# BLOCKING SETS OF CERTAIN LINE SETS IN $PG(3, q)$


***By***

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As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by **BIKRAMADITYA SAHU** entitled "**Blocking Sets of Certain Line Sets in PG(3,q)**" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

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# DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Bikramaditya Sahu

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# List of Publications arising from the thesis

## Journal

1. "Minimum size blocking sets of certain line sets related to a conic in $PG(2, q)$", K. L. Patra, B. K. Sahoo and Bikramaditya Sahu, *Discrete Math.* **339** (2016), 1716–1721.
2. "Blocking sets of tangent and external lines to a hyperbolic quadric in $PG(3, q)$", B. De Bruyn, B. K. Sahoo and Bikramaditya Sahu, *Discrete Math.* **341** (2018), 2820–2826.
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Bikramaditya Sahu

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*Dedicated to My Family*

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# Contents

# Synopsis

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**Synopsis of Ph.D. Thesis**

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## 1 Introduction

Throughout the synopsis, $q$ is assumed to be a prime power. For a positive integer $n$, let $PG(n, q)$ be the $n$-dimensional Desarguesian projective space over a finite field of order $q$. Let $L$ be a given non-empty subset of the line set of $PG(n, q)$. A *blocking set with respect to* $L$ (or simply, an $L$-*blocking set*) in $PG(n, q)$ is a subset $B$ of the point set of $PG(n,q)$ such that every line in $L$ contains at least one point of $B$. An $L$-blocking set $B$ is said to be *minimal* if no proper subset of $B$ is an $L$-blocking set in $PG(n,q)$.

Blocking sets in $PG(n,q)$ are combinatorial objects in finite geometry with several applications, and have been the subject of investigation by several researchers with respect to varying sets of lines. An important issue in this context is to determine the minimum size of an $L$-blocking set and if possible, to describe all $L$-blocking sets of that minimum cardinality. Every minimum size $L$-blocking set is also a minimal $L$-blocking set.

If $L$ is the set of all lines of $PG(n,q)$, then the point set of a hyperplane of $PG(n,q)$ is a classical example of an $L$-blocking set of minimum size. In fact, the following fundamental result was proved by Bose and Burton [11, Theorem 1] which is stated in the language of blocking sets.

> *If $B$ is a blocking set in $PG(n,q)$ with respect to all its lines, then $|B| \geqslant (q^n-1)/(q-1)$. Further, equality holds if and only if $B$ is the point set of a hyperplane of $PG(n,q)$.*

A blocking set $B$ in $PG(n,q)$ with respect to all its lines is called a *nontrivial blocking set* if $B$ does not contain any hyperplane of $PG(n,q)$, or equivalently, if every hyperplane of $PG(n,q)$ contains a point which is not in $B$.

In view of the above result of Bose and Burton, there are two aspects to the study of blocking sets in $PG(n,q)$:

(1) Find the possible sizes of minimal nontrivial blocking sets in $PG(n,q)$ with respect to all its lines and describe such blocking sets corresponding to those cardinalities.

(2) For proper subsets $L$ of the line set of $PG(n,q)$, find the minimum size of an

$L$-blocking set and describe all blocking sets of that minimum cardinality.

There has been a considerable amount of interest in the first problem by several researchers and many results are available in the literature in this regard. In this thesis we contribute to the second problem for $n \in \{2, 3\}$ and for certain proper subsets $L$ which have nice geometric descriptions.

In $PG(2, q)$, a classical result of P. Erdős and L. Lovász [15] says that if $B$ is a minimum size $L$-blocking set with $|L| \geqslant q^2 - q$, then $B$ must be of linear type in the sense that $B$ can be obtained from a line of $PG(2, q)$ by deleting and adding a few points. Results of this kind are viewed as stability theorems in the investigations by Z. Weiner and T. Szőnyi [42].

For a given non-degenerate quadric $\mathcal{Q}$ in $PG(n, q)$, let $L$ be the set of all lines of $PG(n, q)$ which are contained in $\mathcal{Q}$. In this case, the minimum size $L$-blocking sets in $PG(n, q)$ are extensively studied by K. Metsch in a series of papers. Such blocking sets are obtained as sets consisting of the non-singular points of quadrics $\mathcal{Q} \cap H$ for suitable hyperplanes $H$ of $PG(n, q)$. One can refer to the papers [23, 24, 25, 26, 27] for the details and more general results on blocking sets.

The main objective of this thesis is to investigate the minimum size blocking sets in $PG(3, q)$ of certain line sets defined with respect to a given hyperbolic quadric in it. In the process of the investigation, we discuss the minimum size blocking sets in $PG(2, q)$ of similar line sets defined with respect to an irreducible conic in it and prove some new results.

There are four chapters in the thesis. In Chapter 1, we recall the basic definitions and properties of point-line geometries such as projective planes, affine planes and generalized quadrangles. In this chapter, we also discuss substructures like ovals and conics in $PG(2, q)$, and ovoids and quadrics in $PG(3, q)$. The contents of the other three chapters are described in the following sections.

# 2 Blocking sets in $PG(2,q)$

Let $\mathcal{C}$ be a fixed irreducible conic in $PG(2,q)$. There are $q+1$ points in $\mathcal{C}$, no three of which are on the same line of $PG(2,q)$. We denote by $\mathcal{E}$ (respectively, $\mathcal{T}$, $\mathcal{S}$) the set of all lines of $PG(2,q)$ which are external (respectively, tangent, secant) with respect to $\mathcal{C}$.

In Chapter 2, we discuss the minimum size $L$-blocking sets in $PG(2,q)$, where the line set $L$ is one of $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E}\cup\mathcal{T}$, $\mathcal{E}\cup\mathcal{S}$ and $\mathcal{T}\cup\mathcal{S}$. In a series of papers [1, 2, 3, 4, 10, 17], the minimum size $L$-blocking sets with $L\in\{\mathcal{E},\mathcal{S},\mathcal{E}\cup\mathcal{T},\mathcal{T}\cup\mathcal{S}\}$ were completely characterized. It was proved that all such blocking sets are of linear type, except for some sporadic examples which occur in a few planes of small orders. We give a brief survey of the known results available in the literature. If $q$ is even, it is clear that the singleton set consisting of the nucleus of $\mathcal{C}$ is the only minimum size $\mathcal{T}$-blocking set in $PG(2,q)$. Regarding the minimum size $\mathcal{T}$-blocking sets for odd $q$ and $(\mathcal{E}\cup\mathcal{S})$-blocking sets in $PG(2,q)$, we prove the following three results.

**Theorem 2.1.** *Let $A$ be a $\mathcal{T}$-blocking set in $PG(2,q)$, where $q$ is odd. Then $|A|\geqslant(q+1)/2$. Further, equality holds if and only if $A$ consists of $(q+1)/2$ exterior points to $\mathcal{C}$ such that for any two distinct points $a_1,a_2$ in $A$, the two tangent lines through $a_1$ are different from the two tangent lines through $a_2$.*

**Theorem 2.2.** *Let $A$ be an $(\mathcal{E}\cup\mathcal{S})$-blocking set in $PG(2,q)$, where $q$ is even. Then $|A|\geqslant q$. Further, equality holds if and only if $A$ consists of all the points of a tangent line other than the nucleus of $\mathcal{C}$.*

**Theorem 2.3.** *Let $A$ be an $(\mathcal{E}\cup\mathcal{S})$-blocking set in $PG(2,q)$, where $q$ is odd. Then the following statements hold:*

*(i) If* $q = 3$*, then* $|A| \geqslant 3$*, and equality holds if and only if* $A$ *consists of all the three interior points of* $PG(2,3)$ *with respect to* $\mathcal{C}$*.*

*(ii) If* $q \geqslant 5$*, then* $|A| \geqslant q+1$*.*

*(iii) If* $q \geqslant 7$*, then* $|A| = q+1$ *if and only if* $A$ *is a line of* $PG(2,q)$*.*

*(iv) If* $q = 5$*, then* $|A| = 6$ *if and only if one of the following two cases occurs:*

*(a)* $A$ *is a line of* $PG(2,5)$*.*

*(b)* $A = \mathcal{I} \setminus \{a_1, a_2, a_3, a_4\}$*, where* $\mathcal{I}$ *is the set of all interior points of* $PG(2,5)$ *with respect to* $\mathcal{C}$ *and* $\{a_1, a_2, a_3, a_4\} \subseteq \mathcal{I}$ *is a quadrangle such that the line determined by any two distinct* $a_i, a_j$ *is external to* $\mathcal{C}$*.*

# 3 Blocking sets in $PG(3,q)$

Let $\mathcal{H}$ be a fixed hyperbolic quadric in $PG(3,q)$. For any line $l$ of $PG(3,q)$, we have $|l \cap \mathcal{H}| \in \{0, 1, 2, q+1\}$. We denote by $\mathbb{E}$ (respectively, $\mathbb{T}_1$, $\mathbb{S}$, $\mathbb{T}_0$) the set of all lines of $PG(3,q)$ that intersect $\mathcal{H}$ in 0 (respectively, 1, 2, $q+1$) points. The elements of $\mathbb{E}$ are called *external lines*, those of $\mathbb{S}$ *secant lines* and those of $\mathbb{T} := \mathbb{T}_0 \cup \mathbb{T}_1$ *tangent lines.* If $l \in \mathbb{T}_i$ with $i \in \{0,1\}$, then $l$ is also called a $\mathbb{T}_i$-*line.* The $\mathbb{T}_0$-lines are precisely the lines of $PG(3,q)$ which are contained in $\mathcal{H}$. The quadric $\mathcal{H}$ contains $(q+1)^2$ points and $2(q+1)$ lines of $PG(3,q)$.

In Chapter 3, we recall the basic properties of points, lines and planes of $PG(3,q)$ with respect to the quadric $\mathcal{H}$. In addition to a few other results, we prove the following theorem which is needed to study the minimum size $\mathbb{S}$- and $(\mathbb{E} \cup \mathbb{S})$-blocking sets in $PG(3,3)$.

**Theorem 3.1.** *Suppose $q = 3$. Then there exists a subset $B$ of the point set of $PG(3,3)$ satisfying the following three conditions:*

*(i)* $|B| = 12$ *and $B$ is disjoint from $\mathcal{H}$;*

*(ii) Every external line to $\mathcal{H}$ meets $B$ in two points;*

*(iii) Every secant line to $\mathcal{H}$ meets $B$ in one point.*

*In particular, $B$ is an $L$-blocking set in $PG(3,3)$ for $L \in \{\mathbb{S}, \mathbb{E}, \mathbb{S} \cup \mathbb{E}\}$.*

In Chapter 4, we investigate the minimum size $L$-blocking sets in $PG(3,q)$, where the line set $L$ is one of $\mathbb{E}$, $\mathbb{T}$, $\mathbb{S}$, $\mathbb{E} \cup \mathbb{T}$, $\mathbb{E} \cup \mathbb{S}$ and $\mathbb{T} \cup \mathbb{S}$.

## 3.1 $\mathbb{S}$-blocking sets

We prove the following theorem which characterizes the minimum size $\mathbb{S}$-blocking sets in $PG(3,q)$ for all $q$.

**Theorem 3.2.** *Let $B$ be an $\mathbb{S}$-blocking set in $PG(3,q)$. Then $|B| \geqslant q^2 + q$. Further, equality holds if and only if one of the following occurs:*

*(i) If $B \subseteq \mathcal{H}$, then $B = \mathcal{H} \setminus l$ for some $\mathbb{T}_0$-line $l$.*

*(ii) If $B \setminus \mathcal{H} \neq \emptyset$ and $B \cap \mathcal{H} \neq \emptyset$, then $B = (\mathcal{H} \setminus (l_0 \cup l_1)) \cup (l \setminus \{w\})$, where $l_0, l_1$ are two $\mathbb{T}_0$-lines intersecting at the point $w$ and $l$ is a $\mathbb{T}_1$-line with $l \cap \mathcal{H} = \{w\}$.*

*(iii) If $B \cap \mathcal{H} = \emptyset$, then $q \in \{2, 3\}$ and the following statements hold:*

*(a) $q = 2 :$ $B$ consists of all the six points of $PG(3,2)$ outside $\mathcal{H}$.*

*(b) $q = 3 :$ $B$ satisfies the three conditions (i)–(iii) of Theorem 3.1.*

## 3.2 $\mathbb{E}$-blocking sets

A plane of $PG(3,q)$ which is generated by two intersecting $\mathbb{T}_0$-lines is called a *tangent plane*. The following theorem describes the minimum size $\mathbb{E}$-blocking sets in $PG(3,q)$ for all $q$.

**Theorem 3.3.** *Let $B$ be an $\mathbb{E}$-blocking set in $PG(3,q)$. Then $|B| \geqslant q^2 - q$, and equality holds if and only if $B = \pi \setminus \mathcal{H}$ for some tangent plane $\pi$ of $PG(3,q)$.*

We note that Theorem 3.3 was proved by Biondi et al. in [6, Theorem 1.1] for $q$ even and in [7, Theorem 2.4] for $q$ odd, with exception of the equality case for some small values of $q$, namely for $q \in \{2,3,4,5,7\}$. In this thesis we give alternate proof of the equality case in Theorem 3.3 which works for all $q$, in particular for $q \in \{2,3,4,5,7\}$. The description of the minimum size $\mathbb{E}$-blocking sets in $PG(3,q)$ for all $q$ is necessary while studying the minimum size $(\mathbb{T} \cup \mathbb{E})$-blocking sets in $PG(3,q)$.

## 3.3 $(\mathbb{T} \cup \mathbb{S})$-blocking sets

We prove the following theorem which characterizes the minimum size $(\mathbb{T} \cup \mathbb{S})$-blocking sets in $PG(3,q)$ for all $q$.

**Theorem 3.4.** *Let $B$ be a $(\mathbb{T} \cup \mathbb{S})$-blocking set in $PG(3,q)$. Then $|B| \geqslant q^2+q+1$, and equality holds if and only if $B$ is a plane of $PG(3,q)$.*

## 3.4 $(\mathbb{E} \cup \mathbb{S})$-blocking sets

We prove the following result regarding the minimum size $(\mathbb{E} \cup \mathbb{S})$-blocking sets in $PG(3,q)$ for all $q$.

**Theorem 3.5.** *Let $B$ be an $(\mathbb{E} \cup \mathbb{S})$-blocking set in $PG(3,q)$. Then the following statements hold:*

*(i) If* $q \in \{2,3\}$*, then* $|B| \geqslant q^2+q$.

*(ii) If* $q=2$*, then* $|B|=6$ *if and only if one of the following two cases occurs:*

*(a)* $B$ *consists of all the six points of* $PG(3,2)$ *outside* $\mathcal{H}$.

*(b)* $B=(\mathcal{H}\setminus(l_0\cup l_1))\cup(l\setminus\{w\})$*, where* $l_0, l_1$ *are two* $\mathbb{T}_0$*-lines intersecting at the point* $w\in\mathcal{H}$ *and* $l$ *is the unique* $\mathbb{T}_1$*-line through* $w$.

*(iii) If* $q=3$*, then* $|B|=12$ *if and only if* $B$ *satisfies the three conditions (i)–(iii) of Theorem 3.1.*

*(iv) If* $q\geqslant 4$*, then* $|B|\geqslant q^2+q+1$*, and equality holds if and only if* $B$ *is a plane of* $PG(3,q)$.

Let $\mathbb{P}$ denote the point set of $PG(3,q)$. The point-line geometry with point set $\mathbb{P}$ and line set consisting of the totally isotropic lines of $PG(3,q)$ with respect to a symplectic polarity is a generalized quadrangle of order $q$, denoted by $W(q)$. If $q$ is even, then the point-line geometry $\mathcal{X}=(\mathbb{P},\mathbb{T})$ with point set $\mathbb{P}$ and line set $\mathbb{T}$ is a generalized quadrangle of order $q$ which is isomorphic to $W(q)$. We note that Theorem 3.5 was proved in [36, Theorem 1.3] for all even $q\geqslant 4$ using properties of the generalized quadrangle $W(q)$. In this thesis we give a proof of Theorem 3.5 which works for all $q$ irrespective of $q$ even or odd.

## 3.5 $(\mathbb{T}\cup\mathbb{E})$-blocking sets

Recall that if $\pi$ is a tangent plane of $PG(3,q)$ with respect to $\mathcal{H}$, then $\pi\cap\mathcal{H}$ is the union of two $\mathbb{T}_0$-lines intersecting at some point $w\in\mathcal{H}$. The point $w$ is called the *pole* of the tangent plane $\pi$. An *ovoid* of $W(q)$ is a set of points which meets each line of $W(q)$ at a unique point. An ovoid of $W(q)$ contains $q^2+1$ points. Further, $W(q)$ has ovoids if and only if $q$ is even.

We prove the following theorem which characterizes the minimum size $(\mathbb{T}\cup\mathbb{E})$-blocking sets in $PG(3,q)$ for all $q$.

**Theorem 3.6.** *Let $B$ be a $(\mathbb{T}\cup\mathbb{E})$-blocking set in $PG(3,q)$. Then $|B| \geqslant q^2+q$ and the following statements hold for the equality case:*

*(i) If $q=2$, then $|B|=6$ if and only if one of the following occurs:*

*(a) $B=\pi\setminus\{x\}$ for some tangent plane $\pi$ of $PG(3,2)$ with pole $x\in\mathcal{H}$.*

*(b) $B=\mathcal{O}\cup\{\alpha\}$, where $\mathcal{O}$ is a an ovoid of the generalized quadrangle $\mathcal{X}=(\mathbb{P},\mathbb{T})\simeq W(2)$ of order 2 and $\alpha\in\mathbb{P}\setminus\mathcal{H}$ is such that the unique external line through $\alpha$ is disjoint from $\mathcal{O}$.*

*(ii) If $q\geqslant 3$, then $|B|=q^2+q$ if and only if $B=\pi\setminus\{x\}$ for some tangent plane $\pi$ of $PG(3,q)$ with pole $x\in\mathcal{H}$.*

## 3.6 $\mathbb{T}$-blocking sets

In the last section of Chapter 4, we discuss the minimum size $\mathbb{T}$-blocking sets in $PG(3,q)$. A simple counting argument gives the following:

**Theorem 3.7.** *If $B$ is a $\mathbb{T}$-blocking set in $PG(3,q)$ of minimum size, then $q^2+1\leqslant |B|\leqslant q^2+q$.*

In the case $q$ even, using the facts that $\mathcal{X}=(\mathbb{P},\mathbb{T})\simeq W(q)$ and that $W(q)$ has ovoids, the following theorem characterizes the minimum size $\mathbb{T}$-blocking sets in $PG(3,q)$.

**Theorem 3.8.** *Let $B$ be a $\mathbb{T}$-blocking set in $PG(3,q)$, where $q$ is even. Then $|B|=q^2+1$ if and only if $B$ is an ovoid of $\mathcal{X}=(\mathbb{P},\mathbb{T})\simeq W(q)$.*

For a general $q$ which is odd, other than the bounds given in Theorem 3.7, not much is known for the minimum size $\mathbb{T}$-blocking sets in $PG(3, q)$. However, for $q = 3$, we are able to prove the following theorem which characterizes the minimum size $\mathbb{T}$-blocking sets in $PG(3, 3)$.

**Theorem 3.9.** *There is no* $\mathbb{T}$*-blocking set of size* 10 *in* $PG(3, 3)$*. Up to isomorphism, there are two* $\mathbb{T}$*-blocking sets of size* 11 *in* $PG(3, 3)$*.*

In order to prove Theorem 3.9, we construct two nonisomorphic $\mathbb{T}$-blocking sets in $PG(3, 3)$ each of size 11. Then we go on to prove the nonexistence of $\mathbb{T}$-blocking sets of size 10 in $PG(3, 3)$ and classify the $\mathbb{T}$-blocking sets of size 11 in $PG(3, 3)$.

# Chapter 1

# Preliminaries

In this chapter, we recall the definitions and basic properties of those point-line geometries that are needed in the subsequent chapters.

## 1.1 Point-line geometry

A *point-line geometry* is a pair $\mathcal{X} = (P, L)$, where $P$ is a non-empty set and $L$ is a collection of subsets of $P$ each of size at least two. The elements of $P$ are called *points* and that of $L$ are called *lines* of $\mathcal{X}$. If $l \in L$ is a line containing a point $x \in P$, then we also say that $x$ *lies on* $l$ or $l$ *passes through* $x$. If $P$ is a finite set, then $\mathcal{X}$ is called a *finite point-line geometry*. If any two distinct points of $\mathcal{X}$ are contained in at most one line, then $\mathcal{X}$ is called a *partial linear space*. If any two distinct points of $\mathcal{X}$ are contained in exactly one line, then $\mathcal{X}$ is called a *linear space*. Every linear space is also a partial linear space.

Let $\mathcal{X} = (P, L)$ be a partial linear space. Two points of $\mathcal{X}$ are said to be *collinear* if there is a line of $\mathcal{X}$ containing them. If $x$ and $y$ are two distinct collinear points of $\mathcal{X}$, then we denote by $xy$ the unique line of $\mathcal{X}$ containing both $x$ and $y$. A subset $P_0$ of $P$ is called a *subspace* of $\mathcal{X}$ if each line of $\mathcal{X}$ containing at least two points of $P_0$ is entirely contained in $P_0$. The empty set, singletons, lines and $P$ are examples of subspaces of $\mathcal{X}$. Observe that intersection of subspaces is again a subspace. For a subset $Y$ of $P$, the *subspace generated by* $Y$, denoted by $\langle Y \rangle$, is the intersection of all subspaces of $\mathcal{X}$ containing $Y$. Thus $\langle Y \rangle$ is the smallest subspace of $\mathcal{X}$ containing $Y$ as a subset.

Let $\mathcal{X} = (P, L)$ and $\mathcal{X}' = (P', L')$ be two partial linear spaces. A map $\phi : P \longrightarrow P'$ is called an *isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ if it is a bijection preserving collinearity of points and induces a bijection from $L$ to $L'$. In that case, $\mathcal{X}$ and $\mathcal{X}'$ are called *isomorphic* and written as $\mathcal{X} \simeq \mathcal{X}'$. An isomorphism from $\mathcal{X}$ to itself is called an *automorphism* of $\mathcal{X}$.

## 1.2 The projective space $PG(n,q)$

A linear space $\mathcal{X} = (P, L)$ is called a *projective space* if every line contains at least three points and the following axiom of Veblen-Young is satisfied:

> If $x, y, a, b$ are four distinct points of $\mathcal{X}$ and the lines $xy$ and $ab$ intersect in some point, then the lines $xa$ and $yb$ also intersect in a point.

Throughout the thesis, *$q$ is assumed to be a prime power and the finite field of order $q$ is denoted by $\mathbb{F}_q$*. Every finite dimensional vector space over $\mathbb{F}_q$ gives rise to a finite projective space as described below.

Let $V$ be a vector space of finite dimension $n+1$ over $\mathbb{F}_q$. Let $P$ be the set of all 1-dimensional subspaces of $V$ and $L$ be the set of all 2-dimensional subspaces of $V$. If $U \in L$ is a 2-dimensional subspace of $V$, then we identify $U$ as the subset of $P$ consisting of those 1-dimensional subspaces of $V$ which are contained in $U$. Then the point-line geometry $(P, L)$ is a finite projective space, denoted by $PG(n,q)$ and is called *the $n$-dimensional projective space over $\mathbb{F}_q$*.

For a subspace $W$ of $V$, the points and the lines of $PG(n, q)$ which are contained in $W$ form a subspace of $PG(n, q)$. Conversely, if $S$ is a subspace of $PG(n, q)$, then there exists a $(k+1)$-dimensional subspace $W$ of $V$ such that $S$ is the $k$-dimensional projective space over $\mathbb{F}_q$ corresponding to $W$. In this case, $S$ is called a *$k$-dimensional subspace of* $PG(n, q)$.

The subspaces of $PG(n, q)$ of dimensions $0, 1$ are precisely the points and the lines, respectively, of $PG(n, q)$. The 2-dimensional subspaces of $PG(n, q)$ are called *planes* and the $(n-1)$-dimensional subspaces of $PG(n, q)$ are called *hyperplanes*. The number of points in $PG(n, q)$ is equal to $(q^{n+1}-1)/(q-1)$. In general, for $0 \leqslant k \leqslant n$, the number of $k$-dimensional subspaces of $PG(n, q)$ is

$$\frac{(q^{n+1}-1)(q^{n+1}-q)\cdots(q^{n+1}-q^k)}{(q^{k+1}-1)(q^{k+1}-q)\cdots(q^{k+1}-q^k)}.$$

Every invertible linear transformation $\phi$ of $V$ induces an automorphism of $PG(n, q)$ mapping points $\langle v \rangle$ to $\langle \phi(v) \rangle$ for $0 \neq v \in V$. An automorphism of $PG(n, q)$ obtained in this way is called a *projective transformation*. Note that two invertible linear transformations of $V$ define the same projective transformation if and only if $\phi_2 = \lambda\phi_1$ for some non-zero element $\lambda \in \mathbb{F}_q$. Two geometric figures in the projective space $PG(n, q)$ are said to be *projectively equivalent* if one of them can be carried into the other by a projective transformation.

A *duality* of $PG(n, q)$ is a bijection from the point set of $PG(n, q)$ to the set of hyperplanes of $PG(n, q)$ that induces a bijection from the set of $k$-dimensional subspaces to the set of $(n-k-1)$-dimensional subspaces of $PG(n, q)$ for $0 \leqslant k \leqslant n-1$. For example, a duality in $PG(3, q)$ maps points to planes, lines to lines and planes to points. A *polarity* of $PG(n, q)$ is a duality that has order 2 as a map.

## 1.3 Projective planes

A linear space is called a *projective plane* if any two distinct lines meet at exactly one point and there are four points such that no three of them are contained in the same line.

Let $\mathcal{X} = (P, L)$ be a finite projective plane. Then there exists a positive integer $s \geqslant 2$ such that every line of $\mathcal{X}$ contains precisely $s+1$ points and every point of $\mathcal{X}$ is contained in precisely $s+1$ lines. The integer $s$ is called the *order* of $\mathcal{X}$. There are $s^2+s+1$ points and $s^2+s+1$ lines of $\mathcal{X}$. Let $P_0$ be a subset of $P$ and define $L_0 := \{l \cap P_0 : l \in L \text{ and } |l \cap P_0| \geqslant 2\}$. If $\mathcal{X}_0 = (P_0, L_0)$ is a projective plane, then $\mathcal{X}_0$ is called a *subplane* of $\mathcal{X}$ and it is *proper* if $P_0 \neq P$.

Let $\mathcal{X} = (P, L)$ be a finite projective plane of order $s$. If $\mathcal{X}_0 = (P_0, L_0)$ is a proper subplane of $\mathcal{X}$ of order $s_0$, then $s_0^2 \leqslant s$. Further, $s_0^2 = s$ if and only if every line of $\mathcal{X}$ contains some point of $\mathcal{X}_0$. In the equality case, $\mathcal{X}_0$ is called a *Baer subplane* of $\mathcal{X}$ and the lines of $\mathcal{X}_0$ are called *Baer lines*. Thus a Baer subplane in a projective plane of order $s$ can not exist unless $s$ is a perfect square. If $\mathcal{X}_0$ is a Baer subplane of $\mathcal{X}$, then for every point $x$ of $P \setminus P_0$, there is one line of $\mathcal{X}$ through $x$ meeting $\mathcal{X}_0$ in a Baer line and each of the other lines of $\mathcal{X}$ through $x$ meets $\mathcal{X}_0$ in a single point, see [12].

The projective space $PG(2, q)$ is a projective plane of order $q$. For every perfect square $q$, $PG(2, q)$ has Baer subplanes necessarily isomorphic to $PG(2, \sqrt{q})$. We refer to [28] for the basic properties mentioned in the rest of this section.

### 1.3.1 Ovals in $PG(2, q)$

A $k$-*arc* in $PG(2, q)$ is a set of $k$ points such that no three of them are contained in the same line. If $PG(2, q)$ has a $k$-*arc*, then $k \leqslant q+1$ for $q$ odd, and $k \leqslant q+2$

for $q$ even. Any $(q+1)$-arc in $PG(2,q)$ is called an *oval*. If $q$ is even, then any $(q+2)$-arc in $PG(2,q)$ is called a *hyperoval*.

Let $\mathcal{O}$ be an oval in $PG(2,q)$. A line $l$ of $PG(2,q)$ is called *external*, *tangent* or *secant* to $\mathcal{O}$ according as $|l \cap \mathcal{O}| = 0, 1$ or 2. Every point of $\mathcal{O}$ lies on a unique tangent line, giving exactly $q+1$ tangent lines to $\mathcal{O}$.

Suppose that $q$ is even. Then all the $q+1$ tangent lines meet in a point and this common point of intersection is called the *nucleus* of $\mathcal{O}$. Every point of $\mathcal{O}$ lies on $q$ secant lines. Every point outside $\mathcal{O}$ and different from the nucleus lies on $q/2$ secant lines and $q/2$ external lines. In this case, $\mathcal{O}$ is contained in a unique hyperoval which is obtained as the union of $\mathcal{O}$ and its nucleus. Conversely, one can obtain an oval from a hyperoval by removing just one point from it.

Now, suppose that $q$ is odd. Then no three tangent lines to $\mathcal{O}$ share a common point. So every point of $PG(2,q)$ is contained in at most two tangent lines. A point $x$ of $PG(2,q)$ is called *interior*, *absolute* or *exterior* with respect to $\mathcal{O}$ according as $x$ is on 0, 1 or 2 tangent lines to $O$. Absolute points of $PG(2,q)$ are precisely the points contained in $\mathcal{O}$, giving $q+1$ absolute points. There are $q(q-1)/2$ interior points, $q(q+1)/2$ exterior points, $q(q-1)/2$ external lines and $q(q+1)/2$ secant lines. Every absolute point lies on one tangent line and $q$ secant lines. Every interior point lies on $(q+1)/2$ secant lines and $(q+1)/2$ external lines. Every exterior point lies on 2 tangent lines, $(q-1)/2$ secant lines and $(q-1)/2$ external lines. Every tangent line contains one absolute point and $q$ exterior points. Every external line contains $(q+1)/2$ interior points and $(q+1)/2$ exterior points. Every secant line contains 2 absolute points, $(q-1)/2$ interior points and $(q-1)/2$ exterior points.

### 1.3.2 Conics in $PG(2,q)$

A *conic* in $PG(2,q)$ is the set $\mathcal{C}$ of points $\langle(\alpha,\beta,\gamma)\rangle$ of $PG(2,q)$ satisfying a nonzero homogeneous quadratic polynomial equation in three variables of the form $Q(X,Y,Z)=0$, where

$$Q(X,Y,Z)=aX^2+bY^2+cZ^2+dXY+eXZ+fYZ$$

with $a,b,c,d,e,f\in\mathbb{F}_q$. The conic $\mathcal{C}$ is said to be *irreducible* if the polynomial $Q(X,Y,Z)$ is irreducible in $\mathbb{F}_q[X,Y,Z]$. Any irreducible conic in $PG(2,q)$ is projectively equivalent to the conic defined by the equation $Y^2=XZ$.

If $\mathcal{C}$ is an irreducible conic in $PG(2,q)$, then $\mathcal{C}$ contains exactly $q+1$ points of $PG(2,q)$ and no three points of $\mathcal{C}$ are contained in the same line. Thus every irreducible conic in $PG(2,q)$ is an oval. A famous theorem by Segre [37] says that if $q$ is odd, then every oval in $PG(2,q)$ is an irreducible conic. For $q\in\{2,4\}$, every oval of $PG(2,q)$ is an irreducible conic [14, Theorem 4.9]). However, if $q\geqslant 8$ is even, then $PG(2,q)$ has ovals that are not irreducible conics [14, Theorem 4.11].

## 1.4 Bilinear and Quadratic forms

Let $V$ be a finite dimensional vector space over $\mathbb{F}_q$. A *bilinear form* on $V$ is a function $b:V\times V\to\mathbb{F}_q$ satisfying the following conditions:

- $b(u+v,w)=b(u,w)+b(v,w)$ and $b(\lambda u,w)=\lambda b(u,w)$;
- $b(u,v+w)=b(u,v)+b(u,w)$ and $b(u,\lambda w)=\lambda b(u,w)$

for all $u,v,w\in V$ and $\lambda\in\mathbb{F}_q$.

Let $b$ be a bilinear form on $V$. We say that $b$ is *symmetric* if $b(u,v)=b(v,u)$ for all $u,v\in V$, and $b$ is *alternating* if $b(u,u)=0$ for all $u\in V$. If $b$ is alternating, then $b(v,u) = -b(u,v)$ for all $u, v \in V$. The bilinear form $b$ is said to be *non-degenerate* if for any non-zero vector $u \in V$, there exist vectors $v, w \in V$ such that $b(u,v) \neq 0$ and $b(w,u) \neq 0$.

Now, let $b$ be a symmetric/alternating bilinear form on $V$. For a subspace $U$ of $V$, define $U^{\perp} = \{v \in V : b(u,v) = 0 \text{ for all } u \in U\}$. A subspace $U$ of $V$ is said to be *totally isotropic* if $U$ is contained in $U^{\perp}$, that is, if $b(u,u') = 0$ for all $u, u' \in U$. Note that $b$ is non-degenerate if and only if $V^{\perp} = \{0\}$. If $b$ is non-degenerate and alternating, then the dimension of $V$ must be even. A non-degenerate alternating bilinear form is also called a *symplectic form*.

A *quadratic form* on $V$ is a map $f : V \to \mathbb{F}_q$ satisfying the following two conditions:

- $f(\lambda v) = \lambda^2 f(v)$ for all $v \in V$ and $\lambda \in \mathbb{F}_q$.
- The map $b : V \times V \to \mathbb{F}_q$, defined by $b(u,v) = f(u+v) - f(u) - f(v)$ for $u, v \in V$, is a symmetric bilinear form on $V$.

Let $f$ be a quadratic form on $V$ with associated symmetric bilinear form $b$. We say that $f$ is *non-degenerate* if $V^{\perp} \cap f^{-1}(0) = \{0\}$, that is, if for every non-zero $u \in V$ with $f(u) = 0$, there exists $w \in V$ with $b(u,w) \neq 0$. Clearly, if $b$ is non-degenerate, then so is $f$. The converse is also true if $q$ is odd. A subspace $U$ of $V$ is *totally singular* if $f(u) = 0$ for all $u \in U$. If $f$ is non-degenerate, then every totally singular subspace has dimension at most half of the dimension of $V$.

The set $\mathcal{Q}$ consisting of all one-dimensional subspaces $\langle v \rangle$ of $V$ with $f(v) = 0$ is called a *quadric* in $PG(n,q)$ with respect to $f$. If the dimension of $V$ is 3, then the quadric $\mathcal{Q}$ is a conic. If the quadratic form $f$ on $V$ is non-degenerate, then $\mathcal{Q}$ is called a *non-degenerate quadric*. If $\mathcal{Q}$ is non-degenerate, then the *Witt index* of $\mathcal{Q}$ is the maximum (vector space) dimension of a totally singular subspace contained in $\mathcal{Q}$.

### 1.4.1 Non-degenerate quadrics in $PG(3,q)$

Consider $V = \mathbb{F}_q^4$ and let $f : V \to \mathbb{F}_q$ be a non-degenerate quadratic form. After a suitable linear change of coordinates, $f$ is given by either

(H) $f(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$; or

(E) $f(x_1, x_2, x_3, x_4) = x_1x_2 + ax_3^2 + bx_3x_4 + cx_4^2$

for $(x_1, x_2, x_3, x_4) \in V$, where $a, b, c \in \mathbb{F}_q$ such that the quadratic polynomial $aX^2 + bX + c \in \mathbb{F}_q[X]$ is irreducible over $\mathbb{F}_q$. In the first case, the associated quadratic in $PG(3,q)$ is of Witt index 2 and is called a *hyperbolic quadric*. In the latter case, the quadric is of Witt index 1 and is called an *elliptic quadric*. A hyperbolic quadric contains $(q+1)^2$ points and $2(q+1)$ lines. An elliptic quadric contains $q^2+1$ points but no lines. One can refer to [18] for the basic properties of points, lines and planes of $PG(3,q)$ with respect to a quadric.

If $b$ is the symmetric bilinear form associated with $f$, then the map $U \mapsto U^\perp = \{v \in V : b(u,v) = 0 \text{ for all } u \in U\}$ for subspaces $U$ of $V$, defines a polarity of $PG(3,q)$. Such a polarity is called an *orthogonal polarity* or a *pseudo polarity* according as $q$ is odd or even.

## 1.5 Ovoids of $PG(3,q)$

Let $\mathbb{P}$ and $\mathbb{L}$ denote the set of points and lines, respectively, of $PG(3,q)$. A subset $\mathcal{O}$ of $\mathbb{P}$ is called an *ovoid* of $PG(3,q)$ if the following two conditions are satisfied:

(i) Each line of $PG(3,q)$ meets $\mathcal{O}$ in at most two points.

(ii) For every point $x \in \mathcal{O}$, the union of all the lines which meet $\mathcal{O}$ at $x$ is a plane of $PG(3, q)$.

Let $\mathcal{O}$ be an ovoid of $PG(3, q)$. A line of $PG(3, q)$ is called *tangent* to $\mathcal{O}$ if it meets $\mathcal{O}$ in one point. For $x \in \mathcal{O}$, there are $q + 1$ lines through $x$ which are tangent to $\mathcal{O}$. Each of the remaining $q^2$ lines through $x$ must meet $\mathcal{O}$ in exactly one more point. It follows that $|\mathcal{O}| = q^2 + 1$ and there are $(q+1)(q^2+1)$ lines in $\mathbb{L}$ which are tangent to $\mathcal{O}$.

We recall a few basic facts related to ovoids of $PG(3, q)$. Every plane of $PG(3, q)$ meets an ovoid in a single point or in an oval. When $q > 2$, the ovoids of $PG(3, q)$ are precisely the subsets of $\mathbb{P}$ of the largest possible size, no three points of which are on the same line in $\mathbb{L}$. This is not true in $PG(3, 2)$. In this case, the complement of a plane is a subset of maximum size 8 in which no three points are on the same line, but such a set is not an ovoid of $PG(3, 2)$. For $q$ odd, the ovoids of $PG(3, q)$ are precisely the elliptic quadrics. This was proved independently by Barlotti [5] and Panella [32]. For $q$ even, say $q = 2^r$, the known ovoids of $PG(3, q)$ are of two types:

(i) the elliptic quadrics which exist for all $r \geqslant 1$,

(ii) the Tits ovoids which exist for odd $r \geqslant 3$, that is, when $q > 2$ is a non-square.

One can refer to [18, Section 16.4] for more on Tits ovoids. For small values of the even prime power $q$, namely when $r \in \{1, 2, 3, 4, 5\}$, a complete classification of all ovoids of $PG(3, q)$ has been obtained. By [16, 29, 30, 31], we know that every ovoid of $PG(3, q)$ is an elliptic quadric if $q \in \{2, 4, 16\}$, and either an elliptic quadric or a Tits ovoid if $q \in \{8, 32\}$. However, for a general even prime power $q$, classifying all ovoids of $PG(3, q)$ is still an open problem.

## 1.6 Generalized quadrangles

We refer to [33] for the basics on finite generalized quadrangles. Let $s$ and $t$ be positive integers. A (finite) *generalized quadrangle of order* $(s,t)$ is a partial linear space $\mathcal{X} = (P, L)$ with point set $P$ and line set $L$ satisfying the following three axioms:

(Q1) Every line contains $s+1$ points.

(Q2) Every point is contained in $t+1$ lines.

(Q3) For every point-line pair $(x,l) \in P \times L$ with $x \notin l$, there exists a unique line $m \in L$ containing $x$ and intersecting $l$.

Let $\mathcal{X} = (P, L)$ be a generalized quadrangle of order $(s,t)$. Then $|P| = (s+1)(st+1)$ and $|L| = (t+1)(st+1)$ [33, 1.2.1]. If $s = t$, then $\mathcal{X}$ is said to have order $s$. If $P$ is a subset of the point set of some projective space $PG(n,q)$, $L$ is a set of lines of $PG(n,q)$ and $P$ is the union of all lines in $L$, then $\mathcal{X}$ is called a *projective generalized quadrangle*. An *ovoid* of $\mathcal{X}$ is a set $\mathcal{O}$ of points with the property that each line of $\mathcal{X}$ contains exactly one point of $\mathcal{O}$. If $\mathcal{X}$ has an ovoid $\mathcal{O}$, then counting in two ways the number of point-line pairs $(x,l)$, where $x \in \mathcal{O}$ and $l$ is a line of $\mathcal{X}$ containing $x$, it follows that $|\mathcal{O}| = st+1$.

The points and the lines contained in a hyperbolic quadric in $PG(3,q)$ form a generalized quadrangle of order $(q,1)$ and it has ovoids.

Two points of $\mathcal{X}$ are said to be *collinear* if there exists a line of $\mathcal{X}$ containing both of them. For a subset $Z$ of $P$, $Z^{\perp}$ denotes the set of all elements of $P$ which are collinear with each element of $Z$ (our two different uses of the notation '$\perp$' should be clear from the context). Note that, for $a \in P$, $a^{\perp} := \{a\}^{\perp}$ contains $a$. For two distinct points $a, b \in P$, we have $|\{a,b\}^{\perp}| = s+1$ or $t+1$ according as $a$

is collinear with $b$ or not. For two non-collinear points $a, b \in P$, the set $\{a, b\}^{\perp\perp}$ is called a *hyperbolic line* defined by $a$ and $b$. A point $x \in P$ is said to be *regular* if $|\{x, y\}^{\perp\perp}| = t + 1$ for every point $y \notin x^{\perp}$.

### 1.6.1 The generalized quadrangle $W(q)$

Let $V$ be a 4-dimensional vector space over $\mathbb{F}_q$ and $b : V \times V \to \mathbb{F}_q$ be a symplectic form on $V$ (that is, $b$ is non-degenerate and alternating). The map $U \mapsto U^{\perp} = \{v \in V : b(u, v) = 0 \text{ for all } u \in U\}$ for subspaces $U$ of $V$, defines a polarity of $PG(3, q)$. Such a polarity is called a *symplectic polarity* of $PG(3, q)$.

The point-line geometry whose point set is the set of all points of $PG(3, q)$ and line set consisting of all the totally isotropic lines of $PG(3, q)$ with respect to $b$ is a generalized quadrangle of order $q$, denoted by $W(q)$. All points of $W(q)$ are regular [33, 3.2.1, 3.3.1]. Further, $W(q)$ has ovoids if and only if $q$ is even, see [33, 3.2.1, 3.4.1]. By a result of Segre [38], every ovoid of $PG(3, q)$ with $q$ even, is an ovoid of some $W(q)$. In [41], Thas proved the converse statement that every ovoid of $W(q)$, $q$ even, is an ovoid of the ambient space $PG(3, q)$.

## 1.7 Blocking sets

Let $\mathcal{X}$ be a point-line geometry and $L$ be a given non-empty subset of the line set of $\mathcal{X}$. A *blocking set with respect to* $L$ (or simply, an $L$*-blocking set*) in $\mathcal{X}$ is a subset $B$ of the point set of $\mathcal{X}$ such that each line in $L$ contains at least one point of $B$. Blocking sets in various point-line geometries with respect to varying sets of lines have been studied by several authors. The first step in this regard has been to determine the minimum size of an $L$-blocking set and to give, if possible, geometric description of all $L$-blocking sets of that minimum cardinality.

An $L$-blocking set $B$ is said to be *minimal* if no proper subset of $B$ is an $L$-blocking set in $\mathcal{X}$. Every minimum size $L$-blocking set in $\mathcal{X}$ is also a minimal $L$-blocking set.

When $\mathcal{X} = PG(n, q)$, blocking sets in $PG(n, q)$ are combinatorial objects in finite geometry with several applications and have been the subject of investigation by many researchers. The following fundamental result is due to Bose and Burton [11, Theorem 1] which is stated in the language of blocking sets.

**Proposition 1.7.1.** *[11] If $B$ is a blocking set in $PG(n, q)$ with respect to the set of all lines of $PG(n, q)$, then $|B| \geqslant (q^n - 1)/(q - 1)$. Further, equality holds if and only if $B$ is the point set of a hyperplane of $PG(n, q)$.*

We shall use the above proposition frequently in the subsequent chapters while studying minimum size blocking sets in $PG(n, q)$, $n \in \{2, 3\}$, with respect to certain proper subsets of the line set of $PG(n, q)$.

A blocking set $B$ in $PG(n, q)$ with respect to all its lines is called *nontrivial* if $B$ does not contain any hyperplane of $PG(n, q)$, or equivalently, if every hyperplane of $PG(n, q)$ contains a point which is not in $B$. In view of the above result of Bose and Burton, one can study minimal nontrivial blocking sets in $PG(n, q)$. When $n = 2$, many results on minimal nontrivial blocking sets in $PG(2, q)$ are available in the literature, for example, see [8, 22, 34, 35, 40] and the references therein.

# Chapter 2

# Blocking sets in $PG(2, q)$

Let $\mathcal{C}$ be an irreducible conic in $PG(2, q)$. We denote by $\mathcal{E}$ (respectively, $\mathcal{T}$, $\mathcal{S}$) the set of all lines of $PG(2, q)$ which are external (respectively, tangent, secant) with respect to $\mathcal{C}$. In this chapter, we discuss the minimum size $L$-blocking sets in $PG(2, q)$, where the line set $L$ is one of $\mathcal{E}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{S} \cup \mathcal{T}$ and $\mathcal{E} \cup \mathcal{S}$.

## 2.1 Results from the literature

In this section, we give a brief survey of the known results regarding the minimum size $L$-blocking sets in $PG(2, q)$ for $L \in \{\mathcal{E}, \mathcal{S}, \mathcal{T} \cup \mathcal{E}, \mathcal{T} \cup \mathcal{S}\}$. If $q$ is even, we shall denote by $n$ the nucleus of $\mathcal{C}$.

### 2.1.1 $\mathcal{E}$-blocking sets

For $q$ odd, Aguglia and Korchmáros studied the minimum size $\mathcal{E}$-blocking sets in $PG(2, q)$ [3, Theorem 1.1]. Using a result on the linear system of polynomials vanishing at every interior point to $\mathcal{C}$ and a corollary to the classification theorem of all subgroups of the projective general linear group $PGL(2, q)$, they proved the following.

**Theorem 2.1.1.** *[3] Let $A$ be an $\mathcal{E}$-blocking set in $PG(2,q)$, where $q$ is odd. Then $|A| \geqslant q-1$ and the following hold for equality case:*

*(i) If $q \geqslant 9$, then $|A| = q-1$ if and only if $A = l \setminus \mathcal{C}$ for some line $l$ of $PG(2,q)$ secant to $\mathcal{C}$.*

*(ii) If $q \in \{5,7\}$, then $|A| = q-1$ if and only if one of the following two cases occurs:*

*(a) $A = l \setminus \mathcal{C}$ for some line $l$ of $PG(2,q)$ secant to $\mathcal{C}$.*

*(b) $A$ is a suitable set of $q-1$ interior points with respect to $\mathcal{C}$.*

*(iii) If $q = 3$, then $|A| = 2$ if and only if one of the following two cases occurs:*

*(a) $A = l \setminus \mathcal{C}$ for some line $l$ of $PG(2,3)$ secant to $\mathcal{C}$.*

*(b) $A$ consists of any two interior points with respect to $\mathcal{C}$.*

When $q = 3$, the possibility stated in Theorem 2.1.1(iii)(b) was not included in the statement of [3, Theorem 1.1]. We give a proof of Theorem 2.1.1(iii) below. For $q = 5$, an $\mathcal{E}$-blocking set consisting of 4 interior points is given in Example 2.2.1.

**Proof of Theorem 2.1.1(iii).** There are three interior points and three external lines in $PG(2,3)$ with respect to $\mathcal{C}$. Every external line in $PG(2,3)$ contains exactly two interior points. So any two interior points will block all the three external lines in $PG(2,3)$, justifying the statement in Theorem 2.1.1(iii)(b).

Conversely, let $A = \{x, y\}$ be a blocking set of minimum size 2 of the external lines in $PG(2,3)$ with respect to $\mathcal{C}$. Then the minimality of $|A|$ implies $A \cap \mathcal{C} = \emptyset$. Let $l := xy$ be the line of $PG(2,q)$ through $x$ and $y$. We may assume that $l$ is not secant to $\mathcal{C}$. Suppose that $l$ is tangent to $\mathcal{C}$. Let $z \in l$ be the unique point such that $l \setminus \mathcal{C} = \{x, y, z\}$. Then the unique external line through $z$ would not meet $A$, a contradiction. So $l$ is external to $\mathcal{C}$. If at least one of $x$ and $y$ is not interior to $\mathcal{C}$, then there exists a point $b \in l \setminus \{x, y\}$ which is interior to $\mathcal{C}$. Then the external line through $b$, different from $l$, would be disjoint from $A$, again a contradiction. Thus both $x$ and $y$ are interior with respect to $\mathcal{C}$. $\square$

The case $q$ even was considered by Giulietti in [17, Theorems 1.1, 1.2], where he provided two more possibilities for the $\mathcal{E}$-blocking sets in $PG(2, q)$ of smallest cardinality. Using results on the linear system of polynomials vanishing at points uncovered by the lines of a line-conic in $PG(2, q)$ together with the classification of all subgroups of $PSL(2, q)$, the following was proved.

**Theorem 2.1.2.** *[17] Let $A$ be an $\mathcal{E}$-blocking set in $PG(2, q)$, where $q$ is even. Then $|A| \geqslant q-1$, and equality holds if and only if one of the following three cases occurs:*

*(i)* $A = l \setminus \mathcal{C}$ *for some line $l$ of $PG(2, q)$ secant to $\mathcal{C}$.*

*(ii)* $A = l \setminus (\mathcal{C} \cup \{n\})$ *for some line $l$ of $PG(2, q)$ tangent to $\mathcal{C}$.*

*(iii) $q$ is a square and $A = \Pi \setminus (\{n\} \cup (\Pi \cap \mathcal{C}))$, where $\Pi$ is a Baer subplane of $PG(2, q)$ such that $\Pi \cap \mathcal{C}$ is an irreducible conic in $\Pi$.*

As a consequence of Theorems 2.1.1 and 2.1.2 above, we have the following.

**Corollary 2.1.3.** *Any $\mathcal{E}$-blocking set in $PG(2, q)$ of minimum size $q-1$ is disjoint from $\mathcal{C}$.*

### 2.1.2 $\mathcal{S}$-blocking sets

A *quadrangle* in $PG(2,q)$ is a set of four points, no three of which are collinear. If $a, b, c, d$ are the points of a quadrangle in $PG(2,q)$, define the three points $x, y, z$ to be the intersections of the lines $ab$ and $cd$, $ac$ and $bd$, $ad$ and $bc$, respectively. The points $x, y, z$ are called the *diagonal points* of the given quadrangle. Note that a quadrangle in $PG(2,q)$ has collinear diagonal points if and only if $q$ is even [21, 9.63, p.501].

For $q$ odd, Aguglia and Giulietti studied the minimum size $\mathcal{S}$-blocking sets in $PG(2,q)$ [1, Theorem 1.1]. They showed that any such blocking set contains at least $q$ points and a blocking set with exactly $q$ points necessarily consists of $q-k$ points of $\mathcal{C}$ and $k$ other points for some $k \in \{0,1,3\}$. More precisely, they proved the following.

**Theorem 2.1.4.** *[1] Let $A$ be an $\mathcal{S}$-blocking set in $PG(2,q)$, where $q$ is odd. Then $|A| \geqslant q$, and equality holds if and only if one of the following three cases occurs:*

*(i)* $A = \mathcal{C} \setminus \{x\}$ *for some* $x \in \mathcal{C}$.

*(ii)* $A = (\mathcal{C} \setminus \{x,y\}) \cup \{a\}$ *for distinct points* $x, y \in \mathcal{C}$, *where $a$ is a point (different from $x$ and $y$) on the secant line through $x$ and $y$.*

*(iii)* $A = (\mathcal{C} \setminus \{w,x,y,z\}) \cup \{a,b,c\}$ *for some quadrangle* $\{w,x,y,z\} \subseteq \mathcal{C}$ *with diagonal points* $a, b, c$.

Note that if $|A| = q$, then $A \cap \mathcal{C}$ is nonempty, except when $q = 3$ and Theorem 2.1.4(iii) holds, in which case $A$ consists of all the three interior points with respect to $\mathcal{C}$.

In order to prove the above theorem, the authors used a result that is interesting in its own right: For any set $K$ consisting of $k > 4$ points of $\mathcal{C}$, there is no set of $k-1$ points disjoint from $\mathcal{C}$ that blocks all secant lines to $K$ (that is, lines of $PG(2,q)$ intersecting $K$ at two points).

When $q$ is even, Aguglia et al. proved in [4, Theorem 1.1] that the minimum size of an $\mathcal{S}$-blocking set in $PG(2,q)$ is $q$. However, their characterization for the $\mathcal{S}$-blocking sets of size $q$ is quite different. In the following, we describe their procedure to construct several examples of $\mathcal{S}$-blocking sets of size $q$. We first recall some definitions.

Let $\tau$ be a collineation of $PG(2,q)$, that is, an automorphism of $PG(2,q)$. A point $z$ of $PG(2,q)$ is called a *centre* of $\tau$ if every line through $z$ is fixed by $\tau$. A line $l$ of $PG(2,q)$ is called an *axis* of $\tau$ if every point of $l$ is fixed by $\tau$. If $\tau$ has a centre and an axis, then $\tau$ is called a *central collineation*. A central collineation of $PG(2,q)$ is called an *elation* if the center is contained in the axis. Let $l_0$ be a fixed line of $PG(2,q)$ and $AG(2,q)$ be the classical affine plane obtained from $PG(2,q)$ by deleting the line $l_0$ and all its points. A *translation* of $AG(2,q)$ is an automorphism $\phi$ of $AG(2,q)$ such that either $\phi$ is the identity map, or $\phi$ has no fixed point and $\phi$ fixes every line of some parallel class. Every translation of $AG(2,q)$ can be extended to an elation of $PG(2,q)$ with axis $l_0$. Conversely, every elation of $PG(2,q)$ with axis $l_0$ induces a translation of $AG(2,q)$.

Now, assume that the conic $\mathcal{C}$ is given with its affine equation $Y = X^2$, that is, $\mathcal{C}$ is a parabola in the affine plane $AG(2,q)$. For every $a \in \mathbb{F}_q$, the map $\varphi_a : (X,Y) \mapsto (X+a, Y+a^2)$ is a translation of $AG(2,q)$. Viewing $\varphi_a$ as an elation of $PG(2,q)$, the centre of $\varphi_a$ is the infinite point $(1,a,0)$. The translation group of $\mathcal{C}$ is $T = \{\varphi_a : a \in \mathbb{F}_q\}$ and it is isomorphic to the additive group $(\mathbb{F}_q, +)$ of $\mathbb{F}_q$. For a subgroup $H$ of $(\mathbb{F}_q, +)$, consider the subgroup $G = \{\varphi_a : a \in H\}$ of $T$ and define $\Gamma$ to be the set of all centres of the nontrivial translations in $G$. For an affine point $(u,u^2)$ in $\mathcal{C}$, the orbit of $(u,u^2)$ under $G$ is $\Delta_u = \{(a+u,(a+u)^2) : a \in H\}$.

Then, the set $A(G,u) = (\mathcal{C} \setminus \Delta_u) \cup \Gamma$ is an $\mathcal{S}$-blocking set in $PG(2,q)$ of size $q$.

**Theorem 2.1.5.** *[4] Let $A$ be an $\mathcal{S}$-blocking set in $PG(2,q)$, where $q$ is even. Then $|A| \geqslant q$. Further, equality holds if and only if $A = B(G,u)$ described above, for some affine point $(u,u^2) \in \mathcal{C}$ and some subgroup $G$ of $T$ arising from a subgroup $H$ of $(\mathbb{F}_q,+)$.*

From the above, we have the following.

**Corollary 2.1.6.** *Let $A$ be an $\mathcal{S}$-blocking set in $PG(2,q)$. Then $|A| \geqslant q$. If $|A| = q$, then $A \cap \mathcal{C} \neq \emptyset$, except when $q = 3$ and $A$ consists of all the three interior points with respect to $\mathcal{C}$.*

### 2.1.3 $(\mathcal{T} \cup \mathcal{S})$-blocking sets

The following lower bound for the sizes of $(\mathcal{T} \cup \mathcal{S})$-blocking sets in $PG(2,q)$ is easily obtained.

**Theorem 2.1.7.** *Let $A$ be a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $PG(2,q)$. Then $|A| \geqslant q+1$.*

*Proof.* Clearly, the result follows if $A$ contains $\mathcal{C}$. Suppose that $x$ is a point of $\mathcal{C}$ which is not in $A$. Each line through $x$ is either a tangent or a secant line and hence meets $A$. It follows that $|A| \geqslant q+1$. □

Clearly, every line of $PG(2,q)$ and the conic $\mathcal{C}$ itself are minimal $(\mathcal{T} \cup \mathcal{S})$-blocking sets in $PG(2,q)$ each of size $q+1$. The following result was proved by Bruen and Thas in [13] for $q$ even and by Segre and Korchmáros in [39] for all $q$.

**Theorem 2.1.8.** *[13, 39] If $A$ is a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $PG(2,q)$ of size $q+1$ such that $A$ is disjoint from $\mathcal{C}$, then $A$ is an exterior line with respect to $\mathcal{C}$.*

All $(\mathcal{T} \cup \mathcal{S})$-blocking sets of size $q+1$ that are different from $\mathcal{C}$ and the lines of $PG(2,q)$ were described by Boros et al. in [10]. We present their description below.

Consider a line $l$ of $PG(2,q)$. Let $c_1, c_2, \cdots, c_m$, $m \in \{q-1, q, q+1\}$, be the points of $\mathcal{C}$ which are not on $l$. Denote by $l(c_i, c_j)$ the line passing through $c_i$ and $c_j$ (if $i = j$, then $l(c_i, c_i)$ is the tangent line at $c_i$). Let the $m$ points $x_1, x_2, \cdots, x_m$ of $l$ which are not in $\mathcal{C}$ be indexed so that the point $x_i$ lies on the line $l(c_1, c_i)$. As a consequence of Pascal's theorem, the set $\{1, 2, \cdots, m\}$ forms an abelian group under the multiplication rule:

$$ij = k \quad \text{if } x_k \text{ lies on the line } l(c_i, c_j).$$

Denote this group by $G(l, c_1)$. It is known [20, 19] that $G(l, c_1)$ is cyclic if $m \in \{q-1, q+1\}$ and elementary abelian if $m = q$.

Now consider a proper subgroup $H$ of $G(l, c_1)$ and a coset $Hk$ of $H$ in $G(l, c_1)$. Let $A$ be the set of points obtained by deleting from $\mathcal{C}$ the points of $\mathcal{C} \setminus l$ corresponding to $Hk$ and adding the points of $l \setminus \mathcal{C}$ corresponding to the coset $Hk^2$. Then $|A| = q+1$ and $A$ is a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $PG(2,q)$. Note that replacing $c_1$ by any other $c_i$ gives the same $(\mathcal{T} \cup \mathcal{S})$-blocking set as the system of cosets does not change. The following was proved in [10, Theorem 2.5].

**Theorem 2.1.9.** *[10] If $A$ is a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $PG(2,q)$ of minimum size $q+1$ which is different from $\mathcal{C}$ and the lines of $PG(2,q)$, then $A$ arises from a proper subgroup $H$ of $G(l, c_1)$ as described above.*

### 2.1.4 $(\mathcal{T} \cup \mathcal{E})$-blocking sets

For $q$ odd, Aguglia and Korchmáros studied the minimum size $(\mathcal{T} \cup \mathcal{E})$-blocking sets in $PG(2, q)$ [2, Theorem 1.1]. Using results on the linear system of polynomials vanishing at every point of $\mathcal{C}$ and at every interior point to $\mathcal{C}$ together with a corollary to the classification theorem of all subgroups of $PGL(2, q)$, they proved the following.

**Theorem 2.1.10.** *[2] Let $A$ be a $(\mathcal{T} \cup \mathcal{E})$-blocking set in $PG(2, q)$, where $q$ is odd. Then $|A| \geqslant q$, and equality holds if and only if one of the following three cases occurs:*

*(i) $A$ consists of all points of a tangent line, minus the tangency point.*

*(ii) $A$ consists of all points of a secant line different from the two intersecting points with $\mathcal{C}$, plus the pole[1] of this secant line with respect to $\mathcal{C}$.*

*(iii) $q$ is a square and $A = \Pi \setminus (\Pi \cap \mathcal{C})$, where $\Pi$ is a Baer subplane of $PG(2, q)$ such that $\Pi \cap \mathcal{C}$ is an irreducible conic in $\Pi$.*

When $q$ is even, Aguglia and Giulietti studied the minimum size $(\mathcal{T} \cup \mathcal{E})$-blocking sets in $PG(2, q)$ [1, Theorem 1.2]. As an application of the equality case of Theorem 2.1.2, they proved the following which is similar to Theorem 2.1.10.

**Theorem 2.1.11.** *[1] Let $A$ be a $(\mathcal{T} \cup \mathcal{E})$-blocking set in $PG(2, q)$, where $q$ is even. Then $|A| \geqslant q$, and equality holds if and only if one of the following three cases occurs:*

*(i) $A$ consists of all points of a tangent line, minus the tangency point.*

[1] Suppose that $q$ is odd and $x_1, x_2$ are two distinct points of $\mathcal{C}$. If $l_i$ is the tangent line through $x_i$ for $i \in \{1, 2\}$, then the point of intersection of $l_1$ and $l_2$ is called the *pole* of the secant line $x_1 x_2$.

*(ii)* *A consists of all points of a secant line different from the two intersecting points with $\mathcal{C}$, plus the nucleus of $\mathcal{C}$.*

*(iii)* *$q$ is a square and $A = \Pi \setminus (\Pi \cap \mathcal{C})$, where $\Pi$ is a Baer subplane of $PG(2,q)$ such that $\Pi \cap \mathcal{C}$ is an irreducible conic in $\Pi$.*

## 2.2 New results

In this section, the other two line sets are considered, that is, we study the minimum size $\mathcal{T}$- and $(\mathcal{E} \cup \mathcal{S})$-blocking sets in $PG(2,q)$.

### 2.2.1 $\mathcal{T}$-blocking sets

For $q$ even, it is clear that the singleton set $\{n\}$ consisting of the nucleus of $\mathcal{C}$ is the only $\mathcal{T}$-blocking set in $PG(2,q)$ of minimum size one.

Assume that $q$ is odd. Let $\mathcal{C} = \{x_1, x_2\} \cup \cdots \cup \{x_q, x_{q+1}\}$ be a partition of $\mathcal{C}$ into subsets of size two and let $l_i$ denote the unique tangent line to $\mathcal{C}$ through $x_i$, $1 \leqslant i \leqslant q+1$. For $1 \leqslant k \leqslant (q+1)/2$, each pair $\{x_{2k-1}, x_{2k}\}$ corresponds to an exterior point $a_k$ which is the point of intersection of the tangent lines $l_{2k-1}$ and $l_{2k}$. Then the set $\{a_1, a_2, \cdots, a_{(q+1)/2}\}$ is a $\mathcal{T}$-blocking set in $PG(2,q)$ of size $(q+1)/2$.

Conversely, let $A$ be any $\mathcal{T}$-blocking set in $PG(2,q)$. Every point of $A$ lies on at most two tangent lines. Counting the size of the set $Z = \{(l,x) \in \mathcal{T} \times A : x \text{ lies on } l\}$ in two ways, we get $q+1 = |\mathcal{T}| \leqslant |Z| \leqslant 2|A|$ and hence $|A| \geqslant (q+1)/2$. If $|A| = (q+1)/2$, then each point of $A$ lies on exactly two tangent lines. So $A$ consists of $(q+1)/2$ exterior points. Since $A$ is a $\mathcal{T}$-blocking set, it follows that for any two distinct points $a, b \in A$, the two tangent lines through $a$ are different from the two tangent lines through $b$. Thus, we have

**Theorem 2.2.1.** *Let $A$ be a $\mathcal{T}$-blocking set in $PG(2,q)$, where $q$ is odd. Then $|A| \geqslant (q+1)/2$. Further, equality holds if and only if $A$ consists of $(q+1)/2$ exterior points to $\mathcal{C}$ such that for any two distinct points $a, b \in A$, the two tangent lines through $a$ are different from the two tangent lines through $b$.*

### 2.2.2 $(\mathcal{E} \cup \mathcal{S})$-blocking sets

The only case left is the study of the minimum size blocking sets in $PG(2,q)$ with respect to all the secant and external lines to $\mathcal{C}$. We now consider this case. For $q$ even, we prove the following.

**Theorem 2.2.2.** *Let $A$ be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,q)$, where $q$ is even. Then $|A| \geqslant q$. Further, equality holds if and only if $A = l \setminus \{n\}$ for some tangent line $l$.*

Before stating the result for odd $q$, we give an example of a minimal $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,5)$ of size 6 which is different from a line. Using homogeneous coordinates, we write a point of $PG(2,5)$ as $\langle(a,b,c)\rangle$, the one dimensional subspace generated by a non-zero vector $(a,b,c)$ in $\mathbb{F}_5^3$. Each line of $PG(2,5)$ is a two dimensional subspace of the form $\alpha X + \beta Y + \gamma Z = 0$, where $\alpha, \beta, \gamma \in \mathbb{F}_5$ are not all zero, which we coordinatize as $\langle(\alpha,\beta,\gamma)^T\rangle$. The point $\langle(a,b,c)\rangle$ lies on the line $\langle(\alpha,\beta,\gamma)^T\rangle$ if and only if $\alpha a + \beta b + \gamma c = 0$. Without loss, we may assume that the conic $\mathcal{C}$ in $PG(2,5)$ has the equation $Y^2 = XZ$. Let $\mathcal{I}$ be the set of all interior points of $PG(2,5)$ with respect to $\mathcal{C}$. Then, we have

$$\begin{aligned}
\mathcal{C} &= \{\langle(1,0,0)\rangle, \langle(0,0,1)\rangle, \langle(1,1,1)\rangle, \langle(1,2,4)\rangle, \langle(1,3,4)\rangle, \langle(1,4,1)\rangle\}\,; \\
\mathcal{I} &= \{\langle(1,0,2)\rangle, \langle(1,0,3)\rangle, \langle(1,2,1)\rangle, \langle(1,3,1)\rangle, \langle(1,1,3)\rangle, \langle(1,2,2)\rangle, \\
&\quad \langle(1,1,4)\rangle, \langle(1,4,4)\rangle, \langle(1,3,2)\rangle, \langle(1,4,3)\rangle\};
\end{aligned}$$

$$\begin{aligned}
\mathcal{T} &= \left\{\langle(0,0,1)^T\rangle, \langle(1,0,0)^T\rangle, \langle(1,3,1)^T\rangle, \langle(1,4,4)^T\rangle, \langle(1,1,4)^T\rangle, \langle(1,2,1)^T\rangle\right\};\\
\mathcal{E} &= \{\langle(1,4,2)^T\rangle, \langle(1,1,1)^T\rangle, \langle(1,0,3)^T\rangle, \langle(1,3,3)^T\rangle, \langle(1,2,4)^T\rangle, \langle(1,2,3)^T\rangle,\\
&\quad \langle(1,0,2)^T\rangle, \langle(1,4,1)^T\rangle, \langle(1,3,4)^T\rangle, \langle(1,1,2)^T\rangle\};\\
\mathcal{S} &= \{\langle(0,1,0)^T\rangle, \langle(1,1,0)^T\rangle, \langle(1,0,1)^T\rangle, \langle(0,1,1)^T\rangle, \langle(1,2,0)^T\rangle, \langle(1,3,0)^T\rangle,\\
&\quad \langle(1,4,0)^T\rangle, \langle(1,3,2)^T\rangle, \langle(1,0,4)^T\rangle, \langle(0,1,2)^T\rangle, \langle(0,1,3)^T\rangle, \langle(0,1,4)^T\rangle,\\
&\quad \langle(1,1,3)^T\rangle, \langle(1,2,2)^T\rangle, \langle(1,4,3)^T\rangle\}.
\end{aligned}$$

Consider the following set consisting of six interior points:

$$A = \left\{\langle(1,0,2)\rangle, \langle(1,2,1)\rangle, \langle(1,2,2)\rangle, \langle(1,1,4)\rangle, \langle(1,3,2)\rangle, \langle(1,4,3)\rangle\right\}.$$

It can be verified that $A$ is a blocking set in $PG(2,5)$ with respect to all the secant and external lines to $\mathcal{C}$. The minimality of the blocking set $A$ follows from Theorem 2.2.3(ii). Observe that the four points $\langle(1,0,3)\rangle$, $\langle(1,3,1)\rangle$, $\langle(1,1,3)\rangle$, $\langle(1,4,4)\rangle$ of $\mathcal{I}\setminus A$ form a quadrangle and they have the property that any two of them determine an external line.

The following is an example of an $\mathcal{E}$-blocking set in $PG(2,5)$ consisting of 4 interior points, see Theorem 2.1.1(ii)(b).

**Example 2.2.1.** *Let $A$ be the set consisting of the three interior points $\langle(1,0,2)\rangle$, $\langle(1,3,1)\rangle$, $\langle(1,4,4)\rangle$ lying on the external line $\langle(1,4,2)^T\rangle$, together with the interior point $\langle(1,4,3)\rangle$. Then it can be seen that $A$ is an $\mathcal{E}$-blocking set in $PG(2,5)$ of minimum size* 4.

For $q$ odd, we prove the following for the minimum size blocking sets of $PG(2,q)$ with respect to all the secant and external lines to $\mathcal{C}$.

**Theorem 2.2.3.** *Let $A$ be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,q)$, where $q$ is odd. Then the following hold:*

*(i) If $q = 3$, then $|A| \geqslant 3$ and equality holds if and only if $A$ consists of all the three interior points to $\mathcal{C}$.*

*(ii) If $q \geqslant 5$, then $|A| \geqslant q+1$.*

*(iii) If $q \geqslant 7$, then $|A| = q+1$ if and only if $A$ is a line of $PG(2,q)$.*

*(iv) If $q = 5$, then $|A| = 6$ if and only if one of the following two cases occurs:*

*(a) $A$ is a line of $PG(2,5)$.*

*(b) $A = \mathcal{I} \setminus \{a_1, a_2, a_3, a_4\}$, where $\{a_1, a_2, a_3, a_4\} \subseteq \mathcal{I}$ is a quadrangle such that the line determined by any two distinct $a_i, a_j$ is external to $\mathcal{C}$.*

We note that four interior points satisfying the condition in Theorem 2.2.3(iv)(b) do exist, which follows from the above discussion.

**Corollary 2.2.4.** *For $q \geqslant 7$ odd, every $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,q)$ of minimum size $q+1$ is a blocking set with respect to all lines of $PG(2,q)$.*

### Proof of Theorem 2.2.2

Here $q$ is even. Let $A$ be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,q)$. Since every secant line meets $A$, Theorem 2.1.5 implies that $|A| \geqslant q$.

For every tangent line $l$, clearly the set $l \setminus \{n\}$ is an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,q)$ of size $q$. Conversely, let $A$ be an $(\mathcal{E} \cup \mathcal{S})$-blocking set of minimum size $q$. Then $n \notin A$ by the minimality of $|A|$.

**Lemma 2.2.5.** $|A \cap \mathcal{C}| = 1$.

*Proof.* Since $A \setminus (A \cap \mathcal{C})$ blocks every external line, we have $|A \setminus (A \cap \mathcal{C})| \geqslant q - 1$ by Theorem 2.1.2. Then $|A| = q$ implies that $|A \cap \mathcal{C}| \leqslant 1$. We show that $A \cap \mathcal{C}$ is nonempty.

If $A \cap \mathcal{C}$ is empty, then every point of $A$ lies on $q/2$ secant lines to $\mathcal{C}$. It follows that $A$ blocks at most $q \times q/2$ secant lines. But $q \times q/2 < \frac{1}{2}q(q+1) = |\mathcal{S}|$, contradicting that every secant line meets $A$. □

**Lemma 2.2.6.** *Any two distinct points of $A \setminus \mathcal{C}$ lie on a tangent line.*

*Proof.* We have $|A \setminus \mathcal{C}| = q - 1$ by Lemma 2.2.5. Let $x, y$ be two distinct points of $A \setminus \mathcal{C}$ (so $q \geqslant 4$). If the line $xy$ is external to $\mathcal{C}$, then $A$ blocks at most $\frac{q}{2}(q-1) - 1$ external lines. But $\frac{q}{2}(q-1) - 1 < \frac{q}{2}(q-1) = |\mathcal{E}|$, contradicting that every external line meets $A$. If $xy$ is secant to $\mathcal{C}$, then $A$ blocks at most $\frac{q}{2}(q-1) - 1 + q$ secant lines. Since $\frac{q}{2}(q-1) - 1 + q < \frac{q}{2}(q+1) = |\mathcal{S}|$, again we get a contradiction to that every secant line meets $A$. So $xy$ must be tangent to $\mathcal{C}$. □

**Corollary 2.2.7.** *All the $q - 1$ points of $A \setminus \mathcal{C}$ lie on the same tangent line.*

*Proof.* Every point of $PG(2, q)$, different from the nucleus, lies on a unique tangent line. If $q = 2$, then $A \setminus \mathcal{C}$ is a singleton and so the result is true. If $q \geqslant 4$, then the result follows from Lemma 2.2.6. □

Now, let $l$ be the tangent line containing all points of $A \setminus \mathcal{C}$. Let $w$ be the tangency point of $l$, that is, $\{w\} = l \cap \mathcal{C}$. If $w \notin A$, then for any $z \in \mathcal{C} \setminus (A \cup \{w\})$, the secant line $wz$ would not contain any point of $A$. So $w \in A$ and hence $A = l \setminus \{n\}$ for the tangent line $l$. This completes the proof of Theorem 2.2.2. □

**Remark 2.2.1.** *While studying the minimum size blocking sets in $PG(2, q)$ with respect to the secant lines, a stronger result was implicitly proved by Aguglia et al. (see the argument after the proof of Lemma 2.1 in [4]), from which Corollary*

*2.2.7 can be derived. However, our proof is much simpler using the fact that $A$ blocks both the secant and external lines to $\mathcal{C}$.*

**Proof of Theorem 2.2.3**

Here $q$ is odd. Let $A$ be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2, q)$. The following elementary result is useful.

**Lemma 2.2.8.** *If $x$ is an interior point and $x \notin A$, then $|A| \geqslant q + 1$.*

*Proof.* Every line through $x$, being either a secant or an external line, must meet $A$. Since $x \notin A$, it follows that $|A| \geqslant q + 1$. □

First assume that $q = 3$. Considering $A$ as an $\mathcal{S}$-blocking set, Theorem 2.1.4 implies that $|A| \geqslant 3$. Every secant line contains one interior point and every external line contains two interior points. So the set consisting of all the interior points to $\mathcal{C}$ is an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2, q)$ of size three. Conversely, if $|A| = 3$, then $A$ contains all the three interior points to $\mathcal{C}$. Otherwise, Lemma 2.2.8 would imply that $|A| \geqslant 4$. This proves Theorem 2.2.3(i).

Now assume that $q \geqslant 5$. We show that $|A| \geqslant q+1$. This is clear if $A$ contains all the interior points to $\mathcal{C}$, since the number of such points is $q(q-1)/2 > q+1$. If $A$ does not contain some interior point, then it follows from Lemma 2.2.8. This proves Theorem 2.2.3(ii).

In order to prove Theorem 2.2.3(iii), we need the following generalization of Theorem 2.1.8 given by Blokhuis and Wilbrink in [9].

**Theorem 2.2.9.** *[9] Let $X$ and $Y$ be two disjoint sets of points in $PG(2, q)$, where $|X| \geqslant q$ and $|Y| = q + 1$. If each line through a point of $X$ meets $Y$, then $Y$ is a line.*

Assume that $q \geqslant 7$ and that $|A| = q+1$. The number of interior points is equal to $q(q-1)/2$. It may happen that all the $q+1$ elements of $A$ are interior points. Since $q \geqslant 7$, we have $q(q-1)/2 - (q+1) > q$. So there exists a set $X$ consisting of at least $q$ interior points such that $X \cap A$ is empty. Each line through a point of $X$ is either secant or external to $\mathcal{C}$ and so meets $A$. Applying Theorem 2.2.9 to the sets $X$ and $A$, it follows that $A$ is a line. This proves Theorem 2.2.3(iii).

Finally, assume that $q = 5$. Recall that $\mathcal{I}$ is the set of all interior points in $PG(2,5)$ with respect to $\mathcal{C}$. Let $X \subseteq \mathcal{I}$ be a quadrangle such that any two points of $X$ determine an external line. Then clearly every external line meets $A = \mathcal{I} \setminus X$ as any such line contains three interior points. Also, every secant line meets $A$ as any such line contains two interior points one of which must be from $A$ by the given condition. Hence $A$ is an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,5)$ of size 6.

Conversely, let $A$ be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $PG(2,5)$ of size 6. If $A$ contains at most five interior points, then take $X = \mathcal{I} \setminus A$. We have $|X| \geqslant 5$ and $|A| = 6$. As in the above proof of Theorem 2.2.3(iii), applying Theorem 2.2.9 to the sets $X$ and $A$, we get that $A$ is a line. Suppose that all the 6 elements of $A$ are interior points. Let $Z = \mathcal{I} \setminus A$. Then $|Z| = 4$. If $l$ is a line containing three (interior) points of $Z$, then $l$ is an external line which would not contain any further interior point from $A$. So $Z \subseteq \mathcal{I}$ must be a quadrangle. It also follows that any two points of $Z$ determine an external line. This proves Theorem 2.2.3(iv).

## 2.3 Miscellaneous results

We conclude this chapter with the following three results.

**Lemma 2.3.1.** *Let $x$ be a point of $PG(2,q)$ and $L$ be the set of all lines of $PG(2,q)$ not passing through $x$. If $A$ is an $L$-blocking set in $PG(2,q)$, then*

*$|A| \geqslant q$, and equality holds if and only if $A = l \setminus \{x\}$ for some line $l$ through $x$.*

*Proof.* Since $A \cup \{x\}$ is a blocking set in $PG(2,q)$ with respect to all its lines, we have $|A \cup \{x\}| \geqslant q+1$ by Proposition 1.7.1 and so $|A| \geqslant q$.

Clearly, $l \setminus \{x\}$ is an $L$-blocking set of size $q$ for every line $l$ through $x$. Conversely, let $|A| = q$. Then $x \notin A$, otherwise, $A \setminus \{x\}$ would be an $L$-blocking set of size $q-1$. Let $l$ be a line through $x$ containing at least one point of $A$. If some point $y$ of $l \setminus \{x\}$ is not in $A$, then it follows that $A$ contains at least one point from each line through $y$. This gives $|A| \geqslant q+1$, which is a contradiction. So $A = l \setminus \{x\}$. □

**Lemma 2.3.2.** *Let $A$ be a $(\mathcal{T} \cup \mathcal{E})$-blocking set in $PG(2,q)$, $q$ even. If $|A| = q$, then the following hold:*

*(a) $A$ is disjoint from $\mathcal{C}$.*

*(b) If $q \geqslant 4$, then there exists at least three secant lines through some point $\alpha$ of $\mathcal{C}$ which are disjoint from $A$.*

*Proof.* Part (a) directly follows from Theorem 2.1.11. Part (b) can be seen as follows. Since $|A| = q$, one of the three cases (i)–(iii) of Theorem 2.1.11 occurs. If Theorem 2.1.11(i) holds for some tangent line $l$, then take $\alpha$ to be the tangency point of $l$ in $\mathcal{C}$. If Theorem 2.1.11(ii) holds for some secant line $l$, then take $\alpha$ to be one of the two points in $l \cap \mathcal{C}$.

Finally, suppose that $q$ is a square and Theorem 2.1.11(iii) holds for some Baer subplane $\Pi$ of $PG(2,q)$. Then $A = \Pi \setminus (\Pi \cap \mathcal{C})$. Take $\alpha$ to be any point of $\mathcal{C} \setminus \Pi$. Let $\beta$ be a point in the Baer conic $\Pi \cap \mathcal{C}$ in $\Pi$ and $l$ be the secant line (to $\mathcal{C}$) through $\alpha$ and $\beta$. We claim that $l$ is disjoint from $A$. Otherwise, $l$ would be a Baer line which is tangent to the conic $\Pi \cap \mathcal{C}$ in $\Pi$ with tangency point $\beta$.

Then the nucleus of $\Pi \cap \mathcal{C}$ is contained in the Baer line $l$. Since $\mathcal{C}$ and $\Pi \cap \mathcal{C}$ share the same nucleus, it follows that the secant line $l$ to $\mathcal{C}$ contains the nucleus $n$, a contradiction. Thus the secant lines through $\alpha$ and meeting $\Pi \cap \mathcal{C}$ are disjoint from $A$. The rest follows from the fact that $|\Pi \cap \mathcal{C}| = \sqrt{q} + 1 \geqslant 3$. □

**Lemma 2.3.3.** *Let $k$ be a secant line to $\mathcal{C}$. If $A$ is an $(\mathcal{E} \cup \mathcal{T})$-blocking set in $PG(2,q)$ that is disjoint from $k$, then $A$ contains at least $q$ points of $\overline{\mathcal{C}} := PG(2,q) \setminus \mathcal{C}$.*

*Proof.* Suppose to the contrary that $A$ contains at most $q-1$ points of $\overline{\mathcal{C}}$. Since $A \cap \overline{\mathcal{C}}$ is a blocking set with respect to the external lines, Theorems 2.1.1 and 2.1.2 then imply that $|A \cap \overline{\mathcal{C}}| = q - 1$. Moreover, one of the following cases occurs for $A \cap \overline{\mathcal{C}}$.

(1) $A \cap \overline{\mathcal{C}} = l \setminus \mathcal{C}$ for some secant line $l$.

If we put $l \cap \mathcal{C} = \{x_1, x_2\}$, then the fact that $k \cap A = \emptyset$ implies that $k \cap l = \{x_i\}$ for some $i \in \{1, 2\}$. The tangent line through the point $x_i \notin A$ would then be disjoint from $A$, a contradiction.

(2) $q \in \{3, 5, 7\}$ and $A \cap \overline{\mathcal{C}}$ is a suitable set of $q-1$ interior points.

In this case, the tangent line through a point of $k \cap \mathcal{C}$ (which cannot contain interior points) would be disjoint from $A$, a contradiction.

(3) $q$ is even and $A \cap \overline{\mathcal{C}} = l \setminus (\mathcal{C} \cup \{n\})$ for some tangent line $l$.

As $k$ is a secant line, it does not contain the nucleus $n$ of $\mathcal{C}$. As the line $k$ is disjoint from $A$, it follows that the unique point $x$ of $\mathcal{C}$ on $l$ must belongs to $k$. If $y$ denotes the other point of $\mathcal{C}$ on the line $k$, then the tangent line through $y$ meets $l$ at $n$ and so would be disjoint from $A$, a contradiction.

(4) $q$ is an even square and $A \cap \overline{\mathcal{C}} = \Pi \setminus \Big((\Pi \cap \mathcal{C}) \cup \{n\}\Big)$, where $\Pi$ is a Baer subplane of $PG(2,q)$ such that $\Pi \cap \mathcal{C}$ is an irreducible conic in $\Pi$.

Since $k$ is a secant line, we have $n \notin k$. Every line of $\Pi$ contains a point of $A$, implying that $k$ cannot intersect $\Pi$ in a Baer subline. So, $k$ intersects $\Pi$ in a unique point, say $x$. Since $k$ is disjoint from $A$, we have that $x \in \Pi \cap \mathcal{C}$. If $y$ denotes the other point of $\mathcal{C}$ on the line $k$, then the tangent line through $y$ would intersect $\Pi$ at the point $n$ and hence be disjoint from $A$, which is again impossible.

Therefore, $A$ contains at least $q$ points of $\overline{\mathcal{C}}$. This completes the proof. □

# Chapter 3

# Hyperbolic quadric in $PG(3, q)$

In this chapter, we recall the basic properties of points, lines and planes of $PG(3, q)$ with respect to a given hyperbolic quadric and prove a few results which are needed in the next chapter.

## 3.1 Properties of points, lines and planes

One can refer to [18] for the following basic properties of the points, lines and planes of $PG(3, q)$ with respect to a hyperbolic quadric in it. We shall denote by $\mathbb{P}$ the point set of $PG(3, q)$ and by $\mathbb{L}$ the line set of $PG(3, q)$. We have

$$|\mathbb{P}| = (q + 1)(q^2 + 1) \text{ and } |\mathbb{L}| = (q^2 + 1)(q^2 + q + 1).$$

Let $\mathcal{H}$ be a hyperbolic quadric in $PG(3, q)$, that is, a non-degenerate quadric of Witt index two. For $l \in \mathbb{L}$, there are 0, 1, 2 or $q + 1$ points in $l \cap \mathcal{H}$. We denote by $\mathbb{E}$ (respectively, $\mathbb{T}_1$, $\mathbb{S}$, $\mathbb{T}_0$) the set of lines of $PG(3, q)$ that intersect $\mathcal{H}$ in 0 (respectively, 1, 2, $q + 1$) points. The elements of $\mathbb{E}$ are called *external lines*, those of $\mathbb{S}$ *secant lines* and those of $\mathbb{T} := \mathbb{T}_0 \cup \mathbb{T}_1$ *tangent lines*. If $l \in \mathbb{T}_i$

with $i \in \{0,1\}$, then $l$ is also called a $\mathbb{T}_i$-*line*. The $\mathbb{T}_0$-lines are precisely the lines contained in $\mathcal{H}$. For a $\mathbb{T}_1$-line $l$, the unique point of $l \cap \mathcal{H}$ is called the *tangency point* of $l$ and denoted by $x_l$.

The quadric $\mathcal{H}$ consists of $(q+1)^2$ points and $2(q+1)$ $\mathbb{T}_0$-lines. Every point $x \in \mathbb{P}$ is contained in $q^2+q+1$ lines, and $q+1$ of them are tangent to $\mathcal{H}$. If $x$ is a point of $\mathcal{H}$, then $x$ is contained in two $\mathbb{T}_0$-lines, $q-1$ $\mathbb{T}_1$-lines and the remaining $q^2$ lines through $x$ are secant to $\mathcal{H}$. If $x$ is a point of $\mathbb{P} \setminus \mathcal{H}$, then $x$ is contained in $q(q+1)/2$ secant lines and $q(q-1)/2$ external lines. We have

$$|\mathbb{T}_1| = (q-1)(q+1)^2,$$

$$|\mathbb{T}| = (q+1)(q^2+1),$$

$$|\mathbb{S}| = \tfrac{1}{2}q^2(q+1)^2,$$

$$|\mathbb{E}| = \tfrac{1}{2}q^2(q-1)^2.$$

With the quadric $\mathcal{H}$, there is naturally associated a polarity $\zeta$ which is symplectic if $q$ is even and orthogonal if $q$ is odd. Thus the planes of $PG(3,q)$ are precisely $x^\zeta$ as $x$ runs over all points of $PG(3,q)$. For a point $x$ of $PG(3,q)$, $x^\zeta$ is called a *tangent plane* or a *secant plane* according as $x$ is a point of $\mathcal{H}$ or not.

For every point $x$ of $\mathcal{H}$, the tangent plane $x^\zeta$ intersects $\mathcal{H}$ in the union of two $\mathbb{T}_0$-lines through $x$. The $q+1$ tangent lines through $x$ are precisely the lines through $x$ contained in $x^\zeta$. In this case, we shall denote by $\pi_x$ the tangent plane $x^\zeta$ and call $x$ the *pole* of $\pi_x = x^\zeta$. Now let $x$ be a point of $\mathbb{P} \setminus \mathcal{H}$. Then the secant plane $x^\zeta$ intersects $\mathcal{H}$ in an irreducible conic $\mathcal{C}_x$. If $q$ is even, then $x$ is a point of $x^\zeta$ and is the nucleus of the conic $\mathcal{C}_x$ in $x^\zeta$. The $q+1$ tangent lines through $x$ are precisely the lines through $x$ contained in $x^\zeta$. In this case also, we shall denote by $\pi_x$ the secant plane $x^\zeta$. If $q$ is odd, then $x$ is not a point of $x^\zeta$. In this case, the tangent lines through $x$ are precisely the lines through $x$ meeting $\mathcal{C}_x$.

Thus, when $q$ is even, the planes of $PG(3,q)$ are precisely $\pi_x$ as $x$ runs over all the points of $PG(3,q)$. When $q$ is odd, the tangent planes of $PG(3,q)$ are $\pi_x$ with $x \in \mathcal{H}$ and the secant planes are $x^\zeta$ with $x \in \mathbb{P} \setminus \mathcal{H}$.

Considering $\mathcal{H}$ as a point-line geometry, an *ovoid* of $\mathcal{H}$ is a set of points intersecting each $\mathbb{T}_0$-line in a unique point. Every ovoid of $\mathcal{H}$ has exactly $q+1$ points. For every point $y$ of $\mathbb{P} \setminus \mathcal{H}$, the conic $\mathcal{C}_y$ in $y^\zeta$ is an ovoid of $\mathcal{H}$. The map $y \mapsto \mathcal{C}_y$ defines a bijection between $\mathbb{P} \setminus \mathcal{H}$ and the set of conics contained in $\mathcal{H}$ (and hence an injection between $\mathbb{P} \setminus \mathcal{H}$ and the set of ovoids of $\mathcal{H}$). When $q = 3$, an easy calculation shows that there are 24 ovoids of $\mathcal{H}$ and so it follows that the set of conics contained in $\mathcal{H}$ coincides with the set of ovoids of $\mathcal{H}$.

There are $(q+1)^2$ tangent planes and $q^3 - q$ secant planes of $PG(3,q)$. Every point of $\mathcal{H}$ is contained in $2q+1$ tangent planes and $q(q-1)$ secant planes. Every point of $\mathbb{P} \setminus \mathcal{H}$ is contained in $q+1$ tangent planes and $q^2$ secant planes. Every external line is contained in $q+1$ secant planes, every secant line is contained in two tangent planes and $q-1$ secant planes, every $\mathbb{T}_0$-line is contained in $q+1$ tangent planes, and every $\mathbb{T}_1$-line is contained in one tangent plane and $q$ secant planes.

If $\pi$ is a secant plane with $\pi = y^\zeta$ for some point $y$ in $\mathbb{P}\setminus\mathcal{H}$, then we also denote by $\mathcal{C}_\pi$ the conic $\mathcal{C}_y$ in $\pi = y^\zeta$ and thus $\mathcal{C}_\pi = \pi \cap \mathcal{H}$. Let $l$ be a line of $PG(3,q)$ and $\pi_0, \pi_1, \cdots, \pi_q$ be the $q+1$ planes of $PG(3,q)$ containing $l$. If $l$ is external to $\mathcal{H}$, then $\mathcal{C}_{\pi_0}, \mathcal{C}_{\pi_1}, \cdots, \mathcal{C}_{\pi_q}$ constitute a *linear flock* of $\mathcal{H}$ (that is, partitioning $\mathcal{H}$ into mutually disjoint conics). If $l$ is a $\mathbb{T}_1$-line with tangency point $x_l \in \mathcal{H}$, then $\mathcal{C}_{\pi_0} \setminus \{x_l\}, \mathcal{C}_{\pi_1} \setminus \{x_l\}, \cdots, \mathcal{C}_{\pi_q} \setminus \{x_l\}$ give a partition of the points in $\mathcal{H} \setminus (l_0 \cup l_1)$, where $l_0, l_1$ are the two $\mathbb{T}_0$-lines through $x_l$.

## 3.2 Basic results

If $q$ is odd and $x$ is a point of $\mathbb{P} \setminus \mathcal{H}$, then we denote by $\mathbb{E}(x)$ (respectively, $\mathbb{S}(x)$) the set of all lines of $PG(3,q)$ through $x$ which are external (respectively, secant) to $\mathcal{H}$, and by $I_x$ (respectively, $E_x$) the set of all interior (respectively, exterior) points in the plane $x^\zeta$ with respect to the conic $\mathcal{C}_x$.

**Lemma 3.2.1.** *Suppose that $x$ is a point of $PG(3,q) \setminus \mathcal{H}$, where $q$ is odd. Then the following hold:*

*(i) Each line in $\mathbb{E}(x)$ meets the plane $x^\zeta$ in a point of $I_x$.*

*(ii) The map from $\mathbb{E}(x)$ to $I_x$, sending each line in $\mathbb{E}(x)$ to its point of intersection with $I_x$, is bijective.*

*Proof.* Consider a line $l \in \mathbb{E}(x)$. Since $x$ is not a point of $x^\zeta$, the line $l$ contains exactly one point of $x^\zeta$. Denote this point by $z_l$. Since $l$ is an external line, we have $z_l \in x^\zeta \setminus \mathcal{C}_x$.

(i) We show that $z_l \in I_x$. Suppose this is not true. Then $z_l$ is exterior to $\mathcal{C}_x$. Let $m$ be a $\mathbb{T}_1$-line through $z_l$ contained in $x^\zeta$ and $\pi$ be the plane of $PG(3,q)$ generated by $l$ and $m$. Then $\pi$ is a secant plane, as it contains the external line $l$. On the other hand, if $y$ is the unique point of $m \cap \mathcal{C}_x$, then the $\mathbb{T}_1$-line $m_1 := yx$ is contained in $\pi$. So $\pi$ is also the plane generated by the two $\mathbb{T}_1$-lines $m$ and $m_1$ intersecting at $y \in \mathcal{C}_x$, implying that $\pi$ is the tangent plane with pole $y$, which is a contradiction.

(ii) Let $f : \mathbb{E}(x) \to I_x$ be the map defined by $f(l) = z_l$ for $l \in \mathbb{E}(x)$. By (i), $f$ is well-defined. Clearly, $f$ is injective. Since $|\mathbb{E}(x)| = q(q-1)/2 = |I_x|$, $f$ is also surjective. □

As a consequence of Lemma 3.2.1, we have the following.

**Lemma 3.2.2.** *Suppose that $x$ is a point of $PG(3,q) \setminus \mathcal{H}$, where $q$ is odd. Then the following hold:*

*(i) Each line in $\mathbb{S}(x)$ meets the plane $x^\zeta$ in a point of $E_x$.*

*(ii) The map from $\mathbb{S}(x)$ to $E_x$, sending each line in $\mathbb{S}(x)$ to its point of intersection with $E_x$, is bijective.*

*Proof.* (i) Let $l \in \mathbb{S}(x)$. Then $l$ contains exactly one point of $x^\zeta$ as $x \notin x^\zeta$. Since any line through $x$ and a point of $\mathcal{C}_x$ is tangent to $\mathcal{H}$, it follows that $l$ meets $x^\zeta$ in a point outside $\mathcal{C}_x$. Then Lemma 3.2.1(ii) implies that $l$ must meet $x^\zeta$ in a point exterior to $\mathcal{C}_x$, that is, in a point of $E_x$.

(ii) By (i), the map is well-defined and is clearly injective. Since $|\mathbb{S}(x)| = q(q+1)/2 = |E_x|$, the map is also surjective. □

The following two results are related to the special case that $q = 3$.

**Lemma 3.2.3.** *Suppose that $q = 3$ and $w$ is a point of $\mathcal{H}$. Let $l, m$ be the two $\mathbb{T}_1$-lines through $w$. Then, for $x \in m \setminus \{w\}$, the secant plane $x^\zeta$ contains $l$.*

*Proof.* Note that $w$ is a point of $x^\zeta$, as $w \in \mathcal{C}_x$. Since $q = 3$, the tangent line in $x^\zeta$ through $w$ is either $l$ or $m$. Since $x$ is not a point of $x^\zeta$, it follows that $m$ is not a line of $x^\zeta$ and hence $l$ is the tangent line through $w$ in $x^\zeta$. □

**Lemma 3.2.4.** *Suppose that $q = 3$. Let $\pi_1$ be a secant plane and $\mathcal{C}_1$ be the conic $\pi_1 \cap \mathcal{H}$ in $\pi_1$. Fix a line $l$ of $\pi_1$ which is external with respect to $\mathcal{C}_1$. Then there exists exactly one more secant plane $\pi_2$ satisfying the following:*

(1) *$l$ is an external line of $\pi_2$ with respect to the conic $\mathcal{C}_2 := \pi_2 \cap \mathcal{H}$.*

(2) *If a point $a \in l$ is exterior (respectively, interior) to $\mathcal{C}_1$ in $\pi_1$, then it is also exterior (respectively, interior) to $\mathcal{C}_2$ in $\pi_2$.*

*In fact, if $a \in l$ is exterior to $\mathcal{C}_1$ in $\pi_1$, then the two $\mathbb{T}_1$-lines through $a$ not in $\pi_1$ are contained in $\pi_2$.*

*Proof.* Let $x$ be the point of $PG(3,3) \setminus \mathcal{H}$ such that $\mathcal{C}_x = \mathcal{C}_1$. Such a point $x$ exists, since the map $\alpha \mapsto \mathcal{C}_\alpha := \alpha^\zeta \cap \mathcal{H}$ is a bijection between $PG(3,3) \setminus \mathcal{H}$ and the set of conics contained in $\mathcal{H}$. We have $\pi_1 = x^\zeta$. Write $l = \{a, b, z_1, z_2\}$, where $a, b$ (respectively, $z_1, z_2$) are exterior (respectively, interior) with respect to $\mathcal{C}_1$ in $\pi_1$. By Lemma 3.2.1(ii), the lines $t_1 := xz_1$ and $t_2 := xz_2$ are external lines.

Let $\pi_2$ be the plane generated by the line $l$ and the point $x$. Then $\pi_2$ is a secant plane in which $l$ is external to the conic $\mathcal{C}_2 := \pi_2 \cap \mathcal{H}$. The lines $t_1$ and $t_2$ in $\pi_2$ are external to $\mathcal{C}_2$. Thus, for $i \in \{1, 2\}$, $l$ and $t_i$ are two external lines in $\pi_2$ through $z_i$. It follows that both the points $z_1$ and $z_2$ are interior to $\mathcal{C}_2$ in $\pi_2$. This implies that the points $a$ and $b$ must be exterior to $\mathcal{C}_2$ in $\pi_2$. Hence $\pi_2$ satisfies the conditions (1) and (2).

Out of the four $\mathbb{T}_1$-lines through $a$ (respectively, through $b$), two are contained in $\pi_1$ and the other two are in $\pi_2$ (as $\pi_1 \cap \pi_2 = l$ is not a $\mathbb{T}_1$-line). This must hold for any secant plane satisfying the conditions (1) and (2). This fact implies the uniqueness of $\pi_2$ satisfying (1) and (2). □

**Proposition 3.2.5.** *If $q$ is even, then the point-line geometry $\mathcal{X} = (\mathbb{P}, \mathbb{T})$ with point set $\mathbb{P}$ and line set $\mathbb{T}$ is a generalized quadrangle of order $q$ which is isomorphic to $W(q)$.*

*Proof.* We shall verify the axioms (Q1), (Q2) and (Q3) in the definition of an $(s, t)$-generalized quadrangle. Clearly, the axioms (Q1) and (Q2) are satisfied with $s = q = t$. We verify (Q3). Let $(x, l) \in \mathbb{P} \times \mathbb{T}$ be a point-line pair with $x \notin l$. We show that there exists a unique line in $\mathbb{T}$ which contains $x$ and intersects $l$. Observe that there are four cases depending on $x \in \mathcal{H}$ or not, and $l$ is a $\mathbb{T}_0$-line or not. We consider all cases together.

Let $\pi = \langle x, l \rangle$ be the plane of $PG(3,q)$ generated by $x$ and $l$. Since $q$ is even, we have $\pi = \pi_w$ for some $w \in l$. If $\pi$ is a tangent plane, then the only tangent line containing $x$ and intersecting $l$ is the line through $x$ and the pole $w$ of $\pi$ (the remaining $q$ lines through $x$ in $\pi$ are secant to $\mathcal{H}$). If $\pi$ is a secant plane, then the unique tangent line through $x$ and meeting $l$ is the line through $x$ and the nucleus $w$ of the conic $\mathcal{C}_\pi$ in $\pi$. This verifies the axiom (Q3).

Thus $\mathcal{X} = (\mathbb{P}, \mathbb{T})$ is a projective generalized quadrangle of order $q$ with ambient space $PG(3,q)$. Then it follows from [33, 4.4.8] that $\mathcal{X}$ is isomorphic to $W(q)$. □

**Remark 3.2.1.** *When $q$ is odd and $\pi = \langle x, l \rangle$ is a secant plane as in the proof of Theorem 3.2.5, the uniqueness (respectively, existence) of the tangent line trough $x$ and intersecting $l$ fails if $x$ is an exterior (respectively, interior) point to the conic $\mathcal{C}_\pi$ in $\pi$. So axiom (Q3) fails for $q$ odd.*

Another way to prove Proposition 3.2.5 is the following, see [18, Theorem 15.3.16]: Consider the polarity $\zeta$ of $PG(3,q)$ induced by the quadric $\mathcal{H}$. So $\zeta$ is an inclusion reversing bijection of order two on the set of all subspaces of $PG(3,q)$. It fixes the line set $\mathbb{L}$ of $PG(3,q)$, and interchanges $\mathbb{P}$ and the set of all planes of $PG(3,q)$. A subspace $U$ of $PG(3,q)$ is called *absolute* with respect to $\zeta$ if it is incident with $U^\zeta$. Since $q$ is even, $\zeta$ is a null polarity, that is, each point (and so each plane) of $PG(3,q)$ is absolute with respect to $\zeta$. The point set $\mathbb{P}$ and the line set consisting of all the absolute lines of $PG(3,q)$ with respect to $\zeta$ form a generalized quadrangle $W(q)$ of order $q$. Since $\mathcal{H}$ is a (geometric) hyperplane of $W(q)$, each absolute line is either contained in $\mathcal{H}$ or meets $\mathcal{H}$ at exactly one point. Then it follows that $\mathbb{T}$ is precisely the set of all absolute lines of $PG(3,q)$ with respect to $\zeta$.

## 3.3 A set in $PG(3,3)$

We need the following result in the next chapter while studying the minimum size $\mathbb{S}$- and $(\mathbb{E} \cup \mathbb{S})$-blocking sets in $PG(3,3)$.

**Theorem 3.3.1.** *Suppose that $q = 3$. Then there exists a subset $B$ of the point set of $PG(3,3)$ satisfying the following conditions:*

*(i) $|B| = 12$ and $B$ is disjoint from $\mathcal{H}$;*

*(ii) Every external line to $\mathcal{H}$ meets $B$ in two points;*

*(iii) Every secant line to $\mathcal{H}$ meets $B$ in one point.*

*In particular, $B$ is an $L$-blocking set in $PG(3,3)$ for $L \in \{\mathbb{S}, \mathbb{E}, \mathbb{S} \cup \mathbb{E}\}$.*

We first give an example of a subset $B$ of the point set of $PG(3,3)$ satisfying the conditions (i)–(iii) of Theorem 3.3.1. Using homogeneous coordinates, we write a point of $PG(3,3)$ as $\langle(x_0, x_1, x_2, x_3)\rangle$, the one dimensional subspace generated by a non-zero vector $(x_0, x_1, x_2, x_3)$ in $\mathbb{F}_3^4$. Without loss, we may assume that the quadric $\mathcal{H}$ has the equation $X_0X_3 = X_1X_2$. Then, we have

$$\begin{aligned} \mathcal{H} &= \{\langle(1,0,0,0)\rangle, \langle(0,1,0,0)\rangle, \langle(1,1,0,0)\rangle, \langle(1,2,0,0)\rangle, \\ &\quad \langle(0,0,1,0)\rangle, \langle(0,0,0,1)\rangle, \langle(0,0,1,1)\rangle, \langle(0,0,1,2)\rangle, \\ &\quad \langle(1,0,1,0)\rangle, \langle(0,1,0,1)\rangle, \langle(1,1,1,1)\rangle, \langle(1,2,1,2)\rangle, \\ &\quad \langle(1,0,2,0)\rangle, \langle(0,1,0,2)\rangle, \langle(1,1,2,2)\rangle, \langle(1,2,2,1)\rangle\}. \end{aligned}$$

The lines contained in $\mathcal{H}$ form a regulus $R = \{l_0, l_1, l_2, l_3\}$ together with its opposite regulus $\overline{R} = \{m_0, m_1, m_2, m_3\}$, where

$$l_0 = \{\langle(1,0,0,0)\rangle, \langle(0,1,0,0)\rangle, \langle(1,1,0,0)\rangle, \langle(1,2,0,0)\rangle\};$$

$$
\begin{aligned}
l_1 &= \{\langle(0,0,1,0)\rangle, \langle(0,0,0,1)\rangle, \langle(0,0,1,1)\rangle, \langle(0,0,1,2)\rangle\};\\
l_2 &= \{\langle(1,0,1,0)\rangle, \langle(0,1,0,1)\rangle, \langle(1,1,1,1)\rangle, \langle(1,2,1,2)\rangle\};\\
l_3 &= \{\langle(1,0,2,0)\rangle, \langle(0,1,0,2)\rangle, \langle(1,1,2,2)\rangle, \langle(1,2,2,1)\rangle\};\\
m_0 &= \{\langle(1,0,0,0)\rangle, \langle(0,0,1,0)\rangle, \langle(1,0,1,0)\rangle, \langle(1,0,2,0)\rangle\};\\
m_1 &= \{\langle(0,1,0,0)\rangle, \langle(0,0,0,1)\rangle, \langle(0,1,0,1)\rangle, \langle(0,1,0,2)\rangle\};\\
m_2 &= \{\langle(1,1,0,0)\rangle, \langle(0,0,1,1)\rangle, \langle(1,1,1,1)\rangle, \langle(1,1,2,2)\rangle\};\\
m_3 &= \{\langle(1,2,0,0)\rangle, \langle(0,0,1,2)\rangle, \langle(1,2,1,2)\rangle, \langle(1,2,2,1)\rangle\}.
\end{aligned}
$$

Consider the following set

$$
\begin{aligned}
B = \{&\langle(0,1,1,0)\rangle, \langle(1,1,1,0)\rangle, \langle(1,2,2,0)\rangle, \langle(1,0,0,2)\rangle,\\
&\langle(1,1,0,2)\rangle, \langle(1,2,0,2)\rangle, \langle(0,1,1,1)\rangle, \langle(1,2,1,1)\rangle,\\
&\langle(1,0,2,2)\rangle, \langle(1,1,2,1)\rangle, \langle(1,0,1,2)\rangle, \langle(0,1,1,2)\rangle\};
\end{aligned}
$$

which contains 12 points. Starting with a $\mathbb{T}_1$-line $l$ through the point $w = \langle(1,0,0,0)\rangle \in \mathcal{H}$, say

$$l = \{\langle(1,0,0,0)\rangle, \langle(0,1,1,0)\rangle, \langle(1,1,1,0)\rangle, \langle(1,2,2,0)\rangle\},$$

we have constructed $B$ as the following. Every secant plane $\pi$ through $l$ contains six exterior points with respect to the conic $\mathcal{C}_\pi$, three of them are the points of $l$ different from its tangency point $w$. Then $B$ is taken as the union of the exterior points from all the three secant planes through $l$. We have verified that $B$ satisfies the conditions (i)–(iii) of Theorem 3.3.1. Based on the construction of the set $B$, we shall give a theoretical proof of Theorem 3.3.1 in the rest of this section.

### 3.3.1 Proof of Theorem 3.3.1

For a secant plane $\pi$, we denote by $E(\pi)$ the set of all the exterior points in $\pi$ with respect to the conic $\mathcal{C}_\pi$. Fix a $\mathbb{T}_1$-line $l$. Let $\pi_1, \pi_2, \pi_3$ be the three secant planes of $PG(3,3)$ through $l$ and define the following set:

$$B := E(\pi_1) \cup E(\pi_2) \cup E(\pi_3).$$

Let $\{w\} = l \cap \mathcal{H}$. Then $\pi_w \cap B = l \setminus \{w\}$. We claim that $B$ satisfies the Conditions (i)–(iii) of Theorem 3.3.1 and conversely, any set of points in $PG(3,3)$ satisfying the three conditions of Theorem 3.3.1 is obtained in this way.

Set $\mathcal{C}_i := \mathcal{C}_{\pi_i}$ for $i \in \{1,2,3\}$. Note that the conics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ pairwise intersect at the point $w$ only.

(I) Since each $E(\pi_i)$ is disjoint from $\mathcal{C}_i$, it follows that $B$ is disjoint from $\mathcal{H}$. Each $E(\pi_i)$ contains six points and $E(\pi_j) \cap E(\pi_k) = l \setminus \{w\}$ for $1 \leqslant j \neq k \leqslant 3$. This gives $|B| = 12$ and so the Condition (i) of Theorem 3.3.1 is satisfied.

(II) Consider a line $l_1$ of $PG(3,3)$ external to $\mathcal{H}$. If $l_1$ is contained in some $\pi_i$, then $l_1 \cap B = l_1 \cap E(\pi_i)$ consists of the two exterior points on $l_1$ with respect to $\mathcal{C}_i$. So we assume that none of $\pi_1, \pi_2, \pi_3$ contains $l_1$. Then $l \cap l_1$ is empty, as each of the three external lines through a point of $l \setminus \{w\}$ is contained in some $\pi_i$. Since $l_1$ meets each of $\pi_1, \pi_2, \pi_3$ and $\pi_w$, we may suppose that $l_1 := \{w_1, x_1, x_2, x_3\}$, where $w_1 \in \pi_w \setminus l$ and $x_i \in \pi_i \setminus l$ for $1 \leqslant i \leqslant 3$. Then $w_1 \notin B$, as $\pi_w \cap B = l \setminus \{w\}$. Let $m := \{w, w_1, w_2, w_3\}$ be the $\mathbb{T}_1$-line in $\pi_w$ through $w$ and $w_1$. By Lemma 3.2.3, each of the three secant planes $w_1^\zeta, w_2^\zeta, w_3^\zeta$ contains $l$. So we must have $\{w_1^\zeta, w_2^\zeta, w_3^\zeta\} = \{\pi_1, \pi_2, \pi_3\}$.

Without loss of generality, we may assume that $w_i^\zeta = \pi_i$ for $i \in \{1,2,3\}$. Since $w_1 \notin w_1^\zeta$ and the external line $l_1$ through $w_1$ meets $w_1^\zeta$ at $x_1$, Lemma

3.2.1(i) implies that the point $x_1$ in $\pi_1 = w_1^\zeta$ is interior with respect to $\mathcal{C}_1 = \mathcal{C}_{w_1}$. So $x_1 \notin B$ by the construction of $B$. We claim that $B$ contains $x_2$ and $x_3$. It is enough to show that the point $x_j$ is exterior in $\pi_j = w_j^\zeta$ with respect to $\mathcal{C}_j = \mathcal{C}_{w_j}$, where $j \in \{2, 3\}$.

Suppose that $x_j$ is interior to $\mathcal{C}_{w_j}$ in $w_j^\zeta$ for some $j$. Then the line $w_j x_j$ must be external to $\mathcal{H}$, see Lemma 3.2.1. There are two external lines in $w_j^\zeta = \pi_j$ through $x_j$ different from $w_j x_j$ and $l_1$, thus giving four external lines through $x_j$, a contradiction. This verifies the Condition (ii) of Theorem 3.3.1.

(III) Consider a line $l_2$ of $PG(3,3)$ secant to $\mathcal{H}$. First suppose that $l_2 \cap l$ is nonempty. Then $l_2$ is contained in some plane through $l$. If $l_2$ is contained in some $\pi_i$, $1 \leqslant i \leqslant 3$, then one point of $l_2 \backslash \mathcal{H}$ is interior and the other one is exterior in $\pi_i$ with respect to $\mathcal{C}_i$ and so $l_2 \cap B = l_2 \cap E(\pi_i)$ is a singleton. If $l_2$ is contained in the tangent plane $\pi_w$, then $l_2$ meets $l$ in a point of $l \setminus \{w\}$ and it follows that $l_2 \cap B = l_2 \cap (\pi_w \cap B) = l_2 \cap (l \setminus \{w\})$ is a singleton.

Now, suppose that $l_2 \cap l$ is empty. Then $l_2$ is not contained in $\pi_w$ nor in any $\pi_i$ and so each of $\pi_w, \pi_1, \pi_2, \pi_3$ contains exactly one point of $l_2$. Let $l_2 := \{a, b, y, z\}$, where $l_2 \cap \mathcal{H} = \{a, b\}$. We have that $a$ and $b$ both can not be in $\pi_w$.

**Case 1**. None of $a$ and $b$ is in $\pi_w$. Then, without loss, we may suppose that $y \in \pi_w$. Since $y \notin l$ and $\pi_w \cap B = l \setminus \{w\}$, we have $y \notin B$. We show that $z \in B$. By Lemma 3.2.2(i), the secant line $l_2$ through $y$ meets the plane $y^\zeta$ in a point exterior to $\mathcal{C}_y$. Since $y \notin y^\zeta$, we must have $l_2 \cap y^\zeta = \{z\}$ and so $z$ is exterior to $\mathcal{C}_y$ in $y^\zeta$. Since $l$ and $wy$ are the two $\mathbb{T}_1$-lines through $w$, Lemma 3.2.3 implies that $y^\zeta$ contains $l$ and so $y^\zeta \in \{\pi_1, \pi_2, \pi_3\}$. Thus $z$ is an exterior point in one of the $\pi_i$'s and hence $z \in B$.

**Case 2**. One of $a$ and $b$ is in $\pi_w$. Without loss, we may assume that $a \in \pi_w$, $b \in \pi_1$, $y \in \pi_2$ and $z \in \pi_3$. Then $wb$ is a secant line. Let $\alpha, \beta$ be the two points of $\mathcal{H}$ satisfying the following:

(1) $\pi_\alpha, \pi_\beta$ are the two tangent planes through $l_2$.

(2) The line $w\beta$ is a $\mathbb{T}_0$-line (and so $w\alpha$ is necessarily a secant line).

Then $\alpha, \beta \in \mathcal{C}_y \cap \mathcal{C}_z$. Since the sets $\mathcal{C}_i \setminus \{w\}$, $1 \leqslant i \leqslant 3$, partition the points of $\mathcal{H}$ which are not on the $\mathbb{T}_0$-lines through $w$, one of the conics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ contains $\alpha$. But $b \in \mathcal{C}_1$ as $b$ is a point of $\pi_1$. So $\alpha$ is in either $\mathcal{C}_2$ or $\mathcal{C}_3$. Without loss, we may assume that $\alpha \in \mathcal{C}_2$. Since $y \in \pi_2$, $y\alpha$ is then a $\mathbb{T}_1$-line in $\pi_2$. So the point $y$ in $\pi_2$ is exterior to $\mathcal{C}_2$ and hence is in $B$. We show that $z \notin B$. Note that $b, \alpha \notin \mathcal{C}_3$. Let $\gamma$ be the point on the $\mathbb{T}_0$-line through $\alpha$ and $b$ which is in $\mathcal{C}_3$. Then the four points of $\mathcal{C}_3$ are $w$, $\gamma$, one point $\tau$ from the $\mathbb{T}_0$-line $b\beta$ and one point $\theta$ from the $\mathbb{T}_0$-line $a\alpha$. Since $\alpha, \beta \in \mathcal{C}_z$, it follows that none of $w, \gamma, \tau, \theta$ is in $\mathcal{C}_z$ and so $C_3 \cap \mathcal{C}_z$ is empty. This implies that there is no tangent line in $\pi_3$ through $z$. So $z$ is interior in $\pi_3$ with respect to $\mathcal{C}_3$ and hence $z \notin B$. This completes the verification of Condition (iii) of Theorem 3.3.1.

Conversely, let $B$ be a set of points in $PG(3,3)$ satisfying the three conditions (i)–(iii) of Theorem 3.3.1. Consider a tangent plane $\pi_w$ for some point $w$ of $\mathcal{H}$. By condition (i), $\pi_w \cap B$ is disjoint from $\mathcal{H}$. By condition (iii), $\pi_w \cap B$ blocks every secant line contained in $\pi_w$. Since the secant lines contained in $\pi_w$ are precisely the lines of $\pi_w$ not passing through $w$, we have $|\pi_w \cap B| \geqslant 3$ by Lemma 2.3.1. If $|\pi_w \cap B| > 3$, then there would be a secant line contained in $\pi_w$ meeting $B$ in two points, violating condition (iii). So $|\pi_w \cap B| = 3$. Then, using the fact that $\pi_w \cap B$ is disjoint from $\mathcal{H}$, the equality case of Lemma 2.3.1 implies that $\pi_w \cap B = l \setminus \{w\}$ for some $\mathbb{T}_1$-line $l$ in $\pi_w$.

Let $\pi_1, \pi_2, \pi_3$ be the three secant planes through $l$. Then the points of $B$ are contained in $(\pi_1 \cup \pi_2 \cup \pi_3) \setminus \mathcal{H}$. We claim that $B = E(\pi_1) \cup E(\pi_2) \cup E(\pi_3)$. Since

$|B| = 12$, it is enough to show that no point of $\pi_i$ interior to $\mathcal{C}_i := \mathcal{C}_{\pi_i}$ is in $B$ for any $i \in \{1,2,3\}$. Suppose that, for some $i$, $B$ contains a point $z$ of $\pi_i$ which is interior to $\mathcal{C}_i$. Then each of the three lines through $z$ and a point of $l \setminus \{w\}$ contains at least two points of $B$. This follows from the facts that $z \in B$, $l$ is a line of $\pi_i$ and $l \setminus \{w\}$ is contained in $B$. Since $z$ is an interior point, each such line can not be tangent to $\mathcal{C}_i$ and so must be an external line by conditions (ii) and (iii). This gives three external lines through $z$ in $\pi_i$, which is a contradiction.

# Chapter 4

# Blocking sets in $PG(3, q)$

Let $\mathcal{H}$ be a fixed hyperbolic quadric in $PG(3, q)$. As in the previous chapter, we denote by $\mathbb{E}$ (respectively, $\mathbb{T}$, $\mathbb{S}$) the set of all lines of $PG(3, q)$ which are external (respectively, tangent, secant) with respect to $\mathcal{H}$. Further, $\mathbb{T} = \mathbb{T}_0 \cup \mathbb{T}_1$, where $\mathbb{T}_0$ is the set of tangent lines contained $\mathcal{H}$ and $\mathbb{T}_1$ is the set of tangent lines meeting $\mathcal{H}$ in singletons. In this chapter, we discuss the minimum size $L$-blocking sets in $PG(3, q)$, where the line set $L$ is one of $\mathbb{E}$, $\mathbb{T}$, $\mathbb{S}$, $\mathbb{T} \cup \mathbb{E}$, $\mathbb{T} \cup \mathbb{S}$ and $\mathbb{E} \cup \mathbb{S}$.

Consider a line set $L \in \{\mathbb{E}, \mathbb{T}, \mathbb{S}, \mathbb{T} \cup \mathbb{E}, \mathbb{T} \cup \mathbb{S}, \mathbb{E} \cup \mathbb{S}\}$. Two $L$-blocking sets $B_1$ and $B_2$ in $PG(3, q)$ are said to be *isomorphic* if there is an automorphism of $PG(3, q)$ which stabilizes the quadric $\mathcal{H}$ and maps $B_1$ to $B_2$. For a given plane $\pi$ of $PG(3, q)$, we set

$$L_\pi := \pi \cap L,$$

the set of lines of $L$ which are contained in $\pi$. Note that if $B$ is an $L$-blocking set in $PG(3, q)$, then $\pi \cap B$ is an $L_\pi$-blocking set in $\pi$. We shall use the following fact mostly without mention.

**Proposition 4.0.2.** *Let $B$ be an $L$-blocking set in $PG(3, q)$. If $\pi_0, \pi_1, \cdots, \pi_q$ are* *the $q+1$ planes of $PG(3,q)$ through a given line $l$, then*

$$|B| = |l \cap B| + \sum_{i=0}^{q} (|\pi_i \cap B| - |l \cap B|).$$

## 4.1 $\mathbb{S}$-blocking sets

In this section, we prove the following theorem which characterizes the minimum size $\mathbb{S}$-blocking sets in $PG(3,q)$ for all $q$.

**Theorem 4.1.1.** *Let $B$ be an $\mathbb{S}$-blocking set in $PG(3,q)$. Then $|B| \geqslant q^2+q$. Further, equality holds if and only if one of the following occurs:*

*(i) If $B \subseteq \mathcal{H}$, then $B = \mathcal{H} \setminus l$ for some $\mathbb{T}_0$-line $l$.*

*(ii) If $B \setminus \mathcal{H} \neq \emptyset$ and $B \cap \mathcal{H} \neq \emptyset$, then $B = (\mathcal{H} \setminus (l_0 \cup l_1)) \cup (l \setminus \{w\})$, where $l_0, l_1$ are two $\mathbb{T}_0$-lines intersecting at the point $w$ and $l$ is a $\mathbb{T}_1$-line with tangency point $x_l = w$.*

*(iii) If $B \cap \mathcal{H} = \emptyset$, then $q \in \{2,3\}$ and the following hold:*

*(a) $q = 2$ : $B$ consists of all the six points outside $\mathcal{H}$.*

*(b) $q = 3$ : $B$ satisfies the three conditions (i)–(iii) of Theorem 3.3.1.*

### 4.1.1 Proof of Theorem 4.1.1

Let $B$ be an $\mathbb{S}$-blocking set in $PG(3,q)$. For any secant plane $\pi$ of $PG(3,q)$, $\mathbb{S}_\pi$ is the set of all secant lines contained in $\pi$ with respect to the conic $\mathcal{C}_\pi = \pi \cap \mathcal{H}$. The following simple result is used frequently.

**Lemma 4.1.2.** *Let $\pi$ be any plane of $PG(3,q)$. Then the following hold:*

*(i) $|\pi \cap B| \geqslant q$.*

*(ii) If $\pi$ is a tangent plane with pole $x$, then $|\pi \cap B| = q$ if and only if $\pi \cap B = l \setminus \{x\}$ for some tangent line $l$ through $x$ (contained in $\pi$).*

*Proof.* If $\pi$ is a secant plane, then $\pi \cap B$ is an $\mathbb{S}_\pi$-blocking set in $\pi$. So $|\pi \cap B| \geqslant q$ by Theorems 2.1.4 and 2.1.5. Assume that $\pi = \pi_x$ is a tangent plane for some point $x$ of $\mathcal{H}$. Then all the lines of $\pi_x$ not passing through $x$ are secant to $\mathcal{H}$ and so $\pi \cap B$ blocks each such line. The rest follows from Lemma 2.3.1. □

Now on, suppose that the $\mathbb{S}$-blocking set $B$ is of minimum possible size. Clearly, each of the sets defined in Theorem 4.1.1 is an $\mathbb{S}$-blocking set of size $q^2 + q$. So $|B| \leqslant q^2 + q$. Then, by Proposition 1.7.1, there exists at least one line of $PG(3, q)$ which is disjoint from $B$. We have the following.

**Lemma 4.1.3.** $|B| = q^2 + q$

*Proof.* Let $l$ be a line of $PG(3, q)$ which is disjoint from $B$ and $\pi_0, \pi_1, \cdots, \pi_q$ be the planes of $PG(3, q)$ through $l$. By Lemma 4.1.2(i), we have $|\pi_i \cap B| \geqslant q$ for every $i$. Then

$$q^2 + q \geqslant |B| = \sum_{i=0}^{q} |\pi_i \cap B| \geqslant q(q + 1)$$

and so $|B| = q^2 + q$. □

As a consequence of the proof of Lemma 4.1.3, we have the following.

**Corollary 4.1.4.** *If $l$ is a line of $PG(3, q)$ disjoint from $B$, then every plane through $l$ contains exactly $q$ points of $B$.*

**Corollary 4.1.5.** *Let $l \in \mathbb{T}$ with $l \cap B = \emptyset$ and $x \in l \cap \mathcal{H}$. If $m$ $(\neq l)$ is a tangent line through $x$ containing at least one point of $B$, then $\pi_x \cap B = m \setminus \{x\}$.*

*Proof.* By Corollary 4.1.4, we have $|\pi_x \cap B| = q$. Since $m \cap B \neq \emptyset$, Lemma 4.1.2(ii) implies that $\pi_x \cap B = m \setminus \{x\}$. □

**Lemma 4.1.6.** *There exists a $\mathbb{T}_0$-line which is disjoint from $B$.*

*Proof.* Let $l$ be a line of $PG(3,q)$ which is disjoint from $B$. We may assume that $l$ is not a $\mathbb{T}_0$-line. Then $l$ is necessarily an external line or a $\mathbb{T}_1$-line. Suppose that $l \in \mathbb{E}$. Let $\pi$ be a secant plane through $l$. Then $|\pi \cap B| = q$ by Corollary 4.1.4. Since $\pi \cap B$ is an $\mathbb{S}_\pi$-blocking set in $\pi$, Theorem 2.1.7 implies that there exists a $\mathbb{T}_1$-line in $\pi$ disjoint from $\pi \cap B$ (and hence from $B$).

So, without loss, we may assume that $l$ itself is a $\mathbb{T}_1$-line. Then $|\pi_{x_l} \cap B| = q$ again by Corollary 4.1.4. So $\pi_{x_l} \cap B = l_1 \setminus \{x_l\}$ for some tangent line $l_1$ through $x_l$ by Lemma 4.1.2(ii). Then it follows that at least one of the two $\mathbb{T}_0$-lines through $x_l$ is disjoint from $B$. □

**Lemma 4.1.7.** *Suppose that $B \nsubseteq \mathcal{H}$. Then the following hold:*

*(i) There exist two intersecting $\mathbb{T}_0$-lines $l_0$ and $l_1$ which are disjoint from $B$.*

*(ii) If $q \geqslant 4$, then some point of $\mathcal{H} \setminus (l_0 \cup l_1)$ is in $B$.*

*Proof.* Let $x$ be a point of $B \setminus \mathcal{H}$. By Lemma 4.1.6, let $l_0$ be a $\mathbb{T}_0$-line disjoint from $B$. One of the $q+1$ tangent planes through $l_0$ contains $x$, say $x \in \pi_w$ for some point $w \in l_0$. Let $l_1$ be the other $\mathbb{T}_0$-line through $w$, and $l$ be the $\mathbb{T}_1$-line through $w$ and $x$. Since $l_0 \cap B = \emptyset$ and $x \in l \cap B$, applying Corollary 4.1.5 to the triple $(l_0, w, l)$, we have $\pi_w \cap B = l \setminus \{w\}$. It follows that $l_1 \cap B = \emptyset$. This proves (i).

Let $m$ be a $\mathbb{T}_1$-line through $w$ different from $l$. Such a line $m$ exists for $q \geqslant 3$. Consider a secant plane $\pi$ through $m$. Since $m \cap B = \emptyset$, we have $|\pi \cap B| = q$ by Corollary 4.1.4. Thus $\pi \cap B$ is an $\mathbb{S}_\pi$-blocking set in $\pi$ of size $q$. Since $q \geqslant 4$, Corollary 2.1.6 implies that $C_\pi$ contains some point, say $z$, of $\pi \cap B$. Since $w \in C_\pi \setminus B$, we have $z \neq w$ and it follows that $z$ is a point in $B \cap (\mathcal{H} \setminus (l_0 \cup l_1))$. This proves (ii). □

**Lemma 4.1.8.** *Let $l_0$ and $l_1$ be two intersecting $\mathbb{T}_0$-lines which are disjoint from $B$. If $B$ contains some point of $\mathcal{H} \setminus (l_0 \cup l_1)$, then all points of $\mathcal{H} \setminus (l_0 \cup l_1)$ are in $B$.*

*Proof.* Let $x_1$ be a point in $B \cap (\mathcal{H} \setminus (l_0 \cup l_1))$ and $m$ be the $\mathbb{T}_0$-line through $x_1$ intersecting $l_0$ at some point, say $x_0$. Since $l_0 \cap B = \emptyset$ and $m \cap B \neq \emptyset$, applying Corollary 4.1.5 to the triple $(l_0, x_0, m)$, we get that $m \setminus \{x_0\} \subseteq B$.

Let $m = \{x_0, x_1, x_2, \cdots, x_q\}$. For $1 \leqslant i \leqslant q$, let $m_i$ be the $\mathbb{T}_0$-line through $x_i$ intersecting $l_1$ at the point, say $y_i$. Since $l_1 \cap B = \emptyset$ and $x_i \in m_i \cap B$, again applying Corollary 4.1.5 to the triple $(l_1, y_i, m_i)$, we get that $m_i \setminus \{y_i\} \subseteq B$ for every $i$. Since $\mathcal{H} \setminus (l_0 \cup l_1)$ is a disjoint union of the sets $m_i \setminus \{y_i\}$, it follows that $\mathcal{H} \setminus (l_0 \cup l_1) \subseteq B$. □

**Lemma 4.1.9.** *Assume that $q = 3$ and $B \cap \mathcal{H} = \emptyset$. Let $\pi$ be a secant plane. Then the following hold:*

*(i) For every point $x$ of $\mathcal{H}$, one of the two $\mathbb{T}_1$-lines through $x$ is disjoint from $B$ and the other one intersects $B$ in three points.*

*(ii) If $\pi \cap B$ contains some point of $\pi$ exterior to $\mathcal{C}_\pi$, then every tangent line in $\pi$ meets $B$ in three points. In particular, $\pi \cap B$ contains all the six exterior points to $\mathcal{C}_\pi$.*

*(iii) $\pi \cap B$ consists of either the three interior points or the six exterior points to $\mathcal{C}_\pi$.*

*Proof.* (i) Each $\mathbb{T}_0$-line through $x$ is disjoint from $B$. So the statement follows from Corollary 4.1.4 and Lemma 4.1.2(ii) applied to the tangent plane $\pi_x$.

(ii) This follows from (i), using the fact that every exterior point in $\pi$ with respect to $\mathcal{C}_\pi$ lies on two tangent lines contained in $\pi$.

(iii) By Lemma 4.1.2(i), $|\pi \cap B| \geqslant 3$. If $|\pi \cap B| = 3$, then $B \cap \mathcal{C}_\pi = \emptyset$ implies that $\pi \cap B$ consists of all the three interior points to $\mathcal{C}_\pi$ by Corollary 2.1.6. If $|\pi \cap B| \geqslant 4$, then $\pi \cap B$ contains some point exterior to $\mathcal{C}_\pi$ and so $\pi \cap B$ conatins all the six exterior points to $\mathcal{C}_\pi$ by (ii). We show that $|\pi \cap B| = 6$. Let $l$ be a $\mathbb{T}_1$-line in $\pi$ with tangency point $x_l \in \mathcal{H}$. Then $|l \cap B| = 3$ by (ii). Since $x_l \notin B$, the three secant lines through $x_l$ contained in $\pi$ meet $B$ at different points. So every secant plane through $l$ contains at least three points of $B$ other than those of $l \cap B$. Since $|B| = 12$, it follows that every secant plane contains exactly six points of $B$. In particular, $|\pi \cap B| = 6$. □

We now prove Theorem 4.1.1 for the equality case.

**Lemma 4.1.10.** *If $|B| = q^2 + q$ and $B \subseteq \mathcal{H}$, then $B = \mathcal{H} \setminus l$ for some $\mathbb{T}_0$-line $l$.*

*Proof.* By Lemma 4.1.6, there is a $\mathbb{T}_0$-line $l$ disjoint from $B$. So $B \subseteq \mathcal{H} \setminus l$. Since $|B| = q^2 + q = |\mathcal{H} \setminus l|$, we must have $B = \mathcal{H} \setminus l$. □

Lemma 4.1.10 can also be seen without using Lemma 4.1.6. Let $x, y$ be two distinct points of $\mathcal{H} \setminus B$. Since $B \subseteq \mathcal{H}$ and $B$ blocks every secant line, the line of $PG(3, q)$ through $x$ and $y$ is not secant to $\mathcal{H}$ and so must be a $\mathbb{T}_0$-line. Thus any two distinct points of $\mathcal{H} \setminus B$ lie on a $\mathbb{T}_0$-line. Since $|\mathcal{H} \setminus B| = q + 1$, it follows that $\mathcal{H} \setminus B$ itself is a $\mathbb{T}_0$-line $l$ and hence $B = \mathcal{H} \setminus l$.

**Lemma 4.1.11.** *If $|B| = q^2 + q$, $B \setminus \mathcal{H} \neq \emptyset$ and $B \cap \mathcal{H} \neq \emptyset$, then $B = (\mathcal{H} \setminus (l_0 \cup l_1)) \cup (l \setminus \{w\})$, where $l_0, l_1$ are two $\mathbb{T}_0$-lines intersecting at the point $w$ and $l$ is a $\mathbb{T}_1$-line through $w$.*

*Proof.* Since $B \nsubseteq \mathcal{H}$, there exist two intersecting $\mathbb{T}_0$-lines $l_0, l_1$ which are disjoint from $B$, by Lemma 4.1.7(i). Let $l_0 \cap l_1 = \{w\}$. By Corollary 4.1.4, we have $|\pi_w \cap B| = q$. So, by Lemma 4.1.2(ii), $\pi_w \cap B = l \setminus \{w\}$ for some $\mathbb{T}_1$-line $l$ through

$w$. Since $B \cap \mathcal{H} \neq \emptyset$, it follows that $\mathcal{H} \setminus (l_0 \cup l_1)$ contains some point of $B$ and so $\mathcal{H} \setminus (l_0 \cup l_1)$ is contained in $B$ by Lemma 4.1.8. Thus $(\mathcal{H} \setminus (l_0 \cup l_1)) \cup (l \setminus \{w\})$ is contained in $B$. Since both sets have size $q^2 + q$, we must have $B = (\mathcal{H} \setminus (l_0 \cup l_1)) \cup (l \setminus \{w\})$. □

**Lemma 4.1.12.** *Let $|B| = q^2 + q$ and $B \cap \mathcal{H} = \emptyset$. Then $q \in \{2, 3\}$ and the following hold:*

*(i) If $q = 2$, then $B$ consists of all the six points outside $\mathcal{H}$.*

*(ii) If $q = 3$, then $B$ satisfies the three conditions (i)–(iii) of Theorem 3.3.1.*

*Proof.* Since $B \cap \mathcal{H} = \emptyset$, Lemma 4.1.7(ii) implies that $q \in \{2, 3\}$. Clearly, (i) holds if $q = 2$. Assume that $q = 3$. Obviously $B$ satisfies the condition (i) of Theorem 3.3.1. Since $B \cap \mathcal{H} = \emptyset$, every point of $B$ lies on 6 secant lines. Counting the number of elements in $\{(x, l) : x \in B, l \in \mathbb{S}, x \in l\}$, we get $12 \times 6 = |B| \times 6 \geqslant |\mathbb{S}| = 72$ and so equality should hold everywhere. It follows that each secant line contains one point of $B$ and so $B$ satisfies the condition (iii) of Theorem 3.3.1.

Now let $l$ be an external line and $\pi$ be a secant plane through $l$. By Lemma 4.1.9(iii), $\pi \cap B$ consists of either the three interior points or the six exterior points of $\pi$ with respect to $\mathcal{C}_\pi$. Since $l$ contains two interior points and two exterior points to $\mathcal{C}_\pi$, it follows that $|l \cap B| = |l \cap (\pi \cap B)| = 2$. So $B$ satisfies the condition (ii) of Theorem 3.3.1. □

## 4.2 $(\mathbb{T} \cup \mathbb{S})$-blocking sets

In this section, we prove the following theorem which characterizes the minimum size $(\mathbb{T} \cup \mathbb{S})$-blocking sets in $PG(3, q)$ for all $q$. For any secant plane $\pi$ of $PG(3, q)$, $(\mathbb{T} \cup \mathbb{S})_\pi$ is the set of all tangent and secant lines contained in $\pi$ with respect to the conic $\mathcal{C}_\pi = \pi \cap \mathcal{H}$.

**Theorem 4.2.1.** *Let $B$ be a $(\mathbb{T} \cup \mathbb{S})$-blocking set in $PG(3, q)$. Then $|B| \geqslant q^2 + q + 1$, and equality holds if and only if $B$ is a plane of $PG(3, q)$.*

*Proof.* Let $B$ be an $(\mathbb{T} \cup \mathbb{S})$-blocking set in $PG(3, q)$ of minimum possible size. Then $|B| \leqslant q^2+q+1$ by Proposition 1.7.1. Let $x$ be a point of $\mathcal{H} \setminus B$. There are $q^2+q+1$ lines of $PG(3, q)$ through $x$. Each of the lines through $x$, being a tangent or a secant to $\mathcal{H}$, contains at least one point of $B$. This gives $|B| \geqslant q^2 + q + 1$ and hence $|B| = q^2 + q + 1$.

We show that $B$ is plane of $PG(3, q)$. By Proposition 1.7.1, it is enough to show that every external line $l$ meets $B$. For each secant plane $\pi$ through $l$, the set $\pi \cap B$ is an $(\mathbb{T} \cup \mathbb{S})_\pi$-blocking set in $\pi$. So $|\pi \cap B| \geqslant q + 1$ by Theorem 2.1.7. If $l$ is disjoint from $B$, then the $q + 1$ secant planes through $l$ together contain at least $(q + 1)^2$ points of $B$, giving $|B| \geqslant (q + 1)^2 > q^2 + q + 1$, which is a contradiction. □

## 4.3 $\mathbb{E}$-blocking sets

In this section, we prove the following theorem which characterizes the minimum size $\mathbb{E}$-blocking sets in $PG(3, q)$.

**Theorem 4.3.1.** *Let $B$ be an $\mathbb{E}$-blocking set in $PG(3, q)$. Then $|B| \geqslant q^2 - q$, and equality holds if and only if $B = \pi \setminus \mathcal{H}$ for some tangent plane $\pi$ of $PG(3, q)$.*

We note that Theorem 4.3.1 was proved by Biondi et al. in [6, Theorem 1.1] for $q$ even and in [7, Theorem 2.4] for $q$ odd, with exception of the equality case for some small values of $q$, namely $q \in \{2, 3, 4, 5, 7\}$.

Here our aim is to give alternate proof of the equality case in Theorem 4.3.1 which works for all $q$, in particular for $q \in \{2, 3, 4, 5, 7\}$, so that we can use them in the next two sections while studying the minimum size $(\mathbb{E} \cup \mathbb{S})$- and

$(\mathbb{T} \cup \mathbb{E})$-blocking sets in $PG(3,q)$. We give two separate proofs of Theorem 4.3.1 depending on $q$ even or odd.

As in the proof of [6, Proposition 2.1] and [7, Proposition 2.1], by counting in two ways the cardinality of the set $\{(x,l) : x \in B, l \in \mathbb{E}, x \in l\}$, it follows that $|B| \geqslant q^2 - q$ with equality if and only if $B \cap \mathcal{H} = \emptyset$ and each external line contains exactly one point of $B$. Thus the following hold:

**Lemma 4.3.2.** *If $B$ is an $\mathbb{E}$-blocking set in $PG(3,q)$ of minimum size $q^2 - q$, then $B \cap \mathcal{H} = \emptyset$ and each external line contains exactly one point of $B$.*

For any secant plane $\pi$ of $PG(3,q)$, $\mathbb{E}_\pi$ is the set of all external lines contained in $\pi$ with respect to the conic $\mathcal{C}_\pi = \pi \cap \mathcal{H}$. If $q$ is even and $\pi = \pi_x$ for some $x \in \mathbb{P} \setminus \mathcal{H}$, then recall that the conic $\mathcal{C}_\pi = \mathcal{C}_{\pi_x}$ is also denoted by $\mathcal{C}_x$.

### 4.3.1 Proof of Theorem 4.3.1 for $q$ even

Let $B$ be an $\mathbb{E}$-blocking set in $PG(3,q)$, $q$ even, of minimum size $q^2 - q$. For any secant plane $\pi_x$, $x \in \mathbb{P} \setminus \mathcal{H}$, the set $\pi_x \cap B$ is an $\mathbb{E}_{\pi_x}$-blocking set in $\pi_x$. As in the proof of [6, Proposition 2.3], by counting in two ways the cardinality of the set $\{(y,l) : y \in \pi_x \cap B, l \in \mathbb{E}_{\pi_x}, y \in l\}$, the following hold:

**Lemma 4.3.3.** *Let $\pi_x$, $x \in \mathbb{P} \setminus \mathcal{H}$, be a secant plane of $PG(3,q)$. Then the following hold.*

*(i)* $|\pi_x \cap B| = q - 1$ *if and only if* $x \notin B$.

*(ii)* $|\pi_x \cap B| = q$ *if and only if* $x \in B$.

Note that if Lemma 4.3.3(ii) holds, then $(\pi_x \cap B) \setminus \{x\}$ is an $\mathbb{E}_{\pi_x}$-blocking set in $\pi_x$ of minimum size $q - 1$. For any $l \in \mathbb{T}_1$, the tangency point $x_l \in \mathcal{H}$ of $l$ is not in $B$ as $B \cap \mathcal{H}$ is empty by Lemma 4.3.2.

**Lemma 4.3.4.** *Let $l \in \mathbb{T}_1$ be such that $|l \cap B| \geqslant 1$. Then every line in $\mathbb{T}_1$ through $x_l$ meets $B$.*

*Proof.* Suppose that there exists a line $l_1 \in \mathbb{T}_1$ through $x_l$ such that $l_1 \cap B = \emptyset$. We count the points of $B$ contained in the planes through $l_1$. The tangent plane $\pi_{x_l}$ through $l_1$ contains at least one point of $B$ (comes from $l \cap B$). Since $l_1 \cap B = \emptyset$, each of the $q$ secant planes through $l_1$ contains $q-1$ points of $B$ by Lemma 4.3.3(i). It follows that $B$ contains at least $1 + q(q-1) = q^2 - q + 1$ points, a contradiction to the fact that $|B| = q^2 - q$. □

By Proposition 3.2.5, the point-line geometry $\mathcal{X} = (\mathbb{P}, \mathbb{T})$ is a generalized quadrangle of order $q$ which is isomorphic to $W(q)$. The lines of $PG(3, q)$ are of the form $\{x, y\}^\perp$ for distinct $x, y \in \mathbb{P}$, where $\{x, y\}^\perp$ is in $\mathbb{T}$ or $\mathbb{S} \cup \mathbb{E}$ according as $y \in x^\perp$ or not. For $l \in \mathbb{E}$ (respectively, $l \in \mathbb{S}$), we have $l^\perp \in \mathbb{E}$ (respectively, $l^\perp \in \mathbb{S}$).

**Lemma 4.3.5.** *There exists a line in $\mathbb{T}_1$ containing at least two points of $B$.*

*Proof.* Consider a line $l$ in $\mathbb{E}$. Then $l^\perp$ is also in $\mathbb{E}$. Since every external line meets $B$ at exactly one point, let $\{a\} = l \cap B$ and $\{b\} = l^\perp \cap B$. Then the line $l_1$ through $a$ and $b$ is in $\mathbb{T}_1$ and $|l_1 \cap B| \geqslant 2$. □

**Lemma 4.3.6.** *Let $l$ be a line in $\mathbb{T}_1$ with $|l \cap B| \geqslant 2$. Then $l \cap B = l \setminus \{x_l\}$.*

*Proof.* Since $x_l \notin B$, we have $l \cap B \subseteq l \setminus \{x_l\}$. Clearly the lemma holds for $q = 2$. Assume that $q \geqslant 4$. Suppose that there exists a point $w$ in $l \setminus \{x_l\}$ which is not in $B$. Then $|\pi_w \cap B| = q - 1$ by Lemma 4.3.3(i). So one of the three cases (i)–(iii) of Theorem 2.1.2 holds for the $\mathbb{E}_{\pi_w}$-blocking set $\pi_w \cap B$ in $\pi_w$ of size $q - 1$. Since at least two points of $\pi_w \cap B$ are on the same tangent line $l$ in $\pi_w$, Theorem 2.1.2(i) does not occur for $\pi_w \cap B$.

Suppose that Theorem 2.1.2(ii) holds for $\pi_w \cap B$. Then the fact that $|l \cap B| \geqslant 2$ implies $\pi_w \cap B = l \setminus \{w, x_l\}$. Let $z \in \pi_w \cap B = l \cap B$. Then $(\pi_z \cap B) \setminus \{z\}$ is an $\mathbb{E}_{\pi_z}$-blocking set in $\pi_z$ of minimum size $q-1$. So again one of the three possibilities (i)–(iii) of Theorem 2.1.2 holds for $(\pi_z \cap B) \setminus \{z\}$. Since the line $l$ in $\pi_z$ contains $q-1$ points of $B$ (which are outside the conic $\mathcal{C}_z$), it follows that Theorem 2.1.2(ii) must occur for $(\pi_z \cap B) \setminus \{z\}$ and that $(\pi_z \cap B) \setminus \{z\} = l \setminus \{z, x_l\}$. This gives $w \in B$, a contradiction.

Now, suppose that $q$ is a square and that Theorem 2.1.2(iii) holds for $\pi_w \cap B$. Then $\pi_w \cap B = \Pi \setminus (\{w\} \cup (\Pi \cap \mathcal{C}_w))$, where $\Pi$ is a Baer subplane of $\pi_w$ such that $\Pi \cap \mathcal{C}_w$ is an irreducible conic in $\Pi$ (with nucleus $w$). Note that $l$ contains at least three points of $\Pi \setminus (\Pi \cap \mathcal{C}_w)$, namely, $w$ and at least two points of $\pi_w \cap B$. This implies that $l_1 := l \cap \Pi$ is a tangent line of $\Pi$ and so $q \geqslant 16$. Thus $l \cap B = l_1 \setminus \{w, x_l\}$ is of size $\sqrt{q} - 1$.

Let $u \in l \cap B$. Then $(\pi_u \cap B) \setminus \{u\}$ is an $\mathbb{E}_{\pi_u}$-blocking set in $\pi_u$ of minimum size $q-1$. Since $q \geqslant 16$ and $|l \cap B| = \sqrt{q} - 1$, it can be seen that Theorem 2.1.2(iii) must hold for $(\pi_u \cap B) \setminus \{u\}$. Let

$$(\pi_u \cap B) \setminus \{u\} = \Pi' \setminus (\{u\} \cup (\Pi' \cap \mathcal{C}_u)),$$

where $\Pi'$ is a Baer subplane of $\pi_u$ such that $\Pi' \cap \mathcal{C}_u$ is an irreducible conic in $\Pi'$ (with nucleus $u$). Since $l_2 := l \cap \Pi'$ is a tangent line of $\Pi'$ (as $l \cap B \subseteq \Pi' \setminus (\Pi' \cap \mathcal{C}_u)$) and $u \in l \cap B$, it follows that $l_2 \cap B = l_2 \setminus \{x_l\}$ is of size $\sqrt{q}$. This implies that $|l \cap B| = \sqrt{q}$, a contradiction. Therefore, $l \cap B = l \setminus \{x_l\}$. □

The following lemma proves the equality case of Theorem 4.3.1 for all even $q$.

**Lemma 4.3.7.** *$B = \pi \setminus \mathcal{H}$ for some tangent plane $\pi$.*

*Proof.* By Lemmas 4.3.5 and 4.3.6, consider a line $l \in \mathbb{T}_1$ such that $l \cap B = l \setminus \{x_l\}$. We claim that $B = \pi_{x_l} \setminus \mathcal{H}$. This is clear if $q = 2$. Assume that $q \geqslant 4$.

Let $l = l_1, l_2, \cdots, l_{q-1}$ be the $q-1$ lines in $\mathbb{T}_1$ through $x_l$. Since $|B| = q^2 - q$, it is enough to show that $l_i \cap B = l_i \setminus \{x_l\}$ for each $i$, $2 \leqslant i \leqslant q-1$. By Lemma 4.3.4, we have $|l_i \cap B| \geqslant 1$. Let

$$t = \min\{|l_i \cap B| : 2 \leqslant i \leqslant q-1\}.$$

Then $1 \leqslant t \leqslant q$ as $x_l \notin B$. We show that $t = q$ and this would complete the proof. Consider a line $l_k$, $2 \leqslant k \leqslant q-1$, such that $|l_k \cap B| = t$. Let

$$l_k = \{x_l = x_0, x_1, \cdots, x_t, x_{t+1}, \cdots, x_q\},$$

where $l_k \cap B = \{x_1, \cdots, x_t\}$. We count the points of $B$ contained in the planes through $l_k$. The tangent plane $\pi_{x_l}$ through $l_k$ contains at least $q + t(q-2)$ points of $B$. By Lemma 4.3.3(ii), each of the $t$ secant planes $\pi_{x_i}$, $1 \leqslant i \leqslant t$, contains at least $q - t$ points of $B$ different from the points of $l_k \cap B$. Again, by Lemma 4.3.3(i), each of the remaining $q - t$ secant planes $\pi_{x_i}$, $t + 1 \leqslant i \leqslant q$, contains at least $q - 1 - t$ points of $B$ different from that of $l_k \cap B$. Thus all the planes through $l_k$ together contain at least

$$q + t(q-2) + t(q-t) + (q-t)(q-1-t) = q^2 - t$$

points of $B$ and so $q^2 - t \leqslant |B| = q^2 - q$. This gives $t \geqslant q$ and hence $t = q$. □

## 4.3.2 Proof of Theorem 4.3.1 for $q$ odd

Let $B$ be an $\mathbb{E}$-blocking set in $PG(3, q)$, $q$ odd, of minimum size $q^2 - q$.

**Lemma 4.3.8.** *For any external line $l$ of $PG(3,q)$, exactly one of the planes through $l$ contains $q$ points of $B$ and each of the remaining planes contains $q-1$ points of $B$.*

*Proof.* Let $\pi_0, \pi_1, \cdots, \pi_q$ be the $q+1$ planes through $l$. Then each $\pi_i$ is a secant plane and $\pi_i \cap B$ is an $\mathbb{E}_{\pi_i}$-blocking set in $\pi_i$. By Theorem 2.1.1, $|\pi_i \cap B| \geqslant q-1$ for each $i$. Now the lemma follows from the three facts that $B = \bigcup_{i=0}^{q} (\pi_i \cap B)$, $|l \cap B| = 1$ and $|B| = q^2 - q$. □

**Lemma 4.3.9.** *Let $\pi$ be a secant plane of $PG(3,q)$. Then the following hold:*

*(i)* $|\pi \cap B| = q-1$ *or* $q$.

*(ii) If* $|\pi \cap B| = q-1$, *then* $\pi \cap B = l \setminus \mathcal{C}_\pi$ *for some secant line $l$ contained in $\pi$.*

*(iii) If* $|\pi \cap B| = q$, *then each point of $\pi \cap B$ is exterior in $\pi$ with respect to $\mathcal{C}_\pi$.*

*Proof.* Considering an external line $l$ contained in $\pi$, (i) follows from Lemma 4.3.8. We prove (ii) and (iii).

Let $\alpha$ (respectively, $\beta$) denote the number of points of $\pi \cap B$ which are interior (respectively, exterior) in $\pi$ with respect to $\mathcal{C}_\pi$. Since $B \cap \mathcal{C}_\pi = B \cap \mathcal{H} = \emptyset$, we have $\alpha + \beta = |\pi \cap B|$. Consider the following set of point-line pairs:

$$X = \{(x,l) : x \in \pi \cap B, l \in \mathbb{E}_\pi, x \in l\}.$$

Counting $|X|$ in two ways, we get

$$\alpha \binom{\frac{q+1}{2}}{} + \beta \binom{\frac{q-1}{2}}{} = |X| = \frac{q(q-1)}{2}.$$

This gives

$$(\alpha+\beta)q + \alpha - \beta = q(q-1). \tag{4.3.1}$$

If $|\pi \cap B| = q-1$, then putting $\alpha+\beta = q-1$ in equation (4.3.1), we get $\alpha = \beta$. Thus, half of the points of $\pi \cap B$ are interior and the other half are exterior with respect to $\mathcal{C}_\pi$. Then (ii) follows from Theorem 2.1.1, since $\pi \cap B$ is an $\mathbb{E}_\pi$-blocking set in $\pi$ of minimum size $q-1$.

If $|\pi \cap B| = q$, then we have $\alpha + \beta = q$. Then equation (4.3.1) implies that $\alpha - \beta = -q$. It follows that $\alpha = 0$ and $\beta = q$. Thus, all the points of $\pi \cap B$ are exterior with respect to $\mathcal{C}_\pi$, implying (iii). $\square$

As a consequence of Lemmas 4.3.8 and 4.3.9(ii), we have the following.

**Corollary 4.3.10.** *There exists a secant line $l$ such that $l \setminus \mathcal{H}$ is contained in $B$.*

The following lemma proves the equality case of Theorem 4.3.1 for all odd $q$.

**Lemma 4.3.11.** *$B = \pi \setminus \mathcal{H}$ for some tangent plane $\pi$.*

*Proof.* Consider a secant $l$ such that $l \setminus \mathcal{H}$ is contained in $B$ and the planes through it. Let $\pi$ be a secant plane through $l$. By Lemma 4.3.9(i), we have $|\pi \cap B| = q-1$ or $q$. Since half of the points of $l \setminus \mathcal{H} = l \setminus \mathcal{C}_\pi$ are interior in $\pi$ with respect to $\mathcal{C}_\pi$, Lemma 4.3.9(iii) implies that $|\pi \cap B| \neq q$. So $|\pi \cap B| = q-1$ and hence $\pi \cap B = l \setminus \mathcal{H}$.

Thus $\pi \cap B = l \setminus \mathcal{H}$ for every secant plane $\pi$ through $l$. It follows that the points of $B \setminus l$ are contained in the two tangent planes through $l$. Since $|B \setminus l| = q^2 - q - (q-1) = (q-1)^2$, one of the tangent planes through $l$, say $\pi_0$, contains at least $(q-1)^2/2$ points of $B \setminus l$. Then $\pi_0$ contains at least $q - 1 + (q-1)^2/2 = (q^2-1)/2$ points of $B$ and so $|B \setminus \pi_0| \leqslant (q-1)^2/2$.

We claim that $B = \pi_0 \setminus \mathcal{H}$. It is enough to show that each point of $\pi_0 \setminus \mathcal{H}$ is in $B$. On the contrary, suppose that there exists a point $x \in \pi_0 \setminus \mathcal{H}$ which is not in $B$. There are $q(q-1)/2$ external lines through $x$ and each of them meets $B$ at a unique point outside $\pi_0$. This defines an injective map from the set of external lines through $x$ to the set $B\setminus\pi_0$. But such a map is not possible, since $q(q-1)/2>(q-1)^2/2\geqslant|B\setminus\pi_0|$, a contradiction. Therefore $B=\pi_0\setminus\mathcal{H}$. □

## 4.4 $(\mathbb{E}\cup\mathbb{S})$-blocking sets

In this section, we prove the following theorem which characterizes the minimum size $(\mathbb{E}\cup\mathbb{S})$-blocking sets in $PG(3,q)$ for all $q$. For any secant plane $\pi$ of $PG(3,q)$, $(\mathbb{E}\cup\mathbb{S})_\pi$ denotes the set of all external and secant lines of $PG(3,q)$ contained in $\pi$. Then $(\mathbb{E}\cup\mathbb{S})_\pi$ is precisely the set of all external and secant lines in $\pi$ with respect to the conic $\mathcal{C}_\pi=\pi\cap\mathcal{H}$.

**Theorem 4.4.1.** *Let $B$ be an $(\mathbb{E}\cup\mathbb{S})$-blocking set in $PG(3,q)$. Then the following hold:*

*(i) If $q\in\{2,3\}$, then $|B|\geqslant q^2+q$.*

*(ii) If $q=2$, then $|B|=6$ if and only if one of the following two cases occurs:*

*(a) $B$ consists of all the six points outside $\mathcal{H}$.*

*(b) $B=(\mathcal{H}\setminus(l_0\cup l_1))\cup(l\setminus\{w\})$, where $l_0,l_1$ are two $\mathbb{T}_0$-lines intersecting at the point $w\in\mathcal{H}$ and $l$ is the unique $\mathbb{T}_1$-line through $w$.*

*(iii) If $q=3$, then $|B|=12$ if and only if $B$ satisfies the three conditions (i)–(iii) of Theorem 3.3.1.*

*(iv) If $q\geqslant 4$, then $|B|\geqslant q^2+q+1$, and equality holds if and only if $B$ is a plane of $PG(3,q)$.*

**Remark 4.4.1.** *Theorem 4.4.1 was proved by Sahoo and Sastry in [36, Theorem 1.3] for even $q\geqslant 4$ using the properties of the generalized quadrangle $\mathcal{X}=(\mathbb{P},\mathbb{T})\simeq$*

*$W(q)$. Here we give an alternate proof which works for all $q$ irrespective of $q$ even or odd.*

### 4.4.1 Proof of Theorem 4.4.1

Let $B$ be an ($\mathbb{S}\cup\mathbb{E}$)-blocking set in $PG(3,q)$ of minimum possible size. Considering $B$ as an $\mathbb{S}$-blocking set, we have $|B| \geqslant q^2+q$ by Theorem 4.1.1 and so

$$q^2+q \leqslant |B| \leqslant q^2+q+1$$

by Proposition 1.7.1. Since $B \setminus \mathcal{H}$ is an $\mathbb{E}$-blocking set, we have

$$|B \setminus \mathcal{H}| \geqslant q^2 - q \tag{4.4.1}$$

by Theorem 4.3.1. First assume that $q \in \{2,3\}$. Observe that the $\mathbb{S}$-blocking sets of size $q^2+q$ in Theorem 4.1.1(iii) are also ($\mathbb{S}\cup\mathbb{E}$)-blocking sets. So $|B| = q^2+q$, proving Theorem 4.4.1(i). Then again Theorem 4.1.1 together with the fact that $B$ blocks every external line imply Theorem 4.4.1(ii) and (iii).

For the rest of this section, we assume that $q \geqslant 4$. Then $|B| = q^2+q+1$. Otherwise, $B$ is a set of the form as defined in Theorem 4.1.1(i) and (ii) and this would imply $|B \setminus \mathcal{H}| \leqslant q$, contradicting the Inequality (4.4.1) as $q \geqslant 4$.

We prove two results for $q$ even, which are needed to show that any secant plane contains at least $q+1$ points of $B$.

**Lemma 4.4.2.** *Suppose that $q$ is even. Let $l$ be a tangent line containing a point $y$ which is not in $B$. Then for every $x \in l \setminus \{y\}$, $\pi_x \cap B$ contains at least $q$ points different from the points of $l \cap B$.*

*Proof.* Let $m$ be one of the $q$ lines through $y$ in $\pi_x$ different from $l$. If $\pi_x$ is a secant plane with $l \cap \mathcal{H} \neq \{y\}$, then $m$ is secant or external to $\mathcal{C}_{\pi_x}$ in $\pi_x$ (and hence to $\mathcal{H}$). If $\pi_x$ is a secant plane with $l \cap \mathcal{H} = \{y\}$ or if $\pi_x$ is a tangent plane, then $m$ is secant to $\mathcal{H}$. In all cases, $B$ blocks each such line $m$. Since $y \notin B$, it follows that $\pi_x \cap B$ contains at least $q$ points other than those of $l \cap B$. □

**Lemma 4.4.3.** *Suppose that $q$ is even and let $x \in B \cap \mathcal{H}$. If there exists a tangent line $l$ through $x$ with $|l \cap B| = q$, then every tangent line through $x$ contains at least $q$ points of $B$. In particular, $|\pi_x \cap B| \geqslant q^2$.*

*Proof.* Let $m$ be a tangent line through $x$ different from $l$. Since $l$ has a point not in $B$, Lemma 4.4.2 implies that the tangent plane $\pi_x$ through $l$ (and hence through $m$) contains at least $2q$ points of $B$. Suppose that $|m \cap B| \leqslant q-1$. Then applying Lemma 4.4.2 again carefully to the line $m$, it follows that each of the $q$ planes $\pi_z$, $z \in m \setminus \{x\}$, through $m$ contains at least $q$ points of $B$ different from the points of $m \cap B$. Thus all the planes through $m$ together contain at least $2q + q^2$ points of $B$. This gives $|B| \geqslant q^2 + 2q > q^2 + q + 1$, which is a contradiction. □

**Lemma 4.4.4.** *Let $\pi$ be any plane of $PG(3, q)$. Then the following hold:*

*(i) If $\pi$ is a tangent plane with pole $x$, then $|\pi \cap B| \geqslant q$, and equality holds if and only if $\pi \cap B = l \setminus \{x\}$ for some tangent line $l$ through $x$.*

*(ii) If $\pi$ is a secant plane, then $|\pi \cap B| \geqslant q + 1$.*

*Proof.* (i) This follows from Lemma 4.1.2, considering $B$ simply as an $\mathbb{S}$-blocking set in $PG(3, q)$.

(ii) Here $\pi \cap B$ is an $(\mathbb{S} \cup \mathbb{E})_\pi$-blocking set in $\pi$. If $q \geqslant 5$ is odd, then $|\pi \cap B| \geqslant q + 1$ by Theorem 2.2.3(ii). Assume that $q \geqslant 4$ is even. Let $\pi = \pi_x$ for some point $x \in \mathbb{P} \setminus \mathcal{H}$. By Theorem 2.2.2, we have $|\pi_x \cap B| \geqslant q$. We show that $|\pi_x \cap B| \geqslant q + 1$.

Suppose that $|\pi_x \cap B| = q$. Then $\pi_x \cap B = l \setminus \{x\}$ for some $\mathbb{T}_1$-line $l$ through $x$, again by Proposition 2.2.2. Let $l = \{x_0, x_1, \cdots, x_{q-1}, x_q = x\}$ with tangency point $x_l = x_0 \in \mathcal{H}$. By Lemma 4.4.2, each of the secant planes $\pi_{x_i}$, $1 \leqslant i \leqslant q-1$, through $l$ contains at least $q$ points of $B$ other than those of $l \cap B$. By Lemma 4.4.3, we have $|\pi_{x_0} \cap B| \geqslant q^2$. It follows that all the planes through $l$ together contain at least $q^2+(q-1)q$ points of $B$. This gives $|B| \geqslant q^2+(q-1)q > q^2+q+1$ as $q \geqslant 4$, which is a contradiction. □

**Lemma 4.4.5.** *Every* $\mathbb{T}_1$*-line meets* $B$.

*Proof.* Consider a $\mathbb{T}_1$-line $l$. Let $\pi_0$ be the tangent plane and $\pi_1, \pi_2, ..., \pi_q$ be the secant planes through $l$. By Lemma 4.4.4, we have $|\pi_0 \cap B| \geqslant q$ and $|\pi_i \cap B| \geqslant q+1$ for $1 \leqslant i \leqslant q$. If $l$ is disjoint from $B$, then it follows that $|B| \geqslant q + q(q+1) > q^2 + q + 1$, which is a contradiction. □

**Lemma 4.4.6.** *Every* $\mathbb{T}_0$*-line meets* $B$.

*Proof.* Let $m$ be a $\mathbb{T}_0$-line. Suppose that $m$ is disjoint from $B$. By Lemma 4.4.4(i), each of the $q+1$ tangent planes through $m$ contains at least $q$ points of $B$. Since $|B| = q^2+q+1$, one of them contains $q+1$ points of $B$ and each of the remaining $q$ planes meets $B$ in $q$ points. Consider a plane $\pi_x$, $x \in m$, containing $q$ points of $B$. By Lemma 4.4.4(i) again, $\pi_x \cap B = l \setminus \{x\}$ for some tangent line $l$ through $x$. Since $q \geqslant 4$, it follows that there exists a $\mathbb{T}_1$-line through $x$ which is disjoint from $B$, which is a contradiction to Lemma 4.4.5. □

The following lemma proves Theorem 4.4.1(iv).

**Lemma 4.4.7.** $B$ *is a plane of* $PG(3, q)$.

*Proof.* By Lemmas 4.4.5 and 4.4.6, every tangent lines meets $B$. Since $B$ blocks every external and secant lines, it follows that $B$ blocks every line of $PG(3, q)$. Then $|B| = q^2+q+1$ implies that $B$ is plane of $PG(3, q)$ by Proposition 1.7.1. □

## 4.5 $(\mathbb{T} \cup \mathbb{E})$-blocking sets

In this section, we prove the following theorem which characterizes the minimum size $(\mathbb{T} \cup \mathbb{E})$-blocking sets in $PG(3, q)$ for all $q$.

**Theorem 4.5.1.** *Let $B$ be a $(\mathbb{T} \cup \mathbb{E})$-blocking set in $PG(3, q)$. Then $|B| \geqslant q^2 + q$ and the following hold for the equality case:*

*(i) If $q = 2$, then $|B| = 6$ if and only if one of the following occurs:*

*(a) $B = \pi \setminus \{x\}$ for some tangent plane $\pi$ with pole $x \in \mathcal{H}$.*

*(b) $B = \mathcal{O} \cup \{\alpha\}$, where $\mathcal{O}$ is a an ovoid of the generalized quadrangle $\mathcal{X} = (\mathbb{P}, \mathbb{T}) \simeq W(2)$ of order 2 and $\alpha \in \mathbb{P} \setminus \mathcal{H}$ is such that the unique external line through $\alpha$ is disjoint from $\mathcal{O}$.*

*(ii) If $q \geqslant 3$, then $|B| = q^2 + q$ if and only if $B = \pi \setminus \{x\}$ for some tangent plane $\pi$ with pole $x \in \mathcal{H}$.*

We give a proof of Theorem 4.5.1 for all even $q$. However, the arguments used for $q$ even can not be extended to odd $q$. We then give a proof of Theorem 4.5.1 for all $q \geqslant 3$ irrespective of $q$ even or odd. We first prove a few basic results which are independent of the parity of $q$.

Let $B$ be an $(\mathbb{E} \cup \mathbb{T})$-blocking set in $PG(3, q)$ of minimum possible size. Observe that, for every point $x$ of $\mathcal{H}$, the set $\pi_x \setminus \{x\}$ is an $(\mathbb{E} \cup \mathbb{T})$-blocking set of size $q^2 + q$. So

$$|B| \leqslant q^2 + q. \tag{4.5.1}$$

Since $B \cap \mathcal{H}$ blocks every $\mathbb{T}_0$-line, we have $|B \cap \mathcal{H}| \geqslant q + 1$. Every external line meets $B$ outside $\mathcal{H}$. So $B \setminus \mathcal{H}$ is an $\mathbb{E}$-blocking set in $PG(3, q)$ and hence

$|B \setminus \mathcal{H}| \geqslant q^2 - q$ by Theorem 4.3.1. Thus, we have

$$q^2 - q \leqslant |B \setminus \mathcal{H}| \leqslant q^2 - 1 \tag{4.5.2}$$

and

$$q + 1 \leqslant |B \cap \mathcal{H}| \leqslant 2q. \tag{4.5.3}$$

For a given $\mathbb{T}_1$-line $l$, recall that $x_l$ is the tangency point of $l$ in $\mathcal{H}$. Let $\overline{\mathbb{T}}$ denote the set of all $\mathbb{T}_1$-lines $l$ such that $|l \cap B| = 1$ and $x_l \notin B$.

**Lemma 4.5.2.** *$\overline{\mathbb{T}}$ is nonempty.*

*Proof.* Let $R$ be the set of all $\mathbb{T}_1$-lines $l$ for which $x_l \notin B$. For every point $x$ of $\mathcal{H}$, there are $q - 1$ $\mathbb{T}_1$-lines through $x$. Using the upper bound for $|B \cap \mathcal{H}|$ given in the inequality (4.5.3), we get

$$|R| = \left[(q+1)^2 - |B \cap \mathcal{H}|\right](q-1) \geqslant (q^2+1)(q-1). \tag{4.5.4}$$

Suppose that $\overline{\mathbb{T}}$ is empty. Then each line of $R$ meets $B$ in at least two points. Consider the set

$$Z = \{(x, l) : x \in B,\ l \in \mathbb{T},\ x \in l\}.$$

Counting $|Z|$ in two ways, we get

$$|B|(q+1) = |Z| \geqslant 2|R| + |\mathbb{T} \setminus R| = |R| + |\mathbb{T}|.$$

Since $|B| \leqslant q^2 + q$ and $|\mathbb{T}| = (q+1)(q^2+1)$, it follows that $|R| \leqslant q^2 - 1$, which is a contradiction to the inequality (4.5.4). So $\overline{\mathbb{T}}$ is nonempty. □

For any secant plane $\pi$ of $PG(3, q)$, $(\mathbb{T} \cup \mathbb{E})_\pi$ denotes the set of all tangent and external lines of $PG(3,q)$ contained in $\pi$. Then $(\mathbb{T} \cup \mathbb{E})_\pi$ is precisely the set of all tangent and external lines in $\pi$ with respect to the conic $\mathcal{C}_\pi = \pi \cap \mathcal{H}$.

**Lemma 4.5.3.** *Let $\pi$ be any plane of $PG(3,q)$. Then the following hold:*

*(i) If $\pi$ is a secant plane, then $|\pi \cap B| \geqslant q$.*

*(ii) If $\pi = \pi_x$ is a tangent plane for some point $x$ in $\mathcal{H} \setminus B$, then $|\pi \cap B| \geqslant q+1$.*

*Proof.* (i) The set $\pi \cap B$ is a $(\mathbb{T} \cup \mathbb{E})_\pi$-blocking set in $\pi$. So $|\pi \cap B| \geqslant q$ by Theorems 2.1.10 and 2.1.11.

(ii) This follows from the facts that $x \notin B$ and that each of the $q+1$ tangent lines through $x$ in $\pi_x$ meets $B$. □

**Lemma 4.5.4.** *For every $l \in \overline{\mathbb{T}}$, there exists a secant plane through $l$ containing exactly $q$ points of $B$.*

*Proof.* Suppose that this is not the case. By Lemma 4.5.3, we then know that each of the $q$ secant planes through $l$ contains at least $q+1$ points of $B$ and the tangent plane through $l$ contains at least $q+1$ points of $B$ (as $x_l \notin B$). Using the fact that $|l \cap B| = 1$, it follows that all the planes through $l$ together contain at least $(q+1)q + 1 = q^2 + q + 1$ points of $B$, which is a contradiction to the inequality (4.5.1). □

As a consequence of Lemmas 4.5.2 and 4.5.4, we have the following.

**Corollary 4.5.5.** *There exists a secant plane containing exactly $q$ points of $B$.*

Note that there are lines secant to $\mathcal{H}$ which are disjoint from $B$. Otherwise, $|B| \geqslant q^2 + q + 1$, as $B$ would be a blocking set with respect to all the lines of $PG(3,q)$.

**Lemma 4.5.6.** *Let $l$ be a secant line disjoint from $B$. If $\pi_c$ and $\pi_d$, for points $c, d$ of $\mathcal{H}$, are the two tangent planes through $l$, then at least one of $c$ and $d$ is in $B$.*

*Proof.* Suppose that none of $c$ and $d$ is in $B$. By Lemma 4.5.3, each of $\pi_c$ and $\pi_d$ contains at least $q+1$ points of $B$ and each of the $q-1$ secant planes through $l$ contains at least $q$ points of $B$. Since $l \cap B$ is empty, we get $|B| \geqslant 2(q+1) + q(q-1) = q^2 + q + 2$, which is a contradiction to the inequality (4.5.1). □

### 4.5.1 Proof of Theorem 4.5.1 for $q$ even

Here, we prove Theorem 4.5.1 for all $q$ even. We need the following lemmas.

**Lemma 4.5.7.** *If $x \in \mathbb{P} \setminus B$, then $|\pi_x \cap B| \geqslant q+1$.*

*Proof.* This follows, since $x \notin B$ and each of the $q+1$ tangent lines in $\pi_x$ through $x$ meets $B$. □

**Lemma 4.5.8.** $|B| = q^2 + q$.

*Proof.* By Lemma 4.5.2, consider a line $l = \{x_l = x_0, x_1, \cdots, x_q\}$ of $\overline{\mathbb{T}}$. We count the points of $B$ contained in the planes through $l$. We may assume that $l \cap B = \{x_q\}$. By Lemma 4.5.7, each of the planes $\pi_{x_i}$ $(0 \leqslant i \leqslant q-1)$ contains at least $q+1$ points of $B$. By Lemma 4.5.3, the secant plane $\pi_{x_q}$ contains at least $q$ points of $B$. This gives

$$q^2 + q \geqslant |B| = 1 + \sum_{i=0}^{q} (|\pi_{x_i} \cap B| - 1) \geqslant 1 + q^2 + q - 1 = q^2 + q. \qquad (4.5.5)$$

The first equality holds, since $\pi_{x_i} \cap \pi_{x_j} = l$ for $0 \leqslant i \neq j \leqslant q$ and $|l \cap B| = 1$. It follows that equality should hold everywhere in the inequality (4.5.5) and so $|B| = q^2 + q$. □

Using the fact that $|B| = q^2 + q$, we have the following as a consequence of the proof of Lemma 4.5.8.

**Corollary 4.5.9.** *Let $l \in \overline{\mathbb{T}}$ and $x \in l$. Then $|\pi_x \cap B| = q$ or $q+1$ according as $\{x\} = l \cap B$ or not. In particular, the folowing hold:*

*(i) If $\{x\} \neq l \cap B$, then each of the tangent lines through $x$ contains exactly one point of $B$.*

*(ii) If $\{x\} = l \cap B$, then the conic $\mathcal{C}_x = \pi_x \cap \mathcal{H}$ in $\pi_x$ is disjoint from $B$.*

*Proof.* Only (ii) needs a proof. The set $\pi_x \cap B$ is a $(\mathbb{T} \cup \mathbb{E})_{\pi_x}$-blocking set in $\pi_x$ of minimum size $q$. So $\mathcal{C}_x$ is disjoint from $\pi_x \cap B$ (and hence from $B$) by Lemma 2.3.2(a). □

**Lemma 4.5.10.** *Let $l$ and $m$ be two $\mathbb{T}_0$-lines intersecting at a point $x_0$. If $|l \cap B| = 1$ and $|m \cap B| \geqslant 2$, then $x_0 \in B$.*

*Proof.* Suppose that $x_0 \notin B$. Let $l = \{x_0, x_1, \cdots, x_q\}$. We may assume that $l \cap B = \{x_q\}$. There are $(q-1)$ $\mathbb{T}_1$-lines through $x_0$. Since $|m \cap B| \geqslant 2$, Corollary 4.5.9(i) implies that none of these lines through $x_0$ is in $\overline{\mathbb{T}}$, and so each such line meets $B$ in at least two points. This gives that the tangent plane $\pi_{x_0}$ through $l$ contains at least $2q+1$ points of $B$. By Lemma 4.5.3(ii), each of the tangent planes $\pi_{x_i}$, $1 \leqslant i \leqslant q-1$, through $l$ contains at least $q+1$ points of $B$. It follows that the planes $\pi_{x_i}$, $0 \leqslant i \leqslant q-1$, through $l$ together contain at least

$$2q + 1 + (q-1)q = q^2 + q + 1$$

points of $B$, a contradiction to Lemma 4.5.8. So $x_0 \in B$. □

**Lemma 4.5.11.** *Every $\mathbb{T}_0$-line contains one or $q$ points of $B$.*

*Proof.* Let $m = \{x_0, x_1, \cdots, x_q\}$ be a $\mathbb{T}_0$-line containing at least two points of $B$. Fix a line $l = \{y_0, y_1, \cdots, y_q\}$ of $\overline{\mathbb{T}}$. We may assume that the tangency point of $l$ is $y_0$ and $l \cap B = \{y_q\}$. Since $|m \cap B| \geqslant 2$, by Corollary 4.5.9(i), $m$ is different from the two $\mathbb{T}_0$-lines through $y_0$. Let $l_1$ be the $\mathbb{T}_0$-line through $y_0$ which intersects $m$. Then $|l_1 \cap B| = 1$, again by Corollary 4.5.9(i). We may assume that $m \cap l_1 = \{x_0\}$. Note that the conics $\mathcal{C}_{y_j} = \pi_{y_j} \cap \mathcal{H}$, $1 \leqslant j \leqslant q$, pairwise intersect at $y_0$ and each of them contains a unique point, say $x_j$, of $m \setminus \{x_0\}$. Since $l \cap B = \{y_q\}$, the conic $\mathcal{C}_{y_q}$ is disjoint from $B$ by Corollary 4.5.9(ii). So $x_q \notin B$.

We claim that $m \setminus \{x_q\}$ is contained in $B$. Since $|l_1 \cap B| = 1$, $|m \cap B| \geqslant 2$ and $m \cap l_1 = \{x_0\}$, Lemma 4.5.10 implies that $x_0 \in B$. We next show that $x_i \in B$ for $1 \leqslant i \leqslant q-1$. Let $m_i$ be the tangent line through $x_i$ and $y_i$. Since $l \cap B \neq \{y_i\}$, $m_i$ contains exactly one point of $B$ by Corollary 4.5.9(i). If $m_i \cap B \neq \{x_i\}$, then $m_i \in \overline{\mathbb{T}}$ as $x_i$ is the tangency point of $m_i$. Applying Corollary 4.5.9(i) again to the line $m_i$, it follows that every tangent line through $x_i$, and in particular, $m$ contains exactly one point of $B$. This leads to a contradiction to our assumption that $|m \cap B| \geqslant 2$. Thus $m_i \cap B = \{x_i\}$ and so $x_i \in B$. □

**Lemma 4.5.12.** *If $q \geqslant 4$, then there exists a $\mathbb{T}_0$-line containing $q$ points of $B$.*

*Proof.* By Lemma 4.5.11, it is enough to show that there exists a $\mathbb{T}_0$-line containing at least two points of $B$.

Let $\pi_x$, $x \in \mathbb{P} \setminus \mathcal{H}$, be a secant plane containing $q$ points of $B$. The existence of such a plane follows from Corollary 4.5.5. We have $x \in B$, otherwise $|\pi_x \cap B| \geqslant q+1$ by Lemma 4.5.7. The set $\pi_x \cap B$ is a $(\mathbb{T}\cup\mathbb{E})_{\pi_x}$-blocking set in $\pi_x$ of minimum size $q$. So, for some $w \in \mathcal{C}_x$, there exists three lines $l_1, l_2, l_3$ in $\pi_x$ through $w$ which are secant to $\mathcal{C}_x$ (and hence to $\mathcal{H}$) and disjoint from $\pi_x \cap B$ (and hence from $B$). This is possible by Lemma 2.3.2(b) as $q \geqslant 4$. By Lemma 4.5.6, $B$ contains a point $z_i \in \mathcal{H}$, where $\pi_{z_i}$ is one of the two tangent planes through $l_i$ for $i \in \{1, 2, 3\}$. The points $z_1, z_2, z_3$ are pair-wise distinct and each of them lies on a $\mathbb{T}_0$-line through $w$. It follows that one of the two $\mathbb{T}_0$-lines through $w$ contains at least two points of $B$. $\square$

**Lemma 4.5.13.** *If $q \geqslant 4$, then $B \cap \mathcal{H} = (l \cup m) \setminus \{x\}$ for some $\mathbb{T}_0$-lines $l$ and $m$ intersecting at $x$. In particular, $|B \cap \mathcal{H}| = 2q$ and $|B \setminus \mathcal{H}| = q^2 - q$.*

*Proof.* By Lemma 4.5.12, let $l$ be a $\mathbb{T}_0$-line containing $q$ points of $B$. Let $\{x\} = l \setminus B$ and $m$ be the other $\mathbb{T}_0$-line though $x$. We have $|m \cap B| \geqslant 1$. Since $x \notin B$, Lemma 4.5.10 implies that $|m \cap B| \geqslant 2$ and so $|m \cap B| = q$ by Lemma 4.5.11. Since $|B \cap \mathcal{H}| \leqslant 2q$, it follows that $B \cap \mathcal{H} = (l \cup m) \setminus \{x\}$. Thus $|B \cap \mathcal{H}| = 2q$ and hence $|B \setminus \mathcal{H}| = q^2 - q$ as $|B| = q^2 + q$. $\square$

**Proof of Theorem 4.5.1(ii)**

**Lemma 4.5.14.** *If $q \geqslant 4$, then $B = \pi_x \setminus \{x\}$ for some tangent plane $\pi_x$, $x \in \mathcal{H}$.*

*Proof.* By Lemma 4.5.13, let $B \cap \mathcal{H} = (l_0 \cup l_1) \setminus \{x\}$, where $l_0$ and $l_1$ are two $\mathbb{T}_0$-lines intersecting at a point $x$. We claim that $B \setminus \mathcal{H} = \pi_x \setminus \mathcal{H}$. Let $l_2, \cdots, l_q$ be the $\mathbb{T}_1$-lines through $x$. Since $|B \setminus \mathcal{H}| = q^2 - q$ by Lemma 4.5.13, it is enough to show that $l_i \cap B = l_i \setminus \{x\}$ for each $2 \leqslant i \leqslant q$. We shall apply a similar argument as in the proof of Lemma 4.3.7.

We have $1 \leqslant |l_i \cap B| \leqslant q$. If $|l_j \cap B| = 1$ for some $j$ with $2 \leqslant j \leqslant q$, then Corollary 4.5.9(i) would imply that each of the $\mathbb{T}_0$-lines $l_0$ and $l_1$ meets $B$ at exactly one point, which is not possible. So $|l_i \cap B| \geqslant 2$. Let

$$t = \min\{|l_i \cap B| : 2 \leqslant i \leqslant q\}.$$

We show that $t = q$ and this would complete the proof. Consider a line $l_k$,

$2 \leqslant k \leqslant q$, such that $|l_k \cap B| = t$. Let

$$l_k = \{x = x_0, x_1, \cdots, x_t, x_{t+1}, \cdots, x_q\},$$

where $l_k \cap B = \{x_1, \cdots, x_t\}$. Now consider the planes through $l_k$ and count the points of $B$ contained in them. The tangent plane $\pi_{x_0}$ contains at least $2q+t(q-1)$ points of $B$. By Lemma 4.5.3(i), each of the secant planes $\pi_{x_i}$, $1 \leqslant i \leqslant t$, contains at least $q - t$ points of $B$ different from the points of $l_k \cap B$. By Lemma 4.5.7, each of the remaining planes $\pi_{x_i}$, $t + 1 \leqslant i \leqslant q$, contains at least $q + 1 - t$ points of $B$ different from that of $l_k \cap B$. Thus the planes through $l_k$ together contain at least

$$2q + t(q - 1) + t(q - t) + (q - t)(q + 1 - t) = q^2 + 3q - 2t$$

points of $B$ and so $q^2 + 3q - 2t \leqslant |B| = q^2 + q$. This gives $t \geqslant q$ and so $t = q$. This completes the proof of Therem 4.5.1(ii). □

**Proof of Theorem 4.5.1(i)**

We first justify the statement of Theorem 4.5.1(i)(b). We have $|\mathbb{E}| = 2$. Let $l_1$ and $l_2$ be the two lines external to $\mathcal{H}$. Then $\mathbb{P} \setminus \mathcal{H} = l_1 \cup l_2$ and $l_i^{\zeta} = l_j$ for $\{i, j\} = \{1, 2\}$, where $\zeta$ is the symplectic polarity associated with $\mathcal{H}$. Let $\mathcal{O}$ be an ovoid of the generalized quadrangle $\mathcal{X} = (\mathbb{P}, \mathbb{T}) \simeq W(2)$ (Proposition 3.2.5). Then $|\mathcal{O}| = 5$. By [33, 1.8.4], $(|l_1 \cap \mathcal{O}|, |l_2 \cap \mathcal{O}|) = (0, 2)$ or $(2, 0)$. We may assume that $(|l_1 \cap \mathcal{O}|, |l_2 \cap \mathcal{O}|) = (2, 0)$. Then, for any $\alpha \in l_2$, it is clear that the set $B_2 = \mathcal{O} \cup \{\alpha\}$ is a $(\mathbb{T} \cup \mathbb{E})$-blocking set in $PG(3, 2)$ of size 6. Since $\mathcal{O} \cap \mathcal{H}$ is an ovoid of $\mathcal{H}$, we have $|B_2 \cap \mathcal{H}| = 3$ and so $|B_2 \setminus \mathcal{H}| = 3$.

We now prove Theorem 4.5.1(i). So assume that $q = 2$ and $|B| = 6$. We have $(|B \setminus \mathcal{H}|, |B \cap \mathcal{H}|) = (2, 4)$ or $(3, 3)$.

First assume that $(|B \setminus \mathcal{H}|, |B \cap \mathcal{H}|) = (2, 4)$. Since $q = 2$ and $|B \cap \mathcal{H}| = 4$, it is clear that there exists a $\mathbb{T}_0$-line $l_0$ containing exactly two points of $B$. Let $\{x\} = l_0 \setminus B$ and $l_1$ be the other $\mathbb{T}_0$-line though $x$. We have $1 \leqslant |l_1 \cap B| \leqslant 2$. Since $x \notin B$, Lemma 4.5.10 implies that $|l_1 \cap B| = 2$. So $B \cap \mathcal{H} = (l_0 \cup l_1) \setminus \{x\}$. Let $l_2$ be the unique $\mathbb{T}_1$-line through $x$. Then $1 \leqslant |l_2 \cap B| \leqslant 2$. Since $|\pi_x \cap B| \geqslant 5 > q + 1$, Corollary 4.5.9 implies that $l_2 \notin \overline{\mathbb{T}}$ and so $|l_2 \cap B| = 2$. Thus $B \setminus \mathcal{H} = l_2 \setminus \{x\}$ as $|B \setminus \mathcal{H}| = 2$ and hence $B = \pi_x \setminus \{x\}$.

Now assume that $(|B \setminus \mathcal{H}|, |B \cap \mathcal{H}|) = (3, 3)$. Then $B \cap \mathcal{H}$ is an ovoid of $\mathcal{H}$. There exists a unique ovoid $\mathcal{O}$ of $\mathcal{X} = (\mathbb{P}, \mathbb{T}) \simeq W(2)$ containing $B \cap \mathcal{H}$. Note that $\mathcal{O} \cap \mathcal{H} = B \cap \mathcal{H}$. We show that $\mathcal{O}$ is contained in $B$. Let $x \in \mathcal{O} \setminus (B \cap \mathcal{H})$. Suppose that $x \notin B$. Since $|\pi_x \cap B| \geqslant 3$ and the conic $\mathcal{C}_x$ is disjoint from $B \cap \mathcal{H}$, it follows that $|\pi_x \cap B| = 3$ and so $\pi_x \cap B$ is the unique line in $\pi_x$ which is external with respect to $\mathcal{C}_x$ (and hence to $\mathcal{H}$). Since $|B| = 6$, we must have $B = (B \cap \mathcal{H}) \cup (\pi_x \cap B)$. Then the other external line through $x$ is disjoint from $B$, contradicting that $B$ blocks every external line. Thus $\mathcal{O} \subseteq B$. As mentioned in the above paragraph in which the statement of Theorem 4.5.1(i)(b) is justified, exactly one of the two external lines to $\mathcal{H}$, say $l_1$, is disjoint from $\mathcal{O}$. Since $l_1$ meets $B$, let $\alpha \in l_1 \cap B$. Then, $|B| = 6$ implies that $B = \mathcal{O} \cup \{\alpha\}$. This completes the proof of Therem 4.5.1(i).

### 4.5.2 Proof of Theorem 4.5.1 for all $q \geqslant 3$

Here, we prove Theorem 4.5.1 for all $q \geqslant 3$ irrespective of $q$ even or odd.

By Corollary 4.5.5, let $\pi$ be any secant plane of $PG(3, q)$ containing exactly $q$ points of $B$. Since $\pi \cap B$ is a minimum size $(\mathbb{T} \cup \mathbb{E})_\pi$-blocking set in $\pi$, there are three possibilities for $\pi \cap B$ by Theorems 2.1.10 and 2.1.11:

(I) $\pi \cap B = l \setminus \{x_l\}$ for some tangent line $l$ contained in $\pi$;

(II) $\pi \cap B = (l \setminus \mathcal{C}_\pi) \cup \{\alpha\}$ for some secant line $l$ in $\pi$, where $\alpha$ is the pole of $l$ if $q$ is odd and the nucleus of $\mathcal{C}_\pi$ if $q$ is even;

(III) $q$ is a square and $\pi \cap B = \Pi \setminus (\Pi \cap \mathcal{C}_\pi)$, where $\Pi$ is a Baer subplane of $\pi$ such that $\Pi \cap \mathcal{C}_\pi$ is an irreducible conic in $\Pi$.

**Lemma 4.5.15.** *Possibility (I) occurs for every secant plane that contains $q$ points of $B$.*

*Proof.* Suppose that $\pi$ is a secant plane containing $q$ points of $B$ for which possibility (I) does not occur. The number of secant lines in $\pi$ that are disjoint from $\pi \cap B$ is then equal to $2(q-1)$ or $(\sqrt{q}+1)(q-\sqrt{q}) = \sqrt{q}(q-1)$ depending on whether possibility (II) or (III) occurs. Each of these secant lines is contained in exactly two tangent planes, implying that the number of points $a \in \mathcal{H} \setminus \mathcal{C}_\pi$ for which $\pi_a \cap \pi$ is a secant line disjoint from $\pi \cap B$ is equal to $4(q-1)$ or $2\sqrt{q}(q-1)$. Since $|B \cap \mathcal{H}| \leqslant 2q$ by equation (4.5.3) and $2q < \min\{4(q-1), 2\sqrt{q}(q-1)\}$ for $q \geqslant 3$, there exists a point $a^* \in \mathcal{H} \setminus (B \cup \mathcal{C}_\pi)$ such that $l^* := \pi_{a^*} \cap \pi$ is a secant line disjoint from $\pi \cap B$.

There are $q-1$ secant planes through $l^*$. For each such plane $\pi'$, the set $\pi' \cap B$ (disjoint from $l^*$) is a $(\mathbb{T} \cup \mathbb{E})_{\pi'}$-blocking set in $\pi'$ and so $\pi' \cap B$ contains at least $q$ points of $B \setminus \mathcal{H}$ by Lemma 2.3.3. As $a^* \notin B$, the tangent plane $\pi_{a^*}$ through $l^*$ contains at least $q-1$ points of $B \setminus \mathcal{H}$. Hence, $|B \setminus \mathcal{H}| \geqslant (q-1)q + q - 1 = q^2 - 1$. As $|B \setminus \mathcal{H}| \leqslant q^2 - 1$ by equation (4.5.2), we thus have that $|B \setminus \mathcal{H}| = q^2 - 1$. As $|B| \leqslant q^2 + q$ and $q + 1 \leqslant |B \cap \mathcal{H}|$ by equations (4.5.1) and (4.5.3), we then know that $|B| = q^2 + q$ and $|B \cap \mathcal{H}| = q + 1$.

As mentioned above, there are $2(q-1)$ or $\sqrt{q}(q-1)$ secant lines in $\pi$ disjoint from $\pi \cap B$. For each such secant line $l$, we know from Lemma 4.5.6 that there exists a point $a \in B \cap \mathcal{H}$ for which $\pi_a \cap \pi = l$. In this way, we get a collection of $N \in \{2(q-1), \sqrt{q}(q-1)\}$ points of $B \cap \mathcal{H}$. Since $N \leqslant |B \cap \mathcal{H}| = q+1$ and $q \geqslant 3$, we find that $q = 3$ and that possibility (II) occurs for the secant plane $\pi$.

We thus have that $q = 3$, $|B| = q^2 + q = 12$, $|B \setminus \mathcal{H}| = q^2 - 1 = 8$ and $|B \cap \mathcal{H}| = q + 1 = 4$ (so $B \cap \mathcal{H}$ is an ovoid of $\mathcal{H}$). Moreover, for each of the four secant lines $l$ contained in $\pi$ and disjoint from $\pi \cap B$, there exists a unique point $a \in B \cap \mathcal{H}$ for which $\pi_a \cap \pi = l$ and a unique point $b \in \mathcal{H} \setminus (B \cup \mathcal{C}_\pi)$ for which $\pi_b \cap \pi = l$. Among the eight points of $B \setminus \mathcal{H}$, there are two contained in $\pi_b$ and six contained in the two secant planes through $l$ (recall Lemma 2.3.3). So, the tangent plane $\pi_a$ through $l$ cannot contain further points of $B \setminus \mathcal{H}$.

As $l$ ranges over all four secant lines of $\pi$ disjoint from $\pi \cap B$, the point $a$ will range over all four points of $B \cap \mathcal{H}$. As none of the four tangent planes $\pi_a$, $a \in B \cap \mathcal{H}$, contains points of $B \setminus \mathcal{H}$, we thus have:

(∗) any $\mathbb{T}_1$-line through a point of $B \cap \mathcal{H}$ does not contain points of $B \setminus \mathcal{H}$.

For every point $x$ of $PG(3,3) \setminus \mathcal{H}$, the conic $\mathcal{C}_x$ in $x^\zeta$ is an ovoid of $\mathcal{H}$. The map $x \mapsto \mathcal{C}_x$ from $PG(3,3) \setminus \mathcal{H}$ to the set of ovoids of $\mathcal{H}$ is a bijection (see the first section of Chapter-III). Any two distinct ovoids of $\mathcal{H}$ intersect in at most two points. If $x_1$ and $x_2$ are two distinct points of $PG(3,3)\setminus\mathcal{H}$, then $x_1x_2$ is a tangent, secant or external line whenever $|\mathcal{C}_{x_1} \cap \mathcal{C}_{x_2}|$ is equal to 1, 2 or 0, respectively. By (∗), we have

For every $x \in B \setminus \mathcal{H}$, the ovoid $\mathcal{C}_x$ is disjoint from $B \cap \mathcal{H}$.

We can now label the points of $\mathcal{H}$ by $x_{ij}$, where $i, j \in \{1,2,3,4\}$, such that two distinct points $x_{ij}$ and $x_{i'j'}$ of $\mathcal{H}$ are incident with a $\mathbb{T}_0$-line if either $i = i'$ or $j = j'$.

Without loss of generality, we may suppose that $B \cap \mathcal{H} = \{x_{11}, x_{22}, x_{33}, x_{44}\}$. Then the ovoids of $\mathcal{H}$ disjoint from $B \cap \mathcal{H}$ are the following:

$$O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\}, O_2 = \{x_{13}, x_{31}, x_{24}, x_{42}\}, O_3 = \{x_{14}, x_{41}, x_{23}, x_{32}\},$$

$$O_4 = \{x_{12}, x_{24}, x_{31}, x_{43}\}, O_5 = \{x_{12}, x_{23}, x_{34}, x_{41}\}, O_6 = \{x_{13}, x_{24}, x_{32}, x_{41}\},$$

$$O_7 = \{x_{13}, x_{21}, x_{34}, x_{42}\}, O_8 = \{x_{14}, x_{21}, x_{32}, x_{43}\}, O_9 = \{x_{14}, x_{23}, x_{31}, x_{42}\}.$$

The collection $\{\mathcal{C}_x \mid x \in B \setminus \mathcal{H}\}$ consists of eight of these nine ovoids. So, one of the above ovoids is missing in this collection.

Suppose one of the ovoids $O_1$, $O_2$ and $O_3$ is missing in the above collection. Without loss of generality, we may suppose that $O_1$ is the ovoid that is missing. Since $O_4 \cap O_5 = \{x_{12}\}$ is a singleton, the two points of $PG(3,3) \setminus \mathcal{H}$ corresponding to $O_4$ and $O_5$ lie on the same $\mathbb{T}_1$-line through $x_{12}$. Then the other $\mathbb{T}_1$-line through $x_{12}$ would not contain any point of $B$, a contradiction.

Suppose one of the ovoids $O_4, O_5, \ldots, O_9$ is missing in the above collection. Without loss of generality, we may suppose that $O_4$ is the ovoid that is missing. The ovoids $O_4 = \{x_{12}, x_{24}, x_{31}, x_{43}\}$ and $O' := \{x_{11}, x_{23}, x_{34}, x_{42}\}$ are disjoint and hence correspond to points $y_4$ and $y'$ of $PG(3,3) \setminus \mathcal{H}$ such that the line $y_4 y'$ is external. Denote by $y''$ and $y'''$ the other two points of the line $y_4 y'$, and by $O''$ and $O'''$ the corresponding ovoids of $\mathcal{H}$. Then $\{O_4, O', O'', O'''\}$ is a partition of the point set of $\mathcal{H}$ in ovoids. So, these ovoids determine a partition of $B \cap \mathcal{H}$. Since $(B \cap \mathcal{H}) \cap O_4 = \emptyset$ and $|(B \cap \mathcal{H}) \cap O'| = 1$, each of the ovoids $O''$ and $O'''$ intersects $B \cap \mathcal{H}$ in 1 or 2 points. It follows that none of the points $y_4, y', y'', y'''$ belongs to $B$. This would imply that the external line $y_4 y'$ is disjoint from $B$, a contradiction. $\square$

By recycling some of the arguments in the proof of Lemma 4.5.15, we show the following.

**Lemma 4.5.16.** *We have $|B| = q^2 + q$, $|B \cap \mathcal{H}| = 2q$ and $|B \setminus \mathcal{H}| = q^2 - q$. Moreover, there exist two intersecting $\mathbb{T}_0$-lines $l_0$ and $l_1$ such that $B \cap \mathcal{H} = (l_0 \cup l_1) \setminus (l_0 \cap l_1)$.*

*Proof.* By Corollary 4.5.5 and Lemma 4.5.15, there exists a secant plane $\pi$ containing $q$ points of $B$ for which possibility (I) occurs. So, there exists a $\mathbb{T}_1$-line $m$ contained in $\pi$ such that $\pi \cap B = m \setminus \{x_m\}$. Let $l_0$ and $l_1$ denote the two $\mathbb{T}_0$-lines through $x_m$. In the plane $\pi$, there are exactly $q$ secant lines disjoint from $\pi \cap B$, and each of these lines contains the point $x_m$. For each of these secant lines $l$, there exists (by Lemma 4.5.6) a point $a \in B \cap \mathcal{H}$, necessarily belonging to $(l_0 \cup l_1) \setminus \{x_m\}$, for which $\pi_a \cap \pi = l$. In this way, we get a collection of $q \geqslant 3$ points belonging to $(l_0 \cup l_1) \cap B$. So, $B \cap \mathcal{H}$ cannot be an ovoid of $\mathcal{H}$ and hence $|B \cap \mathcal{H}| > q + 1$. As $|B| \leqslant q^2 + q$ by equation (4.5.1), this implies that $|B \setminus \mathcal{H}| < q^2 - 1$.

Suppose now there exists a point $a$ of $(l_0 \cup l_1) \setminus \{x_m\}$ which is not in $B$. Then $\pi_a \cap \pi$ is a secant line $l$ disjoint from $\pi \cap B$. Applying a similar argument as in the proof of Lemma 4.5.15, each of the $q - 1$ secant planes through $l$ contains at least $q$ points of $B \setminus \mathcal{H}$ and the tangent plane $\pi_a$ contains at least $q - 1$ points of $B \setminus \mathcal{H}$. This would again lead to the inequality $|B \setminus \mathcal{H}| \geqslant q^2 - 1$, which is impossible.

Hence, all the $2q$ points of $(l_0 \cup l_1) \setminus \{x_m\}$ belong to $B$. As $|B| \leqslant q^2 + q$, $q^2 - q \leqslant |B \setminus \mathcal{H}|$ and $|B \cap \mathcal{H}| \leqslant 2q$ by equations (4.5.1), (4.5.2) and (4.5.3) respectively, this implies that $|B \cap \mathcal{H}| = 2q$, $|B \setminus \mathcal{H}| = q^2 - q$ and $|B| = q^2 + q$. Moreover, the $2q$ points of $B \cap \mathcal{H}$ are precisely the points of $(l_0 \cup l_1) \setminus \{x_m\}$. $\square$

Invoking Lemma 4.5.16, we can now prove the following.

**Proposition 4.5.17.** *There exists a point $x$ of $\mathcal{H}$ such that $B = \pi_x \setminus \{x\}$.*

*Proof.* Since $B \setminus \mathcal{H}$ is a set of size $q^2 - q$ blocking all external lines, Theorem 4.3.1 implies that there exists a point $x \in \mathcal{H}$ such that $B \setminus \mathcal{H} = \pi_x \setminus \mathcal{H}$. Every point $y$ of $\pi_x \cap \mathcal{H}$ distinct from $x$ is contained in a $\mathbb{T}_1$-line that is not contained in $\pi_x$. As this $\mathbb{T}_1$-line contains a point of $B$, we must have $y \in B$. So, $\pi_x \setminus \{x\}$ is contained in and hence equal to $B$ (as both sets contain $q^2 + q$ points). □

## 4.6 $\mathbb{T}$-blocking sets

In this section, we shall discuss the minimum size $\mathbb{T}$-blocking sets in $PG(3, q)$. A lower bound for the sizes of $\mathbb{T}$-blocking sets is easily derived.

**Lemma 4.6.1.** *Let $B$ be a $\mathbb{T}$-blocking set in $PG(3, q)$. Then $|B| \geqslant q^2 + 1$, with equality if and only if every tangent line contains a unique point of $B$.*

*Proof.* Count the cardinality of the set $D = \{(x, l) \mid x \in B, l \in \mathbb{T}, x \in l\}$ in two ways. Each point of $B$ is contained in $q + 1$ tangent lines. This gives $|B|(q + 1) = |D|$. Since $|\mathbb{T}| = (q + 1)(q^2 + 1)$ and each tangent line contains at least one point of $B$, we get $|D| \geqslant (q + 1)(q^2 + 1)$. It follows that

$$|B| \geqslant \frac{(q + 1)(q^2 + 1)}{q + 1} = q^2 + 1.$$

Clearly, equality holds if and only if every tangent line contains a unique point of $B$. □

If $\pi$ is a tangent plane of $PG(3, q)$ with pole $x \in \mathcal{H}$, then observe that $\pi \setminus \{x\}$ is a $\mathbb{T}$-blocking set of size $q^2 + q$. As a consequence of Lemmas 4.6.1, we thus have the following.

**Corollary 4.6.2.** *If $B$ is a $\mathbb{T}$-blocking set in $PG(3, q)$ of minimum size, then $q^2 + 1 \leqslant |B| \leqslant q^2 + q$.*

The following theorem characterizes the minimum size $\mathbb{T}$-blocking sets in $PG(3,q)$, $q$ even. By Proposition 3.2.5, the point-line geometry $\mathcal{X} = (\mathbb{P}, \mathbb{T})$ is a generalized quadrangle of order $q$ isomorphic to $W(q)$.

**Theorem 4.6.3.** *Let $B$ be a $\mathbb{T}$-blocking set in $PG(3,q)$, $q$ even. Then $|B| = q^2+1$ if and only if $B$ is an ovoid of $\mathcal{X} = (\mathbb{P}, \mathbb{T}) \simeq W(q)$.*

*Proof.* We know that $W(q)$ has ovoids, each of which is of size $q^2+1$. By Lemma 4.6.1, $|B| = q^2+1$ if and only if every tangent line contains a unique point of $B$. The latter statement is equivalent to that $B$ is an ovoid of $\mathcal{X} = (\mathbb{P}, \mathbb{T}) \simeq W(q)$. $\square$

In the $q$ odd case, other than the bounds given in Corollary 4.6.2, not much general theory is known for the minimum size $\mathbb{T}$-blocking sets in $PG(3,q)$. We shall consider the case $q = 3$ in the next section.

## 4.7 $\mathbb{T}$-blocking sets in $PG(3,3)$

Throughout this section, we assume that $q = 3$. We prove the following theorem which characterizes the minimum size $\mathbb{T}$-blocking sets in $PG(3,3)$.

**Theorem 4.7.1.** *There is no $\mathbb{T}$-blocking set of size 10 in $PG(3,3)$. Up to isomorphism, there are two $\mathbb{T}$-blocking sets of size 11 in $PG(3,3)$.*

We first construct two nonisomorphic $\mathbb{T}$-blocking sets in $PG(3,3)$ each of size 11. Then we prove the nonexistence of $\mathbb{T}$-blocking sets of size 10 in $PG(3,3)$ and classify the $\mathbb{T}$-blocking sets of size 11 in $PG(3,3)$.

Let $\zeta$ be the orthogonal polarity of $PG(3,3)$ associated with $\mathcal{H}$. For a point $x \in PG(3,3) \setminus \mathcal{H}$, recall that $\mathcal{C}_x$ denotes the conic $x^\zeta \cap \mathcal{H}$ in the plane $x^\zeta$. We shall denote by $I_x$ the set of all interior points in $x^\zeta$ with respect to $\mathcal{C}_x$ and by $\mathbb{E}(x)$ the set of all lines of $PG(3,3)$ through $x$ which are external to $\mathcal{H}$.

### 4.7.1 Construction of the $\mathbb{T}$-blocking set $B_1$

Consider a point $x \in PG(3,3) \setminus \mathcal{H}$ and let $I_x = \{z_1, z_2, z_3\}$. Fix a line $l$ of $x^\zeta$ which is external to $\mathcal{C}_x$. Then $l$ contains exactly two points of $I_x$, say $z_2$ and $z_3$. Let $\bar{l}$ be the unique line in $\mathbb{E}(x)$ such that $\bar{l}$ meets $x^\zeta$ in $z_1$, see Lemma 3.2.1(ii). Define the following set:

$$B_1 := \mathcal{C}_x \cup l \cup \Big(\bar{l} \setminus \{x\}\Big).$$

We prove the following:

**Proposition 4.7.2.** *$B_1$ is a $\mathbb{T}$-blocking set of size 11 in $PG(3,3)$.*

*Proof.* Clearly, $|B_1| = 11$. Let $A = x^\zeta \setminus B_1$. Then $A$ consists of four exterior points, each of which is different from the two exterior points contained in $l$. Since every tangent line meets $x^\zeta$, it is enough to prove that each $\mathbb{T}_1$-line through a point of $A$ meets $B_1$.

Take a point $a \in A$ and a $\mathbb{T}_1$-line $t$ through $a$. If $t$ is contained in $x^\zeta$, then observe that $t$ meets $B_1$ in two points, one from $\mathcal{C}_x$ and the other one is an exterior point contained in $l$. So assume that $t$ is not contained in $x^\zeta$. We show that $t$ contains a point of $B_1 \setminus x^\zeta = \bar{l} \setminus \{x, z_1\}$.

Let $m$ be the line of $x^\zeta$ through $a$ and $z_1$. Then $m$ is either external or secant to $\mathcal{C}_x$ in $x^\zeta$, as it contains the interior point $z_1$. Since $m$ has to intersect the external line $l$ of $x^\zeta$ in a point different from $a$ and $z_1$, it follows that $m$ can not be secant to $\mathcal{C}_x$. So $m$ is external to $\mathcal{C}_x$ in $x^\zeta$ and hence contains an interior point different from $z_1$. Without loss, we may assume that $m$ contains $z_2$ as the second interior point.

Setting $\pi_1 = x^\zeta$ and taking the external line $m$ of $\pi_1$ in Lemma 3.2.4, we get a secant plane $\pi_2$ containing $m$ such that $z_1, z_2$ are interior points and $a$ is an exterior point in $\pi_2$ with respect to the conic $\mathcal{C}_{\pi_2}$. Note that $t$ is a $\mathbb{T}_1$-line through $a$ in $\pi_2$.

Let $\overline{m}$ ($\neq m$) be the second line of $\pi_2$ through $z_1$ which is external to $\mathcal{C}_{\pi_2}$. Out of the three lines through $z_1$ external to $\mathcal{H}$, the line $m$ is common to both the planes $\pi_1 = x^\zeta$ and $\pi_2$. The plane $x^\zeta$ contains one more external line through $z_1$. So $\overline{m}$ must be the external line through $x$ which corresponds to the point $z_1$ under the map defined in Lemma 3.2.1(ii). It follows that $\overline{m} = \bar{l}$. As $xz_1$ and $xz_2$ are external lines of $\pi_2$ (by Lemma 3.2.1(ii)), $x$ must be interior to $\mathcal{C}_{\pi_2}$ in $\pi_2$. Since the $\mathbb{T}_1$-line $t$ and the external line $\bar{l}$ of $\pi_2$ meet in a point exterior to $\mathcal{C}_{\pi_2}$, it follows that $t$ contains a point of $\bar{l} \setminus \{x, z_1\}$. This completes the proof. □

### 4.7.2 Construction of the $\mathbb{T}$-blocking set $B_2$

Fix a point $x \in PG(3,3) \setminus \mathcal{H}$ and let $I_x = \{z_1, z_2, z_3\}$. Let $y$ be a point in $x^\zeta$ exterior to $\mathcal{C}_x$. Let $l_1$ and $l_2$ be the two $\mathbb{T}_1$-lines through $y$ which are not contained in $x^\zeta$. For $i \in \{1,2\}$, let $w_i$ be the tangency point of $l_i$ in $\mathcal{H}$. Define the following set:

$$B_2 := \mathcal{C}_x \cup I_x \cup \Big(l_1 \setminus \{y, w_1\}\Big) \cup \Big(l_2 \setminus \{y, w_2\}\Big).$$

We prove the following:

**Proposition 4.7.3.** *$B_2$ is a $\mathbb{T}$-blocking set of size 11 in $PG(3,3)$.*

*Proof.* Clearly, $|B_2| = 11$. Let $D = x^\zeta \setminus B_2$. Then $D$ consists of the six exterior points in $x^\zeta$ with respect to $\mathcal{C}_x$. Since every tangent line meets $x^\zeta$, it is enough to prove that each $\mathbb{T}_1$-line through a point of $D$ meets $B_2$.

Take a point $a \in D$ and a $\mathbb{T}_1$-line $t$ through $a$. If $t$ is contained in $x^\zeta$, then $t$ meets $B_2$ in the unique point of $t \cap \mathcal{C}_x$. So assume that $t$ is not contained in $x^\zeta$. If $a = y$, then $t$ is either $l_1$ or $l_2$ and hence meets $B_2$ at two points. Assume that $a \neq y$. Since both $a$ and $y$ are exterior to $\mathcal{C}_x$, the line $m := ay$ in $x^\zeta$ is either tangent or external to $\mathcal{C}_x$.

**Case I**: $m$ is tangent to $\mathcal{C}_x$. Let $\pi$ be the secant plane generated by the lines $t$ and $m$. Then the point $y$ in $\pi$ is exterior with respect to the conic $\mathcal{C}_\pi$. So there exists one more $\mathbb{T}_1$-line in $\pi$ (different from $m$) containing $y$. Since $\pi \cap x^\zeta = m$, it follows that either $l_1$ or $l_2$ is a line of $\pi$. Without loss, we may assume that $l_1$ is a line of $\pi$. The lines $t$ and $l_1$ intersect in $\pi$ in a point different from $y$ and $w_1$. So $t$ meets $B_2$ at a point of $l_1 \setminus \{y, w_1\}$.

**Case II**: $m$ is external to $\mathcal{C}_x$. Setting $\pi_1 = x^\zeta$ and taking the external line $m$ of $\pi_1$ in Lemma 3.2.4, we get a secant plane $\pi_2$ through $m$ containing the lines $t, l_1$ and $l_2$. Now it can be seen that $t$ intersects $l_1$ (respectively, $l_2$) in $\pi_2$ at a point different from $y$ and $w_1$ (respectively, $w_2$). So $t$ meets $B_2$ at two points, one from $l_1 \setminus \{y, w_1\}$ and one from $l_2 \setminus \{y, w_2\}$.

Thus $B_2$ is a $\mathbb{T}$-blocking set in $PG(3,3)$ of size 11. □

### 4.7.3 $B_1$ and $B_2$ are nonisomorphic

The following proposition proves that the $\mathbb{T}$-blocking sets $B_1$ and $B_2$ in $PG(3,3)$ of size 11 each constructed in Sections 4.7.1 and 4.7.2 are nonisomorphic.

**Proposition 4.7.4.** *The two blocking sets $B_1$ and $B_2$ are nonisomorphic.*

*Proof.* Write $B_2$ as a disjoint union $B_2 = (B_2 \cap x^\zeta) \cup (B_2 \setminus x^\zeta)$. Observe that any line meets $B_2 \setminus x^\zeta$ in at most two points. Let $k$ be a line external to $\mathcal{H}$. If $k$ is a line of $x^\zeta$, then $k$ meets $B_2$ at exactly two points of $B_2 \cap x^\zeta$ (which come from $I_x$) and is disjoint from $B_2 \setminus x^\zeta$. Suppose that $k$ is not a line of $x^\zeta$. Then $k$ contains at most one point from $B_2 \cap x^\zeta$ and at most two points from $B_2 \setminus x^\zeta$. So $k$ is not contained in $B_2$. Thus every external line meets $B_2$ in at most three points.

However, from the construction of $B_1$, it is clear that $B_1$ contains a line external to $\mathcal{H}$. So $B_1$ and $B_2$ are nonisomorphic. □

### 4.7.4 $\mathbb{T}$-blocking sets of sizes 10 and 11 in $PG(3,3)$

We label the points of the hyperbolic quadric $\mathcal{H}$ in $PG(3,3)$ by $x_{ij}$ where $i,j \in \{1,2,3,4\}$ such that two distinct points $x_{ij}$ and $x_{i'j'}$ of $\mathcal{H}$ are incident with a $\mathbb{T}_0$-line if either $i=i'$ or $j=j'$. Recall that, since $q=3$, *the set of conics contained in $\mathcal{H}$ coincides with the set of ovoids of $\mathcal{H}$.* This allows us to use the words 'ovoid' and 'conic' interchangeably.

We denote by $O^*$ the ovoid $\{x_{11},x_{22},x_{33},x_{44}\}$ of $\mathcal{H}$. There are nine ovoids of $\mathcal{H}$ that are disjoint from $O^*$. These are:

$$O_1=\{x_{12},x_{21},x_{34},x_{43}\}, O_2=\{x_{13},x_{31},x_{24},x_{42}\}, O_3=\{x_{14},x_{41},x_{23},x_{32}\},$$

$$O_4=\{x_{12},x_{24},x_{43},x_{31}\}, O_5=\{x_{12},x_{23},x_{34},x_{41}\}, O_6=\{x_{13},x_{24},x_{32},x_{41}\},$$

$$O_7=\{x_{13},x_{21},x_{34},x_{42}\}, O_8=\{x_{14},x_{21},x_{32},x_{43}\}, O_9=\{x_{14},x_{23},x_{31},x_{42}\}.$$

**Lemma 4.7.5.** *There are four collections, each of six ovoids from $\{O_1,O_2,\dots,O_9\}$, such that every point of $\mathcal{H}\setminus O^*$ is contained in precisely two ovoids of a given collection. These four collections are*

$$\mathcal{G}^* = \{O_4,O_5,O_6,O_7,O_8,O_9\}, \{O_1,O_2,O_5,O_6,O_8,O_9\},$$
$$\{O_1,O_3,O_4,O_6,O_7,O_9\} \text{ and } \{O_2,O_3,O_4,O_5,O_7,O_8\}.$$

*Proof.* It is easily verified that each of these four collections satisfies the required condition. Conversely, suppose that $\mathcal{G}\neq\mathcal{G}^*$ is a collection of six ovoids satisfying the condition of the lemma. As $\mathcal{G}\neq\mathcal{G}^*$, at least one of $O_1,O_2,O_3$ is contained in $\mathcal{G}$. Now, any partition of $\mathcal{H}\setminus O^*$ in three ovoids must contain either one or three ovoids of the set $\{O_1, O_2, O_3\}$, implying that at least one of $O_1, O_2, O_3$ is not contained in $\mathcal{G}$.

Suppose $O_1 \in \mathcal{G}$ and $O_2 \notin \mathcal{G}$. As each of $x_{13}, x_{31}$ should be contained in two ovoids of $\mathcal{G}$, we then must have $O_4, O_6, O_7, O_9 \in \mathcal{G}$. At this stage, $x_{12}$ and $x_{21}$ are already contained in two ovoids of the collection $\mathcal{G}$, implying that $O_5$ and $O_8$ do not belong to $\mathcal{G}$. So, $\mathcal{G}$ is necessarily equal to $\{O_1, O_3, O_4, O_6, O_7, O_9\}$.

By symmetry we then see that $\mathcal{G}$ always contains precisely two ovoids of the set $\{O_1, O_2, O_3\}$. If $O_1, O_2 \in \mathcal{G}$ and $O_3 \notin \mathcal{G}$, then a similar reasoning as above shows that $\mathcal{G} = \{O_1, O_2, O_5, O_6, O_8, O_9\}$. Similarly, if $O_2, O_3 \in \mathcal{G}$ and $O_1 \notin \mathcal{G}$, then $\mathcal{G} = \{O_2, O_3, O_4, O_5, O_7, O_8\}$. □

Invoking Lemma 4.7.5, the verification of the following lemma is straightforward.

**Lemma 4.7.6.** *Suppose $\mathcal{G}$ is a collection of six ovoids from $\{O_1, O_2, \dots, O_9\}$ such that every point of $\mathcal{H} \setminus O^*$ is contained in precisely two ovoids of $\mathcal{G}$. Let $S$ denote the set of all points $x \in \mathcal{H} \setminus O^*$ such that $\{x\}$ is the intersection of two distinct ovoids of $\mathcal{G}$. Then $S = \mathcal{H} \setminus O^*$ if $\mathcal{G} = \mathcal{G}^*$, and $S = O$ if $\mathcal{G} \neq \mathcal{G}^*$, where $O$ is the unique element of $\{O_1, O_2, O_3\}$ not contained in $\mathcal{G}$.*

**Lemma 4.7.7.** *Let $x$ be a point of $\mathcal{H}$ and let $l_1 = \{x, y_1, y_2, y_3\}$ and $l_2 = \{x, z_1, z_2, z_3\}$ be the two $\mathbb{T}_1$-lines through $x$. Then the following hold:*

(1) *$\{\mathcal{C}_{y_1}, \mathcal{C}_{y_2}, \mathcal{C}_{y_3}\}$ (resp. $\{\mathcal{C}_{z_1}, \mathcal{C}_{z_2}, \mathcal{C}_{z_3}\}$) is a set of ovoids of $\mathcal{H}$ through $x$ partitioning the set of points of $\mathcal{H}$ noncollinear with $x$.*

(2) *If $i, j \in \{1, 2, 3\}$, then $\mathcal{C}_{y_i} \cap \mathcal{C}_{z_j}$ contains precisely two points (one of which is $x$).*

*Proof.* (1) As $l_1$ is a $\mathbb{T}_1$-line, we see that $x \in \mathcal{C}_{y_i}$ for every $i \in \{1, 2, 3\}$. Now, take an arbitrary point $u \in \mathcal{H}$ noncollinear with $x$. Then $u^\zeta$ does not contain $x$ and so intersects $l_1$ in a unique point $y_i$. The point $y_i$ is the unique point $v$ of $l_1 \setminus \{x\}$ for which $u \in v^\zeta$. So, $\{\mathcal{C}_{y_1}, \mathcal{C}_{y_2}, \mathcal{C}_{y_3}\}$ partitions the set of points of $\mathcal{H}$ noncollinear with $x$. A similar argument holds for the line $l_2$.

(2) There are six ovoids through the point $x$. One coincides with $\mathcal{C}_{y_i}$, two ($\mathcal{C}_{y_r}$, $\mathcal{C}_{y_s}$) intersect $\mathcal{C}_{y_i}$ in $\{x\}$ where $\{i, r, s\} = \{1, 2, 3\}$, and the remaining three (necessarily $\mathcal{C}_{z_1}$, $\mathcal{C}_{z_2}$, $\mathcal{C}_{z_3}$) intersect $\mathcal{C}_{y_i}$ in two points (one of which is $x$). □

### Nonexistence of $\mathbb{T}$-blocking sets of size 10

The following result proves the nonexistence of $\mathbb{T}$-blocking sets of size 10 in $PG(3,3)$.

**Proposition 4.7.8.** *There are no $\mathbb{T}$-blocking sets of size 10 in $PG(3,3)$.*

*Proof.* Suppose $X$ is a $\mathbb{T}$-blocking sets of size 10 in $PG(3,3)$. By Lemma 4.6.1, we then know that each tangent line contains a unique point of $X$. In particular, $O := X \cap \mathcal{H}$ is an ovoid of $\mathcal{H}$ and $Y := X \setminus \mathcal{H}$ is a set of 6 points outside $\mathcal{H}$. Without loss of generality, we may suppose that $O = O^* = \{x_{11}, x_{22}, x_{33}, x_{44}\}$. We show the following properties for the collection $\mathcal{G} = \{\mathcal{C}_y \mid y \in Y\}$ of six ovoids:

(a) all ovoids of $\mathcal{G}$ are disjoint from $O$;

(b) any two ovoids of $\mathcal{G}$ cannot intersect in a singleton;

(c) every point of $\mathcal{H} \setminus O$ is contained in precisely two ovoids of $\mathcal{G}$.

If $\mathcal{C}_y$ with $y \in Y$ contains a point $x \in O$, then the tangent line $xy$ would contain two points of $X = O \cup Y$, namely $x$ and $y$, a contradiction. If $\mathcal{C}_{y_1} \cap \mathcal{C}_{y_2}$ is a singleton $\{x\}$, where $y_1, y_2 \in Y$ with $y_1 \neq y_2$, then Lemma 4.7.7 would imply that there is a $\mathbb{T}_1$-line through $x$ containing $y_1$ and $y_2$, a contradiction. Finally, every point $x \in \mathcal{H} \setminus O$ is contained in two $\mathbb{T}_1$-lines, each containing exactly one point of $Y$, showing that $x$ is contained in precisely two ovoids of $\mathcal{G}$.

By Lemmas 4.7.5 and 4.7.6, we however know that there are no collections $\mathcal{G}$ of six ovoids that satisfy the above properties (a), (b) and (c). Therefore, there is no $\mathbb{T}$-blocking set of size 10 in $PG(3,3)$. □

**Classification of the $\mathbb{T}$-blocking sets of size 11**

In the rest of this section, we classify the $\mathbb{T}$-blocking sets of size 11 in $PG(3,3)$. We show that there are only two such $\mathbb{T}$-blocking sets up to isomorphism, necessarily isomorphic to the blocking sets $B_1$ and $B_2$ constructed in Sections 4.7.1 and 4.7.2.

**Lemma 4.7.9.** *If $X$ is a $\mathbb{T}$-blocking set of size 11 in $PG(3,3)$, then $|X \setminus \mathcal{H}| \in \{6,7\}$ and $|X \cap \mathcal{H}| \in \{4,5\}$.*

*Proof.* Since $|X \cap \mathcal{H}| \leqslant |X| < 12$, there exists a line $l$ in $\mathcal{H}$ meeting $X$ in either 1 or 2 points. Suppose every line of $\mathcal{H}$ meets $X$ in 2 points. Then $|X \cap \mathcal{H}| = 8$. If $l$ is a line of $\mathcal{H}$ and $l \setminus X = \{a,b\}$, then each of the four $\mathbb{T}_1$-lines meeting $\{a,b\}$ contains at least one point of $X \setminus \mathcal{H}$. Any collection of four points of $X \setminus \mathcal{H}$ that arise in this way are mutually distinct, implying that $|X| = |X \cap \mathcal{H}| + |X \setminus \mathcal{H}| \geqslant 8 + 4 = 12$, which is a contradiction.

Hence, there exists a line $l$ in $\mathcal{H}$ meeting $X$ in a unique point. If $l \setminus X = \{a,b,c\}$, then there are six $\mathbb{T}_1$-lines meeting $\{a,b,c\}$ and each of these six $\mathbb{T}_1$-lines contains at least one point of $X \setminus \mathcal{H}$. Any collection of six points of $X \setminus \mathcal{H}$ that arise in this way are mutually distinct, implying that $|X \setminus \mathcal{H}| \geqslant 6$. As $|X \cap \mathcal{H}| \geqslant 4$, we thus have that $|X \setminus \mathcal{H}| \in \{6,7\}$ and $|X \cap \mathcal{H}| \in \{4,5\}$. □

**Proposition 4.7.10.** *If $X$ is a $\mathbb{T}$-blocking set of size 11 in $PG(3,3)$, then $|X \cap \mathcal{H}| = 4$ and $|X \setminus \mathcal{H}| = 7$.*

*Proof.* Suppose that this is not the case. Then $|X \cap \mathcal{H}| = 5$ and $|X \setminus \mathcal{H}| = 6$ by Lemma 4.7.9. As each $\mathbb{T}_0$-line contains a point of $X$, there are precisely two $\mathbb{T}_0$-lines $l_1$ and $l_2$ that contain exactly two points of $X$ (while every other $\mathbb{T}_0$-line intersects $X$ in a singleton). The lines $l_1$ and $l_2$ belong to distinct parallel classes of lines of $\mathcal{H}$. We distinguish two cases.

**Case I.** The singleton $l_1 \cap l_2$ is not contained in $X$. Without loss of generality, we may suppose that $X \cap \mathcal{H} = \{x_{12}, x_{13}, x_{21}, x_{31}, x_{44}\}$. The reasoning in Lemma 4.7.9 leading to the inequality $|X \setminus \mathcal{H}| \geqslant 6$ shows that if $l$ is a $\mathbb{T}_0$-line meeting $X$ in a singleton, then any $\mathbb{T}_1$-line meeting $l \setminus X$ cannot contain more than one point of $X$, and any $\mathbb{T}_1$-line meeting $l \cap X$ cannot contain a point of $X \setminus \mathcal{H}$. As any point of $\mathcal{H} \setminus \{x_{11}\}$ is contained in a $\mathbb{T}_0$-line intersecting $X$ in a singleton, we thus see from Lemma 4.7.7 that any two ovoids $\mathcal{C}_{y_1}$ and $\mathcal{C}_{y_2}$, where $y_1, y_2 \in X \setminus \mathcal{H}$, cannot intersect in a singleton distinct from $\{x_{11}\}$. Also, no ovoid $\mathcal{C}_y$ with $y \in X \setminus \mathcal{H}$ can contain a point of $X \cap \mathcal{H}$. It can be seen that there are exactly six ovoids of $\mathcal{H}$ disjoint from $X \cap \mathcal{H}$ and so these ovoids are precisely the six ovoids $\mathcal{C}_y$, where $y \in X \setminus \mathcal{H}$. But that is impossible as two of these ovoids, namely $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ and $\{x_{14}, x_{23}, x_{32}, x_{41}\}$, intersect in the singleton $\{x_{23}\} \neq \{x_{11}\}$.

**Case II.** The singleton $l_1 \cap l_2$ is contained in $X$. Without loss of generality, we may suppose that $X \cap \mathcal{H} = O^* \cup \{x_{12}\}$. The reasoning in Lemma 4.7.9 leading to the inequality $|X \setminus \mathcal{H}| \geqslant 6$ shows that if $l$ is a $\mathbb{T}_0$-line meeting $\mathcal{H} \cap X$ in a singleton, then each of the $\mathbb{T}_1$-lines meeting $l \setminus X$ cannot contain more than one point of $X$. As any point of $\mathcal{H} \setminus \{x_{12}\}$ is contained in a line of $\mathcal{H}$ intersecting $X$ in a singleton, we thus see from Lemma 4.7.7 that any two ovoids $\mathcal{C}_{y_1}$ and $\mathcal{C}_{y_2}$, where $y_1, y_2 \in X \setminus \mathcal{H}$, cannot intersect in a singleton distinct from $\{x_{12}\}$.

Put $\mathcal{G} = \{\mathcal{C}_y \,|\, y \in X \setminus \mathcal{H}\}$. Then $\mathcal{G}$ is a set of six ovoids of $\mathcal{H}$, no two of which intersect in a singleton distinct from $\{x_{12}\}$. Moreover, each point $x \in \mathcal{H} \setminus X$ is contained in precisely two $\mathbb{T}_1$-lines and hence in precisely two ovoids of $\mathcal{G}$.

We count the number of line-point pairs $(l, x)$, where $l$ is a $\mathbb{T}_1$-line disjoint from $X \cap \mathcal{H}$ and $x \in l \cap X$. There are $|\mathcal{H} \setminus X| \cdot 2 = 22$ possibilities for $l$, and each such $l$ contains a unique point of $X$, implying that there are 22 such pairs. On the other hand, there are 6 possibilities for $x \in X \setminus \mathcal{H}$.

Since $6 \cdot 3 = 18$, there are at least $22 - 18 = 4$ points of $X \setminus \mathcal{H}$ whose induced ovoids are disjoint from $\mathcal{H} \cap X$. The following are six ovoids of $\mathcal{H}$ that are disjoint from $X \cap \mathcal{H}$:

$$A_1 = \{x_{13}, x_{24}, x_{31}, x_{42}\}, \qquad A_2 = \{x_{14}, x_{23}, x_{32}, x_{41}\},$$

$$A_3 = \{x_{13}, x_{21}, x_{34}, x_{42}\}, \qquad A_4 = \{x_{13}, x_{24}, x_{32}, x_{41}\},$$

$$A_5 = \{x_{14}, x_{23}, x_{31}, x_{42}\}, \qquad A_6 = \{x_{14}, x_{21}, x_{32}, x_{43}\}.$$

Among the six ovoids that we have to choose for the set $\mathcal{G}$, at least four come from the collection $\{A_1, A_2, \ldots, A_6\}$. As exactly two of the six ovoids of $\mathcal{G}$ contain $x_{13}$, at most two of $A_1, A_3, A_4$ can occur in $\mathcal{G}$. Similarly, by considering the point $x_{14}$, we see that at most two of $A_2$, $A_5$, $A_6$ can occur in $\mathcal{G}$. We can conclude that precisely two of $A_1, A_3, A_4$, as well as precisely two of $A_2, A_5, A_6$ belong to $\mathcal{G}$. As $A_3 \cap A_4$ and $A_5 \cap A_6$ are singletons distinct from $\{x_{12}\}$, the ovoids $A_1$ and $A_2$ must belong to $\mathcal{G}$. Then the fact that $A_3 \cap A_5$, $A_3 \cap A_6$ and $A_4 \cap A_6$ are singletons distinct from $\{x_{12}\}$ implies that $A_3$ and $A_6$ cannot belong to $\mathcal{G}$. So, $\mathcal{G}$ certainly contains the ovoids $A_1$, $A_2$, $A_4$ and $A_5$.

We still need to find two additional ovoids for $\mathcal{G}$. As the points $x_{21}$, $x_{34}$ and $x_{43}$ are not contained in $A_1 \cup A_2 \cup A_4 \cup A_5$ and need to be covered twice, each of these two ovoids should contain these points. But that is impossible as there is only one ovoid containing these three points, namely $\{x_{12}, x_{21}, x_{34}, x_{43}\}$. This completes the proof. □

In the sequel, we suppose that $X$ is a set of 11 points of $PG(3,3)$ that is a $\mathbb{T}$-blocking set. Then $|X \cap \mathcal{H}| = 4$ and $|X \setminus \mathcal{H}| = 7$ by Proposition 4.7.10. In fact, $U_1 := X \cap \mathcal{H}$ is an ovoid of $\mathcal{H}$. Denote by $U_2$ the subset of $\mathcal{H}$ consisting of the following points:

- points of $X \cap \mathcal{H}$ contained in a $\mathbb{T}_1$-line that contains points of $X \setminus \mathcal{H}$,
- points of $\mathcal{H} \setminus X$ contained in a $\mathbb{T}_1$-line that contains at least two points of $X \setminus \mathcal{H}$.

**Lemma 4.7.11.** *The set $U_2$ is an ovoid of $\mathcal{H}$.*

*Proof.* Let $l$ be a line of $\mathcal{H}$ and put $\{z_l\} := l \cap U_1$. For every $y \in X \setminus \mathcal{H}$ denote by $y'$ the unique point of $l \cap \mathcal{C}_y$, that is, the unique point $y'$ of $l$ for which $yy'$ is a $\mathbb{T}_1$-line. Each $\mathbb{T}_1$-line meeting $l \setminus \{z_l\}$ contains at least one point of $X \setminus \mathcal{H}$, and so each point of $l \setminus \{z_l\}$ is the image of at least two points of $X \setminus \mathcal{H}$ under the map $y \mapsto y'$. So, precisely one of the following two cases occurs:

(a) The point $z_l$ is the image of precisely one point of $X \setminus \mathcal{H}$ and each of the three points of $l \setminus \{z_l\}$ is the image of precisely two points of $X \setminus \mathcal{H}$.

(b) There exists a unique point $z_l'$ on $l \setminus \{z_l\}$ which is the image of precisely three points of $X \setminus \mathcal{H}$, each of the two remaining points of $l \setminus \{z_l\}$ is the image of precisely two points of $X \setminus \mathcal{H}$. In this case, the point $z_l$ itself is not the image of any point of $X \setminus \mathcal{H}$.

In case (a), we see that $z_l$ is the unique point of $U_2$ on $l$. In case (b), we see that $z_l'$ is the unique point of $U_2$ on $l$. Since $l \cap U_2$ is always a singleton, we conclude that $U_2$ must be an ovoid of $\mathcal{H}$. □

Now, if $\mathcal{G}$ is the collection of the seven ovoids $\mathcal{C}_y$, where $y \in X \setminus \mathcal{H}$, then the following properties hold:

(P1) No point of $U_1 \setminus U_2$ is contained in an ovoid of $\mathcal{G}$.

(P2) Every point of $U_1 \cap U_2$ is contained in precisely one ovoid of $\mathcal{G}$.

(P3) Every point of $\mathcal{H} \setminus (U_1 \cup U_2)$ is contained in precisely two ovoids of $\mathcal{G}$.

(P4) Every point of $U_2 \setminus U_1$ is contained in precisely three ovoids of $\mathcal{G}$.

(P5) No two ovoids of $\mathcal{G}$ intersect in a singleton $\{x\}$, where $x \in \mathcal{H} \setminus (U_1 \cup U_2)$.

(P6) No three ovoids of $\mathcal{G}$ can mutually intersect in the same singleton $\{x\}$, where $x \in U_2 \setminus U_1$.

**Proposition 4.7.12.** *Suppose that $U_1$ and $U_2$ are two (not necessarily distinct) ovoids of $\mathcal{H}$. Let $Y$ be a set of seven points of $PG(3,3) \setminus \mathcal{H}$ and put $\mathcal{G} := \{\mathcal{C}_y \,|\, y \in Y\}$. If $\mathcal{G}$ satisfies the properties (P1) – (P6) above, then $U_1 \cup Y$ is a $\mathbb{T}$-blocking set of size 11 in $PG(3,3)$.*

*Proof.* We have $|U_1 \cup Y| = 11$. Since $U_1$ is an ovoid of $\mathcal{H}$, every $\mathbb{T}_0$-line meets $U_1$ at a unique point. Every $\mathbb{T}_1$-line through a point of $U_1$ obviously meets $U_1$. By (P4) and (P6), every $\mathbb{T}_1$-line through a point of $U_2 \setminus U_1$ contains a point of $Y$. By (P3) and (P5), every $\mathbb{T}_1$-line through a point of $\mathcal{H} \setminus (U_1 \cup U_2)$ contains a point of $Y$. □

We now use the above result to classify the $\mathbb{T}$-blocking sets of size 11 in $PG(3,3)$. We assume that $U_1$ and $U_2$ are two ovoids of $\mathcal{H}$ and that $\mathcal{G}$ is a collection of seven ovoids of $\mathcal{H}$ satisfying the properties (P1) – (P6) above. If $Y$ is the set of seven points of $PG(3,3) \setminus \mathcal{H}$ for which the collection $\{\mathcal{C}_y \,|\, y \in Y\}$ coincides with $\mathcal{G}$, then $X = U_1 \cup Y$ is a $\mathbb{T}$-blocking set of size 11 by Proposition 4.7.12.

Without loss of generality, we may suppose that $U_1 = O^* = \{x_{11}, x_{22}, x_{33}, x_{44}\}$. Then the nine ovoids disjoint from $U_1 = \{x_{11}, x_{22}, x_{33}, x_{44}\}$ are $O_1, O_2, \ldots, O_9$ as defined earlier.

The ovoid $U_2$ can have five positions with respect to $U_1$ (up to isomorphism):

I: $U_2 = \{x_{11}, x_{22}, x_{33}, x_{44}\} = U_1$,

II: $U_2 = \{x_{11}, x_{22}, x_{34}, x_{43}\}$,

III: $U_2 = \{x_{11}, x_{23}, x_{34}, x_{42}\}$,

IV: $U_2 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$,

V: $U_2 = \{x_{12}, x_{23}, x_{34}, x_{41}\}$.

**Treatment of Case I**

In this case, (P2) implies that the points of $U_1 \cap U_2 = U_1 = U_2$ are partitioned by certain ovoids of $\mathcal{G}$. The partition has shape 4, $2+2$, $2+1+1$ or $1+1+1+1$, leading to four subcases.

I(a) Suppose the mentioned partition has shape 4. Then $U_1 = U_2 \in \mathcal{G}$. Again (P2) implies that every ovoid of $\mathcal{G} \setminus \{U_1\}$ is disjoint from $U_1 = U_2$. By (P3), $\mathcal{G} \setminus \{U_1\}$ is a collection of six ovoids as in Lemma 4.7.5. A contradiction is then readily obtained from Lemma 4.7.6 and property (P5).

I(b) Suppose the mentioned partition has shape $2+2$. Without loss of generality, we may suppose that $\{x_{11}, x_{22}, x_{34}, x_{43}\}$ and $\{x_{33}, x_{44}, x_{12}, x_{21}\}$ belong to $\mathcal{G}$. By (P2), each of the remaining five ovoids of $\mathcal{G}$ is disjoint from $U_1 = U_2$. So we need to find five additional ovoids from the collection $\{O_1, O_2, \ldots, O_9\}$. By (P3) and (P5), the second ovoid of $\mathcal{G}$ through $x_{12}$ must contain $x_{21}$ and therefore be equal to $O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. As $x_{12}$, $x_{21}$, $x_{34}$ and $x_{43}$ have already been covered twice, the remaining four ovoids should be contained in $\{x_{13}, x_{14}, x_{23}, x_{24}, x_{31}, x_{32}, x_{41}, x_{42}\}$ and hence equal to $O_2$, $O_3$, $O_6$ and $O_9$. One readily verifies that the collection consisting of the seven ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, $O_1$, $O_2$, $O_3$, $O_6$ and $O_9$ satisfies the properties (P1) – (P6).

I(c) Suppose the mentioned partition has shape $2+1+1$. Without loss of generality, we may suppose that $\{x_{11}, x_{22}, x_{34}, x_{43}\}$ is present in $\mathcal{G}$. Then the ovoid $\{x_{12}, x_{21}, x_{33}, x_{44}\}$ is not in $\mathcal{G}$. By (P3) and (P5), the second ovoid of $\mathcal{G}$ through $x_{34}$ must contain $x_{43}$ and hence coincides with $O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. Note that each of $x_{34}, x_{43}$ has now been covered twice, while each of $x_{12}$ and $x_{21}$ only once. Therefore the second ovoid of $\mathcal{G}$ through $x_{12}$, which cannot intersect $\{x_{12}, x_{21}, x_{34}, x_{43}\}$ in a singleton, must also contain $x_{21}$. But that is impossible as the two ovoids through $\{x_{12}, x_{21}\}$, namely $O_1$ and $\{x_{12}, x_{21}, x_{33}, x_{44}\}$ are already forbidden.

I(d) Suppose the mentioned partition has shape $1+1+1+1$. Without loss of generality, we may suppose that $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ belongs to $\mathcal{G}$. Each $y \in \{x_{23}, x_{34}, x_{42}\}$ is contained in a second ovoid of $\mathcal{G}$ which meets $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ in a second point $y' \in \{x_{23}, x_{34}, x_{42}\}$. But then the pairs $\{y, y'\}$ would partition $\{x_{23}, x_{34}, x_{42}\}$, an obvious contradiction.

### Treatment of Case II

We have $U_2 = \{x_{11}, x_{22}, x_{34}, x_{43}\}$. If $U_2 \in \mathcal{G}$, then by (P1) – (P4), $\mathcal{G} \setminus \{U_2\}$ is a collection of six ovoids as in Lemma 4.7.5. A contradiction is then readily obtained from Lemma 4.7.6 and property (P5). So, $U_2 \notin \mathcal{G}$. By (P1) and (P2), it follows that the unique ovoid of $\mathcal{G}$ containing $x_{11}$ is either $\{x_{11}, x_{23}, x_{34}, x_{42}\}$

or $\{x_{11}, x_{24}, x_{32}, x_{43}\}$. In view of the symmetry $3 \leftrightarrow 4$, we may without loss of generality suppose that $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ is the unique ovoid of $\mathcal{G}$ containing $x_{11}$. There are still six ovoids to choose for $\mathcal{G}$, one of them contains $x_{22}$ and the other five are contained in the collection $\{O_1, O_2, \ldots, O_9\}$. None of these six ovoids can intersect $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ in the singleton $\{x_{23}\}$ or the singleton $\{x_{42}\}$, implying that $O_2$ and $O_3$ do not belong to $\mathcal{G}$. So, we need to take five ovoids among the seven ovoids $O_1, O_4, O_5, O_6, O_7, O_8, O_9$. Since $O_4 \cap O_5 = \{x_{12}\}$, $O_5 \cap O_6 = \{x_{41}\}$, $O_4 \cap O_6 = \{x_{24}\}$ and $O_7 \cap O_9 = \{x_{42}\}$, (P5) implies that none of the pairs $\{O_4, O_5\}$, $\{O_5, O_6\}$, $\{O_4, O_6\}$, $\{O_7, O_9\}$ can be contained in $\mathcal{G}$. So, two among $O_4, O_5, O_6$ cannot be in $\mathcal{G}$, as well as one among $O_7, O_9$. So, it is impossible to find the five required ovoids from the collection $\{O_1, O_4, O_5, \cdots, O_9\}$.

**Treatment of Case III**

We have $U_2 = \{x_{11}, x_{23}, x_{34}, x_{42}\}$. If $U_2 \in \mathcal{G}$, then by (P1) – (P4), $\mathcal{G} \setminus \{U_2\}$ is a collection of six ovoids as in Lemma 4.7.5. A contradiction is then readily obtained from Lemma 4.7.6 and property (P5). So, $U_2 \notin \mathcal{G}$. Then, using (P1) and (P2), the unique ovoid of $\mathcal{G}$ containing $x_{11}$ must be $\{x_{11}, x_{24}, x_{32}, x_{43}\}$. Each point $y \in \{x_{24}, x_{32}, x_{43}\}$ is contained in a second ovoid of the collection $\mathcal{G}$ which meets $\{x_{11}, x_{24}, x_{32}, x_{43}\}$ in a second point $y' \in \{x_{24}, x_{32}, x_{43}\}$. Then the pairs $\{y, y'\}$ would partition $\{x_{24}, x_{32}, x_{43}\}$, an obvious contradiction.

**Treatment of Case IV**

We have $U_2 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. By (P1), all ovoids of $\mathcal{G}$ are disjoint from $U_1$. So we have to choose seven ovoids for $\mathcal{G}$ among the nine ovoids $O_1, O_2, \ldots, O_9$. By (P4), there are three ovoids of $\mathcal{G}$ containing $x_{12}$. So the ovoids $O_1$, $O_4$ and $O_5$ belong to $\mathcal{G}$. As $O_4 \cap O_6 = \{x_{24}\}$ and $O_4 \cap O_9 = \{x_{31}\}$, the ovoids $O_6$ and $O_9$ are not in $\mathcal{G}$ by (P5). Hence, $\mathcal{G} = \{O_1, O_2, O_3, O_4, O_5, O_7, O_8\}$. One readily verifies that this collection of ovoids satisfies the properties (P1) – (P6).

**Treatment of Case V**

Here $U_2 = \{x_{12}, x_{23}, x_{34}, x_{41}\}$. By (P1), all ovoids of $\mathcal{G}$ are disjoint from $U_1$. So we have to choose seven ovoids for $\mathcal{G}$ among the nine ovoids $O_1, O_2, \ldots, O_9$. Since $O_4 \cap O_6 = \{x_{24}\}$, $O_4 \cap O_8 = \{x_{43}\}$ and $O_4 \cap O_9 = \{x_{31}\}$, $O_4$ cannot occur in $\mathcal{G}$ by (P5). Since $O_6 \cap O_7 = \{x_{13}\}$ and $O_6 \cap O_8 = \{x_{32}\}$, we then know that also $O_6$ cannot occur in $\mathcal{G}$. So, we should have that $\mathcal{G} = \{O_1, O_2, O_3, O_5, O_7, O_8, O_9\}$. But that is impossible again by (P5) as $O_7 \cap O_8 = \{x_{21}\}$.

Let $X_1 = U_1 \cup Y_1 = O^* \cup Y_1$, where $Y_1$ is the set of seven points of $PG(3,3) \setminus \mathcal{H}$ for which the collection $\{\mathcal{C}_y \,|\, y \in Y_1\}$ consists of the ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, $O_1$, $O_2$, $O_3$, $O_6$ and $O_9$ of $\mathcal{H}$. Similarly, let $X_2 = U_1 \cup Y_2 = O^* \cup Y_2$, where $Y_2$ is the set of seven points of $PG(3,3) \setminus \mathcal{H}$ for which the collection $\{\mathcal{C}_y \,|\, y \in Y_2\}$ coincides with $\{O_1, O_2, O_3, O_4, O_5, O_7, O_8\}$. Note that $X_1$ is associated with the seven ovoids corresponding to subcase I(b) in the treatment of Case I and $X_2$ is associated with the seven ovoids in the treatment of Case IV.

By the above discussion, we thus know:

**Proposition 4.7.13.** *Up to isomorphism, $X_1$ and $X_2$ are the two $\mathbb{T}$-blocking sets of size 11 in $PG(3,3)$.*

Our intention is now to identify the two $\mathbb{T}$-blocking sets $X_1$ and $X_2$ with that of $B_1$ and $B_2$ constructed, respectively, in Sections 4.7.1 and 4.7.2. We shall rely on the following lemma.

**Lemma 4.7.14.** *Every ovoid $O$ of $\mathcal{H}$ is contained in precisely four partitions of $\mathcal{H}$ into ovoids. Three of these are induced by external lines.*

*Proof.* Without loss of generality, we may suppose that $O = O^* = \{x_{11}, x_{22}, x_{33}, x_{44}\}$. The partitions then have the form $\{O^*, O_i, O_j, O_k\}$, where $i, j, k \in \{1, 2, \ldots, 9\}$ with $i < j < k$. It is straightforward to verify that these partitions are $\{O^*, O_1, O_2, O_3\}$, $\{O^*, O_1, O_6, O_9\}$, $\{O^*, O_2, O_5, O_8\}$ and $\{O^*, O_3, O_4, O_7\}$. Now, let $x$ denote the unique point of $PG(3,3) \setminus \mathcal{H}$ for which $\mathcal{C}_x = O = O^*$. There are three external lines through $x$. If $\{x, u_1, u_2, u_3\}$, $\{x, u_4, u_5, u_6\}$ and $\{x, u_7, u_8, u_9\}$ are these external lines, then the nine ovoids $\{\mathcal{C}_{u_1}, \mathcal{C}_{u_2}, \ldots, \mathcal{C}_{u_9}\}$ are mutually distinct. So, $\{O^*, O_1, O_6, O_9\}$, $\{O^*, O_2, O_5, O_8\}$ and $\{O^*, O_3, O_4, O_7\}$ must be the partitions among the four that are induced by external lines. □

**Proposition 4.7.15.** *There exist two mutually disjoint external lines $k$, $l$ and a point $x \in k$ such that $X_1 = \mathcal{C}_x \cup (k \setminus \{x\}) \cup l$.*

*Proof.* Let $k$ denote the external line determined by the ovoids $O^*$, $O_1$, $O_6$, $O_9$, and denote by $x$ the unique point of $k$ for which $\mathcal{C}_x = O^*$. Among the four partitions of $\mathcal{H}$ into ovoids containing $O_2$, $\{O^*, O_1, O_2, O_3\}$ is not induced by any external line (see the proof of Lemma 4.7.14). So, again by Lemma 4.7.14, the partition of $\mathcal{H}$ by the four ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, $O_2$ and $O_3$ is induced by some external line, say $l$. Then we have $k \cap l = \emptyset$ and $X_1 = \mathcal{C}_x \cup \Big(k \setminus \{x\}\Big) \cup l$. □

By Proposition 4.7.4, we know that the two blocking sets $B_1$ and $B_2$ constructed in Sections 4.7.1 and 4.7.2 are nonisomorphic. In fact, by the proof of Proposition 4.7.4, we know that $B_2$ does not contain any external line, while $B_1$ does. Comparing this with Propositions 4.7.13 and 4.7.15, we then conclude that the blocking set $X_1$ is isomorphic to $B_1$ and that the blocking set $X_2$ is isomorphic to $B_2$.

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