RANKIN-COHEN TYPE OPERATORS AND SOME PROPERTIES OF FOURIER COEFFICIENTS OF MODULAR FORMS

By
MONI KUMARI
MATH11201304004

National Institute of Science Education and research, Bhubaneswar

A thesis submitted to the
Board of Studies in Mathematical Sciences
In partial fulfillment of requirements
for the Degree of
DOCTOR OF PHILOSOPHY
of
HOMI BHABHA NATIONAL INSTITUTE

December, 2018

Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Moni Kumari entitled "Rankin-Cohen type operators and some properties of Fourier coefficients of modular forms" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman - Prof. Anil Karn		**Date:** 6-12-2018
Guide / Convener - Prof. Brundaban Sahu		**Date:** 06.12.2018
Co-guide - N/A		**Date:**
Examiner - Prof. Soumya Das		**Date:** 06/12/2018
Member 1 - Prof. Binod Kumar Sahoo		**Date:** 06/12/2018
Member 2 - Prof. Jaban Meher		**Date:** 06/12/2018
Member 3 - Prof. K. Senthil Kumar		**Date:** 06/12/2018

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I/We hereby certify that I/we have read this thesis prepared under my/our direction and recommend that it may be accepted as fulfilling the thesis requirement.

06.12.2018

Date: **06/12/2018**
Place: **NISER, Bhubaneswar**

Prof. Brundaban Sahu
Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Moni Kumari

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Moni Kumari

List of Publications arising from the thesis

Journal

1. Rankin-Cohen brackets on Hilbert Modular forms and special values of certain Dirichlet series (with Brundaban Sahu), *Funct. Approx. Comment. Math.*, ***58*** *(2018), no. 2*, 257-268.

2. Simultaneous non-vanishing and sign changes of Fourier coefficients of Modular forms (Moni Kumari and M. Ram Murty), *Int. J. Number Theory,* ***14*** *(2018), no. 8,* 2291-2301.

Communicated

1. Rankin-Cohen type operators for Hilbert-Jacobi forms (Brundaban Sahu).

Moni Kumari

Dedicated to My Family

ACKNOWLEDGEMENTS

I would like to take this opportunity to thank, first of all, my supervisor Prof. Brundaban Sahu. It is impossible to measure the debt I owe to him. Over the past five years he has been an indefatigable source of encouragement, wisdom, and inspiration, and his help and patience guidance have been invaluable in the writing of this thesis.

I thank Prof. Ram Murty for teaching me various courses during my Ph.D. He has been an extremely inspiring teacher and a very encouraging collaborator. I would also like to thank Prof. J. Sengupta for various mathematical discussion and his kind of support.

I am also grateful to many other mathematicians Prof. Soumya Das, Prof. Satadal Ganguly, Prof. M. Manickam, Prof. Ritabrata Munshi, Prof. Kumar Murty, Prof. Ameaya Pitale and Prof. B. Ramakrishnan, who have helped me and taught me.

I wish to thank the other members of my doctoral committee, Prof. A. Karn, Prof. Senthil Kumar, Prof. Jaban Meher and Prof. B. K. Sahoo. I also sincerely thank the faculty members of NISER for their patience and encouragement during my course work.

I take this opportunity to thank Abhash Bhaiya, Priyo, Sujeet and my batch mates, Anoop, Anindya, Bikaram, Nitu for their academic and non-academic helps, during my stay at NISER.

I would like to thank my family, specially my mother, for their continual support and encouragement, and for allowing me the freedom to pursue my own interests. I am really grateful to God for giving me sisters like Sonu di and Suman.

Words are not enough to express my eternal gratitude to my life partner Arvind, for his kind of care, love, support and inspiration. Thank you so much for making my dreams come true.

Contents

Synopsis

Homi Bhabha National Institute

Synopsis of Ph.D. Thesis

1. Name of the Student:	**Moni Kumari**
2. Name of the Constituent Institution:	**National Institute of Science Education and Research, Bhubaneswar**
3. Enrolment No. :	**MATH11201304004**
4. Title of the Thesis:	**Rankin-Cohen type operators and some properties of Fourier coefficients of modular forms**
5. Board of Studies:	**Mathematical Sciences**

The thesis deals with some problems in the theory of classical automorphic forms. There are four chapters in the thesis. Chapter 1 contains all the preliminaries on elliptic modular forms, Hilbert modular forms and Hilbert-Jacobi forms, which are needed for the thesis. In the following sections, we describe our results chapter-wise.

1 Simultaneous non-vanishing and sign changes of Fourier coefficients of modular forms

In chapter 2, we study certain problems related to simultaneous non-vanishing and sign-changes of Fourier coefficients of two different cusp forms. For positive integers N and k, we denote $M_k(N)$ and $S_k(N)$ the space of modular forms and cusp forms respectively, of weight k for the group $\Gamma_0(N)$.

1.1 Simultaneous non-vanishing of Fourier coefficients

The *Delta-function* $\Delta(z)$, defined by

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

is a cusp form of weight 12 for the group $SL_2(\mathbb{Z})$. The function $n \mapsto \tau(n)$ is studied by Ramanujan in 1916, which is called the Ramanujan's Tau function. After some numerical evidence, Lehmer in 1939, conjectured that $\tau(n) \neq 0$ for all $n \geqslant 1$. This conjecture is an important open problem in number theory. Motivated from this conjecture, Serre [48], defined an arithmetic function

$$i_f(n) := \min\{j \geqslant 0 : a(n+j) \neq 0\},$$

for a cusp form $f(z)=\sum_{n=1}^{\infty} a(n)q^n$, and proved that $i_f(n) \ll_f n$, provided f is not a linear combination of CM forms. Then he asked, whether there exists some $\delta < 1$ such that, $i_f(n) \ll_f n^{\delta}$. Existence of a δ follows from the work of Rankin [43] and Selberg [47], which states that

$$\sum_{n \leqslant x} a(n)^2 = cx + O(x^{\frac{3}{5}}),$$

i.e., $i_f(n) \ll_f n^{\frac{3}{5}}$. After that many mathematicians including, Alkan, Balog and Ono improved the value of δ. The best known bound of δ so far is $\frac{1}{4}$, which is proved by Das and Ganaguly [12].

Let f and g be two cusp forms with Fourier coefficients $a(n)$ and $b(n)$ respectively. Then we investigate non-vanishing of the sequence $\{a(n)b(n)\}_{n\in\mathbb{N}}$.

Theorem 1.1. *[26] Suppose $f(z) = \sum_{n=1}^{\infty} a(n)n^{\frac{k-1}{2}}q^n \in S_k(N)$ and $g(z) = \sum_{n=1}^{\infty} b(n)n^{\frac{k-1}{2}}q^n \in S_k(N)$ are two newforms which are not CM forms, then there exist infinitely many primes p such that, $a(p)b(p) \neq 0$.*

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=1}^{\infty} b(n)q^n$ be two cusp forms of weight k for the group $\Gamma_0(N)$. We introduce an analogous concept of a gap function $i_{f,g}$ for simultaneous non-vanishing of Fourier coefficients and obtain a bound for $i_{f,g}$. More precisely we define, for $n \geqslant 1$,

$$i_{f,g}(n) := \text{min}\{m \geqslant 0 : a(n+m)b(n+m) \neq 0\}.$$

From the work of Lu [33], we notice that $i_{f,g}(n) \ll_{f,g} n^{\frac{7}{8}+\varepsilon}$. Using the results of Chen and Wu [5] about the distribution of $\mathcal{B}$-free numbers in short intervals as well as in an arithmetic progression, we give an improvement of the above estimate.

Theorem 1.2. *[26] Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)n^{\frac{k-1}{2}}q^n \in S_k(N)$ and*

$g(z) = \sum_{n=1}^{\infty} b(n) n^{\frac{k-1}{2}} q^n \in S_k(N)$ *are two newforms with* $k > 2$ *which are not a linear combination of CM forms. Then the following results hold.*

(i) For every $\varepsilon > 0$, $x > x_0(f, g, \varepsilon)$ *and* $x^{\frac{7}{17}+\varepsilon} \leqslant y$ *we have*

$$|\{x < n < x + y : a(n)b(n) \neq 0\}| \gg_{f,g,\varepsilon} y.$$

In particular, we get that $i_{f,g}(n) \ll_{f,g,\varepsilon} n^{\frac{7}{17}+\varepsilon}$.

(ii) For every $\varepsilon > 0$, $x \geqslant x_0(f, g, \varepsilon)$, $y \geqslant x^{\frac{17}{38}+100\varepsilon}$ *and* $1 \leqslant a \leqslant q \leqslant x^{\varepsilon}$ *with* $(a, q) = 1$, *we have*

$$|\{x < n \leqslant x + y : \; n \equiv a \pmod{q} \text{ and } a(n)b(n) \neq 0\}| \gg_{f,g,\varepsilon} y/q.$$

1.2 Simultaneous sign-changes of Fourier coefficients

Information of the signs of Fourier coefficients of modular forms give lots of implications. For example, recently Kowalski, Lau, Soundarajan and Wu [24] and later Matomӑki [34] showed that any normalized Hecke eigenform f is uniquely determined by the signs of their Hecke eigenvalues at primes.

Kohnen and Sengupta [23] considered a problem related to the simultaneous sign changes and proved the following.

Theorem 1.3. *[23] For* $k_1 \neq k_2$, *let* $f(z) = \sum_{n=1}^{\infty} a(n) q^n \in S_{k_1}(N)$ *and* $g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k_2}(N)$ *be normalized so that* $a(1) = b(1) = 1$ *and* $a(n), b(n)$ *are totally real algebraic numbers. Then there exists* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *such that the sequence* $\{\sigma(a(n))\sigma(b(n))\}_{n \in \mathbb{N}}$ *has infinitely many sign changes.*

Gun, Kohnen and Rath [16] removed the dependency on the Galois conjugacy and proved the following.

Theorem 1.4. *[16] Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=1}^{\infty} b(n)q^n$ be non-zero cusp forms of level N and weights $1 < k_1 < k_2$ respectively, with real Fourier coefficients. Furthermore, if $a(1)b(1) \neq 0$, then the sequence $\{a(n)b(n)\}_{n\in\mathbb{N}}$ has infinitely many sign changes.*

In chapter 2, we prove the following quantitative result for the simultaneous sign changes.

Theorem 1.5. *[26] Let $k \geqslant 2$ be an integer. Assume that*

$$f(z) = \sum_{n\geqslant 1} a(n)n^{\frac{k-1}{2}}q^n \text{ and } g(z) = \sum_{n\geqslant 1} b(n)n^{\frac{k-1}{2}}q^n$$

are two distinct newforms of weight k on $\Gamma_0(N)$. Further, let $a(n), b(n)$ be real numbers, then for any $\delta > \frac{7}{8}$, the sequence $\{a(n)b(n)\}_{n\in\mathbb{N}}$ has at least one sign change for $n \in (x, x + x^{\delta}]$ for sufficiently large x. In particular, the number of sign changes for $n \leqslant x$ is $\gg x^{1-\delta}$.

2 Rankin-Cohen brackets and construction of Hilbert cusp forms

Let $f(z) = \sum_{m=1}^{\infty} a_m q^m \in S_k(1)$ and $g(z) = \sum_{m=1}^{\infty} b_m q^m \in S_l(1)$. For a positive integer n, define the shifted Dirichlet series as follows:

$$L_{f,g;n}(s) := \sum_{m=1}^{\infty} \frac{a_{m+n}\overline{b_m}}{(n+m)^s}. \tag{1}$$

Using Deligne's estimate one can see that the series $L_{f,g;n}(s)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{k+l}{2}$. Using the existence of adjoint map and property of Poincaré series, Kohnen [22] constructed cusp forms whose Fourier coefficients involve special values of the Dirichlet series (3.1). More precisely,

Theorem 2.1. *[22] Let k and l be a positive integers with $k > l+2$. Let $f(z) = \sum_{m=1}^{\infty} a_m q^m \in S_{k+l}(1)$ and $g(z) = \sum_{m=1}^{\infty} b_m q^m \in S_l(1)$. Then the function*

$$T_g^*(f)(z) := \sum_{m=1}^{\infty} m^{k-1} L_{f,g:m}(k+l-1) q^m$$

is a cusp form of weight k for $SL_2(\mathbb{Z})$. In fact, the map $S_{k+l}(1) \to S_k(1)$ defined by $f \mapsto \dfrac{\Gamma(k+l-1)}{\Gamma(k-1)(4\pi)^l} T_g^*(f)$ is the adjoint of the map $T_g : S_k(1) \to S_{k+l}(1)$, $h \mapsto gh$, with respect to the Petersson scalar product.

This result has been generalized by several authors to other automorphic forms. In particular, Lee [29], Pei and Wang [41], Wang [50] have analogous result for Hilbert modular forms.

There are many interesting connections between differential operators and modular forms and many interesting results have been found. Rankin [44] gave a general description of the differential operators which send modular forms to modular forms. Cohen [11] constructed bilinear operators and obtained elliptic modular forms with interesting Fourier coefficients. Zagier [52] studied the algebraic properties of these bilinear operators and called them *Rankin-Cohen brackets*.

Theorem 2.2. *[11] For $f \in M_k(1)$ and $g \in M_l(1)$ and for every $n \geqslant 0$, the function $[f,g]_n$ defined by*

$$[f,g]_n := \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{\Gamma(k+n)\Gamma(l+n)}{\Gamma(k+r)\Gamma(l+n-r)} D^r f D^{n-r} g,$$

belongs to $M_{k+l+2n}(1)$, where $D^r f = \frac{1}{(2\pi i)^r} \frac{d^r f}{dz^r}$.

Recently the work of Kohnen has been generalized by Herrero [18] where he constructed the adjoint map using the Rankin–Cohen brackets by a fixed cusp form instead of product map. More precisely, for a fixed $g \in M_l(1)$ and an integer $n \geqslant 0$, consider the linear map

$$T_{g,n} : S_k(1) \longrightarrow S_{k+l+2n}(1)$$

defined by $f \longmapsto [f, g]_n$ (the n-th Rankin–Cohen bracket). Let $T^*_{g,n}$ be its adjoint map with respect to the Petersson inner product.

Theorem 2.3. *[18] Suppose k, l, n are non-negative integers with $k \geqslant 6$. Let $g(z) = \sum_{m=0}^{\infty} b_m q^m \in M_l(1)$. Suppose that either g is a cusp form or $l < k-3$. Then the image of any cusp form $f(z) = \sum_{m=1}^{\infty} a_m q^m \in S_{k+l+2n}(1)$ under $T^*_{g,n}$ is given by*

$$T^*_{g,n}(f)(z) = \frac{\Gamma(k+l+2n-1)}{(4\pi)^{l+2n}\Gamma(k-1)} \sum_{m=1}^{\infty} m^{k-1} \left(\sum_{r=0}^{\infty} \frac{a_{m+r}\overline{b}_r}{(m+r)^{k+l+2n-1}} \varepsilon^{k,l,n}_{m,r} \right) q^m,$$

where

$$\varepsilon^{k,l,n}_{m,r} = \sum_{t \in \mathbb{N}_0, 0 \leqslant t \leqslant n} (-1)^t \binom{k+n-1}{n-t} \binom{l+n-1}{t} m^t r^{n-t}.$$

In chapter 3, we generalize the work of Herrero to the case of Hilbert modular forms. Let K be a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$. For $k \in \mathbb{N}_0^n$, let $M_k(\Gamma_K)$ (resp. $S_k(\Gamma_K)$) be the space of Hilbert modular forms (resp. Hilbert cusp forms) of weight k for the group $SL_2(\mathcal{O}_K)$. Rankin-Cohen brackets for Hilbert modular forms are studied by Choie, Kim and Richter [10].

Theorem 2.4. *[10] For $f_1 \in M_k(\Gamma_K)$, $f_2 \in M_l(\Gamma_K)$ and $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}_0^n$, the function $[f_1, f_2]_\nu$ defined by*

$$[f_1, f_2]_\nu := \sum_{\substack{t \in \mathbb{N}_0^n \\ 0 \leqslant t_i \leqslant \nu_i}} (-1)^{|t|} \binom{k+\nu-\overrightarrow{1}}{\nu - t} \binom{l+\nu-\overrightarrow{1}}{t} f_1^{(t)}(z) f_2^{(\nu-t)}(z),$$

belongs to $M_{k+l+2\nu}(\Gamma_K)$. Here for $t = (t_1, \cdots, t_n) \in \mathbb{N}_0^n$, $f^{(t)} := \frac{\partial^{|t|}}{\partial z_1^{t_1} \partial z_2^{t_2} \ldots \partial z_n^{t_n}} f(z)$.

Now, for a fixed $g \in M_l(\Gamma_K)$ and $\nu \in \mathbb{N}_0^n$, consider the linear map

$$\mathcal{T}_{g,\nu} : S_k(\Gamma_K) \longrightarrow S_{k+l+2\nu}(\Gamma_K)$$

defined by $f \longmapsto [f,g]_\nu$ (the ν-th Rankin-Cohen bracket). Let $\mathcal{T}_{g,\nu}^*$ be its adjoint map with respect to Petersson inner product. In chapter 3, we compute explicitly the Fourier coefficients of $\mathcal{T}_{g,\nu}^*(f)$, which involves special values of certain shifted Dirichlet series associated to f and g.

Theorem 2.5. *[27] Suppose $k, l, \nu \in \mathbb{N}_0^n$ with $k_i \geqslant 4n+2$ for some i. Let $g(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} b_m e[tr(mz)] \in M_l(\Gamma_K)$. Suppose that either g is a cusp form or $k_i - l_i > 4n$ for some i. Then the image of any cusp form $f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} a_m e[tr(mz)] \in S_{k+l+2\nu}(\Gamma_K)$ under $\mathcal{T}_{g,\nu}^*$ is given by*

$$\mathcal{T}_{g,\nu}^*(f)(z) = \sum_{\substack{\mu \in \mathcal{O}_K^* \\ \mu \gg 0}} c_\mu e[tr(\mu z)],$$

where

$$c_\mu = \frac{\Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{l+2\nu}\Gamma(k-\overrightarrow{1})}\mu^{k-\overrightarrow{1}} \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} \frac{a_{m+\mu}\bar{b}_m}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}} \varepsilon_{\mu,m}^{k,l,\nu}$$

and

$$\varepsilon_{\mu,m}^{k,l,\nu} = \sum_{\substack{t \in \mathbb{N}_0^n \\ 0 \leqslant t_i \leqslant \nu_i}} (-1)^{|t|} \binom{k+\nu-\overrightarrow{1}}{\nu-t} \binom{l+\nu-\overrightarrow{1}}{t} \mu^t m^{\nu-t}.$$

Let $f \in S_{k+l+2n}(1)$, $g \in M_l(1)$ and $n \geqslant 0$ be an integer. Zagier [51] computed explicitly the Petersson scalar product $\langle f, [E_k, g]_n \rangle$ in terms of special values of a certain Dirichlet series associated to f and g. The main idea of his proof is to express $[E_k, g]_n$ as a linear combination of elliptic Poincaré series and then use the characterization property of the Poincaré series. Choie, Kim and Richter [10] generalized the above work of Zagier for Hilbert modular forms and proved the following.

Theorem 2.6. *[10] Let $k > 2$ be a natural number and $l, \nu \in N_0^n$ with $k - l_i > 2n$ for some $i, 1 \le i \le n$. Suppose that $f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} a_m e[tr(mz)] \in S_{k+l+2\nu}(\Gamma_K)$ and $g(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} b_m e[tr(mz)] \in M_l(\Gamma_K)$. Then*

$$\langle f, [E_k, g]_\nu \rangle = vol(\mathcal{O}_K/\mathbb{R}^n)(2i\pi)^{|\nu|} \frac{(\overrightarrow{k} + l + 2\nu - \overrightarrow{2})!(\overrightarrow{k} + \nu - \overrightarrow{1})!}{(4\pi)^{|\overrightarrow{k} + l + 2\nu - \overrightarrow{1}|}(\overrightarrow{k} - \overrightarrow{1})!\nu!} \sum_{\substack{n \in \mathcal{O}_K^* \\ n \gg 0}} \frac{a_n \overline{b_n}}{n^{k+l+\nu-\overrightarrow{1}}}.$$

As an application, we note that one can give a different proof of this result following our method.

3 Rankin-Cohen type operators for Hilbert-Jacobi forms

The theory of Jacobi forms is an interesting and fruitful area of research in the recent past, which is systematically studied by Eichler and Zagier [13] in their monograph, "The Theory of Jacobi Forms". In this monograph, they gave a method, using differential operators, to construct Jacobi forms of odd weight from forms of even weight. More precisely,

Theorem 3.1. *[13] Let $\phi(\tau, z)$ and $\phi'(\tau, z)$ be Jacobi forms of weight k and k' and m and m', respectively. Then*

$$m'(\partial_z \phi)\phi' - m\phi(\partial_z \phi')$$

is a Jacobi form of weight $k + k' + 1$ and index $m + m'$.

Using the heat operator on the space of Jacobi forms, Choie [6, 7] studied such construction which increase the weight by even. Further, Choie and Eholzer [8] studied Rankin-Cohen type operators for Jacobi forms in more generality. More precisely, they proved the following.

Theorem 3.2. *[8] Let $\phi(\tau, z)$ and $\phi'(\tau, z)$ be Jacobi forms of weight k and k' and m and m', respectively. Then for $X \in \mathbb{C}$, $[\phi, \phi']_{X,2l}^{k,k',m,m'}$ and $[\phi, \phi']_{X,2l+1}^{k,k',m,m'}$, defined by*

$$[\phi, \phi']_{X,2l}^{k,k',m,m'} := \sum_{r+s+p=l} C_{r,s,p}(k,k')(1-mX)^s(1+m'X)^r L_{m+m'}^p(L_m^r(\phi)L_{m'}^s(\phi')).$$

and

$$[\phi, \phi']_{X,2l+1}^{k,k',m,m'} := m[\phi, \partial_z\phi']_{X,2l}^{k,k',m,m'} - m'[\partial_z\phi, \phi']_{X,2l}^{k,k',m,m'}.$$

are Jacobi forms of weight $k+k'+2l$ and $k+k'+2l+1$ of the same index $m+m'$, respectively. The coefficients $C_{r,s,p}(k,k')$ are given by

$$C_{r,s,p}(k,k') = \frac{(\alpha+r+s+p)_{s+p} \quad (\beta+r+s+p)_{r+p} \quad (-(\gamma+r+s+p))_{r+s}}{r! \quad s! \quad p!}$$

with $(x)_n = \prod_{0 \leqslant i \leqslant n-1}(x-i), \alpha = k-3/2, \beta = k'-3/2, \gamma = k+k'-3/2.$

Skogman [49] in his thesis generalized the theory of Jacobi forms (over $\mathbb{Q}$) to Hilbert-Jacobi forms (over any totally real number field). In chapter 4, we construct Rankin-Cohen type operators for Hilbert-Jacobi forms. Let K be a totally real number field of degree g over $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$. Let $J_{k,m}^K$ denote the space of Hilbert-Jacobi forms of weight $k = (k_1, k_2, ..., k_g) \in \mathbb{Z}^g$ and index $m \in \mathcal{O}_K$ for the field K.

We first define the heat operator L_m for $m \in \mathcal{O}_K$, which is as follows.

$$L_m := \left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)^{e_1} o \left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)^{e_2} o\cdots o \left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)^{e_g}$$

where e_j is j-th unit vector in $\mathbb{R}^g$ for $1 \leqslant j \leqslant g$. Now using this operator, we construct Rankin-Cohen type differential operators for the space of Hilbert-Jacobi forms.

Definition 3.3. *Let $\phi, \phi' : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$ be two holomorphic functions and let k, k', m, m' be complex numbers. Then for any $X \in \mathbb{C}^g$, $\nu \in \mathbb{N}_0^g$ and $l \in \mathbb{N}_0^g$ with $l_i \in \{0,1\}$ for all $1 \leqslant i \leqslant g$, define*

$$[\phi, \phi']_{X,2\nu+l}^{k,k',m,m'} = \sum_{\substack{j\in\mathbb{N}_0^g \\ j\leqslant l}} (-1)^j m^{l-j} m'^j [\partial_z^j \phi, \partial_z^{l-j}\phi']_{X,2\nu}^{k,k',m,m',l},$$

where for any two holomorphic functions f and f' on $\mathbb{H}^g \times \mathbb{C}^g$

$$[f, f']_{X,2\nu}^{k,k',m,m',l} := \sum_{\substack{r,s,p\in\mathbb{N}_0^g, \\ r+s+p=\nu}} A_{r,s,p}(k,k',l)(1+mX)^r(1-m'X)^s L_{m+m'}^p(L_m^r(f)L_{m'}^s(f')),$$

with

$$A_{r,s,p}(k,k',l) = \frac{(-(k+k'+l-3/2+\nu))_{r+s}}{r!\ s!\ p!\ (k-3/2+r)!\ (k'-3/2+s)!}.$$

We now state the main result of chapter 4.

Theorem 3.4. *[28] Let ϕ, ϕ' be Hilbert-Jacobi forms of weight and index k, m and k', m' respectively. Then for any $X \in \mathbb{C}^g$, $\nu \in \mathbb{N}_0^g$ and $l \in \mathbb{N}_0^g$ with $l_i \in \{0,1\}$ for all $1 \leqslant i \leqslant g$,*

$$[\phi, \phi']_{X,2\nu+l}^{k,k',m,m'}$$

is a Hilbert-Jacobi form of weight $k+k'+2\nu+l$ and index $m+m'$.

Publications in Refereed Journal:

a. **Published**:

[KS1] Rankin-Cohen brackets on Hilbert Modular forms and special values of certain Dirichlet series, (with Brundaban Sahu), *Funct. Approx. Comment. Math.* Volume 58, Number 2 (2018), 257–268.

[KM] Simultaneous non-vanishing and sign changes of Fourier coefficients of Modular forms, (with M. Ram Murty), *Int. J. Number Theory*, **14** (2018), no. 8, 2291–2301.

c. **Communicated**:

[KS2] Rankin-Cohen type operators for Hilbert-Jacobi forms, (with Brundaban Sahu).

d. **Other Publications:**

[KS] On the parity of the Fourier coefficients of hauptmoduln $j_N(z)$ and $j_N^+(z)$, (with Sujeet Kumar Singh), Accepted in *Acta Arithmetica*.

Participation in Conference/Symposium:

- Attended "School on modular forms" during 12-24 Feb 2018 at KSOM, Kerala, India and delivered a talk on "Rankin-Cohen type operators for Hilbert-Jacobi forms".
- Attended "Artin L-functions, Artin's primitive roots conjecture and applications" during 29 May-09 June 2017 at Nesin mathematics village, Sirince, Turkey and delivered a talk on "Simultaneous non-vanishing and sign changes of Fourier coefficients of Modular forms".

Chapter 1

Preliminaries

1.1 Notations

Let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ be the set of positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. For $a, b, c, d \in \mathbb{Z}$, we write $a|b$ when b is divisible by a and $c \pmod{d}$ means that c varies over a complete set of residue classes modulo d. For $z \in \mathbb{C}$, $\mathrm{Re}(z)$ denotes the real part of z and $\mathrm{Im}(z)$ denotes the imaginary part of z. For any $z \in \mathbb{C}$, we denote $e^{2\pi i z}$ by $e[z]$. Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane. We denote by $q = e[\tau]$, for $\tau \in \mathbb{H}$.

We use interchangeably Landau's notation $f = O(g)$ and Vinogradov's $f \ll g$, both to mean that $|f| \leqslant C|g|$ for a suitable positive constant C, which may be absolute or depend upon various parameters, in which case the dependence may be indicated in a subscript. Moreover, we write $f \asymp g$ to indicate that $f \ll g$ and $g \ll f$ hold simultaneously.

For $n \in \mathbb{N}$ and $x \in \mathbb{N}_0$, we write $\overrightarrow{x} = (x, x, \ldots, x) \in \mathbb{N}_0^n$. For $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $l = (l_1, \ldots, l_n) \in \mathbb{N}_0^n$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, we denote

$$|\nu| = \sum_{i=1}^{n} \nu_i, \;\; \nu! = \prod_{i=1}^{n} \nu_i! \text{ and } z^{\nu} = \prod_{i=1}^{n} z_i^{\nu_i}.$$

Whenever we write $\nu \leqslant l$ means $\nu_j \leqslant l_j$ for all $1 \leqslant j \leqslant n$.

1.2 Modular forms

Consider the group

$$GL_2^+(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}.$$

The subgroup $SL_2(\mathbb{Z})$ of $GL_2^+(\mathbb{R})$, defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1 \right\},$$

is known as the full modular group. For a positive integer N, the principal congruence subgroup of level N, denoted by $\Gamma(N)$, is defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Note that $\Gamma(1) = SL_2(\mathbb{Z})$. A subgroup Γ of $SL_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subset \Gamma$ for some N. For $N \geqslant 1$,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

The group $GL_2^+(\mathbb{R})$ acts on $\mathbb{H}$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}, \qquad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}) \text{ and } z \in \mathbb{H}. \tag{1.2.1}$$

Using the action (1.2.1), we define an action of the group $GL_2^+(\mathbb{R})$ on the set of functions defined on $\mathbb{H}$. For a complex valued function f on $\mathbb{H}$ and $k \in \mathbb{Z}$, we define

$$(f|_k\,\gamma)(z) := (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

Definition 1.2.1 (Modular form)**.** *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and $k \in \mathbb{Z}$. A modular form of weight k for Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying*

1. $f|_k\,\gamma = f, \qquad for\ all \quad \gamma \in \Gamma,$

2. *f is holomorphic at every cusp of Γ.*

A modular form f is called a cusp form if it vanishes at each cusp of Γ.

Let $M_k(\Gamma)$ be the $\mathbb{C}$-vector space of modular forms of weight k for Γ and $S_k(\Gamma)$ be the subspace of cusp forms in $M_k(\Gamma)$. These are finite-dimensional $\mathbb{C}$-vector space. For simplicity, we write $M_k(N)$ and $S_k(N)$ for the space of modular forms and cusp forms of weight k for the group $\Gamma_0(N)$, respectively.

Example 1. Let $k \geqslant 4$ be an even integer. The Eisenstein series E_k defined by

$$E_k(z) = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (m,n)=1}} \frac{1}{(mz+n)^k}, \qquad (z \in \mathbb{H}) \tag{1.2.2}$$

is a modular form of weight k for the group $SL_2(\mathbb{Z})$. It has the following q-expansion at the cusp ∞.

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$ and B_k's are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Example 2. The Ramanujan delta function is defined as

$$\Delta(z) := \frac{1}{1728}(E_4^3(z) - E_6^2(z)).$$

$\Delta(z)$ is a cusp form of weight 12 for $SL_2(\mathbb{Z})$ with the following Fourier expansion:

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n, \tag{1.2.3}$$

where $\tau(n)$ is known as the Ramanujan's Tau function.

$S_k(N)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$\langle f, g \rangle = \frac{1}{i_N} \int_{\mathcal{F}_N} f(z) g(z) y^k \frac{dxdy}{y^2}, \tag{1.2.4}$$

where $\mathcal{F}_N$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$, i_N is the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$ and $z = x + iy$.

We now state a lemma which gives the growth of Fourier coefficients of a modular form.

Lemma 1.2.2. *[40, p. 30] If $f \in M_k(N)$ with Fourier coefficients $a(n)$, then for any $\epsilon > 0$*

$$a(n) \ll n^{k-1+\epsilon},$$

and moreover, if f is a cusp form, then

$$a(n) \ll n^{\frac{k-1}{2}+\epsilon}. \tag{1.2.5}$$

The estimate (1.2.5) is known as Deligne's bound. For more details on the theory of modular forms, we refer to [21] and [40].

1.2.1 Newforms in $S_k(N)$

There is a family of linear operators on the space $S_k(N)$, called Hecke operators, which are used to develop the theory of newforms (see, [40]). Hecke introduced these operators in 1937 by means of "averaging" techniques.

Definition 1.2.3 (Hecke operators). *For $f \in S_k(N)$, the action of n-th Hecke operator T_n on f is defined by*

$$(T_n f)(z) = \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \leqslant b < d} f\left(\frac{az+b}{d}\right).$$

The Hecke operator T_n maps the space $S_k(N)$ into itself. Moreover, if $(n, N) = 1$, then T_n is a hermitian operator with respect to the Petersson inner product defined by (1.2.4).

Definition 1.2.4 (Hecke eigenform). *A cusp form $f \in S_k(N)$ is called a Hecke eigenform if for each $m \geqslant 1$ there is a complex number $\lambda(m)$ for which*

$$(T_m f)(z) = \lambda(m) f(z).$$

The Fourier coefficients of a Hecke eigenform satisfy the following properties.

Proposition 1.2.5. *[40, Prop. 2.6] Suppose that $f(z) = \sum_{n=1}^{\infty} a(n) q^n \in S_k(N)$ is a Hecke eigenform with*

$$(T_m f)(z) = \lambda(m) f(z).$$

1. *If f is non-constant, then $a(1) \neq 0$.*

2. *If f is a normalized cusp form, i.e., $a(1) = 1$, then*

$$a(m) = \lambda(m).$$

Moreover, If $(m, n) = 1$, then

$$a(m)a(n) = a(mn). \tag{1.2.6}$$

Definition 1.2.6 (Artin-Lehner operators)**.** *For a prime divisor p of N with $\mathrm{ord}_p(N) = \ell$ (i.e., $p^\ell || N$), let $Q_p := p^\ell$. The Atkin-Lehner operator $W(Q_p)$ on $S_k(N)$ is defined by any matrix*

$$W(Q_p) := \begin{pmatrix} Q_p\alpha & \beta \\ N\gamma & Q_p\delta \end{pmatrix}$$

with determinant Q_p, where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. Furthermore, the Fricke involution $W(N)$ on $M_k(N)$ is defined by the matrix

$$W(N) := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

It is a well known fact (see, [1]) that if $f(z) \in S_k(N)$ and $d > 1$ then $f(dz) \in S_k(dN)$. We denote $S_k^{\mathrm{old}}(N)$, the linear subspace of $S_k(N)$ spanned by all forms of the type $f(dz)$ where $d|N$ and $f(z) \in S_k(M)$ for some $M < N$ such that $dM|N$. The subspace $S_k^{\mathrm{old}}(N)$ of $S_k(N)$ is called the space of old forms in $S_k(N)$. The space of newforms $S_k^{\mathrm{new}}(N)$ is defined by the orthogonal complement of $S_k^{\mathrm{old}}(N)$ in $S_k(N)$ with respect to the Petersson inner product.

Definition 1.2.7 (Newform)**.** *A newform in* $S_k^{\mathrm{new}}(N)$ *is a normalized cusp form that is an eigenfunction of all the Hecke operators and all of the Atkin-Lehner operators* $W(Q_p)$*, for primes* $p|N$ *and* $W(N)$*.*

Definition 1.2.8 (CM form)**.** *A newform* $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ *of level* N *and weight* k *is said to be a CM form if there is a quadratic imaginary field* K *such that* $a(p) = 0$ *if* p *is a prime which is inert in* K*.*

For more details on the theory of newforms, we refer to [1] and [40].

1.2.2 L-functions

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a newform of weight k for the group $\Gamma_0(N)$. Define the function

$$L(s, f) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \qquad s \in \mathbb{C}.$$

Using the Deligne's estimate (1.2.5) we see that the function $L(s, f)$ converges absolutely for $\mathrm{Re}(s) > \frac{k+1}{2}$. This L-function has analytic continuation to $\mathbb{C}$ and satisfy the following functional equation.

Lemma 1.2.9. *[40, p. 150] Let* $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ *be a newform of weight* k *for the group* $\Gamma_0(N)$ *and*

$$\Lambda(s, f) := (2\pi)^{-s} N^{s/2} \Gamma(s) L(s, f).$$

Then there is an $\epsilon \in \{\pm 1\}$ *such that*

$$\Lambda(s, f) = \epsilon\, \Lambda(k - s, f).$$

1.3 Hilbert modular forms

Let K be a totally real number field of degree n over $\mathbb{Q}$ and $\mathcal{O}_K$ be the ring of integers. Assume that $\sigma_1, \sigma_2, ..., \sigma_n$ denote the real embeddings of K. We write $\alpha_i = \sigma_i(\alpha)$ for $\alpha \in K$ and $1 \leqslant i \leqslant n$. The trace and norm of $\alpha \in K$ is defined by $\mathrm{tr}(\alpha) = \sum_{i=1}^n \alpha_i$ and $N(\alpha) = \prod_{i=1}^n \alpha_i$ respectively. For $\alpha \in \mathbb{C}^n$, the trace and norm are defined by the sum and the product of its components respectively. More generally, if $c = (c_1, c_2, \ldots, c_n), d = (d_1, d_2, \ldots, d_n), z = (z_1, z_2, \ldots, z_n)$ and $m = (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n$, then the norm and trace are defined by

$$N(cz+d) := \prod_{i=1}^{n}(c_i z_i + d_i) \qquad \text{and} \qquad \mathrm{tr}(mz) := \sum_{i=1}^{n} m_i z_i.$$

For $\alpha \in \mathcal{O}_K$, we write $\alpha \succeq 0$ to demonstrate that either $\alpha = 0$ or α is totally positive (means all the conjugates of α are positive) and $\alpha \gg 0$ for α to be totally positive. Let

$$\Gamma_K = SL_2(\mathcal{O}_K) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\}$$

be the Hilbert modular group which can be embedded into $SL_2(\mathbb{R})^n$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \left(\begin{pmatrix} \sigma_1(a) & \sigma_1(b) \\ \sigma_1(c) & \sigma_1(d) \end{pmatrix}, \ldots, \begin{pmatrix} \sigma_n(a) & \sigma_n(b) \\ \sigma_n(c) & \sigma_n(d) \end{pmatrix} \right).$$

For $\gamma = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) \in SL_2(\mathbb{R})^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$, we define

$$\gamma \circ z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right).$$

This action deduces an action of $SL_2(\mathcal{O}_K)$ on $\mathbb{H}^n$. Let $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$.

For $\gamma = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) \in SL_2(\mathbb{R})^n$ and for $f : \mathbb{H}^n \to \mathbb{C}$ define the slash operator

$$(f\mid_k \gamma)(z) = j(\gamma, z)^{-k} f(\gamma \circ z), \qquad \text{where } \ j(\gamma, z) = (cz + d).$$

Definition 1.3.1 (Hilbert modular form)**.** *A Hilbert modular form of weight $k \in \mathbb{N}_0^n$ for Γ_K is a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ such that*

$$f\mid_k \gamma = f, \qquad \textit{for all } \gamma \in \Gamma_K.$$

Note that for $n = 1$, we need holomorphicity condition at the cusps of Γ_K. In addition, f is called a Hilbert cusp form if it vanishes at all the cusps of Γ_K.

Let $M_k(\Gamma_K)$ denotes the space of Hilbert modular forms of weight $k \in \mathbb{N}_0^n$ for the group Γ_K and $S_k(\Gamma_K)$ be the corresponding subspace of cusp forms. These are finite-dimensional complex vector spaces and $S_k(\Gamma_K)$ is a Hilbert space with respect to the Petersson inner product

$$\langle f, g \rangle := \int_{\Gamma_K \backslash \mathbb{H}^n} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2},$$

where $z = x + iy$, $dx = dx_1 \ldots dx_n$ and $dy = dy_1 \ldots dy_n$. By Koecher principle, $f \in M_k(\Gamma_K)$ has a Fourier expansion at the cusp ∞ of the form

$$f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \succeq 0}} a_m e[\mathrm{tr}(mz)],$$

where $\mathcal{O}_K^* := \{\mu \in K \mid \mathrm{tr}(\mu\lambda) \in \mathbb{Z} \text{ for all } \lambda \in \mathcal{O}_K\}$ is the dual space of $\mathcal{O}_K$.

Example (Hilbert Poincaré series). For a totally positive element ν of $\mathcal{O}_K^*$ and weight $k = (k_1, k_2, \ldots, k_n)$ $(k_j > 2,\ j = 1, 2, \ldots, n)$, we define the ν-th Hilbert Poincaré series as follows:

$$\mathcal{P}_{k,\nu}(z) = \sum_{M \in \Gamma_\infty \backslash \Gamma_K} j(M, z)^{-k} e[\mathrm{tr}(\nu(Mz))], \tag{1.3.1}$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} : \mu \in \mathcal{O}_K \right\}$. It is well known that if $\nu \neq \overrightarrow{0}$ then $\mathcal{P}_{k,\nu} \in S_k(\Gamma_K)$. For $\nu = \overrightarrow{0}$, $\mathcal{P}_{k,\overrightarrow{0}}(z) := E_k(z)$ is known as Hilbert Eisenstein series of weight k. Moreover, we have the following characterization property of Hilbert Poincaré series.

Theorem 1.3.2. *[15] If $f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} a_m e[\mathrm{tr}(mz)] \in S_k(\Gamma_K)$, then*

$$\langle f, \mathcal{P}_{k,\nu} \rangle = vol(\Lambda \setminus \mathbb{R}^n) \frac{(k - \overrightarrow{2})!}{(4\pi\nu)^{k - \overrightarrow{1}}}\ a_\nu, \tag{1.3.2}$$

where

$$\Lambda = \left\{ \mu \in \mathbb{R}^n : \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \Gamma_K \right\}.$$

One has the following growth of Fourier coefficients of a Hilbert modular form.

Proposition 1.3.3. *(Hecke) Let $f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \succeq 0}} a_m e[\mathrm{tr}(mz)] \in M_k(\Gamma_K)$, then*

$$a_m \ll m^{k - \overrightarrow{1}}. \tag{1.3.3}$$

If f is a cusp form, then

$$a_m \ll m^{\frac{k}{2}}. \tag{1.3.4}$$

For more details on the theory of Hilbert modular forms, we refer to [15].

1.3.1 Rankin-Cohen brackets

There are many interesting connections between differential operators and modular forms and many interesting results have been found. Rankin [44] gave a general description of the differential operators which send modular forms to modular form. Cohen [11] constructed bilinear operators and obtained elliptic modular forms with interesting Fourier coefficients. Zagier [52] studied the algebraic properties of these bilinear operators and called them Rankin-Cohen brackets.

Definition 1.3.4 (Rankin-Cohen bracket)**.** *Let k, l and $n \geqslant 0$ be integers. Let f and g be two holomorphic functions on $\mathbb{H}$. Then the n-th Rankin-Cohen bracket of f and g is defined by*

$$[f,g]_n := \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{\Gamma(k+n)\Gamma(l+n)}{\Gamma(k+r)\Gamma(l+n-r)} D^r f D^{n-r} g, \tag{1.3.5}$$

where $D^r f = \dfrac{1}{(2\pi i)^r} \dfrac{d^r f}{dz^r}$.

Proposition 1.3.5. *[11] For $f \in M_k(1)$ and $g \in M_l(1)$ and for every $n \geqslant 0$, the function $[f,g]_n$ defined by (1.3.5) belong to $M_{k+l+2n}(1)$.*

In 1997, Choie, Kim and Richter [10] generalized the theory of Rankin-Cohen brackets to the space of Hilbert modular forms. For $t = (t_1, \ldots, t_n) \in \mathbb{N}_0^n$, let $f^{(t)}(z) := \dfrac{\partial^{|t|}}{\partial z_1^{t_1} \partial z_2^{t_2} \cdots \partial z_n^{t_n}} f(z)$.

Definition 1.3.6 (Rankin-Cohen bracket of Hilbert modular forms)**.** *Suppose $f_i : \mathbb{H}^n \to \mathbb{C}$ be holomorphic for $i = 1, 2$ and $k = (k_1, \ldots, k_n), l = (l_1, \ldots, l_n) \in \mathbb{N}_0^n$. For all $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$, define the ν-th Rankin-Cohen bracket by*

$$[f_1, f_2]_\nu := \sum_{\substack{t \in \mathbb{N}_0^n \\ 0 \leqslant t_i \leqslant \nu_i}} (-1)^{|t|} \binom{k+\nu-\overrightarrow{1}}{\nu - t} \binom{l+\nu-\overrightarrow{1}}{t} f_1^{(t)}(z) f_2^{(\nu-t)}(z). \tag{1.3.6}$$

Theorem 1.3.7. *[10] For all $M \in SL_2(\mathbb{R})^n$,*

$$[f_1|_k M,\ f_2|_l M]_\nu = [f_1, f_2]_\nu|_{k+l+2\nu} M. \tag{1.3.7}$$

In particular, if $f_1 \in M_k(\Gamma_K)$ and $f_2 \in M_l(\Gamma_K)$ then

$$[f_1, f_2]_\nu \in M_{k+l+2\nu}(\Gamma_K),$$

and if $\nu \neq 0$, then

$$[f_1, f_2]_\nu \in S_{k+l+2\nu}(\Gamma_K).$$

Remark 1.3.1. *For each $\nu \in \mathbb{N}_0^n$, $[. \ , \ .]_\nu$ is a bilinear operator on the space of Hilbert modular forms.*

1.4 Jacobi forms and Hilbert-Jacobi forms

The Jacobi group $\Gamma^J := SL_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$ acts on $\mathbb{H} \times \mathbb{C}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Let k, m be fixed positive integers. For a complex valued function ϕ on $\mathbb{H} \times \mathbb{C}$ and $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma^J$, we define

$$\big(\phi|_{k,m}\gamma\big)(\tau, z) := (c\tau + z)^{-k} e\left[m \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z \right) \right] \phi\big(\gamma \circ (\tau, z)\big).$$

Definition 1.4.1 (Jacobi form)**.** *A Jacobi form of weight k and index m on Γ^J is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ which satisfying the following conditions:*

1. $\phi|_{k,m}\gamma = \phi,$ for all $\gamma \in \Gamma^J$,

2. ϕ *has a Fourier expansion of the form,*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm - r^2 \geqslant 0}} c(n,r)e[(n\tau + rz)].$$

Further, we say ϕ *is a cusp form if* $c(n,r) \neq 0 \Rightarrow r^2 < 4nm$.

The theory of Jacobi forms is systematically developed by Eichler and Zagier [13]. In 2001, Skogman [49] extended the theory of Jacobi forms over any totally real number field K which is known as "Hilbert-Jacobi forms". We now recall some preliminaries about Hilbert-Jacobi forms.

1.4.1 Hilbert-Jacobi forms

Let K be a totally real number field of degree g over $\mathbb{Q}$ with the ring of integers $\mathcal{O}_K$ and we denote its g real embeddings by $\sigma_1, \ldots, \sigma_g$. We denote i-th embedding of an element $\alpha \in K$ by $\alpha^{(i)} := \sigma_i(\alpha)$ for any $1 \leqslant i \leqslant g$. An element $\alpha \in K$ is said to be totally positive, $\alpha > 0$, if all its embeddings $\alpha^{(i)}$ into $\mathbb{R}$ are positive. The trace and norm of $\alpha \in K$ are defined by $\mathrm{tr}(\alpha) = \sum_{i=1}^{g} \alpha^{(i)}$ and $N(\alpha) = \prod_{i=1}^{g} \alpha^{(i)}$. The trace and norm of an element $\alpha \in \mathbb{C}^g$ are given by the sum and by the product of its components, respectively. More generally, for $c = (c_1, \ldots, c_g), d = (d_1, \ldots, d_g), k = (k_1, \ldots, k_g)$ and $m = (m_1, \ldots, m_g) \in \mathbb{C}^g$, we define the following:

$$\mathrm{tr}(mz) := \sum_{i=1}^{g} m_i z_i \quad \text{and} \quad (cz + d)^k := \prod_{i=1}^{g} (c_i z_i + d_i)^{k_i}.$$

Let $\Gamma_K := SL_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\}$. We denote the Hilbert-Jacobi group as $\Gamma^J(K)$ which is defined by

$$\Gamma^J(K) := SL_2(\mathcal{O}_K) \rtimes (\mathcal{O}_K \times \mathcal{O}_K),$$

with the group multiplication

$$\gamma_1.\gamma_2 := \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, (\lambda_1, \mu_1) \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + (\lambda_2, \mu_2) \right),$$

where $\gamma_i := \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, (\lambda_i, \mu_i) \right)$ for $i = 1, 2$. The Hilbert-Jacobi group $\Gamma^J(K)$ acts on the space $\mathbb{H}^g \times \mathbb{C}^g$ by

$$\begin{aligned} &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau_1, \ldots, \tau_g, z_1, \ldots, z_g) \\ &= \left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{a^{(g)}\tau_g + b^{(g)}}{c^{(g)}\tau_g + d^{(g)}}, \frac{z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{z_g + \lambda^{(g)}\tau_g + \mu^{(g)}}{c^{(g)}\tau_g + d^{(g)}} \right), \end{aligned}$$

where $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma^J(K)$ and $(\tau_1, \ldots, \tau_g, z_1, \ldots, z_g) \in \mathbb{H}^g \times \mathbb{C}^g$. For a holomorphic function $\phi : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$, we define the following two slash operators. Let $k \in \mathbb{N}_0^g$ and $m \in \mathcal{O}_K$. We define

$$\big(\phi|_{k,m} M\big)(\tau, z) := (cz + d)^{-k} e\left[\mathrm{tr}\left(-\frac{mcz^2}{c\tau + d}\right)\right] \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (\tau, z) \right), \quad (1.4.1)$$

and

$$\left(\phi|_m(\lambda,\mu)\right)(\tau,z) := e[\mathrm{tr}\left(m(\lambda^2\tau+2\lambda z)\right)]\phi\left((\lambda,\mu)\circ(\tau,z)\right) \tag{1.4.2}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K)$ and $(\lambda,\mu)\in\mathcal{O}_K\times\mathcal{O}_K$.

Definition 1.4.2 (Hilbert-Jacobi form)**.** *A Hilbert-Jacobi form of weight k and index m for the field K is a holomorphic function $\phi:\mathbb{H}^g\times\mathbb{C}^g\to\mathbb{C}$ which satisfies the following conditions:*

1. $\phi|_{k,m}M=\phi,$ for all $M\in SL_2(\mathcal{O}_K)$,

2. $\phi|_m(\lambda,\mu)=\phi,$ for all $(\lambda,\mu)\in\mathcal{O}_K\times\mathcal{O}_K$,

3. *ϕ has a Fourier expansion of the form,*

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}} c_\phi(n,r)e[\mathrm{tr}(n\tau+rz)].$$

We note that $\mathcal{O}_K^*$ is δ_K^{-1}, the inverse of the different ideal of the number field K. Moreover, such a form ϕ is called Hilbert-Jacobi cusp form if $c_\phi(n,r)=0$ whenever $4nm-r^2=0$. Let $J_{k,m}^K$ ($J_{k,m}^{K,\mathrm{cusp}}$) denote the space of Hilbert-Jacobi forms (Hilbert-Jacobi cusp forms) of weight k and index m for the field K. For more details on the theory of Hilbert-Jacobi forms, we refer to [49].

1.4.2 Rankin-Cohen type operators

Using differential operators, Eichler and Zagier [13] constructed Jacobi forms which increase the weight by 1. More precisely,

Theorem 1.4.3. *[13] Let $\phi(\tau, z)$ and $\phi'(\tau, z)$ be Jacobi forms of weight k and k' and m and m', respectively. Then*

$$m'(\partial_z\phi)\phi' - m\phi(\partial_z\phi')$$

is a Jacobi form of weight $k + k' + 1$ and index $m + m'$.

Using the heat operators on the space of Jacobi forms, Choie [6, 7] studied such construction which increase the weight by even.

Theorem 1.4.4. *[6, 7] Let k, k', m and m' be positive integers and $\nu \geqslant 0$ be an integer. Let $\phi(\tau, z)$ and $\phi'(\tau, z)$ be Jacobi forms of weight k and k' and m and m', respectively. Then the function $[\phi, \phi']_\nu$, defined by*

$$[\phi, \phi']_\nu := \sum_{t=0}^{\nu} (-1)^t \binom{k + \nu - \frac{3}{2}}{\nu - t} \binom{k' + \nu - \frac{3}{2}}{t} m^{\nu-t} m'^t L_m^t(\phi) L_{m'}^{\nu-t}(\phi'), \quad (1.4.3)$$

is a Jacobi form of weight $k + k' + 2\nu$ and index $m + m'$. Here for $m \in \mathbb{Z}$, L_m is the m-th heat operator which is defined by

$$L_m := \frac{1}{(2\pi i)^2}\left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right).$$

Further, Choie and Eholzer [8] studied Rankin-Cohen type operators for Jacobi forms in a more generality. More precisely, they proved the following.

Theorem 1.4.5. *[8] Let $\phi(\tau, z)$ and $\phi'(\tau, z)$ be Jacobi forms of weight k and k' and m and m', respectively. Then for $X \in \mathbb{C}$, the functions $[\phi, \phi']_{X,2l}^{k,k',m,m'}$ and $[\phi, \phi']_{X,2l+1}^{k,k',m,m'}$, defined by*

$$[\phi, \phi']_{X,2l}^{k,k',m,m'} := \sum_{r+s+p=l} C_{r,s,p}(k, k')(1 - mX)^s (1 + m'X)^r L_{m+m'}^p (L_m^r(\phi) L_{m'}^s(\phi')).$$

and

$$[\phi,\phi']^{k,k',m,m'}_{X,2l+1} := m[\phi,\partial_z\phi']^{k,k',m,m'}_{X,2l} - m'[\partial_z\phi,\phi']^{k,k',m,m'}_{X,2l}.$$

are Jacobi forms of weight $k+k'+2l$ *and* $k+k'+2l+1$ *of same index* $m+m'$*, respectively. The coefficients* $C_{r,s,p}(k,k')$ *are given by*

$$C_{r,s,p}(k,k') = \frac{(\alpha+r+s+p)_{s+p}\ \ (\beta+r+s+p)_{r+p}\ \ (-(\gamma+r+s+p))_{r+s}}{r!\ \ s!\ \ p!}$$

with $(x)_n = \prod_{0\leqslant i\leqslant n-1}(x-i), \alpha = k-3/2, \beta = k'-3/2, \gamma = k+k'-3/2.$

Using the Maass operators on the space of nearly holomorphic Jacobi forms, Böcherer [3] showed that the space of bilinear holomorphic differential operators raising the weight ν is in general of dimension $1+[\nu/2]$. Furthermore, he noticed that one can give a basis of such space in terms of the operators defined by Theorem 1.4.5. In chapter 4, we generalize the above construction of Choie and Eholzer for Hilbert-Jacobi forms.

Chapter 2

Simultaneous non-vanishing and sign changes of Fourier coefficients of modular forms

In this chapter, we study simultaneous non-vanishing and simultaneous sign changes for Fourier coefficients of two distinct modular forms.

2.1 Introduction and statement of the theorems

Throughout the chapter, let k, N be positive integers and p be a prime. We recall that $S_k(N)$ denotes the space of cusp forms of weight k for the group $\Gamma_0(N)$. Let $\tau(n)$ be the n-th Fourier coefficient of the Ramanujan delta function $\Delta(z)$, defined by $(1.2.3)$. The function $n \mapsto \tau(n)$ is studied by Ramanujan in 1916, which is called the Ramanujan Tau function. After some numerical evidence, in 1939 Lehmer [31] conjectured that $\tau(n) \neq 0$ for all $n \geqslant 1$. This conjecture is an important open problem in number theory. Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n, \ g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_k(N)$

be two non-zero cusp forms which are not a linear combination of CM forms. One of the goals of this chapter is to study simultaneous non-vanishing of $a(n)$ and $b(n)$, partially inspired from the above Lehmer's conjecture.

Motivated from the Lehmer's conjecture, Serre [48] defined an arithmetic function

$$i_f(n) := \min\{j \geqslant 0 : a(n+j) \neq 0\},$$

for a cusp form $f(z) = \sum_{n=1}^{\infty} a(n)q^n$, and proved that $i_f(n) \ll_f n$, provided f is not a linear combination of CM forms. In the same paper [48], he posed the question whether one can prove an estimate of the form

$$i_f(n) \ll_f n^{\delta},$$

where $\delta < 1$. In his paper [39], Kumar Murty first pointed out that $i_f(n) \ll n^{3/5}$ follows immediately from the celebrated work of Rankin [43] and Selberg [47] done in 1939 and 1940, respectively. After that many authors improved the value of δ (for more detail see [12]).

In the case of level 1 and f is an eigenform, Das and Ganguly [12] discovered a clever argument to show $i_f(n) \ll n^{1/4}$ by combining a classical result of Bambah and Chowla [2] with a congruence of Hatada [17] along with a basic lemma of Murty and Murty [38]. Here is a synopsis of their elegant proof. In 1947, Bambah and Chowla showed using an elementary argument that in any interval of length $x^{1/4}$ there is a number n (say) which can be written as a sum of two squares. As f is an eigenform, $a(n)$ is multiplicative. Hatada's theorem [17] implies that $a(p) \equiv 2 \pmod 4$, for $p \equiv 1 \pmod 4$ and $a(p^r) \equiv 1 \pmod 4$ if r is even and $p \equiv 3 \pmod 4$. The one of the lemma in [38] shows that $a(p^r) \neq 0$ for $p \equiv 1 \pmod 4$ provided p is sufficiently large. These congruences combined with the classical theorem about factorization of natural numbers that can be written as a sum of two integral squares now imply $a(n) \neq 0$ provided n is coprime to a given finite set of primes. Thus, one now needs the Bambah-Chowla theorem with n coprime to a finite set of primes. One can tweak the argument in [2] to accommodate this extra condition and thus deduce the non-vanishing result as done in [12]. Actually, the argument of Bambah and Chowla can be generalized with considerable latitude. At the end of the chapter, we generalize the idea of Bambah and Chowla in more generality.

We hasten to highlight that the argument of Das and Ganguly allows for simultaneous non-vanishing. In fact, if $f_1, ..., f_r$ are normalized eigenforms of level 1, with corresponding Fourier coefficients $a_{f_j}(n)$, then one can find an i with $i \ll n^{1/4}$ such that

$$a_{f_j}(n+i) \neq 0, \qquad \text{for all} \quad 1 \leqslant j \leqslant r.$$

It has been suggested that perhaps $i_f(n) \ll n^\epsilon$ for any $\epsilon > 0$. Perhaps even the stronger conjecture $i_f(n) \ll 1$ is true (see for example, [39]).

Let f and g be two cusp forms with Fourier coefficients $a(n)$ and $b(n)$ respectively. Then we investigate non-vanishing of the sequence $\{a(n)b(n)\}_{n\in\mathbb{N}}$.

Theorem 2.1.1. *[26] Suppose that* $f(z) = \sum_{n=1}^{\infty} a(n)n^{\frac{k-1}{2}}q^n \in S_k(N)$ *and* $g(z) = \sum_{n=1}^{\infty} b(n)n^{\frac{k-1}{2}}q^n \in S_k(N)$ *are two newforms which are not CM forms, then there exist infinitely many primes* p *such that* $a(p)b(p) \neq 0$.

Now, we introduce the concept of gap function $i_{f,g}$ for simultaneous non-vanishing analogous to that of i_f and then we derive a bound for $i_{f,g}$ as small as possible, based on current knowledge. For $n \in \mathbb{N}$, define

$$i_{f,g}(n) := \min\{m \geqslant 0 : a(n+m)b(n+m) \neq 0\},$$

which is well-defined from the above theorem. We are interested to find the growth of the function $i_{f,g}(n)$ as $n \to \infty$. In 2014, Lu [33] by using the result of Chandrasekharan and Narasimhan [4] proved the following.

$$\sum_{n \leqslant x} a(n)^2 b(n)^2 = cx + O(x^{\frac{7}{8}+\varepsilon}),$$

where c is a non-zero constant. It then follows that $i_{f,g}(n) \ll n^{\frac{7}{8}+\varepsilon}$. Here we give a better estimate than above.

Theorem 2.1.2. *[26] Suppose that* $f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} q^n \in S_k(N)$ *and* $g(z) = \sum_{n=1}^{\infty} b(n) n^{\frac{k-1}{2}} q^n \in S_k(N)$ *are two newforms with* $k > 2$ *which are not a linear combination of CM forms. Then the following results hold.*

(i) For every $\varepsilon > 0$, $x > x_0(f,g,\varepsilon)$ *and* $x^{\frac{7}{17}+\varepsilon} \leqslant y$ *we have*

$$|\{x < n < x + y : a(n)b(n) \neq 0\}| \gg_{f,g,\varepsilon} y. \tag{2.1.1}$$

In particular, we get that $i_{f,g}(n) \ll_{f,g,\varepsilon} n^{\frac{7}{17}+\varepsilon}$.

(ii) For every $\varepsilon > 0$, $x \geqslant x_0(f,g,\varepsilon)$, $y \geqslant x^{\frac{17}{38}+100\varepsilon}$ *and* $1 \leqslant a \leqslant q \leqslant x^{\varepsilon}$ *with* $(a,q) = 1$, *we have*

$$|\{x < n \leqslant x + y : \ n \equiv a \pmod q \text{ and } a(n)b(n) \neq 0\}| \gg_{f,g,\varepsilon} y/q. \tag{2.1.2}$$

In 2009, Kohnen and Sengupta [23] considered a problem related with the simultaneous sign changes. They proved that given two normalized cusp forms f and g of the same level and different weights with totally real algebraic Fourier coefficients, there exists a Galois automorphism σ such that f^{σ} and g^{σ} have infinitely many Fourier coefficients of the opposite sign. Recently Gun, Kohnen and Rath [16] removed the dependency on the Galois conjugacy. In fact, they extended their result to arbitrary cusp forms with arbitrary real Fourier coefficients but they assumed that both f and g should have first Fourier coefficient to be non-zero. More precisely, they proved the following.

Theorem 2.1.3. *[16] Let*

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \text{ and } g(z) = \sum_{n=1}^{\infty} b(n)q^n$$

be non-zero cusp forms of level N and weights $1 < k_1 < k_2$ respectively. Suppose that $a(n), b(n)$ are real numbers. If $a(1)b(1) \neq 0$, then there exist infinitely many n such that $a(n)b(n) > 0$ and infinitely many n such that $a(n)b(n) < 0$.

If f and g are newforms then we have the following quantitative result for the simultaneous sign changes.

Theorem 2.1.4. *[26] Let $k \geqslant 2$ be an integer. Assume that*

$$f(z) = \sum_{n\geqslant 1} a(n)n^{\frac{k-1}{2}}q^n \text{ and } g(z) = \sum_{n\geqslant 1} b(n)n^{\frac{k-1}{2}}q^n$$

are two distinct newforms of weight k on $\Gamma_0(N)$. Further, let $a(n), b(n)$ be real numbers, then for any $\delta > \frac{7}{8}$, the sequence $\{a(n)b(n)\}_{n\in\mathbb{N}}$ has at least one sign change for $n \in (x, x + x^\delta]$ for sufficiently large x. In particular, the number of sign changes for $n \leqslant x$ is $\gg x^{1-\delta}$.

2.2 Useful results

In this section we state some results from the literature which are used in our proofs. In 1982, Serre [48, p.174, Cor.2] proved the following result.

Lemma 2.2.1. *[48] Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N)$ be a newform with weight $k \geqslant 2$ which does not have complex multiplication. For every $\epsilon > 0$ we have*

$$|\{p \leqslant x : a(p) = 0\}| \ll_{f,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}.$$

To prove Theorem 2.1.2, we shall use the concept of $\mathcal{B}$-free numbers which was introduced by Erdös [14] in 1966 and later many authors studied the distribution of $\mathcal{B}$-free numbers.

Definition 2.2.2 ($\mathcal{B}$-free numbers)**.** *Let $\mathcal{B} = \{b_i : 1 < b_1 < b_2 < \cdots\}$ be a sequence of mutually coprime positive integers for which $\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty$. A positive integer n is called $\mathcal{B}$-free if it is not divisible by any element in $\mathcal{B}$.*

By using sieve theory and estimates for multiple exponential sums, Chen and Wu [5] studied the distribution of $\mathcal{B}$-free numbers in short intervals as well as in an arithmetic progression and proved the following.

Proposition 2.2.3. *[5] Let $\mathcal{B}$ be a sequence of positive integers satisfying the conditions in the definition of $\mathcal{B}$-free numbers. Then,*

(i) for any $\varepsilon > 0,\ x > x_o(\mathcal{B}, \varepsilon)$ and $y \geqslant x^{\frac{7}{17}+\varepsilon}$, we have

$$|\{x < n \leqslant x + y : n \text{ is } \mathcal{B}\text{-free }\}| \gg_{\mathcal{B},\varepsilon} y, \tag{2.2.1}$$

(ii) for any $\varepsilon > 0,\ x > x_o(\mathcal{B}, \varepsilon)$ and $y \geqslant x^{\frac{17}{38}+100\varepsilon},\ 1 \leqslant a \leqslant q \leqslant x^{\varepsilon}$ with $((a, q), b) = 1,$ for all $b \in \mathcal{B}$, we have

$$|\{x < n \leqslant x + y : n \equiv a \pmod q \text{ and } n \text{ is } \mathcal{B}\text{-free }\}| \gg_{\mathcal{B},\varepsilon} y/q. \tag{2.2.2}$$

Here the implied constants depend only on $\mathcal{B}$ and ε.

Recently Meher and Ram Murty [35], gave a general criteria for the sign changes of any sequence of real numbers $\{a(n)\}_{n\in\mathbb{N}}$. More precisely, they proved the following.

Theorem 2.2.4. *[35] Let $\{a(n)\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

(i) $a(n) = O(n^{\alpha})$,

(ii) $\sum_{n\leqslant x} a(n) = O(n^{\beta})$,

(iii) $\sum_{n\leqslant x} a(n)^2 = cx + O(n^{\gamma})$,

with $\alpha, \beta, \gamma, c \geqslant 0$. If $\alpha + \beta < 1$, then for any δ satisfying, $\max\{\alpha+\beta, \gamma\} < \delta < 1$, the sequence $\{a(n)\}_{n\in\mathbb{N}}$ has at least one sign change for $n \in [x, x + x^{\delta}]$. Consequently, the number of sign changes of $a(n)$ for $n \leqslant x$ is $\gg x^{1-\delta}$ for sufficiently large x.

2.3 Proof of Theorem 2.1.1

From Lemma 2.2.1, we have

$$|\{p \leqslant x : a(p) = 0\}| \ll_{f,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}},$$

and

$$|\{p \leqslant x : b(p) = 0\}| \ll_{g,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}.$$

Since $a(p)b(p) = 0$, we have either $a(p) = 0$ or $b(p) = 0$. Hence

$$|\{p \leqslant x : a(p)b(p) = 0\}| \ll_{f,g,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}.$$

By the prime number theorem, we have

$$\pi(x) := |\{p \leqslant x\}| \sim \frac{x}{\log x}.$$

Hence

$$|\{p \leqslant x : a(p)b(p) \neq 0\}| = \pi(x) - |\{p \leqslant x : a(p)b(p) = 0\}| \sim \frac{x}{\log x}.$$

Thus there exist infinitely many primes p such that $a(p)b(p) \neq 0$. Hence we complete the proof.

Actually, the theorem is true without the constraint that the forms are not CM. We make some remarks in the case that either f or g is of CM type. Suppose first that f has CM by an order in an imaginary quadratic field K and g does not. Then, for primes p coprime to the level of f, $a(p) = 0$ if and only if p is inert in K. The density of such primes is $1/2$ and so

$$|\{p \leqslant x : a(p)b(p) \neq 0\}| = \pi(x) - |\{p \leqslant x : a(p)b(p) = 0\}| \gtrsim \frac{x}{2\log x}.$$

Hence, in this case also there are infinitely many primes p such that $a(p)b(p) \neq 0$. If both f and g have CM by two imaginary quadratic fields K_1, K_2 (say, respectively), then we need only choose primes p coprime to the level which split in K_1 and K_2. This density is either $1/2$ (if $K_1 = K_2$) or $1/4$ (if $K_1 \neq K_2$). Thus, in all cases, Theorem 2.1.1 is valid in general.

2.4 Proof of Theorem 2.1.2

Let $S = \{p : a(p)b(p) = 0\} \cup \{p|N\}$. Put $\mathcal{B} = S \cup \{p^2 : p \notin S\}$. Clearly $\mathcal{B}$ is a sequence of mutually coprime integers and if n is $\mathcal{B}$-free, then n is square-free and

$a(n)b(n) \neq 0$ by using the multiplicative properties of $a(n)$ and $b(n)$. Thus $(2.2.1)$ and $(2.2.2)$ imply the first and second assertions of Theorem 1.2 respectively, if we can show that $\sum_{p\in\mathcal{B}} \frac{1}{p} < \infty$. Since $\sum_p 1/p^2 < \infty$, it suffices to show that

$$\sum_{p\in S} \frac{1}{p} < \infty.$$

We know, from Lemma 2.2.1 that

$$\sum_{\substack{p\leqslant x\\ p\in S}} 1 \ll_{f,g} \frac{x}{(\log x)^{1+\delta}}, \quad \text{for some} \quad \delta > 0.$$

Hence, by partial summation formula, we have

$$\sum_{\substack{p\leqslant x\\ p\in S}} \frac{1}{p} = \frac{1}{x}\sum_{\substack{p\leqslant x\\ p\in S}} 1 + \int_2^x \frac{1}{t^2}(\sum_{\substack{p\leqslant t\\ p\in S}} 1)dt \ll_{f,g} \frac{1}{(\log x)^{1+\delta}} + \int_2^x \frac{dt}{t(\log t)^{1+\delta}}$$
$$\ll_{f,g} 1.$$

This completes the proof of Theorem 2.1.2.

2.5 Proof of Theorem 2.1.4

We prove Theorem 2.1.4, as an application of the Theorem 2.2.4, for which we have to analyse the stated conditions for the sequence $\{a(n)b(n)\}_{n\in\mathbb{N}}$.

(i) Ramanujan-Deligne:

$$a(n)b(n) = O(n^{\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{2.5.1}$$

From the paper of Lu [33], one can deduce the following results.

(ii)

$$\sum_{n\leqslant x} a(n)b(n) \ll x^{\frac{3}{5}}(\log x)^{-\frac{2\theta}{3}}, \tag{2.5.2}$$

where $\theta = 0.1512....$

(iii)

$$\sum_{n\leqslant x} a(n)^2 b(n)^2 = cx + O(x^{\frac{7}{8}+\varepsilon}).$$

Hence from Theorem 2.2.4, we immediately deduce Theorem 2.1.4.

2.6 Generalization of a result of Bambah and Chowla

By a straightforward elementary construction, Bambah and Chowla [2] proved the existence of a constant C such that for $x > 0$ there is at least one integer between x and $x + Cx^{\frac{1}{4}}$ which can be expressed as a sum of two squares. They conjectured that there should be an integer expressible as sums of two squares in any interval of the form $(X, X + X^{\epsilon})$ for any $\epsilon > 0$ for all X sufficiently large. Here we give a generalization of Bambah and Chowla result.

Theorem 2.6.1. *Let r and s be natural numbers and set $\alpha = (r-1)(s-1)/rs$. There is an effectively computable C (depending only on r and s) such that in any interval of the form $[n, n + Cn^{\alpha}]$, there is a number m which can be written as*

$$m = A^r + B^s,$$

with A and B integers.

Proof. We essentially follow Bambah and Chowla [2] and modify their argument to our setting. Let $t = [n^{1/s}] = n^{1/s} - \theta$ with $0 \leqslant \theta < 1$. Let x_1, x_2 be positive real numbers such that

$$x_1^r + t^s = n,$$

$$x_2^r + t^s = n + Cn^\alpha$$

with C to be chosen later. Thus, $x_2^r - x_1^r = Cn^\alpha$. Now,

$$x_1 = (n - t^s)^{1/r} \ll n^{(s-1)/rs}, \qquad x_2 \ll n^{(s-1)/rs},$$

by a simple application of the binomial theorem. Hence,

$$x_2^{r-1} + x_2^{r-2}x_1 + \cdots + x_1^{r-1} \ll n^{(s-1)(r-1)/rs} = n^\alpha.$$

Now writing

$$(x_2 - x_1)(x_2^{r-1} + \cdots + x_1^{r-1}) = x_2^r - x_1^r = Cn^\alpha,$$

we immediately see that

$$x_2 - x_1 > 1,$$

for a suitable choice of C (in fact, $C = 2^{rs}rs$ will work). Therefore, there is a natural number N in the interval $[x_1, x_2]$ so that

$$n = x_1^r + t^s < N^r + t^s < x_2^r + t^s = n + Cn^\alpha,$$

as desired. This completes the proof of Theorem 2.6.1. □

We remark that there are several variations of this theorem that can be derived from this proof. For example, if $f(x)$ is a monotonic, continuous function for x sufficiently large, and $f(x) \asymp x^r$, then there is a natural number m such that $m = f(A) + B^s$ for some natural numbers A, B and with $m \in [n, n + Cn^\alpha]$. In particular, this can be applied to the norm form $a^2 + Db^2$, with D squarefree. We record these remarks with the view that the result may have potential applications in other contexts.

2.7 Concluding remarks

It would be interesting to extend Hatada's congruence to modular forms of higher level. This is a research problem of independent interest and is accessible since there have been significant advances in the theory of congruences of modular forms. If one assumes standard conjectures about distribution of primes such as Cramér's conjecture, then it is easy to deduce that $i_f(n) = O(\log^2 n)$. The other problem that suggests itself is to obtain estimates with their dependence on level and weight made explicit. An initiation into such an enterprise can be found in the methods of [36] and [37].

The analogues of these questions for modular forms of half-integral weight takes us into a parallel universe of ideas. There is, of course, a link between these two worlds provided by Waldspurger's theorem and the question is equivalent to the simultaneous non-vanishing of quadratic twists of L-series attached to modular forms. A modest beginning in this line of research was initiated in [25].

Chapter 3

Rankin-Cohen brackets and construction of Hilbert cusp forms

3.1 Introduction

Let $f(z) = \sum_{m=1}^{\infty} a_m q^m \in S_k(1)$ and $g(z) = \sum_{m=1}^{\infty} b_m q^m \in S_l(1)$. For a positive integer n, define the shifted Dirichlet series of Rankin-Selberg type as follows:

$$L_{f,g;n}(s) := \sum_{m=1}^{\infty} \frac{a_{m+n}\overline{b_m}}{(n+m)^s}, \qquad s \in \mathbb{C}.$$

Using Deligne's estimate $(1.2.5)$ one can see that the series $L_{f,g;n}(s)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{k+l}{2}$. Using the existence of adjoint map and property of Poincaré series, Kohnen [22] constructed cusp forms whose Fourier coefficients involve special values of the above Dirichlet series.

Theorem 3.1.1. *[22] Let k and l be a positive integers with $k > l + 2$. Let*

$f(z)=\sum_{m=1}^{\infty} a_m q^m \in S_{k+l}(1)$ and $g(z)=\sum_{m=1}^{\infty} b_m q^m \in S_l(1)$. Then the function

$$T_g^*(f)(z) := \sum_{m=1}^{\infty} m^{k-1} L_{f,g:m}(k+l-1) q^m$$

is a cusp form of weight k for $SL_2(\mathbb{Z})$. In fact, the map $S_{k+l}(1) \to S_k(1)$ defined by $f \mapsto \frac{\Gamma(k+l-1)}{\Gamma(k-1)(4\pi)^l} T_g^*(f)$ is the adjoint of the map $T_g : S_k(1) \to S_{k+l}(1),\ h \mapsto gh$, with respect to the Petersson scalar product.

This result has been generalized by Choie, Kim and Knoop [9] and Sakata [46] for Jacobi forms and then by Lee [30] for Siegel modular forms. Moreover, Lee [29], Pei and Wang [41], Wang [50] have analogous results for Hilbert modular forms.

Recently the work of Kohnen has been generalized by Herrero [18] where he constructed the adjoint map using the Rankin-Cohen brackets by a fixed cusp form instead of product map. More precisely, for a fixed $g \in M_l(1)$ and an integer $n \geqslant 0$, consider the linear map

$$T_{g,n} : S_k(1) \longrightarrow S_{k+l+2n}(1)$$

defined by $f \longmapsto [f,g]_n$ (the n-th Rankin-Cohen bracket defined by $(1.3.5)$). Let $T_{g,n}^*$ be its adjoint map with respect to the Petersson inner product defined by $(1.2.4)$. Herrero [18] computed the explicitly expression for the map $T_{g,n}^*$, which is as follows.

Theorem 3.1.2. *[18] Suppose k, l, n are non-negative integers with $k \geqslant 6$. Let $g(z)=\sum_{m=0}^{\infty} b_m q^m \in M_l(1)$. Suppose that either g is a cusp form or $l < k-3$. Then the image of any cusp form $f(z)=\sum_{m=1}^{\infty} a_m q^m \in S_{k+l+2n}(1)$ under $T_{g,n}^*$ is given by*

$$T_{g,n}^*(f)(z) = \frac{\Gamma(k+l+2n-1)}{(4\pi)^{l+2n}\Gamma(k-1)} \sum_{m=1}^{\infty} m^{k-1} \left(\sum_{r=0}^{\infty} \frac{a_{m+r}\overline{b}_r}{(m+r)^{k+l+2n-1}} \varepsilon_{m,r}^{k,l,n} \right) q^m,$$

where

$$\varepsilon_{m,r}^{k,l,n} = \sum_{t \in \mathbb{N}_0, 0 \leqslant t \leqslant n} (-1)^t \binom{k+n-1}{n-t} \binom{l+n-1}{t} m^t r^{n-t}.$$

The result of Hererro has been generalized for Jacobi forms [19] and Siegel modular forms [20] by Jha and Sahu. In this chapter, we generalize the work of Herrero to the case of Hilbert modular forms. First we state our main theorem and then prove some intermediate results which we need for our proof. In the last section, we give an application of the main result.

3.2 Statement of the theorem

For a fixed $g \in M_l(\Gamma_K)$ and $\nu \in \mathbb{N}_0^n$, consider the linear map

$$T_{g,\nu} : S_k(\Gamma_K) \longrightarrow S_{k+l+2\nu}(\Gamma_K),$$

defined by

$$f \longmapsto [f, g]_\nu, \tag{3.2.1}$$

where $[f, g]_\nu$ is the ν-th Rankin-Cohen bracket of f and g defined by $(1.3.6)$. Since $S_k(\Gamma_K)$ is a finite-dimensional Hilbert space, there exists the adjoint map

$$T_{g,\nu}^* : S_{k+l+2\nu}(\Gamma_K) \longrightarrow S_k(\Gamma_K) \tag{3.2.2}$$

satisfying

$$\langle T_{g,\nu}^*\ f, h\rangle = \langle f, T_{g,\nu} h\rangle, \quad \forall\ f \in S_{k+l+2\nu}(\Gamma_K) \text{ and } h \in S_k(\Gamma_K).$$

We compute the Fourier coefficients of $T_{g,\nu}^*(f)$ explicitly which involve special values of certain Dirichlet series associated to f and g. More precisely, we prove the following result.

Theorem 3.2.1. *[27] Suppose $k, l, \nu \in \mathbb{N}_0^n$ with $k_i \geqslant 4n+2$ for some i. Let $g \in M_l(\Gamma_K)$ with Fourier expansion*

$$g(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \succeq 0}} b_m e[\mathrm{tr}(mz)].$$

Suppose that either (a) g is a cusp form or (b) g is not cusp form and $k_i - l_i > 4n$ for some i. Then the image of any cusp form $f \in S_{k+l+2\nu}(\Gamma_K)$ with Fourier expansion

$$f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} a_m e[\mathrm{tr}(mz)],$$

under $T^*_{g,\nu}$ is given by

$$T^*_{g,\nu}(f)(z) = \sum_{\substack{\mu \in \mathcal{O}_K^* \\ \mu \gg 0}} c_\mu e[\mathrm{tr}(\mu z)],$$

where

$$c_\mu = \frac{\Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{l+2\nu}\Gamma(k-\overrightarrow{1})}\mu^{k-\overrightarrow{1}} \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} \frac{a_{m+\mu}\overline{b}_m}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}} \varepsilon^{k,l,\nu}_{\mu,m} \tag{3.2.3}$$

and

$$\varepsilon^{k,l,\nu}_{\mu,m} = \sum_{\substack{t \in \mathbb{N}_0^n \\ 0 \leqslant t_i \leqslant \nu_i}} (-1)^{|t|} \binom{k+\nu-\overrightarrow{1}}{\nu-t} \binom{l+\nu-\overrightarrow{1}}{t} \mu^t m^{\nu-t}. \tag{3.2.4}$$

Notice that our result generalises the work of Pei and Wang [41] and Wang [50] where the authors computed the adjoint map for $\nu = \overrightarrow{0}$. We now state some useful facts.

Facts: **(a)** Let $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$. For $w \in K$, we denote w^s to be the product

$\prod_{i=1}^{n} \sigma_i(w)^{s_i}$. Then the series

$$\sum_{\substack{w \in \mathcal{O}_K^* \\ w \gg 0}} \frac{1}{w^s},$$

converges absolutely if $\mathrm{Re}(s_i) > n$ for some $i,\ 1 \leq i \leq n$.

Using this fact and the estimates for the Fourier coefficients of f and g from Proposition 1.3.3, one can prove that the series in $(3.2.3)$ converges absolutely.

(b) Let $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$. Then the series

$$\sum_{\substack{\mu,\nu \in \mathcal{O}_K^* \\ \nu > 0, \mu \gg 0}} \frac{1}{(\mu + \nu)^s},$$

converges absolutely if $\mathrm{Re}(s_i) > 2n$ for some $i,\ 1 \leq i \leq n$.

3.3 Intermediate results

We need the following lemma to prove the Theorem 3.2.1.

Lemma 3.3.1. *[27] Let f and g be Hilbert modular forms with Fourier coefficients a_n and b_m respectively as in Theorem 3.2.1. Then the series*

$$\sum_{\substack{n,m \in \mathcal{O}_K^* \\ n \gg 0, m \succeq 0}} \frac{|a_n b_m m^\nu|}{(n + m + \mu)^{k+l+2\nu-\overrightarrow{1}}} \tag{3.3.1}$$

converges.

Proof. Using Proposition 1.3.3, we have $a_n \ll n^{\frac{k+l+2\nu}{2}}$ and $b_m \ll m^{\frac{l}{2}}$ (if g is a Hilbert cusp form). Hence the series (3.3.1) satisfies

$$\ll \sum_{\substack{n,m \in \mathcal{O}_K^* \\ n \gg 0, m \succeq 0}} \frac{1}{(n + m + \mu)^{k/2-\overrightarrow{1}}},$$

which converges absolutely using fact (a) (see, section 3.2) as $k_i \geq 4n+2$ for some i. If g is not a cusp form, then $b_m \ll m^{l-1}$ and the series (3.3.1) satisfies

$$\ll \sum_{\substack{n,m \in \mathcal{O}_K^* \\ n \gg 0, m \succeq 0}} \frac{1}{(n+m+\mu)^{k/2-l/2}},$$

which converges absolutely using fact (b) (see, section 3.2) as $k_i - l_i > 4n$ for some i. □

Proposition 3.3.2. *[27] Let f and g be Hilbert modular forms as in Theorem 3.2.1 and $\mu(\gg 0) \in \mathcal{O}_K^*$. Then the series*

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_K} \int_{\Gamma_K \backslash \mathbb{H}^n} |f(z)\overline{[e[\mathrm{tr}(\mu z)]|_k \gamma, g]_\nu}(z)\ y^{k+l+2\nu}| \frac{dxdy}{y^2} \tag{3.3.2}$$

converges.

Proof. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$, changing the variable z to $\gamma^{-1} \circ z$ for each integral, the sum (3.3.2) equals to

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_K} \int_{\gamma(\Gamma_K \backslash \mathbb{H}^n)} |f(\gamma^{-1} \circ z)\overline{[e[\mathrm{tr}(\mu z)]|_k \gamma, g]_\nu}(\gamma^{-1} \circ z)| \frac{y^{k+l+2\nu}}{|j(\gamma^{-1}, z)|^{2(k+l+2\nu)}} \frac{dxdy}{y^2}.$$

By (1.3.7), the sum is equal to

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_K} \int_{\gamma(\Gamma_K \backslash \mathbb{H}^n)} |f(z)\overline{[e[\mathrm{tr}(\mu z)], g]_\nu(z)}| y^{k+l+2\nu} \frac{dxdy}{y^2}.$$

Now using the Rankin-Selberg unfolding argument, the above sum is equal to

$$\int_{\Gamma_\infty \backslash \mathbb{H}^n} |f(z)\overline{[e[tr(\mu z)], g]_\nu(z)}| y^{k+l+2\nu} \frac{dxdy}{y^2}.$$

Replacing $f(z)$ and $g(z)$ with their Fourier expansions and using the definition of Rankin-Cohen brackets, the last integral is majorized by

$$\sum_{\substack{t\in\mathbb{N}_0^n\\ 0\leqslant t_i\leqslant\nu_i}} \alpha_{\nu,\mu}(t)\int_{\Gamma_\infty\backslash\mathbb{H}^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0\\ m\succeq 0}} |a_n\overline{b_m}m^{\nu-t}e[\mathrm{tr}(nz)]\overline{e[\mathrm{tr}((m+\mu)z)]}|y^{k+l+2\nu}\frac{dxdy}{y^2},$$

where

$$\alpha_{\nu,\mu}(t)=|(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t|.$$

The above sum is a finite sum and now it suffices to show that the integral

$$\mathcal{I}_t=\int_{\Gamma_\infty\backslash\mathbb{H}^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0\\ m\succeq 0}} |a_n\overline{b_m}m^{\nu-t}e[\mathrm{tr}(nz)]\overline{e[\mathrm{tr}((m+\mu)z)]}|y^{k+l+2\nu}\frac{dxdy}{y^2}$$

is finite for each t. We choose $\mathbb{R}^n\setminus\mathcal{O}_K^*\times(0,\infty)^n$ as a fundamental domain for the action of Γ_∞ on $\mathbb{H}^n$ and integrating over it, we have

$$\ll\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0,m\succeq 0}}\frac{|a_nb_mm^\nu|}{(n+m+\mu)^{k+l+2\nu-\overrightarrow{1}}}.$$

Using Lemma 3.3.1, the above series converges. □

3.4 Proof of Theorem 3.2.1

Let $T^*_{g,\nu}(f)(z)=\sum_{\substack{\mu\in\mathcal{O}_K^*\\ \mu\gg 0}} c_\mu e[\mathrm{tr}(\mu z)]$. Using Theorem 1.3.2 we have

$$\begin{aligned} vol(\mathcal{O}_K/\mathbb{R}^n)(4\pi\mu)^{\overrightarrow{1}-k}(k-\overrightarrow{2})!\ c_\mu &= \langle T^*_{g,\nu}f\ ,\mathcal{P}_{k,\mu}\rangle\\ &=\langle f\ ,[\mathcal{P}_{k,\mu},\ g]_\nu\rangle\end{aligned}$$

$$= \int_{\Gamma_K\backslash\mathbb{H}^n} f(z)\overline{[\mathcal{P}_{k,\mu},\ g]_\nu(z)}y^{k+l+2\nu}\frac{dxdy}{y^2}$$
$$=\int_{\Gamma_K\backslash\mathbb{H}^n}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} f(z)\overline{[e[\mathrm{tr}(\mu z)]|_k\gamma, g]_\nu}(z)y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

By Proposition 3.3.2, the above expression is absolutely convergent, hence one can interchange the summation and the integration. The change of variable z to $\gamma^{-1}\circ z$ for each integral gives

$$vol(\mathcal{O}_K/\mathbb{R}^n)(4\pi\mu)^{\overrightarrow{1}-k}(k-\overrightarrow{2})!c_\mu = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K}\int_{\gamma(\Gamma_K\backslash\mathbb{H}^n)} f(z)\overline{[e[\mathrm{tr}(\mu z)], g]_\nu}(z)\ y^{k+l+2\nu}\frac{dxdy}{y^2}. \tag{3.4.1}$$

Using the Rankin-Selberg unfolding argument, the right hand side of $(3.4.1)$ is equal to

$$\int_{\Gamma_\infty\backslash\mathbb{H}^n} f(z)\overline{[e[\mathrm{tr}(\mu z)], g]_\nu}(z)\ y^{k+l+2\nu}\frac{dxdy}{y^2}. \tag{3.4.2}$$

Using the definition of Rankin-Cohen bracket $(1.3.6)$, the above integral is equal to

$$\sum_{\substack{t\in\mathbb{N}_0^n\\ 0\leqslant t_i\leqslant\nu_i}}(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}\int_{\Gamma_\infty\backslash\mathbb{H}^n} f(z)\overline{e[\mathrm{tr}(\mu z)]^{(t)}g^{(\nu-t)}}(z)\ y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Substituting $f(z)$ and $g(z)$ by their Fourier expansions and observing the repeated action of differential operators,

$$e[\mathrm{tr}(\mu z)]^{(t)} = (2\pi i\mu)^t e[\mathrm{tr}(\mu z)]$$
$$g^{(\nu-t)}(z) = \sum_{\substack{m\in\mathcal{O}_K^*\\ m\succeq 0}}(2\pi i m)^{\nu-t}b_m e[\mathrm{tr}(mz)],$$

the integral $(3.4.2)$ equals

$$\sum_{\substack{t\in\mathbb{N}_0^n\\0\leqslant t_i\leqslant\nu_i}}(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t\int_{\mathbb{R}^n\setminus\mathcal{O}_K^*\times(0,\infty)^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\n\gg 0\\m\succeq 0}}a_n\overline{b_m}$$
$$\times(2i\pi m)^{\nu-t}e[\mathrm{tr}(nz)]\overline{e[\mathrm{tr}((m+\mu)z)]}y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Writing $z=x+iy$ and choosing $\mathbb{R}^n\setminus\mathcal{O}_K^*\times(0,\infty)^n$ as a fundamental domain for $\Gamma_\infty\setminus\mathbb{H}^n$ (see, [15]) the above expression equals

$$\sum_{\substack{t\in\mathbb{N}_0^n\\0\leqslant t_i\leqslant\nu_i}}(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t\int_{\mathbb{R}^n\setminus\mathcal{O}_K^*\times(0,\infty)^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\n\gg 0\\m\succeq 0}}a_n\overline{b_m}$$
$$\times(2i\pi m)^{\nu-t}e[\mathrm{tr}((n-(m+\mu))x)]e[\mathrm{tr}((n+(m+\mu))y)]y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Integrating over x first, we have (see, [15, section 1.13])

$$\int_{\mathbb{R}^n\setminus O_K^*}e[\mathrm{tr}((n-(m+\mu))x)]dx=vol(\mathbb{R}^n\setminus O_K^*)\tag{3.4.3}$$

if $n=m+\mu$ and zero, otherwise. Using $(3.4.3)$ in the previous integral, the integral $(3.4.2)$ equals

$$vol(\mathbb{R}^n\setminus O_K^*)\sum_{\substack{t\in\mathbb{N}_0^n\\0\leqslant t_i\leqslant\nu_i}}(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t$$

$$\times\int_{(0,\infty)^n}\sum_{\substack{m\in\mathcal{O}_K^*\\m\gg 0}}a_{(m+\mu)}\overline{b_m}(\overline{2\pi im})^{\nu-t}e^{-4\pi tr((m+\mu)y)}\ y^{k+l+2\nu}\frac{dy}{y^2}.$$

Integrating over y, we have

$$\int_{(0,\infty)^n} e^{-4\pi tr((m+\mu)y)}\ y^{k+l+2\nu}\frac{dy}{y^2} = \frac{\Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{k+l+2\nu-\overrightarrow{1}}}\frac{1}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}}, \quad (3.4.4)$$

where $\Gamma(k+l+2\nu-\overrightarrow{1}) = \prod_{i=1}^{n}\Gamma(k_i+l_i+2\nu_i-1)$. Finally, substituting $(3.4.4)$ in the previous integral, the integral $(3.4.2)$ is equal to

$$\sum_{\substack{t\in\mathbb{N}_0^n\\ 0\leqslant t_i\leqslant \nu_i}} (-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t vol(\mathcal{O}_K/\mathbb{R}^n)$$
$$\times\frac{\Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{k+l+2\nu-\overrightarrow{1}}}\sum_{\substack{m\in\mathcal{O}_K^*\\ m\gg 0}}\frac{a_{m+\mu}\overline{b_m}(\overline{2\pi i m})^{\nu-t}}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}}.$$

Hence,

$$c_\mu = \frac{(2\pi i)^{|\nu|}\Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{l+2\nu}\Gamma(k-\overrightarrow{1})}\mu^{k-\overrightarrow{1}}\sum_{\substack{m\in\mathcal{O}_K^*\\ m\gg 0}}\frac{a_{m+\mu}\overline{b}_m}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}}\varepsilon_{\mu,m}^{k,l,\nu}, \quad (3.4.5)$$

where $\varepsilon_{\mu,m}^{k,l,\nu}$ is given by $(3.2.4)$. This completes the proof.

3.5 Application

Let $f \in S_{k+l+2n}(1),\ g \in M_l(1)$ and n be a non-negative integer. Zagier [51] computed explicitly the Petersson scalar product $\langle f, [E_k, g]_n\rangle$ in terms of special values of a certain Dirichlet series associated to f and g. Note that, here E_k is the usual Eisenstein series defined by $(1.2.2)$. The main idea of his proof is to express $[E_k, g]_n$ as a linear combination of elliptic Poincaré series and then use the characterization property of the Poincaré series. A particular case of his result gives the following interesting identity [51].

Let $f(z) = \sum_{n\geqslant 1} a_n q^n \in S_k(1)$ be a normalized eigenform and r be an even integer with $\frac{k}{2}+2 \leqslant r \leqslant k-4$. Then one has

$$L_f^*(r)L_f^*(k-1) = (-1)^{r/2}2^{k-3}\frac{B_r}{r}\frac{B_{k-r}}{k-r}\langle f, E_rE_{k-r}\rangle,$$

where L_f^* is the completed L-function defined by $L_f^*(s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geqslant 1}\dfrac{a_n}{n^s}$ and B_r is the r-th Bernoulli number.

Following the method of Zagier [51], Choie, Kim and Richter [10] computed the Petersson scalar product $\langle f, [E_k, g]_\nu\rangle$ in the case of Hilbert modular forms. More precisely, they proved the following.

Theorem 3.5.1. *[10] Let $k > 2$ be a natural number and $l, \nu \in N_0^n$ with $k - l_i > 2n$ for some $i, 1 \leq i \leq n$. Suppose that $f \in S_{k+l+2\nu}(\Gamma_K)$ with Fourier expansion*

$$f(z) = \sum_{\substack{m\in\mathcal{O}_K^* \\ m\gg 0}} a_m e[\mathrm{tr}(mz)],$$

and $g \in M_l(\Gamma_K)$ with Fourier expansion

$$g(z) = \sum_{\substack{m\in\mathcal{O}_K^* \\ m\succeq 0}} b_n e[\mathrm{tr}(mz)].$$

Then

$$\langle f, [E_k, g]_\nu\rangle = vol(\mathcal{O}_K/\mathbb{R}^n)(2i\pi)^{|\nu|}\frac{(\overrightarrow{k}+l+2\nu-\overrightarrow{2})!(\overrightarrow{k}+\nu-\overrightarrow{1})!}{(4\pi)^{|\overrightarrow{k}+l+2\nu-\overrightarrow{1}|}(\overrightarrow{k}-\overrightarrow{1})!\nu!}\sum_{\substack{n\in\mathcal{O}_K^* \\ n\gg 0}}\frac{a_n\overline{b_n}}{n^{k+l+\nu-\overrightarrow{1}}}.$$

As an application, we observe that by following the method of proof of Theorem

3.2.1, one can give a different proof of Theorem 3.5.1 by evaluating the integral

$$\int_{\Gamma_K \backslash \mathbb{H}^n} f(z)\overline{[E_k,\ g]_\nu} y^{k+l+2\nu} \frac{dxdy}{y^2}$$

using the Rankin-Selberg unfolding argument.

Chapter 4

Rankin-Cohen type operators for Hilbert-Jacobi forms

In this chapter, we construct Rankin-Cohen type operators for Hilbert-Jacobi forms which generalize the work of Choie and Eholzer (see, [8, Theorem 1.3]) and Zagier (see, [13, Theorem 9.5]).

4.1 Heat operators

Let K be a totally real number field of degree g over $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$. Let $J^K_{k,m}$ denotes the space of Hilbert-Jacobi forms of weight $k = (k_1, k_2, ..., k_g) \in \mathbb{Z}^g$ and index $m \in \mathcal{O}_K$ for the field K. We introduce the notion of heat operator for the space $J^K_{k,m}$.

Definition 4.1.1. *For $1 \leqslant j \leqslant g$, let e_j be j-th unit vector in $\mathbb{R}^g$. For a given $m \in \mathcal{O}_K$, we define the m-th heat operator,*

$$L_m := \prod_{j=1}^{g} \left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \right)^{e_j}, \tag{4.1.1}$$

where $\tau \in \mathbb{H}^g$ *and* $z \in \mathbb{C}^g$*.*

In the above definition, we denote "$\prod$" for the composition of operators. Now we state some properties of these operators which can be proved as in the case of Jacobi forms (see, [6]).

Lemma 4.1.2. *[28] Let* $\phi(\tau, z)$ *be a holomorphic function on the space* $\mathbb{H}^g \times \mathbb{C}^g$*,* $k \in \mathbb{Z}^g$ *and* $m \in \mathcal{O}_K$*. Then*

1. *for* $X \in \mathcal{O}_K \times \mathcal{O}_K$*,*
$$(L_m\phi)|_m X = L_m(\phi|_m X), \tag{4.1.2}$$

2. *for any* $\nu \in \mathbb{N}_0^g$ *and* $M \in SL_2(\mathcal{O}_K)$*, we have*
$$L_m{}^{\nu}(\phi)|_{k+2\nu,m} M = \sum_{\substack{l \in \mathbb{N}_0^g \\ l \leqslant \nu}} \binom{\nu}{l} \frac{(8\pi imc)^{\nu-l}(\alpha+\nu-1)!}{(c\tau+d)^{\nu-l}(\alpha+l-1)!} L^l{}_m(\phi|_{k,m}M), \tag{4.1.3}$$

where $\alpha = k - \frac{1}{2}$*.*

4.2 Statement of the theorem

In this section, we define Rankin-Cohen type differential operators on the space of Hilbert-Jacobi forms using the heat operator (4.1.1).

Definition 4.2.1. *Let* $\phi, \phi' : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$ *be two holomorphic functions and let* k, k', m, m' *be vectors in* $\mathbb{C}^g$*. Then for any* $X \in \mathbb{C}^g$*,* $\nu \in \mathbb{N}_0^g$ *and* $l \in \mathbb{N}_0^g$ *with* $l_i \in \{0, 1\}$ *for all* $1 \leqslant i \leqslant g$*, define*

$$[\phi, \phi']_{X,2\nu+l}^{k,k',m,m'} = \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j [\partial_z^j \phi, \partial_z^{l-j} \phi']_{X,2\nu}^{k,k',m,m',l}, \tag{4.2.1}$$

where for any two holomorphic functions f and f' on $\mathbb{H}^g \times \mathbb{C}^g$

$$[f, f']_{X,2\nu}^{k,k',m,m',l} := \sum_{\substack{r,s,p \in \mathbb{N}_0^g, \\ r+s+p=\nu}} A_{r,s,p}(k,k',l)(1+mX)^r(1-m'X)^s L_{m+m'}^p(L_m^r(f)L_{m'}^s(f')),$$

with

$$A_{r,s,p}(k,k',l) = \frac{(-(k+k'+l-3/2+\nu))_{r+s}}{r!\ s!\ p!\ (k-3/2+r)!\ (k'-3/2+s)!}.$$

Here for $x \in \mathbb{C}^g$ and $n \in \mathbb{N}_0^g$, $(x)_n := \prod_{i=1}^g \prod_{0 \leqslant j_i \leqslant (n_i-1)} (x_i - j_i)$.

Remark 4.2.1. *In the above definition we have the following convention.*

$$[\phi, \phi']_{X,2\nu}^{k,k',m,m',0} = [\phi, \phi']_{X,2\nu}^{k,k',m,m'}.$$

Remark 4.2.2. *Note that the constants $A_{r,s,p}(k,k',l)$ are different than $C_{r,s,p}(k,k')$, which appeared in Theorem 1.3 of [8] for the field $K = \mathbb{Q}$.*

We now state the main result of this chapter.

Theorem 4.2.2. *[28] Let ϕ, ϕ' be Hilbert-Jacobi forms of weight and index k, m and k', m' respectively. Then for any $X \in \mathbb{C}^g$, $\nu \in \mathbb{N}_0^g$ and $l \in \mathbb{N}_0^g$ with $l_i \in \{0,1\}$ for all $1 \leqslant i \leqslant g$,*

$$[\phi, \phi']_{X,2\nu+l}^{k,k',m,m'} \tag{4.2.2}$$

is a Hilbert-Jacobi form of weight $k+k'+2\nu+l$ and index $m+m'$.

There are two known methods to prove result like Theorem 4.2.2. First one, by showing that $[\phi, \phi']_{X,2\nu+l}^{k,k',m,m'}$ satisfy all the required conditions to be a Hilbert-Jacobi form (see, [8, section 4]) and second one, by using generating series (see, [13, Theorem 3.2], [8, section 5]). We prove our result by using generating series. In the next section we shall develop some tools for a proof of Theorem 4.2.2.

4.3 Intermediate results

Proposition 4.3.1. *Let $\phi(\tau,z) \in J^K_{k,m}$ and $\alpha = k - \frac{1}{2}$. Then the formal power series associated with the Jacobi form ϕ defined by*

$$\widetilde{\phi}(\tau, z; W) := \sum_{\nu \in \mathbb{N}_0^g} \frac{L_m^\nu(\phi)(\tau, z)}{\nu!(\alpha+\nu-1)!} W^\nu, \tag{4.3.1}$$

satisfies the following functional equation,

$$\widetilde{\phi}\left(M\tau, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right) = (cz+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] e\left[4\mathrm{tr}\left(\frac{mcW}{c\tau+d}\right)\right] \widetilde{\phi}(\tau, z; W), \tag{4.3.2}$$

for all $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K)$.

Proof. From the definition of $\widetilde{\phi}$, we have

$$\widetilde{\phi}\left(M\tau, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right) = \sum_{\nu \in \mathbb{N}_0^g} \frac{L_m^\nu(\phi)\left(M\tau, \frac{z}{c\tau+d}\right)}{\nu!(\alpha+\nu-1)!} \frac{W^\nu}{(c\tau+d)^{2\nu}}$$

$$= \sum_{\nu \in \mathbb{N}_0^g} \frac{(c\tau+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] (L_m^\nu \phi)|_{k+2\nu,m} M(\tau,z)}{\nu!(\alpha+\nu-1)!} W^\nu.$$

Using (4.1.3) and the assumption that $\phi \in J^K_{k,m}$, the right hand side of the above equation is equal to

$$(c\tau+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] \sum_{\nu \in \mathbb{N}_0^g} \frac{1}{\nu!(\alpha+\nu-1)!} \left(\sum_{\substack{l \in \mathbb{N}_0^g \\ l \leqslant \nu}} \binom{\nu}{l} \frac{(8\pi imc)^{\nu-l}(\alpha+\nu-1)!}{(c\tau+d)^{\nu-l}(\alpha+l-1)!} {L^l}_m(\phi|_{k,m}M) \right) W^\nu$$

$$= (c\tau+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] \sum_{\nu \in \mathbb{N}_0^g} \left(\sum_{\substack{l \in \mathbb{N}_0^g \\ l \leqslant \nu}} \frac{1}{l!(\nu-l)!(\alpha+l-1)!} \frac{(8\pi imc)^{\nu-l}}{(c\tau+d)^{\nu-l}} {L^l}_m(\phi) \right) W^\nu$$

$$= (cz+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] e\left[4\mathrm{tr}\left(\frac{mcW}{c\tau+z}\right)\right]\widetilde{\phi}(\tau,z,W).$$

This completes the proof. □

Let $\widetilde{f}(\tau,z;W)$ be a power series in W whose coefficients are holomorphic functions on $\mathbb{H}^g\times\mathbb{C}^g$ i.e., $\widetilde{f}(\tau,z;W)=\sum_{\nu\in\mathbb{N}_0^g}\chi_\nu(\tau,z)W^\nu$. For $M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathcal{O}_K)$, we define

$$\big(\widetilde{f}|_{k,m}M\big)(\tau,z;W) := (c\tau+d)^{-k} e\left[-\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] e\left[-4\mathrm{tr}\left(\frac{mcW}{c\tau+d}\right)\right] \times \widetilde{f}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d};\frac{W}{(c\tau+d)^2}\right).$$

Next we show that for a given formal power series satisfying certain conditions, one can construct a family of Hilbert-Jacobi forms like in the case of Jacobi forms [8, Theorem 5.1].

Theorem 4.3.2. *Let $\widetilde{\phi}(\tau,z;W)$ be a formal power series in W, i.e.,*

$$\widetilde{\phi}(\tau,z;W)=\sum_{\nu\in\mathbb{N}_0^g}\chi_\nu(\tau,z)W^\nu, \tag{4.3.3}$$

satisfying the functional equation

$$\big(\widetilde{\phi}|_{k,m}M\big)(\tau,z;W)=\widetilde{\phi}(\tau,z;W),\quad \textit{for all } M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathcal{O}_K), \tag{4.3.4}$$

for some $k\in\mathbb{N}_0^g$ and $m\in\mathcal{O}_K$. Furthermore, assume that the coefficients $\chi_\nu(\tau,z)$

are holomorphic functions on $\mathbb{H}^g \times \mathbb{C}^g$ *with Fourier expansion of the form,*

$$\chi_\nu(\tau, z) = \sum_{\substack{n,r \in \mathcal{O}_K^* \\ 4nm - r^2 \geqslant 0}} c(n, r) e[\mathrm{tr}(n\tau + rz)], \tag{4.3.5}$$

satisfying

$$\chi_\nu|_m Y = \chi_\nu \quad \textit{for all} \quad Y \in \mathcal{O}_K \times \mathcal{O}_K. \tag{4.3.6}$$

Then for each $\nu \in \mathbb{N}_0^g$*, the function* $\xi_\nu(\tau, z)$ *defined by*

$$\xi_\nu(\tau, z) := \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant \nu}} \frac{(-(k - 3/2 + \nu))_{\nu - j}}{j!} L_m^j(\chi_{\nu - j}), \tag{4.3.7}$$

is a Hilbert-Jacobi form of weight $k + 2\nu$ *and index* m*.*

Remark 4.3.1. *We call* k *and* m *appeared in the equation* (4.3.4) *is the weight and index of the power series* $\widetilde{\phi}$ *respectively.*

Proof. We show that $\xi_\nu(\tau, z)$, defined by (4.3.7) is invariant under $SL_2(\mathcal{O}_K)$ action. For $1 \leqslant j \leqslant g$, let e_j be the j-th unit vector in $\mathbb{R}^g$. Define the j-th differential operator

$$\widetilde{L}_{k,m}^{e_j} := 8\pi i m^{(j)} \frac{\partial}{\partial \tau_j} - \frac{\partial^2}{\partial z_j^2} - (k_j - 1/2) \frac{\partial}{\partial W_j} - W_j \frac{\partial^2}{\partial W_j^2},$$

where $k = (k_1, k_2, \ldots, k_g)$ and $m \in \mathcal{O}_K$. Let $\widetilde{\mathcal{M}}_{k,m}$ be the collection of all functions $\widetilde{f}(\tau, z; W) = \sum_{\nu \in \mathbb{N}_0^g} \chi'_\nu(\tau, z) W^\nu$ which satisfy the condition:

$$\big(\widetilde{f}|_{k,m} M\big)(\tau, z; W) = \widetilde{f}(\tau, z; W), \qquad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K).$$

We note that the constant term $\chi'_0(\tau, z)$ in the power series expansion of $\widetilde{f}(\tau, z; W) \in$

$\widetilde{\mathcal{M}}_{k,m}$ satisfy the following.

$$\left(\chi'_0|_{k,m}M\right)(\tau,z)=\chi'_0(\tau,z),\qquad \text{for all } M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathcal{O}_K). \tag{4.3.8}$$

Then using the definition of slash operator (4.3.4) one can show that

$$\widetilde{L}^{e_j}_{k,m}(\widetilde{\phi}|_{k,m}M)=(\widetilde{L}^{e_j}_{k,m}\widetilde{\phi})|_{k+2e_j,m}M,$$

for all $M\in SL_2(\mathcal{O}_K)$. We note that $\prod_{j=1}^{g}\widetilde{L}^{e_j}_{k,m}$ (the composition of all $\widetilde{L}^{e_j}_{k,m}$ for $1\leqslant j\leqslant g$), denoted by $\widetilde{L}_{k,m}$ satisfy

$$\widetilde{L}_{k,m}(\widetilde{\phi}|_{k,m}M)=(\widetilde{L}_{k,m}\widetilde{\phi})|_{k+2,m}M,\quad \text{for all } M\in SL_2(\mathcal{O}_K).$$

In other word, $\widetilde{L}_{k,m}$ is a map from $\widetilde{\mathcal{M}}_{k,m}$ to $\widetilde{\mathcal{M}}_{k+2,m}$ which is given in terms of power series by

$$\widetilde{L}_{k,m}:\sum_{\lambda\in\mathbb{N}_0^g}\chi_\lambda(\tau,z)W^\lambda\mapsto\sum_{\lambda\in\mathbb{N}_0^g}\left(\sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant 1}}\frac{(-1)^{1+j}\binom{1}{j}(\lambda+1-j)!(\lambda+\alpha-j)!L^j_m(\chi_{\lambda+1-j})}{\lambda!(\lambda+\alpha-1)!}\right)W^\lambda,$$

with $\alpha=k-1/2$. Composing the maps $\widetilde{L}_{k+i,m}$ for $1\leqslant i\leqslant\nu-1$,

$$\widetilde{\mathcal{M}}_{k,m}\xrightarrow{\widetilde{L}_{k,m}}\widetilde{\mathcal{M}}_{k+2,m}\xrightarrow{\widetilde{L}_{k+2,m}}\cdots\xrightarrow{\widetilde{L}_{k+2\nu-2,m}}\widetilde{\mathcal{M}}_{k+2\nu,m}$$

then it maps $\sum_{\lambda\in\mathbb{N}_0^g}\chi_\lambda(\tau,z)W^\lambda$ to

$$\sum_{\lambda\in\mathbb{N}_0^g}\left(\sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant\nu}}\frac{(-1)^{\nu+j}\binom{\nu}{j}(\lambda+\nu-j)!(\lambda+2\nu+\alpha-j-2)!L^j_m(\chi_{\lambda+\nu-j})}{\lambda!(\lambda+\alpha+\nu-2)!}\right)W^\lambda.$$

We note that the constant term, i.e., $\lambda = \overrightarrow{0}$ in the above series, is $\nu!$ times ξ_ν. Hence from (4.3.8), ξ_ν is invariant under $SL_2(\mathcal{O}_K)$ action. The other conditions hold easily from the given hypothesis on function $\chi_\nu(\tau, z)$. □

In the next two lemmas we show how the operator ∂_z behaves under the group and lattice actions.

Lemma 4.3.3. *Let ϕ be a Hilbert-Jacobi form of weight k and index m. For $j \in \mathbb{N}_0^g$ with $j_i \in \{0, 1\}$ for all $1 \leqslant i \leqslant g$, we have*

$$\partial^j_{z/c\tau+d}\widetilde{\phi}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right)$$
$$= (c\tau+d)^{k+j} e\left[\operatorname{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] e\left[4\operatorname{tr}\left(\frac{mcW}{c\tau+d}\right)\right] \sum_{\substack{a\in\mathbb{N}_0^g \\ a\leqslant j}} \left(\frac{4\pi imcz}{c\tau+d}\right)^a \partial_z^{j-a}\widetilde{\phi}(\tau, z; W). \tag{4.3.9}$$

Proof. This is an easy consequence of Proposition 4.3.1. □

Lemma 4.3.4. *Suppose $f(z)$ is a holomorphic function on the space $\mathbb{H}^g$ and $Y = (\lambda, \mu) \in \mathcal{O}_K \times \mathcal{O}_K$. Then for $j \in \mathbb{N}_0^g$ with $j_i \in \{0, 1\}$ for all $1 \leqslant i \leqslant g$, we have*

$$(\partial_z^j f)|_m Y = \sum_{\substack{a\in\mathbb{N}_0^g \\ a\leqslant j}} (-4\pi im\lambda)^a \partial_z^{j-a}(f|_m Y). \tag{4.3.10}$$

Proof. One can prove this result using the definition of the action "$|_m Y$". □

4.4 Proof of Theorem 4.2.2

First we prove for case $l = \overrightarrow{0}$ and then for general case $l \neq \overrightarrow{0}$.

Case I: $l = \overrightarrow{0}$. For a fixed $X \in \mathbb{C}^g$, consider the series $F_X(\tau, z; W)$ defined by

$$F_X(\tau, z; W) = \widetilde{\phi}\big(\tau, z; (1 + m'X)W\big)\ \widetilde{\phi}'\big(\tau, z; (1 - mX)W\big),$$

where $\widetilde{\phi}$ and $\widetilde{\phi}'$ are defined by the equation (4.3.1). Here we show that the function $F_X(\tau, z; W)$ satisfy all the necessary conditions for Theorem 4.3.2 and consequently deduce the result.

Using the corresponding functional equation for $\widetilde{\phi}$ and $\widetilde{\phi}'$ given in the Proposition 4.3.1, one can easily show that the function $F_X(\tau, z; W)$ also satisfy the same functional equation as (4.3.4) with weight $k + k'$ and index $m + m'$.

Now we look the power series expansion of F_X. Replacing $\widetilde{\phi}$ and $\widetilde{\phi}'$ with their corresponding expressions (4.3.1) in F_X, we get

$$\begin{aligned} F_X(\tau, z; W) &= \Bigg(\sum_{\nu \in \mathbb{N}_0^g} \frac{(1 + m'X)^\nu L_m^\nu(\phi)}{\nu!\ (k - 3/2 + \nu)!} W^\nu\Bigg) \Bigg(\sum_{\nu \in \mathbb{N}_0^g} \frac{(1 - mX)^\nu L_m^\nu(\phi')}{\nu!\ (k' - 3/2 + \nu)!} W^\nu\Bigg) \\ &= \sum_{\nu \in \mathbb{N}_0^g} \Bigg(\sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1 + m'X)^a (1 - mX)^{\nu - a}}{a!\ (\nu - a)!\ (k - 3/2 + a)!\ (k' - 3/2 + \nu - a)!} L_m^a(\phi) L_{m'}^{\nu - a}(\phi')\Bigg) W^\nu \\ &= \sum_{\nu \in \mathbb{N}_0^g} \chi_{\nu, F}(\tau, z) W^\nu, \end{aligned}$$

where

$$\chi_{\nu, F}(\tau, z) := \sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1 + m'X)^a (1 - mX)^{\nu - a}}{a!\ (\nu - a)!\ (k - 3/2 + a)!\ (k' - 3/2 + \nu - a)!} L_m^a(\phi) L_{m'}^{\nu - a}(\phi'). \tag{4.4.1}$$

Clearly $\chi_{\nu, F}(\tau, z)$ is holomorphic on $\mathbb{H}^g \times \mathbb{C}^g$ for all $\nu \in \mathbb{N}_0^g$. We note that if ϕ has the Fourier expansion $\phi(\tau, z) = \sum_{\substack{n, r \in \mathcal{O}_K^* \\ 4nm - r^2 \geqslant 0}} c_\phi(n, r) e[\mathrm{tr}(n\tau + rz)]$, then the function $L_m^l(\phi)$

has the Fourier expansion

$$L_m^l(\phi)(\tau,z) = \sum_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}} c_\phi(n,r)(4nm-r^2)^l e[\mathrm{tr}(n\tau+rz)]. \tag{4.4.2}$$

Replacing ϕ and ϕ' by their Fourier expansions and using the repeated action of the heat operator from (4.4.2), we have

$$\begin{aligned}
\chi_{\nu,F}(\tau,z) &= \sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant\nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!}\\
&\times\Bigg(\sum_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}} (4nm-r^2)^a c_\phi(n,r)e[\mathrm{tr}(n\tau+rz)]\Bigg)\\
&\times\Bigg(\sum_{\substack{n',r'\in\mathcal{O}_K^*\\ 4n'm'-r'^2\geqslant 0}} (4n'm'-r'^2)^{\nu-a} c_{\phi'}(n',r')e[\mathrm{tr}(n'\tau+r'z)]\Bigg)\\
&= \sum_{\substack{N,R\in\mathcal{O}_K^*\\ 4N(m+m')-R^2\geqslant 0}} \Bigg(\sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant\nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!}\\
&\times\sum_{\substack{n,n',r,r'\in\mathcal{O}_K^*\\ n+n'=N,\\ r+r'=R,\\ 4nm-r^2\geqslant 0,\\ 4n'm'-r'^2\geqslant 0}} (4nm-r^2)^a(4n'm'-r'^2)^{\nu-a}c_\phi(n,r)c_{\phi'}(n',r')\Bigg)e[\mathrm{tr}(N\tau+Rz)].
\end{aligned}$$

One can check that $4N(m+m')-R^2\geqslant 0$ for the above choices of N and R and the last sum is a finite sum for a given N and R. From (4.1.2), it is clear that $\chi_{\nu,F}|_{m+m'}Y = \chi_{\nu,F}$ for all $Y\in\mathcal{O}_K\times\mathcal{O}_K$. Hence from Theorem 4.3.2, $\xi_{\nu,F}(\tau,z)$ is a Hilbert-Jacobi form of weight $k+k'+2\nu$ and index $m+m'$. This completes the proof in this case because $[\phi,\phi']_{X,2\nu}^{k,k',m,m'}(\tau,z)=\xi_{\nu,F}(\tau,z)$.

Case II: $l \neq \overrightarrow{0}$. For a fixed $X \in \mathbb{C}^g$, consider the function $G_X(\tau, z; W)$ defined by

$$G_X(\tau, z; W) = \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j \partial_z^j \widetilde{\phi}\big(\tau, z; (1+m'X)W\big)\ \partial_z^{l-j} \widetilde{\phi}'\big(\tau, z; (1-mX)W\big). \tag{4.4.3}$$

We show that the function G_X satisfy the same functional equation as (4.3.4) with weight $k + k' + l$ and index $m + m'$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K)$. Using (4.4.3), we have

$$G_X\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right) = \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j \partial^j_{z/c\tau+d} \widetilde{\phi}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{(1+m'X)W}{(c\tau+d)^2}\right)$$
$$\times\ \partial^{l-j}_{z/c\tau+d} \widetilde{\phi}'\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{(1-mX)W}{(c\tau+d)^2}\right).$$

Using Lemma 4.3.3, the above equation becomes

$$\begin{aligned}
& G_X\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right) \\
&= (c\tau+d)^{k+k'+l} e\left[\operatorname{tr}\left((m+m')\frac{cz^2}{c\tau+d}\right)\right] e\left[4\operatorname{tr}\left((m+m')\frac{cW}{c\tau+d}\right)\right] \\
&\times \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j \Bigg(\sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant j}} \left(\frac{4\pi i m c z}{c\tau+d}\right)^a \partial_z^{j-a} \widetilde{\phi}\big(\tau, z; (1+m'X)W\big) \\
&\times \sum_{\substack{b \in \mathbb{N}_0^g \\ b \leqslant l-j}} \left(\frac{4\pi i m' c z}{c\tau+d}\right)^b \partial_z^{l-j-b} \widetilde{\phi}'\big(\tau, z; (1-mX)W\big) \Bigg).
\end{aligned}$$

Now we split the above sum into two parts,

$$\begin{aligned}
& G_X\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right) \\
&= (c\tau+d)^{k+k'+l} e\left[\operatorname{tr}\left((m+m')\frac{cz^2}{c\tau+d}\right)\right] e\left[4\operatorname{tr}\left((m+m')\frac{cW}{c\tau+d}\right)\right]
\end{aligned}$$

$$\times \Bigg(\sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^j m'^{l-j} \partial_z^j \widetilde{\phi}\big(\tau, z; (1+m'X)W\big)\ \partial_z^{l-j}\widetilde{\phi}'\big(\tau, z; (1-mX)W\big)$$

$$+ \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^g\\ \alpha+\beta<l}} \Bigg(\sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^{l-j} m'^j \left(\frac{4\pi imcz}{c\tau+d}\right)^{j-\alpha} \left(\frac{4\pi im'cz}{c\tau+d}\right)^{l-j-\beta} \Bigg)$$

$$\times \partial_z^{\alpha}\widetilde{\phi}\big(\tau, z; (1+m'X)W\big)\partial_z^{\beta}\widetilde{\phi}'\big(\tau, z; (1-mX)W\big)\Bigg).$$

An easy computation shows that for any pair of $\alpha, \beta \in \mathbb{N}_0^g$ with $\alpha + \beta < l$, the coefficient of $\partial_z^{\alpha}\widetilde{\phi}\ \partial_z^{\beta}\widetilde{\phi}'$ in the second sum of the above equation is zero, which prove our claim. Now replacing the corresponding power series expression for $\widetilde{\phi}$ and $\widetilde{\phi}'$ from (4.3.1) in (4.4.3), we note that the function G_X has power series expansion of the form

$$G_X(\tau, z; W) = \sum_{\nu\in\mathbb{N}_0^g} \chi_{\nu,G}(\tau, z) W^{\nu},$$

where $\chi_{\nu,G}(\tau, z)$ is given by

$$\sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant\nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} \sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^{l-j} m'^j L_m^a(\partial_z^j\phi) L_{m'}^{\nu-a}(\partial_z^{l-j}\phi'). \tag{4.4.4}$$

As mentioned in the previous case one can show that for each $\nu \in \mathbb{N}_0^g$, the corresponding function $\chi_{\nu,G}(\tau, z)$ has the following Fourier expansion.

$$\chi_{\nu,G}(\tau, z) = \sum_{\substack{N,R\in\mathcal{O}_K^*,\\ 4N(m+m')-R^2\geqslant 0}} \Bigg(\sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant\nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!(\nu-a)!(k-3/2+a)!(k'-3/2+\nu-a)!} \sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^{l-j} m'^j$$

$$\times \sum_{\substack{n,n',r,r' \in \mathcal{O}_K^* \\ n+n'=N, \\ r+r'=R, \\ 4nm-r^2 \geqslant 0, \\ 4n'm'-r'^2 \geqslant 0}} (4nm-r^2)^a (4n'm'-r'^2)^{\nu-a} r^j r'^{l-j} c_\phi(n,r) c_{\phi'}(n',r') \Bigg) e[\mathrm{tr}(N\tau + Rz)].$$

Using Theorem 4.3.2 one can deduce that $[\phi,\phi']_{X,2\nu+l}^{k,k',m,m'} \in J_{k+k'+2\nu+l,m+m'}^K$ as $[\phi,\phi']_{X,2\nu+l}^{k,k',m,m'} = \xi_{\nu,G}(\tau,z)$ once we prove $\chi_{\nu,G}(\tau,z)|_{m+m'}Y = \chi_{\nu,G}(\tau,z)$ for all $\nu \in \mathbb{N}_0^g$ and $Y \in \mathcal{O}_K \times \mathcal{O}_K$. From $(4.4.4)$ we have

$$\chi_{\nu,G}(\tau,z)|_{m+m'}Y = \sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1+m'X)^a (1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!}$$
$$\times \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j \partial_z^j (L_m^a \phi)|_m Y\ \partial_z^{l-j} (L_{m'}^{\nu-a}\phi')|_{m'}Y.$$

From Lemma 4.3.4 the right hand side of the above equation is equal to

$$\sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1+m'X)^a (1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j$$
$$\times \Bigg(\sum_{\substack{t \in \mathbb{N}_0^g \\ t \leqslant j}} (-4\pi i m\lambda)^t \partial_z^{j-t} (L_m^a \phi|_m Y) \Bigg) \Bigg(\sum_{\substack{s \in \mathbb{N}_0^g \\ s \leqslant l-j}} (-4\pi i m'\lambda)^s \partial_z^{l-j-s} (L_{m'}^{\nu-a} \phi'|_{m'}Y) \Bigg).$$

Now using the assumption that ϕ and ϕ' are Hilbert-Jacobi forms and $(L_m\phi)|_mY = L_m(\phi|_mY)$, the above expression is equal to

$$= \sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1+m'X)^a (1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j$$
$$\times \Bigg(\sum_{\substack{t \in \mathbb{N}_0^g \\ t \leqslant j}} (-4\pi i m\lambda)^t \partial_z^{j-t} L_m^a \phi \Bigg) \Bigg(\sum_{\substack{s \in \mathbb{N}_0^g \\ s \leqslant l-j}} (-4\pi i m'\lambda)^s \partial_z^{l-j-s} L_{m'}^{\nu-a} \phi' \Bigg).$$

For a fixed $a \in \mathbb{N}_0^g$ we note the following. For $\alpha, \beta \in \mathbb{N}_0^g$ with $\alpha + \beta < l$, the coefficient of $\partial_z^\alpha(L_m^a\phi)\ \partial_z^\beta(L_m^{\nu-a}\phi')$ in the above expression is zero. Thus $\chi_{\nu,G}$ is invariant under the lattice action and this completes the proof.

4.5 Concluding Remarks

Theorem 4.2.2 gives justification to expect that the space of bilinear holomorphic differential operators raising the weight $\nu = (\nu_1, \ldots, \nu_g) \in \mathbb{N}_0^g$ is at least $\prod_{i=1}^g(1 + [\nu_i/2])$ for the space of Hilbert-Jacobi forms over a totally real number field of degree g over $\mathbb{Q}$ on $\mathbb{H}^g \times \mathbb{C}^g$. It would be interesting to prove the generalization of the result of Böcherer [3] in case of Hilbert-Jacobi forms that the dimension is exactly equal to $\prod_{i=1}^g(1 + [\nu_i/2])$.

Bibliography

[1] A. O. L. Atkin and J. Lehner, Hecke operators on $\Gamma_0(m)$, *Math. Ann.* **185** (1970), 134–160.

[2] R. P. Bambah and S. Chowla, On numbers which can be expressed as a sum of two squares, *Proc. Nat. Inst. Sci. India* **13** (1947), 101–103.

[3] S. Böcherer, Bilinear holomorphic differential operators for the Jacobi group, *Comment. Math. Univ. St. Pauli* **47** (1998), 135–154.

[4] K. Chandrasekharan and R. Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetic functions, *Ann. of Math.* **76** (1962), 93–136.

[5] B. Chen and J. Wu, Non-vanishing and sign changes of Hecke eigenvalues for Half-integral weight cusp forms, *Indag. Math.* **27** (2016), no. 2, 488–505.

[6] Y. Choie, Jacobi forms and the heat operator, *Math. Z.* **1** (1997), 95–101.

[7] Y. Choie, Jacobi forms and the heat operator II, *Illinois J. Math.* **42** (1998), 179–186.

[8] Y. Choie and W. Eholzer, Rankin-Cohen operators for Jacobi and Siegel Forms, *J. Number Theory* **68** (1998), 160–177.

[9] Y. Choie, H. Kim and M. Knopp, Construction of Jacobi forms. *Math. Z.* **219** (1995), 71–76.

[10] Y. Choie, H. Kim and O. K. Richter, Differential operators on Hilbert modular forms, *J. Number Theory* **122** (2007), 25–36.

[11] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.* **217** (1975), 271–285.

[12] S. Das and S. Ganguly, Gaps between nonzero Fourier coefficients of cusp forms, *Proc. Amer. Math. Soc.* **142** (2014), 3747–3755.

[13] M. Eichler and D. Zagier, *The Theory of Jacobi forms*, Progr. in Math., vol. 55, Birkhäuser Boston, 1985.

[14] P. Erdös, On the difference of consecutive terms of sequences defined by divisibility properties, *Acta Arith.* **12** (1966/1967), 175–182.

[15] P. Garrette, *Holomorphic Hilbert Modular forms*, The Wadsworth and Brooks/Cole Mathematics Series, 1990.

[16] S. Gun, W. Kohnen and P. Rath, Simultaneous sign change of Fourier-coefficients of two cusp forms, *Arch. Math.* **105** (2015), 413–424.

[17] K. Hatada, Eigenvalues of Hecke operators on $SL(2,\mathbb{Z})$, *Math. Ann.* **239** (1979), no. 1, 75–96.

[18] S. D. Herrero, The adjoint of some linear maps constructed with the Rankin-Cohen brackets, *Ramanujan J.* **36** (2015), no. 3, 529–536.

[19] A. K. Jha and B. Sahu, Rankin-Cohen brackets on Jacobi forms and the adjoint of some linear maps, *Ramanujan J.* **39** (2016), no. 3, 533–544.

[20] A. K. Jha and B. Sahu, Rankin-Cohen brackets on Siegel modular forms and special values of certain Dirichlet series, *Ramanujan J.* **44** (2017), no. 1, 63–73.

[21] N. Koblitz, *Introduction to elliptic curves and modular forms*, Graduate Texts in Mathematics, 97 Springer-Verlag, New York, 1993.

[22] W. Kohnen, Cusp forms and special value of certain Dirichlet Series, *Math. Z.* **207** (1991), 657–660.

[23] W. Kohnen and J. Sengupta, Signs of Fourier coefficients of two cusp forms of different weights, *Proc. Amer. Math. Soc.* **137** (2009), 3563–3567.

[24] E. Kowalski, Y. K. Lau, K. Soundararajan and J. Wu, On modular signs, *Math. Proc. Cambridge Philos. Soc.* **149** (2010), no. 3, 389–411.

[25] S. Krishnamoorthy and M. Ram Murty, On sign changes for almost prime coefficients of half-integral weight modular forms, *Mathematika* **62** (2016), 801–810.

[26] M. Kumari and M. Ram Murty, Simultaneous non-vanishing and sign changes of Fourier coefficients of Modular forms, *Int. J. Number Theory* **14** (2018), no. 8, 2291–2301.

[27] M. Kumari and B. Sahu, Rankin-Cohen brackets on Hilbert Modular forms and special values of certain Dirichlet series, *Funct. Approx. Comment. Math.* **58** (2018), no. 2, 257–268.

[28] M. Kumari and B. Sahu, Rankin-Cohen type operators for Hilbert-Jacobi forms, Submitted.

[29] M. H. Lee, Hilbert Cusp forms and special values of Dirichlet series of Rankin type, *Glasgow Math. J.* **40** (1998), no. 1, 71–77.

[30] M. H. Lee, Siegel cusp forms and special values of Dirichlet series of Rankin type, *Complex Var. Theory Appl.* **31** (1996), no. 2, 97–103.

[31] D. H. Lehmer, The vanishing of Ramanujan's function $\tau(n)$, *Duke Math. J.* **14** (1947), 429–433.

[32] W. C. W. Li, L-series of Rankin type and their functional equations, *Math. Ann.* **244** (1979), 135–166.

[33] G. Lu, Sums of absolute values of cusp form coefficients and their application, *J. Number Theory* **139** (2014), 29–43.

[34] K. Matomäki, On signs of Fourier coefficients of cusp forms, *Math. Proc. Cambridge Philos. Soc.* **152** (2012), no. 2, 207–222.

[35] J. Meher and M. Ram Murty, Sign changes of Fourier coefficients of half-integral weight cusp forms, *Int. J. Number Theory* **10** (2014), 905–914.

[36] M. Ram Murty, *Congruences between modular forms*, in Analytic Number Theory, edited by Y. Motohashi, London Mathematical Society **247** (1997), 313–320.

[37] M. Ram Murty, *Bounds for congruence primes*, in Automorphic Forms, Automorphic Representations and Arithmetic, edited R. Boran, Ze-Li Dou and G. Gilbert, Proc. Symp. Pure Math. **66** (1999), Part 1, pp. 177–192, American Mathematical Society.

[38] M. Ram Murty and V. Kumar Murty, Odd values of Fourier coefficients of certain modular forms, *Int. J. Number Theory* **3** (2007), no. 3, 455–470.

[39] V. Kumar Murty, Lacunarity of modular forms, *Journal of the Indian Math. Society* **52** (1987), 127–146.

[40] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*. In: CBMS Regional Conference Series in Mathematics, vol. 102, 2003.

[41] D. Pei and X. Wang, Hilbert modular forms and special values of some Dirichlet series, *Acta. Math. Sin.* **38** (1995), no. 3, 336–343.

[42] W. Pribitkin, On the sign changes of coefficients of general Dirichlet series, *Proc. Amer. Math. Soc.* **9** (2008), 3089–3094.

[43] R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions, I, *Proc. Cambridge Phil. Soc.* **35** (1939), 351–372.

[44] R. A. Rankin, The construction of automorphic forms from the derivatives of a given form, *J. Indian Math. Soc.* **20** (1956), 103–116.

[45] R. A. Rankin, The construction of automorphic forms from the derivatives of a given forms, *Michigan Math. J.* **4** (1957), 181–186.

[46] H. Sakata, Construction of Jacobi cusp forms, *Proc. Japan Acad. Ser. A Math. Sci.* **74** (1998), 117–119.

[47] A. Selberg, Bemerkung über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, *Archiv for Math. og Naturvid.* **43** (1940), 47–50.

[48] J.-P. Serre, Quelques applications du théorème de densité de Chebotarev, *Inst. Hautes Etudes Sci. Publ. Math.* **54** (1981), 323–401.

[49] H. Skogman, Jacobi forms over totally real number fields, *Results Math.* **39** (2001), no. 1-2, 169–182.

[50] X. Wang, Special values of some Dirichlet series associated to Hilbert modular forms, *JP. J. Algebra Number Theory Appl.* **3** (2003), no. 1, 1–12.

[51] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*, in:. Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), pp. 105–169. Lecture Notes in Math., Vol. 627 (Springer, Berlin, 1977).

[52] D. Zagier, Modular forms and differential operators, *Proc. Indian Acad. Sci. Math. Sci.* **104** (1994), 57–75.