# Adjoint of Some Linear Maps Constructed Using Rankin-Cohen Brackets and Special Values of Certain Dirichlet Series


By

**ABHASH KUMAR JHA**

**Enrolment Number : MATH11201104002**

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**Abhash Kumar Jha**

# DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

**Abhash Kumar Jha**

# List of Publications arising from the thesis

### Journal

1. Rankin-Cohen brackets on Jacobi Forms and the adjoint of some linear maps (with B. Sahu), *The Ramanujan J.*, 39 (2016), no. 3, 533–544 .

2. Rankin-Cohen brackets on Siegel modular forms and Special values of Certain Dirichlet series (with B. Sahu), *The Ramanujan J.*, doi:10.1007/s11139-016-9783-3.

### Others

1. Construction of cusp forms using Rankin-Cohen brackets (with A. Kumar), arXiv:1607.03511v1.

**Abhash Kumar Jha**

*Dedicated to My Family*

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# Contents

# Chapter 0

# Synopsis

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# 1. Introduction

We denote the space of modular forms and the subspace of cusp forms of weight $k$ for $SL_2(\mathbb{Z})$ by $M_k$ and $S_k$, respectively. Suppose $f \in S_k$ and $g \in S_l$ with Fourier coefficients $a(n)$ and $b(n)$, respectively. For a positive integer $n$, define a Dirichlet series

$$L_{f,g;n}(s) = \sum_{m=1}^{\infty} \frac{a(m+n)\overline{b(m)}}{(m+n)^s}. \tag{1}$$

Then $L_{f,g;n}(s)$ is absolutely convergent for $Re(s) > \dfrac{k+l}{2}$. Kohnen [21] constructed certain cusp forms, whose Fourier coefficients involve special values of the Dirichlet series (1). More precisely, he proved the following:

**Theorem 1.** *[21] Let $k > 2, l \geqslant 0$ and $f \in S_{k+l}$ and $g \in S_l$ with Fourier coefficients $a(n)$ and $b(n)$, respectively. Then*

$$T_g^*(f)(z) := \sum_{n=1}^{\infty} n^{k-1} L_{f,g;n}(k+l-1) q^n \in S_k.$$

*In fact, the map $S_{k+l} \to S_k$ defined by $f \mapsto \dfrac{\Gamma(k+l-1)}{\Gamma(k-1)(4\pi)^l} T_g^*(f)$ is the adjoint of the map $T_g : S_k \longrightarrow S_{k+l},\ h \mapsto gh$, with respect to the Petersson scalar product.*

There are many interesting connections between differential operators and modular forms and many interesting results have been found. Rankin [31, 32], gave a general description of the differential operators which send modular forms to modular forms. Cohen [10] explicitly constructed certain bilinear operators using differential operators and obtained elliptic modular forms. Zagier [38, 37] studied algebraic properties of these bilinear operators and called them Rankin–Cohen brackets. Explicitly, for a given $f \in M_k,\ g \in M_l$ and an integer $n \geqslant 0$, the $n$-th Rankin-Cohen bracket $[f, g]_n \in M_{k+l+2n}$ and in fact, $[\ ,\ ]_n : M_k \times M_l \longrightarrow M_{k+l+2n}$ is a bilinear map. Also, the 0-th Rankin-Cohen bracket of $f$ and $g$ is the usual product of $f$ and $g$. Recently, the work of Kohnen has been generalized by Herrero [14], where he computed the adjoint of the map constructed using Rankin-Cohen brackets instead of the product map $T_g : S_k \longrightarrow S_{k+l},\ h \mapsto gh$, for a fixed cusp form $g \in S_l$. More precisely, for a fixed $g \in M_l$ and an integer $n \geqslant 0$, he considered the linear map $T_{g,n} : S_k \longrightarrow S_{k+l+2n}$ defined by $T_{g,n}(f) = [f, g]_n$ and computed its adjoint map with respect to the Petersson scalar product. This map involves special values of certain Dirichlet series of Rankin-Selberg type similar to (1) with additional factors arising due to binomial coefficients appearing in the Rankin-Cohen bracket. The result of Kohnen has been generalized to other automorphic forms (e.g., Jacobi forms, Siegel modular forms, Hilbert modular forms, etc.). The main objective of this thesis is to extend the work of Herrero to the case of Jacobi forms, Siegel modular forms of genus two and modular forms of half-integral weight, which are discussed in Chapters 2, 3 and 4 respectively. In Chapter 5, we give some remarks on Rankin's method for certain automorphic forms.

# 2. Jacobi Forms

Let $k, m$ be fixed positive integers and $\Gamma^J = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. We denote the space of Jacobi forms and the subspace of Jacobi cusp forms of weight $k$ and index $m$ on $\Gamma^J$ by $J_{k,m}$ and $J_{k,m}^{cusp}$ respectively. Choie, Kim and Knopp [5] constructed Jacobi cusp forms whose Fourier coefficients involve special value of certain Dirichlet series of Rankin type. More precisely, for a fixed $\phi \in J_{l,0}$, they considered the linear map $T_\phi : J_{k,m}^{cusp} \longrightarrow J_{k+l,m}^{cusp}$, $\psi \mapsto \phi\psi$ and computed its adjoint with respect to the Petersson scalar product. The Fourier coefficients of the image of a cusp form $\psi$ under the adjoint map involves special values of certain Dirichlet series of Rankin type attached to $\phi$ and $\psi$. Sakata [33] also constructed Jacobi cusp forms with similar Fourier coefficients and generalized the result of Choie, Kim and Knopp by computing the adjoint of the map $T_\phi$ for $\phi \in J_{l,n}$. Rankin-Cohen brackets for Jacobi forms were studied by Choie [6, 7] by using the heat operator $L_m := \frac{1}{(2\pi i)^2}\left(8\pi i m \frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)$. For $\nu \geqslant 0$, $\phi \in J_{k_1,m_1}$ and $\psi \in J_{k_2,m_2}$, the $\nu$-th Rankin–Cohen brackets $[\phi,\psi]_\nu \in J_{k_1+k_2+2\nu,m_1+m_2}$ and $[\ ,\ ]_\nu : J_{k_1,m_1} \times J_{k_2,m_2} \longrightarrow J_{k_1+k_2+2\nu,m_1+m_2}$ is a bilinear map.

## 2.1. Statement of the Theorem

For a fixed $\psi \in J_{k_2,m_2}^{cusp}$ and an integer $\nu \geq 0$, consider the map $T_{\psi,\nu} : J_{k_1,m_1}^{cusp} \to J_{k_1+k_2+2\nu,m_1+m_2}^{cusp}$ defined by $T_{\psi,\nu}(\phi) = [\phi,\psi]_\nu$. Then $T_{\psi,\nu}$ is a $\mathbb{C}$-linear map between two finite dimensional Hilbert spaces and therefore has an adjoint map

$T^*_{\psi,\nu} : J^{cusp}_{k_1+k_2+2\nu,m_1+m_2} \rightarrow J^{cusp}_{k_1,m_1}$ such that $\langle \phi, T_{\psi,\nu}(\omega)\rangle = \langle T^*_{\psi,\nu}(\phi), \omega\rangle, \ \forall\ \phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$ and $\omega \in J^{cusp}_{k_1,m_1}$. In the main result, we exhibit the Fourier coefficients of $T^*_{\psi,\nu}(\phi)$ for $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$. These coefficients involve special values of certain Dirichlet series associated to $\phi$ and $\psi$. We first prove the following lemma to ensure the convergence of Dirichlet series which appears as Fourier coefficients.

**Lemma 1.** *Let $k_1 > 4, k_2 > 3, m_1$ and $m_2$ be natural numbers. Let $\psi \in J^{cusp}_{k_2,m_2}$ with Fourier expansion*

$$\psi(\tau, z) = \sum_{\substack{n_1,r_1\in\mathbb{Z},\\ 4m_2n_1-r_1^2>0}} a(n_1, r_1)q^{n_1}\zeta^{r_1},$$

*and $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} b(n_2, r_2)q^{n_2}\zeta^{r_2}.$$

*Then the sum*

$$\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} \int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau, z)\overline{[e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} \mid dV_J$$

*converges. Here $dV_J = \dfrac{dudvdxdy}{v^3}$ with $\tau = u + iv$ and $z = x + iy$, is an invariant measure under the action on $\Gamma^J$ on $\mathcal{H}\times\mathbb{C}$.*

We now state the main theorem.

**Theorem 2.** *[17] Let $k_1 > 4, k_2 > 3, m_1$ and $m_2$ be natural numbers. Let $\psi \in J_{k_2,m_2}^{cusp}$ with Fourier expansion*

$$\psi(\tau, z) = \sum_{\substack{n_1, r_1 \in \mathbb{Z}, \\ 4m_2n_1 - r_1^2 > 0}} a(n_1, r_1) q^{n_1} \zeta^{r_1}.$$

*Then the image of any cusp form $\phi \in J_{k_1+k_2+2\nu,m_1+m_2}^{cusp}$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n_2, r_2 \in \mathbb{Z}, \\ 4(m_1+m_2)n_2 - r_2^2 > 0}} b(n_2, r_2) q^{n_2} \zeta^{r_2}$$

*under $T_{\psi,\nu}^*$ is given by*

$$T_{\psi,\nu}^*(\phi)(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z}, \\ 4m_1n - r^2 > 0}} c_\nu(n, r) q^n \zeta^r,$$

*where*

$$c_\nu(n, r) = \frac{(4m_1n - r^2)^{k_1 - \frac{3}{2}}}{\pi^{k_2+2\nu}} \frac{(m_1 + m_2)^{k_1+k_2+2\nu-2}}{m_1^{k_1-2}} \frac{\Gamma(k_1 + k_2 + 2\nu - \frac{3}{2})}{\Gamma(k_1 - \frac{3}{2})}$$

$$\times \sum_{l=0}^{\nu} A_l(k_1, m_1, k_2, m_2; \nu)(4m_1n - r^2)^l \sum_{\substack{n_1, r_1 \in \mathbb{Z} \\ 4m_2n_1 - r_1^2 > 0 \\ 4(m_1+m_2)(n+n_1) - (r+r_1)^2 > 0}} \frac{(4m_2n_1 - r_1^2)^{\nu-l} \overline{a(n_1, r_1)} b(n + n_1, r + r_1)}{(4(n + n_1)(m_1 + m_2) - (r + r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}},$$

*and*

$$A_l(k_1, m_1, k_2, m_2; \nu) \ = \ (-1)^l \binom{k_1 + \nu - \frac{3}{2}}{\nu - l} \binom{k_2 + \nu - \frac{3}{2}}{l} m_1^{\nu-l} m_2^l.$$

*Remark* 1. Fix $\psi \in J^{cusp}_{k_2,m_2}$ and suppose that $J^{cusp}_{k_1,m_1}$ is one dimensional space generated by $f(\tau,z)$. Then $T^*_{\psi,\nu}(\phi)(\tau,z) = \alpha_\phi f(\tau,z)$ for some constant $\alpha_\phi$ and for all $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$. In particular, for $\psi = \phi_{10,1} = \frac{1}{144}(E_6E_{4,1} - E_4E_{6,1}) \in J^{cusp}_{10,1}$ and $k_1 = 12, m_1 = 1$ ($J^{cusp}_{12,1} = \langle \phi_{12,1} \rangle$, $\phi_{12,1} := \frac{1}{144}(E_4^2E_{4,1} - E_6E_{6,1})$), we have

$$\sum_{l=0}^{\nu} A_l(12,1,10,1;\nu)(4n-r^2)^l \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4n_1-r_1^2>0\\ 8(n+n_1)-(r+r_1)^2>0}} \frac{(4n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(8(n+n_1)-(r+r_1)^2)^{22+2\nu-\frac{3}{2}}}$$

$$= \alpha_\phi c(n,r)$$

for all $n, r \in \mathbb{Z}$ such that $4n - r^2 > 0$, where $a(p,q), b(p,q)$ and $c(p,q)$ are the $(p,q)$-th Fourier coefficients of $\phi_{10,1}, \phi$ and $\phi_{12,1}$ respectively. If we take $\nu = 0$ in the above example, we have

$$\sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4n_1-r_1^2>0\\ 8(n+n_1)-(r+r_1)^2>0}} \frac{\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(8(n+n_1)-(r+r_1)^2)^{\frac{41}{2}}} = \alpha_\phi c(n,r).$$

## 3. Siegel Modular forms

We denote the space of Siegel modular forms and the subspace of Siegel cusp forms of weight $k$ and genus $g$ on $\Gamma_g := Sp_{2g}(\mathbb{Z})$ by $M_k(\Gamma_g)$ and $S_k(\Gamma_g)$ respectively. Lee [25] constructed Siegel cusp forms of genus $g$ by computing the adjoint map of the product map by a fixed Siegel cusp form of genus $g$

with respect to the Petersson scalar product. In the proof, he used Poincaré series of two variables and the holomorphic projection operator developed by Panchishkin [28]. Let $F \in S_k(\Gamma_2)$ and $G \in S_l(\Gamma_2)$ with Fourier coefficients $A(T)$ and $B(T)$ respectively. For a fixed positive definite $2 \times 2$ matrix $S$ and a non-negative integer $m$, define a Dirichlet series $L_{F,G;S,m}$ as

$$L_{F,G;S,m}(\sigma) = \sum_{T>0} \frac{det\ (T)^m A(T+S)\overline{B(T)}}{(det\ (T+S))^\sigma}. \tag{2}$$

Then $L_{F,G;S,m}(\sigma)$ converges for $Re(\sigma) > \frac{k+l}{2} - m + \frac{5}{18}$. We construct certain Siegel cusp form of genus two whose Fourier coefficients involve special values of the series (2). The Rankin-Cohen type operators for Siegel modular forms of genus two were studied by Choie and Eholzer [3] explicitly and the existence of such operators for general genus were established by Eholzer and Ibukiyama [12].

## 3.1. Statement of the Theorem

For a fixed $G \in S_l(\Gamma_2)$ and an integer $\nu \geq 0$, consider the map $T_{G,\nu} : S_k(\Gamma_2) \to S_{k+l+2\nu}(\Gamma_2)$ defined by $T_{G,\nu}(F) = [F,G]_\nu$, where $[F,G]_\nu$ is the $\nu$-th Rankin-Cohen bracket of $F$ and $G$. Then $T_{G,\nu}$ is a $\mathbb{C}$-linear map between two finite dimensional Hilbert spaces and therefore has an adjoint map $T^*_{G,\nu} : S_{k+l+2\nu}(\Gamma_2) \to S_k(\Gamma_2)$ given by $\langle F, T_{G,\nu}(H)\rangle = \langle T^*_{G,\nu}(F), H\rangle,\ \forall\ F \in S_{k+l+2\nu}(\Gamma_2)$ and $H \in S_k(\Gamma_2)$. We compute the Fourier coefficients of $T^*_{G,\nu}(F)$ in terms of the special values of Dirichlet series defined in (2). We prove the following lemma.

**Lemma 2.** *Let $k \geqslant 6, l$ be natural numbers and $\nu \geq 0$ be a fixed integer. Let $G \in S_l(\Gamma_2)$ with Fourier expansion*

$$G(Z) = \sum_{T_1 > 0} A(T_1) e^{2\pi i (tr(T_1 Z))}$$

*and $F \in S_{k+l+2\nu}(\Gamma_2)$ with Fourier expansion*

$$F(Z) = \sum_{T_2 > 0} B(T_2) e^{2\pi i (tr(T_2 Z))}.$$

*Then the sum $\sum\limits_{M \in \Delta \backslash \Gamma_2} \int\limits_{\Gamma_2 \backslash \mathcal{H}_2} |F(Z) \overline{[e^{2\pi i (tr(TZ))} |_k M, G]_\nu (Z)} \, (det\ Y)^{k+l+2\nu} | dZ$ converges.*

**Theorem 3.** *[18] Let $k \geqslant 6, l$ be natural numbers and $\nu \geq 0$ be a fixed integer. Let $G \in S_l(\Gamma_2)$ with Fourier expansion*

$$G(Z) = \sum_{T_1 > 0} A(T_1) e^{2\pi i (tr(T_1 Z))}.$$

*Then the image of any cusp form $F \in S_{k+l+2\nu}(\Gamma_2)$ with Fourier expansion*

$$F(Z) = \sum_{T_2 > 0} B(T_2) e^{2\pi i (tr(T_2 Z))}$$

*under* $T^*_{G,\nu}$ *is given by*

$$T^*_{G,\nu}(F)(Z) = \sum_{T>0} C(T) e^{2\pi i (tr(TZ))}.$$

*Here*

$$C(T) = \alpha(k,l,\nu) \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^{k+r-3/2}\ L_{F,G;T,s}(k+l+2\nu-(p+3/2)),$$

*with*

$$\alpha(k,l,\nu) = \frac{(-1)^\nu \Gamma_2(k+l+2\nu-\frac{3}{2})}{2\sqrt{\pi}\Gamma(k-\frac{3}{2})\Gamma(k-2)(4\pi)^{2(l+\nu)}},$$

*and*

$$C_{r,s,p}(k,l) = \frac{(k+\nu-3/2)_{s+p}}{r!} \frac{(l+\nu-3/2)_{r+p}}{s!} \frac{(-(k+l+\nu-3/2))_{r+s}}{p!},$$

*where*

$$(x)_m = \prod_{0 \leqslant i \leqslant m-1} (x-i),$$

*and the Gamma function* $\Gamma_2(\sigma)$ *is defined as*

$$\Gamma_2(\sigma) = \int_{\mathbb{Y}} e^{-trY} (det\ Y)^{\sigma-3/2} dY, \quad \text{for } Re(\sigma) > 3/2,$$

*where* $\mathbb{Y} = \{Y \in M_{2\times 2}(\mathbb{C}) \mid Y^t = Y > 0\}$.

# 4. Modular Forms of Half Integral Weight

Let $k \in \mathbb{Z}$ and $\Gamma = \Gamma_0(4)$. We denote the space of modular forms and the subspace of cusp forms of weight $k + \frac{1}{2}$ and Dirichlet character $\chi$ for $\Gamma$ by $M_{k+\frac{1}{2}}(\Gamma, \chi)$ and $S_{k+\frac{1}{2}}(\Gamma, \chi)$respectively. Consider the following three linear maps:

(I) $T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+1}(\Gamma, \chi_2\chi),$ defined by $T_{g,\nu}(f) = [f, g]_\nu,$ with $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2),$

(II) $T_{g,\nu} : S_k(\Gamma) \to S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi),$ defined by $T_{g,\nu}(f) = [f, g]_\nu,$ with $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2),$

(III) $T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi),$ defined by $T_{g,\nu}(f) = [f, g]_\nu,$ with $g \in M_l(\Gamma, \chi_2).$

We compute the adjoint of these maps with respect to the Petersson inner product. We exhibit explicitly the Fourier coefficients of $T^*_{g,\nu}(f)$ for $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ and these coefficients involve special values of certain Dirichlet series of Rankin-Selberg type associated to $f$ and $g$. We now state the result for the map in (I).

**Theorem 4.** *[19] Let $k$ and $l$ be natural numbers and $\nu \geqslant 0$. Let $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$ with Fourier expansion*

$$g(z) = \sum_{m=0}^{\infty} b(m) q^m.$$

*Suppose that either (a) $g$ is a cusp form and $k \geqslant 2$, or (b) $g$ is not a cusp form and $l < k - \frac{3}{2}$. Then the image of any cusp form $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ with Fourier expansion*

$$f(z) = \sum_{m=1}^{\infty} a(m) q^m$$

*under $T^*_{g,\nu}$ is given by*

$$T^*_{g,\nu}(f)(z) = \sum_{n=1}^{\infty} \beta(k,l,\nu;n) L_{f,g,\nu,n}(\gamma) q^n,$$

*where $\gamma = k + l + 2\nu$, $\beta(k,l,\nu;n) = \dfrac{\Gamma(k+l+2\nu)\ n^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})(4\pi)^{l+2\nu+\frac{1}{2}}}$, and $L_{f,g,\nu,n}$ is the $L$-function associated with $f$ and $g$, defined by*

$$L_{f,g,\nu,n}(s) = \sum_{m=1}^{\infty} \frac{a(n+m)\overline{b(m)}\ \ \alpha(k,l,\nu,n,m)}{(n+m)^s},\ \ s \in \mathbb{C}$$

*with*

$$\alpha(k,l,\nu,n,m) \ = \ \sum_{r=0}^{\nu} (-1)^{\nu-r} \binom{\nu}{r} \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)} n^r m^{\nu-r}.$$

*Remark* 2. We have similar results for the map in (II) with

$$\gamma = k+l+2\nu-\tfrac{1}{2},\ \text{and}\ \beta(k,l,\nu;n) = \frac{\Gamma(k+l+2\nu-\frac{1}{2})\ n^{k-1}}{\Gamma(k-1)\ (4\pi)^{l+2\nu+\frac{1}{2}}},$$

and for the map in (III) with

$$\gamma = k+l+2\nu-\frac{1}{2},\ \text{and}\ \beta(k,l,\nu;n) = \frac{\Gamma(k+l+2\nu-\frac{1}{2})\ n^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})\ (4\pi)^{l+2\nu}},$$

with the assumption that either (a) $g$ is a cusp form and $k > 3$, or (b) $g$ is not a cusp form and $l < k-2$.

*Remark* 3. Consider the linear map $T^*_{g,\nu} \circ T_{g,\nu}$ on $S_k(\Gamma)$ with $g(z) \in M_l(\Gamma, \chi_2)$. If $\lambda$ is an eigenvalue of $T^*_{g,\nu} \circ T_{g,\nu}$, then $\lambda \geqslant 0$. Suppose that $S_k(\Gamma)$ is a one-dimensional space generated by $f(z) = \sum_{m=1}^{\infty} a(n)q^n$. Then $T^*_{g,\nu} \circ T_{g,\nu}(h) = \lambda f, \ \forall\ h \in S_k(\Gamma)$. In particular, $T^*_{g,\nu} \circ T_{g,\nu}(f) = \lambda f$ with $\lambda \geqslant 0$ and if we write $T^*_{g,\nu} \circ T_{g,\nu}(f) = \sum_n c(n)q^n$ then

$$c(n) = \frac{\Gamma(k+l+2\nu-1)}{\Gamma(k-1)} \frac{n^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(n+m)\overline{b(m)} \ \ \alpha(k,l,\nu,n,m)}{(n+m)^{k+l+2\nu-1}},$$

where $a_{T_{g,\nu}(f)}(n)$ is the $n$-th Fourier coefficient of $T_{g,\nu}(f) = [f,g]_\nu$. If $a(m_0)$ is the first non-zero Fourier coefficient of $f$ then we have

$$\lambda = \frac{\Gamma(k+l+2\nu-1)}{a(m_0)\Gamma(k-1)} \frac{m_0^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(m_0+m)\overline{b(m)} \ \ \alpha(k,l,\nu,m_0,m)}{(m_0+m)^{k+l+2\nu-1}} \geqslant 0.$$

Further, if we take $l = 0, k = 6, \ \nu = 0$ with $g(z) = \theta(z) = \sum_n q^{n^2}$ and $\Delta_{4,6}(z) = \sum_n \tau_{4,6}(n)q^n \in S_6(\Gamma_0(4))$ in map (II), then

$$\sum_{m=1}^{\infty} \frac{a_{T_{\theta,0}(\Delta_{4,6})}(m+1)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0. \tag{3}$$

Now $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$ is the $(m+1)$-th Fourier coefficient of $\theta(z)\Delta_{4,6}(z)$ and equals to $\sum_{r=1}^{m+1} b(r)\tau_{4,6}(m+1-r)$. Putting the value of $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$ in

(4.4.1), we have

$$\sum_{m=1}^{\infty} \frac{\left(\sum_{r=1}^{m^2+1} \tau_{4,6}(m^2+1-r^2)\right)}{(m^2+1)^{\frac{11}{2}}} > 0.$$

# 5. Remarks on Rankin's Method

Rankin [30] showed that for any normalized eigenform $f \in S_k$ with Fourier coefficients $a(n)$ and any even integer $l$ with $\frac{k}{2}+2 \leqslant l \leqslant k-4$, one has the following identity

$$L_f^*(l)L_f^*(k-1) = (-1)^{\frac{l}{2}} 2^{k-3} \frac{B_l}{l} \frac{B_{k-l}}{k-l} \langle f, E_l E_{k-l} \rangle,$$

where $L_f^*(s) = (2\pi)^s \Gamma(s) L_f(s)$ with $L_f(s) = \sum_{n=1}^{\infty} \dfrac{a(n)}{n^s}$.
Zagier generalized the result of Rankin by taking any modular form instead of Eisenstein series i.e., he computed $\langle f, gE_l \rangle$, for $f \in S_{k+l}$ and $g \in M_k$. He also considered the Rankin-Cohen bracket instead of product and proved the following theorem.

**Theorem 5.** *[37] Let $l \geqslant k+2 > 2$ and $\nu \geqslant 0$ be integers. Let $f \in S_{k+l+2\nu}$ and $g \in M_k$ with Fourier coefficients $a(n)$ and $b(n)$ respectively. Then*

$$\langle f, [g, E_l]_\nu \rangle = \frac{\Gamma(k+l+2\nu-1)\Gamma(l+\nu)}{(4\pi)^{k+l+2\nu-1}\Gamma(l)} \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^{k+l-1}}.$$

To prove it he writes $[g, E_l]_\nu$ as a linear combination of Poincaré series and then computes the Petersson scalar product. We observed that the method of Herrero can be used to give a different proof of Theorem 5. We prove a similar theorem for the case of Siegel modular forms of genus 2.

**Theorem 6.** *Let $k \geqslant 4, l$ be natural numbers and $\nu \geq 0$ be a fixed integer. Let $G \in S_l(\Gamma_2)$ with Fourier coefficients $A(T)$ and $F \in S_{k+l+2\nu}(\Gamma_2)$ with Fourier coefficients $B(T)$. Then*

$$\langle F, [G, E_k^{(2)}]_\nu \rangle = \frac{(-1)^\nu \Gamma_2(k+l+2\nu-\frac{3}{2}) \sum\limits_{r+p=\nu} C_{r,0,p}(k,l)}{(4\pi)^{2(k+l+\nu-\frac{3}{2})}} \sum_{T>0} \frac{\overline{A(T)}B(T)}{(detT)^{k+l+\nu-\frac{3}{2}}}.$$

# Publications:

a. **Published**: [17] Rankin-Cohen brackets on Jacobi Forms and the adjoint of some linear maps (with B. Sahu), *The Ramanujan Journal*, **39** (2016), no. 3, 533–544.

b. **Accepted**: [18] Rankin-Cohen brackets on Siegel modular forms and special values of certain Dirichlet series (with B. Sahu), *The Ramanujan Journal*, DOI 10.1007/s11139-016-9783-3.

c. **Communicated**: [19] Construction of cusp forms using Rankin-Cohen brackets (with A. Kumar).

# Chapter 1

# Preliminaries

In this chapter we give basic definitions and some properties of modular forms, Jacobi forms and Siegel modular forms.

## 1.1 Notations

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. For $z \in \mathbb{C}$, $Re\ z$ denotes the real part of $z$ and $Im\ z$ denotes the imaginary part of $z$. For any complex number $z$ and a non-zero real number $c$, we denote by $e_c(z) = e^{2\pi i z/c}$. If $c = 1$, we simply write $e(z)$ instead of $e_1(z)$. Let $\mathcal{H} = \{\tau \in \mathbb{C} : Im\ \tau > 0\}$ be the complex upper half-plane. We denote by $q = e(\tau)$, for $\tau \in \mathcal{H}$. For a complex number $z$, the square root is defined as follows:

$$\sqrt{z} = \mid z \mid^{\frac{1}{2}} e^{\frac{i}{2} arg\ z}, \text{ with } -\pi < arg\ z \leqslant \pi.$$

We set $z^{\frac{k}{2}} = (\sqrt{z})^k$ for any $k \in \mathbb{Z}$. The full modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

For a positive integer $N$, we denote the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ as follows:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \ (mod\ N) \right\}.$$

## 1.2 Modular forms for $SL_2(\mathbb{Z})$

The group $GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc > 0 \right\}$ acts on $\mathcal{H}$ via fractional linear transformations, i.e., for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and $\tau \in \mathcal{H}$

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

Let $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$. For a complex valued function $f$ define the slash operator as follows:

$$(f \mid_k \gamma)(\tau) := (det\ \gamma)^{\frac{k}{2}}\ (c\tau + d)^{-k} f(\gamma \cdot \tau).$$

**Definition 1.2.1.** *A modular form of weight $k$ for $SL_2(\mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ satisfying*

1. $f \mid_k \gamma = f, \ \forall \gamma \in SL_2(\mathbb{Z})$, *i.e.,*
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \ \forall \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } \forall \tau \in \mathcal{H}.$$
2. $f$ *is holomorphic at infinity, i.e.,* $f$ *has a Fourier series expansion of the form* $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$.

   *If we further have* $a(0) = 0$, *then* $f$ *is called a cusp form.*

We denote the space of all modular forms of weight $k$ for $SL_2(\mathbb{Z})$ and the subspace of all cusp forms of weight $k$ for $SL_2(\mathbb{Z})$ by $M_k$ and $S_k$, respectively.

For $f, \ g \in M_k$ such that at least one of them a cusp form, the Petersson scalar product of $f$ and $g$ is defined as:

$$\langle f, g \rangle \ = \int\limits_{SL_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\overline{g(\tau)}(Im(\tau))^k d^*\tau,$$

where $SL_2(\mathbb{Z}) \setminus \mathcal{H}$ is a fundamental domain and $d^*\tau = \dfrac{dudv}{v^2}$ $(\tau = u + iv)$ is an invariant measure under the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$.

**Example.** Let $k$ be an even integer greater than 2. The normalized Eisenstein series $E_k$ of weight $k$ for $SL_2(\mathbb{Z})$ is defined as:

$$E_k(\tau) := \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2-\{(0,0)\} \\ (m,n)=1}} \frac{1}{(mz+n)^k}.$$

Then $E_k$ is a modular form of weight $k$ for $SL_2(\mathbb{Z})$ with Fourier expansion

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and $B_k$'s are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k=o}^{\infty} B_k \frac{x^k}{k!}.$$

The Fourier expansions of $E_k$ for $k = 4, 6, 8, 10$ and 12 are as follows:

$$\begin{aligned}
E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\
E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \\
E_8(\tau) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \\
E_{10}(\tau) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n, \\
E_{12}(\tau) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n.
\end{aligned}$$

**Example.** The Ramanujan delta function is defined as

$$\Delta(\tau) := \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2).$$

$\Delta$ is a cusp form of weight 12 for $SL_2(\mathbb{Z})$ with Fourier expansion

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \ = \ \sum_{n=1}^{\infty}\tau(n)q^n,$$

where $\tau(n)$ is called the Ramanujan tau function.

**Example.** Let $n$ be a be a positive integer. The $n$-th Poincaŕe series of weight $k$ for $SL_2(\mathbb{Z})$ is defined by

$$P_{k,n}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} e^{2\pi i n\tau}|_k\gamma, \tag{1.2.1}$$

where $\Gamma_\infty := \left\{\pm\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} : \ t\in\mathbb{Z}\right\}$. $P_{k,m}$ is a cusp form of weight $k>2$ for $SL_2(\mathbb{Z})$ with Fourier expansion

$$P_{k,m}(\tau) = \sum_{n=1}^{\infty} g_m(n)q^n,$$

where

$$g_m(n) = \delta_{m,n} + (-1)^{\frac{k}{2}+1}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\pi\sum_{c=1}^{\infty}K_c(m,n)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

and $K_c(m,n)$ is the Kloosterman sum defined by

$$\frac{1}{c}\sum_{\substack{d \pmod{c}\\ dd^{-1}\equiv 1 \pmod{c}}} e^{2\pi i\left(\dfrac{nd+md^{-1}}{c}\right)},$$

and $J_{k-1}(x)$ is the Bessel function of order $k-1$. The Poincaŕe series has the following property: If $f \in S_k$ with Fourier expansion $f(\tau) = \sum_{m=1}^{\infty} a(m)q^m$, then

$$\langle f, P_{k,n} \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a(n). \tag{1.2.2}$$

We now define Hecke operators which send modular forms to modular forms. Let $n$ be a positive integer. For $f(\tau) = \sum_m a(m)q^m \in M_k$, the $n$-th Hecke operator is defined by

$$(T_n f)(\tau) := \sum_m a_n(m)q^m,$$

where $a_n(m) = \sum_{d|(m,n)} \chi(d)d^{k-1}a(\frac{mn}{d^2})$. If $f \in M_k$ (or $S_k$), then $T_n f \in M_k$ (or $S_k$). The family $\{T_n : n \in \mathbb{N}\}$ of Hecke operators is commuting. The Hecke operators $T_n$ acting on $S_k$ are self-adjoint with respect to the Petersson inner product.

**Definition 1.2.2.** *A cusp form is said to be an eigenform if it is simultaneous eigenfunction for all the Hecke operators.*

In the space of cusp forms $S_k$, there exists an orthonormal basis consisting of eigenforms of all the Hecke operators $T_n$.

Let $f(z) \in S_k$, with Fourier expansion $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n$. We associate a $L$-function to $f$ defined by

$$L_f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{1.2.3}$$

This series converges for $Re\ s > c+1$, where $c = \frac{k}{2} + \epsilon$. Define $\Lambda(s)$ by

$$\Lambda(s) := \frac{1}{(2\pi)^s}\Gamma(s)L_f(s),$$

then $\Lambda(s)$ extends to an entire function of $s$, and satisfies the functional equation

$$\Lambda(s) = \pm\,\Lambda(k-s).$$

Further, if $f$ is an eigenform with Fourier coefficient $a(n)$ then $L_f(s)$ has an Euler product

$$L_f(s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

## 1.3 Modular forms for $\Gamma_0(N)$

**Definition 1.3.1.** *Let $k$ be an integer and $\chi$ a Dirichlet character modulo $N$. A holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight $k$, level $N$ and character $\chi$ if*

1. $(f\,|_k\,\gamma)(\tau) = \chi(d)f(\tau),\ \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, *i.e.,*

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau),\ \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

2. *$f$ is holomorphic at all the cusps of $\Gamma_0(N)$.*

*Further, we say $f$ is a cusp form if $f$ vanishes at all the cusps of $\Gamma_0(N)$.*

We denote the space of all modular forms and the subspace of all cusp forms of weight $k$, level $N$ with character $\chi$ on $\Gamma_0(N)$ by $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, respectively. If $\chi$ is the trivial character, then we denote the spaces as $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$, respectively.

For $f,\ g \in M_k(\Gamma_0(N), \chi)$ such that at least one of them a cusp form, the Petersson scalar product of $f$ and $g$ is defined as:

$$\langle f, g \rangle \ = \ \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \int\limits_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)\overline{g(\tau)}(Im(\tau))^k d^*\tau,$$

where $\Gamma_0(N) \setminus \mathcal{H}$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$.

The following lemma tells about the growth of the Fourier coefficients of a modular form.

**Lemma 1.3.2.** *[16] If $f \in M_k(\Gamma_0(N), \chi)$ with Fourier coefficients $a(n)$, then*

$$a(n) \ll |n|^{k-1+\epsilon},$$

*and moreover, if $f$ is a cusp form, then*

$$a(n) \ll |n|^{\frac{k}{2}-\frac{1}{4}+\epsilon}.$$

For more details on the theory of modular forms of integral weight, we refer to [16, 20].

## 1.4 Modular forms of half-integral weight

Let $\Gamma = \Gamma_0(4)$. For $k \in \mathbb{Z} + \frac{1}{2}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ define the slash operator as follows:

$$\left(f\tilde{|}_k\gamma\right)(\tau) := \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^k (c\tau + d)^{-k} f(\gamma \cdot \tau),$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol.

**Definition 1.4.1.** *Let $k$ be an integer and $\chi$ a Dirichlet character modulo 4. A holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight $k + \frac{1}{2}$ and character $\chi$ for $\Gamma$ if*

1. $\left(f\tilde{|}_{k+\frac{1}{2}}\gamma\right)(\tau) \; = \; \chi(d)f(\tau), \; \forall \; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$
2. *$f$ is holomorphic at all the cusps of $\Gamma$.*

*Further, we say $f$ is a cusp form if $f$ vanishes at all the cusps of $\Gamma$.*

We denote the space of all modular forms and the subspace of all cusp forms of weight $k + \frac{1}{2}$ with character $\chi$ on $\Gamma$ by $M_{k+\frac{1}{2}}(\Gamma, \chi)$ and $S_{k+\frac{1}{2}}(\Gamma, \chi)$, respectively. The Petersson scalar product on $S_{k+\frac{1}{2}}(\Gamma, \chi)$ is defined as follows:

$$\langle f, g\rangle \; = \; \int_{\Gamma\backslash\mathcal{H}} f(\tau)\overline{g(\tau)}(Im(\tau))^{k+\frac{1}{2}} d^*\tau.$$

The space $S_{k+\frac{1}{2}}(\Gamma, \chi)$ is a finite dimensional Hilbert space.

**Definition 1.4.2.** *Let $n$ be a be a positive integer. The $n$-th Poincaŕe series of weight $k+\frac{1}{2}$, where $k \in \mathbb{Z}$ is defined by*

$$P_{k+\frac{1}{2},n}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i n \tau} \tilde{|}_{k+\frac{1}{2}} \gamma. \tag{1.4.1}$$

It is well-known that $P_{k+\frac{1}{2},n} \in S_{k+\frac{1}{2}}(\Gamma)$ for $k > 2$ . This series has the following property:

**Lemma 1.4.3.** *Let $f \in S_{k+\frac{1}{2}}(\Gamma)$ with Fourier expansion*

$$f(\tau) = \sum_{m=1}^{\infty} a(m) q^m.$$

*Then*

$$\langle f, P_{k+\frac{1}{2},n} \rangle = \frac{\Gamma(k-\frac{1}{2})}{(4\pi n)^{k-\frac{1}{2}}} a(n). \tag{1.4.2}$$

The following lemma tells about the growth of the Fourier coefficients of a modular form.

**Lemma 1.4.4.** *If $f \in M_{k+\frac{1}{2}}(\Gamma, \chi)$ with Fourier coefficients $a(n)$, then*

$$a(n) \ll |n|^{k-\frac{1}{2}+\epsilon},$$

*and moreover, if $f \in S_{k+\frac{1}{2}}(\Gamma, \chi)$ is a cusp form, then*

$$a(n) \ll |n|^{\frac{k}{2}+\epsilon}.$$

For more details on the theory of modular forms of half-integral weight, we refer to [20, 34].

## 1.5 Jacobi forms

The Jacobi group $\Gamma^J := SL_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$ acts on $\mathcal{H} \times \mathbb{C}$ by

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \cdot (\tau, z) = \left( \frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Let $k, m$ be fixed positive integers. For a complex valued function $\phi$ on $\mathcal{H} \times \mathbb{C}$ and $\gamma = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma^J$ define

$$\left(\phi|_{k,m}\gamma\right)(\tau, z) := (c\tau + d)^{-k} e^{2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z\right)} \phi(\gamma \cdot (\tau, z)).$$

**Definition 1.5.1.** *A Jacobi form of weight $k$ and index $m$ on $\Gamma^J$ is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following:*

$$\phi|_{k,m}\gamma = \phi, \ \forall \gamma \in \Gamma^J$$

*and having a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ r^2 \leq 4nm}} c(n, r) q^n \zeta^r \qquad (q = e^{2\pi i \tau}, \ \zeta = e^{2\pi i z}).$$

*Further, we say $\phi$ is a cusp form if $c(n,r) \neq 0 \implies r^2 < 4nm$.*

We denote the space of all Jacobi forms and the subspace of all Jacobi cusp forms by $J_{k,m}$ and $J_{k,m}^{cusp}$ respectively. The Petersson scalar product of $\phi,\ \psi \in J_{k,m}^{cusp}$ is defined as :

$$\langle \phi, \psi \rangle = \int\limits_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} \phi(\tau, z)\overline{\psi(\tau, z)} v^k e^{\frac{-4\pi m y^2}{v}} dV_J,$$

where $dV_J = \dfrac{dudvdxdy}{v^3}$ is an invariant measure under the action on $\Gamma^J$ on $\mathcal{H} \times \mathbb{C}$ with $\tau = u + iv, z = x + iy$ and $\Gamma^J \setminus \mathcal{H} \times \mathbb{C}$ is a fundamental domain for the action of $\Gamma^J$ on $\mathcal{H} \times \mathbb{C}$. The space $(J_{k,m}^{cusp}, \langle , \rangle)$ is a finite dimensional Hilbert space.

**Example 1.5.1.** *Let $k \geqslant 4$ be an even integer. The Jacobi Eisenstein series of weight $k$ and index $m$ is defined as*

$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e^m \left( \lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right),$$

*where $a$ and $b$ are such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then $E_{k,m}$ is a Jacobi form of weight $k$ and index $m$ for $\Gamma^J$.*

**Example 1.5.2.** *Let $m, n$ and $r$ be fixed integers with $r^2 < 4mn$. The $(n, r)$-*

*th Jacobi-Poincaré series of weight $k$ and index $m$ is defined as*

$$P_{k,m;(n,r)}(\tau,z) := \sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} e^{2\pi i(n\tau+rz)}|_{k,m}\gamma. \tag{1.5.1}$$

*Here* $\Gamma^J_\infty := \left\{\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, (0,\mu)\right) : t,\mu\in\mathbb{Z}\right\}$. *It is well-known that* $P_{k,m;(n,r)} \in J^{cusp}_{k,m}$ *for* $k>2$ *(see [13]).*

This series has the following property:

**Lemma 1.5.2.** *Let* $\phi\in J^{cusp}_{k,m}$ *with Fourier expansion*

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},\\ r^2<4nm}} c(n,r)q^n\zeta^r.$$

*Then*

$$\langle\phi, P_{k,m;(n,r)}\rangle = \alpha_{k,m}(4mn-r^2)^{-k+\frac{3}{2}}c(n,r), \tag{1.5.2}$$

*where*

$$\alpha_{k,m} = \frac{m^{k-2}\Gamma(k-\frac{3}{2})}{2\pi^{k-\frac{3}{2}}}.$$

The following lemma tells about the growth of the Fourier coefficients of a Jacobi form.

**Lemma 1.5.3** (Choie, Kohnen [9])**.** *If* $\phi\in J_{k,m}$ *and* $k>3$ *with Fourier coefficients* $c(n,r)$, *then*

$$c(n,r) \ll |r^2-4nm|^{k-\frac{3}{2}},$$

*and moreover, if $\phi$ is a Jacobi cusp form, then*

$$c(n,r) \ll |r^2 - 4nm|^{\frac{k}{2}-\frac{1}{2}}.$$

For more details on the theory of Jacobi forms, we refer to [11].

## 1.6 Siegel modular forms

Let $\mathcal{H}_g := \{Z = X + iY \in M_{2g}(\mathbb{C}) \mid Z^t = Z,\ Y > 0\}$ be the Siegel upper half-plane. Let $\Gamma_g$ be the symplectic group $Sp_{2g}(\mathbb{Z})$ of genus $2g$ defined as $\Gamma_g := \{M \in M_{2g}(\mathbb{Z}) : MJ_{2g}M^t = J_{2g}\}$, $J_{2g} = \begin{pmatrix} O_g & -I_g \\ -I_g & O_g \end{pmatrix}$, where $O_g$ and $I_g$ are zero matrix and identity matrix of order $g \times g$, respectively. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ with $A, B, C, D \in M_g(\mathbb{Z})$, then $AB^t = BA^t, CD^t = DC^t, AD^t - BC^t = I_g$. The group $\Gamma_g$ acts on $\mathcal{H}_g$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Let $k$ be a fixed positive integer and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$. For a complex valued function $F$ on $\mathcal{H}_g$, define the slash operator

$$(F|_k M)(Z) := det(CZ + D)^{-k} F(M \cdot Z),\ Z \in \mathcal{H}_g.$$

**Definition 1.6.1.** *A Siegel modular form of weight $k$ and genus $g$ is a holomorphic function $F : \mathcal{H}_g \to \mathbb{C}$ satisfying $F|_k M = F,\ \forall M \in \Gamma_g$ i.e., $F((AZ+B)(CZ+D)^{-1}) = (det(CZ+D))^k\ F(Z)$ and having a Fourier expansion of the form*

$$F(Z) = \sum_{T \geq 0} A(T) e^{2\pi i(tr(TZ))}, \tag{1.6.1}$$

*where the summation runs over positive semi-definite half-integral (i.e., $2t_{ij},\ t_{ii} \in \mathbb{Z}$) $g \times g$ matrices $T$.*

We denote the space of Siegel modular forms of weight $k$ and genus $g$ on $\Gamma_g$ by $M_k(\Gamma_g)$. Further, we say $F$ is a cusp form if the summation in (1.6.1) runs over positive definite half-integral matrices $T$. We denote the space of Siegel cusp forms by $S_k(\Gamma_g)$.

We now restrict to the case $g = 2$. If $Z \in \mathcal{H}_2$, then $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathcal{H},\ z \in \mathbb{C}$, and $T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$ with $n, r, m \in \mathbb{Z},\ n, m \geqslant 0$, and $r^2 \leqslant 4nm$. Write $F(\tau,\ z,\ \tau')$ for $F(Z)$ and $A(n,\ r,\ m)$ for $A(T)$, then Fourier expansion of $F$ becomes

$$F(\tau,\ z,\ \tau') \ = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m \geqslant 0 \\ 4nm - r^2 \geqslant 0}} A(n,\ r,\ m) e^{2\pi i(n\tau + rz + m\tau')}.$$

**Theorem 1.6.2.** *Let $F$ be a Siegel modular form of weight $k$ and genus 2 and write the Fourier expansion of $F$ in the form*

$$F(\tau,\ z,\ \tau') \ = \ \sum_{m=0}^{\infty} \phi_m(\tau,\ z) e^{2\pi i m \tau'}.$$

*Then $\phi_m$ is a Jacobi form weight $k$ and index $m$, for each $m$.*

One has the following estimate for the Fourier coefficients of Siegel cusp forms of genus 2.

**Theorem 1.6.3.** *[22, 23, 1] Let $F$ be a Siegel cusp form of weight $k$ and genus 2 with Fourier coefficients $A(T)$. Then*

$$A(T) \ll_{F,\epsilon} (det\ T)^{k/2-13/36+\epsilon} \qquad\qquad (\epsilon > 0). \qquad\qquad (1.6.2)$$

The Petersson scalar product on $S_k(\Gamma_2)$ is defined as

$$\langle F, G\rangle = \int\limits_{\Gamma_2\backslash\mathcal{H}_2} F(Z)\overline{G(Z)}(det\ Y)^k dZ,$$

where $F,\ G \in S_k(\Gamma_2),\ Z = X + iY$ and $dZ = (det\ Y)^{-3} dX dY$ is an invariant measure under the action of $\Gamma_2$ on $\mathcal{H}_2$. The space $(S_k(\Gamma_2), \langle,\rangle)$ is a finite dimensional Hilbert space.

**Example 1.6.1.** *Let $k \geqslant 4$ be a fixed even integer. Then the Eisenstein* *series of weight $k$ and genus 2 is defined as*

$$E_k^{(2)}(Z) := \sum_{M \in \Delta \backslash \Gamma_2} 1|_k M.$$

*Here* $\Delta := \left\{ \begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix} : S \in M_{2\times 2}(\mathbb{Z}),\ S^t = S \right\}$ *is a subgroup of* $\Gamma_2$. *It is well-known that* $E_k^{(2)} \in M_k(\Gamma_2)$.

**Example 1.6.2.** *Let $k \geqslant 6$ be a fixed positive integer and $T$ be a fixed symmetric positive definite half-integral $2 \times 2$ matrix. Then the $T$-th Poincaŕe series of weight $k$ and genus 2 is defined as*

$$P_{k,T}(Z) := \sum_{M \in \Delta \backslash \Gamma_2} e^{2\pi i (tr(TZ))}|_k M. \tag{1.6.3}$$

*Then $P_{k,T}$ is a Siegel cusp form of weight $k$ and genus 2.*

The Poincaŕe series has the following property:

**Lemma 1.6.4.** *[24] Let $F \in S_k(\Gamma_2)$ with Fourier expansion*

$$F(Z) = \sum_{T>0} A(T) e^{2\pi i (tr(TZ))}.$$

*Then*

$$\langle F, P_{k,T} \rangle = c_k (det\ T)^{-k+\frac{3}{2}} A(T), \tag{1.6.4}$$

*where*

$$c_k = 2\sqrt{\pi}(4\pi)^{3-2k}\Gamma(k-3/2)\Gamma(k-2).$$

For basic theory on Siegel modular forms, we refer to [2, 27].

## 1.7 Rankin-Cohen brackets

There are many interesting connections between differential operators, and modular forms and many interesting results have been found. Rankin [31, 32] gave a general description of the differential operators which send modular forms to modular forms. Cohen [10] explicitly constructed certain bilinear operators using differential operators and obtained elliptic modular forms with interesting Fourier coefficients. Zagier [38, 37] studied algebraic properties of these bilinear operators and called them Rankin–Cohen brackets.

Let $k$ and $l$ be real numbers and $\nu \geq 0$ be an integer. Let $f$ and $g$ be two complex-valued holomorphic functions on $\mathcal{H}$. Define the $\nu$-th Rankin-Cohen bracket of $f$ and $g$ by

$$[f,g]_\nu := \sum_{r=0}^{\nu}(-1)^{\nu-r}\binom{\nu}{r}\frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)}D^r f D^{\nu-r} g, \tag{1.7.1}$$

where $D^r f = \dfrac{1}{(2\pi i)^r}\dfrac{d^r f}{d\tau^r}$ and $\Gamma(x)$ is the usual Gamma function.

*Remark* 1.7.1. We note that the 0-th Rankin-Cohen bracket is the usual product, i.e., $[f,g]_0 = fg$.

*Remark* 1.7.2. One has the following property:

$$[f|_k\gamma, g|_l\gamma]_\nu = [f,g]|_{k+l+2\nu}\gamma, \quad \forall \gamma \in \Gamma. \tag{1.7.2}$$

**Theorem 1.7.1** (Cohen [10]). *Let $\nu \geq 0$, be an integer and $f \in M_k(\Gamma, \chi_1)$ and $g \in M_l(\Gamma, \chi_2)$. Then $[f,g]_\nu \in M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$,*

$$\text{where } \chi = \begin{cases} 1, & if \text{ both } k,l \in \mathbb{Z}, \\ \chi_{-4}^k, & if\ k \in \mathbb{Z} \text{ and } l \in \mathbb{Z}+\frac{1}{2}, \\ \chi_{-4}^l, & if\ k \in \mathbb{Z}+\frac{1}{2} \text{ and } l \in \mathbb{Z}, \\ \chi = \chi_{-4}^{k+l} & if \text{ both } k,l \in \mathbb{Z}+\frac{1}{2}. \end{cases}$$

*Moreover, if $\nu > 0$, then $[f,g]_\nu \in S_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$. In fact, $[\ ,\ ]_\nu$ is a bilinear map from $M_k(\Gamma,\chi_1) \times M_l(\Gamma,\chi_2)$ to $M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)$. Here $\chi_{-4}$ is the character defined by $\chi_{-4}(\cdot) = \left(\dfrac{-4}{\cdot}\right)$.*

### 1.7.1 Rankin-Cohen brackets on Jacobi forms

Rankin-Cohen brackets for Jacobi forms were studied by Choie [6, 7] by using the heat operator. For an integer $m$, we define the heat operator

$$L_m := \frac{1}{(2\pi i)^2}\left(8\pi i m\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right).$$

Let $k_1, k_2, m_1$ and $m_2$ be positive integers and $\nu \geq 0$ be an integer. Let $\phi$ and $\psi$ be two complex-valued holomorphic functions on $\mathcal{H} \times \mathbb{C}$. Define the $\nu$-th Rankin-Cohen bracket of $\phi$ and $\psi$ by

$$[\phi, \psi]_\nu := \sum_{l=0}^{\nu} (-1)^l \binom{k_1 + \nu - \frac{3}{2}}{\nu - l} \binom{k_2 + \nu - \frac{3}{2}}{l} m_1^{\nu-l} m_2^l L_{m_1}^l(\phi) L_{m_2}^{\nu-l}(\psi). \tag{1.7.3}$$

We note here that $x! = \Gamma(x+1)$.

*Remark* 1.7.3. Using the action of heat operator, one can verify that

$$[\phi|_{k_1,m_1}\gamma, \psi|_{k_2,m_2}\gamma]_\nu = [\phi, \psi]|_{k_1+k_2+2\nu, m_1+m_2}\gamma, \quad \forall \gamma \in \Gamma^J. \tag{1.7.4}$$

*Remark* 1.7.4. If $\nu \geq 0$ and $\phi_i \in J_{k_i,m_i}$ (or $J_{k_i,m_i}^{cusp}$), $i = 1, 2$ then

$$[\phi_1, \phi_2]_\nu \in J_{k_1+k_2+2\nu, m_1+m_2} \text{ (or } J_{k_1+k_2+2\nu, m_1+m_2}^{cusp}),$$

and if $\nu > 0$, then

$$[\phi_1, \phi_2]_\nu \in J_{k_1+k_2+2\nu, m_1+m_2}^{cusp}.$$

In fact, $[\ ,\ ]_\nu$ is a bilinear map from $J_{k_1,m_1} \times J_{k_2,m_2}$ to $J_{k_1+k_2+2\nu, m_1+m_2}$.

*Remark* 1.7.5. We note that the 0-th Rankin-Cohen bracket is the usual product of Jacobi forms i.e., $[\phi, \psi]_0 = \phi\psi$.

### 1.7.2 Rankin-Cohen brackets on Siegel modular forms of genus two

Rankin-Cohen brackets for Siegel modular forms of genus two were studied in [3] explicitly and existence of recursion formula in [12] for general genus. Let $k,\ l$ be positive integers and $\nu \geq 0$ be an integer. Let $F$ and $G$ be two complex-valued holomorphic functions on $\mathcal{H}_2$. Let $\mathbb{D}$ be the differential operator defined by

$$\mathbb{D} := 4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'} - \frac{\partial^2}{\partial z^2}, \quad \text{for} \quad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2.$$

Define the $\nu$-th Rankin-Cohen bracket of $F$ and $G$ by

$$[F,G]_\nu := \sum_{r+s+p=\nu} C_{r,s,p}(k,l)\ \mathbb{D}^p(\mathbb{D}^r(F)\mathbb{D}^s(G)), \tag{1.7.5}$$

where

$$C_{r,s,p}(k,l) = \frac{(k+\nu-3/2)_{s+p}}{r!}\frac{(l+\nu-3/2)_{r+p}}{s!}\frac{(-(k+l+\nu-3/2))_{r+s}}{p!},$$

and

$$(x)_m = \prod_{0\leqslant i\leqslant m-1}(x-i).$$

*Remark* 1.7.6. One has the following relation

$$[F|_k M, G|_l M]_\nu = [F, G]|_{k+l+2\nu} M, \quad \forall M \in \Gamma_2. \tag{1.7.6}$$

*Remark* 1.7.7. If $F \in M_k(\Gamma_2)$ and $G \in M_l(\Gamma_2)$, then

$$[F, G]_\nu \in M_{k+l+2\nu}(\Gamma_2).$$

Moreover, if $\nu > 0$, then

$$[F, G]_\nu \in S_{k+l+2\nu}(\Gamma_2).$$

In fact, $[\ ,\ ]_\nu : M_k(\Gamma_2) \times M_l(\Gamma_2) \longrightarrow M_{k+l+2\nu}(\Gamma_2)$ is a bilinear map.

*Remark* 1.7.8. The 0-th Rankin-Cohen bracket is the usual product i.e., $[F, G]_0 = FG$.

# Chapter 2

# Adjoint of some linear maps on Jacobi forms

## 2.1 Introduction

Let $k$ and $l$ be positive integers. Let $f(\tau) = \sum_m a(m)q^m \in S_k$ and $g(\tau) = \sum_m b(m)q^m \in S_l$. For a positive integer $n$, define a Dirichlet series as follows:

$$L_{f,g;n}(s) := \sum_{m=1}^{\infty} \frac{a(m+n)\overline{b(m)}}{(m+n)^s}. \tag{2.1.1}$$

Using Deligne's estimate one can see that the series $L_{f,g;n}(s)$ is absolutely convergent for $Re(s) > \frac{k+l}{2}$. Using the existence of adjoint map and property of Poincaŕe series, Kohnen [21] constructed certain cusp forms whose Fourier coefficients involve special values of the Dirichlet series (2.1.1). More precisely:

**Theorem 2.1.1** (Kohnen [21]). *Let $k$ and $l$ be a positive integers with $k > l+2$. Let $f \in S_{k+l}$ and $g \in S_l$ with Fourier expansions*

$$f(\tau) = \sum_{m=0}^{\infty} a(m)q^m \text{ and } g(\tau) = \sum_{m=0}^{\infty} b(m)q^m.$$

*Then the function*

$$T_g^*(f)(\tau) := \sum_{n=1}^{\infty} n^{k-1} L_{f,g;n}(k+l-1)q^n$$

*is a cusp form of weight $k$ for $SL_2(\mathbb{Z})$. In fact, the map $S_{k+l} \to S_k$ defined by $f \mapsto \dfrac{\Gamma(k+l-1)}{\Gamma(k-1)(4\pi)^l} T_g^*(f)$ is the adjoint of the map $T_g : S_k \longrightarrow S_{k+l},\ h \mapsto gh$, with respect to the Petersson scalar product.*

Recently the work of Kohnen has been generalized by Herrero [14], where he computed the adjoint of the linear map constructed using Rankin-Cohen brackets instead of product by a fixed modular form. More precisely, for a fixed $g \in M_l$ and an integer $\nu \geqslant 0$, consider the linear map

$$T_{g,\nu} : S_k \longrightarrow S_{k+l+2\nu},$$

defined by

$$T_{g,\nu}(f) = [f, g]_n.$$

Herrero computed the adjoint map of $T_{g,\nu}$ with respect to the Petersson scalar product which involves special values of certain Dirichlet series of Rankin-Selberg type similar to (2.1.1) with additional factors arising due to the binomial coefficients appearing in the Rankin-Cohen bracket.

**Theorem 2.1.2** (Herrero [14]). *Let* $k \geqslant 6$ *and* $l$ *be natural numbers and* $\nu \geqslant 0$. *Let* $g(\tau) = \sum_{m=0}^{\infty} b(m)q^m \in M_l$. *Suppose that either (a)* $g$ *is a cusp form or (b)* $g$ *is not cusp form and* $l < k-3$. *Then the image of any cusp form* $f \in S_{k+l+2\nu}$ *with Fourier expansion* $f(\tau) = \sum_{m=1}^{\infty} a(m)q^m$ *under* $T^*_{g,\nu}$ *is given by*

$$T^*_{g,\nu}(f)(\tau) = \sum_{n=1}^{\infty} \beta(k,l,\nu;n) L_{f,g,\nu,n}(\gamma) q^n,$$

*where* $\gamma = k+l+2\nu-1$, $\beta(k,l,\nu;n) = \dfrac{\Gamma(k+l+2\nu-1)\ n^{k-1}}{\Gamma(k-1)(4\pi)^{l+2\nu}}$ *and* $L_{f,g,\nu,n}$ *is the L-function associated with* $f$ *and* $g$, *defined by*

$$L_{f,g,\nu,n}(s) = \sum_{m=1}^{\infty} \frac{a(n+m)\overline{b(m)}\alpha(k,l,\nu,n,m)}{(n+m)^s},\ s \in \mathbb{C} \tag{2.1.2}$$

*with* $\alpha(k,l,\nu,n,m) = \sum_{r=0}^{\nu} (-1)^{\nu-r} \binom{\nu}{r} \dfrac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)} n^r m^{\nu-r}$.

*Remark* 2.1.1. The result of Kohnen (Theorem 2.1.1) and Herrero (Theorem 2.1.2) can be generalized to modular forms for congruence subgroups.

The work of Kohnen (Theorem 2.1.1) has been generalized by Choie, Kim and Knopp [5] and Sakata [33] to the case of Jacobi forms. Choie, Kim and Knopp [5] constructed Jacobi cusp forms whose Fourier coefficients involve special values of certain Dirichlet series of Rankin type. For a fix $\phi \in J_{l,0}$

(modular form of weight $l$), consider the linear map

$$T_\phi : J_{k,m}^{cusp} \longrightarrow J_{k+l,m}^{cusp},$$

defined by

$$T_\phi(\psi) = \phi\psi.$$

Choie, Kim and Knopp [5] computed the adjoint of $T_\phi$ with respect to the Petersson scalar product. The Fourier coefficients of the image of a cusp form $\psi$ under the adjoint of $T_\phi$ involves special values of certain Dirichlet series of Rankin-Selberg type attached to $\phi$ and $\psi$.

**Theorem 2.1.3.** *[5] Suppose that $k > 5$ and $l \geqslant 0$ and $\phi(\tau, z) \in J_{k+l,m}^{cusp}$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2<4nm}} a(n, r)q^n\zeta^r,$$

*and $g \in J_{l,0}$ with Fourier expansion*

$$g(\tau) = \sum_{n=1}^{\infty} b(n)q^n.$$

*Then*

$$T_g^*(\phi) := \sum_{\substack{n,r\in\mathbb{Z} \\ r^2<4nm}} c(n, r)q^n\zeta^r,$$

*is a Jacobi cusp form of weight $k$ and index $m$, where*

$$c(n,r) = \frac{(4mn-r^2)^{k-3/2} m^l \Gamma(k+l-3/2)}{\pi^l \Gamma(k-3/2)} \sum_{i \geqslant 1} \frac{a(i+n,r)\overline{b(i)}}{(4nm+4im-r^2)^{k+l-3/2}}.$$

Sakata [33] generalized Theorem 2.1.3 by computing the adjoint of the map $T_\phi$ for any Jacobi form $\phi$. More precisely:

**Theorem 2.1.4.** *[33] Suppose that $k_1 > 4$ and $k_2 > 3$ and $m_1, m_2 \in \mathbb{N}$. Let $\phi_1(\tau,z) \in J^{cusp}_{k_1+k_2, m_1+m_2}$ with Fourier expansion*

$$\phi_1(\tau,z) = \sum_{\substack{n_1, r_1 \in \mathbb{Z} \\ r_1^2 < 4n(m_1+m_2)}} a(n_1,r_1)\ q^n \zeta^r,$$

*and $\phi_2(\tau,z) \in J^{cusp}_{k_2,m_2}$ with Fourier expansion*

$$\phi_2(\tau,z) = \sum_{\substack{n_2, r_2 \in \mathbb{Z} \\ r_2^2 < 4n_2 m_2}} b(n_2,r_2)\ q^n \zeta^r.$$

*Then*

$$T^*_{\phi_2}(\phi_1)(\tau,z) := \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 < 4nm_1}} c(n,r) q^n \zeta^r$$

*is a Jacobi cusp form of weight $k_1$ and index $m_1$, where*

$$c(n,r) = \frac{(4m_1 n - r^2)^{k_1-3/2} (m_1+m_2)^{k_1+k_2-2} \Gamma(k_1+k_2-3/2)}{\pi^{k_2} m_1^{k_1-2} \Gamma(k_1-3/2)}$$

$$\times \sum_{\substack{n_1, r_1 \in \mathbb{Z} \\ r_1^2 < 4n_1 m_2 \\ (r+r_1)^2 < 4(m_1+m_2)(n+n_1)}} \frac{a(n+n_1, r+r_1)\overline{b(n_2, r_2)}}{(4(m_1+m_2)(n+n_1) - (r+r_1)^2)^{k_1+k_2-3/2}}.$$

In this chapter, we generalize the work of Herrero to the case of Jacobi forms, which generalises the work of Sakata [33]. First we state the theorem and prove a lemma which is needed for the convergence and then we give a proof of the main theorem.

## 2.2 Statement of the result

For a fixed $\psi \in J_{k_2,m_2}^{cusp}$ and an integer $\nu \geq 0$, consider the linear map

$$T_{\psi,\nu} : J_{k_1,m_1}^{cusp} \to J_{k_1+k_2+2\nu,m_1+m_2}^{cusp},$$

defined by

$$T_{\psi,\nu}(\phi) = [\phi, \psi]_\nu,$$

where $[\phi, \psi]_\nu$ is the $\nu$-th Rankin-Cohen bracket of $\phi$ and $\psi$ defined in (1.7.3). Then $T_{\psi,\nu}$ is a $\mathbb{C}$-linear map between two finite dimensional Hilbert spaces and therefore has an adjoint map

$$T_{\psi,\nu}^* : J_{k_1+k_2+2\nu,m_1+m_2}^{cusp} \to J_{k_1,m_1}^{cusp}$$

such that

$$\langle \phi, T_{\psi,\nu}(\omega) \rangle = \langle T^*_{\psi,\nu}(\phi), \omega \rangle, \ \forall \phi \in J^{cusp}_{k_1+k_2+2\nu, m_1+m_2} \text{ and } \omega \in J^{cusp}_{k_1,m_1}.$$

We exhibit the Fourier coefficients of $T^*_{\psi,\nu}(\phi)$ for $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$. These coefficients involve special values of certain Dirichlet series associated to $\phi$ and $\psi$. Now we state the main theorem.

**Theorem 2.2.1.** *[17] Let $k_1 > 4, k_2 > 3, m_1$ and $m_2$ be natural numbers. Let $\psi \in J^{cusp}_{k_2,m_2}$ with Fourier expansion*

$$\psi(\tau, z) = \sum_{\substack{n_1, r_1 \in \mathbb{Z}, \\ r_1^2 < 4m_2 n_1}} a(n_1, r_1) q^{n_1} \zeta^{r_1}.$$

*Then the image of any cusp form $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n_2, r_2 \in \mathbb{Z}, \\ r_2^2 < 4(m_1+m_2) n_2}} b(n_2, r_2) q^{n_2} \zeta^{r_2},$$

*under $T^*_{\psi,\nu}$ is given by*

$$T^*_{\psi,\nu}(\phi)(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z}, \\ r^2 < 4m_1 n}} c_\nu(n, r) q^n \zeta^r,$$

*where*

$$c_\nu(n,r) = \frac{(4m_1n-r^2)^{k_1-\frac{3}{2}}}{\pi^{k_2+2\nu}}\frac{(m_1+m_2)^{k_1+k_2+2\nu-2}}{m_1^{k_1-2}}\frac{\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{\Gamma(k_1-\frac{3}{2})}$$
$$\times \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)(4m_1n-r^2)^l$$
$$\times \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1\\ (r+r_1)^2<4(m_1+m_2)(n+n_1)}} \frac{(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}},$$

*and*

$$A_l(k_1,m_1,k_2,m_2;\nu) \;=\; (-1)^l\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}m_1^{\nu-l}m_2^l.$$

*Remark* 2.2.1. Using Lemma 1.5.3 (as given in Remark 3.1 in [33]) one can show that the inner sum of the series converges for $k_1 > 4$ and $k_2 > 3$.

## 2.3 Proof of Theorem 2.2.1

We need the following lemma to prove the Theorem 2.2.1.

**Lemma 2.3.1.** *Using the same notation as in Theorem 2.2.1, we have*

$$\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}\int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)}\mid_{k_1,m_1}\gamma,\ \psi]_\nu}v^{k_1+k_2+2\nu}e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} \mid dV_J$$

*converges.*

*Proof.* Changing the variable $(\tau, z)$ to $\gamma^{-1}.(\tau, z)$ and using Remark 1.7.3, the sum equals to

$$\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J}\int_{\gamma.\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)},\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} \mid dV_J$$

and using Rankin unfolding argument, the last sum equals to

$$\int_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)}, \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} \mid dV_J.$$

Now replacing $\phi$ and $\psi$ with their Fourier expansions and using the definition of Rankin-Cohen brackets, the last integral is majorized by

$$\sum_{l=0}^{\nu} A_l(k_1, m_1, k_2, m_2; \nu)\ \mathcal{I}_l(k_1, k_2, m_1, m_2, \nu; n, r),$$

where

$$\mathcal{I}_l(k_1, k_2, m_1, m_2, \nu; n, r) = \int_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} \sum_{\substack{n_2, r_2\in\mathbb{Z},\\ r_2^2<4(m_1+m_2)n_2}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1}} |(4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}$$

$$\times\, a(n_1, r_1)b(n_2, r_2)e^{2\pi i((n+n_1+n_2)\tau+(r+r_1+r_2)z)} \mid v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J.$$

Now it suffices to show that the integral $\mathcal{I}_l(k_1, k_2, m_1, m_2, \nu; n, r)$ is finite for each $l$. We choose a fundamental domain for the action of $\Gamma^J_\infty$ on $\mathcal{H}\times\mathbb{C}$

which is given by $([0,1]\times[0,\infty])\times([0,1]\times\mathbb{R})$ and integrating over it, we have

$$\mathcal{I}_l(k_1,k_2,m_1,m_2,\nu;n,r)\leq\frac{4(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-3/2)}{\pi^{k_1+k_2+2\nu-3/2}}$$

$$\times\sum_{\substack{n_2,r_2\in\mathbb{Z},\\ r_2^2<4(m_1+m_2)n_2}}\sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1}}\frac{(4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}|a(n_1,r_1)b(n_2,r_2)|}{(8(m_1+m_2)(n+n_1+n_2)-(r+r_1+r_2)^2)^{k_1+k_2+2\nu-3/2}}.$$

Using the growth of the Fourier coefficients given in Lemma 1.5.3, the above series converges absolutely, which proves the lemma. □

We now give a proof of Theorem 2.2.1. Write

$$T^*_{\psi,\nu}(\phi)(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z},\\ 4m_1n-r^2>0}}c_\nu(n,r)q^n\zeta^r.$$

Now, we consider the $(n,r)$-th Poincaŕe series of weight $k_1$ and index $m_1$ as given in (1.5.1). Using Lemma 1.5.2, we have

$$\langle T^*_{\psi,\nu}\phi,P_{k_1,m_1;(n,r)}\rangle=\alpha_{k_1,m_1}(4m_1n-r^2)^{\frac{3}{2}-k_1}c_\nu(n,r),$$

where

$$\alpha_{k_1,m_1}=\frac{m_1^{k_1-2}\Gamma(k_1-\frac{3}{2})}{2\pi^{k_1-\frac{3}{2}}}.$$

On the other hand, by definition of the adjoint map we have

$$\langle T^*_{\psi,\nu}\phi,P_{k_1,m_1;(n,r)}\rangle=\langle\phi,T_{\psi,\nu}(P_{k_1,m_1;(n,r)})\rangle=\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle.$$

Hence we get

$$c_\nu(n,r) = \frac{(4m_1n - r^2)^{k_1 - \frac{3}{2}}}{\alpha_{k_1,m_1}} \langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle. \tag{2.3.1}$$

By definition, $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\begin{aligned}
&\int\limits_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{[P_{k_1,m_1;(n,r)},\psi]_\nu} v^{k_1+k_2+2\nu}\, e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J \\
&= \int\limits_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{[\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J_1} e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J \\
&= \int\limits_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z) \sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} \overline{[e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J \\
&= \int\limits_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J.
\end{aligned}$$

By Lemma 2.3.1, we can interchange the sum and integration in $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$. Therefore $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ equals to

$$\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} \int\limits_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J.$$

Using the change of variable $(\tau, z)$ to $\gamma^{-1} \cdot (\tau, z)$ and using Remark 1.7.3,

$\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu$ equals

$$\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} \int_{\gamma.\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)},\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J.$$

Now using the Rankin-Selberg unfolding argument, $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\begin{aligned}
& \int_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)}, \psi]_\nu} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J \\
= & \int_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z) \sum_{l=0}^{\nu} (-1)^l \binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l} m_1^{\nu-l} m_2^l \\
& \times \ \overline{L^l_{m_1}(e^{2\pi i(n\tau+rz)}) L^{\nu-l}_{m_2}(\psi)} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J
\end{aligned}$$

$$\begin{aligned}
= \ & \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \int_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{L^l_{m_1}(e^{2\pi i(n\tau+rz)})} \\
& \times \ \overline{L^{\nu-l}_{m_2}(\psi)} v^{k_1+k_2+2\nu} e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J. \qquad (2.3.2)
\end{aligned}$$

The repeated action of heat operators $L_{m_1}$ and $L_{m_2}$ give

$$\begin{aligned}
L^l_{m_1}(e^{2\pi i(n\tau+rz)}) &= (4m_1n-r^2)^l e^{2\pi i(n\tau+rz)}, \\
L^{\nu-l}_{m_2}(\psi) &= \sum_{\substack{n_1,r_1\in\mathbb{Z} \\ 4m_2n_1-r_1^2>0}} a(n_1,r_1)(4m_2n_1-r_1^2)^{\nu-l} e^{2\pi i(n_1\tau+r_1z)}.
\end{aligned}$$

Now replacing $\phi$ and $\psi$ by their Fourier series in (2.3.2), $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \int\limits_{\Gamma_\infty^J \backslash \mathcal{H}\times\mathbb{C}} \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ r_2^2<4(m_1+m_2)n_2}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}$$

$$\overline{a(n_1,r_1)}b(n_2,r_2)e^{2\pi i(n_2\tau+r_2z)}\overline{e^{2\pi i(n_1\tau+r_1z)}e^{2\pi i(n\tau+rz)}}v^{k_1+k_2+2\nu}e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}}dV_J \tag{2.3.3}$$

$$= \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ r_2^2<4(m_1+m_2)n_2}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}$$

$$b(n_2,r_2) \int\limits_{\Gamma_\infty^J \backslash \mathcal{H}\times\mathbb{C}} e^{2\pi i(n_2\tau+r_2z)}\overline{e^{2\pi i(n_1\tau+r_1z)}e^{2\pi i(n\tau+rz)}}\ v^{k_1+k_2+2\nu}e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}}dV_J. \tag{2.3.4}$$

Putting $\tau = u+iv, z = x+iy$, $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ r_2^2<4(m_1+m_2)n_2}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}\ b(n_2,r_2)$$

$$\int\limits_{\Gamma_\infty^J \backslash \mathcal{H}\times\mathbb{C}} e^{-2\pi v(n_2+n+n_1)}e^{-2\pi y(r_2+r+r_1)}e^{2\pi i(r_2-r-r_1)x}e^{2\pi i(n_2-n-n_1)u}v^{k_1+k_2+2\nu}e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}}dV_J. \tag{2.3.5}$$

A fundamental domain for the action of the group $\Gamma^J_\infty$ on $\mathcal{H}\times\mathbb{C}$ is given by $([0,1]\times[0,\infty])\times([0,1]\times\mathbb{R})$. Integrating over this region, $\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle$ equals

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ r_2^2<4(m_1+m_2)n_2}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}$$

$$\times\ \overline{a(n_1,r_1)}\ b(n_2,r_2)\int_0^1\int_0^\infty\int_0^1\int_{-\infty}^\infty e^{-2\pi v(n_2+n+n_1)}e^{-2\pi y(r_2+r+r_1)}e^{2\pi i(r_2-r-r_1)x}$$

$$\times\ e^{2\pi i(n_2-n-n_1)u}v^{k_1+k_2+2\nu-3}e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}}dudvdxdy.$$

Integrating on $x$ and $u$, $\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle$ equals

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)\sum_{\substack{n_1,r_1\in\mathbb{Z}\\ r_1^2<4m_2n_1\\ (r+r_1)^2<4(m_1+m_2)(n+n_1)}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}b(n+n_1,r+r_1)$$

$$\overline{a(n_1,r_1)}\int_0^\infty\int_{-\infty}^\infty e^{-4\pi v(n+n_1)}e^{-4\pi y(r+r_1)}v^{k_1+k_2+2\nu-3}e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}}dydv.$$

Integrating over $y$, we have

$$\int_{-\infty}^\infty e^{-4\pi\left((r_1+r)y+\frac{(m_1+m_2)y^2}{v}\right)}dy=\frac{\sqrt{v}\ e^{\pi\dfrac{(r+r_1)^2v}{m_1+m_2}}}{2\sqrt{m_1+m_2}}. \tag{2.3.6}$$

Substituting the value on (2.3.6) and integrating over $v$, we have

$$\begin{aligned}&\int_0^\infty e^{-4\pi v(n+n_1)} v^{k_1+k_2+2\nu-3} \frac{\sqrt{v}\; e^{\pi\frac{(r_1+r)^2 v}{m_1+m_2}}}{2\sqrt{m_1+m_2}} dv \\ &= \frac{1}{2\pi^{k_1+k_2+2\nu-\frac{3}{2}}} \frac{(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}}.\end{aligned} \tag{2.3.7}$$

Putting the value of integral (2.3.7) in (2.3.6), $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\begin{aligned}&\frac{(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{2\pi^{k_1+k_2+2\nu-\frac{3}{2}}} \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \\ &\times \sum_{\substack{n_1,r_1\in\mathbb{Z} \\ r_1^2<4m_2n_1 \\ (r+r_1)^2<4(m_1+m_2)(n+n_1)}} \frac{(4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}}.\end{aligned} \tag{2.3.8}$$

Now substituting $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ from (2.3.8) in (2.3.1), we get the required expression for $c_\nu(n,r)$ given in Theorem 2.2.1.

## 2.4 Applications

In this section we give some applications of Theorem 2.2.1. Fix $\psi \in J^{cusp}_{k_2,m_2}$ and suppose that $J^{cusp}_{k_1,m_1}$ is one-dimensional space generated by $f(\tau,z)$, then applying the above theorem we get $T^*_{\psi,\nu}(\phi)(\tau,z) = \alpha_\phi f(\tau,z)$ for some constant $\alpha_\phi$ and for all $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$. Now equating the $(n,r)$-th Fourier coefficients both the sides, we get relation among the special values of Rankin-

Selberg type convolution of the Jacobi forms $\phi$ and $\psi$ with the Fourier coefficients of $f(\tau, z)$. For example, taking $\psi = \phi_{10,1} = \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1}) \in J_{10,1}^{cusp}$ and $k_1 = 12, m_1 = 1$ ($J_{12,1}^{cusp}$ is one-dimensional space generated by $\phi_{12,1}$) where $\phi_{12,1} := \frac{1}{144}(E_4^2 E_{4,1} - E_6 E_{6,1})$, we have the following relation:

$$\sum_{l=0}^{\nu} A_l(12, 1, 10, 1; \nu)(4n - r^2)^l \sum_{\substack{n_1, r_1 \in \mathbb{Z} \\ 4n_1 - r_1^2 > 0 \\ 8(n+n_1) - (r+r_1)^2 > 0}} \frac{(4n_1 - r_1^2)^{\nu - l}\overline{a(n_1, r_1)} b(n + n_1, r + r_1)}{(8(n + n_1) - (r + r_1)^2)^{22 + 2\nu - \frac{3}{2}}}$$

$$= \alpha_\phi c(n, r)$$

for all $n, r \in \mathbb{Z}$ such that $4n - r^2 > 0$, where $a(p, q), b(p, q)$ and $c(p, q)$ are the $(p, q)$-th Fourier coefficients of $\phi_{10,1}, \phi$ and $\phi_{12,1}$, respectively. In particular taking $\nu = 0$ in the above example, we get the special value of Rankin-Selberg type convolution of $\phi_{10,1}$ and $\phi$ in terms of Fourier coefficients of $\phi_{12,1}$, i.e.,

$$\sum_{\substack{n_1, r_1 \in \mathbb{Z} \\ 4n_1 - r_1^2 > 0 \\ 8(n+n_1) - (r+r_1)^2 > 0}} \frac{\overline{a(n_1, r_1)} b(n + n_1, r + r_1)}{(8(n + n_1) - (r + r_1)^2)^{\frac{41}{2}}} = \alpha_\phi c(n, r).$$

# Chapter 3

# Adjoint of some linear maps on Siegel modular forms

## 3.1 Introduction

The work of Kohnen [21] has been generalized by Lee [25] to the case of Siegel modular forms to construct certain Siegel cusp forms. Lee computed the adjoint map with respect to the Petersson scalar product of the product map by a fixed Siegel cusp form. The proof uses Poincaŕe series of two variables and the holomorphic projection operator developed by Panchishkin [28].

**Theorem 3.1.1.** *[25] Let $F \in S_{k+l}(\Gamma_g)$ with Fourier expansion*

$$F(Z) = \sum_{T>0} B(T) e^{2\pi i (tr(TZ))},$$

*and* $G \in S_l(\Gamma_g)$ *with Fourier expansion*

$$G(Z) = \sum_{T>0} A(T) e^{2\pi i (tr(TZ))}.$$

*Then*

$$T_G^*(F) = \sum_{T>0} C(T) e^{2\pi i (tr(TZ))},$$

$$with \quad C(T) = \frac{(det(T))^{k-\frac{g+1}{2}} \Gamma_g(k+l-\frac{g+1}{2})}{(4\pi)^{g(l-(g-1)/2} \Gamma_g(k-\frac{g+1}{2})} \sum_{S>0} \frac{A(S+T)\overline{B(T)}}{(det(S+T))^{k+l-g}},$$

*is a Siegel cusp form of weight* $k$ *and genus* $g$, *where*

$$\Gamma_g(\sigma) = \int_{\mathbb{Y}} e^{-trY} (det\ Y)^{\sigma-(g+1)/2} dY, \quad for\ Re(\sigma) > (g+1)/2, \tag{3.1.1}$$

*and* $\mathbb{Y} = \{Y \in M_{g\times g}(\mathbb{C}) \mid Y^t = Y > 0\}$.

In this chapter, we generalize the work of Herrero [14] and Lee [25] to the case of Siegel modular forms of genus two.

Let $F \in S_k(\Gamma_2)$ and $G \in S_l(\Gamma_2)$ with Fourier coefficients $A(T)$ and $B(T)$ respectively, then for a fixed positive definite $2 \times 2$ matrix $S$ and a non-negative integer $m$, define the Dirichlet series $L_{F,G;S,m}$ as follows:

$$L_{F,G;S,m}(\sigma) = \sum_{T>0} \frac{det\ (T)^m A(T+S)\overline{B(T)}}{(det\ (T+S))^\sigma}. \tag{3.1.2}$$

The above series converges for $Re(\sigma) > \frac{k+l}{2} - m + \frac{5}{18}$. We use the Rankin-Cohen bracket of Siegel modular forms of genus 2 and special values of the above series to construct Siegel cusp forms. First, we state the theorem and give the proof, then we give some applications.

## 3.2 Statement of the result

For a fixed $G \in S_l(\Gamma_2)$ and an integer $\nu \geq 0$, consider the map

$$T_{G,\nu} : S_k(\Gamma_2) \to S_{k+l+2\nu}(\Gamma_2),$$

defined by

$$T_{G,\nu}(F) = [F, G]_\nu,$$

where $[F, G]_\nu$ is the $\nu$-th Rankin-Cohen bracket of $F$ and $G$ defined in (1.7.5). Then $T_{G,\nu}$ is a $\mathbb{C}$-linear map between finite dimensional Hilbert spaces and therefore has an adjoint map

$$T^*_{G,\nu} : S_{k+l+2\nu}(\Gamma_2) \to S_k(\Gamma_2)$$

given by

$$\langle F, T_{G,\nu}(H)\rangle = \langle T^*_{G,\nu}(F), H\rangle, \quad \forall F \in S_{k+l+2\nu}(\Gamma_2) \quad \text{and} \quad H \in S_k(\Gamma_2).$$

We explicitly compute the Fourier coefficients of $T^*_{G,\nu}(F)$ in terms of the special values of Dirichlet series defined in (3.1.2).

**Theorem 3.2.1.** *[18] Let $k \geqslant 6, l$ be natural numbers and $\nu \geq 0$ be a fixed* *integer. Let $G \in S_l(\Gamma_2)$ with Fourier expansion*

$$G(Z) = \sum_{T_1 > 0} A(T_1) e^{2\pi i (tr(T_1 Z))}.$$

*Then the image of any cusp form $F \in S_{k+l+2\nu}(\Gamma_2)$ with Fourier expansion*

$$F(Z) = \sum_{T_2 > 0} B(T_2) e^{2\pi i (tr(T_2 Z))},$$

*under $T^*_{G,\nu}$ is given by*

$$T^*_{G,\nu}(F)(Z) = \sum_{T > 0} C(T) e^{2\pi i (tr(TZ))},$$

*where*

$$C(T) = \alpha(k,l,\nu) \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\, T)^{k+r-3/2}\, L_{F,G;T,s}(k+l+2\nu-(p+3/2)),$$

*with*

$$\alpha(k,l,\nu) = \frac{(-1)^\nu \Gamma_2(k+l+2\nu-3/2)}{2\sqrt{\pi}\Gamma(k-\frac{3}{2})\Gamma(k-2)(4\pi)^{2(l+\nu)}}.$$

*Remark* 3.2.1. Using the estimate given in Theorem 1.6.3, one can show that the above series converges absolutely.

## 3.3 Proof of Theorem 3.2.1

We need the following lemma.

**Lemma 3.3.1.** *Using the same notation as in Theorem 3.2.1, the sum*

$$\sum_{M \in \Delta \backslash \Gamma_2} \int_{\Gamma_2 \backslash \mathcal{H}_2} |F(Z)\overline{[e^{2\pi i(tr(TZ))} \mid_k M, G]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}|dZ$$

*converges.*

*Proof.* The proof is similar to Lemma 2.3.1. □

Now we give a proof of Theorem 3.2.1. Write

$$T^*_{G,\nu}(F)(Z) = \sum_{T>0} C(T)e^{2\pi i(tr(TZ))}.$$

Using Lemma 1.6.4, we have

$$\langle T^*_{G,\nu}F, P_{k,T}\rangle = c_k(det\ T)^{-k+\frac{3}{2}}C(T),$$

where

$$c_k = 2\sqrt{\pi}(4\pi)^{3-2k}\Gamma(k-3/2)\Gamma(k-2).$$

On the other hand, by the definition of the adjoint map we have

$$\langle T^*_{G,\nu}F, P_{k,T}\rangle = \langle F, T_{G,\nu}(P_{k,T})\rangle = \langle F, [P_{k,T}, G]_\nu\rangle.$$

Hence we get

$$C(T) = \frac{(det\ T)^{k-\frac{3}{2}}}{c_k}\langle F, [P_{k,T}, G]_\nu\rangle. \tag{3.3.1}$$

By definition,

$$\begin{aligned}\langle F, [P_{k,T}, G]_\nu \rangle &= \int\limits_{\Gamma_2\backslash\mathcal{H}_2} F(Z)\overline{[P_{k,T}, G]_\nu\,(Z)}\;(det\ Y)^{k+l+2\nu} dZ \\ &= \int\limits_{\Gamma_2\backslash\mathcal{H}_2} \sum_{M\in\Delta\backslash\Gamma_2} F(Z)\overline{[e^{2\pi i(tr(TZ))}\mid_k M, G]_\nu(Z)}\;(det\ Y)^{k+l+2\nu} dZ.\end{aligned}$$

By Lemma 3.3.1, we interchange the sum and the integration in $\langle F, [P_{k,T}, G]_\nu \rangle$, which gives

$$\langle F, [P_{k,T}, G]_\nu \rangle = \sum_{M\in\Delta\backslash\Gamma_2} \int\limits_{\Gamma_2\backslash\mathcal{H}_2} F(Z)\overline{[e^{2\pi i(tr(TZ))}\mid_k M, G]_\nu(Z)}\;(det\, Y)^{k+l+2\nu} dZ.$$

Using the change of variable $Z$ to $M^{-1}\cdot Z$ in each integral and using Remark 1.7.6, we get

$$\langle F, [P_{k,T}, G]_\nu \rangle = \sum_{M\in\Delta\backslash\Gamma_2} \int\limits_{M\cdot\Gamma_2\backslash\mathcal{H}_2} F(Z)\overline{[e^{2\pi i(tr(TZ))}, G]_\nu(Z)}\;(det\ Y)^{k+l+2\nu} dZ.$$

Using the Rankin-Selberg unfolding argument,

$$\langle F, [P_{k,T}, G]_\nu \rangle = \int\limits_{\Delta\backslash\mathcal{H}_2} F(Z)\overline{[e^{2\pi i(tr(TZ))}, G]_\nu(Z)}\;(det\ Y))^{k+l+2\nu} dZ$$

$$= \int\limits_{\Delta\backslash\mathcal{H}_2} F(Z) \sum_{r+s+p=\nu} C_{r,s,p}(k,l)\overline{\mathbb{D}^p(\mathbb{D}^r(e^{2\pi i(tr(TZ))})\mathbb{D}^s(G(Z)))}\;(det\ Y)^{k+l+2\nu} dZ$$

$$= \sum_{r+s+p=\nu} C_{r,s,p}(k,l) \int_{\Delta\backslash\mathcal{H}_2} F(Z)\, \overline{\mathbb{D}^p(\mathbb{D}^r(e^{2\pi i(tr(TZ))})\mathbb{D}^s(G(Z)))}\, (det\ Y)^{k+l+2\nu} dZ. \tag{3.3.2}$$

The repeated action of the operator $\mathbb{D}$ gives

$$\mathbb{D}^r(e^{2\pi i(tr(TZ))}) = (4\pi i)^{2r}(det\ T)^r e^{2\pi i(tr(TZ))},$$

$$\mathbb{D}^s(G(Z)) = (4\pi i)^{2s} \sum_{T_1>0} A(T_1)(det\ T_1)^s e^{2\pi i(tr(T_1 Z))}.$$

Now replacing $F$ and $G$ by their Fourier expansions in (3.3.2), we have

$$\langle F, [P_{k,T}, G]_\nu\rangle = (4\pi i)^{2\nu} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \int_{\Delta\backslash\mathcal{H}_2} \left( \sum_{T_2>0} B(T_2) e^{2\pi i(tr(T_2 Z))} \right)$$
$$\times \left( \sum_{T_1>0} (det\ T_1)^s (det(T+T_1))^p \overline{A(T_1)} e^{-2\pi i \overline{tr(T+T_1)Z}} \right) (det\ Y)^{k+l+2\nu} dZ$$

$$= (4\pi i)^{2\nu} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \sum_{T_2>0} \sum_{T_1>0} (det\ T_1)^s (det(T+T_1))^p\ \overline{A(T_1)} B(T_2)$$
$$\times \int_{\Delta\backslash\mathcal{H}_2} e^{2\pi i(tr(T_2 Z))}\ e^{-2\pi i \overline{tr(T+T_1)Z}} (det\ Y)^{k+l+2\nu} dZ$$

$$= (4\pi i)^{2\nu} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \sum_{T_2>0} \sum_{T_1>0} (det\ T_1)^s (det(T+T_1))^p\ \overline{A(T_1)} B(T_2)$$
$$\times \int_{\Delta\backslash\mathcal{H}_2} e^{2\pi i(tr(T_2-(T+T_1))X)))} e^{-2\pi(tr(T_2+T+T_1)Y)} (det\ Y)^{k+l+2\nu} \frac{dXdY}{det\ (Y)^3}. \tag{3.3.3}$$

We know that the set $\mathbb{F} := \{Z = X + iY \in \mathcal{H}_2 \mid X \in \mathbb{X}, Y \in \mathbb{Y}\}$ is a fundamental domain for the action of $\Delta$ on $\mathcal{H}_2$, where

$$\mathbb{X} = \left\{ X = \left( \begin{array}{cc} u & x \\ x & u' \end{array} \right) : \ \frac{-1}{2} \leqslant u \leqslant \frac{1}{2}, \ \frac{-1}{2} \leqslant x \leqslant \frac{1}{2}, \ \frac{-1}{2} \leqslant u' \leqslant \frac{1}{2} \right\},$$

and

$$\mathbb{Y} = \left\{ Y \in M_{2\times 2}(\mathbb{C}) \mid Y^t = Y > 0 \right\}.$$

Integrating over this fundamental domain, $\langle F, [P_{k,T}, G]_\nu \rangle$ equals

$$(4\pi i)^{2\nu} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \sum_{T_2>0} \sum_{T_1>0} (det\ T_1)^s (det(T+T_1))^p\ \overline{A(T_1)} B(T_2)$$
$$\int_{\mathbb{X}} \int_{\mathbb{Y}} e^{2\pi i (tr(T_2 - (T+T_1))X)))}\ e^{-2\pi (tr(T_2+T+T_1)Y)} (det\ Y)^{k+l+2\nu-3} dX dY.$$

Integrating on $\mathbb{X}$ first, $\langle F, [P_{k,T}, G]_\nu \rangle$ equals

$$(4\pi i)^{2\nu} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \sum_{T_1>0} (det\ T_1)^s (det(T+T_1))^p\ \overline{A(T_1)} B(T+T_1)$$
$$\int_{\mathbb{Y}} e^{-4\pi (tr(T+T_1)Y)} (det\ Y)^{k+l+2\nu-3} dY. \tag{3.3.4}$$

Now integration over $\mathbb{Y}$ gives

$$\int_{\mathbb{Y}} e^{-4\pi (tr(T+T_1)Y)} (det\ Y)^{k+l+2\nu-3} dY \ = \ \frac{\Gamma_2\left(k+l+2\nu-\frac{3}{2}\right)}{(det\ (4\pi(T+T_1)))^{k+l+2\nu-\frac{3}{2}}}. \tag{3.3.5}$$

Substituting the value from (3.3.5) in (3.3.4), $\langle F, [P_{k,T}, G]_\nu \rangle$ equals

$$(-1)^\nu \frac{\Gamma_2\left(k+l+2\nu-\frac{3}{2}\right)}{(4\pi)^{2(k+l+\nu-\frac{3}{2})}} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\, T)^r \sum_{T_1>0} \frac{(det\, T_1)^s \overline{A(T_1)} B(T+T_1)}{(det(T+T_1))^{k+l+2\nu-(p+\frac{3}{2})}}. \tag{3.3.6}$$

Now substituting $\langle F, [P_{k,T}, G]_\nu \rangle$ from (3.3.6) in (3.3.1), we get the required expression for $C(T)$ as given in Theorem 3.2.1, which completes the proof.

## 3.4 Applications

Fix $G(Z) \in S_l(\Gamma_2)$ and suppose that $S_k(\Gamma_2)$ is the one-dimensional space generated by $F(Z)$. Then by Theorem 3.2.1, $T^*_{G,\nu}(H)(Z) = \alpha_G F(Z)$ for some constant $\alpha_G$ and for all $H \in S_{k+l+2\nu}(\Gamma_2)$. Now equating the $T$-th Fourier coefficients both the sides, we get a relation among the special values of the associated Dirichlet series $L_{H,G;T,m}$ (3.1.2) with the $T$-th Fourier coefficients of $F(Z)$. For example taking $G = \chi_{10}$ the Igusa cusp form of weight 10 (see [15], p.195) and $H = \chi_{10}^2$, then $T^*_{\chi_{10},0}(\chi_{10}^2) = \alpha_{\chi_{10}}\chi_{10}$ for some constant $\alpha_{\chi_{10}}$ and then equating the $T$-th Fourier coefficients on both sides, we get a relation among the special values of the associated Dirichlet series $L_{\chi_{10}^2,\chi_{10};T,0}$ with the $T$-th Fourier coefficients of $\chi_{10}$. Similarly taking $G = \chi_{10}$ and $H = \Upsilon_{20}$ (the Hecke eigenform of weight 20, see [35], p.390) then $T^*_{\chi_{10},0}(\Upsilon_{20}) = \beta_{\chi_{10}}\chi_{10}$ for some constant $\beta_{\chi_{10}}$ and then equating the $T$-th Fourier coefficients on both sides, we get a relation among the special values of the associated Dirichlet series $L_{\Upsilon_{20},\chi_{10};T,0}$ with the $T$-th Fourier coefficients of $\chi_{10}$.

# Chapter 4

# Adjoint of some linear maps on modular forms of half-integral weight

## 4.1 Introduction

The modular forms of half-integral weight was developed by Shimura [34] and the Rankin-Cohen bracket was studied by Cohen [10]. In this chapter we extend the result of Herrero to the case of modular forms of half-integral weight. We first state the main theorem and give a proof, then we give an application to the non-vanishing of special values of certain Rankin-Selberg convolution of modular forms.

## 4.2 Statement of the result

Let $\Gamma = \Gamma_0(4)$, $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$ and $h \in M_l(\Gamma, \chi_2)$ Consider the following linear maps:

(I) $T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$, defined by $T_{g,\nu}(f) = [f, g]_\nu$,

(II) $T_{g,\nu} : S_k(\Gamma) \to S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi)$, defined by $T_{g,\nu}(f) = [f, g]_\nu$,

(III) $T_{h,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi)$, defined by $T_{h,\nu}(f) = [f, h]_\nu$,

(IV) $T_{h,\nu} : S_k(\Gamma) \to S_{k+l+2\nu}(\Gamma, \chi_2)$ defined by $T_{h,\nu}(f) = [f, h]_\nu$,

where $[\ ,\ ]_\nu$ is the $\nu$-th Rankin-Cohen bracket defined in (1.7.1).

Herrero [14] computed the adjoint of the map in (IV) for $\Gamma = SL_2(\mathbb{Z})$ and $\chi_2$ the trivial character. We exhibit explicitly the Fourier coefficients of $T^*_{g,\nu}(f)$ for $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ in (I). The analogous results for the maps in (II) and (III) are given in remark 4.2.1. These involve special values of certain Dirichlet series of Rankin- Selberg type associated to $f$ and $g$. We now state the theorem for the map in case (I).

**Theorem 4.2.1.** *[19] Let $k$ and $l$ be natural numbers and $\nu \geqslant 0$. Let $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$ with Fourier expansion $g(\tau) = \sum_{m=0}^{\infty} b(m)q^m$. Suppose that either (a) $g$ is a cusp form and $k > 2$ or (b) $g$ is not cusp form and $l < k - \frac{3}{2}$. Then the image of any cusp form $f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi)$ with Fourier expansion*

$$f(\tau) = \sum_{m=1}^{\infty} a(m)q^m,$$

*under* $T^*_{g,\nu}$ *is given by*

$$T^*_{g,\nu}(f)(\tau) = \sum_{n=1}^{\infty} \beta(k,l,\nu;n) L_{f,g,\nu,n}(\gamma) q^n, \tag{4.2.1}$$

*where*

$$\gamma = k + l + 2\nu, \quad \beta(k,l,\nu;n) = \frac{\Gamma(k+l+2\nu)\ n^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})(4\pi)^{l+2\nu+\frac{1}{2}}}$$

*and* $L_{f,g,\nu,n}(\gamma)$ *is defined in* (2.1.2).

*Remark* 4.2.1. We have the similar results for the map in (II) with

$$\gamma = k + l + 2\nu - \frac{1}{2}, \text{ and } \beta(k,l,\nu;n) = \frac{\Gamma(k+l+2\nu-\frac{1}{2})\ n^{k-1}}{\Gamma(k-1)\ (4\pi)^{l+2\nu+\frac{1}{2}}},$$

and for the map in (III) with

$$\gamma = k + l + 2\nu - \frac{1}{2}, \text{ and } \beta(k,l,\nu;n) = \frac{\Gamma(k+l+2\nu-\frac{1}{2})\ n^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})\ (4\pi)^{l+2\nu}},$$

with the assumption that either $(a)$ $g$ is a cusp form and $k > 3$ or $(b)$ $g$ is not cusp form and $l < k - 2$.

*Remark* 4.2.2. Using Lemma 1.3.2 and Lemma 1.4.4 one can show that the series appearing in (4.2.1) converges.

## 4.3 Proof of Theorem 4.2.1

We need the following lemma to prove Theorem 4.2.1.

**Lemma 4.3.1.** *Using the same notation in Theorem 4.2.1, we have*

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash\mathcal{H}} \mid f(\tau)\ \overline{[e^{2\pi i n\tau}\tilde{|}_k\gamma, g]_\nu}\ (Im(\tau))^{k+l+2\nu+1} \mid\ d^*\tau$$

*converges.*

*Proof.* The proof is similar to Lemma 1 in [14]. □

Now we give a proof of Theorem 4.2.1. Put

$$T^*_{g,\nu}(f)(\tau) = \sum_{n=1}^{\infty} c(n)q^n.$$

Consider the $n$-th Poincaré series of weight $k+\frac{1}{2}$ as given in (1.4.1). Then using Lemma 1.4.3, we have

$$\langle T^*_{g,\nu}f, P_{k+\frac{1}{2},n}\rangle = \frac{\Gamma(k-\frac{1}{2})}{(4\pi n)^{k-\frac{1}{2}}}c(n).$$

On the other hand, by the definition of the adjoint map we have

$$\langle T^*_{g,\nu}f, P_{k+\frac{1}{2},n}\rangle = \langle f, T_{g,\nu}(P_{k+\frac{1}{2},n})\rangle = \langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle.$$

Hence we get

$$c(n) = \frac{(4\pi n)^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})}\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle. \tag{4.3.1}$$

By definition,

$$\begin{aligned}
\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle &= \int\limits_{\Gamma\backslash\mathcal{H}} f(\tau)\ \overline{\left[P_{k+\frac{1}{2},n}, g\right]_\nu (\tau)}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau \\
&= \int\limits_{\Gamma\backslash\mathcal{H}} f(\tau)\ \overline{[\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e^{2\pi i n\tau}\tilde{|}_{k+\frac{1}{2}}\gamma, g]_\nu(\tau)}\ (Im(\tau))^{k+l+2\nu+\ 1}\ d^*\tau \\
&= \int\limits_{\Gamma\backslash\mathcal{H}} f(\tau) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{[e^{2\pi i n\tau}\tilde{|}_{k+\frac{1}{2}}\gamma, g]_\nu(\tau)}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau \\
&= \int\limits_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\tau)\ \overline{[e^{2\pi i n\tau}\tilde{|}_{k+\frac{1}{2}}\gamma, g]_\nu(\tau)}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau.
\end{aligned}$$

By Lemma 4.3.1, we can interchange the sum and integration in $\langle f, [P_{k,n}, g]_\nu\rangle$. Hence we get,

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int\limits_{\Gamma\backslash\mathcal{H}} f(\tau)\ \overline{[e^{2\pi i n\tau}\tilde{|}_{k+\frac{1}{2}}\gamma, g]_\nu(\tau)}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau.$$

Since $g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2)$, $g\tilde{|}_{l+\frac{1}{2}}\gamma = \chi_2(d)g$, for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Therefore, $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle$ equals to

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int\limits_{\Gamma\backslash\mathcal{H}} f(\tau)\overline{[e^{2\pi i n\tau}\tilde{|}_{k+\frac{1}{2}}\gamma, \frac{1}{\chi_2(d)}\ g\tilde{|}_{l+\frac{1}{2}}\gamma]_\nu(\tau)}\ Im(\tau)^{k+l+2\nu+1} d^*\tau$$

$$= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)} \int\limits_{\Gamma\backslash\mathcal{H}} f(\tau)\ \overline{[e^{2\pi i n\tau}\ |_{k+\frac{1}{2}}\ \gamma, g\ |_{l+\frac{1}{2}}\ \gamma]_\nu(\tau)}\ Im(\tau)^{k+l+2\nu+1} d^*\tau.$$

Using the change of variable $\tau$ to $\gamma^{-1}\cdot\tau$ in each integral, $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle$ equals

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)} \int\limits_{\Gamma\backslash\mathcal{H}} f(\gamma^{-1}\cdot\tau)\ \overline{[e^{2\pi i n\tau}\ |_{k+\frac{1}{2}}\ \gamma, g\ |_{l+\frac{1}{2}}\ \gamma]_\nu(\gamma^{-1}\cdot\tau)}$$
$$\times\ (Im(\gamma^{-1}\cdot\tau))^{k+l+2\nu+1}\ d^*(\gamma^{-1}\cdot\tau).$$

Since $f\in S_{k+l+2\nu+1}(\Gamma,\chi_2\chi)$, $f(\gamma^{-1}\cdot\tau) = \chi_2(d)\chi(d)(cz+d)^{k+l+2\nu+1}f(\tau)$, for every $\gamma\in\Gamma$. Therefore $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle$ equals

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)} \int\limits_{\Gamma\backslash\mathcal{H}} \chi_2(a)\chi(a)(-c\tau+a)^{k+l+2\nu+1} f(\tau)\overline{(-c\tau+a)^{k+l+2\nu+1}}$$

$$\times\ \overline{([e^{2\pi i n\tau}\ |_{k+\frac{1}{2}}\ \gamma, g\ |_{l+\frac{1}{2}}\ \gamma]_\nu\ |_{k+l+2\nu+1}\ \gamma^{-1})(\tau)} \left(\frac{Im(\tau)}{|-c\tau+a|^2}\right)^{k+l+2\nu+1}\ d^*\tau.$$

Now using Remark 1.7.2, $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle$ equals

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\left(\frac{(\frac{-4}{d})^{k+l+1}}{\chi_2(d)}\right)} \chi_2(a)\chi(a) \int\limits_{\gamma\Gamma\backslash\mathcal{H}} f(\tau)\ \overline{[e^{2\pi i n\tau}, g]_\nu}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau.$$

We note that the term appearing before integral is equal to 1 for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty\setminus\Gamma$. Therefore we get

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle\ =\ \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\ \int\limits_{\gamma\Gamma\backslash\mathcal{H}} f(\tau)\ \overline{[e^{2\pi i n\tau}, g]_\nu}\ (Im(z))^{k+l+2\nu+1}\ d^*\tau.$$

Now using Rankin-Selberg unfolding argument, $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle$ equals to

$$\begin{aligned}
&\int\limits_{\Gamma_\infty \backslash \mathcal{H}} f(\tau)\ \overline{[e^{2\pi i n\tau}, g]_\nu}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau \\
&= \int\limits_{\Gamma_\infty \backslash \mathcal{H}} f(\tau) \sum_{r=0}^{\nu} C_r(k,l;\nu)\ \overline{D^r(e^{2\pi i n\tau}) D^{\nu-r}(g)} (Im(\tau))^{k+l+2\nu+1}\ d^*\tau \\
&= \sum_{r=0}^{\nu} C_r(k,l;\nu) \int\limits_{\Gamma_\infty \backslash \mathcal{H}} f(\tau)\ \overline{D^r(e^{2\pi i n\tau}) D^{\nu-r}(g)} (Im(\tau))^{k+l+2\nu+1}\ d^*\tau. \quad (4.3.2)
\end{aligned}$$

Now replacing $f$ and $g$ by their Fourier series in (4.3.2), $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle$ equals

$$\begin{aligned}
&\sum_{r=0}^{\nu} C_r(k,l;\nu) \int\limits_{\Gamma_\infty \backslash \mathcal{H}} \left( \sum_s a(s) e^{2\pi i s\tau} \right) n^r \overline{e^{2\pi i n\tau}} \left( \sum_m m^{\nu-r} \overline{b(m) e^{2\pi i m\tau}} \right) Im(\tau)^{k+l+2\nu+1} d^*\tau \\
&= \int\limits_{\Gamma_\infty \backslash \mathcal{H}} \sum_s \sum_m \alpha(k,l,\nu,n,m) a(s) \overline{b(m)}\ e^{2\pi i s\tau}\ \overline{e^{2\pi i n\tau}}\ \overline{e^{2\pi i m\tau}}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau \\
&= \sum_s \sum_m \alpha(k,l,\nu,n,m) a(s) \overline{b(m)} \int\limits_{\Gamma_\infty \backslash \mathcal{H}} e^{2\pi i s\tau}\ \overline{e^{2\pi i n\tau}}\ \overline{e^{2\pi i m\tau}}\ (Im(\tau))^{k+l+2\nu+1}\ d^*\tau.
\end{aligned}$$

Putting $\tau = u + iv$, $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu \rangle$ equals

$$\sum_s \sum_m \alpha(k,l,\nu,n,m) a(s) \overline{b(m)} \int\limits_{\Gamma_\infty \backslash \mathcal{H}} e^{2\pi i (s-n-m)u} e^{-2\pi(s+n+m)v} v^{k+l+2\nu+1} \frac{du dv}{v^2}.$$

A fundamental domain for the action of $\Gamma_\infty$ on $\mathcal{H}$ is given by $[0,1] \times [0,\infty)$.

Integrating over this region, $\langle f, [P_{k,n}, g]_\nu\rangle$ equals

$$\sum_s \sum_m \alpha(k,l,\nu,n,m) a(s)\overline{b(m)} \int_0^1 \int_0^\infty e^{2\pi i(s-n-m)u} e^{-2\pi(s+n+m)v} v^{k+l+2\nu-1} du dv.$$

Integrating on $u$ first, $\langle f, [P_{k,n}, g]_\nu\rangle$ equals

$$\sum_m \alpha(k,l,\nu,n,m) a(n+m)\overline{b(m)} \int_0^\infty e^{-4\pi(n+m)v} v^{k+l+2\nu-1} dv. \quad (4.3.3)$$

Integrating over $v$, we have

$$\int_0^\infty e^{-4\pi(n+m)v} v^{k+l+2\nu-1} dv = \frac{\Gamma(k+l+2\nu)}{(4\pi(n+m))^{k+l+2\nu}}. \quad (4.3.4)$$

Putting the value of integral (4.3.4) in (4.3.3), we have

$$\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle = \frac{\Gamma(k+l+2\nu)}{(4\pi)^{k+l+2\nu}} \sum_m \frac{a(n+m)\overline{b(m)}\alpha(k,l,\nu,n,m)}{(n+m)^{k+l+2\nu}}. \quad (4.3.5)$$

Now substituting $\langle f, [P_{k+\frac{1}{2},n}, g]_\nu\rangle$ from (4.3.5) in (4.3.1), we get the required expression for $c(n)$ as given in Theorem 4.2.1.

## 4.4 Applications

Consider the linear map $T^*_{g,\nu} \circ T_{g,\nu}$ on $S_k(\Gamma)$ with $g(\tau) \in M_l(\Gamma, \chi_2)$. If $\lambda$ is a eigenvalue of $T^*_{g,\nu} \circ T_{g,\nu}$, then $\lambda \geqslant 0$. Suppose that $S_k(\Gamma)$ is one-dimensional space generated by $f(\tau) = \sum_m a(n)q^n$. Then $T^*_{g,\nu} \circ T_{g,\nu}(h) = \lambda f, \ \forall\ h \in S_k(\Gamma)$. In particular, $T^*_{g,\nu} \circ T_{g,\nu}(f) = \lambda f$ with $\lambda \geqslant 0$ and if we write $T^*_{g,\nu} \circ T_{g,\nu}(f) = \sum_n c(n)q^n$ then

$$c(n) = \frac{\Gamma(k+l+2\nu-1)}{\Gamma(k-1)} \frac{n^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(n+m)\overline{b(m)} \ \ \alpha(k,l,\nu,n,m)}{(n+m)^{k+l+2\nu-1}},$$

where $a_{T_{g,\nu}(f)}(n)$ is the $n$-th Fourier coefficient of $T_{g,\nu}(f) = [f,g]_\nu$. If $a(m_0)$ is the first non-zero Fourier coefficient of $f$ then by comparing the Fourier coefficients in $T^*_{g,\nu} \circ T_{g,\nu}(f) = \lambda f$, we have

$$\lambda = \frac{\Gamma(k+l+2\nu-1)}{a(m_0)\Gamma(k-1)} \frac{m_0^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} \frac{a_{T_{g,\nu}(f)}(m_0+m)\overline{b(m)} \ \ \alpha(k,l,\nu,m_0,m)}{(m_0+m)^{k+l+2\nu-1}} \geqslant 0.$$

In particular, if we take $l = 0, \ k = 6$ and $\nu = 0$ with $g(\tau) = \theta(\tau) = \sum\limits_n q^{n^2}$ and the unique newform $\Delta_{4,6}(\tau) = \eta(2\tau)^{12} = \sum_n \tau_{4,6}(n)q^n \in S_6(\Gamma_0(4))$ in the case (II), then $m_0 = 1, \ \alpha(k,l,\nu,m_0,m) = 1$, and

$$\lambda = \frac{\Gamma(11/2)}{\Gamma(5)2\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{a_{T_{\theta,0}(\Delta_{4,6})}(m+1)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0,$$

or equivalently

$$\sum_{m=1}^{\infty} \frac{a_{T_{\theta,0}(\Delta_{4,6})}(m+1)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0. \tag{4.4.1}$$

Now $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$ is the $(m+1)$-th Fourier coefficient of $\theta(z)\Delta_{4,6}(\tau)$ and equals to $\sum\limits_{r=1}^{m+1} b(r)\tau_{4,6}(m+1-r)$. Putting the value of $a_{T_{\theta,0}(\Delta_{4,6})}(m+1)$ in

(4.4.1), we have

$$\sum_{m=1}^{\infty} \frac{\left(\sum_{r=1}^{m+1} b(r)\tau_{4,6}(m+1-r)\right)\overline{b(m)}}{(m+1)^{\frac{11}{2}}} > 0,$$

equivalently

$$\sum_{m=1}^{\infty} \frac{\left(\sum_{r=1}^{m^2+1} \tau_{4,6}(m^2+1-r^2)\right)}{(m^2+1)^{\frac{11}{2}}} > 0.$$

# Chapter 5

# Remarks on Rankin's method

## 5.1 Introduction

In this chapter we give some remarks on Rankin's method for modular forms, Jacobi forms and Siegel modular forms of genus 2. Rankin [30] showed that for any normalized eigenform $f \in S_k$ and any even integer $l$ with $\frac{k}{2}+2 \leqslant l \leqslant k-4$ one has the following identity:

$$L_f^*(l)L_f^*(k-1) = (-1)^{\frac{l}{2}} 2^{k-3} \frac{B_l}{l} \frac{B_{k-l}}{k-l} \langle f, E_l E_{k-l} \rangle, \tag{5.1.1}$$

where $L_f^*(s) = (2\pi)^s \Gamma(s) L_f(s)$ is the completed $L$-function. Zagier generalized the result of Rankin by considering any modular form instead of Eisenstein series and computed the Petersson scalar product $\langle f, gE_l \rangle$, for $f \in S_{k+l}$ and $g \in M_k$. Further more, Zagier also computed the Petersson scalar product $\langle f, [g, E_l]_\nu \rangle$ (where $[g, E_l]_\nu$ is the $\nu$-th Rankin-Cohen bracket of $g$ and $E_l$) and expressed in terms of special values of Rankin-Selberg type $L$-function associated with $f$ and $g$. More precisely:

**Theorem 5.1.1.** *[38] Let $l \geqslant k+2 > 2$ and $\nu \geqslant 0$ be integers. Let $f \in S_{k+l+2\nu}$ with Fourier expansion*

$$f(\tau) = \sum_{n=1}^{\infty} a(n) q^n,$$

*and $g \in M_k$ with Fourier expansion*

$$g(\tau) = \sum_{n=0}^{\infty} b(n) q^n.$$

*Then*

$$\langle f, [g, E_l]_\nu \rangle = \frac{\Gamma(k+l+2\nu-1)\Gamma(l+\nu)}{(4\pi)^{k+l+2\nu-1}\Gamma(l)} \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^{k+l+\nu-1}}.$$

To prove the above theorem one writes $[g, E_l]_\nu$ as linear combination of Poincaré series as follows:

$$[g, E_l] = \frac{\Gamma(l+\nu)}{\Gamma(l)} \sum_{n=0}^{\infty} n^\nu b(n) P_{k+l+2\nu,n}$$

where $P_{k+l+2\nu,n}$ is the $n$-th Poincaré series of weight $k+l+2\nu$, and then use the characterization property of Poincaré series given in Lemma 1.2.2 to compute the inner product $\langle f, [g, E_l]_\nu \rangle$.

*Remark* 5.1.1. If we take $\nu = 0$ in the above theorem, we have the following:

$$\langle f, gE_l \rangle = \frac{\Gamma(k+l-1)}{(4\pi)^{k+l-1}} \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^{k+l-1}}. \tag{5.1.2}$$

*Remark* 5.1.2. In particular, if we take $g = E_k$ (the Eisenstein series of weight $k$) in (5.1.2), we get Rankin's identity (5.1.1).

*Remark* 5.1.3. Following the proof of Theorem 2.1.2 (Herrero [14]), one can give a different proof of Theorem 5.1.1 by evaluating the integral

$$\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\overline{[g, E_l]}(Im\ z)^{k+l+2\nu} d^*\tau$$

explicitly using Rankin-Selberg unfolding argument.

Choie and Kohnen [9] generalized the work of Zagier to the case of Jacobi forms and computed the Petersson scalar product $\langle \phi, [\psi, E_{k_2,m_2}]_\nu \rangle$ in terms of special values of a certain Rankin-Selberg convolution of Jacobi forms $\phi$ and $\psi$.

**Theorem 5.1.2.** *[9] Let $k_1 > 3$, $k_2 > k_1 + 3$ and $\nu \geqslant 0$ be integers. Let $\phi \in J^{cusp}_{k_1+k_2+2\nu, m_1+m_2}$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ r^2 < 4(m_1+m_2)n}} a(n,r) q^n \zeta^r,$$

*and $\psi \in J_{k_1,m_1}$ with Fourier expansion*

$$\psi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z},\\ r^2\leq 4m_1 n}} b(n,r) q^n \zeta^r.$$

*Then*

$$\langle \phi, [\psi, E_{k_2,m_2}]_\nu \rangle = c_{k_1,k_2,m_1,m_2;\nu} \sum_{\substack{n\geqslant 1, r\in\mathbb{Z}\\ r^2\leqslant 4m_1 n}} \frac{(4m_1 n - r^2)^\nu a(n,r)\overline{b(n,r)}}{(4(m_1+m_2)n - r^2)^{k_1+k_2+2\nu-\frac{3}{2}}},$$

*where* $c_{k_1,k_2,m_1,m_2;\nu} = 2^{2\nu-1} m_2^\nu (m_1+m_2)^{k_1+k_2+2\nu-2} \binom{k_2+\nu-\frac{3}{2}}{\nu} \dfrac{\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{\pi^{k_1+k_2-\frac{3}{2}}}$.

Following the method of Zagier, one writes $[\psi, E_{k_2,m_2}]_\nu$ as a linear combination of Jacobi-Poincaré series as follows:

$$[\psi, E_{k_2,m_2}]_\nu = c_{k_2,m_2;\nu} \sum_{n,r\in\mathbb{Z}, r^2\leqslant 4m_1 n} (4m_1 n - r^2)^\nu b(n,r) P_{k_1+k_2+2\nu, m_1+m_2; n,r},$$

where $c_{k_2,m_2;\nu} = (2\pi)^{2\nu} m_2^\nu \binom{k_2+\nu-\frac{3}{2}}{\nu}$, and then use the characterization property of Jacobi-Poincaré series given in Lemma 1.5.2 to compute the inner product $\langle \phi, [\psi, E_{k_2,m_2}]_\nu \rangle$.

*Remark* 5.1.4. Following the method of proof of Theorem 2.2.1, one can give a different proof of Theorem 5.1.2 by evaluating the integral

$$\int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{\left[\psi, E_{k_2,m_2;(n,r)}\right]_\nu} v^{k_1+k_2+2\nu}\, e^{\dfrac{-4\pi(m_1+m_2)y^2}{v}} dV_J$$

using Rankin-Selberg unfolding argument.

Similar results have been studied for Jacobi forms of higher degree [29], Hilbert modular forms [8] and other automorphic forms.

## 5.2 Rankin's method on Siegel modular forms

In this section we generalize the result of Zagier to the case of Siegel modular forms of genus 2, following the method of Herrero.

**Theorem 5.2.1.** *Let $k \geqslant 4, l$ and $\nu \geq 0$ be natural numbers and $E_k^{(2)}$ be the Siegel Eisenstein series of weight $k$ and genus 2. Let $G \in S_l(\Gamma_2)$ with Fourier expansion*

$$G(Z) = \sum_{T>0} A(T) e^{2\pi i (tr(TZ))},$$

*and $F \in S_{k+l+2\nu}(\Gamma_2)$ with Fourier expansion*

$$F(Z) = \sum_{T>0} B(T) e^{2\pi i (tr(TZ))}.$$

*Then*

$$\langle F, [G, E_k^{(2)}]_\nu \rangle \ = \ \alpha(k,l,\nu) \sum_{T>0} \frac{\overline{A(T)} B(T)}{(detT)^{k+l+\nu-\frac{3}{2}}} \tag{5.2.1}$$

*with*

$$\alpha(k,l,\nu) = \frac{(-1)^\nu \Gamma_2(k+l+2\nu-\frac{3}{2}) \sum\limits_{r+p=\nu} C_{r,0,p}(k,l)}{(4\pi)^{2(k+l+\nu-\frac{3}{2})}}$$

*and $C_{r,0,p}(k,l)$ is the coefficients $C_{r,s,p}(k,l)$ with $s=0$ as in (1.7.5).*

*Proof.* Following the method of proof of Theorem 3.2.1 one can explicitly compute the integral

$$\int_{\Gamma_2\backslash\mathcal{H}_2} F(Z)\overline{[G,\quad E_k^{(2)}]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}dZ.$$

$\square$

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