C_p = p - 1$ for $p > 2$. Now it is not hard to see that $\mathcal{R}_2 \leqslant C_p \int_{|x|<1} |x|^{\alpha - d_k} d\mu_k(x) < \infty$. The case $d_k < ps$ can be treated similarly using the sequence of trial functions described in [16] taking $\alpha = (d_k - ps)/p$.

3.4 Fractional Hardy Inequality for Half-Space

Let R_1 be a root system on $\mathbb{R}^{N-1}$ and a k_1 be a multiplicity function on R_1. Extend R_1 to a root system R of $\mathbb{R}^N$ as $R = R_1 \times \{0\} = \{(x, 0) : x \in R_1\}$. Clearly it is a root system on $\mathbb{R}^N$ and the multiplicity function k_1 can be extended to k which acts on R by $k(x_1, x_2, ...x_{N-1}, x_N) = k_1(x_1, x_2, ...x_{N-1})$. Let $R_{1,+}$ be a positive subsystem of R_1 with $R_1 = R_{1,+} \cup (-R_{1,+})$. Then we can write $R = R_+ \cup (-R_+)$ where the positive subsystem R_+ of R given by $R_+ = \{(x, 0) : x \in R_{1,+}\}$. γ_k remains the same as $\gamma_k = \sum_{\nu \in R_+} k(\nu) = \sum_{\nu \in R_{1,+}} k_1(\nu) = \gamma_{k_1}$. The Dunkl measure corresponding to the root system R and the multiplicity function k will be

$$\begin{aligned} d\mu_k(x) &= \prod_{\nu \in R_+} |\langle x, \nu\rangle|^{2k(\nu)} dx \\ &= \prod_{\nu \in R_{1,+}} |\langle x', \nu\rangle|^{2k_1(\nu)} dx'.dx_N = d\mu_{k_1}(x') dx_N, \end{aligned}$$

where $x = (x', x_N) \in \mathbb{R}^N$.

Theorem 3.4.1. *Let $N \geqslant 1$, $1 \leqslant p < \infty$, and $0 < s < 1$ with $ps \neq 1$. Then for all $u \in \dot{W}^p_s(\mathbb{R}^N_+)$*

$$\int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} |u(x) - u(y)|^p \Phi_{ps}(x, y) d\mu_k(x) d\mu_k(y) \geqslant D_{N,\gamma_k,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x), \tag{3.4.1}$$

where

$$D_{N,\gamma_k,p,s} := c_{k_1}^{-1} 2^{-\lambda_{k_1}} \frac{\Gamma((1 + ps)/2)}{\Gamma((d_k + ps)/2)} \int_0^1 |1 - r^{(ps-1)/p}|^p \frac{dr}{(1 - r)^{1+ps}}. \tag{3.4.2}$$

and the constant $D_{N,\gamma_k,p,s}$ is optimal. If $p = 1$ and $N = 1$, equality holds iff u is proportional to a non-increasing function. If $p = 1$ or if $p = 1$ and $N \geqslant 2$, the inequality strict for any non zero function in $\dot{W}^s_p(\mathbb{R}^N_+)$. Further for $p \geqslant 2$ we also have

$$\begin{aligned} &\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) \qquad (3.4.3)\\ \geqslant\ & D_{N,\gamma_k,p,s}\int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x)\\ &\qquad + c_p \int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} |v(x)-v(y)|^p \Phi_{ps}(x,y) \frac{d\mu_k(x)}{|x_N|^{(1-ps)/2}} \frac{d\mu_k(y)}{|y_N|^{(1-ps)/2}}, \end{aligned}$$

where $v := x_N^{(1-ps)/p} u$, Φ is as in (3.3.3), $D_{N,\gamma_k,p,s}$ is given in (3.4.2) and c_p is given in (3.2.19). $c_2 = 1$ and the equality holds in $p = 2$ case.

Proof. Let $x = (x', x_N)$ and $y = (y', y_N)$ are elements of $\mathbb{R}^N$. Choose $w(x) = |x_N|^{(1-ps)/p}$ and $V(x) = D_{N,\gamma_k.p,s}|x_N|^{-ps}$. Since for the fixed root system R

$$\tau^k_y(e^{-s|.|^2})(x) = e^{-s|x_N - y_N|^2}\tau^{k_1}_{y'}(e^{-s|.|^2})(x').$$

the definition of $\Phi_{ps}(x,y)$ in (3.3.3) takes the form

$$\begin{aligned} \Phi_{ps}(x,y) &:= \frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1}\tau^k_y\left(e^{-s|.|^2}\right)(x)ds\\ &= \frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1} e^{-s|x_N-y_N|^2}\tau^{k_1}_{y'}\left(e^{-s|.|^2}\right)(x')ds. \end{aligned}$$

We start with the Euler - Lagrange equation corresponding to (3.4.1) and let us verify that $w(x) = |x_N|^{-\frac{1-ps}{p}}$ solves it.

$$\begin{aligned}
&\int\limits_{\substack{y\in\mathbb{R}^N_+\\ |x_N-y_N|>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y) \qquad (3.4.4)\\
=\ & \frac{1}{\Gamma((d_k+ps)/2)} \int\limits_{\substack{y\in\mathbb{R}^N_+\\ |x_N-y_N|>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2} \times \\
& \int_0^\infty s^{\frac{d_k+ps}{2}-1}\tau_y(e^{-s|.|^2})(x)dsd\mu_k(y)\\
=\ & \frac{1}{\Gamma((d_k+ps)/2)} \int_{\mathbb{R}^{N-1}}\int_{|x_N-y_N|>\epsilon} (w(x)-w(y))|w(x)-w(y)|^{p-2}\\
& \int_0^\infty s^{\frac{d_k+ps}{2}-1} e^{-s|x_N-y_N|^2}\tau^{k_1}_{y'}(e^{-s|.|^2})(x')dsdy_Nd\mu_{k_1}(y').
\end{aligned}$$

Property of translation of a radial function[37, Theorem 3.8] gives that

$$\int_{\mathbb{R}^{N-1}} \tau^{k_1}_{y'}(e^{-s|.|^2})(x')d\mu_{k_1}(y') = \int_{\mathbb{R}^{N-1}} e^{-s|y'|^2}d\mu_{k_1}(y') \qquad (3.4.5)$$

From the definition of Gamma function we get

$$\begin{aligned}
&\frac{1}{\Gamma((d_k+ps)/2)}\int_0^\infty s^{\frac{d_k+ps}{2}-1}e^{-s(|x_N-y_N|^2+|y'|^2)}ds \qquad (3.4.6)\\
&= \frac{1}{(|x_N-y_N|^2+|y'|^2)^{\frac{d_k+ps}{2}}}.
\end{aligned}$$

Applying (3.4.5) and (3.4.6) to (3.4.4 we find

$$\begin{aligned}
&\int\limits_{\substack{y\in\mathbb{R}^N_+,\\ |x_N-y_N|>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y)\\
&= \int\limits_{\substack{y\in\mathbb{R}^N_+,\\ |x_N-y_N|>\epsilon}} \frac{(w(x)-w(y))|w(x)-w(y)|^{p-2}}{(|x_N-y_N|^2+|y'|^2)^{\frac{d_k+ps}{2}}}d\mu_k(y).
\end{aligned}$$

Let us calculate the following integral separately for convenience, and let us call $m = |x_N - y_N|^2$ and keep in mind that $d_{k_1} = d_k - 1$

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \frac{1}{(m^2+|y'|^2)^{\frac{d_k+ps}{2}}} d\mu_k(y') &= \|\mathbb{S}^{N-2}\|_{k_1} \int_0^\infty \frac{1}{(m^2+r^2)^{\frac{d_k+ps}{2}}} r^{d_k-2} dr \\ &= \|\mathbb{S}^{N-2}\|_{k_1} \frac{1}{m^{1+ps}} \int_0^\infty \frac{t^{d_k-2}}{(1+t^2)^{\frac{d_k+ps}{2}}} dt \\ &= \|\mathbb{S}^{N-2}\|_{k_1} \frac{1}{2m^{1+ps}} \frac{\Gamma((d_k-1)/2)\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)}.\end{aligned}$$

Now come back to the the equation and use the [15, Theorem 1.1] for $N = 1$ to conclude. Also substitute the value of $\|\mathbb{S}^{N-2}\|_{k_1} = (c_{k_1}^{-1} 2^{-\lambda_{k_1}})/\Gamma(d_{k_1}/2)$. We use the same notation w for the function $w(x_N) = |x_N|^{-(1-ps)/p}$;

$$\begin{aligned}&\int_{y\in\mathbb{R}^N_+, |x_N-y_N|>\epsilon} \frac{(w(x)-w(y))|w(x)-w(y)|^{p-2}}{(|x_N-y_N|^2+|y'|^2)^{\alpha/2}} d\mu_k(y) \\ &= \frac{c_{k_1}^{-1} 2^{-\lambda_{k_1}} \Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)} \\ &\qquad \int_{|x_N-y_N|>\epsilon} \frac{(w(x_N)-w(y_N))|w(x_N)-w(y_N)|^{p-2}}{|x_N-y_N|^{1+ps}} dy_N.\end{aligned} \tag{3.4.7}$$

From [15, Lemma 3.1], considering $x_N, y_N \in \mathbb{R}$, we can write

$$\begin{aligned}&\frac{C_{1,p,s}}{|x_N|^{ps}} w(x_N)^{p-1} \\ =\ & 2 \lim_{\epsilon\to 0} \int_{\substack{\mathbb{R}, \\ ||x_N|-|y_N||>\epsilon}} \frac{(w(x_N)-w(y_N)||w(x_N)-w(y_N)|^{p-2}}{|x_N-y_N|^{1+ps}} dy_N \\ &= 2\int_0^\infty (w(x_N)-w(y_N)|w(x_N)-w(y_N)|^{p-2} \\ &\qquad \left(\frac{1}{|x_N-y_N|^{1+ps}+|x_N+y_N|^{1+ps}}\right) dy_N.\end{aligned} \tag{3.4.8}$$

This gives the constant in [15, Theorem 1.1] as

$$C_{1,p,s} = 2\int_0^1 \left|1 - r^{(1-ps)/p}\right|^p \left(\frac{1}{(1-r)^{1+ps}} + \frac{1}{(1+r)^{1+ps}}\right) dr. \tag{3.4.9}$$

But in our case we are only interested in the case $y_N > 0$, so (3.4.8) and (3.4.9) implies that

$$\begin{aligned} &2\lim_{\epsilon\to 0} \int\limits_{\substack{0, \\ |x_N - y_N|>\epsilon}}^{\infty} (w(x_N) - w(y_N)) \frac{|w(x_N) - w(y_N)|^{p-2}}{|x_N - y_N|^{1+ps}} dy_N \\ &= \frac{\tilde{C}_{1,p,s}}{|x_N|^{ps}} w(x)^{p-1}, \end{aligned} \tag{3.4.10}$$

where

$$\tilde{C}_{1,p,s} := 2\int_0^1 \frac{\left|1 - r^{(1-ps)/p}\right|^p}{(1-r)^{1+ps}} dr. \tag{3.4.11}$$

Now by using (3.4.10) and (3.4.7) we can conclude

$$\begin{aligned} &2\lim_{\epsilon\to 0} \int_{y\in\mathbb{R}^N_+, |x_N - y_N|>\epsilon} \frac{(w(x) - w(y))|w(x) - w(y)|^{p-2}}{(|x_N - y_N|^2 + |y'|^2)^{\alpha/2}} d\mu_k(y) \\ &= \frac{c_{k_1}^{-1} 2^{-\lambda_{k_1} - 1} \Gamma((1+ps)/2)}{\Gamma((d_k + ps)/2)} \frac{\tilde{C}_{1,p,s}}{|x_N|^{ps}} w(x)^{p-1}. \end{aligned} \tag{3.4.12}$$

We can see that the constant appearing in (3.4.2) and $\frac{c_{k_1}^{-1} 2^{-\lambda_{k_1} - 1} \Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)} \tilde{C}_{1,p,s}$ are same.

The Hardy inequalities (3.4.1) and (3.4.3), the strictness for $p > 1$ and the equality in case of $p = 1$ follow from the proof of [16, Theorem 1.1]. Optimality comes from the optimality of the Theorem 3.3.1. □

3.5 Fractional Hardy Inequality for Cone

For $0 \leqslant l \leqslant l$ a cone $\mathbb{R}^N_{l_+}$ is defined as a subset of $\mathbb{R}^N$ which is precisely the set $\{x = (x_1, ...x_N) \in \mathbb{R}^N : x_{N-l+1} > 0, ..., x_N > 0\}$. In the case of half-space we extended a root system of $\mathbb{R}^{N-1}$ to a root system of $\mathbb{R}^N$ and we found a corresponding multiplicity function and Dunkl weighted measure on $\mathbb{R}^N_+$. In the case of cone we write $\mathbb{R}^N = \mathbb{R}^{N-l} \times \mathbb{R}^l$ and we extend a root system of $\mathbb{R}^{N-l}$ to $\mathbb{R}^N$. For an element $x \in \mathbb{R}^N$ we write $x = (x', x_{N-l+1}, x_{N-l+2}, ...x_N)$ where $x' \in \mathbb{R}^{N-l}$. Let R_1 be a root system on $\mathbb{R}^{N-l}$ and k_1, $d\mu_{k_1} := h_k^2(x')$ be the corresponding multiplicity function and Dunkl weighted measure. Define $R := \{(x, 0) \in \mathbb{R}^N : x \in R_1\}$. It is easy to verify that R is a root system on $\mathbb{R}^N$. Now as in the case of upper half-space extend the multiplicity function to k of $\mathbb{R}^N$ as $k(x', 0) = k_1(x)$ and the corresponding Dunkl weighted measure $d\mu_k(x) = d\mu_{k_1}(x')dx_{N-l+1}...dx_N$. For the convenience of the calculations we write $x \in \mathbb{R}^N$ as $x = (x', x'')$ with $x' \in \mathbb{R}^{N-l}$ and $x'' \in \mathbb{R}^l$.

Theorem 3.5.1. *Let $N \in \mathbb{N}$, $1 \leqslant p < \infty$. Further $0 < s < 1$ with a condition $ps \neq 1$. Then for all $u \in \dot{W}^p_s(\mathbb{R}^N_{l_+})$ the following inequality holds:*

$$\int_{\mathbb{R}^N_{l_+}} \int_{\mathbb{R}^N_{l_+}} |u(x) - u(y)|^p \Phi_{ps}(x, y) d\mu_k(x) d\mu_k(y) \geqslant D_{N_l,\gamma_k,p,s} \int_{\mathbb{R}^N_{l_+}} \frac{|u(x)|^2}{x^2_{N-l+1} + \cdots + x^2_N} d\mu_k(x). \tag{3.5.1}$$

Here

$$D_{N_l,\gamma_k,p,s} = \frac{c_{k_1}^{-1} 2^{-\lambda_k} \Gamma((l + ps)/2)}{\Gamma((d_k + ps)/2)} \int_0^1 r^{ps-1} |1 - r^{(l-ps)/p}|^p \tilde{\Phi}_{l_+,s,p}(r) dr \tag{3.5.2}$$

with

$$\tilde{\Phi}_{l_+,s,p}(r) = \int_{\mathbb{S}^{l-1}_{l_+}} \frac{1}{|\tilde{x} - r\tilde{y}|^{l+ps}} d\sigma(\tilde{y}),$$

where $\tilde{x} \in \mathbb{S}^{l-1}_{l_+}$ *and* $\mathbb{S}^{l-1}_{l_+} = \mathbb{S}^{l-1} \cap \mathbb{R}^l_{l_+}$. *The constant* $D_{N_l,\gamma_k,p,s}$ *is optimal. If* $p = 1$ *and* $N = l$, *equality holds iff* u *is proportional to a non-increasing function. Also for* $p \geqslant 2$ *the following inequality holds:*

$$\begin{aligned}
&\int_{\mathbb{R}^N_{l_+}} \int_{\mathbb{R}^N_{l_+}} |u(x) - u(y)|^p \Phi_{ps}(x, y) d\mu_k(x) d\mu_k(y) \\
&\geqslant D_{N_l,\gamma_k,p,s} \int_{\mathbb{R}^N_{l_+}} \frac{|u(x)|^p}{|x''|^{ps}} d\mu_k(x) \\
&+ c_p \int_{\mathbb{R}^N_{l_+}} \int_{\mathbb{R}^N_{l_+}} |v(x) - v(y)|^p \Phi_{ps}(x, y) \frac{d\mu_k(x)}{|x''|^{(1-ps)/2}} \frac{d\mu_k(y)}{|y''|^{(1-ps)/2}}, \qquad (3.5.3)
\end{aligned}$$

where $v := |x''|^{(l-ps)/p} u$, Φ *is as in (3.3.3),* $D_{N,\gamma_k,p,s}$ *is given in (3.5.2) and* c_p *is given in (3.2.19). Moreover* $c_2 = 1$ *and the equality holds in* $p = 2$ *case.*

Proof. The proof is very similar to that of Hardy inequality of the half-space. Similar steps will lead to the desired conclusion. In order to find a positive solution of the Euler Lagrange equation corresponding to (3.5.1) we set $w(x) = |x''|^{-(l-ps)/2}$ and $V(x) = D_{N_l,\gamma_k,p,s}|x''|^{-ps}$. The $\Phi_{ps}(x, y)$ given in (3.3.3) will take the form

$$\begin{aligned}
\Phi_{ps}(x, y) &:= \frac{1}{\Gamma((d_k + ps)/2)} \int_0^\infty s^{\frac{d_k+ps}{2}-1} \tau^k_y \big(e^{-s|.|^2}\big)(x) ds \\
&= \frac{1}{\Gamma((d_k + ps)/2)} \int_0^\infty s^{\frac{d_k+ps}{2}-1} e^{-s\sum_{j=N-l+1}^N |x_j - y_j|^2} \tau_{y'} \big(e^{-s|.|^2}\big)(x') ds,
\end{aligned}$$

since

$$\tau^k_y \big(e^{-s|.|^2}\big)(x) = e^{-s\sum_{j=N-l+1}^N |x_j - y_j|^2} \tau_{y'} \big(e^{-s|.|^2}\big)(x')$$

with our root system R on $\mathbb{R}^N$.

Repeating the same arguments as in the proof of the Theorem 3.4.1 we obtain

$$\int\limits_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y)$$
$$= \int\limits_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}} \frac{(w(x)-w(y))|w(x)-w(y)|^{p-2}}{(|x''-y''|^2+|y'|^2)^{\frac{d_k+ps}{2}}}d\mu_k(y).$$

We evaluate $\int_{\mathbb{R}^{N-l}} \frac{1}{(m^2+|y'|^2)^{\alpha/2}}d\mu_k(y')$ as in the previous proof with $m=|x''-y''|$ and find

$$\int_{\mathbb{R}^{N-l}} \frac{1}{(|x''-y''|^2+|y'|^2)^{\frac{d_k+ps}{2}}}d\mu_k(y') = \pi^{\frac{d_{k_1}}{2}}\frac{\Gamma((l+ps)/2)}{\Gamma((d_k+ps)/2)}\frac{1}{|x''-y''|^{l+ps}},$$

where $d_{k_1}=N-l+2\gamma_{k_1}$. Now the Euler Lagrange equation corresponding to (3.5.1) is of the form

$$2\lim_{\epsilon\to 0}\int\limits_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}} (w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y)$$
$$= \frac{c_{k_1}^{-1}2^{-\lambda_{k_1}}\Gamma((l+ps)/2)}{\Gamma((d_k+ps)/2)}\times \lim_{\epsilon\to 0}\int\limits_{\substack{y\in\mathbb{R}^l_{l_+},\\ ||x''|-|y''||>\epsilon}} \frac{(w(x'')-w(y''))|w(x'')-w(y'')|^{p-2}}{(|x''-y''|)^{l+ps}}dy'', \qquad (3.5.4)$$

with $w(x'')=|x''|^{-(l-ps)/p}$.

If $\mathbb{S}^{l-1}_{l_+}=\mathbb{S}^{l-1}\cap\mathbb{R}^l_{l_+}$, the polar decomposition of right-hand side integral of

(3.5.4) can be written as

$$\lim_{\epsilon\to 0}\int\limits_{\substack{y\in\mathbb{R}^l_{l_+},\\ ||x''|-|y''||>\epsilon}} \frac{(w(x'')-w(y''))|w(x'')-w(y'')|^{p-2}}{(|x''-y''|)^{l+ps}}dy''$$
$$=\int_{|\rho-r|>\epsilon}\int_{\mathbb{S}^{l-1}_{l_+}} \frac{(r^{-\alpha}-\rho^{-\alpha})|r^{-\alpha}-\rho^{-\alpha}|^{p-2}}{|r\tilde{x}-\rho\tilde{y}|^{l+ps}}d\sigma(\tilde{y})d\rho,$$

where $x''=r\tilde{x}$, $y''=\rho\tilde{y}$ and $\alpha=(l-ps)/p$. using similar steps in the proof of [15, Lemma 3.1] we can prove that

$$2\lim_{\epsilon\to 0}\int_{\mathbb{R}^l_{l_+}} \frac{(w(x'')-w(y''))|w(x'')-w(y'')|^{p-2}}{|x''-y''|^{l+ps}}dy''=\frac{\tilde{C}_{l_+,s,p}}{|x''|^ps}w(x'')^{p-1}, \qquad (3.5.5)$$

where for $l\geqslant 2$

$$\tilde{C}_{l_+,s,p}=2\int_0^1 r^{ps-1}|1-r^{(l-ps)/p}|^p\tilde{\Phi}_{l_+,s,p}(r)dr$$

with

$$\tilde{\Phi}_{l_+,s,p}(r)=\int_{\mathbb{S}^{l-1}_{l_+}}\frac{1}{|\tilde{x}-r\tilde{y}|^{l+ps}}d\sigma(\tilde{y})$$

and when $l=1$ then $\tilde{C}_{1_+,s,p}=\tilde{C}_{1,p,s}$ given in equation (3.4.11). The constant $\tilde{C}_{l_+,s,p}$ is different from the constant $C_{l,s,p}$ given in [15, Theorem 1.1] since instead of integrating over the whole sphere $\mathbb{S}^{l-1}$ we are only integrating over the points on the sphere which intersect with the cone, that is only on $\mathbb{S}^{l-1}_{l_+}$.

Define $D_{N_l,\gamma_k,p,s}:=\frac{c_{k_1}^{-1}2^{-\lambda_{k_1}}\Gamma((l+ps)/2)}{\Gamma((d_k+ps)/2)}\tilde{C}_{l_+,s,p}$, from (3.5.4) and (3.5.5), we get w

as the positive solution of the Euler Lagrange equation corresponding to (3.5.1);

$$2\lim_{\epsilon\to 0}\int_{\substack{y\in\mathbb{R}^N_{l_+},\\ ||x''|-|y''||>\epsilon}}(w(x)-w(y))|w(x)-w(y)|^{p-2}\Phi_{ps}(x,y)d\mu_k(y)$$
$$=\frac{D_{N_l,\gamma_k,p,s}}{|x''|^{ps}}w(x)^{p-1}.$$

Proof of the Hardy inequalities (3.5.1) and (3.5.3) and the proof of optimality of the constant $D_{N_l,\gamma_k,p,s}$(it follows from the optimality of $\tilde{C}_{1_+,s,p}$) can be obtained by the same techniques used in the proof of [15, Theorem 1.1, Theorem 1.2]. □

Remark 3.5.2. Since we could not calculate the integral $\int_{\mathbb{S}^{l-1}_{l_+}}\frac{1}{|\tilde{x}-r\tilde{y}|^{l+ps}}d\sigma(\tilde{y})$ explicitly, the expression of the constant $D_{N_l,\gamma_k,p,s}$ in the Theorem 3.5.1 is not explicit compare to the constants given in Theorem 3.3.1 and Theorem 3.4.1 .

3.6 Stein-Weiss Inequality and Some Related Inequalities

In [32], Stein and Weiss have proven the following inequality which is known as Stein-Weiss inequality.

For every $0<\beta<N$ and for every $\varphi\in L^2(\mathbb{R}^N)$ there exists a positive constant such that the following inequality holds:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta}{2}}|x-y|^{N-\beta}|y|^{\frac{\beta}{2}}}dxdy\leqslant C\|\varphi\|_2^2. \tag{3.6.1}$$

Moreover, it has been proven in [21], that $C=\frac{1}{2^\beta}\left(\frac{\Gamma(\frac{N-\beta}{4})}{\Gamma(\frac{N+\beta}{4})}\right)^2$ is the best constant. We will prove the following Dunkl version of (3.6.1) using the ground state representation technique.

Theorem 3.6.1. *Let $0 < \beta < d_k$. Then for every $\varphi \in L^2(\mathbb{R}^N, d\mu_k(x))$ the Stein-Weiss inequality is given by*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}}\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y) \leqslant \frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2 \int_{\mathbb{R}^N} |\varphi(x)|^2 d\mu_k(x), \tag{3.6.2}$$

where the constant appearing on the right hand side is optimal.

Proof. Let $w(x) = |x|^{-\frac{d_k}{2}}$. Then by using Lemma 4.1 and Proposition 4.2 of [38] we have

$$\int_{\mathbb{R}^N\setminus\{0\}} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}} w(x)d\mu_k(x) = \frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2 w(y). \tag{3.6.3}$$

Multiply the left hand side by the test function φ^2/w and integrate over $\mathbb{R}^N$ to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}} w(x)\frac{\varphi^2(y)}{w(y)}d\mu_k(y)d\mu_k(x)\\
&= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}})\left(w(x)\frac{\varphi^2(y)}{w(y)} + w(y)\frac{\varphi^2(x)}{w(x)}\right)d\mu_k(y)d\mu_k(x)\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}}\varphi(x)\varphi(y)d\mu_k(x)d\mu_k(y)\\
&+ \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}})w(x)w(y)\left(\frac{\varphi(x)}{w(x)} - \frac{\varphi(y)}{w(y)}\right)d\mu_k(x)d\mu_k(y).
\end{aligned} \tag{3.6.4}$$

Similarly multiply the right hand side of (3.6.3) by φ^2/w and integrate. Then

$$\frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2 \int_{\mathbb{R}^N} w(y)\frac{\varphi^2(y)}{w(y)} = \frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2 \int_{\mathbb{R}^N} \varphi^2 d\mu_k(x). \tag{3.6.5}$$

Now by putting the equations (3.6.3), (3.6.4) and (3.6.5) together, we get

$$\begin{aligned}
&\frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2\int_{\mathbb{R}^N}\varphi^2(x)d\mu_k(x)\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}}\varphi(x)\varphi(y)d\mu_k(x)d\mu_k(y)\\
&+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}}w(x)w(y)\left(\frac{\varphi(x)}{w(x)}-\frac{\varphi(y)}{w(y)}\right)^2d\mu_k(x)d\mu_k(y)\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}}\varphi(x)\varphi(y)d\mu_k(x)d\mu_k(y)\\
&+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}}\left||x|^{\frac{d_k}{2}}\varphi(x)-|y|^{\frac{d_k}{2}\varphi(y)}\right|^2d\mu_k(x)d\mu_k(y).
\end{aligned} \tag{3.6.6}$$

Since the second integral on the right hand side of (3.6.6) is positive we obtain the Stein-Weiss inequality stated in (3.6.2).

To obtain the optimality it is sufficient to prove that

$$\sup_{\substack{\varphi\in L^2(\mathbb{R}^N)\\ \|\varphi\|_2\leqslant 1}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta}{2}}|y|^{\frac{\beta}{2}}}\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y)=\frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2. \tag{3.6.7}$$

To deduce (3.6.7) from (3.6.6) we will find a family functions $\{u_t\}_{t\geqslant 1}$ in $L^2(\mathbb{R}^N, d\mu_k(x))$ such that

$$\sup_{t\geqslant 1}\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\lambda_k+\beta}{2}}|y|^{\frac{\lambda_k+\beta}{2}}}\left|u_t(x)|x|^{\frac{\lambda_k}{2}}-u_t(y)|y|^{\frac{\lambda_k}{2}}\right|^2d\mu_k(x)d\mu_k(y)}{\int_{\mathbb{R}^N}|u_t^2|d\mu_k(x)}=0. \tag{3.6.8}$$

Let $t\geqslant 1$ and $\eta\in C\big((0,\infty);[0,1]\big)$ be such that $\eta=1$ on $(0,1)$, $\eta=0$ on $(2,\infty)$. Define $u_t(x):=\eta(\frac{|x|}{t})\eta(\frac{1}{t|x|})\frac{1}{|x|^{\frac{d_k}{2}}}$. First we will show that the numerator of (3.6.8) is finite. We find

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}} \left|u_t(x)|x|^{\frac{d_k}{2}} - u_t(y)|y|^{\frac{d_k}{2}}\right|^2 d\mu_k(x)d\mu_k(y)\\
&\leqslant \int_{\mathbb{R}^{2N}\setminus(B_t\setminus B_{1/t})^2\setminus(B_{1/2t}\cup\mathbb{R}^N\setminus B_{2t})^2} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}} d\mu_k(x)d\mu_k(y)\\
&\leqslant 2\int_{B_{2t}}\int_{\mathbb{R}^N\setminus B_t} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}} d\mu_k(x)d\mu_k(y)\\
&\qquad\qquad + 2\int_{B_{1/t}}\int_{\mathbb{R}^N\setminus B_{1/2t}} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}} d\mu_k(x)d\mu_k.
\end{aligned}$$

To prove that the right hand side is finite, by scale invariance, we need to realize that the integral

$$\int_{B_2}\int_{\mathbb{R}^N\setminus B_1} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\alpha}{2}}|y|^{\frac{d_k+\alpha}{2}}} d\mu_k(x)d\mu_k(y)$$

is finite. Since it is sufficient to do for any $t \geqslant 1$ we arrive the conclusion that

$$\sup_{t\geqslant 1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}} \left|u_t(x)|x|^{\frac{d_k}{2}} - u_t(y)|y|^{\frac{d_k}{2}}\right|^2 d\mu_k(x)d\mu_k(y) < \infty.$$

Dividing both sides by $\int_{\mathbb{R}^N}|u_t|^2$ and using the fact that

$$\lim_{t\to\infty}\int_{\mathbb{R}^N}|u_t|^2 = \infty,$$

we arrive to the proof of the Theorem. □

In 2008, W. Beckner found the optimal constant for the Stein Weiss potentials with gradient estimate, [7]. The author has proved that, for every $N \geqslant 3$ and

$0 < \beta < N$ the following inequality holds:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta+2}{2}}|x-y|^{N-\beta}|y|^{\frac{\beta+2}{2}}}dxdy \leqslant \frac{1}{2^{\beta-2}}\left(\frac{\Gamma(\frac{N-\beta}{4})}{(N-2)\Gamma(\frac{N+\beta}{4})}\right)^2 \|\varphi\|^2_{\dot{H}^1} \quad (3.6.9)$$

where $\varphi \in \dot{H}^1(\mathbb{R}^N)$ and the constant $\left(\frac{\Gamma(\frac{N-\beta}{4})}{(N-2)\Gamma(\frac{N+\beta}{4})}\right)^2$ is optimal. The space $\dot{H}^1(\mathbb{R}^N)$ is the homogeneous Sobolev space with the norm $\|\varphi\|^2_{\dot{H}^1} := \int_{\mathbb{R}^N}|\nabla\varphi|^2dx$. Now we state the corresponding result in the Dunkl setting.

Theorem 3.6.2. *Let $d_k \geqslant 3$ and $0 < \beta < d_k$. Then for every G-invariant $\varphi \in \dot{H}^1(\mathbb{R}^N, d\mu_k(x))$ the following inequality*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}}\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y)$$
$$\leqslant \frac{1}{2^{\beta-2}}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{(d_k-2)\Gamma(\frac{d_k+\beta}{4})}\right)^2\int_{\mathbb{R}^N}|\nabla_k\varphi|^2d\mu_k(x) \quad (3.6.10)$$

holds for all φ for which right hand side is finite. Also the constant appearing on the right-hand side of the inequality is optimal.

Proof. Let $w(x) = |x|^{-\frac{d_k-2}{2}}$.

$$\int_{\mathbb{R}^N\setminus\{0\}} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}}w(x)d\mu_k(x) = \frac{1}{2^\beta}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+\beta}{4})}\right)^2\frac{1}{|y|^{\frac{d_k+2}{2}}}. \quad (3.6.11)$$

Using the following expression of Dunkl Laplacian for radial functions,

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{2\lambda_k+1}{r}\frac{\partial}{\partial r}; \qquad \lambda_k = \frac{d_k-2}{2},$$

we can calculate

$$\Delta_k w(y) = \left(\frac{d_k(d_k-2)}{4}\right)|y|^{-(\frac{d_k+2}{2})} - \left((d_k-1)\frac{d_k-2}{2}\right)|y|^{-(\frac{d_k+2}{2})}$$
$$= -\left(\frac{d_k-2}{2}\right)^2 \frac{1}{|y|^{\frac{d_k+2}{2}}}. \tag{3.6.12}$$

Using the expressions (3.6.11) and (3.6.12) together and Integrating both over the test function φ^2/w, we obtain

$$\frac{1}{2^{\beta-2}}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{(d_k-2)\Gamma(\frac{d_k+\beta}{4})}\right)^2 \int_{\mathbb{R}^N}(-\Delta_k)w(x).\frac{\varphi^2(x)}{w(x)}d\mu_k(x)$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N\setminus\{0\}} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}} w(x)\frac{\varphi^2(y)}{w(y)}d\mu_k(x)d\mu_k(y). \tag{3.6.13}$$

By similar calculations as in (3.6.4) the right-hand side of (3.6.13) become

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}} w(x)\frac{\varphi^2(y)}{w(y)}d\mu_k(y)d\mu_k(x)$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}} \varphi(x)\varphi(y)d\mu_k(x)d\mu_k(y)$$

$$+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}})w(x)w(y)\left(\frac{\varphi(x)}{w(x)}-\frac{\varphi(y)}{w(y)}\right)d\mu_k(x)d\mu_k(y). \tag{3.6.14}$$

Using the G-invariance of φ and the equations (3.2), (3.3) and (3.4) of [4] for the functions w and φ/w, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_k \varphi|^2 d\mu_k(x) &= \int_{\mathbb{R}^N} |\nabla_k \Big(\frac{\varphi}{w}\Big)|^2 w^2 d\mu_k(x) - \int_{\mathbb{R}^N} w \frac{\varphi^2}{w} \Delta_k w d\mu_k(x) \\ &= \int_{\mathbb{R}^N} \left|\nabla_k \Big(\frac{\varphi}{w}\Big)(x)\right|^2 w^2(x) d\mu_k(x) \\ &\quad + \int_{\mathbb{R}^N} \nabla_k \Big(\frac{\varphi^2}{w}\Big)(x).\nabla_k w(x) d\mu_k(x). \end{aligned} \tag{3.6.15}$$

The inequality in (3.6.15) allows us to write

$$\begin{aligned}&\frac{1}{2^{\beta-2}}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{(d_k-2)\Gamma(\frac{d_k+\beta}{4})}\right)^2 \int_{\mathbb{R}^N} (-\Delta_k) w(x). \frac{\varphi^2(x)}{w(x)} d\mu_k(x) \\ &= \frac{1}{2^{\beta-2}}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{(d_k-2)\Gamma(\frac{d_k+\beta}{4})}\right)^2 \int_{\mathbb{R}^N} \nabla_k \Big(\frac{\varphi^2}{w}\Big)(x).\nabla_k w(x) d\mu_k(x) \\ &= \frac{1}{2^{\beta-2}}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{(d_k-2)\Gamma(\frac{d_k+\beta}{4})}\right)^2 \\ &\qquad \left(\int_{\mathbb{R}^N} |\nabla_k \varphi|^2 d\mu_k(x) - \int_{\mathbb{R}^N} \left|\nabla_k \Big(\frac{\varphi}{w}\Big)(x)\right|^2 w^2(x) d\mu_k(x)\right). \end{aligned} \tag{3.6.16}$$

Now using the identities in (3.6.13) and (3.6.14) in (3.6.16) and substituting the expression of w gives

$$\begin{aligned}&\frac{1}{2^{\beta-2}}\left(\frac{\Gamma(\frac{d_k-\beta}{4})}{(d_k-2)\Gamma(\frac{d_k+\beta}{4})}\right)^2 \\ &\qquad \left(\int_{\mathbb{R}^N} |\nabla_k \varphi|^2 d\mu_k(x) - \int_{\mathbb{R}^N} \frac{\left|\nabla_k \Big(|x|^{\frac{d_k-2}{2}} \varphi\Big)(x)\right|^2}{|x|^{d_k-2}} d\mu_k(x)\right)\end{aligned}$$

$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}}\varphi(x)\varphi(y)d\mu_k(x)d\mu_k(y)$$
$$+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2}{2}}|y|^{\frac{\beta+2}{2}}})w(x)w(y)\left(\frac{\varphi(x)}{w(x)}-\frac{\varphi(y)}{w(y)}\right)d\mu_k(x)d\mu_k(y). \tag{3.6.17}$$

Since the second integral on both left and right hand side of (3.6.17) are positive we obtain the combination of Stein-Weiss and Hardy inequality stated in (3.6.10).

The optimality can be obtained similar to the idea of Theorem 3.6.1. Choose t and η as in Theorem 3.6.1. Define

$$u_t(x) := \eta\Big(\frac{|x|}{t}\Big)\eta\Big(\frac{1}{tx}\Big)\frac{1}{|x|^{\frac{d_k-2}{3}}}.$$

As in Theorem 3.6.1, we get

$$\lim_{t\to\infty}\int_{\mathbb{R}^N}|\nabla_k u_t|^2=\infty$$

and

$$\sup_{t\geqslant 1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}}\left|u_t(x)|x|^{\frac{d_k-2}{2}}-u_t(y)|y|^{\frac{d_k-2}{2}}\right|^2 d\mu_k(x)d\mu_k(y)<\infty.$$

Also, see that

$$\int_{\mathbb{R}^N}\frac{\left|\nabla_k(|x|^{\frac{d_k-2}{2}}u_t(x))\right|^2}{|x|^{d_k-2}}d\mu_k(x)=\int_{B_{2t}\setminus B_t}\frac{\eta'(|x|/t)^2}{t^2|x|^{d_k-2}}d\mu_k(x)$$
$$+\int_{B_{1/t}\setminus B_{1/2t}}\frac{t^2\eta'(t/|x|)^2}{|x|^{d_k+2}}d\mu_k(x).$$

A change of variable for $|x|/t$ gives the right hand side of the above equation independent of t and conclude as in the Theorem 3.6.1. □

The classical version of the Stein-Weiss inequality for the fractional gradient is proven by Moroz and Schaftingen in [25]. Their result states that for $0 < s < 1$, $s < N/2$ and $0 < \beta < N$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta+2s}{2}}|x-y|^{N-\beta}|y|^{\frac{\beta+2s}{2}}}dxdy \leqslant \frac{1}{2^{\beta+2s}}\left(\frac{\Gamma(\frac{N-2s}{4})\Gamma(\frac{N-\beta}{4})}{\Gamma(\frac{N+2s}{4})\Gamma(\frac{N+\beta}{4})}\right)^2 \|\varphi\|^2_{\dot{H}^s} \tag{3.6.18}$$

holds for every $\varphi \in \dot{H}^s(\mathbb{R}^N)$. Here $\dot{H}^s(\mathbb{R}^N)$ denotes the fractional homogeneous Sobolev space equipped with the norm

$$\|\varphi\|^2_{\dot{H}^s} := \frac{s\Gamma(\frac{N+2s}{2})}{2^{2(1-s)}\pi^{N/2}\Gamma(1-s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}dxdy. \tag{3.6.19}$$

The statement of the Stein-Weiss potential with the Dunkl fractional gradient is as follows:

Theorem 3.6.3. *Let $s \in (0,1)$, $s < d_k/2$ and $0 < \beta < d_k$. Then for all $\varphi \in \dot{W}^{s,2}(\mathbb{R}^N)$ the following inequality holds*

$$\begin{aligned} C_{k,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(y)|^2\Phi_{2s}(x,y)d\mu_k(x)d\mu_k(y) \\ \geqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y)\end{aligned} \tag{3.6.20}$$

and the constant $C_{k,s} = \frac{1}{2^{\beta+s}}\left(\frac{\Gamma(\frac{d_k-2s}{4})\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+2s}{4})\Gamma(\frac{d_k+\beta}{4})}\right)^2$ is optimal.

Proof. Let $w(x) = \frac{1}{|x|^{\frac{d_k-s}{2}}}$. By the definition of fractional power of Dunkl Laplacian and by Plancheral theorem for Dunkl transform we can write

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\varphi(x)-\varphi(y))(w(x)-w(y))\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y)\\
&=\int_{\mathbb{R}^N}\overline{\mathcal{F}_k(\varphi)(\xi)}|\xi|^s\mathcal{F}_k(w)(\xi)d\mu_k(\xi)\\
&=2^s\frac{\Gamma(\frac{d_k+2s}{4})^2}{\Gamma(\frac{d_k-2s}{4})^2}\int_{\mathbb{R}^N}\frac{\varphi(x)}{|x|^{\frac{d_k+2s}{2}}}d\mu_k(x).
\end{aligned}\tag{3.6.21}$$

Now by the semi group properties of Dunkl Riesz potential one can write

$$\int_{\mathbb{R}^N}\frac{|y|^{\frac{d_k+\beta}{2}}}{|x|^{\frac{\beta+2s}{2}}}\Phi_{-\beta}(x,y)d\mu_k(y)=2^{-\beta}\frac{\Gamma(\frac{d_k-\beta}{4})^2}{\Gamma(\frac{d_k+\beta}{4})^2}\frac{1}{|x|^{\frac{d_k+2s}{2}}}.\tag{3.6.22}$$

Now combining the equations (3.6.21) and (3.6.22) will allow us to write

$$\begin{aligned}
&\frac{1}{2^{\beta+s}}\left(\frac{\Gamma(\frac{d_k-2s}{4})\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+2s}{4})\Gamma(\frac{d_k+\beta}{4})}\right)^2\\
&\qquad\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\varphi(x)-\varphi(y))(w(x)-w(y))\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y)\\
&=\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}w(x)d\mu_k(x).
\end{aligned}\tag{3.6.23}$$

Integrating again the integral on the right hand side of (3.6.23) after multiplying with φ^2/w, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}w(x)\frac{\varphi(y)^2}{w(y)}d\mu_k(x)\\
&=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}\left(w(x)\frac{\varphi(y)^2}{w(y)}+w(y)\frac{\varphi(x)^2}{w(x)}\right)d\mu_k(x)d\mu_k(y)
\end{aligned}$$

$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}\varphi(x)\varphi(y)d\mu_k(x)d\mu_k(y)$$
$$+ \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}w(x)w(y)\left(\frac{\varphi(x)}{w(x)}-\frac{\varphi(y)}{w(y)}\right)^2 d\mu_k(x)d\mu_k(y). \tag{3.6.24}$$

Using the equation (6.3) of [4], we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(w(x)-w(y))(\psi(x)-\psi(y))\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y)$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[|\varphi(x)-\varphi(y)|^2 - |\frac{\varphi}{w}(x)-\frac{\varphi}{w}(y)|^2 w(x)w(y)\right]\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y). \tag{3.6.25}$$

Combine the equations (3.6.23), (3.6.24) and (3.6.25) we get the following equality:

$$\frac{1}{2^{\beta+s}}\left(\frac{\Gamma(\frac{d_k-2s}{4})\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+2s}{4})\Gamma(\frac{d_k+\beta}{4})}\right)^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(x)-\varphi(y)|^2\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y)$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}\varphi(x)\varphi(y)d\mu_k(x)d\mu_k(y)$$
$$+ \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}w(x)w(y)\left(\frac{\varphi(x)}{w(x)}-\frac{\varphi(y)}{w(y)}\right)^2 d\mu_k(x)d\mu_k(y)$$
$$+ \frac{1}{2^{\beta+s}}\left(\frac{\Gamma(\frac{d_k-2s}{4})\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+2s}{4})\Gamma(\frac{d_k+\beta}{4})}\right)^2$$
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\frac{\varphi}{w}(x)-\frac{\varphi}{w}(y)|^2 w(x)w(y)\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y). \tag{3.6.26}$$

Since the second and third integrals on the right-hand side of (3.6.26) are positive we arrive at the inequality given in (3.6.20).

To prove the optimality, as in the proof of previous theorems we will consider a family of function $\{u_t\}_{t\geqslant 1}$ in $\dot{H}^s(\mathbb{R}^N)$ whose homogeneous fractional norm converges to infinity as t goes to infinity. Then if we prove that second and third integrals on the right-hand side of (3.6.26) are finite we are done with the proof of optimality.

Define the functions u_t for $t > 1$ as

$$u_t(x) := \eta\Big(\frac{|x|}{t}\Big)\eta\Big(\frac{1}{t|x|}\Big)\frac{1}{|x|^{\frac{d_k-2s}{2}}},$$

where the function η is continuous from the positive reals to the closed interval $[0,1]$ with $\eta(x) = 1$ when $0 < x < 1$ and it vanishes if $x > 2$.

For $0 < \beta < d_k$,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}}|u_\lambda(x)|x|^{\frac{d_k-2s}{2}} - u_t(y)|y|^{\frac{d_k-2s}{2}}|^2 d\mu_k(x)d\mu_k(y) \\
&\leqslant 2\int_{B_{2t}}\int_{\mathbb{R}^N\setminus B_t}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}}d\mu_k(x)d\mu_k(y) \\
&\qquad + 2\int_{B_{1/t}}\int_{\mathbb{R}^N\setminus B_1}\frac{\Phi_{-\beta}}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}}d\mu_k(x)d\mu_k(y).
\end{aligned} \tag{3.6.27}$$

Since

$$\int_{B_2}\int_{\mathbb{R}^N\setminus B_1}\frac{\Phi_{-\beta}(x,y)}{|x|^{\frac{d_k+\beta}{2}}|y|^{\frac{d_k+\beta}{2}}}d\mu_k(x)d\mu_k(y) < \infty$$

it deduces that the left-hand side of (3.6.27) is finite.

Now to finish the proof, we will prove that the following integral is finite. Let us compute the integral

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_t(x)|x|^{\frac{d_k-2s}{2}} - u_t(y)|y|^{\frac{d_k-2s}{2}}|^2}{|x|^{\frac{d_k-2s}{2}}\Phi_{-\beta}(x,y)|y|^{\frac{d_k-2s}{2}}} d\mu_k(x)d\mu_k(y) \\
&\leqslant \int_{B_{2t}}\int_{\mathbb{R}^N\setminus B_t} \frac{1}{|x|^{\frac{d_k-2s}{2}}\Phi_{-\beta}(x,y)|y|^{\frac{d_k-2s}{2}}} d\mu_(x)d\mu_k(y) \\
&\qquad + \int_{B_{1/t}}\int_{\mathbb{R}^N\setminus B_{1/2t}} \frac{1}{|x|^{\frac{d_k-2s}{2}}\Phi_{-\beta}(x,y)|y|^{\frac{d_k-2s}{2}}} d\mu_(x)d\mu_k(y)
\end{aligned} \tag{3.6.28}
$$

Observe that

$$
\int_{B_2}\int_{\mathbb{R}^N\setminus B_1} \frac{1}{|x|^{\frac{d_k-s}{2}}\Phi_{-\beta}(x,y)|y|^{\frac{d_k-s}{2}}} d\mu_k(x)d\mu_k(y) < \infty \tag{3.6.29}
$$

and the right-hand side of (3.6.28) is bounded by the integral of the form (3.6.29) which is independent of t and finite. Finally note that the $\dot{H}^s$ norm of u_t

$$
\begin{aligned}
&\lim_{t\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_t(x)-u_t(y)|^2\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y) \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi_{-\beta}(x,y)\left|\frac{1}{|x|^{\frac{d_k-2s}{2}}} - \frac{1}{|y|^{\frac{d_k-2s}{2}}}\right|^2 d\mu_k(x)d\mu_k(y) = \infty.
\end{aligned}
$$

Now it can be concluded that

$$
\begin{aligned}
&\sup_{\substack{\varphi\in\dot{H}^s(\mathbb{R}^N)\\ \|\varphi\|_{\dot{H}^s}\leqslant 1}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(x)\varphi(y)}{|x|^{\frac{\beta+2s}{2}}|y|^{\frac{\beta+2s}{2}}}\Phi_{-\beta}(x,y)d\mu_k(x)d\mu_k(y) \\
&\qquad\qquad = \frac{1}{2^{\beta+s}}\left(\frac{\Gamma(\frac{d_k-2s}{4})\Gamma(\frac{d_k-\beta}{4})}{\Gamma(\frac{d_k+2s}{4})\Gamma(\frac{d_k+\beta}{4})}\right)^2
\end{aligned}
$$

which gives the optimality of (3.6.20). □

Chapter 4

Improved L^p Fractional Hardy Inequalities in the Dunkl Setting

In this chapter we improve the fractional Hardy inequalities discussed in the last chapter. We will establish improved Hardy inequalities in this chapter which are true for $1 < p < \infty$ as well as the improvement term is coming from a norm associated to a fractional Dunkl gradient.

4.1 Introduction

In a remarkable paper [16], Frank and Seiringer have proven the sharp Hardy inequality with a remainder term. Their result is as follows: for $p \geqslant 2$ and $0 < s < 1$ and for some positive constants $C_{N,s,p}$ and c_p

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - C_{N,s,p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \geqslant c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{(N-ps)/2}} \frac{dy}{|y|^{(N-ps)/2}}, \tag{4.1.1}$$

where $v := |x|^{(N-ps)/2}u$. The result is true for all $u \in C_0^\infty(\mathbb{R}^N)$ if $ps < N$ and for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $ps > N$. The same authors proved a similar fractional Hardy inequality on half-space in [15], which states that: for $p \geqslant 2$, $0 < s < 1$ and $ps \neq 1$

$$\begin{aligned}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - D_{N,p,s}\int_{\mathbb{R}^N_+}\frac{|u(x)|^p}{x_N^{ps}}dx \\ \geqslant c_p \int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\frac{dx}{x_N^{(1-ps)/2}}\frac{dy}{y_N^{(1-ps)/p}},\end{aligned} \tag{4.1.2}$$

where $D_{N,p,s}$ and c_p are positive constants and $v := x_N^{(1-ps)/p}u$. A more generalized version of (4.1.1) and (4.1.2) in the Dunkl settings are proven in [5]. Combining the results due to Abdellaoui et al. in [1, 2, 3] we can get an improved fractional Hardy inequality for $1 < p < \infty$ which is stated below.

Let $0 < s < 1$, $ps < N$, $1 < q < p < \infty$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - C_{N,p,s}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx \\ \geqslant C \int_{\Omega}\int_{\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}dxdy\end{aligned} \tag{4.1.3}$$

for all functions $u \in C_0^\infty(\Omega)$. The constant $C_{N,p,s}$ is the sharp constant in the fractional Hardy inequality obtained by Frank et al. in [16] and the constant C is positive and depends on N, q, s and the domain Ω. Unlike in [16] the result is true for all $1 < p < \infty$ and the remainder term here is a p-norm of a fractional gradient.

In the proof of fractional Hardy inequalities mentioned in (4.1.1), (4.1.2) and (4.1.3), various properties of the kernel of the form $|x-y|^{-(N+\delta)}$ with $\delta > -N$

play an important role. When it comes to the Dunkl case we use a generalized kernel Φ_δ, $\delta > -d_k$ which is defined in (4.2.1).

4.2 Fractional Sobolev Spaces and Some Auxiliary Lemmas

We begin the section by stating three algebraic lemmas which we will use later to prove the main theorems.

Lemma 4.2.1. *[22, P. Lindqvist] For any $1 < p < 2$ there exist a positive constant c depending on p such thar for all $a, b \in \mathbb{R}^N$ we have:*

$$|a|^p - |b|^p - p|b|^{p-2}\langle b, a-b\rangle \geqslant c\frac{|a-b|^2}{(|a|+|b|)^{2-p}}$$

and for $p \geqslant 2$

$$|a|^p - |b|^p - p|b|^{p-2}\langle b, a-b\rangle \geqslant \frac{|a-b|^2}{2^{p-1}-1}.$$

Lemma 4.2.2. *[2, B. Abdellaoui, F. Mahmoudi] Let $1 \leqslant p \leqslant 2$ and $0 \leqslant t \leqslant 1$ and $a \in \mathbb{R}$. Then for some positive constant c depending only on p we have the following inequality:*

$$|a-t|^p - (1-t)^{p-1}(|a|^p - t) \geqslant c\frac{|a-1|^2 t}{(|a-t|+|1-t|)^{2-p}}.$$

4.2.1 Weighted Sobolev Spaces

We recall that Φ_δ with $\delta \neq -d_k$ given by the integral

$$\Phi_\delta(x,y) := \frac{1}{\Gamma((d_k+\delta)/2)}\int_0^\infty s^{\frac{d_k+\delta}{2}-1}\tau_y^k\left(e^{-s|.|^2}\right)(x)ds. \qquad (4.2.1)$$

If the multiplicity function is identically zero, that is $k \equiv 0$, then the kernel $\Phi_\delta(x,y)$ reduces to the Euclidean kernel $|x-y|^{-N-\delta}$. From this understanding we define fractional Sobolev space in the Dunkl setting by using $\Phi_\delta(x,y)$.

Let Ω be an open subset of $\mathbb{R}^N$ containing origin. Let $s \in (0,1)$ and $1 < p < \infty$. Then we define the fractional Sobolev space $W_k^{s,p}(\Omega)$ with the kernel Φ_{ps} as

$$W_k^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega, d\mu_k(x)) : \iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) < \infty \Big\},$$

and the norm is given by

$$\|u\|_{W_k^{s,p}(\Omega)} = \left(\int_\Omega |u|^p d\mu_k(x) \right)^{\frac{1}{p}} + \left(\iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) \right)^{\frac{1}{p}}.$$

Let $C_0^\infty(\Omega)$ be the compactly supported smooth functions on Ω. We define the Sobolev space $W_{k,0}^{s,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with the norm $\|.\|_{W_k^{s,p}(\Omega)}$.

Proposition 4.2.3. *Let $\Omega \subset \mathbb{R}^N$ be open and G-invariant. Let $u \in W_k^{s,p}(\Omega)$ and let $A \subset \Omega$ such that A is compact and u is supported in A. Define an extension $\tilde{u}$ on $\mathbb{R}^N$ as $\tilde{u}(x) = u(x)$ when $x \in \Omega$ and $\tilde{u}(x) = 0$ when $x \in \mathbb{R}^N \setminus \Omega$. Then $\tilde{u}$ belongs to $W_k^{s,p}(\mathbb{R}^N)$ and*

$$\|\tilde{u}\|_{W_k^{s,p}(\mathbb{R}^N)} \leqslant C(\Omega, A, d_k, p, s) \|u\|_{W_k^{s,p}(\Omega)}.$$

Proof. By the definition of $\tilde{u}$ it is clear that $\tilde{u} \in L^p(\mathbb{R}^N, d\mu_k(x))$. Since Φ_{ps} is symmetric on x and y, we can write

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\tilde{u}(x)-\tilde{u}(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)\\&=\int_{\Omega}\int_{\Omega}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)\\&+2\int_{\Omega}\left(\int_{\mathbb{R}^N\setminus\Omega}|u(x)|^p\Phi_{ps}(x,y)d\mu_k(y)\right)d\mu_k(x).\end{aligned}\tag{4.2.2}$$

Since $u\in W_k^{s,p}(\Omega)$

$$\int_{\Omega}\int_{\Omega}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y<\infty.$$

Also u is supported in A and hence for any $y\in\mathbb{R}^N\setminus\Omega$

$$|u(x)|^p\Phi_{ps}(x,y)=|u(x)|^p\chi_A(x)\Phi_{ps}(x,y).$$

Now by [17, Lemma 2.3]

$$\Phi_{ps}(x,y)=\int_{\mathbb{R}^N}\Big(|x|^2+|y|^2-2\langle y,\eta\rangle\Big)^{-\frac{d_k+ps}{2}}d\mu_k^x(\eta),$$

where μ_k^x is a probability Borel measure whose support is contained in $\mathrm{Co}(G)$, the convex hull of G-orbit of x in $\mathbb{R}^N$ (see also [27]). It is easy to see that for any $\eta\in\mathrm{Co}(G)$

$$\Big(|x|^2+|y|^2-2\langle y,\eta\rangle\Big)^{\frac{1}{2}}\geqslant\min_{\sigma\in G}|\sigma y-x|.$$

Using this and the fact that μ_k^x is a probability measure we get

$$\Phi_{ps}(x,y)\leqslant\Big(\min_{\sigma\in G}|\sigma y-x|\Big)^{-(d_k+ps)}.$$

Since Ω is G-invariant we find that $y\in\mathbb{R}^N\setminus\Omega$ implies $\sigma y\in\mathbb{R}^N\setminus\Omega$ for any $\sigma \in G$. Using the fact that A is compact and and Ω is bounded we have $dist(\sigma y, \partial A) \geqslant dist(\partial\Omega, \partial A) > 0$ for all $\sigma \in G$ and $y \in \mathbb{R}^N \setminus \Omega$.

But $\min_{\sigma\in G} |\sigma y - x| \geqslant \min_{\sigma\in G}(dist(\sigma y, \partial A))$ and hence we can write

$$\int_\Omega \left(\int_{\mathbb{R}^N\setminus\Omega} |u(x)|^p \Phi_{ps}(x,y) d\mu_k(y) \right) d\mu_k(x) \\ \leqslant \|u\|^p_{L^p(\Omega, d\mu_k(x))} \int_{\mathbb{R}^N\setminus\Omega} \frac{d\mu_k(y)}{dist(\partial\Omega, \partial A)}.$$

Since $dist(\partial\Omega, \partial A) > 0$ and $d_k + ps > d_k$ the integral

$$\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{dist(\partial\Omega, \partial A)} d\mu_k(y)$$

is finite. Finiteness of the above integral together with (4.2.2) we find that

$$\|\tilde{u}\|_{W_k^{s,p}(\mathbb{R}^N)} \leqslant C(d_k, p, s, A, \Omega) \|u\|_{W_k^{s,p}(\Omega)}$$

. □

For $1 < p < \infty$ and $0 < \beta < \frac{d_k - ps}{2}$ we define the kernel K_p^β as

$$K_p^\beta(x,y) = \frac{\Phi_{ps}(x,y)}{|x|^\beta |y|^\beta}.$$

We also define the weighted fractional Sobolev space $W_k^{s,p,\beta}(\Omega)$ with $0 \in \Omega$ as

$$W_k^{s,p,\beta}(\Omega) \\ := \left\{ u \in L^p(\Omega, \frac{d\mu_k(x)}{|x|^{2\beta}}) : \iint_{\Omega\times\Omega} |u(x) - u(y)|^p K_p^\beta(x,y) d\mu_k(x) d\mu_k(y) < \infty \right\}$$

endowed with the norm

$$\|u\|_{W_k^{s,p,\beta}(\Omega)} := \left(\int_\Omega |u(x)|^p \frac{d\mu_k(x)}{|x|^{2\beta}} \right)^{\frac{1}{p}} + \left(\iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p K_p^\beta(x,y) d\mu_k(x) d\mu_k(y) \right)^{\frac{1}{p}}.$$

For $1 < q < p$ and $0 < \beta < \frac{d_k - qs}{2}$ we define the space $W_k^{s,p,q,\beta}(\Omega)$as follows:

$$W_k^{s,p,q,\beta}(\Omega) := \left\{ u \in L^p(\Omega, \frac{d\mu_k(x)}{|x|^{2\beta}}) : \iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y) < \infty \right\},$$

where the norm is given by

$$\|u\|_{W_k^{s,p,q,\beta}(\Omega)} := \left(\int_\Omega |u(x)|^p \frac{d\mu_k(x)}{|x|^{2\beta}} \right)^{\frac{1}{p}} + \left(\iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y) \right)^{\frac{1}{p}}. \tag{4.2.3}$$

Let us denote $W_{k,0}^{s,p,q,\beta}(\Omega)$ be the completion $C_0^\infty(\Omega)$ with respect to the norm of $W_k^{s,p,q,\beta}(\Omega)$.

Using the similar arguments of Proposition 4.2.3 we can say that, if $u \in C_0^\infty(\Omega)$, with a compact support $A \subset \Omega$, then there exist an extension function $\tilde{u}$ of u belongs to $W_{k,0}^{s,p,q,\beta}(\mathbb{R}^N)$ such that

$$\|\tilde{u}\|_{W_k^{s,p,q,\beta}(\mathbb{R}^N)} \leqslant C\|u\|_{W_k^{s,p,q,\beta}(\Omega)}, \tag{4.2.4}$$

where Ω is G-invariant and $C = C(\Omega, A, d_k, p, q, s)$ is a positive constant.

If Ω is a bounded domain of $\mathbb{R}^N$ we can attach $W_{k,0}^{s,p,q,\beta}(\Omega)$ with an equivalent norm $|||.|||_{W_{k,0}^{s,p,q,\beta}}$,

$$|||u|||_{W_{k,0}^{s,p,q,\beta}(\Omega)} = \left(\iint_{\Omega\times\Omega} |u(x) - u(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y) \right)^{\frac{1}{p}}, \tag{4.2.5}$$

and for positive constants c and C we can write

$$c|||u|||_{W_{k,0}^{s,p,q,\beta}(\Omega)} \leqslant \|u\|_{W_k^{s,p,q,\beta}(\Omega)} \leqslant C|||u|||_{W_{k,0}^{s,p,q,\beta}(\Omega)}. \tag{4.2.6}$$

4.2.2 Picone's inequality

We are going to prove the Picone's Inequality for the Sobolev space $W_k^{s,p,q,\beta}(\Omega)$. Now for $w \in W_{k,0}^{s,p,q,\beta}(\mathbb{R}^N)$, we define

$$L(w)(x) = P.V. \int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2}(w(x) - w(y)) K_q^\beta(x,y) d\mu_k(x) d\mu_k(y)$$

and for $v, w \in W_{k,0}^{s,p,q,\beta}(\mathbb{R}^N)$, we have

$$\begin{aligned} &\langle L(w), v\rangle \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y)) K_q^\beta(x,y) d\mu_k(x) d\mu_k(y). \end{aligned}$$

Theorem 4.2.4. *Let $Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and w be a positive function in $W_{k,0}^{s,p,q,\beta}(\Omega)$ with $L(w)(x) \geqslant 0$ for all x in Ω. Then for all $u \in C_0^\infty(\Omega)$ the following inequality holds:*

$$\frac{1}{2} \iint_Q |u(x) - u(y)|^p \Phi_{qs}(x,y) \frac{d\mu_k(x) d\mu_k(y)}{|x|^\beta |y|^\beta} \geqslant \langle L(w), \frac{|u|^p}{w^{p-1}}\rangle.$$

Proof. Let $v(x) = \frac{|u(x)|^p}{|w(x)|^{p-1}}$,

$$
\begin{aligned}
&\langle L(w), v\rangle \\
&= \int_\Omega v(x) \int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y) \\
&= \int_\Omega \frac{|u(x)|^p}{|w(x)|^{p-1}} \int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y).
\end{aligned}
$$

Since $K_q^\beta(x,y) = K_q^\beta(y,x)$, we can write

$$
\begin{aligned}
&\langle L(w), v\rangle \\
&= \iint\limits_Q \left(\frac{|u(x)|^p}{|w(x)|^{p-1}} - \frac{|u(y)|^p}{|w(y)|^{p-1}}\right) \\
&\qquad\qquad |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y).
\end{aligned}
$$

Define the function $g = u/w$ and obtain

$$
\begin{aligned}
\langle L(w), v\rangle &= \frac{1}{2}\big(|g(x)|^p w(x) - |g(y)|^p w(y)\big) \\
&\qquad |w(x)-w(y)|^{p-2}(w(x)-w(y))K_q^\beta(x,y)d\mu_k(x)d\mu_k(y) \\
&= \frac{1}{2}\iint\limits_Q \big[|u(x)-u(y)|^p - \phi(x,y)\big]K_q^\beta(x,y)d\mu_k(x)d\mu_k(y),
\end{aligned}
$$

where

$$\phi(x,y) = |u(x)-u(y)|^p - (|g(x)|^p w(x) - |g(y)|^p w(y))|w(x)-w(y)|^{p-2}(w(x)-w(y)).$$

It is enough to prove $\phi \geqslant 0$ to get the desired inequality

$$\langle L(w), v\rangle \leqslant \frac{1}{2}\iint\limits_Q |u(x)-u(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y).$$

Since ϕ is symmetric we can assume that $w(x) \geqslant w(y)$. Putting $t = w(y)/w(x)$, $a = u(x)/u(y)$ and applying the inequality (3.3.8) in Lemma 3.3.3, we see that $\phi \geqslant 0$. □

Lemma 4.2.5. *Let $0 < \beta < \frac{d_k - qs}{2}$ and let $0 < \alpha < \frac{d_k - qs - 2\beta}{p-1}$. For $w(x) = |x|^{-\alpha}$ we have the following equality for a.e. non zero x in $\mathbb{R}^N$*

$$L(w) = \Lambda(\alpha)\frac{w^{p-1}}{|x|^{qs+2\beta}},$$

where $\Lambda(\alpha)$ is a positive constant.

Proof. For w given in the statement, we have

$$L(w)(x) = P.V.\int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2}(w(x) - w(y))K_q^\beta(x, y)d\mu_k(y).$$

Let $r = |x|$ and $\rho = |y|$. Also let $x = rx'$ and $y = \rho y'$ with $x', y' \in \mathbb{S}^{N-1}$. With these setting we can write

$$L(w)(x) = \int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{|r^{-\alpha} - \rho^{-\alpha}|^{p-2}(r^{-\alpha} - \rho^{-\alpha})\Phi_{qs}(rx' - \rho y')}{r^\beta \rho^\beta}\rho^{2\lambda_k+1}d\sigma_k(y')d\rho.$$

Let $t = \rho/r$. Using [17, Lemma 2.3] we have the following properties for Φ_δ

$$\Phi_\delta(rx', \rho y') = r^{-d_k-\delta}\Phi_\delta(x', ty')$$

and

$$P(t) := \int_{\mathbb{S}^{N-1}} \Phi_{qs}(x', ty')d\sigma_k(y') = \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})}\int_0^\pi \frac{sin^{d_k-2}\theta}{(1 - 2t\ cos\ \theta + t^2)^{\frac{d_k+qs}{2}}}d\theta. \tag{4.2.7}$$

With these properties we can write

$$L(w)(x) = \frac{r^{-\alpha(p-1)}}{r^{2\beta+qs}} \int_0^\infty |1-t^{-\alpha}|^{p-2}(1-t^{-\alpha})t^{2\lambda_k+1-\beta}P(t)dt = \Lambda(\alpha)\frac{w^{p-1}(x)}{|x|^{2\beta+qs}},$$

where $\Lambda(\alpha) = \int_0^\infty \varphi(t)dt$ with $\varphi(t) = |1-t^{-\alpha}|^{p-2}(1-t^{-\alpha})t^{2\lambda_k+1-\beta}P(t)$. Now we need to check the convergence of the integral $\int_0^\infty \varphi(t)dt$. With $t \to \frac{1}{t}$ and using the fact that $P(\frac{1}{t}) = t^{d_k+qs}P(1/t)$ we can write

$$\int_0^1 \varphi(t)dt = -\int_1^\infty (t^\alpha - 1)^{p-1}t^{\beta+ps-1}P(t)dt$$

and with this, $\Lambda(\alpha)$ becomes

$$\Lambda(\alpha) = \int_1^\infty (t^\alpha-1)^{p-1}P(t)\big(t^{d_k-1-\beta-\alpha(p-1)} - t^{\beta+qs-1}\big)dt. \tag{4.2.8}$$

Observe that $P(t)$ is similar to $\frac{1}{t^{d_k+qs}}$ as t tends to ∞ and $P(t)$ is dominated by a constant multiple of $\frac{1}{|t-1|^{1+qs}}$ as t tends to 1. Together with this understanding and using the assumption on α and β, as $t \to \infty$ we have

$$(t^\alpha-1)^{p-1}P(t)\big(t^{d_k-1-\beta-\alpha(p-1)} - t^{\beta+qs-1}\big) \simeq \frac{1}{t^{1+\beta+qs}} \tag{4.2.9}$$

and as $t \to 1$ we have

$$(t^\alpha-1)^{p-1}P(t)\big(t^{d_k-1-\beta-\alpha(p-1)} - t^{\beta+qs-1}\big) \simeq (t-1)^{p-1-qs}. \tag{4.2.10}$$

One can easily see that the similar function written on the right-hand side of (4.2.9) and (4.2.10) are integrable on the intervals $(2,\infty)$ and $(1,2)$ respectively.

This gives $\Lambda(\alpha)$ is finite. Now since $0 < \alpha(p-1) < d_k - qs - 2\beta$,

$$\left(t^{d_k-1-\beta-\alpha(p-1)} - t^{\beta+qs-1}\right) > 0$$

and hence from the expression of $\Lambda(\alpha)$ in (4.2.8) we conclude $\Lambda(\alpha) > 0$. □

We have just proved above that under the given assumptions

$$L(w) = \Lambda(\alpha)\frac{w^{p-1}}{|x|^{qs+2\beta}}.$$

Now Picone's Theorem 4.2.4 for this w gives that

$$\begin{aligned} 2\Lambda(\alpha)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{qs+2\beta}}d\mu_k(x) &= \langle L(w), \frac{|u|^p}{w^{p-1}}\rangle \\ &\leqslant \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y). \quad (4.2.11) \end{aligned}$$

Remark 4.2.6. Now choose Ω to be a bounded G-invariant domain containing origin and let $u \in C_0^\infty(\Omega)$. Then as we described earlier we have an extension function $\tilde{u}$ of $u \in W_k^{s,p,q,\beta}(\Omega)$. Using (4.2.11) for $\tilde{u}$ together with the equations (4.2.3) and (4.2.4) we find

$$\begin{aligned} 2\Lambda(\alpha)\int_{\mathbb{R}^N}\frac{|\tilde{u}(x)|^p}{|x|^{qs+2\beta}}d\mu_k(x) &\leqslant \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|\tilde{u}(x)-\tilde{u}(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y) \\ &\leqslant \|\tilde{u}\|_{W_k^{s,p,q,\beta}(\mathbb{R}^N)} \leqslant C\|u\|_{W_k^{s,p,q,\beta}(\Omega)}. \end{aligned}$$

Now by restricting $\tilde{u}$ to u and using equations (4.2.5) and (4.2.6), we obtain

$$\begin{aligned} 2\Lambda(\alpha)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{qs+2\beta}} &\leqslant C\|u\|_{W_k^{s,p,q,\beta}(\Omega)} \\ \leqslant C'|||u|||_{W_{k,0}^{s,p,q,\beta}(\Omega)} &= C'\iint\limits_{\Omega\times\Omega}|u(x)-u(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y). \end{aligned} \tag{4.2.12}$$

4.3 Improved Fractional Hardy Inequality on $\mathbb{R}^N$

In this section we give the proof of the Theorem 4.3.2. We start with the following lemma

Lemma 4.3.1. *Fix $\alpha = \frac{d_k-ps}{p}, \beta = \frac{d_k-ps}{2}$ and let $w(x) = |x|^{-\alpha}$. Let $u \in C_0^\infty(\mathbb{R}^N)$ and define $v(x) = u(x)/w(x)$. Then for all $1 < q < p < \infty$ and for a given positive constant C the following inequality holds:*

$$\begin{aligned} &\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|v(x)-v(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y) \\ &\qquad\geqslant C\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y). \end{aligned}$$

Proof. Let

$$\begin{aligned} f_1(x,y) &:= |v(x)-v(y)|^p K_q^\beta(x,y) \\ &= \frac{|w(y)u(x)-w(x)u(y)|^p}{\big(w(x)w(y)\big)^{\frac{p}{2}}}\Phi_{qs}(x,y) \\ &= \left|\big((u(y)-u(x)\big) - \frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\right|^p\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\Phi_{qs}(x,y). \end{aligned}$$

Observing the symmetry of $f_1(x,y)$ we define $f_2(x,y)$ in the following way

$$f_2(x,y) := \left|\Big((u(x)-u(y)\Big) - \frac{u(x)}{w(x)}\Big(w(y)-w(x)\Big)\right|^p \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} \Phi_{qs}(x,y).$$

Now the integral

$$H_k(v) := \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y)$$

can be written as

$$H_k(v) = \frac{1}{2}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} (f_1(x,y)+f_2(x,y))d\mu_k(x)d\mu_k(y).$$

Also let

$$Q(x,y) = \frac{\big(w(x)w(y)\big)^{\frac{p}{2}}}{w(x)^p+w(y)^p} \quad \text{and} \quad D(x,y) = \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} + \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}.$$

It is clear that $Q(x,y)\leqslant C$ and $Q(x,y)D(x,y)=1$ for all x and y. So for $p\geqslant 2$ we can apply the Lemma 4.2.1 to obtain the following inequality

$$\begin{aligned} f_1(x,y) \geqslant CQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \Bigg[&|u(x)-u(y)|^p\Phi_{qs}(x,y) \\ &- p|u(x)-u(y)|^{p-2}\Phi_{qs}(x,y)\langle u(x)-u(y), \frac{u(y)}{w(y)}(w(x)-w(y))\rangle \\ &+ c(p)|\frac{u(y)}{w(y)}(w(x)-w(y))|^p\Phi_{qs}(x,y)\Bigg] \end{aligned} \tag{4.3.1}$$

and for $1 < p < 2$, again by using Lemma 4.2.1, we can write

$$f_1(x,y) \geqslant CQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\Big[|u(x)-u(y)|^p\Phi_{qs}(x,y) + p|u(x)-u(y)|^{p-2}\Phi_{qs}(x,y)\langle u(x)-u(y), \frac{u(y)}{w(y)}(w(x)-w(y))\rangle. \quad (4.3.2)$$

Now combining equations (4.3.1) and (4.3.2), we can write for $1 < p < \infty$,

$$f_1(x,y) \geqslant \left[CQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}|u(x)-u(y)|^p\Phi_{qs}(x,y)\right] - pCQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}|u(x)-u(y)|^{p-1}\Phi_{qs}(x,y)\left|\frac{u(y)}{w(y)}\right||w(x)-w(y)|\Big].$$

Similarly, we can calculate

$$f_2(x,y) \geqslant \left[CQ(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}|u(x)-u(y)|^p\Phi_{qs}(x,y)\right] - pCQ(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}|u(x)-u(y)|^{p-1}\Phi_{qs}(x,y)\left|\frac{u(x)}{w(x)}\right||w(x)-w(y)|\Big].$$

Now by using the estimates of f_1 and f_2 we obtain

$$
\begin{aligned}
H_k(v) \geqslant C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} Q(x,y\Bigg[\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} + \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\Bigg] \\
|u(x-u(y))|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \\
- pC \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \Bigg[Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}|u(x)-u(y)|^{p-1} \\
\Phi_{qs}(x,y)\left|\frac{u(y)}{w(y)}\right|w(x)-w(y)|\Bigg]d\mu_k(x)d\mu_k(y) \\
- pC \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \Bigg[Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}|u(x)-u(y)|^{p-1} \\
\Phi_{qs}(x,y)\left|\frac{u(x)}{w(x)}\right|w(x)-w(y)|\Bigg]d\mu_k(x)d\mu_k(y).
\end{aligned}
$$

So we have

$$
\begin{aligned}
H_k(v) \geqslant C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \\
- C_1 \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \big(h_1(x,y)+h_2(x,y)\big)d\mu_k(x)d\mu_k(y),
\end{aligned}
\tag{4.3.3}
$$

where

$$
h_1(x,y) = Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}|u(x)-u(y)|^{p-1}\Phi_{qs}(x,y)\left|\frac{u(y)}{w(y)}\right||w(x)-w(y)|
$$

and

$$
h_2(x,y) = Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}|u(x)-u(y)|^{p-1}\Phi_{qs}(x,y)\left|\frac{u(x)}{w(x)}\right||w(x)-w(y)|.
$$

Since $h_1(x,y) = h_2(y,x)$ we have

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} h_1(x,y)d\mu_k(x)d\mu_k(y) = \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} h_2(x,y)d\mu_k(x)d\mu_k(y). \tag{4.3.4}$$

Therefore, it is sufficient to estimate one of the integral. Now by Young's inequality we can write

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} h_2(x,y)d\mu_k(x)d\mu_k(y) \leqslant \varepsilon \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)$$
$$+ C(\varepsilon)\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y), \tag{4.3.5}$$

where

$$G(x,y) = Q(x,y)^p\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\left|\frac{u(x)}{w(x)}\right|^p |w(x)-w(y)|^p\Phi_{qs}(x,y).$$

The proof will be completed if we can establish

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y) \leqslant C\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^pK_q^\beta(x,y)d\mu_k(x)d\mu_k(y). \tag{4.3.6}$$

Let us calculate

$$\begin{aligned}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} & G(x,y)d\mu_k(x)d\mu_k(y)\\ &= \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u(x)^pw(x)^{p(p-1)}|w(x)-w(y)|^p}{\big(w(x)^p+w(y)^p\big)^p}\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)\\ &= \int_{\mathbb{R}^N} u(x)^p\int_{\mathbb{R}^N}\frac{||x|^\alpha-|y|^\alpha|^p|y|^{\alpha p(p-1)}}{(|x|^{\alpha p}+|y|^{\alpha p})^p}\Phi_{qs}(x,y)d\mu_k(y)d\mu_k(x).\end{aligned}$$

Let $|x| = r$ and $|y| = \rho$ with $x = rx'$ and $y = \rho y'$. Also write $t = \rho/r$ and

$d\sigma_k(y') = h_k^2(y')d\sigma(y')$ with $d\sigma(y')$ as the Eulidean surface measure on the sphere $\mathbb{S}^{N-1}$. Then we have

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y)\\
&= \int_{\mathbb{R}^N} u(x)^p \int_0^\infty \frac{|r^\alpha-\rho^\alpha|^p\rho^{\alpha p(p-1)+2\lambda_k+1}}{(r^{p\alpha}+\rho^{p\alpha})^p}\int_{\mathbb{S}^{N-1}} \Phi_{qs}(rx',\rho y')d\sigma_k(y')d\rho d\mu_k(x)\\
&= \int_{\mathbb{R}^N} \frac{u(x)^p}{|x|^{qs}} \int_0^\infty \frac{|1-t^\alpha|^p t^{\alpha p(p-1)+2\lambda_k+1}}{(1+t^{\alpha p})^p}\int_{\mathbb{S}^{N-1}} \Phi_{qs}(x',ty')d\sigma_k(y')dtdx\\
&= I\int_{\mathbb{R}^N} \frac{u(x)^p}{|x|^{qs}}d\mu_k(x),
\end{aligned}$$

with

$$I = \int_0^\infty \frac{|1-t^\alpha|^p t^{\alpha p(p-1)+2\lambda_k+1}}{(1+t^{\alpha p})^p}P(t)dt.$$

Here we set

$$P(t) = \int_{\mathbb{S}^{N-1}} \Phi_{qs}(x',ty')d\sigma_k(y')$$

and used the property of the kernel $\Phi_{qs}(rx',\rho y') = r^{-d_k-qs}\Phi_{qs}(x',ty')$ (see [17, Lemma 2.3] for a proof). By proceeding with the similar steps used in Lemma 4.2.5 we get I is finite. Since we chose $w(x) = |x|^{-\frac{d_k-ps}{p}}$ and $u = vw$ we have

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y) = I\int_{\mathbb{R}^N} \frac{|v(x)|^p}{|x|^{qs+(d_k-ps)}}d\mu_k(x).$$

Set $\beta_0 = \frac{d_k-ps}{2} < \frac{d_k-qs}{2}$ and apply (4.2.11) for v, to get

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y)d\mu_k(x)d\mu_k(y) \leqslant C\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^p K_q^\beta(x,y)d\mu_k(x)d\mu_k(y). \tag{4.3.7}$$

Thus we proved our claim in (4.3.6). Now by considering the inequalities (4.3.3), (4.3.4), (4.3.5) and (4.3.7) we get the desired inequality

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)$$
$$\leqslant C \iint_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y).$$

□

Let Ω be a bounded G-invariant domain on $\mathbb{R}^N$ containing origin. Also let $u \in C_0^\infty(\Omega)$ and $\tilde{u}$ be its extension to $\mathbb{R}^N$ as explained earlier (see Proposition 4.2.3). As $u = vw$ we let the extension of v as $\tilde{v}$ and $\tilde{u} = \tilde{v}w$. Now using (4.2.4) and Lemma 4.3.1 together, we get

$$\begin{aligned}\iint_{\Omega\times\Omega} &|v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y)\\ &\geqslant C \iint_{\mathbb{R}^N\times\mathbb{R}^N} |\tilde{v}(x)-\tilde{v}(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y)\\ &\geqslant C \iint_{\mathbb{R}^N\times\mathbb{R}^N} |\tilde{u}(x)-\tilde{u}(y)|\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)\\ &\geqslant C \iint_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \qquad (4.3.8)\end{aligned}$$

Theorem 4.3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded G-invariant domain. Let $1 < q < p < \infty$ and $0 < s < 1$. Then for all $u \in C_0^\infty(\Omega)$*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) - \Lambda_{d_k,s,p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x),$$
$$\geqslant C \iint_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y), \qquad (4.3.9)$$

where

$$\Lambda_{d_k,s,p} = 2\int_0^1 r^{ps-1}|1-r^{(d_k-ps)/p}|^p \Phi(r)dr, \tag{4.3.10}$$

with

$$\Phi(r) = \begin{cases} \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\Gamma(\frac{d_k-1}{2})}\int_0^\pi \frac{\sin^{d_k-2}\theta}{(1-2r\cos\theta+r^2)^{\frac{d_k+ps}{2}}}d\theta & \text{for } N \geqslant 2 \\ \Big(\tau_r^k(|.|^{-d_k-ps}) + \tau_{-r}^k(|.|^{-d_k-ps})\Big)(1) & \text{for } N = 1 \end{cases}$$

and C is a positive constant depending on Ω, d_k, q and s.

Proof. The main idea of the proof is to show that

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) - \Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)$$
$$\geqslant C\iint\limits_{\Omega\times\Omega}|v(x)-v(y)|^pK_q^\beta(x,y)d\mu_k(x)d\mu_k(y), \tag{4.3.11}$$

for some positive constant C. Then by using Lemma 4.3.1 we reach the desired inequality. In order to prove (4.3.11) we need to consider two different cases $p \geqslant 2$ and $1 < p < 2$.

Case 1: $p \geqslant 2$

From [5], we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) - C_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)$$
$$\geqslant c_p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(x)-v(y)|^p\Phi_{ps}(x,y)\frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}}\frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}}.$$

But for $\Omega \subset \mathbb{R}^N$ bounded, we have $\Phi_{ps}(x,y) \geqslant C(\Omega)\Phi_{qs}(x,y)$ on Ω. Using this we can write

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)|^p \Phi_{ps}(x,y) \frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}} \frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}} \geqslant C(\Omega) \iint_{\Omega\times\Omega} |v(x)-v(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y)$$

and it gives the claim given in (4.3.11) for $p \geqslant 2$.

Case 2: $1 < p < 2$

We define f_1 and f_2 same as described in the proof of Lemma 4.3.1. We split the domain $\Omega \times \Omega$ in accordance with the values of $w(x)$ and $w(y)$ as

$$D_1 = \{(x,y) \in \Omega\times\Omega : w(y) \leqslant w(x)\} \text{ and } D_2 = \{(x,y) \in \Omega\times\Omega : w(x) < w(y)\}. \tag{4.3.12}$$

Now

$$\begin{aligned} C(\Omega) H_\Omega(v) &:= \iint_{\Omega\times\Omega} |v(x)-v(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y) \\ &= \iint_{\Omega\times\Omega} f_1(x,y) d\mu_k(x) d\mu_k(y) \\ &= \iint_{D_1} f_1(x,y) d\mu_k(x) d\mu_k(y) + \iint_{D_2} f_2(x,y) d\mu_k(x) d\mu_k(y) \\ &= I_1 + I_2. \end{aligned}$$

We will first estimate the integral in I_1. We can write

$$\begin{aligned} J_1(x,y) &:= \Big|\big((u(y)-u(x)\big) - \frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\Big|^p \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \\ &= \frac{\Big|\big((u(y)-u(x)\big) - \frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\Big|^p}{\big|u(x)-u(y)\big| + \Big|\frac{u(y)}{w(x)}\big(w(x)-w(y)\big)\Big|^{(2-p)\frac{p}{2}}} \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \\ &\quad \times \big|u(x)-u(y)\big| + \Big|\frac{u(y)}{w(x)}\big(w(x)-w(y)\big)\Big|^{(2-p)\frac{p}{2}}. \end{aligned}$$

Now applying the Hölder's inequality, we obtain

$$I_1 \leqslant I_{1,1} \times I_{1,2}. \tag{4.3.13}$$

Here we denote

$$I_{1,1} = \left(\iint\limits_{D_1} \frac{\Big|\big((u(y)-u(x)\big) - \frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\Big|^2}{\big|u(x)-u(y)\big| + \Big|\frac{u(y)}{w(x)}\big(w(x)-w(y)\big)\Big|^{(2-p)}} \frac{w(y)}{w(x)} \Phi_{qs}(x,y)\, d\mu_k(x)\, d\mu_k(y) \right)^{\frac{p}{2}} \tag{4.3.14}$$

and

$$I_{1,2} = \left(\iint\limits_{D_1} \Big|\big((u(y)-u(x)\big) - \frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\Big|^p \Phi_{qs}(x,y)\, d\mu_k(x)\, d\mu_k(y) \right)^{\frac{2-p}{p}}. \tag{4.3.15}$$

From (4.3.8), we get

$$I_{1,2}^{\frac{2}{2-p}} \leqslant C_1 \iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) + \iint\limits_{\Omega\times\Omega} |\frac{u(y)}{w(y)}(w(x)-w(y))|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)$$
$$\leqslant \iint\limits_{\Omega\times\Omega} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y) = C(\Omega)H_\Omega(v). \tag{4.3.16}$$

Thus we arrive at

$$I_{1,2} \leqslant C(\Omega)H_{(\Omega)}^{\frac{2-p}{2}}(v). \tag{4.3.17}$$

An application of Lemma 4.2.2 with $t = \frac{w(y)}{w(x)}$, $a = \frac{v(x)}{v(y)}$ we find for $(x,y) \in D_1$

$$\frac{\big|\big((u(y)-u(x)\big) - \frac{u(y)}{w(y)}\big(w(x)-w(y)\big)\big|^2}{\big|u(x)-u(y)\big| + \big|\frac{u(y)}{w(x)}\big(w(x)-w(y)\big)\big|^{(2-p)}}\frac{w(y)}{w(x)} = \frac{w(x)^p|v(y)|^p|a-1|^2t}{(|a-t|+|1-t|^{2-p})}$$
$$\leqslant w(x)^p|v(y)|^p\big(|a-t|^p - (1-t)^{p-1}(|a|^p - t)\big)$$
$$= w(x)^p|v(y)|^p\left(|\frac{v(x)}{v(y)} - \frac{w(y)}{w(x)}|^p - \big(1-\frac{w(y)}{w(x)}\big)^{p-1}\big(|\frac{v(x)}{v(y)}|^p - \frac{w(y)}{w(x)}\big)\right)$$
$$= |u(x)u(y)|^p - (w(x)-w(y))^{p-2}(w(x)-w(y))\big(\frac{|u(x)|^p}{w(x)^{p-1}} - \frac{|u(y)|^p}{w(y)^{p-1}}\big). \tag{4.3.18}$$

Further using (4.3.16) and (4.3.18) the first integral $I_{1,1}$ in (4.3.13) becomes

$$\begin{aligned}
&C(\Omega)I_{1,1}^{2/p}\\
&\leqslant \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)\\
&-\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \Big(\frac{|u(x)|^p}{w(x)^{p-1}}-\frac{|u(y)|^p}{w(y)^{p-1}}\Big)|w(x)-w(y)|^{p-2}(w(x)-w(y))\Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y)\\
&\qquad=\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x).
\end{aligned}\tag{4.3.19}$$

This gives that

$$\begin{aligned}
I_1&=\iint\limits_{D_1} f_1(x,y)d\mu_k(x)d\mu_k(y)\\
&\leqslant C(\Omega)H_\Omega^{\frac{2-p}{2}}(v)\times\\
&\qquad\Big(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)\Big)^{\frac{p}{2}}.
\end{aligned}\tag{4.3.20}$$

The same arguments allow us to write

$$\begin{aligned}
I_2&=\iint\limits_{D_2} f_2(x,y)d\mu_k(x)d\mu_k(y)\\
&\leqslant C(\Omega)H_\Omega^{\frac{2-p}{2}}(v)\times\\
&\Big(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p\Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y)-\Lambda_{d_k,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}d\mu_k(x)\Big)^{\frac{p}{2}}.
\end{aligned}\tag{4.3.21}$$

Now put (4.3.20) and (4.3.21) together with the fact $C(\Omega)H_\Omega(v) = I_1 + I_2$ to get

$$H_\Omega(v) \leqslant C(\Omega) H_\Omega^{\frac{2-p}{2}}(v) \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x) \right)^{\frac{p}{2}}$$

and hence

$$H_\Omega(v) \leqslant C(\Omega) \iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x).$$

Now the case 1 and case 2 together provide the claim

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x) \geqslant C(\Omega) \iint_{\Omega\times\Omega} |v(x)-v(y)|^p K_q^\beta(x,y) d\mu_k(x) d\mu_k(y). \quad (4.3.22)$$

for all $1 \leqslant q < p < \infty$.

The desired inequality

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x) \geqslant C \iint_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y).$$

will be established by using (4.3.22) together with Lemma 4.3.1. □

4.4 Hardy Inequality on Half-space

Let R be a root system on $\mathbb{R}^{N-1}$ and k be a multiplicity function from R to $(0,\infty)$. Define the root system R_1 on $\mathbb{R}^N_+$ as $R_1 := R \times \{0\}$. We use the same notation G for the corresponding Coxeter group. Also extend the multiplicity function k to k_1 by defining $k_1(x,0) = k(x)$ where $x \in R$. With the root system R_1 and the multiplicity function k_1 on $\mathbb{R}^N_+$ we can write the kernel Φ_{qs} on $\mathbb{R}^N_+$ with $1 < q < \infty$ and $0 < s < 1$ as

$$\Phi_{qs}(x,y) = \frac{1}{\Gamma((d_{k_1}+qs)/2)} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s|x_N-y_N|^2} \tau^k_{y'}(e^{-s|.|^2})(x')ds.$$

For an element $x \in \mathbb{R}^N_+$ we write $x = (x', x_N)$ where $x' \in \mathbb{R}^{N-1}$ and $x_N > 0$. Using the properties of Dunkl translation and gamma function we can perform the following calculations

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \Phi_{qs}(x,y) d\mu_k(y') \\
&= \frac{1}{\Gamma((d_{k_1}+qs)/2)} \int_{\mathbb{R}^{N-1}} \int_0^\infty s^{\frac{d_{k_1}+qs}{2}-1} e^{-s|x_N-y_N|^2} \tau^k_{y'}(e^{-s|.|^2})(x')ds\ d\mu_k(y') \\
&= \frac{1}{\Gamma((d_{k_1}+qs)/2)} \int_{\mathbb{R}^{N-1}} \int_0^\infty s^{\frac{d_{k_1}+qs}{2}-1} e^{-s(|x_N-y_N|^2+|x'-y'|^2)} ds\ d\mu_k(y') \\
&= \int_{\mathbb{R}^{N-1}} \frac{d\mu_k(y')}{(|x_n-y_N|^2+|x'-y'|^2)^{\frac{d_{k_1}+qs}{2}}} \\
&= \|\mathbb{S}^{N-2}\|_k \int_0^\infty \frac{1}{(|x_N-y_N|^2+r^2)^{\frac{d_{k_1}+qs}{2}}} r^{d_k-2} dr \\
&= \|\mathbb{S}^{N-2}\|_k \frac{1}{|x_N-y_N|^{1+qs}} \int_0^\infty \frac{t^{d_k-2}}{(1+t^2)^{\frac{d_{k_1}+qs}{2}}} dt \\
&= \|\mathbb{S}^{N-2}\|_k \frac{1}{|x_N-y_N|^{1+qs}} \frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)}.
\end{aligned} \tag{4.4.1}$$

Let $\Omega \subset \mathbb{R}^N_+$ be an open G-invariant subset and let $w_0 \in W^{s,p,q,\beta}_{k,0}(\Omega)$. Define

$$L_0(w_0)(x) := P.V. \int_{\mathbb{R}^N_+} |w_0(x) - w_0(y)|^{p-2}(w_0(x) - w_0) K^{\beta}_{q,0}(x,y) d\mu_k(x) d\mu_k(y).$$

Also let $\Omega \subset \mathbb{R}^N_+$ be bounded and we denote $Q_0 = \mathbb{R}^N_+ \times \mathbb{R}^N_+ \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. Then by the same arguments in the proof of Theorem 4.2.4 we can conclude a Picone's inequality for half-space, that is

$$\frac{1}{2} \iint_{Q_0} |u(x) - u(y)|^p \Phi_{qs}(x,y) \frac{d\mu_k(x) d\mu_k(y)}{x_N^{\beta} y_N^{\beta}} \geqslant \langle L_0(w_0), \frac{|u|^p}{w_0^{p-1}} \rangle, \tag{4.4.2}$$

for all functions $u \in C_0^{\infty}(\Omega)$ and for all positive function $w \in W^{s,p,q,\beta}_{k,0}(\Omega)$.

Let $0 < \beta < \frac{1-qs}{2}$, $0 < \alpha < \frac{1-qs-2\beta}{p-1}$ and $w_0(x) = x_N^{-\alpha}$. Then for almost every non zero $x \in \mathbb{R}^N$ we have

$$L_0(w_0) = \Lambda_0(\alpha) \frac{w_0^{p-1}}{x_N^{qs+2\beta}} \tag{4.4.3}$$

for a positive constant $\Lambda_0(\alpha)$. The proof of this can be done with similar steps of the proof of the Lemma 4.2.5. Denoting $r = x_N$, $\rho = y_N$ and using the calculations in (4.4.1), we get

$$L_0(w_0)(x) = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)} \int_0^{\infty} \frac{|r^{-\alpha} - \rho^{-\alpha}|^{p-2}(r^{-\alpha} - \rho^{-\alpha})\Phi_{qs}(rx' - \rho y')}{r^{\beta}\rho^{\beta}|r-\rho|^{1+qs}} d\rho.$$

Set $t = r/\rho$,

$$L_0(w_0)(x) = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)} \frac{r^{-\alpha(p-1)}}{r^{2\beta}} \int_0^\infty \frac{|1-t^{-\alpha}|^{p-2}(1-t^{-\alpha})}{t^\beta |1-t|^{1+qs}} dt$$
$$= \Lambda_0(\alpha) \frac{w^{p-1}(x)}{x_N^{2\beta+qs}},$$

where the constant

$$\Lambda_0(\alpha) = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_{k_1}-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_{k_1}+qs)/2)} \int_0^\infty \frac{|1-t^{-\alpha}|^{p-2}(1-t^{-\alpha})}{t^\beta |1-t|^{1+qs}} dt.$$

It remains to show that $\Lambda_0(\alpha)$ is positive. Splitting the integral in to two domains; $(0,1)$ and $(1,\infty)$ and use the change of variable $t \to 1/t$ on $(0,1)$ we can write $\Lambda_0(\alpha)$ as

$$\Lambda_0(\alpha) = \int_1^\infty \frac{(t^\alpha - 1)^{p-1}}{|1-t|^{1+qs}} \left(t^{-\beta-\alpha(p-1)} - t^{\beta+qs-1}\right) dt.$$

A repetition of same arguments in the proof of Lemma 4.2.5 will show that $\Lambda_0(\alpha)$ is positive.

Use the identity (4.4.3) and the Picone's inequality for half-space given in (4.4.2) together to see that

$$2\Lambda_0(\alpha) \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{qs+2\beta}} d\mu_k(x) = \langle L_0(w_0), \frac{|u|^p}{w_0^{p-1}} \rangle$$
$$\leqslant \iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x)-u(y)|^p K^\beta_{q,0}(x,y) d\mu_k(x) d\mu_k(y),$$

where

$$K^\beta_{q,0}(x,y) = \frac{\Phi_{qs}(x,y)}{x_N^\beta y_N^\beta}.$$

Lemma 4.4.1. *Fix $\alpha = \beta = \frac{1-ps}{p}$ and let $w_0(x) = x_N^{-\alpha}$. Let $u \in C_0^\infty(\mathbb{R}^N)$ and* *define* $v(x) := u(x)/w(x)$. *Then for all* $1 < q < p < \infty$ *and for a given positive constant* C *the following inequality holds*

$$\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |v(x) - v(y)|^p K^{\beta}_{q,0}(x,y) d\mu_k(x) d\mu_k(y)$$
$$\geqslant C \iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x) - u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y).$$

Proof. We will prove the lemma by following the proof of Lemma 4.3.1. Replacing K and w by K_0 and w_0 we can define the functions f_1 and f_2 as:

$$\begin{aligned} f_1(x,y) &:= |v(x) - v(y)|^p K^{\beta}_{q,0}(x,y) \\ &= \frac{|w_0(y)u(x) - w_0(x)u(y)|^p}{\big(w_0(x)w_0(y)\big)^{\frac{p}{2}}} \Phi_{qs}(x,y) \\ &= \left| \big((u(y) - u(x)\big) - \frac{u(y)}{w_0(y)} \big(w_0(x) - w_0(y)\big) \right|^p \left(\frac{w_0(y)}{w_0(x)} \right)^{\frac{p}{2}} \Phi_{qs}(x,y); \end{aligned}$$

$$f_2(x,y) := \left| \big((u(x) - u(y)\big) - \frac{u(x)}{w_0(x)} \big(w_0(y) - w_0(x)\big) \right|^p \left(\frac{w_0(x)}{w_0(y)} \right)^{\frac{p}{2}} \Phi_{qs}(x,y).$$

Proceeding with similar steps of the proof of Lemma 4.3.1 we arrive at

$$\begin{aligned} &\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} G(x,y) d\mu_k(x) d\mu_k(y) \\ &\qquad = \iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} \frac{u(x)^p w_0(x)^{p(p-1)} |w_0(x) - w_0(y)|^p}{\big(w_0(x)^p + w_0(y)^p\big)^p} \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y) \\ &\qquad = \int_{\mathbb{R}^N_+} u(x)^p \int_{\mathbb{R}^N_+} \frac{|x_N^{\alpha} - y_N^{\alpha}|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \Phi_{qs}(x,y) d\mu_k(y) d\mu_k(x), \end{aligned} \tag{4.4.4}$$

where

$$G(x,y) = Q(x,y)^p \left(\frac{w_0(x)}{w_0(y)}\right)^{\frac{p}{2}} \left|\frac{u(x)}{w_0(x)}\right|^p |w_0(x) - w_0(y)|^p \Phi_{qs}(x,y).$$

By the definition of the root system we can write

$$\Phi_{qs}(x,y) = \frac{1}{\Gamma(\frac{d_k+qs}{2})} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s|x_N-y_N|^2} \tau_{y'}^{k_1}(e^{-s|.|^2})(x')ds.$$

Using this and the properties of Dunkl translation(see [38, Proposition 2.4]), the integral become

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \Phi_{qs}(x,y) d\mu_k(y) \\
&= \frac{1}{\Gamma(\frac{d_k+qs}{2})} \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \\
&\qquad \int_{\mathbb{R}^{N-1}} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s|x_N-y_N|^2} \tau_{y'}^{k_1}(e^{-s|.|^2})(x') ds d\mu_{k_1}(y') dy_N \\
&= \frac{1}{\Gamma(\frac{d_k+qs}{2})} \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \\
&\qquad \int_{\mathbb{R}^{N-1}} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s|x_N-y_N|^2+|y'|^2} ds d\mu_{k_1}(y') dy_N.
\end{aligned} \tag{4.4.5}$$

Using the polar coordinates and integrating, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \frac{1}{(|x_N - y_N|^2 + |y'|^2)^{\frac{d_k+qs}{2}}} d\mu_k(y') \\
&= \|\mathbb{S}^{N-2}\|_k \int_0^\infty \frac{1}{(|x_N - y_N|^2 + r^2)^{\frac{d_k+qs}{2}}} r^{d_k-2} dr \\
&= \|\mathbb{S}^{N-2}\|_k \frac{1}{|x_N - y_N|^{1+qs}} \int_0^\infty \frac{t^{d_k-2}}{(1+t^2)^{\frac{d_k+qs}{2}}} dt \\
&= \|\mathbb{S}^{N-2}\|_k \frac{1}{|x_N - y_N|^{1+qs}} \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)}.
\end{aligned} \tag{4.4.6}$$

Also by using the gamma function we obtain

$$\begin{aligned}
&\frac{1}{\Gamma((d_k+qs)/2)} \int_0^\infty s^{\frac{d_k+qs}{2}-1} e^{-s(|x_N-y_N|^2+|x'-y'|^2)} ds \\
&\qquad = \frac{1}{(|x_N - y_N|^2 + |x' - y'|^2)^{\frac{d_k+qs}{2}}}.
\end{aligned} \tag{4.4.7}$$

Substitute the equations (4.4.5), (4.4.6) and (4.4.7) in (4.4.4) we get the integral

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} G(x,y) d\mu_k(x) d\mu_k(y) \\
&= \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)}. \\
&\qquad \int_{\mathbb{R}^N_+} u(x)^p \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p} + y_N^{\alpha p})^p} \frac{dy_N d\mu_k(x)}{|x_N - y_N|^{1+qs}}.
\end{aligned}$$

Set $t = y_N/x_N$, then

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} G(x,y)d\mu_k(x)d\mu_k(y) \\
&= \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)} \\
&\qquad\qquad \int_{\mathbb{R}^N_+} u(x)^p \int_0^\infty \frac{|x_N^\alpha - y_N^\alpha|^p y_N^{\alpha p(p-1)}}{(x_N^{\alpha p}+y_N^{\alpha p})^p} \frac{dy_N d\mu_k(x)}{|x_N-y_N|^{1+qs}} \\
&= I\int_{\mathbb{R}^N_+} \frac{u(x)^p}{x_N^{qs}} d\mu_k(x),
\end{aligned}$$

where

$$I = \|\mathbb{S}^{N-2}\|_k \frac{\Gamma((d_k-1)/2)\Gamma((1+qs)/2)}{\Gamma((d_k+qs)/2)} \int_0^\infty \frac{|1-t^\alpha|^p t^{\alpha p(p-1)+2\lambda_k+1}}{(1+t^{\alpha p})^p|1-t|^{1+qs}} dt.$$

Following the similar steps used in proving Lemma 4.2.5 we get

$$\begin{aligned}
\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} G(x,y)d\mu_k(x)d\mu_k(y) &= I\int_{\mathbb{R}^N_+} \frac{|v(x)|^p}{x_N^{qs+1-ps}} \\
&= C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |v(x)-v(y)|^p K^\beta_{q,0}(x,y)d\mu_k(x)d\mu_k(y)
\end{aligned}$$

and the inequality (see the proof of Lemma 4.2.5 and the beginning of Section 4.4 for more understanding)

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{qs}(x,y)d\mu_k(x)d\mu_k(y) \leqslant C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} K^\beta_{q,0}(x,y)d\mu_k(x)d\mu_k(y).$$

□

Theorem 4.4.2. *Let $\Omega \subset \mathbb{R}^N_+$ be a bounded G-invariant domain. Let $1 < q <$*

$p < \infty$ and $0 < s < 1$. Then for all $u \in C_0^\infty(\Omega)$

$$\iint\limits_{\mathbb{R}_+^N \times \mathbb{R}_+^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}_+^N} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x)$$
$$\geqslant C \iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y), \quad (4.4.8)$$

where $\Lambda^0_{d_k,s,p}$ is given as

$$\Lambda^0_{d_k,s,p} := c_{k_1}^{-1} 2^{-\lambda_{k_1}} \frac{\Gamma((1+ps)/2)}{\Gamma((d_k+ps)/2)} \int_0^1 |1 - r^{(ps-1)/p}|^p \frac{dr}{(1-r)^{1+ps}}. \quad (4.4.9)$$

and $C = C(\Omega, d_k, q, s)$ is a positive constant.

Proof. We follow the similar steps of the proof of Theorem 4.3.2. As in that case we have two cases $p \geqslant 2$ and $p < 2$.

Case 1: $p \geqslant 2$

From [5], we have

$$\int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - C_{d_k,s,p} \int_{\mathbb{R}_+^N} \frac{|u(x)|^p}{x|_N^{ps}} d\mu_k(x)$$
$$+ c_p \int_{\mathbb{R}_+^N} \int_{\mathbb{R}^N} |v(x)-v(y)|^p \Phi_{ps}(x,y) \frac{d\mu_k(x)}{x_N^{(1-ps)/2}} \frac{d\mu_k(y)}{y_N^{(1-ps)/2}}.$$

But since $\Omega \subset \mathbb{R}^N$ bounded, we have $\Phi_{ps}(x,y) \geqslant C(\Omega)\Phi_{qs}(x,y)$ on Ω, and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x)-v(y)|^p \Phi_{ps}(x,y) \frac{d\mu_k(x)}{x_N^{(d_k-ps)/2}} \frac{d\mu_k(y)}{y_N^{(d_k-ps)/2}}$$
$$\geqslant C(\Omega) \iint\limits_{\Omega\times\Omega} |v(x)-v(y)|^p K^\beta_{q,0}(x,y) d\mu_k(x) d\mu_k(y).$$

The proof of Theorem 4.4.2 for $p \geqslant 2$ will be completed by applying Lemma 4.4.1.

Case 2: $1 < p < 2$

Let f_1 and f_2 be as in the proof of Lemma 4.4.1 and define D_1 and D_2 as in (4.3.12) just by replacing w by w_0. Now we have

$$\iint\limits_{D_1} f_1(x,y)d\mu_k(x)d\mu_k(y) + \iint\limits_{D_2} f_2(x,y)d\mu_k(x)d\mu_k(y)$$
$$= C(\Omega) \iint\limits_{\Omega\times\Omega} |v(x)-v(y)|^p K_q^{\beta}(x,y)d\mu_k(x)d\mu_k(y) := C(\Omega)H_{\Omega,0}(v).$$

A similar calculations from (4.3.13) to (4.3.19) yield

$$\iint\limits_{D_1} f_1(x,y)d\mu_k(x)d\mu_k(y)$$
$$\leqslant C(\Omega)H_{\Omega,0}^{\frac{2-p}{2}}(v)$$
$$\left(\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) - \Lambda^0_{d_k,s,p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{x_N^{ps}}d\mu_k(x)\right)^{\frac{p}{2}}. \tag{4.4.10}$$

Similarly for f_2

$$\iint\limits_{D_1} f_2(x,y)d\mu_k(x)d\mu_k(y)$$
$$\leqslant C(\Omega)H_{\Omega,0}^{\frac{2-p}{2}}(v)$$
$$\left(\iint\limits_{\mathbb{R}^N_+\times\mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y)d\mu_k(x)d\mu_k(y) - \Lambda^0_{d_k,s,p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{x_N^{ps}}d\mu_k(x)\right)^{\frac{p}{2}}. \tag{4.4.11}$$

Combining (4.4.10) and (4.4.11) we arrive at

$$H_{\Omega,0}(v) \leqslant C(\Omega) \iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x).$$

Now putting both cases together we can write

$$\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x) \geqslant C(\Omega) \iint\limits_{\Omega\times\Omega} |v(x)-v(y)|^p K^{\beta}_{q,0}(x,y) d\mu_k(x) d\mu_k(y). \quad (4.4.12)$$

Now a direct application of Lemma 4.4.1 and (4.4.12) we get the desired improved farctional Hardy inequality

$$\iint\limits_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} |u(x)-u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^0_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{x_N^{ps}} d\mu_k(x) \geqslant C \iint\limits_{\Omega\times\Omega} |u(x)-u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y).$$

□

By choosing the multiplicity function $k \equiv 0$ in Theorem 4.4.2 we obtain the following corollary.

Corollary 4.4.3. *Let $0 < s < 1$ and $ps < N$. Also let Ω be a bounded domain of $\mathbb{R}^N_+$. Then for all $1 < q < p < \infty$ and for all functions $u \in C_0^\infty(\Omega)$ the following* *inequality holds:*

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - D_{N,p,s}\int_{\mathbb{R}^N_+}\frac{|u(x)|^p}{x_N^{ps}}dx \geqslant C\int_{\Omega}\int_{\Omega}\frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}dxdy. \tag{4.4.13}$$

The constant $D_{N,p,s}$ is sharp and is given by

$$D_{N,p,s} = c_{n-1}\frac{\Gamma(\frac{1+ps}{2})}{\Gamma(\frac{N+ps}{2})}\int_0^1 |1-r^{\frac{ps-1}{p}}|^p \frac{dr}{(1-r)^{1+ps}}, \tag{4.4.14}$$

with $c_{n-1} = 2^{\frac{N-3}{2}}\int_{\mathbb{R}^{N-1}} e^{-|x'|^2/2}dx'$. The constant C is positive and depends on N, q, s and the domain Ω.

4.4.1 Fractional Hardy inequality for cone

Let $1 \leqslant l \leqslant N$ and $\mathbb{R}^N_{l_+} \subset \mathbb{R}^N$ be a cone, that is $\mathbb{R}^N_{l_+} := \{(x_1, x_2, ..., x_N) \in \mathbb{R}^N : x_{N-l+1} > 0, ..., x_N > 0\}$. Let R be a root system on $\mathbb{R}^{N-l}$ and k be a multiplicity function on R. Let $R_1 := \{(x,0) \in \mathbb{R}^N : x \in R\}$ and we can check easily that R_1 is a root system on $\mathbb{R}^N$. Similar to the case of half-space we can extend the multiplicity function k to k_1 on R_1 by setting $k_1(x,0) = k(x)$ for $x \in R$. Let us write $x \in \mathbb{R}^N$ as $x = (x', x'')$ with $x' \in \mathbb{R}^{N-l}$ and $x'' \in \mathbb{R}^l$. By using same method as in the case of half-space we can obtain a Picone's inequality and the following theorem for the cone. Since the proof is very similar to that of Theorem 4.4.2 we state the main theorem without proof.

Theorem 4.4.4. *Let $\Omega \subset \mathbb{R}^N_{l_+}$ be open and bounded and G-invariant. Also let*

$1 < p < q < \infty$. *Then for all* $u \in c_0^\infty(\mathbb{R}^N_{l_+})$ *the following inequality holds:*

$$\iint\limits_{\mathbb{R}^N_{l_+} \times \mathbb{R}^N_{l_+}} |u(x) - u(y)|^p \Phi_{ps}(x,y) d\mu_k(x) d\mu_k(y) - \Lambda^l_{d_k,s,p} \int_{\mathbb{R}^N_{l_+}} \frac{|u(x)|^p}{x''^{ps}}$$

$$\geqslant C \iint\limits_{\Omega \times \Omega} |u(x) - u(y)|^p \Phi_{qs}(x,y) d\mu_k(x) d\mu_k(y),$$

where $\Lambda^l_{d_k,s,p}$ *is given in [5] as*

$$\Lambda^l_{d_k,s,p} := \frac{c_k^{-1} 2^{-\lambda_{k_1}} \Gamma((l+ps)/2)}{\Gamma((d_{k_1}+ps)/2)} \int_0^1 r^{ps-1} |1 - r^{(l-ps)/p}|^p \tilde{\Phi}_{l_+,s,p}(r) dr.$$

and $C = C(\Omega, d_k, q, s)$ *is a positive constant.*

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