A STUDY OF STABILITY ISSUES IN SOME INVERSE PROBLEMS

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Ajith Kumar T

Dedicated
to
my parents

Thanga Kumar I and ***Tamil Thangam T***

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ABSTRACT

The Calderón problem has played a central role in the study of inverse problems and partial differential equations (PDEs) since its introduction by Alberto P. Calderón in 1980. Sylvester and Uhlmann made a major breakthrough in this direction for dimensions greater than or equal to three. They introduced the concept of complex geometrical optics (CGO) solutions in their work, which motivated more researchers to explore this area in higher dimensions. Since this inverse problem is mostly ill-posed, a better understanding of the stability becomes very important. However, the optimal stability estimates are of logarithmic type in these settings. This inspired the notion of increasing stability, where it was explored whether an improvement of the stability to Hölder type is possible in the presence of frequency.

In this thesis, we focus on high-frequency stability estimates for inverse boundary value problems associated with the Schrödinger equation, the biharmonic operator, and the polyharmonic operator with constant attenuation, working in the partial data setting where part of the boundary is flat. We first explore stability estimates for the linearized inverse problems related to the Schrödinger and biharmonic operators with constant attenuation for potentials in C^1. We then consider the more general polyharmonic case and derive stability estimates for less regular potentials in H^s with $0 < s < \frac{1}{2}$. This result also generalizes the results obtained in the earlier cases, where C^1 regularity was assumed for the potentials.

Next, we focus on stability estimates for the corresponding nonlinear inverse problems under the same low-regularity assumptions on the potential. While in the linearized problems our stability estimates exhibit a polynomial dependence on the frequency in all cases, for the nonlinear problems we are able to establish such polynomial dependence only for the Schrödinger equation. For the biharmonic and polyharmonic operators, the dependence is exponential, which is consistent with the work of Liu (2020) in the absence of attenuation.

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Summary

In the history of inverse problems and partial differential equations (PDEs), the Calderón problem plays an important role. Calderón first explored it in 1980, where he considered a linearized form of the inverse conductivity problem. Sylvester and Uhlmann (1987) proved the uniqueness result for the nonlinear inverse conductivity problem by reducing it to the Schrödinger formulation for sufficiently smooth conductivities. This work gave a significant impetus to future works in this direction.

Since these inverse problems are mostly ill-posed, the study of stability is essential. In this context, the first stability result was accomplished by Alessandrini (1988), who proved a logarithmic-type stability estimate. This estimate implies an amplification of errors in the reconstruction of potentials and later motivated the concept of increasing stability, in which the improvement of stability from the logarithmic type to the Hölder type is explored with increasing frequency.

In this direction, Isakov and Wang (2014) explored the stability problem with constant attenuation. Isakov, Lai and Wang (2016) extended these results to the case of non-constant attenuation. The linearized inverse problem, where the Dirichlet-to-Neumann map is linearized near the zero potential, was analyzed by Isakov, Zou and Lu (2020). In all these works, it was assumed that the data and measurements are available on the full boundary. Zou, Lu and Xu (2022) studied the stability problem for the linearized case (without attenuation) with the assumption of partial boundary data and measurements where a part of the boundary is flat.

In this thesis, our primary focus is to establish the high-frequency stability for the inverse boundary value problem corresponding to the Schrödinger equation, the biharmonic operator and the polyharmonic operator with constant attenuation under partial boundary measurements where part of the boundary is flat.

We start with the linearized stability problem for the Schrödinger equation with constant attenuation, assuming that the regularity of the potential is C^1. In this analysis, we employ the linearization technique used in [25] and the reflection argument for the partial bound-

ary. In this case, the stability estimate exhibits a polynomial dependence on the frequency, which helps us to understand the improvement in stability better. Also, we study the linearized stability problem for the biharmonic operator with constant attenuation, assuming the same regularity for the potential as in the linearized inverse Schrödinger case. The resulting stability estimate mirrors that of the linearized inverse Schrödinger problem, with the estimate again exhibiting a polynomial dependence on the frequency in the Lipschitz part of the estimate.

Next, we explore the linearized stability problem for the polyharmonic operator with constant attenuation, assuming lower regularity for the potential (that of H^s, $0 < s < \frac{1}{2}$). In this work, we generalize the linearization used in [38] to the polyharmonic operator. To tackle the issue of partial boundary measurements, we employ the reflection argument and the quantitative version of the Riemann-Lebesgue lemma used in [33]. This stability result generalizes the results for the Schrödinger and the biharmonic case under a lower regularity assumption on the potential. As in the previous two cases, we have a polynomial dependence on the frequency in the Lipschitz part of the estimate, while the logarithmic part decays with increasing frequency.

Further, we study the nonlinear inverse problem for the Schrödinger equation and the biharmonic operator in the same setup as in the linearized problem, assuming low regularity for the potential as in the linearized inverse polyharmonic case. Since both problems are nonlinear, we employ complex geometrical optics (CGO) type solutions following the slightly different approach used in [26], and to tackle the issue with partial boundary measurements, we use the same process as in the case of the linearized inverse problem for the polyharmonic operator. The resulting stability estimate, again, exhibits a polynomial dependence on the frequency for the Schrödinger case. For the biharmonic case, the stability result is weaker, since it has an exponential dependence on the frequency in the Lipschitz part. Thus, the stability result is better in the linearized case compared to the nonlinear setup.

Finally, we discuss the nonlinear inverse problem for the polyharmonic operator under the same conditions as in the linearized case. The stability result, in this case, is similar to that obtained in the case of the nonlinear inverse problem for the biharmonic operator but weaker than the linearized case.

Chapter 1

Introduction

A large class of inverse problems in the context of partial differential equations (PDEs) involves determining the coefficients of the underlying equations from certain information about the solution. The Calderón problem, proposed by Alberto P. Calderón in 1980, is a fundamental question in this direction that involves determining the coefficients of the PDEs from boundary measurements of the solution. This is widely regarded as the mathematical foundation of Electrical Impedance Tomography (EIT), a non-invasive imaging technique used in medicine, geophysics, material sciences, engineering for detecting anomalous structures or identifying inhomogeneities in materials.

1.1 The Calderón problem

The original motivation for the Calderón problem came from geophysical prospecting. In his seminal work, Calderón proposed the following problem:
Let $\Omega \subset \mathbb{R}^n$ be a bounded region that represents a physical object which has an electrical conductivity distribution $\gamma(x) > 0$ inside Ω. When an electric voltage is applied to the boundary $\partial\Omega$, the resulting voltage $u(x)$ within the body satisfies the conductivity equation

$$\begin{cases} \nabla \cdot [\gamma(x)\nabla u(x)] &= 0 \quad \text{in } \Omega, \\ \qquad u &= f \quad \text{in } \partial\Omega, \end{cases} \tag{1.1.1}$$

where the conductivity γ is a positive function and f (a voltage distribution) is a known input data along the boundary. The measurement data is the corresponding current flux across the boundary, which is given by the normal derivative of $u(x)$ weighted by the conductivity.

This relationship between the voltage distribution f and the current flux $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ along the boundary is encoded in the Dirichlet-to-Neumann (D-N) map Λ_γ given by

$$\Lambda_\gamma : f \mapsto \gamma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}. \tag{1.1.2}$$

The Calderón problem is to recover $\gamma(x)$ from the knowledge of Λ_γ. In other words, it explores the injectivity of the map

$$\gamma \xmapsto{\Lambda} \Lambda_\gamma. \tag{1.1.3}$$

The following one-dimensional version of the problem helps us understand the context better. We start with a resistor made of metal wire stretched along the interval $0 \leq x \leq l$ and a voltage source attached to the ends of the wire at $x = 0$ and $x = l$. If the voltage at a point x on the wire is denoted by $u(x)$ and the resistance density σ at each point on the wire is continuous, then by Ohm's law, the current I is given by

$$I(x) = -\gamma(x)u'(x), \tag{1.1.4}$$

where $\gamma = \frac{1}{\sigma}$ is the conductivity of the resistor. Since there are no sources inside the wire, the current I is constant along the wire. Hence, we have

$$[\gamma(x)u'(x)]' = 0.$$

Suppose we don't know the conductivity $\gamma(x)$ inside the wire, but we can only measure the applied voltage and the resulting current at the boundary (at the ends of the wire), that is, we only have the knowledge of $u(0)$, $u(l)$, $\gamma(0)u'(0)$ and $\gamma(l)u'(l)$. Then the question becomes whether we can determine the conductivity γ everywhere in the wire.

Unfortunately, the answer is no. From the equation, we can only infer

$$u(l) - u(0) = \gamma(0)u'(0) \int_l^0 \frac{dx}{\gamma(x)},$$

and therefore, the boundary measurements only recover the total resistance of the wire, not the pointwise conductivity. For two and higher dimensions, the determination is indeed possible, although the mathematical techniques used are quite different.

1.2 The uniqueness problem

The fundamental question in uniqueness is whether two conductivities giving rise to same boundary measurements are necessarily equal. A.P. Calderón proved the uniqueness result in [6] for the linearized inverse problem. For the nonlinear case, this question was answered in [35] by Sylvester and Uhlmann for dimension $n \geq 3$, assuming the conductivities to be sufficiently smooth. They reduced the study from the conductivity equation to the Schrödinger equation as follows:
Let $q = \dfrac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. Then $\gamma^{-1/2}u$ is the solution for the conductivity equation, where u is the solution for the Schrodinger equation. That is,

$$-\nabla \cdot \gamma\nabla(\gamma^{-1/2}u) = \gamma^{1/2}(-\Delta + q)u.$$

Here q is an unknown potential and we have the Dirichlet-to-Neumann map Λ_q defined as follows:

$$\Lambda_q : f \mapsto \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}. \tag{1.2.1}$$

Then the question of uniqueness can be reformulated as the question of injectivity of the map $q \overset{\Lambda}{\longmapsto} \Lambda_q$.

Compared to the conductivity equation, the Schrödinger equation turns out to be easier to handle for analysis. It inspired more researchers to explore the Calderón problem and its Schrödinger formulation. Bukhgeim and Uhlmann established the uniqueness result in [5] for the case of partial boundary measurements. This was a major breakthrough compared to the earlier results that required measurements on a large portion of the boundary.

Isakov proved a uniqueness result in [21] for the inverse conductivity problem using partial boundary measurements, where the part of the boundary is flat or a sphere.

Later researchers also extended the study of the uniqueness problem to perturbations of the biharmonic and the polyharmonic operators. Krupchyk, Lassas, and Uhlmann studied the uniqueness problem for the biharmonic operator with a first-order perturbation in [28], aiming to recover the lower-order coefficients from partial boundary measurements. They also proved the uniqueness problem for polyharmonic operators with lower-order perturbations in [29]. Yang studied the uniqueness problem of recovering the first-order perturbation of the biharmonic operator on both bounded and unbounded domains in [37] using partial boundary measurements.

1.3 The question of stability

In practical applications of the Calderón problem such as EIT and imaging, the boundary measurements usually are noisy, discrete and incomplete. Consequently, the uniqueness question is insufficient; we must also understand how errors in boundary measurements impact the reconstruction of the unknown. This brings us to the question of stability, that is, we want to understand the following:

If two different conductivities yield nearly identical boundary measurements, are the conductivities themselves close?

In other words, we want to study the continuous dependence of conductivity γ on Λ_γ.

In the Schrödinger case, this can be mathematically described in the following manner.

Does there exist a constant $C > 0$ and a suitable modulus of continuity ω, that is, a continuous function ω defined in $[0, \infty)$ with $\omega(t) \to 0$ as $t \to 0$ such that for all potentials q_1, q_2,

$$\|q_1 - q_2\|_1 \leq C\omega(\|\Lambda_{q_1} - \Lambda_{q_1}\|_2), \tag{1.3.1}$$

where $\|\cdot\|_j$ $(j = 1, 2)$ denote appropriate norms. The modulus of continuity determines the stability of the problem. A Lipschitz or Hölder continuous function as the modulus of continuity is the ideal stability one seeks for. However, for the Calderón problem, the best possible global stability estimate is of logarithmic type, that is, $\omega(t) = C|\log(t)|^{-\alpha}$ for $\alpha \in (0, 1)$. In other words,

$$\|q_1 - q_2\|_1 \leq C|\log(\|\Lambda_{q_1} - \Lambda_{q_1}\|_2)|^{-\alpha}. \tag{1.3.2}$$

This implies that small errors in the known boundary data and measurements can cause comparatively large errors in the reconstructed unknown.

1.4 Developments in Stability Estimates

The first logarithmic stability estimate for the Schrödinger formulation under full boundary measurements was established by Alessandrini in [2], and later Mandache, in [34], proved that this estimate is optimal. Heck and Wang established the double logarithmic stability in [16] corresponding to the uniqueness result in [5]. The dominating part of the stability estimate, in this case, is of the form

$$\log[|\log(\|\cdot\|)|^{-\alpha}] \quad \text{for} \quad \alpha \in (0, 1). \tag{1.4.1}$$

Choudhury and Krishnan, in [10], obtained the stability estimates for the biharmonic operator with bounded potentials from boundary measurements made on the whole boundary and slightly more than half of the boundary. In the case of partial boundary measurements, when a part of the boundary is inaccessible and flat, Heck and Wang proved in [17] that logarithmic type stability is optimal following the uniqueness result in [21]. It is also applicable to the case when an inaccessible part of the boundary is part of a sphere. Choudhury and Heck, in [8], studied the inverse boundary value problem for the biharmonic operator in the same setup (under the flatness assumption) and established logarithmic type stability estimates for the recovery of the potential.

Since logarithmic type estimates amplify the errors, similar inverse problems, in the presence of a frequency term, were studied to see if the stability estimate can be improved to a Lipschitz/Hölder type with increasing frequency. In [22], Isakov investigated the inverse problem of recovering the potential in the Schrödinger case and established a stability estimate of the form

$$\mathcal{O}_1(k)\, \|\Lambda_{q_1} - \Lambda_{q_1}\|_2^{\alpha} + \frac{1}{[\mathcal{O}_2(k) + |\log(\|\Lambda_{q_1} - \Lambda_{q_1}\|_2)|]^{\beta}}, \tag{1.4.2}$$

where $\alpha \in (0, 1]$, $\beta \in (0, 1)$ and $\mathcal{O}_j(k)$ is an increasing function of k for $j = 1, 2$. As the frequency k increases, the logarithmic part becomes very small and the first (Hölder) part becomes dominant. In other words, the stability improves to a Lipschitz or Hölder type as the frequency increases, thereby increasing the overall stability of the problem. This was referred to as the increasing stability phenomenon and garnered massive attention. Liang, in [32], studied the stability in the recovery of the potential from partial Cauchy data in the Schrödinger case. In [9], Choudhury and Heck studied the same problem but with lower regularity assumption on the potentials. For the same partial measurement case, Liu studied the stability result for the biharmonic operator in [33].

In the presence of an attenuation, Isakov and Wang studied the stability problem for the Schrödinger equation with constant attenuation in [26] for the full boundary case. For the non-constant attenuation, Isakov, Lai, and Wang investigated the stability estimate in [23] for the Schrödinger equation with full boundary measurements.

Recently, the linearized inverse problem for the Schrödinger equation was also analyzed by Isakov, Lu, and Xu in [25] for the case of full boundary measurements. Following that, Zou, Lu, and Xu studied the linearized inverse problem for the Schrödinger equation with partial boundary measurements in [39]. Zhao and Yuan established the stability estimate for linearized inverse problem for the biharmonic operator with constant attenuation in [38] under the assumption of full boundary measurements.

Chapter 2

Preliminaries

In this chapter, we introduce some notations and discuss some preliminaries that will be used throughout the subsequent chapters.

2.1 Domains satisfying a flatness condition

The problems that we consider in this thesis are posed in a specific type of domain Ω described below. Such domains were introduced, in the context of inverse problems, by Isakov in [21].

Let $\Omega \subset \mathbb{R}^n$, $(n \geq 3)$ be a bounded open set with a smooth boundary, satisfying the following conditions:

1. $\Omega \subset \{x := (x', x_n) \in \mathbb{R}^n \ : \ x_n < 0\}$,

2. $\Gamma_0 := \partial\Omega \cap \{x := (x', x_n) \in \mathbb{R}^n \ : \ x_n = 0\}$ is a non-empty subset of $\partial\Omega$ (boundary of Ω),

3. $\Gamma := \partial\Omega \setminus \Gamma_0$ which contains the support of the boundary data.

The flatness condition, stated above, will henceforth be referred to as the condition $(\mathcal{A})$. The flat part Γ_0 is assumed to be inaccessible, and both Γ and Γ_0 have non-empty interiors.

2.2 The Sobolev spaces

Since we shall be working with data and measurements only on a part Γ of the boundary $\partial\Omega$, we shall work with the Sobolev spaces defined as follows:

$$\widetilde{H}^s(\Gamma) := \left\{u \in H^s(\partial\Omega) : \operatorname{supp}(u) \subseteq \overline{\Gamma}\right\}, \quad H^s(\Gamma) := \{u = v|_\Gamma : v \in H^s(\partial\Omega)\}, \; s \geq 0,$$

with the corresponding norms

$$\|u\|_{\widetilde{H}^s(\Gamma)} := \|\tilde{u}\|_{H^s(\partial\Omega)} \quad \text{and} \quad \|u\|_{H^s(\Gamma)} := \inf_{\substack{v \in H^s(\partial\Omega) \\ v|_\Gamma = u}} \|v\|_{H^s(\partial\Omega)}.$$

Here $\tilde{u}$ denotes the extension of u from Γ to $\partial\Omega$ by 0.

The dual of $\widetilde{H}^s(\Gamma)$ is denoted by $H^{-s}(\Gamma)$ and is equipped with the norm

$$\|u\|_{H^{-s}(\Gamma)} := \inf_{\Psi \in H^{-s}(\partial\Omega), \; \Psi|_\Gamma = u} \|\Psi\|_{H^{-s}(\partial\Omega)}.$$

Let $\mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ denote the space of bounded linear operators between $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. We shall denote the norm in $\mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ as $\|\cdot\|_*$.

On the product spaces $\prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$ and $\prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma)$ (which we shall denote as $\widetilde{H}^{\alpha_1,\dots,\alpha_m}(\Gamma)$ and $H^{\alpha_1,\dots,\alpha_m}(\Gamma)$ respectively), for $m \geq 2$, we shall consider the norms

$$\|(u_1, u_2, \dots, u_m)\|_{\widetilde{H}^{\alpha_1,\dots,\alpha_m}(\Gamma)} := \sum_{j=1}^m \|u_j\|_{\widetilde{H}^{\alpha_j}(\Gamma)},$$

$$\|(u_1, u_2, \dots, u_m)\|_{H^{\alpha_1,\dots,\alpha_m}(\Gamma)} := \sum_{j=1}^m \|u_j\|_{H^{\alpha_j}(\Gamma)}.$$

We shall denote the space of bounded linear operators between the product spaces $\prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$ and $\prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma)$ by $\mathcal{L}\left(\prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma), \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma)\right)$ and the corresponding norm by $\|\cdot\|_*$.

2.3 Some preliminary results

In this section, we recall some auxiliary results that will be used in the thesis.

First, recall that a function $F : X \to X$ defined on a Banach space X is called *non-*

expansive if

$$\|F(x) - F(y)\| \leq \|x - y\|, \; x, y \in X.$$

For such mappings, we have the following fixed-point theorem due to Browder (see [4]).

Theorem 2.3.1. ***(Browder fixed-point theorem)*** *Let G be a nonempty convex closed bounded set in a uniformly convex Banach space. Then, any non-expansive function $F : G \to G$ has a fixed point.* □

Consider a constant coefficient linear differential operator P in $\mathbb{R}^n$. We denote the symbol of P by $P(\xi)$ and let $\tilde{P}(\xi)$ be defined by $\tilde{P}(\xi) = \left(\sum_{|\alpha|\geq 0} \left|\partial_\xi^\alpha P(\xi)\right|^2\right)^{\frac{1}{2}}$. Note that, here, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with non-negative components and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The following result (see [20,32]) will be used in the construction of the CGO solutions.

Theorem 2.3.2. *Let P be a linear differential operator of order m in $\mathbb{R}^n$ with constant coefficients and let Ω be a bounded open set in $\mathbb{R}^n$. Then there is a bounded linear operator E on $L^2(\Omega)$ such that*

$$PEf = f \quad \text{for all } f \in L^2(\Omega),$$

and for any differential operator Q with constant coefficients, we have

$$\|QEf\|_{L^2(\Omega)} \leq C_0 \sup_{\xi \in \mathbb{R}^n} \left|\frac{\tilde{Q}(\xi)}{\tilde{P}(\xi)}\right| \|f\|_{L^2(\Omega)},$$

where the positive constant C_0 depends only on m, n and Ω. □

We shall also use the following interior elliptic regularity estimate (see Theorem 11.1, Chapter 5, [36]) in an open, bounded subset M of $\mathbb{R}^n$.

Theorem 2.3.3. *Let P be an elliptic linear differential operator of order m and $u \in \mathcal{D}'(M)$ satisfies $Pu = f \in H^s(M)$ for some $s \in \mathbb{R}$. Then $u \in H^{s+m}_{loc}(M)$, and, for each $U \subset\subset$*

$V \subset\subset M$, $\sigma < s + m$, the following estimate holds:

$$\|u\|_{H^{s+m}(U)} \le C\|Pu\|_{H^s(V)} + C\|u\|_{H^\sigma(V)}. \tag{2.3.1}$$

□

The following quantitative version of the Riemann-Lebesgue lemma, which generalizes the one in [8], [9], [17], was proved in [33].

Lemma 2.3.4. *Let $f \in H^s(\mathbb{R}^n),\ 0 < s < 1$, be compactly supported. Then there exists a constant $C > 0$ and for any $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that for all $\xi \in \mathbb{R}^n$ and $\tau \in (0,1)$, we have*

$$|\mathcal{F}[f](\xi)| \le \frac{C_N}{(1+\tau|\xi|)^N}\|f\|_{H^s(\mathbb{R}^n)} + C\tau^s\|f\|_{H^s(\mathbb{R}^n)}.$$

Here $\mathcal{F}[f]$ denotes the Fourier transform of f.

Before we can discuss the proof of the above lemma, we need an auxiliary result that we describe next. Let $\{\Psi_\tau\}_{\tau>0}$ denote the standard family of mollifiers defined by

$$\Psi_\tau(x) := \tau^{-n}\Psi\left(\frac{x}{\tau}\right),\ \tau > 0,$$

where $\Psi \in C_c^\infty(\mathbb{R}^n),\ 0 \le \Psi \le 1,$ and $\int_{\mathbb{R}^n} \Psi(x)\, dx = 1.$

The following approximation result was established in Proposition B.1 of [30]. Here, we rewrite the estimate to emphasise the appearance of the term $\|f\|_{H^s(\mathbb{R}^n)}$.

Lemma 2.3.5. *Let $f \in H^s(\mathbb{R}^n),\ 0 \le s < 1$. Then $f_\tau := f * \Psi_\tau \in (H^s \cap C^\infty)(\mathbb{R}^n)$ and*

$$\|f - f_\tau\|_{L^2(\mathbb{R}^n)} \le C\tau^s\|f\|_{H^s(\mathbb{R}^n)}.$$

Proof. We note that $\mathcal{F}[\Psi_\tau](\xi) = \mathcal{F}[\Psi](\tau\xi)$, and therefore $\mathcal{F}[f_\tau](\xi) = \mathcal{F}[f](\xi)\cdot\mathcal{F}[\Psi](\tau\xi)$. Using this, and Parseval's identity, we see that

$$\begin{aligned}\tau^{-2s}\|f - f_\tau\|^2_{L^2(\mathbb{R}^n)} &= \tau^{-2s}(2\pi)^{-n}\int_{\mathbb{R}^n} |1-\mathcal{F}[\Psi](\tau\xi)|^2|\mathcal{F}[f](\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}^n} g(\tau\xi)|\xi|^{2s}|\mathcal{F}[f](\xi)|^2\, d\xi,\end{aligned} \tag{2.3.2}$$

where $g(\eta) := (2\pi)^{-n}\frac{|1-\mathcal{F}[\Psi](\eta)|^2}{|\eta|^{2s}}$, $\eta \neq 0$. As $\mathcal{F}[\Psi](0) = 1$, we see that $\mathcal{F}[\Psi](\eta) = 1+\mathcal{O}(|\eta|)$, and therefore, for $0 \leq s < 1$, we have $g(0) = 0$. Also, as $\Psi \in C_c^\infty(\mathbb{R}^n)$, we see that g is continuous and bounded.

Therefore, from (2.3.2), we have

$$\begin{aligned} \tau^{-2s}\|f - f_\tau\|^2_{L^2(\mathbb{R}^n)} &\leq C\int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}[f](\xi)|^2\, d\xi \\ &\leq C\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\mathcal{F}[f](\xi)|^2\, d\xi = C\|f\|^2_{H^s(\mathbb{R}^n)}, \end{aligned}$$

and hence

$$\|f - f_\tau\|_{L^2(\mathbb{R}^n)} \leq C\tau^s\|f\|_{H^s(\mathbb{R}^n)}.$$

□

Using the previous lemma and following [33], we now discuss the proof of the quantitative Riemann-Lebesgue lemma (Lemma 2.3.4).

Proof. (of Lemma 2.3.4)

We note that

$$|\mathcal{F}[f](\xi)| \leq |\mathcal{F}[f_\tau](\xi)| + |\mathcal{F}[f_\tau](\xi) - \mathcal{F}[f](\xi)|,$$

and recall the identity $\mathcal{F}[f_\tau](\xi) = \mathcal{F}[f](\xi)\cdot\mathcal{F}[\Psi](\tau\xi)$. Since f is compactly supported, it follows that $f \in L^1(\mathbb{R}^n)$. Therefore, we have

$$|\mathcal{F}[f_\tau](\xi)| \leq \|f\|_{L^1(\mathbb{R}^n)}|\mathcal{F}[\Psi](\tau\xi)| \leq C\|f\|_{H^s(\mathbb{R}^n)}|\mathcal{F}[\Psi](\tau\xi)|.$$

Using the fact that $\mathcal{F}[\Psi] \in \mathcal{S}(\mathbb{R}^n)$, we get

$$|\mathcal{F}[\Psi](\tau\xi)| \leq \frac{C_N}{(1+\tau|\xi|)^N}$$

for all $\xi \in \mathbb{R}^n$, $\tau > 0$, and $N \in \mathbb{N}$. Here, the constant C_N depends on N. Therefore, we see that

$$|\mathcal{F}[f_\tau](\xi)| \leq \frac{C_N}{(1+\tau|\xi|)^N}\|f\|_{H^s(\mathbb{R}^n)}$$

for all $\xi \in \mathbb{R}^n$, $\tau > 0$, and $N \in \mathbb{N}$.

Next, using Young's inequality, we note that

$$\|f_\tau - f\|_{L^2(\mathbb{R}^n)} \leq 2\|f\|_{L^2(\mathbb{R}^n)} \text{ for } \tau \in (0, \infty).$$

Using this, together with the previous lemma and the fact that f is compactly supported, we see that for all $0 < \tau < 1$,

$$|\mathcal{F}[f_\tau](\xi) - \mathcal{F}[f](\xi)| \leq \|f_\tau - f\|_{L^1(\mathbb{R}^n)} \leq C\|f_\tau - f\|_{L^2(\mathbb{R}^n)} \leq C\tau^s\|f\|_{H^s(\mathbb{R}^n)}.$$

Note that the condition $0 < \tau < 1$ ensures that $f_\tau - f$ is uniformly compactly supported and hence, the constant C appearing in the second inequality above is independent of τ. Therefore, for all $0 < \tau < 1$,

$$\begin{aligned} |\mathcal{F}[f](\xi)| &\leq |\mathcal{F}[f_\tau](\xi)| + |\mathcal{F}[f_\tau](\xi) - \mathcal{F}[f](\xi)| \\ &\leq \frac{C_N}{(1+\tau|\xi|)^N}\|f\|_{H^s(\mathbb{R}^n)} + C\tau^s\|f\|_{H^s(\mathbb{R}^n)}. \end{aligned}$$

□

Chapter 3

Linearized inverse Schrödinger problem with attenuation

In this chapter, based on the work [31], we study the high-frequency stability estimates for the recovery of the potential function in the linearized inverse Schrödinger problem with constant attenuation from partial data. We assume that part of the boundary is inaccessible and flat. Our estimates suggest an improvement of stability from logarithmic to Lipschitz as the frequency increases.

3.1 Introduction

Let us consider the Schrödinger equation with attenuation

$$\begin{cases} -\Delta u - (k^2 - ikb)u + qu = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, where b, the attenuation, is a positive real number, $q \in C^1(\overline{\Omega})$ is a real valued function such that $\operatorname{supp}(q) \subset \Omega$ and $\|q\|_{C^1(\overline{\Omega})} \leq M$ for some $M > 0$. We shall assume that the frequency $k > 1$ and the boundary data $f \in H^{\frac{1}{2}}(\partial\Omega)$.

The attenuation model arises in the modelling of time harmonic wave phenomenon in acoustics (see [27]), particularly in a *lossy* media and accounts for the dissipation of the acoustic energy.

In this work, we assume that $\Omega \subset \{x \in \mathbb{R}^n : x_n < 0\}$ satisfies the flatness condition $(\mathcal{A})$ (see section 2.1) and that f is supported in Γ. Note that $(k^2 - ikb)$ is not a part of the Dirichlet spectrum of $-\Delta + q$ (since its imaginary part is non-zero) in the domain Ω, and therefore, we have a unique solution to the problem (3.1.1).

In order to work with the partial Dirichlet-to-Neumann map, we use the notion of Sobolev spaces defined on a part of the boundary as follows (see section 2.2).

For $s > 0$, we consider the Sobolev spaces (see [18], [39])

$$\widetilde{H}^s(\Gamma) = \{u \in H^s(\partial\Omega) \ : \ \text{supp}(u) \subseteq \Gamma\} \ \text{ and } \ H^s(\Gamma) = \left\{ u\Big|_{\Gamma} \ : \ u \in H^s(\partial\Omega) \right\},$$

with the norms defined by

$$\|u\|_{\widetilde{H}^s(\Gamma)} = \|\widetilde{u}\|_{H^s(\partial\Omega)} \ \text{ and } \ \|u\|_{H^s(\Gamma)} = \inf_{f \in H^s(\partial\Omega),\ f|_\Gamma = u} \|f\|_{H^s(\partial\Omega)}$$

respectively, where $\widetilde{u}$ is extension of u by zero to $\partial\Omega$.

The dual of $\widetilde{H}^s(\Gamma)$ is denoted by $H^{-s}(\Gamma)$ and is equipped with the norm

$$\|u\|_{H^{-s}(\Gamma)} = \inf_{\Psi \in H^{-s}(\partial\Omega),\ \Psi|_\Gamma = u} \|\Psi\|_{H^{-s}(\partial\Omega)}.$$

We denote the space of bounded linear operators between $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ as $\mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$. Recall that the operator norm of T is defined as follows:

$$\|T\|_{\mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))} = \sup_{\|\varphi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} = 1} \|T(\varphi)\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

The partial Dirichlet-to-Neumann (D-N) map is defined as

$$\mathcal{N}_q^S \ : \ \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \quad \text{such that} \quad \mathcal{N}_q^S(f) := \partial_\nu u\Big|_{\Gamma}, \tag{3.1.2}$$

where ν is the unit outer normal vector to $\partial\Omega$ and u is the unique solution to the problem

$$\begin{cases} -\Delta u \ - \ (k^2 - ikb)u \ + \ qu & = 0 \ \text{ in } \Omega, \\ \qquad\qquad u & = f \ \text{ on } \Gamma, \\ \qquad\qquad u & = 0 \ \text{ on } \Gamma_0. \end{cases} \tag{3.1.3}$$

Our aim is to obtain stability estimates for a linearized version of the inverse Schrödinger potential problem, as discussed in [25]. The study of the inverse Schrödinger potential problem dates back to the seminal work [6]. The stability for the inverse Schrödinger potential problem in the full data case when $k = 0$ was shown to be logarithmic in [2], corresponding to the uniqueness result in [35]. Later, in work [34], it was shown that this logarithmic estimate is optimal. In the partial data case, with $k = 0$, a double logarithmic type stability estimate was obtained in [16], corresponding to the uniqueness result in [5]. Also, corresponding to the uniqueness result in [21], logarithmic stability was obtained in [17] for the partial data problem (with $k = 0$) posed in domains where a part of the boundary is inaccessible and is either flat or part of a sphere.

Recently, in order to improve the stability estimates (analytically and numerically) from the logarithmic type stability estimates, the inverse problems with $k \neq 0$ started getting attention. In view of the improvement in the stability estimates when the frequency k is increasing, this phenomenon is also termed as increasing stability. We refer to the works [7, 9, 22–26, 32] and [39] for some studies in this direction. In particular, the works [23] and [26] studied the increasing stability for the Schrödinger equation with attenuation in the full data case. In the work [26], the case with constant attenuation coefficient was studied, while in [23], increasing stability with non-constant attenuation was dealt with. The work [25] studied the increasing stability for the linearized inverse problem for the Schrödinger equation with constant attenuation in the full data case. The work [39] studied the linearized problem in the absence of attenuation for the partial data case for domains similar to that being considered here.

The D-N map is non-linear and poses considerable difficulties, both analytically and from the perspective of numerical reconstructions. The standard approach in dealing with the D-N map for Calderón type inverse problems is to use Complex Geometrical Optics (CGO) solutions. In comparison, the linearized D-N map can be handled using complex exponential solutions which are easier to handle. The numerical solution of the linearized problem is usually faster and more reliable (as observed in [25]).

In this work, we investigate the increasing stability phenomenon for the linearized inverse problem for the Schrödinger equation with attenuation in the partial data case when the domain Ω is as described before. The stability estimate that we obtain clearly suggests the improvement of the estimate with growing frequency.

The plan of the chapter is as follows. In Section 3.2, we introduce the linearized partial D-N map for the Schrödinger equation with attenuation and state our main result on the stability estimate. The construction of the complex exponential solutions and the derivation of the stability estimate are described in Section 3.3, and the justification of the linearization is discussed in Section 3.4.

3.2 Linearized inverse Schrödinger problem

Let us consider the following sub-problems of the original problem (3.1.3), as in [25],

$$\begin{cases} -\Delta u_0 - (k^2 - ikb)u_0 &= 0 \quad \text{in } \Omega, \\ \qquad u_0 &= f \quad \text{on } \Gamma, \\ \qquad u_0 &= 0 \quad \text{on } \Gamma_0, \end{cases} \tag{3.2.1}$$

and

$$\begin{cases} -\Delta u_1 - (k^2 - ikb)u_1 &= -qu_0 \quad \text{in } \Omega, \\ \qquad u_1 &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.2.2}$$

Let u_0 and u_1 denote the solutions to these sub-problems. Note that $(k^2 - ikb)$ is not a part of the Dirichlet spectrum of $-\Delta$ in the domain Ω since its imaginary part is non-zero and therefore, we have a unique solution to the above sub-problems.

We define the linearized partial D-N map $\mathcal{N}_{q,L}^S$ of $\mathcal{N}_q^S$ as

$$\mathcal{N}_{q,L}^S \ : \ \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \quad \text{such that} \quad \mathcal{N}_{q,L}^S(f) := \partial_\nu u_1\Big|_\Gamma. \tag{3.2.3}$$

Let

$$\mathcal{Q} := \{q' \in L^\infty(\Omega) \ : \ 0 \text{ is not a Dirichlet eigenvalue of } -\Delta + q' \text{ in } \Omega\}.$$

Now, let us consider the nonlinear operator $\mathcal{N}_\Gamma : \mathcal{Q} \to \mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$, defined by

$$\mathcal{N}_\Gamma(q') := \widetilde{\mathcal{N}}_{q'}^S, \ \text{ such that } \widetilde{\mathcal{N}}_{q'}^S(f) = \partial_\nu u_f|_\Gamma, \ \text{ for } f \in \widetilde{H}^{\frac{1}{2}}(\Gamma),$$

where u_f is the weak solution of

$$\begin{cases} -\Delta u_f + q' u_f & = 0 \quad \text{in } \Omega, \\ \qquad u_f & = f \quad \text{on } \Gamma, \\ \qquad u_f & = 0 \quad \text{on } \Gamma_0. \end{cases} \tag{3.2.4}$$

Note that for $q' = -(k^2 - ikb) + q$, we have $\widetilde{\mathcal{N}}^S_{q'} = \mathcal{N}^S_q$, where $\mathcal{N}^S_q$ is the original partial D-N map defined in (3.1.2).

Further, we define the operator $\mathcal{N}^1_\Gamma : \mathcal{Q} \to \mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ as follows:

$$\mathcal{N}^1_\Gamma(q) := \mathcal{N}^S_{q,L},$$

which maps the potential q to the linearized partial D-N map $\mathcal{N}^S_{q,L}$ defined in (3.2.3). Note that $\mathcal{N}^1_\Gamma$ is a linear operator.

For all $q' \in \mathcal{Q}$, we define the operator $\mathcal{P}_{q'} : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^1(\Omega)$ such that

$$\mathcal{P}_{q'} f = u_f,$$

where u_f is the solution of the problem (3.2.4). Note that with $q' = -(k^2 - ikb)$ and $q' = -(k^2 - ikb) + q$, we get $\mathcal{P}_{q'} f = u_0$ and $\mathcal{P}_{q'} f = u$, where u_0 and u are the solutions to the equations (3.2.1) and (3.1.3) respectively.

Further, we also define the operator $\mathcal{G}_{q'} : L^2(\Omega) \to H^1_0(\Omega)$ by

$$\mathcal{G}_{q'}(F) = v_F,$$

where v_F is the solution to the problem

$$\begin{cases} -\Delta v_F + q' v_F & = -F \quad \text{in } \Omega, \\ \qquad v_F & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now, for $q \in L^\infty(\Omega)$, using the above notations, we rewrite the original partial D-N map defined in (3.1.2) as

$$\mathcal{N}_\Gamma(-(k^2 - ikb) + q)(f) = \mathcal{N}^S_q(f) = \partial_\nu(\mathcal{P}_{(-(k^2-ikb)+q)} f)|_\Gamma.$$

Similarly, we rewrite the linearized partial D-N map defined in (3.2.3) as

$$\mathcal{N}_\Gamma^1(q)(f) = \mathcal{N}_{q,L}^S(f) = \partial_\nu(\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f)|_\Gamma,$$

where $\mathcal{M}_q$ to be the multiplication operator by q and $\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f = u_1$, which is the solution to the equation (3.2.2).

The following theorem, which can be proved following [39], establishes the relationship between $\mathcal{N}_\Gamma$ and $\mathcal{N}_\Gamma^1$ under the assumption that q is sufficiently small.

Theorem 3.2.1. *Let $k > 1$ and b be a positive real number. Further, let $q \in L^\infty(\Omega)$ satisfy the condition $\|q\|_{L^\infty(\Omega)} < \frac{1}{2}\|\mathcal{G}_{(-(k^2-ikb))}\|^{-1}_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}$. Then*

$$\|\mathcal{N}_\Gamma(-(k^2-ikb)+q) - \mathcal{N}_\Gamma(-(k^2-ikb)) - \mathcal{N}_\Gamma^1(q)\|_{\mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma),H^{-\frac{1}{2}}(\Gamma))} \le C\|q\|^2_{L^\infty(\Omega)},$$

where C only depends on n, Ω, Γ, b and k.

Note that the condition $\|q\|_{L^\infty(\Omega)} < \frac{1}{2}\|\mathcal{G}_{(-(k^2-ikb))}\|^{-1}_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}$ guarantees the fact that

$$\|\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))} < 1,$$

and hence the series $\sum_{j=0}^\infty(\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q)^j$ converges. Using this, given $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$, the solution $u \in H^1(\Omega)$ of the equation (3.1.3) can be expressed as

$$u = \mathcal{P}_{(-(k^2-ikb)+q)}f = \sum_{j=0}^\infty(\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q)^j\mathcal{P}_{(-(k^2-ikb))}f.$$

This gives an expansion of u. In particular, the terms corresponding to $j = 0$ and $j = 1$ are u_0 and u_1, respectively. As observed in the theorem above, the higher order terms (corresponding to $j \ge 2$), can be neglected provided q is sufficiently small.

Then, our main result is the following stability estimate for the recovery of the potential q from the linearized partial D-N map $\mathcal{N}_{q,L}^S$ (3.2.3) defined above.

Theorem 3.2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with a smooth boundary as described above and let $k > 1$. Assume that the potential function $q \in C^1(\overline{\Omega})$ with* $\text{supp}(q) \subset \Omega$ *and $\|q\|_{C^1(\overline{\Omega})} \leq M$ for some $M > 0$, b is a positive constant and $\Omega \subset B(0, R)$ for some $R > 1$. Then, the following estimate*

$$\|q\|_{L^2(\Omega)} \leq C \left[(k + \sqrt{kb})^5 e^{4Rb} \|\mathcal{N}^S_{q,L}\|_* + \frac{e^{4Rb}}{\left(k^2 + \left(\frac{|\ln(\|\mathcal{N}^S_{q,L}\|_*)|}{9R}\right)^2\right)^{\frac{1}{n+2}}} \right]^{\frac{1}{4}} \tag{3.2.5}$$

holds true for the linearized case for $\|\mathcal{N}^S_{q,L}\|_* \leq \frac{1}{e^{9R}}$. Here, the constant C depends only on n, Ω, M.

Remark 3.2.3. As the frequency k increases, the logarithmic term in the stability estimate above decays, thus exhibiting improvement in the stability estimate.

3.3 Complex exponential solutions and the stability estimates

In this section, we construct appropriate complex exponential solutions to $-\Delta u - (k^2 - ikb)u = 0$ and, using them, derive the stability estimates. First, we introduce a change of coordinates as follows.

We denote $\xi := (\xi', \xi_n) \in \mathbb{R}^n$ $(n \geq 3)$, where $\xi' := (\xi_1, ..., \xi_{n-1}) \neq 0$ in $\mathbb{R}^{n-1}$. Define $e_1 = (\frac{\xi'}{|\xi'|}, 0)$, $e_n = (0, ..., 0, 1)$ and choose $e_2, ..., e_{n-1} \in \mathbb{R}^n$ such that the n^{th} component $e_{i,n} = 0$ for $i = 2, ..., n-1$ and the set $e := \{e_1, e_2, ..., e_n\}$ forms an orthonormal basis for $\mathbb{R}^n$. The coordinate representation of the vector ξ is $\xi_e = (|\xi'|, 0, ..., 0, \xi_n)_e$ with respect to this basis. For any two vectors $\tilde{\alpha}, \ \tilde{\beta} \in \mathbb{R}^n$ and their representations $\tilde{\alpha}_e, \ \tilde{\beta}_e$ in the basis $\{e_1, ..., e_n\}$, we have

$$\tilde{\alpha} \cdot \tilde{\beta} = \tilde{\alpha}_e \cdot \tilde{\beta}_e, \ \ \tilde{\alpha}_n = \tilde{\alpha}_{e,n} \text{ and } \tilde{\beta}_n = \tilde{\beta}_{e,n}.$$

Let us choose unit vectors $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_e,\ \beta_e$ have the representations

$$\alpha_e = (0,1,0,...,0)_e,\ \ \beta_e = \left(-\frac{\xi_n}{|\xi|}, 0, ..., 0, \frac{|\xi'|}{|\xi|}\right)_e.$$

Then $\{\xi, \alpha, \beta\}$ forms an orthogonal set. To see this, we note that

$$\begin{aligned}
|\alpha| &= |\alpha_e| = |(0,1,0,...,0)| = 1,\\
|\beta| &= |\beta_e| = \Big|\left(-\frac{\xi_n}{|\xi|}, 0, ..., 0, \frac{|\xi'|}{|\xi|}\right)_e\Big| = \left(\frac{\xi_n^2}{|\xi|^2} + 0 + ... + 0 + \frac{|\xi'|^2}{|\xi|^2}\right)^{\frac{1}{2}}\\
&= \left(\frac{\xi_1^2 + ... + \xi_n^2}{|\xi|^2}\right)^{\frac{1}{2}} = \left(\frac{|\xi|^2}{|\xi|^2}\right)^{\frac{1}{2}} = 1,\\
\alpha\cdot\beta &= \alpha_e\cdot\beta_e = (0,1,0,...,0)_e \cdot \left(-\frac{\xi_n}{|\xi|}, 0, ..., 0, \frac{|\xi'|}{|\xi|}\right)_e = 0,\\
\alpha\cdot\xi &= \alpha_e\cdot\xi_e = (0,1,0,...,0)_e\cdot(|\xi'|, 0, ..., 0, \xi_n)_e = 0,\\
\beta\cdot\xi &= \beta_e\cdot\xi_e = \left(-\frac{\xi_n}{|\xi|}, 0, ..., 0, \frac{|\xi'|}{|\xi|}\right)_e\cdot(|\xi'|, 0, ..., 0, \xi_n)_e\\
&= -\frac{\xi_n|\xi'|}{|\xi|} + 0 + ... + 0 + \frac{\xi_n|\xi'|}{|\xi|} = 0.
\end{aligned}$$

Thus, we get $|\alpha| = 1$, $|\beta| = 1$ and $\alpha\cdot\beta = \alpha\cdot\xi = \beta\cdot\xi = 0$.

Next, let us assume that $\tau > 1$ satisfies $|\xi|^2 \le 3(k^2+\tau^2)$, and denote the principal square root of $\left(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\right)$ by $X + iY$ where $X > 0$. Now, let us choose $\zeta_j \in \mathbb{C}^n,\ j = 1, 2$, as follows:

$$\begin{aligned}
\zeta_1 &= -\frac{\xi}{2} + X\beta + i(Y\beta + \tau\alpha),\\
\zeta_2 &= -\frac{\xi}{2} - X\beta - i(Y\beta + \tau\alpha).
\end{aligned} \tag{3.3.1}$$

Note that by squaring $X + iY$, we get $X^2 - Y^2 + i2XY = k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb$ and by comparing the real and the imaginary parts, we get

$$X^2 - Y^2 = k^2 + \tau^2 - \frac{|\xi|^2}{4} \ \text{ and } \ 2XY = -kb. \tag{3.3.2}$$

Now $\zeta_1 + \zeta_2 = -\xi$ and the product

$$\zeta_1\cdot\zeta_1 = \left(-\frac{\xi}{2}\right)\cdot\left(-\frac{\xi}{2}\right) + \left(-\frac{\xi}{2}\right)\cdot(X\beta) + \left(-\frac{\xi}{2}\right)\cdot(iY\beta) + \left(-\frac{\xi}{2}\right)\cdot(i\tau\alpha)$$

$$
\begin{aligned}
&+ (X\beta)\cdot\left(-\frac{\xi}{2}\right) + (X\beta)\cdot(X\beta) + (X\beta)\cdot(iY\beta) + (X\beta)\cdot(i\tau\alpha) \\
&\quad + (iY\beta)\cdot\left(-\frac{\xi}{2}\right) + (iY\beta)\cdot(X\beta) + (iY\beta)\cdot(iY\beta) + (iY\beta)\cdot(i\tau\alpha) \\
&\qquad + (i\tau\alpha)\cdot\left(-\frac{\xi}{2}\right) + (i\tau\alpha)\cdot(X\beta) + (i\tau\alpha)\cdot(iY\beta) + (i\tau\alpha)\cdot(i\tau\alpha) \\
&= \frac{|\xi|^2}{4} + (X^2 - Y^2)|\beta|^2 + 2iXY|\beta|^2 - \tau^2|\alpha|^2 \quad (\text{since } \alpha\cdot\beta = \alpha\cdot\xi = \beta\cdot\xi = 0) \\
&= \frac{|\xi|^2}{4} + k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb - \tau^2 \qquad (\text{since } |\alpha| = |\beta| = 1 \text{ and by (3.3.2)}) \\
&= k^2 - ikb.
\end{aligned}
$$

Similarly, we get $\zeta_2\cdot\zeta_2 = k^2 - ikb$.

From (3.3.1), we also have $|\zeta_j|^2 = \frac{|\xi|^2}{4} + X^2 + Y^2 + \tau^2$ for $j = 1, 2$, where

$$
\begin{aligned}
X^2 + Y^2 = |X + iY|^2 &= \left|\left(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\right)^{\frac{1}{2}}\right|^2 = \left|k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\right| \\
&= \left(\left(k^2 + \tau^2 - \frac{|\xi|^2}{4}\right)^2 + k^2b^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using the inequality $(c^2 + d^2)^{\frac{1}{2}} \leq (c + d)$ for $c,\ d \geq 0$, with $c = k^2 + \tau^2 - \frac{|\xi|^2}{4}$ and $d = kb$, we get

$$
\begin{aligned}
|\zeta_j|^2 &\leq \frac{|\xi|^2}{4} + k^2 + \tau^2 - \frac{|\xi|^2}{4} + kb + \tau^2 = k^2 + kb + 2\tau^2 \leq 2(k^2 + kb + \tau^2) \\
&\leq 2((k^2 + kb)^{\frac{1}{2}} + \tau)^2.
\end{aligned}
$$

Next, for later use, we find a bound for $\mathrm{Im}\,(\zeta_j)$ when $|\xi|^2 \leq 3(k^2 + \tau^2)$. Recall that

$$
\mathrm{Im}\,(\zeta_1) = Y\beta + \tau\alpha,\ \mathrm{Im}\,(\zeta_2) = -Y\beta - \tau\alpha,
$$

which gives $|\mathrm{Im}\,(\zeta_j)|^2 = Y^2 + \tau^2,\ j = 1, 2$. Now, using (3.3.2), it follows that

$$
\begin{aligned}
|Y| = \frac{kb}{2X} &= \frac{kb}{\sqrt{2}\sqrt{X^2 - Y^2 + \sqrt{(X^2 - Y^2)^2 + 4X^2Y^2}}} \\
&= \frac{kb}{\sqrt{2}\sqrt{k^2 + \tau^2 - \frac{|\xi|^2}{4} + \sqrt{\left(k^2 + \tau^2 - \frac{|\xi|^2}{4}\right)^2 + k^2b^2}}}
\end{aligned}
$$

$$\leq \frac{kb}{\sqrt{2}\sqrt{\frac{1}{4}(k^2+\tau^2)+\sqrt{(\frac{1}{4}(k^2+\tau^2))^2}}}$$

$$= \frac{kb}{\sqrt{2}\sqrt{\frac{1}{4}(k^2+\tau^2)+\frac{1}{4}(k^2+\tau^2)}} = \frac{kb}{\sqrt{2}\sqrt{\frac{1}{2}(k^2+\tau^2)}} = \frac{kb}{\sqrt{k^2+\tau^2}} \leq \frac{kb}{k} = b.$$

Hence,

$$|\text{Im}\,(\zeta_j)|^2 \leq b^2+\tau^2 \leq (b+\tau)^2 \quad \text{for } j=1,2.$$

Now, we use a reflection argument (see [21,25]) to construct complex exponential solutions. We denote $\widetilde{\Omega} := \Omega \cup \Omega^*$, where $\Omega^* := \{(x', x_n) \in \mathbb{R}^n \;:\; (x', -x_n) \in \Omega\}$ is the reflection of Ω by $\{x_n = 0\}$ and q_e, defined by

$$q_e(x) = \begin{cases} q(x', x_n) & \text{if } (x', x_n) \in \Omega, \\ q(x', -x_n) & \text{if } (x', x_n) \in \Omega^* \end{cases}$$

is the extension of q to $\widetilde{\Omega}$ by reflection of q by $\{x_n = 0\}$.

Let $\tilde{q}_e$ denote the extension of q_e to $\mathbb{R}^n$ by zero. Then, it is easy to see that

$$\tilde{u}_0(x) = e^{i\zeta_1\cdot x} \quad \text{and} \quad \tilde{v}(x) = e^{i\zeta_2\cdot x}$$

are solutions to the problems

$$-\Delta\tilde{u}_0 - (k^2 - ikb)\tilde{u}_0 = 0 \quad \text{in } \widetilde{\Omega}, \tag{3.3.3}$$

and

$$-\Delta\tilde{v} - (k^2 - ikb)\tilde{v} = 0 \quad \text{in } \widetilde{\Omega}, \tag{3.3.4}$$

respectively. Now, we define

$$u_0(x) = e^{i\zeta_1\cdot(x',x_n)} - e^{i\zeta_1\cdot(x',-x_n)} \quad \text{and} \quad v(x) = e^{i\zeta_2\cdot(x',x_n)} - e^{i\zeta_2\cdot(x',-x_n)}. \tag{3.3.5}$$

It is easy to check that these are solutions to the problems

$$\begin{cases} -\Delta u_0 - (k^2 - ikb)u_0 &= 0 \quad \text{in } \Omega, \\ \qquad\qquad u_0 &= 0 \quad \text{on } \Gamma_0, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v - (k^2 - ikb)v &= 0 \quad \text{in } \Omega, \\ \qquad\qquad v &= 0 \quad \text{on } \Gamma_0 \end{cases} \tag{3.3.6}$$

respectively. These are the complex exponential solutions that we shall use to derive the stability estimates.

3.3.1 Derivation of the stability estimates.

Now we derive the stability estimate (3.2.5). To do so, we first need an appropriate Green's identity. Let u_1 be the solution to the problem,

$$\begin{cases} -\Delta u_1 \;-\; (k^2 - ikb)u_1 & = -qu_0 \quad \text{in } \Omega, \\ \qquad\qquad u_1 & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.3.7}$$

where u_0 is the complex exponential solution defined in (3.3.5). Multiplying the equation (3.3.7) by v (defined in (3.3.5)) and integrating, we have

$$\begin{aligned} &-\Delta u_1 v - (k^2 - ikb)u_1 v = -qu_0 v \;\; \text{in } \Omega, \\ \text{and } &\int_\Omega qu_0 v dx = \int_\Omega \Delta u_1 v dx + \int_\Omega (k^2 - ikb)u_1 v dx. \end{aligned}$$

By Green's formula

$$\int_\Omega u_1 \Delta v dx - \int_\Omega \Delta u_1 v dx = \int_{\partial\Omega} u_1 \partial_\nu v dS - \int_{\partial\Omega} v \partial_\nu u_1 dS,$$

and using the fact that $u_1 = 0$ on $\partial\Omega$ and $v = 0$ on Γ_0, we have

$$\begin{aligned} \int_\Omega qu_0 v dx &= \int_\Omega u_1 \Delta v dx + \int_\Gamma v \partial_\nu u_1 dS + \int_\Omega (k^2 - ikb)u_1 v dx \\ &= -\int_\Omega u_1(-\Delta v - (k^2 - ikb)v)dx + \int_\Gamma \partial_\nu u_1 v dS = \int_\Gamma \partial_\nu u_1 v dS, \end{aligned}$$

which gives us the required integral identity

$$\int_\Omega qu_0 v dx = \int_\Gamma \partial_\nu u_1 v dS. \tag{3.3.8}$$

Inserting u_0 and v in the left hand side of the integral identity (3.3.8), we have

$$\int_\Omega qu_0 v dx = \int_\Omega q \left(e^{i\zeta_1 \cdot (x', x_n)} - e^{i\zeta_1 \cdot (x', -x_n)} \right) \left(e^{i\zeta_2 \cdot (x', x_n)} - e^{i\zeta_2 \cdot (x', -x_n)} \right) dx$$

$$= \int_\Omega q e^{i(\zeta_1+\zeta_2)\cdot(x',x_n)}dx + \int_\Omega q e^{i(\zeta_1+\zeta_2)\cdot(x',-x_n)}dx \\ - \int_\Omega q e^{i[\zeta_1\cdot(x',x_n)+\zeta_2\cdot(x',-x_n)]}dx - \int_\Omega q e^{i[\zeta_1\cdot(x',-x_n)+\zeta_2\cdot(x',x_n)]}dx.$$

From (3.3.1), we have

$$\begin{aligned}
&\zeta_1\cdot(x',x_n) + \zeta_2\cdot(x',-x_n)\\
&= \left(-\frac{\xi'}{2} + X\beta' + iY\beta' + i\tau\alpha', -\frac{\xi_n}{2} + X\beta_n + iY\beta_n + i\tau\alpha_n\right)\cdot(x',x_n)\\
&\quad + \left(-\frac{\xi'}{2} - X\beta' - iY\beta' - i\tau\alpha', -\frac{\xi_n}{2} - X\beta_n - iY\beta_n - i\tau\alpha_n\right)\cdot(x',-x_n)\\
&= (-\xi', 2(X+iY)\beta_n + 2i\tau\alpha_n)\cdot(x',x_n)\\
&= \left(-\xi', 2(X+iY)\frac{|\xi'|}{|\xi|}\right)\cdot(x',x_n) \quad (\text{ since } \beta_n = \frac{|\xi'|}{|\xi|} \text{ and } \alpha_n = 0)\\
&= \xi_+\cdot x, \quad \text{where } \xi_+ := \left(-\xi', 2(X+iY)\frac{|\xi'|}{|\xi|}\right).
\end{aligned}$$

Similarly, $\zeta_1\cdot(x',-x_n) + \zeta_2\cdot(x',x_n) = \xi_-\cdot x$, where $\xi_- := \left(-\xi', -2(X+iY)\frac{|\xi'|}{|\xi|}\right)$.

Using these and the fact $\zeta_1+\zeta_2 = -\xi$ gives

$$\int_\Omega q u_0 v dx = \int_\Omega q\left(e^{-i\xi\cdot(x',x_n)} + e^{-i\xi\cdot(x',-x_n)}\right)dx - \int_\Omega q\left(e^{i\xi_+\cdot x} + e^{i\xi_-\cdot x}\right)dx. \tag{3.3.9}$$

Now, the first term in the right-hand side of the equation (3.3.9)

$$\int_\Omega q\left[e^{-i\xi\cdot(x',x_n)} + e^{-i\xi\cdot(x',-x_n)}\right]dx = \int_\Omega q(x',x_n)e^{-i\xi\cdot x}dx + \int_{\Omega^*} q(x',-x_n)e^{-i\xi\cdot x}dx \\ = \int_{\widetilde{\Omega}} q_e(x)e^{-i\xi\cdot x}dx = \int_{\mathbb{R}^n}\tilde{q}_e(x)e^{-i\xi\cdot x}dx = \mathcal{F}[\widetilde{q_e}](\xi).$$

Using this and (3.3.9) in the integral identity (3.3.8), we have

$$|\mathcal{F}[\widetilde{q_e}](\xi)| \le \left|\int_\Gamma \partial_\nu u_1 v dS\right| + \left|\int_\Omega q e^{i\xi_+\cdot x}dx\right| + \left|\int_\Omega q e^{i\xi_-\cdot x}dx\right|. \tag{3.3.10}$$

Next, we choose $R > 1$ such that $\Omega \subset B_R := B(0,R)$. Note that, due to the symmetry, $\widetilde{\Omega} \subset B_R$.

We rewrite $\xi_\pm$ as follows:

$$\xi_+ = \left(-\xi', 2X\frac{|\xi'|}{|\xi|}\right) + i\left(0, 2Y\frac{|\xi'|}{|\xi|}\right), \quad \xi_- = \left(-\xi', -2X\frac{|\xi'|}{|\xi|}\right) + i\left(0, -2Y\frac{|\xi'|}{|\xi|}\right).$$

Then, we have $|\mathrm{Im}\,(\xi_\pm)| = 2|Y|\frac{|\xi'|}{|\xi|} \leq 2b$ whenever $|\xi|^2 \leq 3(k^2+\tau^2)$, and

$$\begin{aligned}
|\xi_+|^2 &= \left(-\xi', 2X\frac{|\xi'|}{|\xi|}\right)\cdot\left(-\xi', 2X\frac{|\xi'|}{|\xi|}\right) + \left(0, 2Y\frac{|\xi'|}{|\xi|}\right)\cdot\left(0, 2Y\frac{|\xi'|}{|\xi|}\right) \\
&= |\xi'|^2 + 4X^2\frac{|\xi'|^2}{|\xi|^2} + 4Y^2\frac{|\xi'|^2}{|\xi|^2} = |\xi'|^2 + 4(X^2+Y^2)\frac{|\xi'|^2}{|\xi|^2} \\
&= |\xi'|^2 + 4\left(\left(k^2+\tau^2-\frac{|\xi|^2}{4}\right)^2 + k^2b^2\right)^{\frac{1}{2}}\frac{|\xi'|^2}{|\xi|^2}. \\
&\geq |\xi'|^2 + 4\left(k^2+\tau^2-\frac{|\xi|^2}{4}\right)\frac{|\xi'|^2}{|\xi|^2} = |\xi'|^2 + 4(k^2+\tau^2)\frac{|\xi'|^2}{|\xi|^2} - |\xi'|^2 \\
&= 4(k^2+\tau^2)\frac{|\xi'|^2}{|\xi|^2} \geq (k^2+\tau^2)\frac{|\xi'|^2}{|\xi|^2}.
\end{aligned}$$

Similarly, $|\xi_-|^2 \geq (k^2+\tau^2)\frac{|\xi'|^2}{|\xi|^2}$.

Using the facts that $|\xi_\pm| \neq 0$, $\mathrm{supp}(q) \subset \Omega$ and integration by parts, we have

$$\begin{aligned}
\int_\Omega q(x)e^{i\xi_\pm\cdot x}dx &= \left(\int_\Omega q(x)e^{i\xi_\pm\cdot x}dx\right)\left(\frac{i\langle\xi_\pm,\xi_\pm\rangle}{i|\xi_\pm|^2}\right) \\
&= \frac{1}{i|\xi_\pm|^2}\left\langle\int_\Omega q(x)\left(ie^{i\xi_\pm\cdot x}\xi_\pm\right)dx\ ,\ \xi_\pm\right\rangle \\
&= \frac{1}{i|\xi_\pm|^2}\left\langle\int_\Omega q(x)\,\nabla\left(e^{i\xi_\pm\cdot x}\right)dx\ ,\ \xi_\pm\right\rangle \\
&= \frac{1}{i|\xi_\pm|^2}\left\langle-\int_\Omega \nabla q(x)\,e^{i\xi_\pm\cdot x}dx\ ,\ \xi_\pm\right\rangle.
\end{aligned}$$

This implies, using Cauchy-Schwarz inequality and the fact that $\|q\|_{C^1(\overline{\Omega})} \leq M$,

$$\begin{aligned}
\left|\int_\Omega q(x)e^{i\xi_\pm\cdot x}dx\right| &\leq \left|\frac{1}{i|\xi_\pm|^2}\left\langle-\int_\Omega \nabla q(x)\,e^{i\xi_\pm\cdot x}dx\ ,\ \xi_\pm\right\rangle\right| \\
&\leq \frac{1}{|\xi_\pm|^2}\left|\int_\Omega \nabla q(x)\,e^{i\xi_\pm\cdot x}dx\right|\ \ |\xi_\pm| \leq \frac{1}{|\xi_\pm|}\int_\Omega |\nabla q(x)|\left|e^{i\xi_\pm\cdot x}\right|dx \\
&\leq \frac{e^{R|\mathrm{Im}\,(\xi_\pm)|}}{(k^2+\tau^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}}\|q\|_{C^1(\overline{\Omega})}\left(\int_\Omega dx\right) \\
&\leq C\frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}},
\end{aligned}$$

where C depends on Ω, M. Using this in equation (3.3.10), we have

$$|\mathcal{F}[\widetilde{q}_e](\xi)| \leq \left|\int_\Gamma \partial_\nu u_1 v dS\right| + C\frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\frac{|\xi|}{|\xi'|}. \tag{3.3.11}$$

Next, using the trace theorem and the linearized partial D-N map (3.2.3), we have

$$\begin{aligned}\left|\int_\Gamma \partial_\nu u_1 v dS\right| &\le \|\partial_\nu u_1\|_{H^{-\frac{1}{2}}(\Gamma)}\|v\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} \le C\|\mathcal{N}^S_{q,L}(u_0)\|_{H^{-\frac{1}{2}}(\Gamma)}\|v\|_{H^{\frac{1}{2}}(\partial\Omega)}\\ &\le C\|\mathcal{N}^S_{q,L}\|_* \, \|u_0\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}\|v\|_{H^1(\Omega)} = C\|\mathcal{N}^S_{q,L}\|_* \, \|u_0\|_{H^{\frac{1}{2}}(\partial\Omega)}\|v\|_{H^1(\Omega)}\\ &\le C\|\mathcal{N}^S_{q,L}\|_* \, \|u_0\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},\end{aligned} \tag{3.3.12}$$

where

$$\begin{aligned}\|u_0\|_{H^1(\Omega)} \le \|e^{i\zeta_1\cdot(x',x_n)}\|_{H^1(\Omega)} + \|e^{i\zeta_1\cdot(x',-x_n)}\|_{H^1(\Omega)} \text{ and}\\ \|v\|_{H^1(\Omega)} \le \|e^{i\zeta_2\cdot(x',x_n)}\|_{H^1(\Omega)} + \|e^{i\zeta_2\cdot(x',-x_n)}\|_{H^1(\Omega)}.\end{aligned} \tag{3.3.13}$$

Now suppose w is a function of the form $w = e^{i\zeta\cdot x}$ with $|x| \le R$. Then we have $\nabla w = i\zeta e^{i\zeta\cdot x} = i\zeta w$ and $\|w\|^2_{L^2(\Omega)} = \int_\Omega \left|e^{i\zeta\cdot x}\right|^2 dx \le e^{2R|\text{Im}\,(\zeta)|}\left(\int_\Omega dx\right) = |\Omega|e^{2R|\text{Im}\,(\zeta)|}$, which implies

$$\begin{aligned}\|w\|_{H^1(\Omega)} &= \left(\|w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \le \left(|\Omega|e^{2R|\text{Im}\,(\zeta)|} + \|i\zeta w\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\\ &= \left(|\Omega|e^{2R|\text{Im}\,(\zeta)|} + |\zeta|^2\|w\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \le \left(|\Omega|e^{2R|\text{Im}\,(\zeta)|} + |\zeta|^2|\Omega|e^{2R|\text{Im}\,(\zeta)|}\right)^{\frac{1}{2}}\\ &= |\Omega|^{\frac{1}{2}}e^{R|\text{Im}\,(\zeta)|}(1+|\zeta|^2)^{\frac{1}{2}}.\end{aligned}$$

Using this for u_0 and v defined in (3.3.5) (and using the observation (3.3.13)), and the facts that $|\zeta_j|^2 \le 2((k^2+kb)^{\frac{1}{2}}+\tau)^2$, $|\text{Im}\,(\zeta_j)| \le b+\tau$ for $j = 1, 2$ and $1 < 2((k^2+kb)^{\frac{1}{2}}+\tau)^2$, we have

$$\begin{aligned}\|u_0\|_{H^1(\Omega)} &\le Ce^{R(b+\tau)}((k^2+kb)^{\frac{1}{2}}+\tau)\\ &\le Ce^{Rb}(k+\sqrt{kb})e^{R\tau} + Ce^{Rb}(k+\sqrt{kb})\tau e^{R\tau} \quad \text{(since } k > 1)\\ &\le Ce^{Rb}(k+\sqrt{kb})e^{2R\tau} + Ce^{Rb}(k+\sqrt{kb})e^{2R\tau} \quad (\text{since } R > 1, \text{ gives } \tau \le e^{R\tau})\\ &\le Ce^{Rb}(k+\sqrt{kb})e^{2R\tau},\end{aligned}$$

and

$$\|v\|_{H^1(\Omega)} \le Ce^{Rb}(k+\sqrt{kb})e^{2R\tau}.$$

Using these estimates in (3.3.11) and (3.3.12), we get

$$|\mathcal{F}[\widetilde{q}_e](\xi)| \leq Ce^{2Rb}(k+\sqrt{kb})^2 e^{4R\tau}\|\mathcal{N}^S_{q,L}\|_* + C\frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\frac{|\xi|}{|\xi'|}, \tag{3.3.14}$$

for $0 < |\xi| \leq 3(k^2+\tau^2)$ with $|\xi'| > 0$. Also, note that the set $\{\xi = (\xi', \xi_n) \in \mathbb{R}^n \ : \ |\xi'| = 0\}$ has measure zero. So, the assumption $|\xi'| > 0$ will not affect the derivations that will follow. Let $\rho > 1$ (which we will choose later) and define

$$\mathcal{Z}_\rho := \{(x', x_n) \in \mathbb{R}^n \ : \ 0 < |x'| < \rho \text{ and } |x_n| < \rho\}\,.$$

Then, we can write

$$\|\tilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{\mathcal{Z}_\rho} \left(1+|\xi|^2\right)^{-1} |\mathcal{F}[\widetilde{q}_e](\xi)|^2\, d\xi + \int_{\mathbb{R}^n\setminus\mathcal{Z}_\rho} \left(1+|\xi|^2\right)^{-1} |\mathcal{F}[\widetilde{q}_e](\xi)|^2\, d\xi. \tag{3.3.15}$$

By Parseval's identity, we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathcal{Z}_\rho} \left(1+|\xi|^2\right)^{-1} |\mathcal{F}[\widetilde{q}_e](\xi)|^2\, d\xi &\leq \int_{|\xi|>\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{|\xi|^2} d\xi \leq \frac{1}{\rho^2}\int_{\mathbb{R}^n} |\mathcal{F}[\widetilde{q}_e](\xi)|^2\, d\xi \\ &\leq \frac{1}{\rho^2}\int_{\mathbb{R}^n} |\tilde{q}_e(x)|^2\, dx = \frac{1}{\rho^2}\int_{\widetilde{\Omega}} |q_e(x)|^2\, dx \\ &= \frac{1}{\rho^2}\left[\int_\Omega |q(x)|^2\, dx + \int_{\Omega^*} |q(x', -x_n)|^2\, dx\right].\end{aligned}$$

Applying the transformation $(x', x_n) \mapsto (x', -x_n)$ to the second term in the right-hand side, and using the fact that $|det(J)| = 1$ (where J denotes the Jacobian of the transformation), we get

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathcal{Z}_\rho} \left(1+|\xi|^2\right)^{-1} |\mathcal{F}[\widetilde{q}_e](\xi)|^2\, d\xi &\leq \frac{1}{\rho^2}\left[\int_\Omega |q(x)|^2\, dx + \int_\Omega |q(x)|^2\, dx\right] \\ &= \frac{2}{\rho^2}\|q\|^2_{L^2(\Omega)} \leq \frac{2|\Omega|M^2}{\rho^2} \leq \frac{C}{\rho^2}.\end{aligned} \tag{3.3.16}$$

Let us assume that the inequality (3.3.14) holds true if $\xi \in \mathcal{Z}_\rho$. Then, since $1+|\xi|^2 > 1$, we can write

$$\int_{\mathcal{Z}_\rho} \left(1+|\xi|^2\right)^{-1} |\mathcal{F}[\widetilde{q}_e](\xi)|^2\, d\xi \leq \int_{\mathcal{Z}_\rho} \Bigg| Ce^{2Rb}(k+\sqrt{kb})^2 e^{4R\tau}\|\mathcal{N}^S_{q,L}\|_*$$

$$+ C\frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\frac{|\xi|}{|\xi'|}\Bigg|^2 d\xi$$

$$\leq Ce^{4Rb}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\left(\int_{\mathcal{Z}_\rho} d\xi\right) + C\frac{e^{4Rb}}{k^2+\tau^2}\int_{\mathcal{Z}_\rho}\frac{|\xi'|^2+|\xi_n|^2}{|\xi'|^2}d\xi.$$

Using the inequality $c^2+d^2 \leq (c+d)^2 = 4cd+(c-d)^2 \leq 4cd + c^2$ for $c,\ d \geq 0$, with $c = |\xi'|$ and $d = |\xi_n|$, we further have

$$\int_{\mathcal{Z}_\rho}\left(1+|\xi|^2\right)^{-1}|\mathcal{F}[\widetilde{q}_e](\xi)|^2\,d\xi$$

$$\leq Ce^{4Rb}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\int_{-\rho}^{\rho}\int_{B'(0,\rho)} d\xi' d\xi_n$$

$$+ C\frac{e^{4Rb}}{k^2+\tau^2}\int_{-\rho}^{\rho}\int_{B'(0,\rho)}\frac{4|\xi'||\xi_n|+|\xi'|^2}{|\xi'|^2}d\xi' d\xi_n$$

(where $B'(0,\rho) \subset \mathbb{R}^{n-1}$)

$$\leq Ce^{4Rb}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\rho\int_{|\theta|=1}\int_0^\rho r^{n-2}drd\theta$$

$$+ C\frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\rho\int_{|\theta|=1}\int_0^\rho\left(\frac{4r\rho}{r^2}+\frac{r^2}{r^2}\right)r^{n-2}drd\theta$$

$$= Ce^{4Rb}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\rho\int_{|\theta|=1}\frac{\rho^{n-1}}{n-1}d\theta$$

$$+ C\frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\rho\int_{|\theta|=1}\int_0^\rho\left(4\rho r^{n-3}+r^{n-2}\right)drd\theta$$

$$\leq Ce^{4Rb}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\rho^n\left(\int_{|\theta|=1} d\theta\right)$$

$$+ C\frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\rho\int_{|\theta|=1}\left(4\rho\frac{\rho^{n-2}}{n-2}+\frac{\rho^{n-1}}{n-1}\right)d\theta$$

$$\leq Ce^{4Rb}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\rho^n + C\frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\rho(5\rho^{n-1})\left(\int_{|\theta|=1} d\theta\right).$$

That is

$$\int_{\mathcal{Z}_\rho}\left(1+|\xi|^2\right)^{-1}|\mathcal{F}[\widetilde{q}_e](\xi)|^2\,d\xi \leq Ce^{4Rb}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\rho^n + C\frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\rho^n,$$

where the constant C depends on n, Ω and M. Using this and (3.3.16) in (3.3.15), we have

$$\|\tilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \leq Ce^{4Rb}\left[(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|_*^2\rho^n + \frac{\rho^n}{(k^2+\tau^2)^{\frac{1}{2}}} + \frac{1}{\rho^2}\right],$$

where we have also used the fact that $1 \le e^{4Rb}$.

Now, we choose ρ in such a way that the last two terms in the above estimate becomes comparable. This leads us to the choice $\rho = (k^2+\tau^2)^{\frac{1}{2(n+2)}} > 1$. Then for $\tau > 1$, we have

$$\begin{aligned}
\|\tilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\le Ce^{4Rb}\Bigg[(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|^2_*(k^2+\tau^2)^{\frac{n}{2(n+2)}} \\
&\qquad + \frac{(k^2+\tau^2)^{\frac{n}{2(n+2)}}}{(k^2+\tau^2)^{\frac{1}{2}}} + \frac{1}{(k^2+\tau^2)^{\frac{1}{n+2}}}\Bigg] \\
&\le Ce^{4Rb}\Bigg[(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|^2_*(k^2\tau^2+k^2\tau^2)^{\frac{1}{2}} \\
&\qquad + \frac{1}{(k^2+\tau^2)^{\frac{1}{2}-\frac{n}{2(n+2)}}} + \frac{1}{(k^2+\tau^2)^{\frac{1}{n+2}}}\Bigg] \\
&= Ce^{4Rb}\left[\sqrt{2}(k+\sqrt{kb})^4 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|^2_* k\tau + \frac{1}{(k^2+\tau^2)^{\frac{1}{n+2}}} + \frac{1}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right] \\
&\le Ce^{4Rb}\left[2(k+\sqrt{kb})^5 e^{8R\tau}\|\mathcal{N}^S_{q,L}\|^2_* e^{R\tau} + \frac{2}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right] \\
&\le Ce^{4Rb}\left[(k+\sqrt{kb})^5\|\mathcal{N}^S_{q,L}\|^2_* e^{9R\tau} + \frac{1}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right].
\end{aligned}$$

Let us choose $\tau = \frac{E}{9R}$ ($E = |\ln(\|\mathcal{N}^S_{q,L}\|_*)|$). Then for $\|\mathcal{N}^S_{q,L}\|_* < \frac{1}{e^{9R}}$, we have $\tau > 1$ and

$$\begin{aligned}
\|\tilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\le Ce^{4Rb}\left[(k+\sqrt{kb})^5\|\mathcal{N}^S_{q,L}\|^2_* e^{9R\frac{E}{9R}} + \frac{1}{\left(k^2+\left(\frac{E}{9R}\right)^2\right)^{\frac{1}{n+2}}}\right] \\
&= Ce^{4Rb}\left[(k+\sqrt{kb})^5\|\mathcal{N}^S_{q,L}\|^2_* e^{E} + \frac{1}{\left(k^2+\left(\frac{E}{9R}\right)^2\right)^{\frac{1}{n+2}}}\right] \\
&= Ce^{4Rb}\left[(k+\sqrt{kb})^5\|\mathcal{N}^S_{q,L}\|_* + \frac{1}{\left(k^2+\left(\frac{E}{9R}\right)^2\right)^{\frac{1}{n+2}}}\right],
\end{aligned}$$

which implies

$$\|\tilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} \le C\left[(k+\sqrt{kb})^5 e^{4Rb}\|\mathcal{N}^S_{q,L}\|_* + \frac{e^{4Rb}}{\left(k^2+\left(\frac{E}{9R}\right)^2\right)^{\frac{1}{n+2}}}\right]^{\frac{1}{2}}. \tag{3.3.17}$$

Note that for $\xi \in \mathcal{Z}_\rho$, that is, $0 < |\xi'| < \rho,\ |\xi_n| < \rho$, and $\frac{1}{n+2} < 1$, we have

$$\frac{|\xi|^2}{3(k^2+\tau^2)} \leq \frac{2\rho^2}{3(k^2+\tau^2)} = \frac{2}{3}\frac{(k^2+\tau^2)^{\frac{1}{n+2}}}{(k^2+\tau^2)} = \frac{2}{3}\frac{1}{(k^2+\tau^2)^{1-\frac{1}{n+2}}} < 1.$$

Hence $|\xi|^2 \leq 3(k^2+\tau^2)$, and thus the estimate (3.3.14) is valid for our choice of ρ. Therefore, from (3.3.17), we have the estimate

$$\|q\|_{H^{-1}(\Omega)} \leq C\left[(k+\sqrt{kb})^5 e^{4Rb}\|\mathcal{N}_{q,L}^S\|_* + \frac{e^{4Rb}}{\left(k^2+\left(\frac{E}{9R}\right)^2\right)^{\frac{1}{n+2}}}\right]^{\frac{1}{2}}. \tag{3.3.18}$$

Next, by using the estimate (3.3.18) and the interpolation theorem, we are able to estimate the L^2 norm for q as follows.

For given l_1, l_2, l such that $l_1 < l_2$ and $l = (1-s)l_1 + sl_2$, where $0 < s < 1$, by interpolation theorem, we have

$$\|q\|_{H^l} \leq \|q\|_{H^{l_1}}^{1-s}\|q\|_{H^{l_2}}^{s}.$$

Choosing $l_1 = -1$, $l_2 = 1$ and $l = 0$, we can write $l = (1-s)l_1 + sl_2$ for $s = \frac{1}{2}$. Now, using the interpolation theorem, we have

$$\|q\|_{L^2(\Omega)} = \|q\|_{H^0(\Omega)} \leq \|q\|_{H^{-1}(\Omega)}^{1-\frac{1}{2}}\|q\|_{H^1(\Omega)}^{\frac{1}{2}} \leq \|q\|_{H^{-1}(\Omega)}^{\frac{1}{2}}(2|\Omega|M^2)^{\frac{1}{4}},$$

which implies, using (3.3.18), that

$$\|q\|_{L^2(\Omega)} \leq C\left[(k+\sqrt{kb})^5 e^{4Rb}\|\mathcal{N}_{q,L}^S\|_* + \frac{e^{4Rb}}{\left(k^2+\left(\frac{E}{9R}\right)^2\right)^{\frac{1}{n+2}}}\right]^{\frac{1}{4}}.$$

This gives us the estimate (3.2.5), and the proof is complete.

3.4 Appendix

Proof of the Theorem 3.2.1

Proof. Let $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$, and from problems (3.1.1) and (3.2.1), we see that

$$\begin{aligned}0 &= (-\Delta - (k^2 - ikb) + q)\mathcal{P}_{(-(k^2-ikb)+q)}f - (-\Delta - (k^2-ikb))\mathcal{P}_{(-(k^2-ikb))}f \\ &= (-\Delta - (k^2-ikb))(\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f) + \mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)+q)}f.\end{aligned}$$

It gives,

$$\begin{cases}(-\Delta-(k^2-ikb))(\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f) & =-\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)+q)}f \ \text{ in } \Omega,\\ \mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f & =0 \quad \text{ on } \ \partial\Omega.\end{cases}$$

Using the uniqueness of the problem (3.2.2), we have

$$\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f=\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)+q)}f \ \text{ in } \ \Omega. \tag{3.4.1}$$

Now, we have

$$\begin{aligned}&\left[\mathcal{N}_\Gamma(-(k^2-ikb)+q)-\mathcal{N}_\Gamma(-(k^2-ikb))-\mathcal{N}_\Gamma^1(q)\right](f)\\ &\qquad=\partial_\nu\left(\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f-\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f\right)\big|_\Gamma\\ &\qquad=\partial_\nu\left(\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f\right)\big|_\Gamma\\ &\qquad=\partial_\nu\left(\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f)\right)\big|_\Gamma:=\partial_\nu w|_\Gamma,\end{aligned}$$

where $w=\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f)$.

Note that w satisfies the equation

$$(-\Delta-(k^2-ikb))w=-q(\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f) \ \text{ in } \ \Omega.$$

For any $\phi\in\widetilde{H}^{\frac{1}{2}}(\Gamma)$, by trace theorem, there exists a $\Phi\in H^1(\Omega)$ such that

$$\Phi\big|_{\partial\Omega}=\phi \ \text{ and } \ \|\Phi\|_{H^1(\Omega)}\le c\|\phi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}. \tag{3.4.2}$$

Applying the function ϕ to $\partial_\nu w$ (weak sense), we obtain

$$\begin{aligned}\int_\Gamma\partial_\nu w\phi dS=\int_\Omega\nabla w\cdot\nabla\Phi dx-\int_\Omega(k^2-ikb)w\Phi dx\\ +\int_\Omega q(\mathcal{P}_{(-(k^2-ikb)+q)}f-\mathcal{P}_{(-(k^2-ikb))}f)\Phi dx.\end{aligned}$$

Taking the absolute value of both sides of the equation above and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left|\int_\Gamma \partial_\nu w\phi dS\right| &\leq \left|\int_\Omega \nabla w\cdot\nabla\Phi dx\right| + \left|\int_\Omega (k^2-ikb)w\Phi dx\right| \\
&\qquad + \left|\int_\Omega q(\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f)\Phi dx\right| \\
&\leq \|\nabla w\|_{L^2(\Omega)}\|\nabla\Phi\|_{L^2(\Omega)} + |k^2-ikb|\|w\|_{L^2(\Omega)}\|\Phi\|_{L^2(\Omega)} \\
&\qquad + \|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f\|_{L^2(\Omega)}\|\Phi\|_{L^2(\Omega)} \\
&\leq \|w\|_{H^1(\Omega)}\|\Phi\|_{H^1(\Omega)} + (k^4+k^2b^2)^{\frac{1}{2}}\|w\|_{H^1(\Omega)}\|\Phi\|_{H^1(\Omega)} \\
&\qquad + \|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f\|_{L^2(\Omega)}\|\Phi\|_{H^1(\Omega)} \\
&\leq (1+(k^4+k^2b^2)^{\frac{1}{2}})\|w\|_{H^1(\Omega)}\|\Phi\|_{H^1(\Omega)} \\
&\qquad + \|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f\|_{L^2(\Omega)}\|\Phi\|_{H^1(\Omega)} \\
&\leq \left(C\|w\|_{H^1(\Omega)} + \|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f\|_{L^2(\Omega)}\right)\|\Phi\|_{H^1(\Omega)}.
\end{aligned} \tag{3.4.3}$$

Let us assume that

$$\|q\|_{L^\infty(\Omega)} < \frac{1}{2}\|\mathcal{G}_{(-(k^2-ikb))}\|^{-1}_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}$$

then we have

$$\|\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))} \leq \|q\|_{L^\infty(\Omega)}\|\mathcal{G}_{(-(k^2-ikb))}\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))} < \frac{1}{2} < 1,$$

which implies that $I-\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q$ is invertible. Also, from the following equation

$$\begin{aligned}
&(I-\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q)(\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f) \\
&\qquad = \mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f - \mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)+q)}f \\
&\qquad\qquad + \mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f \\
&\qquad = \mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f \quad \text{(by (3.4.1))},
\end{aligned}$$

we get

$$\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f = (I-\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q)^{-1}(\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f).$$

Now, we calculate that

$$
\begin{aligned}
&\|\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f\|_{L^2(\Omega)} \\
&\quad \le \|(I-\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q)^{-1}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f\|_{H^1(\Omega)} \\
&\quad = \frac{1}{1-\|\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}} \\
&\qquad\qquad \|\mathcal{G}_{(-(k^2-ikb))}\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}\|\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb))}f\|_{L^2(\Omega)} \\
&\quad \le 2\|\mathcal{G}_{(-(k^2-ikb))}\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}\|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb))}f\|_{H^1(\Omega)} \\
&\quad \le 2\|\mathcal{G}_{(-(k^2-ikb))}\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}\|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb))}\|_{\mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma),H^1(\Omega))}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} \\
&\quad \le C\|q\|_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}.
\end{aligned} \tag{3.4.4}
$$

Using the above estimate, we get

$$
\begin{aligned}
\|w\|_{H^1(\Omega)} &= \|\mathcal{G}_{(-(k^2-ikb))}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f)\|_{H^1(\Omega)} \\
&\le \|\mathcal{G}_{(-(k^2-ikb))}\|_{\mathcal{L}(L^2(\Omega),H^1(\Omega))}\|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)+q)}f - \mathcal{P}_{(-(k^2-ikb))}f\|_{L^2(\Omega)} \\
&\le C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)},
\end{aligned} \tag{3.4.5}
$$

where C depends on n, Ω, Γ, k and b.

Inserting the estimates (3.4.2), (3.4.4) and (3.4.5) to estimate (3.4.3), we have

$$
\begin{aligned}
\left|\int_\Gamma \partial_\nu w\phi dS\right| &\le (C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} + C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)})c\|\phi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} \\
&\le C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}\|\phi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}.
\end{aligned}
$$

It gives,

$$
\begin{aligned}
\|\partial_\nu w\|_{H^{-\frac{1}{2}}(\Gamma)} &= \sup_{\|\phi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}=1}\left|\int_\Gamma \partial_\nu w\phi dS\right| \le \sup_{\|\phi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}=1} C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} \\
&= C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}.
\end{aligned}
$$

Now, the norm

$$
\begin{aligned}
&\|\mathcal{N}_\Gamma(-(k^2-ikb)+q)-\mathcal{N}_\Gamma(-(k^2-ikb))-\mathcal{N}_\Gamma^1(q)\|_{\mathcal{L}(\widetilde{H}^{\frac{1}{2}}(\Gamma),H^{-\frac{1}{2}}(\Gamma))}\\
&\qquad=\sup_{\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}\neq 0}\frac{\|(\mathcal{N}_\Gamma(-(k^2-ikb)+q)-\mathcal{N}_\Gamma(-(k^2-ikb))-\mathcal{N}_\Gamma^1(q))(f)\|_{H^{-\frac{1}{2}}(\Gamma)}}{\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}}\\
&\qquad=\sup_{\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}\neq 0}\frac{\|\partial_\nu w\|_{H^{-\frac{1}{2}}(\Gamma)}}{\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}}\leq\sup_{\|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)}\neq 0}C\|q\|^2_{L^\infty(\Omega)}\\
&\qquad=C\|q\|^2_{L^\infty(\Omega)}.
\end{aligned}
$$

Hence proved. □

Chapter 4

Linearized inverse biharmonic problem with attenuation

In this chapter, based on the work [12], high-frequency stability estimates are explored for the determination of the zeroth-order perturbation of the biharmonic operator with constant attenuation from the linearized partial Dirichlet-to-Neumann map when part of the boundary is inaccessible and flat. The results obtained suggest improvement of the stability with an appropriate choice of frequency.

4.1 Introduction

Let us consider the following boundary-value problem for the biharmonic operator with constant attenuation posed in a bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 3$ with a smooth boundary $\partial\Omega$:

$$\begin{cases} \Delta^2 u \ -\ (k^2 - ikb)^2 u \ +\ qu &= 0 \ \text{ in } \Omega, \\ \qquad\qquad\qquad u &= f \ \text{ on } \partial\Omega, \\ \qquad\qquad\quad \Delta u &= g \ \text{ on } \partial\Omega. \end{cases} \tag{4.1.1}$$

Here $b > 0$ is the constant attenuation and q is a real valued function such that $q \in C^1(\overline{\Omega})$ with $\text{supp}(q) \subset \Omega$ and $\|q\|_{C^1(\overline{\Omega})} \leq M$ for some $M > 0$. We shall assume that the frequency $k > 1$ and the boundary data $f \in H^{\frac{7}{2}}(\partial\Omega)$ and $g \in H^{\frac{3}{2}}(\partial\Omega)$. The boundary conditions that we consider in (4.1.1) are known as the Navier boundary conditions (see [15]).

We further assume that the bounded domain Ω satisfies the condition $(\mathcal{A})$ (see section 2.1). We also assume that the supports of f and g are contained in $\Gamma := \partial\Omega \setminus \Gamma_0$ (note that

Γ is an open subset of the boundary $\partial\Omega$) and the boundary measurements $\partial_\nu u$ and $\partial_\nu(\Delta u)$ are available on Γ only, and thus, the flat part Γ_0 is assumed to be inaccessible.

As in the previous chapter, we work with the Sobolev spaces $\widetilde{H}^s(\Gamma)$ and $H^s(\Gamma)$, $s \geq 0$.

Note that the imaginary part of $(k^2 - ikb)^2$ is non-zero and hence, it is not a part of the spectrum of $\Delta^2 + q$. Therefore, there exists a unique solution to (4.1.1) when $(f, g) \in \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma)$.

In this case, the partial Dirichlet-to-Neumann (D-N) map is defined as

$$\mathcal{N}_q^B \;:\; \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma) \to H^{\frac{5}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \quad \text{such that} \quad (f, g) \mapsto \left(\partial_\nu u\Big|_\Gamma, \partial_\nu(\Delta u)\Big|_\Gamma\right).$$

Here ν denotes the exterior unit normal vector to $\partial\Omega$ and u is the solution to the problem

$$\begin{cases} \Delta^2 u \;-\; (k^2 - ikb)^2 u \;+\; qu & = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad u & = f \quad \text{on } \Gamma, \\ \qquad\qquad\qquad \Delta u & = g \quad \text{on } \Gamma. \end{cases} \tag{4.1.2}$$

On the spaces $\widetilde{H}^\alpha(\Gamma) \times \widetilde{H}^\beta(\Gamma)$ and $H^\alpha(\Gamma) \times H^\beta(\Gamma)$ (which we shall denote as $\widetilde{H}^{\alpha,\beta}(\Gamma)$ and $H^{\alpha,\beta}(\Gamma)$ respectively), we consider the norms

$$\|(f,g)\|_{\widetilde{H}^{\alpha,\beta}(\Gamma)} := \|f\|_{\widetilde{H}^\alpha(\Gamma)} + \|g\|_{\widetilde{H}^\beta(\Gamma)}, \quad \|(f,g)\|_{H^{\alpha,\beta}(\Gamma)} := \|f\|_{H^\alpha(\Gamma)} + \|g\|_{H^\beta(\Gamma)},$$

and define

$$\|\mathcal{N}_q^B\|_* := \sup\{\|\mathcal{N}_q^B(f,g)\|_{H^{\frac{5}{2},\frac{1}{2}}(\Gamma)} \;:\; \|(f,g)\|_{\widetilde{H}^{\frac{7}{2},\frac{3}{2}}(\Gamma)} = 1\}.$$

Our aim here is to address the question of stability of the recovery of the potential q from the knowledge of a linearized partial D-N map that we shall describe in the next section (also see [25,38,39]) and to study the dependence of the stability estimate on the frequency k.

For works related to the uniqueness question of determination of the potential from the Dirichlet-to-Neumann map for the biharmonic operator, we refer to the works [19,28,29,37] and for stability results, in the case when $k = 0, b = 0$, we refer to the works [8,10]. In [10], stability estimates were studied in the case of boundary measurements on the whole boundary and slightly more than half of the boundary. In [8], stability estimates were studied for domains satisfying condition $(\mathcal{A})$. In the case $b = 0$, the work [33] studies the biharmonic problem from the perspective of increasing stability in domains satisfying the assumption $(\mathcal{A})$.

In the presence of constant attenuation, the work [38] studied the linearized inverse problem for the biharmonic operator with constant attenuation in the full data case. In this work, we study the linearized inverse problem for the biharmonic operator with constant attenuation in domains satisfying the assumption $(\mathcal{A})$. The results obtained here suggest the improvement of the stability in the recovery of the potential q with an appropriate choice of frequency.

In Section 4.2, we introduce the linearization for the biharmonic operator with attenuation and state our main result on stability. The construction of appropriate complex exponential solutions and the proof of the stability estimate are discussed in Section 4.3.

4.2 Linearization and the main result

We consider the linearization of (4.1.1) around a zero potential function (see [25, 38, 39]). Such a linearization can be justified when the potential q is small compared to the frequency k.

In this direction, we consider the equations

$$\begin{cases} \Delta^2 u - (k^2 - ikb)^2 u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \Gamma, \\ \Delta u &= g \quad \text{on } \Gamma, \end{cases} \tag{4.2.1}$$

and

$$\begin{cases} \Delta^2 u - (k^2 - ikb)^2 u &= -q u_0 \quad \text{in } \Omega, \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.2.2}$$

where u_0 denotes the solution to (4.2.1). Now, if we denote the solution to (4.2.2) by u_1, then the solution u to (4.1.1) can be written as

$$u = u_0 + u_1 + \text{ higher order terms.}$$

Note that the imaginary part of $(k^2 - ikb)^2$ is non-zero and hence, it is not a part of the spectrum of Δ^2. Therefore, there exist unique solutions to (4.2.1) and (4.2.2) when $(f, g) \in \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma)$.

Using this, we define the linearized partial D-N map $\mathcal{N}^B_{q,L}$ as

$$\mathcal{N}^B_{q,L} \;:\; \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma) \to H^{\frac{5}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \quad \text{such that} \quad (f, g) \mapsto \left(\partial_\nu u_1\Big|_\Gamma, \partial_\nu(\Delta u_1)\Big|_\Gamma\right).$$

Let us denote $E := |\log(\|\mathcal{N}^B_{q,L}\|_*)|$. Then we have the following stability result for the recovery of the potential q from $\mathcal{N}^B_{q,L}$.

Theorem 4.2.1. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$ satisfying the assumption $(\mathcal{A})$ described above and $\Omega \subset B(0, R)$ for some $R > 1$. Let b be a non-negative constant and the frequency $k > 1$. Further, suppose that the potential $q \in C^1(\overline{\Omega})$ with $\operatorname{supp}(q) \subset \Omega$ and $\|q\|_{C^1(\overline{\Omega})} \leq M$ for some $M > 0$. Then there exists a constant $C > 0$, depending only on n, Ω and M such that*

$$\|q\|_{L^2(\Omega)} \leq C \left[(k^2 + kb)^9 e^{4Rb} \|\mathcal{N}^B_{q,L}\|_* + \frac{e^{4Rb}}{\left(k^2 + \left(\frac{E}{21R}\right)^2\right)^{\frac{1}{n+2}}} \right]^{\frac{1}{4}}. \tag{4.2.3}$$

Remark 4.2.2. The estimate (4.2.3) suggests an improvement of stability if an appropriate choice of the frequency is made. To see this, let us consider the case when $b \leq 1$. The case when $b > 1$ can be dealt similarly.

Let us denote $\epsilon := \|\mathcal{N}^B_{q,L}\|_*$. Then, since $b \leq 1$, we have $e^{4Rb} \leq e^{4R}$, $kb \leq k^2$ and using this in the estimate (4.2.3), we obtain

$$\begin{aligned}\|q\|_{L^2(\Omega)} &\le C\left[(k^2+k^2)^9 e^{4R}\epsilon + \frac{e^{4R}}{\left(k^2+\left(\frac{E}{21R}\right)^2\right)^{\frac{1}{n+2}}}\right]^{\frac{1}{4}}\\ &\le Ce^R\left[k^{18}\epsilon + \frac{1}{k^{\frac{2}{n+2}}}\right]^{\frac{1}{4}}.\end{aligned}$$

To make the discussion simpler, let us also choose $R = 2$ and $n = 3$. Then given $\epsilon < 1$, depending on the frequency k used, the possible error in estimating q can be as large as

$$Ce^2\left[k^{18}\epsilon + \frac{1}{\left(k^2+\left(\frac{E}{42}\right)^2\right)^{\frac{1}{5}}}\right]^{\frac{1}{4}}. \tag{4.2.4}$$

Now, let us choose $\tilde{k}$ such that $\tilde{k}^{18}\epsilon = \tilde{k}^{-\frac{2}{n+2}}$, that is, $\tilde{k} = \epsilon^{-\frac{n+2}{18n+38}} = \epsilon^{-\frac{5}{92}}$. Note that since $\epsilon < 1$, we have $\tilde{k} = \epsilon^{-\frac{5}{92}} > 1$ and with this choice of frequency $\tilde{k}$, the maximum possible error in estimating q is

$$\begin{aligned}\|q\|_{L^2(\Omega)} &\le Ce^2\left(\epsilon^{\frac{n+2}{18n+38}}\right)^{\frac{1}{2(n+2)}}\\ &\le Ce^2\epsilon^{\frac{1}{2(18n+38)}} = Ce^2\epsilon^{\frac{1}{184}}.\end{aligned}$$

For small ϵ, this error (corresponding to the choice $\tilde{k}$) can be seen to be less than that in (4.2.4) for other choices of the frequency k. Thus, this choice of the frequency minimises the error in the estimation of q.

4.3 Complex exponential solutions and the stability estimate

In this section, we give a proof for the stability estimate (4.2.3). In order to do so, we first construct appropriate complex exponential solutions and then use them and an integral identity to prove Theorem 4.2.1.

4.3.1 Complex exponential solutions

Given a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

For a given $\xi := (\xi', \xi_n) \in \mathbb{R}^n$, where $\xi' \neq 0$, we choose unit vectors α and β in an appropriate way and use these vectors to construct the complex exponential solutions.

To begin with, we define an orthonormal basis of $\mathbb{R}^n$ in the following manner: Let $e_1 = (\frac{\xi'}{|\xi'|}, 0)$ and $e_n = (0, ..., 0, 1)$. Let $e_2, ..., e_{n-1} \in \mathbb{R}^n$ be such that the n^{th} component $e_{i,n} = 0$ for $i = 2, ..., n-1$ and the set $\{e_1, e_2, ..., e_n\}$ forms an orthonormal basis of $\mathbb{R}^n$.

The coordinate representation of the vector ξ with respect to this basis is

$$\xi_e = \left(\frac{\xi' \cdot \xi'}{|\xi'|}, 0, ..., 0, \xi_n\right)_e.$$

Let us choose unit vectors $\alpha, \beta \in \mathbb{R}^n$ in such a way that with respect to this new basis, α and β have the representations

$$\alpha_e = (0, 1, 0, ..., 0)_e, \;\; \beta_e = \Big(-\frac{\xi_n}{|\xi|}, 0, ..., 0, \frac{|\xi'|}{|\xi|}\Big)_e.$$

Then it is easy to see that $\{\xi, \alpha, \beta\}$ forms an orthogonal set. Also, since this coordinate change preserves the scalar product and the n-th coordinate, it follows that

$$\alpha \cdot \beta = \alpha_e \cdot \beta_e, \;\; \alpha_n = \alpha_{e,n} \text{ and } \beta_n = \beta_{e,n}.$$

Using these vectors, let us define

$$\begin{aligned} \zeta_1 &= -\frac{\xi}{2} + \Big(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\Big)^{\frac{1}{2}}\beta + i\tau\alpha, \\ \zeta_2 &= -\frac{\xi}{2} - \Big(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\Big)^{\frac{1}{2}}\beta - i\tau\alpha, \end{aligned} \tag{4.3.1}$$

where $k > 1$ and $\tau > 1$. Then

$$\zeta_1 + \zeta_2 = -\xi, \quad \zeta_j \cdot \zeta_j = k^2 - ikb, \; i = 1, 2. \tag{4.3.2}$$

Next we assume that $|\xi|^2 \leq 3(k^2 + \tau^2)$, and denote the principal square root of $k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb$ by

$$X + iY := \Big(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\Big)^{\frac{1}{2}},$$

where $X > 0$. Using this notation, we rewrite ζ_1 and ζ_2 as

$$\zeta_1 = -\frac{\xi}{2} + X\beta + i(Y\beta + \tau\alpha), \quad \zeta_2 = -\frac{\xi}{2} - X\beta - i(Y\beta + \tau\alpha).$$

Therefore,

$$|\zeta_j|^2 = \frac{|\xi|^2}{4} + X^2 + Y^2 + \tau^2, \quad j = 1, 2,$$

where

$$X^2 + Y^2 = |X + iY|^2 = \left(\left(k^2 + \tau^2 - \frac{|\xi|^2}{4} \right)^2 + k^2 b^2 \right)^{\frac{1}{2}}. \tag{4.3.3}$$

Then

$$|\zeta_j|^2 \le k^2 + kb + 2\tau^2 \le 2((k^2 + kb)^{\frac{1}{2}} + \tau)^2, \tag{4.3.4}$$

where we have used the inequality $(c^2 + d^2)^{\frac{1}{2}} \le (c + d),\ c, d \ge 0$ with $c = k^2 + \tau^2 - \frac{|\xi|^2}{4}$ and $d = kb$.

Also, following [31], we can show that $|\text{Im}\,(\zeta_j)|^2 = Y^2 + \tau^2,\ j = 1, 2$, and $|Y| \le b$ (this is where the condition $|\xi|^2 \le 3(k^2 + \tau^2)$ is used and also to show that $X > 0$), which gives us

$$|\text{Im}\,(\zeta_j)|^2 \le b^2 + \tau^2 \le (b + \tau)^2 \quad \text{for } j = 1, 2. \tag{4.3.5}$$

Next, we construct the complex exponential solutions following a reflection argument, as in [21].

Let us denote $\widetilde{\Omega} := \Omega \cup \Omega^*$, where $\Omega^* := \{(x', x_n) \in \mathbb{R}^n : (x', -x_n) \in \Omega\}$ is the reflection of Ω by $\{x_n = 0\}$. Also, let q_e denote the extension of q to $\widetilde{\Omega}$ by reflection by $\{x_n = 0\}$, that is,

$$q_e(x) = \begin{cases} q(x', x_n), & \text{if } (x', x_n) \in \Omega, \\ q(x', -x_n), & \text{if } (x', x_n) \in \Omega^*. \end{cases}$$

We further extend q_e to $\mathbb{R}^n$ by defining it to be zero outside $\widetilde{\Omega}$ and denote this extension by $\tilde{q}_e$.

Further, we choose $R > 1$ such that $\Omega \subset B_R := B(0, R)$ (by the symmetry of the domain, this also implies that $\widetilde{\Omega} \subset B_R$). Now we consider the equations

$$\Delta^2 \tilde{u}_0 - (k^2 - ikb)^2 \tilde{u}_0 = 0 \quad \text{in } \widetilde{\Omega}, \tag{4.3.6}$$

and

$$\Delta^2 \tilde{v} - (k^2 - ikb)^2 \tilde{v} = 0 \quad \text{in } \widetilde{\Omega}, \tag{4.3.7}$$

and observe that

$$\tilde{u}_0(x) = e^{i\zeta_1 \cdot x} \quad \text{and} \quad \tilde{v}(x) = e^{i\zeta_2 \cdot x}$$

are (complex exponential) solutions to (4.3.6) and (4.3.7) respectively. Using these, we define

$$u_0(x) = e^{i\zeta_1 \cdot (x', x_n)} - e^{i\zeta_1 \cdot (x', -x_n)} \quad \text{and} \quad v(x) = e^{i\zeta_2 \cdot (x', x_n)} - e^{i\zeta_2 \cdot (x', -x_n)}, \tag{4.3.8}$$

which are solutions to

$$\begin{cases} \Delta^2 u_0 - (k^2 - ikb)^2 u_0 & = 0 \quad \text{in } \Omega, \\ u_0 = \Delta u_0 & = 0 \quad \text{on } \Gamma_0 \end{cases} \quad \text{and} \quad \begin{cases} \Delta^2 v - (k^2 - ikb)^2 v & = 0 \quad \text{in } \Omega, \\ v = \Delta v & = 0 \quad \text{on } \Gamma_0 \end{cases} \tag{4.3.9}$$

respectively. Next, we will use these complex exponential solutions to derive the stability estimate (4.2.3).

4.3.2 Derivation of the stability estimate.

In order to derive the stability estimate, we will need the following integral identity.

Lemma 4.3.1. *Let u_0 and v be functions satisfying (4.3.9) and u_1 be the solution to (4.2.2) corresponding to u_0. Then the following integral identity holds true:*

$$\int_\Omega q u_0 v \, dx = -\int_\Gamma \partial_\nu u_1 (\Delta v) \, dS - \int_\Gamma \partial_\nu (\Delta u_1) v \, dS. \tag{4.3.10}$$

Proof. From (4.2.2), multiplying by v and integrating, we have

$$\int_\Omega q u_0 v\, dx = -\int_\Omega \Delta^2 u_1 v\, dx + \int_\Omega (k^2 - ikb)^2 u_1 v\, dx.$$

Using the Green's formula for the biharmonic operator

$$\int_\Omega (\Delta^2 u_1) v\, dx - \int_\Omega u_1 (\Delta^2 v)\, dx = \int_{\partial\Omega} \partial_\nu u_1 (\Delta v)\, dS - \int_{\partial\Omega} (\Delta u_1) \partial_\nu v\, dS + \int_{\partial\Omega} \partial_\nu (\Delta u_1) v\, dS - \int_{\partial\Omega} u_1 \partial_\nu (\Delta v)\, dS,$$

and the facts that $u_1 = 0 = \Delta u_1$ on $\partial\Omega$ and $v = 0 = \Delta v$ on Γ_0, we get

$$\begin{aligned}
\int_\Omega q u_0 v\, dx &= -\int_\Omega u_1 (\Delta^2 v)\, dx + \int_\Omega (k^2 - ikb)^2 u_1 v\, dx - \int_\Gamma \partial_\nu u_1 (\Delta v)\, dS \\
&\quad - \int_\Gamma \partial_\nu (\Delta u_1) v\, dS \\
&= -\int_\Omega u_1 \left(\Delta^2 v - (k^2 - ikb)^2 v\right)\, dx - \int_\Gamma \partial_\nu u_1 (\Delta v)\, dS - \int_\Gamma \partial_\nu (\Delta u_1) v\, dS \\
&= -\int_\Gamma \partial_\nu u_1 (\Delta v)\, dS - \int_\Gamma \partial_\nu (\Delta u_1) v\, dS,
\end{aligned}$$

and hence (4.3.10) follows. □

Now, using (4.3.8) in the left hand side of the integral identity (4.3.10), we have

$$\begin{aligned}
\int_\Omega q u_0 v\, dx &= \int_\Omega q \left(e^{i\zeta_1\cdot(x',x_n)} - e^{i\zeta_1\cdot(x',-x_n)}\right) \left(e^{i\zeta_2\cdot(x',x_n)} - e^{i\zeta_2\cdot(x',-x_n)}\right)\, dx \\
&= \int_\Omega q e^{i(\zeta_1+\zeta_2)\cdot x}\, dx + \int_\Omega q e^{i(\zeta_1+\zeta_2)\cdot(x',-x_n)}\, dx \\
&\quad - \int_\Omega q e^{i[\zeta_1\cdot(x',x_n)+\zeta_2\cdot(x',-x_n)]}\, dx - \int_\Omega q e^{i[\zeta_1\cdot(x',-x_n)+\zeta_2\cdot(x',x_n)]}\, dx.
\end{aligned} \tag{4.3.11}$$

Recall, from (4.3.1) and (4.3.2), that $\zeta_1 + \zeta_2 = -\xi$. Also

$$\zeta_1 \cdot (x', x_n) + \zeta_2 \cdot (x', -x_n) = \xi_+ \cdot x,$$

$$\zeta_1 \cdot (x', -x_n) + \zeta_2 \cdot (x', x_n) = \xi_- \cdot x,$$

where

$$\xi_+ := \left(-\xi', 2\left(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\right)^{\frac{1}{2}} \frac{|\xi'|}{|\xi|}\right),$$

$$\xi_- := \left(-\xi', -2\left(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb\right)^{\frac{1}{2}} \frac{|\xi'|}{|\xi|}\right).$$

Using these notations, we rewrite (4.3.11) as

$$\int_\Omega q u_0 v \, dx = \int_\Omega q \left(e^{-i\xi\cdot(x',x_n)} + e^{-i\xi\cdot(x',-x_n)} \right) \, dx - \int_\Omega q \left(e^{i\xi_+\cdot x} + e^{i\xi_-\cdot x} \right) \, dx. \quad (4.3.12)$$

Note that

$$\int_\Omega q \left[e^{-i\xi\cdot(x',x_n)} + e^{-i\xi\cdot(x',-x_n)} \right] dx = \mathcal{F}[\tilde{q}_e](\xi).$$

Using this and (4.3.12) in the integral identity (4.3.10), we have

$$|\mathcal{F}[\tilde{q}_e](\xi)| \le \left| \int_\Gamma \partial_\nu u_1 (\Delta v) \, dS + \int_\Gamma \partial_\nu (\Delta u_1) v \, dS \right| + \left| \int_\Omega q e^{i\xi_+\cdot x} \, dx \right| + \left| \int_\Omega q e^{i\xi_-\cdot x} \, dx \right|. \quad (4.3.13)$$

To estimate the last two terms above, we proceed as follows. We write

$$\xi_+ = \left(-\xi', 2X\frac{|\xi'|}{|\xi|} \right) + i\left(0, 2Y\frac{|\xi'|}{|\xi|} \right),$$
$$\xi_- = \left(-\xi', -2X\frac{|\xi'|}{|\xi|} \right) + i\left(0, -2Y\frac{|\xi'|}{|\xi|} \right),$$

where $X + iY = \left(k^2 + \tau^2 - \frac{|\xi|^2}{4} - ikb \right)^{\frac{1}{2}}$ (as discussed in subsection 3.1). Then, using the fact that $|Y| \le b$, we see that

$$|\mathrm{Im}\,(\xi_\pm)| = 2|Y|\frac{|\xi'|}{|\xi|} \le 2b,$$

for $\xi \in \mathbb{R}^n$ such that $0 < |\xi|^2 \le 3(k^2 + \tau^2)$. Also for such ξ, using (4.3.3), we have (see [31]) the following lower bound for $|\xi_\pm|$:

$$|\xi_\pm|^2 \ge (k^2 + \tau^2)\frac{|\xi'|^2}{|\xi|^2} > 0.$$

Using these estimates and integration by parts, we have

$$\begin{aligned} \int_\Omega q(x) e^{i\xi_\pm\cdot x} \, dx &= \left(\int_\Omega q(x) e^{i\xi_\pm\cdot x} \, dx \right) \left(\frac{i\langle \xi_\pm, \xi_\pm \rangle}{i|\xi_\pm|^2} \right) \\ &= \frac{1}{i|\xi_\pm|^2} \left\langle \int_\Omega q(x) \left(i e^{i\xi_\pm\cdot x} \xi_\pm \right) \, dx, \xi_\pm \right\rangle \\ &= \frac{1}{i|\xi_\pm|^2} \left\langle - \int_\Omega \nabla q(x) \, e^{i\xi_\pm\cdot x} \, dx, \xi_\pm \right\rangle, \end{aligned}$$

and further using Cauchy-Schwarz inequality, this gives

$$\left|\int_\Omega q(x)e^{i\xi_\pm\cdot x}\,dx\right| \le \frac{1}{|\xi_\pm|^2}\left|\int_\Omega \nabla q(x)\,e^{i\xi_\pm\cdot x}\,dx\right|\;\;|\xi_\pm| \le \frac{1}{|\xi_\pm|}\int_\Omega |\nabla q(x)|\left|e^{i\xi_\pm\cdot x}\right|\,dx$$
$$\le C\frac{e^{R|\mathrm{Im}\,(\xi_\pm)|}}{(k^2+\tau^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}} \le C\frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}},$$

where the constant C depends on Ω and M. Using this in (4.3.13), we get

$$|\mathcal{F}[\tilde{q}_e](\xi)| \le \left|\int_\Gamma \partial_\nu u_1(\Delta v)\,dS + \int_\Gamma \partial_\nu(\Delta u_1)v\,dS\right| + C\frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\frac{|\xi|}{|\xi'|}. \tag{4.3.14}$$

Next, we estimate the first term in the right hand side of (4.3.14) as follows. Using Cauchy-Schwarz inequality, we see that

$$\left|\int_\Gamma \partial_\nu u_1(\Delta v)dS + \int_\Gamma \partial_\nu(\Delta u_1)vdS\right| \le \|\partial_\nu u_1\|_{L^2(\Gamma)}\|\Delta v\|_{L^2(\Gamma)} + \|\partial_\nu(\Delta u_1)\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)},$$

and therefore

$$\left|\int_\Gamma \partial_\nu u_1(\Delta v)dS + \int_\Gamma \partial_\nu(\Delta u_1)vdS\right|$$
$$\le \|\partial_\nu u_1\|_{L^2(\Gamma)}\|\Delta v\|_{L^2(\partial\Omega)} + \|\partial_\nu(\Delta u_1)\|_{L^2(\Gamma)}\|v\|_{L^2(\partial\Omega)}$$
$$\le C\left(\|\partial_\nu u_1\|_{L^2(\Gamma)}\|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\partial_\nu(\Delta u_1)\|_{L^2(\Gamma)}\|v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)$$
$$\le C\left(\|\partial_\nu u_1\|_{L^2(\Gamma)} + \|\partial_\nu(\Delta u_1)\|_{L^2(\Gamma)}\right)\left(\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right).$$

Using the trace theorem and the linearized partial D-N map, we further observe that

$$\left|\int_\Gamma \partial_\nu u_1(\Delta v)dS + \int_\Gamma \partial_\nu(\Delta u_1)vdS\right|$$
$$\le C\left(\|\partial_\nu u_1\|_{H^{\frac{5}{2}}(\Gamma)} + \|\partial_\nu(\Delta u_1)\|_{H^{\frac{1}{2}}(\Gamma)}\right)\left(\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)$$
$$= C\|\mathcal{N}^B_{q,L}(u_0|_\Gamma, \Delta u_0|_\Gamma)\|_{H^{\frac{5}{2},\frac{1}{2}}(\Gamma)}\left(\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)$$
$$\le C\|\mathcal{N}^B_{q,L}\|_*\|(u_0|_\Gamma, \Delta u_0|_\Gamma)\|_{\widetilde{H}^{\frac{7}{2},\frac{3}{2}}(\Gamma)}\left(\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)$$
$$\le C\|\mathcal{N}^B_{q,L}\|_*\left(\|u_0\|_{\widetilde{H}^{\frac{7}{2}}(\Gamma)} + \|\Delta u_0\|_{\widetilde{H}^{\frac{3}{2}}(\Gamma)}\right)\left(\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)$$
$$= C\|\mathcal{N}^B_{q,L}\|_*\left(\|u_0\|_{H^{\frac{7}{2}}(\partial\Omega)} + \|\Delta u_0\|_{H^{\frac{3}{2}}(\partial\Omega)}\right)\left(\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)$$
$$\le C\|\mathcal{N}^B_{q,L}\|_*\left(\|u_0\|_{H^4(\Omega)} + \|\Delta u_0\|_{H^2(\Omega)}\right)\left(\|\Delta v\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}\right). \tag{4.3.15}$$

Now to estimate the terms in the right hand side of (4.3.15), we first make the following observation.

Let $\zeta \in \mathbb{C}^n$ and

$$w(x) := e^{i\zeta\cdot x},\ x \in \Omega.$$

Then,

$$\begin{aligned}
&\nabla w = i\zeta e^{i\zeta\cdot x} = i\zeta w, \quad \Delta w = -(\zeta\cdot\zeta)e^{i\zeta\cdot x} = -(\zeta\cdot\zeta)w,\\
&\nabla(\Delta w) = -i(\zeta\cdot\zeta)\zeta e^{i\zeta\cdot x} = -i(\zeta\cdot\zeta)\zeta w, \quad \Delta^2 w = (\zeta\cdot\zeta)^2 e^{i\zeta\cdot x} = (\zeta\cdot\zeta)^2 w,
\end{aligned} \tag{4.3.16}$$

and

$$\begin{aligned}
\sum_{|\alpha|=1} |\partial^\alpha w|^2 &= \sum_{l=1}^n |\partial_{x_l} w|^2 = \sum_{l=1}^n |i\zeta_l w|^2 = \sum_{l=1}^n |\zeta_l|^2|w|^2 = |\zeta|^2|w|^2,\\
\sum_{|\alpha|=2} |\partial^\alpha w|^2 &= \sum_{l,s=1}^n |\partial_{x_s}\partial_{x_l} w|^2 = \sum_{l,s=1}^n |-\zeta_l\zeta_s w|^2 = \sum_{l,s=1}^n |\zeta_l|^2|\zeta_s|^2|w|^2 = |\zeta|^4|w|^2,\\
\sum_{|\alpha|=3} |\partial^\alpha w|^2 &= \sum_{l,s,t=1}^n |\partial_{x_t}\partial_{x_s}\partial_{x_l} w|^2 = \sum_{l,s,t=1}^n |-i\zeta_l\zeta_s\zeta_t w|^2\\
&= \sum_{l,s,t=1}^n |\zeta_l|^2|\zeta_s|^2|\zeta_t|^2|w|^2 = |\zeta|^6|w|^2,\\
\sum_{|\alpha|=4} |\partial^\alpha w|^2 &= \sum_{l,s,t,r=1}^n |\partial_{x_r}\partial_{x_t}\partial_{x_s}\partial_{x_l} w|^2 = \sum_{l,s,t,r=1}^n |\zeta_l\zeta_s\zeta_t\zeta_r w|^2 = |\zeta|^8|w|^2.
\end{aligned} \tag{4.3.17}$$

Using (4.3.16) and (4.3.17) together with the fact that $\Omega \subset B_R$, we observe the following estimates for the function w:

$$\begin{aligned}
&\|w\|_{L^2(\Omega)} \le |\Omega|^{\frac{1}{2}} e^{R|\mathrm{Im}\,(\zeta)|}, \quad \|w\|_{H^1(\Omega)} \le |\Omega|^{\frac{1}{2}} e^{R|\mathrm{Im}\,(\zeta)|}(1+|\zeta|^2)^{\frac{1}{2}},\\
&\|\Delta w\|_{H^1(\Omega)} \le |\Omega|^{\frac{1}{2}} e^{R|\mathrm{Im}\,(\zeta)|}|\zeta\cdot\zeta|(1+|\zeta|^2)^{\frac{1}{2}},\\
&\|\Delta w\|_{H^2(\Omega)} \le |\Omega|^{\frac{1}{2}} e^{R|\mathrm{Im}\,(\zeta)|}|\zeta\cdot\zeta|(1+|\zeta|^2+|\zeta|^4)^{\frac{1}{2}},\\
&\|w\|_{H^4(\Omega)} \le |\Omega|^{\frac{1}{2}} e^{R|\mathrm{Im}\,(\zeta)|}(1+|\zeta|^2+|\zeta|^4+|\zeta|^6+|\zeta|^8)^{\frac{1}{2}}.
\end{aligned} \tag{4.3.18}$$

Now using the above observation for the functions u_0 and v and the inequalities (4.3.2), (4.3.4), (4.3.5), we get the estimates

$$\begin{aligned}\|\Delta u_0\|_{H^2(\Omega)} &\le Ce^{R(b+\tau)}((k^2+kb)^2+\tau^4),\\ \|u_0\|_{H^4(\Omega)} &\le Ce^{R(b+\tau)}((k^2+kb)^2+\tau^4),\\ \|\Delta v\|_{H^1(\Omega)} &\le Ce^{R(b+\tau)}((k^2+kb)^2+\tau^4),\\ \|v\|_{H^1(\Omega)} &\le Ce^{R(b+\tau)}((k^2+kb)^2+\tau^4),\end{aligned} \tag{4.3.19}$$

where we also use the fact that $2((k^2+kb)^{\frac{1}{2}}+\tau)^2 > 1$ and the inequality $(c+d)^2 \le 2(c^2+d^2),$ for $c, d \in \mathbb{R}$.

With the estimates (4.3.19) at our disposal, from (4.3.15) and the fact that $k > 1$, we infer that

$$\begin{aligned}&\left|\int_\Gamma \partial_\nu u_1(\Delta v)\,dS + \int_\Gamma \partial_\nu(\Delta u_1)v\,dS\right|\\ &\qquad\le C\|\mathcal{N}_{q,L}^B\|_* e^{2R(b+\tau)}((k^2+kb)^4+\tau^8)\\ &\qquad\le Ce^{2Rb}(k^2+kb)^4\|\mathcal{N}_{q,L}^B\|_* e^{2R\tau} + Ce^{2Rb}(k^2+kb)^4\|\mathcal{N}_{q,L}^B\|_*\tau^8 e^{2R\tau}\\ &\qquad\le Ce^{2Rb}(k^2+kb)^4\|\mathcal{N}_{q,L}^B\|_* e^{10R\tau},\end{aligned}$$

where we again use the inequality $(c+d)^2 \le 2(c^2+d^2),$ for $c, d \in \mathbb{R}$.

Then from the inequality (4.3.14), we have

$$|\mathcal{F}[\tilde{q}_e](\xi)| \le Ce^{2Rb}(k^2+kb)^4\|\mathcal{N}_{q,L}^B\|_* e^{10R\tau} + C\frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}}\frac{|\xi|}{|\xi'|}, \tag{4.3.20}$$

for $\xi \in \mathbb{R}^n$ satisfying the conditions $|\xi'| > 0$ and $0 < |\xi| \le 3(k^2+\tau^2)$.

Next we estimate $\|q\|_{H^{-1}(\Omega)}$. Let $\rho > 1$ be a real number to be chosen later and consider the set

$$\mathcal{Z}_\rho := \{(\xi', \xi_n) \in \mathbb{R}^n : \quad 0 < |\xi'| < \rho \text{ and } |\xi_n| < \rho\}.$$

Then, using Parseval's identity, we have the estimate

$$\begin{aligned}\|\tilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &= \int_{\mathcal{Z}_\rho}\left(1+|\xi|^2\right)^{-1}|\mathcal{F}[\tilde{q}_e](\xi)|^2\,d\xi + \int_{\mathbb{R}^n\setminus\mathcal{Z}_\rho}\left(1+|\xi|^2\right)^{-1}|\mathcal{F}[\tilde{q}_e](\xi)|^2\,d\xi\\ &\le \int_{\mathcal{Z}_\rho}\left(1+|\xi|^2\right)^{-1}|\mathcal{F}[\tilde{q}_e](\xi)|^2\,d\xi + \frac{C}{\rho^2},\end{aligned} \tag{4.3.21}$$

where the constant C depends on Ω and M.

Assume that for $\xi \in \mathcal{Z}_\rho$, the inequality (4.3.20) holds. Then the first term in the right hand side of the above inequality satisfies

$$\begin{aligned}
&\int_{\mathcal{Z}_\rho} \left(1+|\xi|^2\right)^{-1} |\mathcal{F}[\tilde{q}_e](\xi)|^2 \, d\xi \\
&\qquad \le \int_{\mathcal{Z}_\rho} \left| C e^{2Rb}(k^2+kb)^4 e^{10R\tau} \|\mathcal{N}^B_{q,L}\|_* + C \frac{e^{2Rb}}{(k^2+\tau^2)^{\frac{1}{2}}} \frac{|\xi|}{|\xi'|} \right|^2 d\xi \\
&\qquad \le C e^{4Rb}(k^2+kb)^8 e^{20R\tau} \|\mathcal{N}^B_{q,L}\|_*^2 \left(\int_{\mathcal{Z}_\rho} d\xi \right) + C \frac{e^{4Rb}}{k^2+\tau^2} \int_{\mathcal{Z}_\rho} \frac{|\xi'|^2+|\xi_n|^2}{|\xi'|^2} d\xi,
\end{aligned}$$

where we have used the fact that $1+|\xi|^2 > 1$. Note that the set $\{\xi \in \mathbb{R}^n : |\xi'| = 0\}$ is of n-dimensional Lebesgue measure zero and therefore, we can ignore it while estimating the integral over $\mathcal{Z}_\rho$.

Now using the inequality $c^2+d^2 \le (c+d)^2 = 4cd + (c-d)^2 \le 4cd + c^2$ for $c,\ d \ge 0$, with $c = |\xi'|$ and $d = |\xi_n|$, we obtain the estimate

$$\begin{aligned}
&\int_{\mathcal{Z}_\rho} \left(1+|\xi|^2\right)^{-1} |\mathcal{F}[\tilde{q}_e](\xi)|^2 \, d\xi \\
&\quad \le C e^{4Rb}(k^2+kb)^8 e^{20R\tau} \|\mathcal{N}^B_{q,L}\|_*^2 \rho^n + C \frac{e^{4Rb}}{k^2+\tau^2} \int_{-\rho}^{\rho} \int_{B'(0,\rho)} \frac{4|\xi'||\xi_n| + |\xi'|^2}{|\xi'|^2} d\xi' d\xi_n \\
&\quad \le C e^{4Rb}(k^2+kb)^8 e^{20R\tau} \|\mathcal{N}^B_{q,L}\|_*^2 \rho^n + C \frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}} \rho \int_{|\theta|=1} \int_0^\rho \left(\frac{4r\rho}{r^2} + \frac{r^2}{r^2} \right) r^{n-2} dr d\theta \\
&\quad \le C e^{4Rb}(k^2+kb)^8 e^{20R\tau} \|\mathcal{N}^B_{q,L}\|_*^2 \rho^n + C \frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}} \rho \int_{|\theta|=1} \int_0^\rho \left(4\rho r^{n-3} + r^{n-2} \right) dr d\theta \\
&\quad \le C e^{4Rb}(k^2+kb)^8 e^{20R\tau} \|\mathcal{N}^B_{q,L}\|_*^2 \rho^n + C \frac{e^{4Rb}}{(k^2+\tau^2)^{\frac{1}{2}}} \rho^n,
\end{aligned}$$

where $B'(0,\rho) := \{y \in \mathbb{R}^{n-1} : |y| < \rho\}$. Using this in (4.3.21), and the fact that $1 \le e^{4Rb}$, we have

$$\|\tilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \le C e^{4Rb} \left[(k^2+kb)^8 e^{20R\tau} \|\mathcal{N}^B_{q,L}\|_*^2 \rho^n + \frac{\rho^n}{(k^2+\tau^2)^{\frac{1}{2}}} + \frac{1}{\rho^2} \right]. \tag{4.3.22}$$

Next, we choose $\rho = (k^2+\tau^2)^{\frac{1}{2(n+2)}}$. Then $\rho > 1$ and for $\xi \in \mathcal{Z}_\rho$, we have

$$\frac{|\xi|^2}{3(k^2+\tau^2)} \le \frac{2\rho^2}{3(k^2+\tau^2)} = \frac{2}{3} \frac{(k^2+\tau^2)^{\frac{1}{n+2}}}{(k^2+\tau^2)} = \frac{2}{3} \frac{1}{(k^2+\tau^2)^{1-\frac{1}{n+2}}} < 1,$$

where we also use the fact that $\frac{1}{n+2} < 1$. Hence $|\xi|^2 \leq 3(k^2 + \tau^2)$ and so, the estimate (4.3.20) is valid for this choice of ρ. Using this in (4.3.22), we get the estimate

$$\begin{aligned}
\|\tilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\leq Ce^{4Rb}\left[(k^2+kb)^8 e^{20R\tau}\|\mathcal{N}^B_{q,L}\|^2_*(k^2+\tau^2)^{\frac{n}{2(n+2)}} + \frac{2}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right]\\
&\leq Ce^{4Rb}\left[(k^2+kb)^8 e^{20R\tau}\|\mathcal{N}^B_{q,L}\|^2_*(k^2\tau^2+k^2\tau^2)^{\frac{1}{2}} + \frac{2}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right]\\
&= Ce^{4Rb}\left[\sqrt{2}(k^2+kb)^8 e^{20R\tau}\|\mathcal{N}^B_{q,L}\|^2_* k\tau + \frac{2}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right]\\
&\leq Ce^{4Rb}\left[2(k^2+kb)^8 e^{20R\tau}\|\mathcal{N}^B_{q,L}\|^2_*(k^2+kb)e^{R\tau} + \frac{2}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right]\\
&\leq Ce^{4Rb}\left[(k^2+kb)^9\|\mathcal{N}^B_{q,L}\|^2_* e^{21R\tau} + \frac{1}{(k^2+\tau^2)^{\frac{1}{n+2}}}\right].
\end{aligned}$$

Now choosing $\tau = \frac{E}{21R}$, we see that

$$\|\tilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} \leq C\left[(k^2+kb)^9 e^{4Rb}\|\mathcal{N}^B_{q,L}\|_* + \frac{e^{4Rb}}{\left(k^2+\left(\frac{E}{21R}\right)^2\right)^{\frac{1}{n+2}}}\right]^{\frac{1}{2}}. \tag{4.3.23}$$

Recall that we need $\tau > 1$ and therefore, this choice of τ imposes the condition that $\|\mathcal{N}^B_{q,L}\|_* < \frac{1}{e^{21R}}$. This immediately implies that

$$\|q\|_{H^{-1}(\Omega)} \leq C\left[(k^2+kb)^9 e^{4Rb}\|\mathcal{N}^B_{q,L}\|_* + \frac{e^{4Rb}}{\left(k^2+\left(\frac{E}{21R}\right)^2\right)^{\frac{1}{n+2}}}\right]^{\frac{1}{2}}$$

for $\|\mathcal{N}^B_{q,L}\|_* < \frac{1}{e^{21R}}$. The case when $\|\mathcal{N}^B_{q,L}\|_* \geq \frac{1}{e^{21R}}$ easily follows from the following fact:

$$\begin{aligned}
\|q\|_{H^{-1}(\Omega)} \leq C\|q\|_{L^\infty(\Omega)} &\leq \frac{CM}{\delta^{\frac{1}{2}}}\delta^{\frac{1}{2}} \leq \frac{CM}{\delta^{\frac{1}{2}}}\|\mathcal{N}^B_{q,L}\|^{\frac{1}{2}}_*\\
&\leq C\left[(k^2+kb)^9 e^{4Rb}\|\mathcal{N}^B_{q,L}\|_* + \frac{e^{4Rb}}{\left(k^2+\left(\frac{E}{21R}\right)^2\right)^{\frac{1}{n+2}}}\right]^{\frac{1}{2}},
\end{aligned}$$

where we denote $\delta := \frac{1}{e^{21R}}$ and the constant C in the last inequality depends only on n, Ω and M.

Now we use the interpolation theorem to estimate the L^2 norm of q. Recall that given

$l_1, l_2, l \in \mathbb{R}$ such that $l_1 < l_2$ and $l = (1-s)l_1 + sl_2$, where $0 < s < 1$, the interpolation theorem gives

$$\|q\|_{H^l} \leq \|q\|_{H^{l_1}}^{1-s} \|q\|_{H^{l_2}}^{s}.$$

Choosing $l_1 = -1$, $l_2 = 1$ and $l = 0$, we have

$$\|q\|_{L^2(\Omega)} = \|q\|_{H^0(\Omega)} \leq \|q\|_{H^{-1}(\Omega)}^{1-\frac{1}{2}} \|q\|_{H^1(\Omega)}^{\frac{1}{2}} \leq C \|q\|_{H^{-1}(\Omega)}^{\frac{1}{2}},$$

which implies

$$\|q\|_{L^2(\Omega)} \leq C \left[(k^2 + kb)^9 e^{4Rb} \|\mathcal{N}_{q,L}^B\|_* + \frac{e^{4Rb}}{\left(k^2 + \left(\frac{E}{21R}\right)^2\right)^{\frac{1}{n+2}}} \right]^{\frac{1}{4}},$$

where the constant C depends only on n, Ω and M. Thus, we have the stability estimate (4.2.3).

Chapter 5

Linearized inverse polyharmonic problem with attenuation

In this chapter, which is based on [13], we explore high-frequency stability estimates for the determination of the zeroth-order perturbation of the polyharmonic operator with constant attenuation from the linearized partial Dirichlet-to-Neumann map in domains satisfying the flatness condition $(\mathcal{A})$. Our result suggests improvement of the stability when the frequency is appropriately chosen. This work extends the results of [25], [39], [31], [12] and [38] to the case of polyharmonic operator with constant attenuation and with lower regularity assumption on the potentials.

5.1 Introduction

Let us consider the inverse boundary-value problem for the determination of the zeroth-order perturbation of the polyharmonic operator with constant attenuation posed in a bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 3$ with a smooth boundary $\partial\Omega$:

$$\begin{cases} (-\Delta)^m u \ - \ (k^2 - ikb)^m u \ + \ qu &= 0 \quad \text{in } \Omega, \\ (u, \Delta u, ..., \Delta^{m-1} u) &= (f_1, f_2, ..., f_m) \ \text{ on } \partial\Omega. \end{cases} \tag{5.1.1}$$

Here m is a positive integer greater than or equal to 3, $b > 0$ is the constant attenuation and the potential q is assumed to be real-valued and

$$q \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}.$$

We shall assume that the frequency $k \geq 1$ and the Navier boundary data

$$(f_1, f_2, ..., f_m) \in H^{2m-\frac{1}{2}}(\partial\Omega) \times H^{2m-\frac{5}{2}}(\partial\Omega) \times \cdots \times H^{\frac{3}{2}}(\partial\Omega) =: \prod_{j=1}^{m} H^{2j-\frac{1}{2}}(\partial\Omega).$$

We further assume that the bounded domain Ω satisfies the flatness condition $(\mathcal{A})$. We also assume that the supports of f_j $(1 \leq j \leq m)$ are contained in the open subset $\Gamma := \partial\Omega \setminus \Gamma_0$ of the boundary and the boundary measurements $\partial_\nu(\Delta^j u),\ 0 \leq j \leq m-1$ (here ν denotes the exterior unit normal vector to $\partial\Omega$) are available on Γ only, and thus, the flat part Γ_0 is also assumed to be inaccessible. Note that the imaginary part of $(k^2 - ikb)^m$ is non-zero and hence, it is not a part of the spectrum of $(-\Delta)^m + q$. Therefore, there exists a unique solution to (5.1.1) when $(f_1, f_2, ..., f_m) \in \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$. In this case, the partial Dirichlet-to-Neumann (D-N) map is defined as

$$\begin{aligned} \mathcal{N}_q^{\mathbb{P}} \ &: \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma) \to \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma) \ \text{ such that} \\ (f_1, f_2, ..., f_m) &\mapsto \left(\partial_\nu u\Big|_\Gamma, \partial_\nu(\Delta u)\Big|_\Gamma, ..., \partial_\nu(\Delta^{m-1} u)\Big|_\Gamma\right). \end{aligned} \tag{5.1.2}$$

We define

$$\begin{aligned} \|\mathcal{N}_q^{\mathbb{P}}\|_* :&= \sup\{\|\mathcal{N}_q^{\mathbb{P}}(f_1, f_2, ..., f_m)\|_{H^{2m-\frac{3}{2}, ..., \frac{1}{2}}(\Gamma)} \\ &: \|(f_1, f_2, ..., f_m)\|_{\widetilde{H}^{2m-\frac{1}{2}, ..., \frac{3}{2}}(\Gamma)} = 1\}. \end{aligned}$$

Our aim, in this work, is to explore the stability of the recovery of the potential q from the knowledge of a linearized version of the partial D-N map (5.1.2) (which we shall describe in detail in the next section) and to study the explicit dependence of the estimate on the frequency k. Using a quantitative Riemann-Lebesgue lemma proved in [33] (see Lemma 2.3.4), we have been able to derive the stability estimate with considerably weaker condition on the potentials (the potential q is assumed to be only in $L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$) as compared to the earlier works. Also, we do not need the potential to be compactly supported inside the domain Ω. Our result also suggests improvement in stability in the recovery of the potential q with an appropriate choice of the frequency k.

The plan of the chapter is as follows. In Section 5.2, we discuss the linearization of the D-N map and our main result on the stability estimate. The construction of appropriate complex exponential type solutions and the derivation of the stability estimate is discussed in Section 5.3. In the appendix, we discuss some auxiliary results that are used in Section 5.3.

5.2 Linearization and the main result

In this section, following the works [25], [39], [31], [12], [38], we discuss the linearization of the partial D-N map and state our main result. Such a linearization can be justified when the potential q is sufficiently small (as we shall see shortly).

In this direction, for $(f_1, f_2, ..., f_m) \in \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$, we consider the equations

$$\begin{cases} (-\Delta)^m u \; - \; (k^2 - ikb)^m u & = 0 \quad \text{in } \Omega, \\ \qquad (u, \Delta u, ..., \Delta^{m-1} u) & = (f_1, f_2, ..., f_m) \;\; \text{on } \partial\Omega, \end{cases} \tag{5.2.1}$$

and

$$\begin{cases} (-\Delta)^m u \; - \; (k^2 - ikb)^m u & = -q u_0 \;\; \text{in } \Omega, \\ \qquad (u, \Delta u, ..., \Delta^{m-1} u) & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.2.2}$$

where u_0 denotes the solution to (5.2.1). We denote the solution to (5.2.2) by u_1. Note that since the imaginary part of $(k^2 - ikb)^m$ is non-zero, it is not a part of the spectrum of $(-\Delta)^m$, and therefore, there exist unique solutions to (5.2.1) and (5.2.2) when $(f_1, f_2, ..., f_m) \in \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$. Using this, we define the linearized partial D-N map $\mathcal{N}_{q,L}^{\mathbb{P}}$ as follows:

$$\begin{aligned} \mathcal{N}_{q,L}^{\mathbb{P}} \; &: \; \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma) \to \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma) \;\; \text{such that} \\ & (f_1, f_2, ..., f_m) \mapsto \left(\partial_\nu u_1\Big|_{\Gamma}, \partial_\nu (\Delta u_1)\Big|_{\Gamma}, ..., \partial_\nu (\Delta^{m-1} u_1)\Big|_{\Gamma} \right). \end{aligned} \tag{5.2.3}$$

Let us denote $E := |\log(\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*)|$. Then, we have the following stability result for the recovery of the potential q from the linearized partial D-N map $\mathcal{N}^{\mathbb{P}}_{q,L}$.

Theorem 5.2.1. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$ satisfying the assumption $(\mathcal{A})$ described before and $\Omega \subset\subset B(0,R)$ for some $R > 1$. Let b be a positive constant and suppose that the frequency $k \geq 1$. Further, suppose that the potential $q \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$ be such that*

$$\|q\|_{L^\infty(\Omega)} + \|q\|_{H^s(\Omega)} \leq M,$$

for some $M > 0$. Then there exists a constant $C > 0$, depending only on n, s, Ω and M such that

$$\|q\|_{H^{-1}(\Omega)} \leq C e^{4Rb} \left[(k^2 + kb)^{4m+1} \|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* + \frac{1}{(k^2 + \left(\frac{E}{5R}\right)^2)^\sigma} \right]^{\frac{1}{2}}, \tag{5.2.4}$$

where $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)} < \frac{1}{2}$.

Remark 5.2.2. In case the potentials have more regularity, we can use (5.2.4) combined with an interpolation argument to derive the stability estimate in the stronger L^2 norm.

Remark 5.2.3. The estimate (5.2.4) implies an improvement in the stability with an appropriate choice of the frequency k. This can be observed by following the arguments in [12].

Remark 5.2.4. Using similar arguments as in the proof of theorem 5.2.1, we can prove stability estimates for the Schrödinger and the biharmonic cases (discussed in the previous two chapters) under the considerably weaker regularity condition on the potentials as in this theorem.

In the rest of this section, we will elaborate on the relation between the D-N map (5.1.2) and the linearized D-N map (5.2.3). Let

$$\mathcal{Q} := \{q' \in L^\infty(\Omega)\ :\ 0 \text{ is not a Dirichlet eigenvalue of } (-\Delta)^m + q' \text{ in } \Omega\}.$$

Now, let us consider the nonlinear operator

$$\mathcal{F}_\Gamma : \mathcal{Q} \to \mathcal{L}\left(\prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma), \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma)\right),$$

defined by

$$\mathcal{F}_\Gamma(q') := \widetilde{\mathcal{N}}_{q'}^{\mathbb{P}}, \quad \text{such that}$$

$$\widetilde{\mathcal{N}}_{q'}^{\mathbb{P}}(f_1, f_2, ..., f_m) = \left(\partial_\nu u\Big|_\Gamma, \partial_\nu(\Delta u)\Big|_\Gamma, ..., \partial_\nu(\Delta^{m-1}u)\Big|_\Gamma\right),$$

$$\text{for } f := (f_1, f_2, ..., f_m) \in \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma),$$

where u_f is the weak solution of

$$\begin{cases} (-\Delta)^m u_f + q' u_f & = 0 \quad \text{in } \Omega, \\ (u_f, \Delta u_f, ..., \Delta^{m-1} u_f) & = f \quad \text{on } \partial\Omega. \end{cases} \tag{5.2.5}$$

Note that for $q' = -(k^2 - ikb)^m + q$, we have $\widetilde{\mathcal{N}}_{q'}^{\mathbb{P}} = \mathcal{N}_q^{\mathbb{P}}$, where $\mathcal{N}_q^{\mathbb{P}}$ is the original partial D-N map defined in (5.1.2).

Further, we define the operator $\mathcal{F}_\Gamma^1 : \mathcal{Q} \to \mathcal{L}\left(\prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma), \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma)\right)$ as follows:

$$\mathcal{F}_\Gamma^1(q) := \mathcal{N}_{q,L}^{\mathbb{P}},$$

which maps the potential q to the linearized partial D-N map $\mathcal{N}_{q,L}^{\mathbb{P}}$ defined in (5.2.3). Note that $\mathcal{F}_\Gamma^1$ is a linear operator.

For all $q' \in \mathcal{Q}$, we define the operator $\mathcal{P}_{q'} : \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma) \to H^{2m}(\Omega)$ such that

$$\mathcal{P}_{q'} f = u_f,$$

where u_f is the solution of the problem (5.2.5). Note that with $q' = -(k^2 - ikb)^m$ and $q' = -(k^2 - ikb)^m + q$, we get $\mathcal{P}_{q'} f = u_0$ and $\mathcal{P}_{q'} f = u$, where u_0 and u are the solutions to (5.2.1) and (5.1.1) respectively.

Further, we also define the operator $\mathcal{G}_{q'} : L^2(\Omega) \to H_0^{2m}(\Omega)$ by

$$\mathcal{G}_{q'}(F) = v_F,$$

where v_F is the solution to the problem

$$\begin{cases} (-\Delta)^m v_F + q' v_F & = -F \quad \text{in } \Omega, \\ (v_F, \Delta v_F, ..., \Delta^{m-1} v_F) & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now, for $q \in L^\infty(\Omega)$, using the above notations, we rewrite the partial D-N map (5.1.2) as

$$\begin{aligned} &\mathcal{F}_\Gamma(-(k^2 - ikb)^m + q)(f) = \mathcal{N}_q^{\mathbb{P}}(f) \\ &= \Big(\partial_\nu \left(\mathcal{P}_{(-(k^2-ikb)^m+q)} f\right)\Big|_\Gamma, \partial_\nu \left(\Delta \left[\mathcal{P}_{(-(k^2-ikb)^m+q)} f\right]\right)\Big|_\Gamma, \\ &\quad ..., \partial_\nu \left(\Delta^{m-1} \left[\mathcal{P}_{(-(k^2-ikb)^m+q)} f\right]\right)\Big|_\Gamma\Big). \end{aligned}$$

Similarly, we rewrite the linearized partial D-N map (5.2.3) as

$$\begin{aligned} &\mathcal{F}_\Gamma^1(q)(f) = \mathcal{N}_{q,L}^{\mathbb{P}}(f) \\ &= \Big(\partial_\nu \left(\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right)\Big|_\Gamma, \\ &\quad \partial_\nu \left(\Delta \left[\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right]\right)\Big|_\Gamma, \\ &\quad ..., \partial_\nu \left(\Delta^{m-1} \left[\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right]\right)\Big|_\Gamma\Big), \end{aligned}$$

where $\mathcal{M}_q$ denotes the multiplication operator by q and $\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f = u_1$, which is the solution to the equation (5.2.2).

The following result, expressed in terms of the operators $\mathcal{F}_\Gamma$ and $\mathcal{F}_\Gamma^1$, establishes the relationship between $\mathcal{N}_q^{\mathbb{P}}$ and $\mathcal{N}_{q,L}^{\mathbb{P}}$ under the assumption that q is sufficiently small. The proof follows by using arguments similar to that in [39].

Theorem 5.2.5. *Let $k \geq 1$ and b be a positive real number. Further, let us assume that the potential $q \in L^\infty(\Omega)$ satisfies the condition*

$$\|q\|_{L^\infty(\Omega)} < \frac{1}{2} \|\mathcal{G}_{(-(k^2-ikb)^m)}\|^{-1}_{\mathcal{L}(L^2(\Omega), H^{2m}(\Omega))}.$$

Then

$$\begin{aligned} &\Big\|\mathcal{F}_\Gamma(-(k^2 - ikb)^m + q) \\ &-\mathcal{F}_\Gamma(-(k^2 - ikb)^m) - \mathcal{F}_\Gamma^1(q)\Big\|_{\mathcal{L}\left(\prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma), \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma)\right)} \\ &\leq C \|q\|^2_{L^\infty(\Omega)}, \end{aligned}$$

where the constant C depends only on n, Ω, Γ, b and k.

Proof. Let $f := (f_1, f_2, ..., f_m) \in \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$. From (5.1.1) and (5.2.1), we see that

$$
\begin{aligned}
0 &= ((-\Delta)^m - (k^2 - ikb)^m + q) \left[\mathcal{P}_{(-(k^2-ikb)^m+q)} f\right] \\
&\quad - ((-\Delta)^m - (k^2 - ikb)^m) \left[\mathcal{P}_{(-(k^2-ikb)^m)} f\right] \\
&= ((-\Delta)^m - (k^2 - ikb)^m) \left[\mathcal{P}_{(-(k^2-ikb)^m+q)} f \right. \\
&\quad \left. -\mathcal{P}_{(-(k^2-ikb)^m)} f\right] + \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m+q)} f.
\end{aligned}
$$

It gives,

$$
\begin{cases}
((-\Delta)^m - (k^2 - ikb)^m) \left[\mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{P}_{(-(k^2-ikb)^m)} f\right] \\
= -\mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m+q)} f \quad \text{in } \Omega, \\
\mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{P}_{(-(k^2-ikb)^m)} f = 0 \qquad \text{on } \partial\Omega.
\end{cases}
$$

Using uniqueness of the problem (5.2.2), we have

$$
\mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{P}_{(-(k^2-ikb)^m)} f = \mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m+q)} f \;\text{ in }\; \Omega. \tag{5.2.6}
$$

Now, using (5.2.6), we have

$$
\begin{aligned}
&\left[\mathcal{F}_\Gamma(-(k^2 - ikb)^m + q) - \mathcal{F}_\Gamma(-(k^2 - ikb)^m) - \mathcal{F}^1_\Gamma(q)\right](f) \\
&= \Big(\partial_\nu \left(\mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{P}_{(-(k^2-ikb)^m)} f - \mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right)\big|_\Gamma, \\
&\quad \partial_\nu \Delta \left(\mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{P}_{(-(k^2-ikb)^m)} f - \mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right)\big|_\Gamma, \\
&\quad ..., \partial_\nu \Delta^{m-1} \left(\mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{P}_{(-(k^2-ikb)^m)} f \right. \\
&\quad \left. -\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right)\big|_\Gamma\Big) \\
&= \Big(\partial_\nu \left(\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right)\big|_\Gamma, \\
&\quad \partial_\nu \Delta \left(\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m+q)} f - \mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right)\big|_\Gamma, \\
&\quad ..., \partial_\nu \Delta^{m-1} \left(\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m+q)} f \right. \\
&\quad \left. -\mathcal{G}_{(-(k^2-ikb)^m)} \mathcal{M}_q \mathcal{P}_{(-(k^2-ikb)^m)} f\right)\big|_\Gamma\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\partial_\nu \left(\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f)\right)\big|_\Gamma, \\
&\quad \partial_\nu \Delta \left(\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f)\right)\big|_\Gamma, \\
&\quad ..., \partial_\nu \Delta^{m-1} \left(\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f)\right)\big|_\Gamma\Big) \\
&:= \left(\partial_\nu w\Big|_\Gamma, \partial_\nu(\Delta w)\Big|_\Gamma, ..., \partial_\nu(\Delta^{m-1} w)\Big|_\Gamma\right),
\end{aligned}
$$

where $w := \mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f)$.

Note that w satisfies the equation

$$
((-\Delta)^m - (k^2-ikb)^m)w = -q(\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f) \quad \text{in } \Omega. \tag{5.2.7}
$$

Let us assume that

$$
\|q\|_{L^\infty(\Omega)} < \frac{1}{2}\|\mathcal{G}_{(-(k^2-ikb)^m)}\|^{-1}_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}. \tag{5.2.8}
$$

Then, we have

$$
\begin{aligned}
&\|\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))} \\
&\le \|\mathcal{G}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}\|\mathcal{M}_q\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \\
&\le \|q\|_{L^\infty(\Omega)}\|\mathcal{G}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))} < \frac{1}{2} < 1,
\end{aligned}
$$

which implies that $I - \mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q$ is invertible. Also, from the following identity

$$
\begin{aligned}
&(I - \mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q)(\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f) \\
&= \mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f - \mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)^m+q)}f \\
&\quad + \mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)^m)}f \\
&= \mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)^m)}f \quad \text{(by (5.2.6))},
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f \\
&= (I - \mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q)^{-1}(\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)^m)}f).
\end{aligned} \tag{5.2.9}
$$

Now, we see that

$$
\begin{aligned}
&\|\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f\|_{L^2(\Omega)} \\
&\le \|(I-\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q)^{-1}\|_{\mathcal{L}(H^{2m}(\Omega),L^2(\Omega))} \\
&\quad \|\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)^m)}f\|_{H^{2m}(\Omega)} \\
&= \frac{1}{1-\|\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}} \\
&\quad \|\mathcal{G}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}\|\mathcal{M}_q\mathcal{P}_{(-(k^2-ikb)^m)}f\|_{L^2(\Omega)} \\
&\le 2\|\mathcal{G}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}\|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)^m)}f\|_{L^2(\Omega)} \\
&\le 2\|\mathcal{G}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}\|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)^m)}f\|_{H^{2m}(\Omega)} \\
&\le 2\|\mathcal{G}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}\|q\|_{L^\infty(\Omega)} \\
&\quad \|\mathcal{P}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}\left(\prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma),H^{2m}(\Omega)\right)}\|f\|_{\widetilde{H}^{2m-\frac{1}{2},\dots,\frac{3}{2}}(\Gamma)} \\
&\le C\|q\|_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{2m-\frac{1}{2},\dots,\frac{3}{2}}(\Gamma)}.
\end{aligned}
\tag{5.2.10}
$$

Using the above estimate, we get

$$
\begin{aligned}
\|w\|_{H^{2m}(\Omega)} &= \|\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q(\mathcal{P}_{(-(k^2-ikb)^m+q)}f - \mathcal{P}_{(-(k^2-ikb)^m)}f)\|_{H^{2m}(\Omega)} \\
&\le \|\mathcal{G}_{(-(k^2-ikb)^m)}\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}\|q\|_{L^\infty(\Omega)}\|\mathcal{P}_{(-(k^2-ikb)^m+q)}f \\
&\quad - \mathcal{P}_{(-(k^2-ikb)^m)}f\|_{L^2(\Omega)} \\
&\le C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{2m-\frac{1}{2},\dots,\frac{3}{2}}(\Gamma)},
\end{aligned}
\tag{5.2.11}
$$

where the constant C depends only on n, Ω, Γ, k and b.

Next, by trace theorem, we have

$$
\|\partial_\nu(\Delta^{m-1-j}w)\|_{H^{2j+\frac{1}{2}}(\Gamma)} \le C\|w\|_{H^{2m}(\Omega)} \quad \text{for } j=0,1,\dots,m-1. \tag{5.2.12}
$$

From the estimates (5.2.11) and (5.2.12) we get,

$$\| \left(\partial_\nu w, \partial_\nu(\Delta w), ..., \partial_\nu(\Delta^{m-1} w)\right) \|_{H^{2m-\frac{3}{2},...,\frac{1}{2}}(\Gamma)} = \sum_{j=0}^{m-1} \|\partial_\nu(\Delta^{m-1-j} w)\|_{H^{2j+\frac{1}{2}}(\Gamma)}$$
$$\leq C\|w\|_{H^{2m}(\Omega)} \leq C\|q\|^2_{L^\infty(\Omega)}\|f\|_{\widetilde{H}^{2m-\frac{1}{2},...,\frac{3}{2}}(\Gamma)}. \tag{5.2.13}$$

Therefore,

$$\|\mathcal{F}_\Gamma(-(k^2-ikb)^m+q)-\mathcal{F}_\Gamma(-(k^2-ikb)^m)-\mathcal{F}^1_\Gamma(q)\|_{\mathcal{L}\left(\prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma), \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma)\right)}$$
$$= \sup_{f\neq 0} \frac{\|(\mathcal{F}_\Gamma(-(k^2-ikb)^m+q)-\mathcal{F}_\Gamma(-(k^2-ikb)^m)-\mathcal{F}^1_\Gamma(q))(f)\|_{H^{2m-\frac{3}{2},...,\frac{1}{2}}(\Gamma)}}{\|f\|_{\widetilde{H}^{2m-\frac{1}{2},...,\frac{3}{2}}(\Gamma)}}$$
$$= \sup_{f\neq 0} \frac{\| \left(\partial_\nu w, \partial_\nu(\Delta w), ..., \partial_\nu(\Delta^{m-1} w)\right) \|_{H^{2m-\frac{3}{2},...,\frac{1}{2}}(\Gamma)}}{\|f\|_{\widetilde{H}^{2m-\frac{1}{2},...,\frac{3}{2}}(\Gamma)}} \leq C\|q\|^2_{L^\infty(\Omega)},$$

which is the estimate we wanted to prove. □

Note that the condition $\|q\|_{L^\infty(\Omega)} < \frac{1}{2}\|\mathcal{G}_{(-(k^2-ikb)^m)}\|^{-1}_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))}$ guarantees the fact that

$$\|\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q\|_{\mathcal{L}(L^2(\Omega),H^{2m}(\Omega))} < 1,$$

and hence the series $\sum_{j=0}^\infty (\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q)^j$ converges. Using this, from (5.2.9), we see that given $f \in \prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$, the solution $u \in H^{2m}(\Omega)$ of (5.1.1) can be expressed as

$$u = \mathcal{P}_{(-(k^2-ikb)^m+q)}f = \sum_{j=0}^\infty (\mathcal{G}_{(-(k^2-ikb)^m)}\mathcal{M}_q)^j \mathcal{P}_{(-(k^2-ikb)^m)}f.$$

In the above expansion of u, the terms corresponding to $j=0$ and $j=1$ are u_0 and u_1, respectively. As observed in the theorem above, the higher order terms (corresponding to $j \geq 2$), can be neglected provided q is sufficiently small.

5.3 Complex exponential solutions and the stability estimate

In this section, we give a proof of our main result on the stability estimate (5.2.4). In order to do so, we first construct appropriate complex exponential solutions and then use an integral identity. This leads us to the estimation of the Fourier transform of $q_1 - q_2$, part of which is handled using the quantitative Riemann-Lebesgue lemma.

5.3.1 Complex exponential solutions

We write $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as $x = (x', x_n)$, where $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

For a given $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, with $\xi' \neq 0$, we choose unit vectors α and β appropriately to construct the complex exponential solutions.

To do so, following [21], we define an orthonormal basis of $\mathbb{R}^n$ as described below: Let $e_1 := (\frac{\xi'}{|\xi'|}, 0)$ and $e_n := (0, ..., 0, 1)$. We choose $e_2, ..., e_{n-1} \in \mathbb{R}^n$ such that the set $\{e_1, e_2, ..., e_n\}$ forms an orthonormal basis of $\mathbb{R}^n$ and the n^{th} component $e_{i,n} = 0$ for $i = 2, ..., n-1$.

Then, the coordinate representation of ξ with respect to this basis is given by

$$\xi_e = \left(\frac{\xi' \cdot \xi'}{|\xi'|}, 0, ..., 0, \xi_n \right)_e .$$

We choose unit vectors $\alpha, \beta \in \mathbb{R}^n$ such that with respect to this new basis, α and β have the representations

$$\alpha_e = (0, 1, 0, ..., 0)_e, \;\; \beta_e = \Big(-\frac{\xi_n}{|\xi|}, 0, ..., 0, \frac{|\xi'|}{|\xi|} \Big)_e .$$

Then $\{\xi, \alpha, \beta\}$ forms an orthogonal set and it follows that

$$\alpha \cdot \beta = \alpha_e \cdot \beta_e, \;\; \alpha_n = \alpha_{e,n} \text{ and } \beta_n = \beta_{e,n}.$$

Using these vectors, let us define (see [31], [12])

$$\begin{aligned} \zeta_1 &= -\frac{\xi}{2} + \Big(k^2 + a^2 - \frac{|\xi|^2}{4} - ikb \Big)^{\frac{1}{2}} \beta + ia\alpha, \\ \zeta_2 &= -\frac{\xi}{2} - \Big(k^2 + a^2 - \frac{|\xi|^2}{4} - ikb \Big)^{\frac{1}{2}} \beta - ia\alpha, \end{aligned} \tag{5.3.1}$$

where $k \geq 1$ and $a \geq 1$. Then

$$\zeta_1 + \zeta_2 = -\xi, \quad \zeta_j \cdot \zeta_j = k^2 - ikb, \; i = 1, 2. \tag{5.3.2}$$

Next assume that $|\xi|^2 \leq 3(k^2+a^2)$. We denote the principal square root of $k^2+a^2-\frac{|\xi|^2}{4}-ikb$ by

$$X+iY := \left(k^2+a^2-\frac{|\xi|^2}{4}-ikb\right)^{\frac{1}{2}},$$

where $X>0$ (the condition $|\xi|^2 \leq 3(k^2+a^2)$ guarantees this). Note that

$$\zeta_1 = -\frac{\xi}{2}+X\beta+i(Y\beta+a\alpha), \quad \zeta_2 = -\frac{\xi}{2}-X\beta-i(Y\beta+a\alpha),$$

and therefore,

$$|\zeta_j|^2 = \frac{|\xi|^2}{4}+X^2+Y^2+a^2, \quad j=1,2,$$

where

$$X^2+Y^2 = |X+iY|^2 = \left(\left(k^2+a^2-\frac{|\xi|^2}{4}\right)^2+k^2b^2\right)^{\frac{1}{2}}. \tag{5.3.3}$$

Also (see [31]) $|\mathrm{Im}\,(\zeta_j)|^2 = Y^2+a^2,\ j=1,2$, and $|Y|\leq b$, and hence

$$|\mathrm{Im}\,(\zeta_j)|^2 \leq b^2+a^2 \leq (b+a)^2 \quad \text{for } j=1,2. \tag{5.3.4}$$

Next, following [21] (see also [12,31,39]), we construct the complex exponential solutions using a reflection argument.

Let $\Omega^* := \{(x',x_n)\in\mathbb{R}^n : (x',-x_n)\in\Omega\}$ be the reflection of Ω by $\{x_n=0\}$ and let $\widetilde{\Omega} := \Omega\cup\Omega^*$.

Further, let q_e denote the extension of q to $\widetilde{\Omega}$ by reflection by $\{x_n=0\}$, that is,

$$q_e(x) = \begin{cases} q(x',x_n), & \text{if } (x',x_n)\in\Omega, \\ q(x',-x_n), & \text{if } (x',x_n)\in\Omega^*. \end{cases}$$

Also, we choose $R>1$ such that $\Omega \subset\subset B(0,R) =: B_R$ (note that, by the symmetry of the domain, this also implies that $\widetilde{\Omega}\subset\subset B_R$). Now we consider the equations

$$(-\Delta)^m\widetilde{u}_0 - (k^2-ikb)^m\widetilde{u}_0 = 0 \ \text{ in } B_R, \tag{5.3.5}$$

and

$$(-\Delta)^m \widetilde{v} - (k^2 - ikb)^m \widetilde{v} = 0 \quad \text{in } \ B_R, \tag{5.3.6}$$

and observe that

$$\widetilde{u}_0(x) = e^{i\zeta_1 \cdot x} \quad \text{ and } \quad \widetilde{v}(x) = e^{i\zeta_2 \cdot x}$$

are (complex exponential) solutions to (5.3.5) and (5.3.6) respectively. Using these, we define

$$u_0(x) = e^{i\zeta_1 \cdot (x', x_n)} - e^{i\zeta_1 \cdot (x', -x_n)} \ \text{ and } \ v(x) = e^{i\zeta_2 \cdot (x', x_n)} - e^{i\zeta_2 \cdot (x', -x_n)}, \tag{5.3.7}$$

which are solutions to

$$\begin{cases} (-\Delta)^m u_0 - (k^2 - ikb)^m u_0 & = 0 \quad \text{in } \Omega, \\ (u_0, \Delta u_0, ..., \Delta^{m-1} u_0) & = 0 \text{ on } \Gamma_0 \end{cases}$$
$$\text{and} \tag{5.3.8}$$
$$\begin{cases} (-\Delta)^m v - (k^2 - ikb)^m v & = 0 \quad \text{in } \Omega, \\ (v, \Delta v, ..., \Delta^{m-1} v) & = 0 \quad \text{on } \Gamma_0 \end{cases}$$

respectively. Next, we will use these complex exponential solutions to derive the stability estimate (5.2.4).

5.3.2 Derivation of the stability estimate.

The following integral identity will be used to derive the stability estimate.

Lemma 5.3.1. *Let u_0 and v be functions satisfying* (5.3.8) *and u_1 be the solution to* (5.2.2) *corresponding to u_0. Then, the following integral identity holds true:*

$$\int_\Omega q u_0 v \, dx = \sum_{j=0}^{m-1} \int_\Gamma \left[\partial_\nu((-\Delta)^j u_1)\right] \left[(-\Delta)^{m-1-j} v\right] \, dS. \tag{5.3.9}$$

Proof. Multiplying (5.2.2) by v and integrating, we have

$$-\int_\Omega q u_0 v\, dx = \int_\Omega (-\Delta)^m u_1 v\, dx - \int_\Omega (k^2 - ikb)^m u_1 v\, dx.$$

By repeatedly applying the Green's formula

$$\int_\Omega u_1 \Delta v dx - \int_\Omega \Delta u_1 v dx = \int_{\partial\Omega} u_1 \partial_\nu v dS - \int_{\partial\Omega} v \partial_\nu u_1 dS,$$

and the facts that $\Delta^j u_1 = 0$ $(0 \le j \le m-1)$ on $\partial\Omega$ and $\Delta^j v = 0$ $(0 \le j \le m-1)$ on Γ_0, we get

$$\begin{aligned}
-\int_\Omega q u_0 v\, dx &= \int_\Omega (-\Delta)^{m-1} u_1 (-\Delta v)\, dx - \int_\Gamma \partial_\nu((-\Delta)^{m-1} u_1) v\, dS \\
&\quad - \int_\Omega (k^2 - ikb)^m u_1 v\, dx \\
&= \int_\Omega (-\Delta)^{m-2} u_1 (-\Delta)^2 v\, dx - \int_\Gamma \partial_\nu((-\Delta)^{m-2} u_1)(-\Delta v)\, dS \\
&\quad - \int_\Gamma \partial_\nu((-\Delta)^{m-1} u_1) v\, dS - \int_\Omega (k^2 - ikb)^m u_1 v\, dx \\
&\quad \vdots \\
&= \int_\Omega (-\Delta u_1)(-\Delta)^{m-1} v\, dx - \int_\Omega (k^2 - ikb)^m u_1 v\, dx \\
&\quad - \sum_{j=1}^{m-1} \int_\Gamma [\partial_\nu((-\Delta)^j u_1)][(-\Delta)^{m-1-j} v]\, dS \\
&= \int_\Omega u_1 (-\Delta)^m v\, dx - \int_\Omega (k^2 - ikb)^m u_1 v\, dx \\
&\quad - \int_\Gamma \partial_\nu u_1 (-\Delta)^{m-1} v\, dS - \sum_{j=1}^{m-1} \int_\Gamma [\partial_\nu((-\Delta)^j u_1)][(-\Delta)^{m-1-j} v]\, dS \\
&= -\sum_{j=0}^{m-1} \int_\Gamma [\partial_\nu((-\Delta)^j u_1)][(-\Delta)^{m-1-j} v]\, dS
\end{aligned}$$

and hence (5.3.9) follows. □

Now, using (5.3.7) in the left hand side of the integral identity (5.3.9), we have

$$\begin{aligned}\int_\Omega q u_0 v \, dx &= \int_\Omega q \left(e^{i\zeta_1\cdot(x',x_n)} - e^{i\zeta_1\cdot(x',-x_n)}\right)\left(e^{i\zeta_2\cdot(x',x_n)} - e^{i\zeta_2\cdot(x',-x_n)}\right) dx \\ &= \int_\Omega q e^{i(\zeta_1+\zeta_2)\cdot x} \, dx + \int_\Omega q e^{i(\zeta_1+\zeta_2)\cdot(x',-x_n)} \, dx \\ &\quad - \int_\Omega q e^{i[\zeta_1\cdot(x',x_n)+\zeta_2\cdot(x',-x_n)]} \, dx - \int_\Omega q e^{i[\zeta_1\cdot(x',-x_n)+\zeta_2\cdot(x',x_n)]} \, dx \\ &= \int_\Omega q \left(e^{-i\xi\cdot x} + e^{-i\xi\cdot(x',-x_n)}\right) dx + \int_\Omega q \left(e^{-i\xi_+\cdot x} + e^{-i\xi_-\cdot x}\right) dx,\end{aligned} \tag{5.3.10}$$

where

$$\xi_- := \left(\xi', -2(X+iY)\frac{|\xi'|}{|\xi|}\right), \; \xi_+ := \left(\xi', 2(X+iY)\frac{|\xi'|}{|\xi|}\right).$$

For the first term in (5.3.10), we have

$$\begin{aligned}\int_\Omega q \left(e^{-i\xi\cdot x} + e^{-i\xi\cdot(x',-x_n)}\right) dx &= \int_\Omega q(x',x_n) e^{-i\xi\cdot x} dx + \int_{\Omega^*} q(x',-x_n) e^{-i\xi\cdot x} dx \\ &= \int_{\widetilde{\Omega}} q_e(x) e^{-i\xi\cdot x} dx = \int_{\mathbb{R}^n} \widetilde{q}_e(x) e^{-i\xi\cdot x} dx = \mathcal{F}[\widetilde{q}_e](\xi),\end{aligned}$$

where $\widetilde{q}_e$ is the extension of q_e from $\widetilde{\Omega}$ to $\mathbb{R}^n$ by zero.

Note that

$$\xi_+ = \left(\xi', 2(X+iY)\frac{|\xi'|}{|\xi|}\right) = \underbrace{\left(\xi', 2X\frac{|\xi'|}{|\xi|}\right)}_{\xi_+^{\text{Re}}} + i\underbrace{\left(0, 2Y\frac{|\xi'|}{|\xi|}\right)}_{\xi_+^{\text{Im}}},$$

$$\xi_- = \left(\xi', -2(X+iY)\frac{|\xi'|}{|\xi|}\right) = \underbrace{\left(\xi', -2X\frac{|\xi'|}{|\xi|}\right)}_{\xi_-^{\text{Re}}} + i\underbrace{\left(0, -2Y\frac{|\xi'|}{|\xi|}\right)}_{\xi_-^{\text{Im}}},$$

and

$$|\text{Im}\,(\xi_\pm)| = 2|Y|\frac{|\xi'|}{|\xi|} \le 2b.$$

Using the lower bound for X, we see that

$$|\xi_\pm^{\text{Re}}|^2 = |\xi'|^2 + 4X^2\frac{|\xi'|^2}{|\xi|^2} \ge |\xi'|^2 + 4\left(k^2 + a^2 - \frac{|\xi|^2}{4}\right)\frac{|\xi'|^2}{|\xi|^2} = 4\left(k^2+a^2\right)\frac{|\xi'|^2}{|\xi|^2}. \tag{5.3.11}$$

Also,

$$\int_\Omega q(x) e^{-i\xi_+\cdot x} \, dx = \int_\Omega q(x) e^{\xi_+^{\text{Im}}\cdot x} e^{-i\xi_+^{\text{Re}}\cdot x} \, dx = \int_\Omega q(x) e^{2Y\frac{|\xi'|}{|\xi|}x_n} e^{-i\xi_+^{\text{Re}}\cdot x} \, dx,$$

and

$$\int_\Omega q(x)e^{-i\xi_-\cdot x}\,dx = \int_\Omega q(x)e^{\xi_-^{\mathrm{Im}}\cdot x}e^{-i\xi_-^{\mathrm{Re}}\cdot x}\,dx = \int_\Omega q(x)e^{-2Y\frac{|\xi'|}{|\xi|}x_n}e^{-i\xi_-^{\mathrm{Re}}\cdot x}\,dx.$$

Consider the functions $g_+, g_- : \Omega\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ defined by

$$g_+(x,\eta) := e^{2Y\frac{|\eta'|}{|\eta|}x_n},\ x\in\Omega,\ \eta\in\mathbb{R}^n\setminus\{0\},$$

$$g_-(x,\eta) := e^{-2Y\frac{|\eta'|}{|\eta|}x_n},\ x\in\Omega,\ \eta\in\mathbb{R}^n\setminus\{0\}.$$

Note that $\forall\eta\in\mathbb{R}^n\setminus\{0\}$, $g_+(\cdot,\eta), g_-(\cdot,\eta)\in H^s(\Omega)\cap L^\infty(\Omega)$ (see Appendix 5.4) and

$$\|g_+(\cdot,\eta)\|_{H^s(\Omega)}\le Ce^{4bR},\quad \|g_+(\cdot,\eta)\|_{L^\infty(\Omega)}\le e^{2bR},$$

$$\|g_-(\cdot,\eta)\|_{H^s(\Omega)}\le Ce^{4bR},\quad \|g_-(\cdot,\eta)\|_{L^\infty(\Omega)}\le e^{2bR}.$$

It is important to note that the bounds above are independent of η.

Next, let us consider the function $q_\pm : \Omega\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ defined by

$$q_\pm(x,\eta) := q(x)g_\pm(x,\eta),\ x\in\Omega,\ \eta\in\mathbb{R}^n\setminus\{0\}.$$

Then $q_\pm(\cdot,\eta)\in H^s(\Omega),\ \forall\eta\in\mathbb{R}^n\setminus\{0\}$ (see Appendix 5.4) and

$$\|q_\pm(\cdot,\eta)\|_{H^s(\Omega)}\le Ce^{4bR},\ \forall\eta\in\mathbb{R}^n\setminus\{0\}.$$

Let $(q_\pm(\cdot,\eta))_z$ denote the zero extensions of $q_\pm(\cdot,\eta)$ from Ω to $\mathbb{R}^n$. Since $q_\pm(\cdot,\eta)\in H^s(\Omega)$ and $0<s<\frac{1}{2}$, it follows from Corollary 5.5 in Chapter 4 of [36] (see also [1]) that the zero extension belongs to $H^s(\mathbb{R}^n)$ and

$$\|(q_\pm(\cdot,\eta))_z\|_{H^s(\mathbb{R}^n)}\le C\|q_\pm(\cdot,\eta)\|_{H^s(\Omega)}\le Ce^{4bR}. \tag{5.3.12}$$

Also, $\forall\eta\in\mathbb{R}^n\setminus\{0\}$, the supports of $(q_\pm(\cdot,\eta))_z$ are contained inside Ω (and therefore, inside the compact set $\overline{B_R}$). By the quantitative Riemann-Lebesgue lemma (Lemma 2.3.4), there exists a constant $C>0$ and for any $N\in\mathbb{N}$, there exists a constant $C_N>0$ such that $\forall p\in\mathbb{R}^n$ and $\tau\in(0,1)$, we have

$$|\mathcal{F}[(q_\pm(\cdot,\eta))_z](p)|\le\frac{C_N}{(1+\tau|p|)^N}\|(q_\pm(\cdot,\eta))_z\|_{H^s(\mathbb{R}^n)}+C\tau^s\|(q_\pm(\cdot,\eta))_z\|_{H^s(\mathbb{R}^n)}. \tag{5.3.13}$$

Using (5.3.12) in (5.3.13), we see that $\forall \eta \in \mathbb{R}^n \setminus \{0\}$ and $\forall p \in \mathbb{R}^n$,

$$|\mathcal{F}[(q_\pm(\cdot,\eta))_z](p)| \leq \frac{Ce^{4bR}}{(1+\tau|p|)^N} + Ce^{4bR}\tau^s, \tag{5.3.14}$$

where the constant C depends only on N, s, Ω and M.

Note that the second term in (5.3.10) can be written as

$$\begin{aligned}\int_\Omega q\left(e^{-i\xi_+\cdot x} + e^{-i\xi_-\cdot x}\right)dx &= \int_{\mathbb{R}^n} (q_+(\cdot,\xi))_z(x)e^{-i\xi_+^{\mathrm{Re}}\cdot x}dx + \int_{\mathbb{R}^n} (q_-(\cdot,\xi))_z(x)e^{-i\xi_-^{\mathrm{Re}}\cdot x}dx \\ &= \mathcal{F}\left[(q_+(\cdot,\xi))_z\right](\xi_+^{\mathrm{Re}}) + \mathcal{F}\left[(q_-(\cdot,\xi))_z\right](\xi_-^{\mathrm{Re}}).\end{aligned}$$

Consequently, from (5.3.10), we get

$$|\mathcal{F}[\widetilde{q}_e](\xi)| \leq \left|\int_\Omega q u_0 \bar{v}\, dx\right| + |\mathcal{F}\left[(q_+(\cdot,\xi))_z\right](\xi_+^{\mathrm{Re}})| + |\mathcal{F}\left[(q_-(\cdot,\xi))_z\right](\xi_-^{\mathrm{Re}})|. \tag{5.3.15}$$

For the second and third terms of the right-hand side of the estimate (5.3.15), using (5.3.14) with $\eta = \xi$ and $p = \xi_\pm^{\mathrm{Re}}$, we obtain

$$|\mathcal{F}\left[(q_\pm(\cdot,\xi))_z\right](\xi_\pm^{\mathrm{Re}})| \leq \frac{Ce^{4bR}}{(1+\tau|\xi_\pm^{\mathrm{Re}}|)^N} + Ce^{4bR}\tau^s.$$

Now, using (5.3.11), we see that

$$|\mathcal{F}\left[(q_\pm(\cdot,\xi))_z\right](\xi_\pm^{\mathrm{Re}})| \leq \frac{Ce^{4bR}}{\left(1+2\tau(k^2+a^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}\right)^N} + Ce^{4bR}\tau^s,$$

where the constant C depends only on N, s, Ω and M.

For the first term in the right-hand side of the estimate (5.3.15), from (5.3.9), using Cauchy-Schwarz inequality, we see that

$$\begin{aligned}\left|\int_\Omega q u_0 v\, dx\right| &= \left|\sum_{j=0}^{m-1}\int_\Gamma \left[\partial_\nu((-\Delta)^j u_1)\right]\left[(-\Delta)^{m-1-j}v\right]\, dS\right| \\ &\leq \sum_{j=0}^{m-1}\int_\Gamma \left|\partial_\nu((-\Delta)^j u_1)\right|\,\left|(-\Delta)^{m-1-j}v\right|\, dS \\ &\leq \sum_{j=0}^{m-1}\|\partial_\nu(\Delta^j u_1)\|_{L^2(\Gamma)}\|\Delta^{m-1-j}v\|_{L^2(\Gamma)}\end{aligned}$$

and therefore

$$\begin{aligned}\left|\int_\Omega q u_0 v\, dx\right| &\leq \left(\sum_{j=0}^{m-1} \|\partial_\nu(\Delta^j u_1)\|_{L^2(\Gamma)}\right)\left(\sum_{j=0}^{m-1}\|\Delta^{m-1-j} v\|_{L^2(\Gamma)}\right)\\ &\leq \left(\sum_{j=0}^{m-1} \|\partial_\nu(\Delta^j u_1)\|_{L^2(\Gamma)}\right)\left(\sum_{j=0}^{m-1}\|\Delta^{m-1-j} v\|_{L^2(\partial\Omega)}\right)\\ &\leq \left(\sum_{j=0}^{m-1} \|\partial_\nu(\Delta^{m-1-j} u_1)\|_{H^{2j+\frac{1}{2}}(\Gamma)}\right)\left(\sum_{j=0}^{m-1}\|\Delta^{j} v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right).\end{aligned}$$

Using the trace theorem and the linearized partial D-N map, we further observe that

$$\begin{aligned}\left|\int_\Omega q u_0 v\, dx\right| &\leq C\left(\sum_{j=0}^{m-1} \|\partial_\nu(\Delta^{m-1-j} u_1)\|_{H^{2j+\frac{1}{2}}(\Gamma)}\right)\left(\sum_{j=0}^{m-1}\|\Delta^{j} v\|_{H^1(\Omega)}\right)\\ &\leq C\|\left(\partial_\nu u_1, \partial_\nu(\Delta u_1), ..., \partial_\nu(\Delta^{m-1}u_1)\right)\|_{H^{2m-\frac{3}{2},...,\frac{1}{2}}(\Gamma)}\|v\|_{H^{2m}(\Omega)}\\ &= C\|\mathcal{N}^{\mathbb{P}}_{q,L}(f_1, f_2, ..., f_m)\|_{H^{2m-\frac{3}{2},...,\frac{1}{2}}(\Gamma)}\|v\|_{H^{2m}(\Omega)}\\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*\|(u_0\big|_\Gamma, \Delta u_0\big|_\Gamma, ..., \Delta^{m-1}u_0\big|_\Gamma)\|_{\widetilde{H}^{2m-\frac{1}{2},...,\frac{3}{2}}(\Gamma)}\|v\|_{H^{2m}(\Omega)}\\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*\left(\sum_{j=1}^{m}\|\Delta^{m-j}u_0\|_{\widetilde{H}^{2j-\frac{1}{2}}(\Gamma)}\right)\|v\|_{H^{2m}(\Omega)}\\ &= C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*\left(\sum_{j=1}^{m}\|\Delta^{m-j}u_0\|_{H^{2j-\frac{1}{2}}(\partial\Omega)}\right)\|v\|_{H^{2m}(\Omega)}\\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*\left(\sum_{j=1}^{m}\|\Delta^{m-j}u_0\|_{H^{2j}(\Omega)}\right)\|v\|_{H^{2m}(\Omega)}\\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*\|u_0\|_{H^{2m}(\Omega)}\|v\|_{H^{2m}(\Omega)}.\end{aligned} \tag{5.3.16}$$

Here, we also used the facts that

$$\|\Delta^j v\|_{H^1(\Omega)} \leq \|v\|_{H^{2m}(\Omega)}, \quad \forall\, j = 0, ..., m-1,$$

$$\|\Delta^{m-j} u_0\|_{H^{2j}(\Omega)} \leq \|u_0\|_{H^{2m}(\Omega)}, \quad \forall\, j = 1, ..., m.$$

Now to estimate the terms in the right hand side of (5.3.16), we first make the following observation. We recall from (5.3.5) and (5.3.6) that $\widetilde{u}_0$ and $\widetilde{v}$ satisfy the PDEs

$$P\widetilde{u}_0 = g_0 \quad \text{and} \quad P\widetilde{v} = g \ \text{ in } B_R, \ \text{ respectively,}$$

where

$$P = (-\Delta)^m \text{ and } g_0 = (k^2 - ikb)^m \widetilde{u}_0,\ g = (k^2 - ikb)^m \widetilde{v}.$$

Therefore, using the interior regularity for elliptic operators (see Theorem 2.3.3) in B_R, we can conclude that

$$\|\widetilde{u}_0\|_{H^{2m}(\widetilde{\Omega})} \leq C\|g_0\|_{L^2(V)} + C\|\widetilde{u}_0\|_{L^2(V)}$$
$$\text{and } \|\widetilde{v}\|_{H^{2m}(\widetilde{\Omega})} \leq C\|g\|_{L^2(V)} + C\|\widetilde{v}\|_{L^2(V)},$$

for any V such that $\widetilde{\Omega} \subset\subset V \subset\subset B_R$. Using the above estimates, we further observe that

$$\begin{aligned} \|\widetilde{u}_0\|_{H^{2m}(\widetilde{\Omega})} &\leq C\|g_0\|_{L^2(B_R)} + C\|\widetilde{u}_0\|_{L^2(B_R)} \\ \text{and } \|\widetilde{v}\|_{H^{2m}(\widetilde{\Omega})} &\leq C\|g\|_{L^2(B_R)} + C\|\widetilde{v}\|_{L^2(B_R)}, \end{aligned} \tag{5.3.17}$$

for some constant C that depends on M and Ω.

Now, we will bound the terms $\|g_0\|_{L^2(B_R)}$ and $\|g\|_{L^2(B_R)}$ by $\|\widetilde{u}_0\|_{L^2(B_R)}$ and $\|\widetilde{v}\|_{L^2(B_R)}$ respectively. For g_0, we have

$$\begin{aligned} \|g_0\|_{L^2(B_R)} &\leq \|(k^2 - ikb)^m \widetilde{u}_0\|_{L^2(B_R)} \leq |k^2 - ikb|^m \|\widetilde{u}_0\|_{L^2(B_R)} \\ &\leq (k^4 + k^2 b^2)^{\frac{m}{2}} \|\widetilde{u}_0\|_{L^2(B_R)} \leq (k^2 + kb)^m \|\widetilde{u}_0\|_{L^2(B_R)}, \end{aligned} \tag{5.3.18}$$

where we have also used the inequality $(c^2 + d^2)^{\frac{1}{2}} \leq c + d,\ c, d \geq 0$.

Similarly, we have

$$\|g\|_{L^2(B_R)} \leq (k^2 + kb)^m \|\widetilde{v}\|_{L^2(B_R)}. \tag{5.3.19}$$

Next, using (5.3.4), we see that

$$\begin{aligned} \|\widetilde{u}_0\|^2_{L^2(B_R)} &= \int_{B_R} |\widetilde{u}_0|^2 dx = \int_{B_R} \left|e^{i\zeta_1 \cdot x}\right|^2 dx \leq \int_{B_R} e^{2|\mathrm{Im}\,(\zeta_1)||x|} dx \\ &\leq e^{2(b+a)R} \int_{B_R} dx = e^{2(b+a)R} |B_R| \leq Ce^{2(b+a)R}. \end{aligned} \tag{5.3.20}$$

Similarly, we have

$$\|\widetilde{v}\|^2_{L^2(B_R)} \leq Ce^{2(b+a)R}. \tag{5.3.21}$$

Now, observe that

$$\|u_0\|_{H^{2m}(\Omega)} \leq 2\|\widetilde{u}_0\|_{H^{2m}(\widetilde{\Omega})} \text{ and } \|v\|_{H^{2m}(\Omega)} \leq 2\|\widetilde{v}\|_{H^{2m}(\widetilde{\Omega})}. \tag{5.3.22}$$

Combining these estimates with those in (5.3.17), (5.3.18), (5.3.19) and the fact that $k \geq 1$, we get

$$\|u_0\|_{H^{2m}(\Omega)} \leq C(k^2+kb)^m e^{(b+a)R} \quad \text{and} \quad \|v\|_{H^{2m}(\Omega)} \leq C(k^2+kb)^m e^{(b+a)R}. \tag{5.3.23}$$

Consequently, from the estimate (5.3.16), we have

$$\left|\int_\Omega q u_0 v \, dx\right| \leq C(k^2+kb)^{2m} \|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* e^{2(b+a)R}. \tag{5.3.24}$$

Then, from the inequality (5.3.15), we have

$$|\mathcal{F}[\tilde{q}_e](\xi)| \leq C(k^2+kb)^{2m} \|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* e^{2(b+a)R} + \frac{Ce^{4bR}}{\left(1+2\tau(k^2+a^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}\right)^N} + Ce^{4bR}\tau^s, \tag{5.3.25}$$

for $0 < |\xi|^2 \leq 3(k^2+a^2)$ with $|\xi'| > 0$, $s \in (0, \frac{1}{2})$, $\tau \in (0,1)$ and $N \in \mathbb{N}$.

Next, we estimate the H^{-1} norm of $\widetilde{q}_e$ using (5.3.25). First, for $\rho > 1$ (to be chosen later), we define

$$\mathcal{K}_\rho := \{(\xi', \xi_n) \in \mathbb{R}^n \;\; : \;\; 0 < |\xi'| < \rho \text{ and } |\xi_n| < \rho\}.$$

Then, for $\xi \in \mathcal{K}_\rho$, using the estimate (5.3.25), we have

$$|\mathcal{F}[\widetilde{q}_e](\xi)| \leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*(k^2+kb)^{2m} e^{2(b+a)R} + \frac{Ce^{4bR}}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^N} + Ce^{4bR}\tau^s, \tag{5.3.26}$$

since $|\xi| < (\rho^2+\rho^2)^{\frac{1}{2}} = \sqrt{2}\rho < 2\rho$.

Now

$$\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi + \int_{\mathbb{R}^n\setminus\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi. \tag{5.3.27}$$

By applying Parseval's identity, the second integral above can be estimated as follows:

$$\int_{\mathbb{R}^n\setminus\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \leq \frac{C}{\rho^2}, \tag{5.3.28}$$

where C depends on M and Ω. For the first integral in (5.3.27), we have

$$\begin{aligned}
\int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi &\leq \int_{\mathcal{K}_\rho} |\mathcal{F}[\widetilde{q}_e](\xi)|^2 \, d\xi \\
&= \int_{\mathcal{K}_\rho} \left| C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*(k^2+kb)^{2m} e^{2(b+a)R} + Ce^{4bR}\tau^s + \frac{Ce^{4bR}}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^N} \right|^2 d\xi \\
&\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*^2 (k^2+kb)^{4m} e^{4(b+a)R} \left(\int_{\mathcal{K}_\rho} d\xi \right) \\
&\quad + Ce^{8bR}\tau^{2s} \left(\int_{\mathcal{K}_\rho} d\xi \right) + Ce^{8bR} \int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{2N}}.
\end{aligned} \tag{5.3.29}$$

We next estimate the integrals in the above estimate. We note that

$$\int_{\mathcal{K}_\rho} d\xi \leq C\rho^n. \tag{5.3.30}$$

Now, using the fact that $\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{2N} > \left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^N$ for $N \in \mathbb{N}$, the second integral can be estimated as follows:

$$\begin{aligned}
\int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{2N}} &\leq \int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{N}} \\
&\leq C\rho \int_0^\rho \frac{r^{n-2}dr}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}r\right)^N}.
\end{aligned}$$

Choosing $N > n-1 > 1$, we see that

$$\int_0^\rho \frac{r^{n-2}dr}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}r\right)^N} \leq \frac{C\rho^{n-1}}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}.$$

Therefore

$$\int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{2N}} \leq \frac{C\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}. \tag{5.3.31}$$

Using the estimates (5.3.30) and (5.3.31) in (5.3.29), we obtain

$$\int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*^2 (k^2+kb)^{4m} e^{4(b+a)R}\rho^n + Ce^{8bR}\tau^{2s}\rho^n + \frac{Ce^{8bR}\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}. \tag{5.3.32}$$

Now, using (5.3.28), (5.3.32) and the fact that $e^{8Rb} \geq 1$ in (5.3.27), we get

$$\begin{aligned} \|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|^2_*(k^2+kb)^{4m}e^{4(b+a)R}\rho^n + Ce^{8Rb}\tau^{2s}\rho^n \\ &\quad + \frac{Ce^{8Rb}\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}} + \frac{Ce^{8Rb}}{\rho^2}. \end{aligned} \tag{5.3.33}$$

Next, we choose $\tau \in (0,1)$ and $\rho > 0$ appropriately. First, we choose $\tau^2 = (k^2+a^2)^{-\frac{n-1}{2s+n-1}}$. Then, given that $k, a \geq 1$, we find that

$$\tau \in (0,1) \quad \text{and} \quad \tau^{2s} = \frac{1}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}.$$

With this choice, we can combine the second and third terms in the right-hand side of (5.3.33) into

$$\frac{e^{8Rb}\rho^n}{(k^2+a^2)^{\frac{s(n-1)}{2s+n-1}}}.$$

Finally, we choose $\rho = (k^2+a^2)^{\frac{s(n-1)}{(n+2)(2s+n-1)}} > 1$. With these choices, the estimate (5.3.33) can be rewritten as

$$\begin{aligned} &\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|^2_*(k^2+kb)^{4m}e^{4(b+a)R}(k^2+a^2)^{\frac{s(n-1)n}{(n+2)(2s+n-1)}} + \frac{Ce^{8Rb}}{(k^2+a^2)^{\frac{2s(n-1)}{(n+2)(2s+n-1)}}}. \end{aligned} \tag{5.3.34}$$

Note that we have derived this estimate under the assumption that if $\xi \in \mathcal{K}_\rho$, then the condition $|\xi|^2 \leq 3(k^2+a^2)$ is satisfied. It is easy to check that with our choice of ρ, this is indeed true. Let us denote $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)}$. Using the fact that $\frac{s(n-1)n}{(n+2)(2s+n-1)} < \frac{1}{2}$ along with the fact $k, a \geq 1$, we obtain

$$\begin{aligned} \|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|^2_*(k^2+kb)^{4m}e^{4(b+a)R}(k^2+a^2)^{\frac{1}{2}} + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|^2_*(k^2+kb)^{4m}e^{4(b+a)R}(k^2a^2+k^2a^2)^{\frac{1}{2}} + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|^2_*(k^2+kb)^{4m}e^{4(b+a)R}ka + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\leq C\|\mathcal{N}^{\mathbb{P}}_{q,L}\|^2_*(k^2+kb)^{4m}e^{4(b+a)R}(k^2+kb)e^{Ra} + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &= Ce^{8Rb}\left[(k^2+kb)^{4m+1}\|\mathcal{N}^{\mathbb{P}}_{q,L}\|^2_*e^{5Ra} + \frac{1}{(k^2+a^2)^\sigma}\right], \end{aligned} \tag{5.3.35}$$

where the second last line follows from the facts that $k \le k^2 + kb$ and $a \le e^{Ra}$.

Next, we choose $a > 1$ suitably to obtain the stability estimate.

Let $a = \frac{E}{5R}$. Then the previous estimate becomes

$$\|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} \le Ce^{4Rb}\left[(k^2+kb)^{4m+1}\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}}. \tag{5.3.36}$$

However, we need to ensure that with our choice of a, the condition $a > 1$ is satisfied. This leads to the following condition on $\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*$. Let

$$\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* < \delta := \frac{1}{e^{5R}}. \tag{5.3.37}$$

Then

$$e^{5R} < \frac{1}{\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*} \iff 5R < \log(\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*^{-1}) = E \iff 1 < \frac{E}{5R} = a. \tag{5.3.38}$$

Hence, our choice of a satisfies the required condition.

From (5.3.36), we obtain the estimate

$$\|q\|_{H^{-1}(\Omega)} \le \|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} \le Ce^{4Rb}\left[(k^2+kb)^{4m+1}\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}}. \tag{5.3.39}$$

The case when $\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* \ge \frac{1}{e^{5R}}$ easily follows from the following fact:

$$\begin{aligned}\|q\|_{H^{-1}(\Omega)} \le C\|q\|_{L^\infty(\Omega)} \le \frac{CM}{\delta^{\frac{1}{2}}}\delta^{\frac{1}{2}} \le \frac{CM}{\delta^{\frac{1}{2}}}\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_*^{\frac{1}{2}} \\ \le Ce^{4Rb}\left[(k^2+kb)^{4m+1}\|\mathcal{N}^{\mathbb{P}}_{q,L}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}},\end{aligned}$$

where the constant C depends only on s, n, Ω and M. Thus, the stability estimate (5.2.4) follows.

5.4 Appendix

In this appendix, we analyse the properties of the functions g_+ and g_- used in section 5.3. Consider the functions $g_+, g_- : B_R \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$g_+(x,\eta) := e^{2Y\frac{|\eta'|}{|\eta|}x_n},\ x \in B_R,\ \eta \in \mathbb{R}^n \setminus \{0\},$$
$$g_-(x,\eta) := e^{-2Y\frac{|\eta'|}{|\eta|}x_n},\ x \in B_R,\ \eta \in \mathbb{R}^n \setminus \{0\}.$$

In section 5.3, we work with these functions restricted to the smaller domain $\Omega \times \mathbb{R}^n \setminus \{0\}$. We note that

$$|g_\pm(x,\eta)| \le e^{2|Y|\frac{|\eta'|}{|\eta|}|x_n|} \le e^{2bR},\ \forall \eta \in \mathbb{R}^n \setminus \{0\},$$

and therefore $\forall \eta \in \mathbb{R}^n \setminus \{0\}$, we have

$$\|g_\pm(\cdot,\eta)\|_{L^\infty(\Omega)} \le \|g_\pm(\cdot,\eta)\|_{L^\infty(B_R)} \le e^{2bR},$$

and

$$\|g_\pm(\cdot,\eta)\|_{L^2(\Omega)} = \left(\int_\Omega |g_\pm(x,\eta)|^2\, dx\right)^{\frac{1}{2}} \le |\Omega|^{\frac{1}{2}} e^{2bR}.$$

Also, since $\frac{|\eta'|}{|\eta|} \le 1$, we have

$$\|\nabla g_\pm(\cdot,\eta)\|_{L^\infty(B_R)} \le 2|Y|\frac{|\eta'|}{|\eta|}e^{2|B||x_n|} \le 2be^{2bR} \le 2bRe^{2bR} \le e^{4bR},\ \forall \eta \in \mathbb{R}^n \setminus \{0\}.$$

Now, since B_R is bounded and $n+2s-2 < n$, we see that, $\forall \eta \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{|g_\pm(x,\eta)-g_\pm(y,\eta)|^2}{|x-y|^{n+2s}}dxdy &\le \int_{B_R}\int_{B_R} \frac{|g_\pm(x,\eta)-g_\pm(y,\eta)|^2}{|x-y|^{n+2s}}dxdy \\
&\le \|\nabla g_\pm(\cdot,\eta)\|^2_{L^\infty(B_R)} \int_{B_R}\int_{B_R} \frac{|x-y|^2}{|x-y|^{n+2s}}dxdy \\
&= \|\nabla g_\pm(\cdot,\eta)\|^2_{L^\infty(B_R)} \int_{B_R}\int_{B_R} \frac{1}{|x-y|^{n+2s-2}}dxdy \\
&\le C\|\nabla g_\pm(\cdot,\eta)\|^2_{L^\infty(B_R)} \le Ce^{8bR}.
\end{aligned}$$

(Here we work with the larger domain B_R owing to the fact that B_R is convex but Ω need not be. The convexity of B_R allows us to apply the mean value theorem).

Therefore, $\forall \eta \in \mathbb{R}^n \setminus \{0\}$, we have

$$\begin{aligned}\|g_\pm(\cdot,\eta)\|_{H^s(\Omega)} &= \|g_\pm(\cdot,\eta)\|_{L^2(\Omega)} + \left(\int_\Omega\int_\Omega \frac{|g_\pm(x,\eta)-g_\pm(y,\eta)|^2}{|x-y|^{n+2s}}dxdy\right)^{\frac{1}{2}}\\ &\le |\Omega|^{\frac{1}{2}}e^{2bR} + Ce^{4bR} \le Ce^{4bR},\end{aligned}$$

where the constant C depends only on n, s and Ω. Note that the constant C depends on B_R as well but B_R in turn depends only on Ω.

Next, let us consider the function $q_\pm : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$q_\pm(x,\eta) := q(x)g_\pm(x,\eta),\ x \in \Omega,\ \eta \in \mathbb{R}^n \setminus \{0\}.$$

We observe that $q_\pm(\cdot,\eta) \in H^s(\Omega),\ \forall \eta \in \mathbb{R}^n \setminus \{0\}$. To see this, we note that

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|q_\pm(x,\eta)-q_\pm(y,\eta)|^2}{|x-y|^{n+2s}}\,dxdy\\ &\le C\left(\int_\Omega\int_\Omega \frac{|q(x)|^2|g_\pm(x,\eta)-g_\pm(y,\eta)|^2}{|x-y|^{n+2s}}\,dxdy + \int_\Omega\int_\Omega \frac{|g_\pm(y,\eta)|^2|q(x)-q(y)|^2}{|x-y|^{n+2s}}\,dxdy\right)\\ &\le C\left(\|q\|^2_{L^\infty(\Omega)}e^{8bR} + M^2\|g_\pm(\cdot,\eta)\|^2_{L^\infty(\Omega)}\right) \le C\left(M^2e^{8bR} + M^2e^{4bR}\right)\\ &\le Ce^{8bR},\ \forall \eta \in \mathbb{R}^n \setminus \{0\},\end{aligned}$$

where the constant C depends only on n, s, M and Ω. Therefore,

$$\left(\int_\Omega\int_\Omega \frac{|q_\pm(x,\eta)-q_\pm(y,\eta)|^2}{|x-y|^{n+2s}}\,dxdy\right)^{\frac{1}{2}} \le Ce^{4bR}.$$

Also,

$$\|q_\pm(\cdot,\eta)\|_{L^2(\Omega)} \le \|q\|_{L^\infty(\Omega)}\|g_\pm(\cdot,\eta)\|_{L^2(\Omega)} \le Ce^{2bR}.$$

Therefore,

$$\|q_\pm(\cdot,\eta)\|_{H^s(\Omega)} \le Ce^{2bR} + Ce^{4bR} \le Ce^{4bR},\ \forall \eta \in \mathbb{R}^n \setminus \{0\}.$$

Let $(q_\pm(\cdot,\eta))_z$ denote the zero extensions of $q_\pm(\cdot,\eta)$ from Ω to $\mathbb{R}^n$. Then, $\forall \eta \in \mathbb{R}^n \setminus \{0\}$, we have

$$\|(q_\pm(\cdot,\eta))_z\|_{L^2(\mathbb{R}^n)} = \|q_\pm(\cdot,\eta)\|_{L^2(\Omega)} \le Ce^{2bR},$$

and

$$\|(q_{\pm}(\cdot,\eta))_z\|_{H^s(\mathbb{R}^n)} \leq C\|q_{\pm}(\cdot,\eta)\|_{H^s(\Omega)} \leq Ce^{4bR},$$

where the generic constant C depends only on n, s, M and Ω.

Chapter 6

Inverse Schrödinger and biharmonic problem with attenuation

In this chapter, following [14], we explore high-frequency stability estimates for the determination of the zeroth-order perturbation of the Schrödinger and the biharmonic operators with constant attenuation from the partial Dirichlet-to-Neumann map in domains satisfying the flatness condition $(\mathcal{A})$. The results are derived under mild regularity assumptions on the potential and extend the results of [26] and [33] in the presence of attenuation in such domains.

6.1 Introduction

Let us consider the following boundary-value problems for the Schrödinger equation:

$$\begin{cases} -\Delta u \;-\; (k^2 - ikb)u \;+\; qu &= 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u &= f \quad \text{on } \partial\Omega, \end{cases} \tag{6.1.1}$$

and the perturbed biharmonic operator:

$$\begin{cases} \Delta^2 u \;-\; (k^2 - ikb)^2 u \;+\; qu &= 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u &= f \quad \text{on } \partial\Omega, \\ \qquad\qquad\qquad\quad \Delta u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{6.1.2}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 3$, with a smooth boundary $\partial\Omega$.

Here $b > 0$ is the constant attenuation and the potential q is real valued and has suitable Sobolev regularity (to be described in due course). We shall assume that the frequency $k \geq 1$. For (6.1.1), we consider the Dirichlet boundary condition and for (6.1.2), we consider the Navier boundary conditions (see [15]) respectively. The regularities of the boundary conditions will be specified in due course.

The attenuation models arise in the modelling of time harmonic wave phenomenon in acoustics (see [27]), particularly while dealing with *lossy* media. The presence of attenuation accounts for the dissipation of the acoustic energy.

In this work, we shall assume that the bounded domain Ω satisfies the flatness condition $(\mathcal{A})$. We also assume that the supports of the boundary data are contained in $\Gamma := \partial\Omega \setminus \Gamma_0$. The boundary measurements are assumed to be available on Γ only, and thus, the flat part Γ_0 is assumed to be inaccessible.

For the Schrödinger case, we shall assume that $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$. Note that the imaginary part of $(k^2 - ikb)$ is non-zero and hence, it is not a part of the spectrum of $-\Delta + q$. Therefore, there exists a unique solution to (6.1.1) when $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$.

The corresponding partial Dirichlet-to-Neumann (D-N) map is defined as

$$\mathcal{N}_q^S \ : \ \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \quad \text{such that} \quad \mathcal{N}_q^S(f) := \partial_\nu u\Big|_{\Gamma},$$

where ν is the unit outer normal vector to $\partial\Omega$ and u is the unique solution to (6.1.1) with boundary data f. The operator norm $\|\mathcal{N}_q^S\|_*$ of $\mathcal{N}_q^S$ is defined accordingly. Note that the above definition of $\mathcal{N}_q^S$ is not rigorous and it has to be suitably interpreted in a weak sense.

For the biharmonic case, we assume that the boundary data $(f, g) \in \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma)$. Since the imaginary part of $(k^2 - ikb)^2$ is non-zero, it is not a part of the spectrum of $\Delta^2 + q$ and therefore, there exists a unique solution to (6.1.2) when $(f, g) \in \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma)$.

In this case, the partial Dirichlet-to-Neumann (D-N) map is defined as

$$\mathcal{N}_q^B \ : \ \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma) \to H^{\frac{5}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \quad \text{such that} \quad (f, g) \mapsto \left(\partial_\nu u\Big|_{\Gamma}, \partial_\nu (\Delta u)\Big|_{\Gamma} \right),$$

where u is the solution to the problem (6.1.2) with boundary data (f, g).

We define

$$\|\mathcal{N}_q^B\|_* := \sup\left\{\|\mathcal{N}_q^B(f,g)\|_{H^{\frac{5}{2},\frac{1}{2}}(\Gamma)} \;:\; \|(f,g)\|_{\widetilde{H}^{\frac{7}{2},\frac{3}{2}}(\Gamma)} = 1\right\}.$$

We have assumed more regularity on the boundary data in the biharmonic case, compared to that in the Schrödinger case, to make the calculations simpler.

Our aim, in this chapter, is to address the question of stability of the recovery of the potential q from the knowledge of the partial D-N map and to study the dependence of the stability estimate on the frequency k. As in [33], we only assume that the potential $q \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$.

The study of inverse boundary value problems of this kind has received vigorous attention following the pioneering work by Calderón on the conductivity equation [6] and the work by Sylvester and Uhlmann [35] on the conductivity and the Schrödinger equations. The stability of the recovery of the potential q in the case of the Schrödinger equation was first considered in [2], wherein logarithmic type stability estimates were obtained for the case $k = 0, b = 0$ with measurements available on the full boundary. The optimality of these logarithmic type estimates was later studied in [34]. In the case of partial boundary measurements, in the case $k = 0, b = 0$, corresponding to the identification result in [5], log-log type stability estimates were derived in [16]. For partial boundary measurements in the case of domains Ω satisfying the assumption $(\mathcal{A})$ or when the inaccessible part of the boundary is part of a sphere, the identification result (when $k = 0, b = 0$) was proved in [21], following which the work [17] showed that the optimal stability is of logarithmic type in this case.

For related works on the question of uniqueness in determination of the potential from the Dirichlet-to-Neumann map for the biharmonic operator (when $k = 0, b = 0$), we refer to the works [19, 28, 29, 37]. Stability results, for the full boundary measurements, measurements on more than half the boundary and for domains $(\mathcal{A})$, were proved in [8, 10] in the case when $k = 0, b = 0$.

Given the fact that logarithmic type stability estimates lead to poor results in reconstruction, the possibility of improving the stability estimate to a Lipschitz type (the increasing stability phenomenon) in the presence of a frequency k started getting attention in the recent years. This phenomenon has been termed as the increasing stability phenomenon in the literature. In particular, the works [9, 22, 32] derive frequency-dependent stability estimates, exhibiting improvements in stability in the case of the Schrödinger equation (in the case $b = 0$) and the work [33] studies the biharmonic case in domains satisfying the assumption $(\mathcal{A})$ and derives similar stability estimates.

In the presence of an attenuation coefficient, the work [26] studied the increasing stability property for full boundary measurements for the Schrödinger equation with constant attenuation. The results were extended to the case of non-constant attenuation in [23]. For related results in the context of inverse source problems, we would like to refer the work [24].

The stability of recovery of the potential from the linearized D-N map was recently studied in [25] for the Schrödinger case with full boundary measurements. This work was followed by [39] (for the case $b = 0$ and the domain satisfying the assumption $(\mathcal{A})$) and the work [31] (which deals with constant attenuation, with the domain satisfying the assumption $(\mathcal{A})$). For the biharmonic case, the work [38] proved similar estimates in the presence of constant attenuation from full boundary measurements, and [12] dealt with the case of constant attenuation in domains satisfying the condition $(\mathcal{A})$.

In this work, we extend the results for the Schrödinger case in [26] to domains of type $(\mathcal{A})$ and potentials q satisfying a considerably lower regularity. As in [33], we only assume that the potentials $q \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$. Our result for the biharmonic case extends the results in [33] to the case of constant attenuation. The main difficulty, in this direction, was to prove the existence of complex geometric optics (CGO) type solutions with such low regularity on the potential. The Schauder fixed point argument in [26] didn't seem to be directly applicable in this case, due to the lower regularity assumptions. Also, due to the presence of the attenuation, the approach taken in [33] to prove the existence of CGO solutions was not directly applicable either. In order to handle this, we used an approach similar to [26] but used Browder fixed point theorem instead to prove the existence of CGO type distributional solutions in a suitably large open ball containing Ω. The required regularity of the solutions could then be inferred from the interior elliptic regularity estimates.

The plan of the chapter is as follows. In Section 6.2, we discuss our main results on the stability estimates. In Section 6.3, some preliminary results are discussed. The construction of appropriate complex geometric optics (CGO) type solutions are discussed in Section 6.4. The stability results for the Schrödinger case and the biharmonic case are derived in Sections 6.5 and 6.6 respectively. In appendix I, we recall some results from [31] that are used in this chapter and in appendix II, we discuss some results related to the construction of CGO type solutions for the biharmonic case.

6.2 Main result and related remarks

In this section, we describe our main results for the Schrödinger equation and the perturbed biharmonic operator.

6.2.1 The Schrödinger case

Let us denote $E := |\log(\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_*)|$. Then we have the following stability result for the recovery of the potential from the D-N map in the Schrödinger case.

Theorem 6.2.1. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$ satisfying the assumption $(\mathcal{A})$*

described above and $\Omega \subset\subset B(0,R)$ *for some* $R > 1$. *Let* b *be a positive constant and suppose that the frequency* $k \geq 1$. *Further, suppose that the potentials* $q_1, q_2 \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$ *be such that*

$$\|q_j\|_{L^\infty(\Omega)} + \|q_j\|_{H^s(\Omega)} \leq M,\ j = 1,2,$$

for some $M > 0$. *Then there exists a constant* $C > 0$*, depending only on* n*,* s*,* Ω *and* M *such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C e^{4Rb} \left[(k^2 + kb)^5 \|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* + \frac{1}{(k^2 + \left(\frac{E}{5R}\right)^2)^\sigma} \right]^{\frac{1}{2}}, \tag{6.2.1}$$

where $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)} < \frac{1}{2}$.

Remark 6.2.2. Using a standard interpolation argument (see [12]) we can estimate the L^2 norm of $q_1 - q_2$ from the H^{-1} estimate above. Also, in case the potentials have more regularity, we can use (6.2.1) and interpolation argument to derive the stability estimates.

Remark 6.2.3. The fact that (6.2.1) implies improvement in stability with an appropriate choice of the frequency k can be observed as in [12].

6.2.2 The Biharmonic case

Let us denote $E := |\log(\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_*)|$. Then, in the biharmonic case, we have the following stability result for the recovery of the potential from the D-N map.

Theorem 6.2.4. *Let* Ω *be a bounded domain in* $\mathbb{R}^n, n \geq 3$ *satisfying the assumption* $(\mathcal{A})$ *described above and* $\Omega \subset\subset B(0,R)$ *for some* $R > 1$. *Let* b *be a positive constant and suppose that the frequency* $k \geq 1$. *Further, suppose that the potentials* $q_1, q_2 \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$ *be such that*

$$\|q_j\|_{L^\infty(\Omega)} + \|q_j\|_{H^s(\Omega)} \leq M,\ j = 1,2,$$

for some $M > 0$. *Then there exists a constant* $C > 0$, *depending only on* n, s, Ω *and* M *such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq Ce^{6Rb}\left[e^{CRk}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_* + \frac{1}{(k^2 + \left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}}, \tag{6.2.2}$$

where $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)} < \frac{1}{2}$.

Remark 6.2.5. The stability estimate (6.2.2) is weaker than in (6.2.1) due the presence of the exponential factor in k. Though this estimate is consistent with the stability estimate in [33], but for the linearized case, it was observed in [12] that a polynomial factor in the frequency k appears. This difference arises in the construction of the CGO solutions as we shall observe shortly (see Remark 6.4.3). A refined construction will possibly help improve this to a polynomial factor.

6.3 Some preliminary results

In this section, we discuss some results that will be used in the construction of the CGO type solutions and the derivations of the stability estimates. We start with the following observations regarding suitable integral identities for the Schrödinger and the biharmonic operator.

For $j = 1, 2$, let $u_j \in H^1(\Omega)$ denote the solutions to the boundary value problems

$$\begin{cases} -\Delta u_j \; - \; (k^2 - ikb)u_j \; + \; q_j u_j & = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\quad u_j & = f \quad \text{on } \partial\Omega, \end{cases} \tag{6.3.1}$$

for potentials q_j as in Theorem 6.2.1 and $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$. Then, $u := u_1 - u_2$ is the unique solution to the problem

$$\begin{cases} -\Delta u \; - \; (k^2 - ikb)u \; + \; q_1 u & = -(q_1 - q_2)u_2 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\quad u & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.3.2}$$

Let $v \in H^1(\Omega)$ be such that

$$\begin{cases} -\Delta v \ - \ (k^2 + ikb)v \ + \ q_1 v & = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad v & = 0 \quad \text{on } \Gamma_0. \end{cases} \tag{6.3.3}$$

Then, from (6.3.2) and Green's formula, proceeding similarly as in [2], we have the following integral identity.

Lemma 6.3.1. *Let $u, v \in H^1(\Omega)$ be as described above (see (6.3.2) and (6.3.3)). Then*

$$\int_\Omega (q_1 - q_2) u_2 \bar{v} \, dx = \int_\Gamma \partial_\nu u \, \bar{v} \, dS. \tag{6.3.4}$$

□

Similarly, as in the case of the Schrödinger equation, for $j = 1, 2$, let $u_j \in H^4(\Omega)$ denote the solutions to the boundary value problems

$$\begin{cases} \Delta^2 u_j \ - \ (k^2 - ikb)^2 u_j \ + \ q_j u_j & = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad u_j & = f \quad \text{on } \partial\Omega, \\ \qquad\qquad\qquad \Delta u_j & = g \quad \text{on } \partial\Omega, \end{cases}$$

where the potentials q_j are as described in Theorem 6.2.4 and $(f, g) \in \widetilde{H}^{\frac{7}{2}}(\Gamma) \times \widetilde{H}^{\frac{3}{2}}(\Gamma)$. Then, $u := u_1 - u_2$ is the unique solution to the boundary value problem

$$\begin{cases} \Delta^2 u \ - \ (k^2 - ikb)^2 u \ + \ q_1 u & = -(q_1 - q_2) u_2 \quad \text{in } \Omega, \\ \qquad\qquad\qquad u & = 0 \quad \text{on } \partial\Omega, \\ \qquad\qquad\qquad \Delta u & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.3.5}$$

Let $v \in H^4(\Omega)$ be such that

$$\begin{cases} \Delta^2 v \ - \ (k^2 + ikb)^2 v \ + \ q_1 v & = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad v & = 0 \quad \text{on } \Gamma_0, \\ \qquad\qquad\qquad \Delta v & = 0 \quad \text{on } \Gamma_0. \end{cases} \tag{6.3.6}$$

Then, from (6.3.5) and Green's formula, we have the following integral identity (that can be proved using arguments similar to that in [8]).

Lemma 6.3.2. *Let $u, v \in H^4(\Omega)$ be as described above (see (6.3.5) and (6.3.6)). Then*

$$\int_\Omega (q_2 - q_1) u_2 \bar{v} \, dx = \int_\Gamma \partial_\nu (\Delta u) \, \bar{v} \, dS + \int_\Gamma \partial_\nu u \left(\overline{\Delta v} \right) \, dS. \tag{6.3.7}$$

□

6.4 Construction of CGO type solutions

6.4.1 The Schrödinger case

In this section, we construct appropriate CGO solutions to the problems

$$\begin{cases} -\Delta u_2 \; - \; (k^2 - ikb)u_2 \; + \; q_2 u_2 & = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u_2 & = 0 \quad \text{on } \Gamma_0, \end{cases} \tag{6.4.1}$$

and

$$\begin{cases} -\Delta v \; - \; (k^2 + ikb)v \; + \; q_1 v & = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad v & = 0 \quad \text{on } \Gamma_0, \end{cases} \tag{6.4.2}$$

for $q_j \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$ with $\|q_j\|_{L^\infty(\Omega)} + \|q_j\|_{H^s(\Omega)} \leq M$, $j = 1, 2$.

In this direction, following [21] (see also [8]), we introduce the following change of coordinates.

Let $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ $(n \geq 3)$ with $\xi' = (\xi_1, \ldots, \xi_{n-1}) \neq 0$ in $\mathbb{R}^{n-1}$. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ satisfying the following conditions:

$$e_1 = \left(\frac{\xi'}{|\xi'|}, 0\right),\ e_n = (0, ..., 0, 1) \text{ and } e_j \in \mathbb{R}^n \text{ (with } e_{j,n} = 0) \text{ for } j = 2, \ldots, n-1.$$

Note that, with respect to this basis, the coordinate representation for ξ is

$$\xi_e = (|\xi'|, 0, ..., 0, \xi_n)_e .$$

Next, we choose unit vectors α, β in $\mathbb{R}^n$ such that $\{\xi, \alpha, \beta\}$ forms an orthogonal set and the coordinate representation for β and α with respect to $\{e_1, e_2, ..., e_n\}$ are

$$\beta_e = \left(-\frac{\xi_n}{|\xi|}, 0, ..., 0, \frac{|\xi'|}{|\xi|}\right)_e \quad \text{and} \quad \alpha_e = (0, 1, 0, ..., 0)_e.$$

Also, we have

$$\alpha_n = \alpha_{e,n}, \quad \beta_n = \beta_{e,n}, \quad \alpha \cdot \beta = \alpha_e \cdot \beta_e.$$

Now, for $k \geq 1$ and $a > 1$, let us denote the principal square root of the complex number $\left(k^2 + a^2 - \frac{|\xi|^2}{4} - ikb\right)$ by $A + iB$. We shall further assume that

$$|\xi|^2 \leq 3(k^2 + a^2),$$

which guaranties that $A > 0$ (see [31]). We shall, henceforth, refer to this condition as the condition (Y).

Note that the relations $A^2 - B^2 = k^2 + a^2 - \frac{|\xi|^2}{4}$ and $2AB = -kb$ follow by squaring $A+iB$ and comparing the square with $\left(k^2 + a^2 - \frac{|\xi|^2}{4} - ikb\right)$.

Next, we choose $\eta_j \in \mathbb{C}^n,\ j = 1, 2$, as follows:

$$\begin{aligned} \eta_1 &= \frac{\xi}{2} - A\beta + i(B\beta + a\alpha), \\ \eta_2 &= -\frac{\xi}{2} - A\beta - i(B\beta + a\alpha). \end{aligned} \tag{6.4.3}$$

Then, $\eta_2 - \overline{\eta_1} = -\xi$, $\eta_1 \cdot \eta_1 = k^2 + ikb$, $\eta_2 \cdot \eta_2 = k^2 - ikb$, and

$$|\eta_j|^2 = \frac{|\xi|^2}{4} + A^2 + B^2 + a^2 \ \text{ for } j = 1, 2, \tag{6.4.4}$$

where $A^2 + B^2 = \left(\left(k^2 + a^2 - \frac{|\xi|^2}{4}\right)^2 + k^2b^2\right)^{\frac{1}{2}}$. Also, we have the following estimates (see Appendix I, also [31]):

$$|\eta_j|^2 \leq 2((k^2 + kb)^{\frac{1}{2}} + a)^2, \ \ |B| < b \ \text{ and } |\mathrm{Im}\,(\eta_j)|^2 \leq (b + a)^2 \quad \text{for } j = 1, 2. \tag{6.4.5}$$

Note that the condition (Y) is used to bound B in the calculation.

Let us denote $\widetilde{\Omega} := \Omega \cup \Omega^*$, where $\Omega^* := \{(x', x_n) \in \mathbb{R}^n : (x', -x_n) \in \Omega\}$ is the reflection of Ω by $\{x_n = 0\}$. Let $q_{z,j}$ denote the extension of q_j to $\mathbb{R}^n$ by zero. Since $s < \frac{1}{2}$ (see [1]), we have $q_{z,j} \in H^s(\mathbb{R}^n)$.

Next, let $q_{re,j}$ denote the reflection of $q_{z,j}$ by the plane $x_n = 0$, that is,

$$q_{re,j}(x', x_n) = q_{z,j}(x', -x_n) \quad \text{for } \ (x', x_n) \in \mathbb{R}^n.$$

Then, owing to the fact that $q_{z,j} \in H^s(\mathbb{R}^n)$, we see that $q_{re,j} \in H^s(\mathbb{R}^n)$. Also,

$$\|q_{z,j}\|_{H^s(\mathbb{R}^n)} \leq C\|q_j\|_{H^s(\Omega)} \leq CM \quad \text{and} \quad \|q_{re,j}\|_{H^s(\mathbb{R}^n)} \leq C\|q_{z,j}\|_{H^s(\mathbb{R}^n)} \leq CM;$$

$$\|q_{z,j}\|_{L^\infty(\mathbb{R}^n)} = \|q_j\|_{L^\infty(\Omega)} \leq M \quad \text{and} \quad \|q_{re,j}\|_{L^\infty(\mathbb{R}^n)} = \|q_{z,j}\|_{L^\infty(\mathbb{R}^n)} \leq M.$$

Note that the constant C appears due to the boundedness of the extension and the reflection operators (see [1]).

Now, let $\widetilde{q}_{e,j} := q_{z,j} + q_{re,j}$. Then $\widetilde{q}_{e,j} \in H^s(\mathbb{R}^n)$ and

$$\|\widetilde{q}_{e,j}\|_{H^s(\mathbb{R}^n)} \leq \|q_{z,j}\|_{H^s(\mathbb{R}^n)} + \|q_{re,j}\|_{H^s(\mathbb{R}^n)} \leq CM$$
$$\text{and} \quad \|\widetilde{q}_{e,j}\|_{L^\infty(\mathbb{R}^n)} \leq \|q_{z,j}\|_{L^\infty(\mathbb{R}^n)} + \|q_{re,j}\|_{L^\infty(\mathbb{R}^n)} \leq CM.$$

Let us denote the restriction of $\widetilde{q}_{e,j}$ to the domain $\widetilde{\Omega}$ by $q_{e,j}$, and note that

$$q_{e,j}(x) := \begin{cases} q_j(x', x_n) & \text{if } (x', x_n) \in \Omega, \\ q_j(x', -x_n) & \text{if } (x', x_n) \in \Omega^*. \end{cases}$$

This restriction $q_{e,j}$ also belongs to $H^s(\widetilde{\Omega})$ for $s < \frac{1}{2}$.

Let $R > 1$ be such that $\widetilde{\Omega} \subset\subset B_R := B(0, R)$, hence $\widetilde{q}_{e,j}|_{B_R} \in H^s(B_R)$ for $s < \frac{1}{2}$. Now, observe that

$$\begin{aligned} \|q_{e,j}\|_{L^\infty(\widetilde{\Omega})} &\leq M, \ \|\widetilde{q}_{e,j}\|_{L^\infty(B_R)} \leq \|\widetilde{q}_{e,j}\|_{L^\infty(\mathbb{R}^n)} \leq CM, \\ \|q_{e,j}\|_{H^s(\widetilde{\Omega})} &\leq \|\widetilde{q}_{e,j}\|_{H^s(\mathbb{R}^n)} \leq CM, \\ \|\widetilde{q}_{e,j}\|_{H^s(B_R)} &\leq \|\widetilde{q}_{e,j}\|_{H^s(\mathbb{R}^n)} \leq CM. \end{aligned} \tag{6.4.6}$$

Let us denote $q := q_1 - q_2$, $q_z := q_{z,1} - q_{z,2}$, $\widetilde{q}_e = \widetilde{q}_{e,1} - \widetilde{q}_{e,2}$ and $q_e := q_{e,1} - q_{e,2}$. Then we have,

$$\begin{aligned} &\|q\|_{L^\infty(\Omega)} \leq CM, \ \|q_z\|_{L^\infty(\mathbb{R}^n)} \leq CM, \ \|q_e\|_{L^\infty(\widetilde{\Omega})} \leq CM, \ \|\widetilde{q}_e\|_{L^\infty(B_R)} \leq CM, \\ &\|q\|_{H^s(\Omega)} \leq CM, \ \|q_z\|_{H^s(\mathbb{R}^n)} \leq CM, \ \|q_e\|_{H^s(\widetilde{\Omega})} \leq CM, \ \|\widetilde{q}_e\|_{H^s(B_R)} \leq CM, \end{aligned} \tag{6.4.7}$$

where $C > 0$ is a generic constant.

Next, we consider the problems

$$-\Delta\widetilde{u}_2 - (k^2 - ikb)\widetilde{u}_2 + \widetilde{q}_{e,2}\widetilde{u}_2 = 0 \quad \text{in} \ B_R, \tag{6.4.8}$$

and

$$-\Delta\widetilde{v} - (k^2 + ikb)\widetilde{v} + \widetilde{q}_{e,1}\widetilde{v} = 0 \quad \text{in } B_R, \tag{6.4.9}$$

and construct CGO solutions of (6.4.8) and (6.4.9) of the form

$$\widetilde{u}_2(x) = e^{i\eta_2\cdot x}(1 + \widetilde{r}_2(x)) \quad \text{and} \quad \widetilde{v}(x) = e^{i\eta_1\cdot x}(1 + \widetilde{r}_1(x)) \tag{6.4.10}$$

respectively. To do so, we first observe that $\tilde{u}_2$ and $\tilde{v}$ as in (6.4.10) are distributional solutions to (6.4.8) and (6.4.9) respectively if and only if for $j = 1, 2$, $\widetilde{r}_j$ is a distributional solution of

$$-\Delta\widetilde{r}_j - 2i\eta_j\cdot\nabla\widetilde{r}_j = -\widetilde{q}_{e,j}(1 + \widetilde{r}_j) \quad \text{in } B_R. \tag{6.4.11}$$

Consider the linear differential operator

$$P_j \equiv -\Delta - 2i\eta_j\cdot\nabla,$$

and the corresponding symbol

$$P_j(\xi^*) = |\xi^*|^2 + 2\eta_j\cdot\xi^*, \quad \xi^* \in \mathbb{R}^n.$$

Then (see [26])

$$\begin{aligned}\left(\widetilde{P}_j(\xi^*)\right)^2 \geq 4|\eta_j|^2 &\geq 4\left(\frac{|\xi|^2}{4} + \left(\left(k^2 + a^2 - \frac{|\xi|^2}{4}\right)^2 + k^2b^2\right)^{\frac{1}{2}} + a^2\right) \\ &\geq 4\left(\frac{|\xi|^2}{4} + k^2 + a^2 - \frac{|\xi|^2}{4}\right) = 4(k^2 + a^2) > 4.\end{aligned} \tag{6.4.12}$$

Then, by Theorem 2.3.2, there is a bounded linear operator E_j on $L^2(B_R)$ such that

$$P_jE_jf = f, \quad \text{for all } f \in L^2(B_R), \tag{6.4.13}$$

and for any linear partial differential operator Q with constant coefficients, we have

$$\|QE_jg\|_{L^2(B_R)} \leq C_0 \sup_{\xi^*\in\mathbb{R}^n}\left|\frac{\widetilde{Q}(\xi^*)}{\widetilde{P}_j(\xi^*)}\right| \|g\|_{L^2(B_R)}, \; g \in L^2(B_R). \tag{6.4.14}$$

Here the positive constant C_0 depends only on n, B_R and the order of P_j (which is two here).

The following lemma, whose proof is an application of the Browder fixed-point theorem, gives us the existence of CGO type solutions to (6.4.8) and (6.4.9).

Lemma 6.4.1. *Let $a > 1$ be such that*

$$C_0 M \leq (k^2 + a^2)^{\frac{1}{2}}. \tag{6.4.15}$$

Then, the functions

$$\widetilde{u}_2(x) = e^{i\eta_2 \cdot x}(1 + \widetilde{r}_2(x)) \quad \text{and} \quad \widetilde{v}(x) = e^{i\eta_1 \cdot x}(1 + \widetilde{r}_1(x)), \tag{6.4.16}$$

are distributional solutions to (6.4.8) and (6.4.9) respectively with

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{(k^2 + a^2)^{\frac{1}{2}}}, \quad j = 1, 2. \tag{6.4.17}$$

Proof. Note that, using (6.4.13), we can reformulate the problem of existence of a distributional solution of (6.4.11) into the following fixed-point problem:

$$\widetilde{r}_j = E_j(-\widetilde{q}_{e,j}(1 + \widetilde{r}_j)) \ \text{ on } B_R. \tag{6.4.18}$$

We shall now use the Browder fixed-point theorem to prove the existence of a fixed point of (6.4.18). In this direction, we define the operator

$$F_j : L^2(B_R) \to L^2(B_R) \ \text{ by } \ h \mapsto E_j(-\widetilde{q}_{e,j}(1 + h)).$$

We recall that $L^2(B_R)$ is a uniformly convex Banach space.

Next, we will show that F_j is non-expansive on a suitable nonempty convex closed bounded subset G. Applying (6.4.14) with $Q = 1$, we observe that for all $g \in L^2(B_R)$,

$$\|E_j g\|_{L^2(B_R)} \leq \frac{C_0}{2(k^2 + a^2)^{\frac{1}{2}}} \|g\|_{L^2(B_R)} = \delta \|g\|_{L^2(B_R)}, \ \text{ where } \delta := \frac{C_0}{2(k^2 + a^2)^{\frac{1}{2}}}. \tag{6.4.19}$$

Let $h_1, h_2 \in L^2(B_R)$. Then using the previous estimate, we have

$$\begin{aligned}\|F_j(h_1) - F_j(h_2)\|_{L^2(B_R)} &= \|E_j(-\widetilde{q}_{e,j}(1+h_1)) - E_j(-\widetilde{q}_{e,j}(1+h_2))\|_{L^2(B_R)} \\ &= \|E_j(-\widetilde{q}_{e,j}(1+h_1) + \widetilde{q}_{e,j}(1+h_2))\|_{L^2(B_R)} \\ &= \|E_j(-\widetilde{q}_{e,j}(h_1-h_2))\|_{L^2(B_R)} \\ &\leq \delta\|\widetilde{q}_{e,j}(h_1-h_2)\|_{L^2(B_R)} \leq \delta M\|h_1-h_2\|_{L^2(B_R)}.\end{aligned}$$

Now, we choose $G := \overline{B(0,\rho_0)} = \{h \in L^2(B_R) \ : \ \|h\| \leq \rho_0\}$, where $\rho_0 := 2|B_R|^{\frac{1}{2}}\delta M$. Then, for every $h \in G$, we have

$$\begin{aligned}\|F_j(h)\|_{L^2(B_R)} \leq \delta M\|\,(1+h)\,\|_{L^2(B_R)} &\leq |B_R|^{\frac{1}{2}}\delta M + \delta M\|h\|_{L^2(B_R)} \\ &\leq |B_R|^{\frac{1}{2}}\delta M + \delta M\rho_0 \\ &\leq \frac{\rho_0}{2} + \frac{C_0 M}{2\,(k^2+a^2)^{\frac{1}{2}}}\rho_0 \leq \rho_0,\end{aligned}$$

where we have used (6.4.15) to get the last inequality. Hence, F_j maps G into itself. Further, using (6.4.15), we see that for any $h_1, h_2 \in G$,

$$\|F_j(h_1) - F_j(h_2)\|_{L^2(B_R)} \leq \delta M\|h_1-h_2\|_{L^2(B_R)} \leq \frac{1}{2}\|h_1-h_2\|_{L^2(B_R)} \leq \|h_1-h_2\|_{L^2(B_R)},$$

and thus, $F_j|_G$ is non-expansive. Hence, by Browder fixed-point theorem (note that G is a nonempty convex closed bounded subset), the map

$$F_j : G \to G$$

has a fixed point, which we call $\widetilde{r}_j$. As discussed before, $\widetilde{r}_j$ is a distributional solution of (6.4.11).

Now, from the fact that $\widetilde{r}_j \in G$, we have the bound

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq \rho_0 = \frac{C_0 M|B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}}.$$

□

Remark 6.4.2. From (6.4.17), using (6.4.15), we also observe that

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}} \leq |B_R|^{\frac{1}{2}}. \tag{6.4.20}$$

Using the CGO solutions (6.4.10), following the reflection technique in [21], we now define the following CGO type solutions:

$$\begin{aligned} u_2(x', x_n) &= e^{i\eta_2\cdot(x',x_n)}(1+\widetilde{r}_2(x',x_n)) - e^{i\eta_2\cdot(x',-x_n)}(1+\widetilde{r}_2(x',-x_n)), \\ v(x', x_n) &= e^{i\eta_1\cdot(x',x_n)}(1+\widetilde{r}_1(x',x_n)) - e^{i\eta_1\cdot(x',-x_n)}(1+\widetilde{r}_1(x',-x_n)),\ (x',x_n)\in\Omega. \end{aligned} \tag{6.4.21}$$

It is easy to check that u_2 and v are distributional solutions to the problems (6.4.1) and (6.4.2) respectively. Now, let us denote

$$r_j := \widetilde{r}_j|_\Omega \ \text{ and } \ r_j^*(x',x_n) := \widetilde{r}_j(x',-x_n),\ (x',x_n)\in\Omega.$$

Observe that,

$$\|r_j\|_{L^2(\Omega)} \leq \|\widetilde{r}_j\|_{L^2(B_R)} \ \text{ and } \ \|r_j^*\|_{L^2(\Omega)} \leq \|\widetilde{r}_j\|_{L^2(B_R)}.$$

Using (6.4.17) and (6.4.20), we get the bounds

$$\begin{gathered} \|r_j\|_{L^2(\Omega)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}} \ \text{ and } \ \|r_j^*\|_{L^2(\Omega)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}}, \\ \|r_j\|_{L^2(\Omega)} \leq |B_R|^{\frac{1}{2}} \ \text{ and } \ \|r_j^*\|_{L^2(\Omega)} \leq |B_R|^{\frac{1}{2}}. \end{gathered} \tag{6.4.22}$$

We shall use both these estimates in the calculations that follow.

Next, we recall from (6.4.8) and (6.4.9) that $\widetilde{u}_2$ and $\widetilde{v}$ satisfy the PDEs

$$P\widetilde{u}_2 = f_1 \ \text{ and } \ P\widetilde{v} = f_2 \ \text{ in } B_R, \ \text{ respectively},$$

where

$$P = -\Delta \ \text{ and } \ f_1 = (k^2 - ikb)\widetilde{u}_2 - \widetilde{q}_{e,2}\widetilde{u}_2,\ f_2 = (k^2+ikb)\widetilde{v} - \widetilde{q}_{e,1}\widetilde{v}.$$

Therefore, using the interior regularity for elliptic operators (see Theorem 2.3.3) in B_R, we can conclude that

$$\|\widetilde{u}_2\|_{H^2(\widetilde{\Omega})} \leq C\|f_1\|_{L^2(V)} + C\|\widetilde{u}_2\|_{L^2(V)} \text{ and } \|\widetilde{v}\|_{H^2(\widetilde{\Omega})} \leq C\|f_2\|_{L^2(V)} + C\|\widetilde{v}\|_{L^2(V)},$$

for any V such that $\widetilde{\Omega} \subset\subset V \subset\subset B_R$. Using the above estimates with the fact that $\|\cdot\|_{H^1(\widetilde{\Omega})} \leq \|\cdot\|_{H^2(\widetilde{\Omega})}$, we obtain

$$\|\widetilde{u}_2\|_{H^1(\widetilde{\Omega})} \leq C\|f_1\|_{L^2(B_R)} + C\|\widetilde{u}_2\|_{L^2(B_R)} \text{ and } \|\widetilde{v}\|_{H^1(\widetilde{\Omega})} \leq C\|f_2\|_{L^2(B_R)} + C\|\widetilde{v}\|_{L^2(B_R)}. \tag{6.4.23}$$

Now, we will bound the terms $\|f_1\|_{L^2(B_R)}$ and $\|f_2\|_{L^2(B_R)}$ by $\|\widetilde{u}_2\|_{L^2(B_R)}$ and $\|\widetilde{v}\|_{L^2(B_R)}$ respectively.

For f_1, we have

$$\begin{aligned}
\|f_1\|_{L^2(B_R)} &\leq \|(k^2 - ikb)\widetilde{u}_2\|_{L^2(B_R)} + \|\widetilde{q}_{e,2}\widetilde{u}_2\|_{L^2(B_R)} \\
&\leq |k^2 - ikb|\|\widetilde{u}_2\|_{L^2(B_R)} + \|\widetilde{q}_{e,2}\|_{L^\infty(B_R)}\|\widetilde{u}_2\|_{L^2(B_R)} \\
&\leq (k^4 + k^2b^2)^{\frac{1}{2}}\|\widetilde{u}_2\|_{L^2(B_R)} + M\|\widetilde{u}_2\|_{L^2(B_R)} \qquad (6.4.24)\\
&\leq (k^4 + k^2b^2)^{\frac{1}{2}}\|\widetilde{u}_2\|_{L^2(B_R)} + (k^4 + k^2b^2)^{\frac{1}{2}}M\|\widetilde{u}_2\|_{L^2(B_R)} \\
&\leq (1 + M)(k^2 + kb)\|\widetilde{u}_2\|_{L^2(B_R)},
\end{aligned}$$

where we have also used the inequality $(a^2 + b^2)^{\frac{1}{2}} \leq a + b,\ a, b \geq 0$.

Similarly, using the fact that $|k^2 - ikb| = |k^2 + ikb|$ for f_2, we have

$$\|f_2\|_{L^2(B_R)} \leq C(k^2 + kb)\|\widetilde{v}\|_{L^2(B_R)} \tag{6.4.25}$$

for some constant C that depends on M and Ω.

Also, from the above discussion, note that u_2 and v defined in (6.4.21) are elements of $H^2(\Omega)$.

6.4.2 The Biharmonic case

Next, we describe the construction of appropriate CGO type solutions to the problems

$$\begin{cases} \Delta^2 u_2 \;-\; (k^2 - ikb)^2 u_2 \;+\; q_2 u_2 & = 0 \;\; \text{in}\ \Omega, \\ \qquad\qquad u_2 & = 0 \;\; \text{on}\ \Gamma_0, \\ \qquad\qquad \Delta u_2 & = 0 \;\; \text{on}\ \Gamma_0, \end{cases} \tag{6.4.26}$$

and

$$\begin{cases} \Delta^2 v \;-\; (k^2 + ikb)^2 v \;+\; q_1 v & = 0 \;\; \text{in}\ \Omega, \\ \qquad\qquad v & = 0 \;\; \text{on}\ \Gamma_0, \\ \qquad\qquad \Delta v & = 0 \;\; \text{on}\ \Gamma_0, \end{cases} \tag{6.4.27}$$

for $q_j \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$ with $\|q_j\|_{L^\infty(\Omega)} + \|q_j\|_{H^s(\Omega)} \leq M$, $j = 1, 2$.

We use the same change of coordinates and define η_j by (6.4.3), as in the case of the Schrödinger equation. Also, we extend the potentials q_j to $\mathbb{R}^n$ as described in the Schrödinger case and denote this extension by $\widetilde{q}_{e,j}$. Hence, the estimates (6.4.6) and (6.4.7) hold true. Next, we consider the problems

$$\Delta^2 \widetilde{u}_2 - (k^2 - ikb)^2 \widetilde{u}_2 + \widetilde{q}_{e,2} \widetilde{u}_2 = 0 \quad \text{in} \;\; B_R, \tag{6.4.28}$$

and

$$\Delta^2 \widetilde{v} - (k^2 + ikb)^2 \widetilde{v} + \widetilde{q}_{e,1} \widetilde{v} = 0 \quad \text{in} \;\; B_R, \tag{6.4.29}$$

and construct CGO solutions of (6.4.28) and (6.4.29) of the form

$$\widetilde{u}_2(x) = e^{i\eta_2 \cdot x}(1 + \widetilde{r}_2(x)) \quad \text{and} \quad \widetilde{v}(x) = e^{i\eta_1 \cdot x}(1 + \widetilde{r}_1(x)) \tag{6.4.30}$$

respectively. We note that (see Appendix II) $\tilde{u}_2$ and $\tilde{v}$ are distributional solutions to (6.4.28) and (6.4.29) respectively if and only if for $j = 1, 2$, $\widetilde{r}_j$ is a distributional solution of

$$\left[(\Delta + 2i\eta_j \cdot \nabla)^2 - 2(\eta_j \cdot \eta_j)\,(\Delta + 2i\eta_j \cdot \nabla) \right] \widetilde{r}_j = -\widetilde{q}_{e,j}(1 + \widetilde{r}_j) \text{ in } B_R. \tag{6.4.31}$$

Now, we consider the linear differential operator

$$P_j \equiv (\Delta + 2i\eta_j \cdot \nabla)^2 - 2(\eta_j \cdot \eta_j)\,(\Delta + 2i\eta_j \cdot \nabla)$$

and the corresponding symbol

$$P_j(\xi^*) = \left(|\xi^*|^2 + 2(\eta_j \cdot \xi^*)\right)^2 + 2(\eta_j \cdot \eta_j)\left(|\xi^*|^2 + 2(\eta_j \cdot \xi^*)\right),\ \xi^* \in \mathbb{R}^n.$$

Then, as discussed in Appendix II, we see that

$$\widetilde{P}_j(\xi^*) \geq 2|\text{Im}\,(\eta_j)| \geq 2a > 1. \tag{6.4.32}$$

Therefore, by Theorem 2.3.2, there is a bounded linear operator E_j on $L^2(B_R)$ such that

$$P_j E_j f = f, \quad \text{for all } f \in L^2(B_R),$$

and for any linear partial differential operator Q with constant coefficients, we have

$$\|QE_j g\|_{L^2(B_R)} \leq C_0 \sup_{\xi^* \in \mathbb{R}^n} \left| \frac{\widetilde{Q}(\xi^*)}{\widetilde{P}_j(\xi^*)} \right| \|g\|_{L^2(B_R)},\ g \in L^2(B_R), \tag{6.4.33}$$

where the positive constant C_0 depends only on n, B_R and the order of P_j (which is four here).

Remark 6.4.3. We note that in contrast to the estimate for the symbol in (6.4.12), we have a weaker estimate in (6.4.32) with η_j replaced by Im (η_j). This weaker estimate, ultimately, leads to the exponential factor in k appearing in (6.2.2) instead of the polynomial factor that appears in (6.2.1).

Using the Browder fixed-point theorem, we can now prove the existence of CGO type solutions to (6.4.28) and (6.4.29), as in the Schrödinger case.

Lemma 6.4.4. *Let $a > 1$ be such that*

$$C_0 M \leq a. \tag{6.4.34}$$

Then, the functions

$$\widetilde{u}_2(x) = e^{i\eta_2 \cdot x}(1 + \widetilde{r}_2(x)) \quad \textit{and} \quad \widetilde{v}(x) = e^{i\eta_1 \cdot x}(1 + \widetilde{r}_1(x)),$$

are distributional solutions to (6.4.28) and (6.4.29) respectively with

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{a}, \quad j = 1, 2. \tag{6.4.35}$$

Proof. As in the Schrödinger case, we reformulate the problem of existence of a distributional solution of (6.4.31) into the fixed-point problem

$$\widetilde{r}_j = E_j(-\widetilde{q}_{e,j}(1+\widetilde{r}_j)) \quad \text{on } B_R, \tag{6.4.36}$$

and use the Browder fixed-point theorem to prove the existence of a fixed point of (6.4.36). To do so, we define the operator

$$F_j : L^2(B_R) \to L^2(B_R) \text{ by } h \mapsto E_j(-\widetilde{q}_{e,j}(1+h)),$$

and show that F_j is non-expansive on a suitable nonempty convex closed bounded subset G. Applying (6.4.33) with $Q = 1$, we observe that for all $g \in L^2(B_R)$,

$$\|E_j g\|_{L^2(B_R)} \leq \frac{C_0}{2a}\|g\|_{L^2(B_R)} = \delta\|g\|_{L^2(B_R)}, \text{ where } \delta := \frac{C_0}{2a}.$$

Let $h_1, h_2 \in L^2(B_R)$. Then using the previous estimate, we have

$$\begin{aligned}\|F_j(h_1) - F_j(h_2)\|_{L^2(B_R)} &= \|E_j(-\widetilde{q}_{e,j}(1+h_1)) - E_j(-\widetilde{q}_{e,j}(1+h_2))\|_{L^2(B_R)} \\ &\leq \delta\|\widetilde{q}_{e,j}(h_1-h_2)\|_{L^2(B_R)} \leq \delta M\|h_1-h_2\|_{L^2(B_R)}.\end{aligned}$$

Now, we choose $G := \overline{B(0,\rho_0)} = \{h \in L^2(B_R) : \|h\| \leq \rho_0\}$, where $\rho_0 := 2|B_R|^{\frac{1}{2}}\delta M$. Then, for every $h \in G$, we have

$$\begin{aligned}\|F_j(h)\|_{L^2(B_R)} \leq \delta M\|\,(1+h)\,\|_{L^2(B_R)} &\leq |B_R|^{\frac{1}{2}}\delta M + \delta M\|h\|_{L^2(B_R)} \\ &\leq |B_R|^{\frac{1}{2}}\delta M + \delta M\rho_0 \\ &\leq \frac{\rho_0}{2} + \frac{C_0 M}{2a}\rho_0 \leq \rho_0,\end{aligned}$$

where we have used (6.4.34) to get the last inequality. Hence, F_j maps G into itself. Further, using (6.4.34), we observe that for any $h_1, h_2 \in G$,

$$\|F_j(h_1) - F_j(h_2)\|_{L^2(B_R)} \leq \delta M\|h_1-h_2\|_{L^2(B_R)} \leq \frac{1}{2}\|h_1-h_2\|_{L^2(B_R)} \leq \|h_1-h_2\|_{L^2(B_R)},$$

and thus, $F_j|_G$ is non-expansive. Hence, by Browder fixed-point theorem, the map

$$F_j : G \to G$$

has a fixed point, which we call $\widetilde{r}_j$. Now, from the fact that $\widetilde{r}_j \in G$, we have the bound

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq \rho_0 = \frac{C_0 M |B_R|^{\frac{1}{2}}}{a}.$$

□

Remark 6.4.5. From (6.4.35), using (6.4.34), we also observe that

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{a} \leq |B_R|^{\frac{1}{2}}. \tag{6.4.37}$$

Using the CGO solutions (6.4.30), we now define the following CGO type solutions:

$$\begin{aligned} u_2(x', x_n) &= e^{i\eta_2\cdot(x',x_n)}(1 + \widetilde{r}_2(x', x_n)) - e^{i\eta_2\cdot(x',-x_n)}(1 + \widetilde{r}_2(x', -x_n)), \\ v(x', x_n) &= e^{i\eta_1\cdot(x',x_n)}(1 + \widetilde{r}_1(x', x_n)) - e^{i\eta_1\cdot(x',-x_n)}(1 + \widetilde{r}_1(x', -x_n)), \ (x', x_n) \in \Omega. \end{aligned} \tag{6.4.38}$$

It is easy to check that u_2 and v are distributional solutions to the problems (6.4.26) and (6.4.27) respectively. Now, we define

$$r_j := \widetilde{r}_j|_\Omega \ \text{ and } \ r_j^*(x', x_n) := \widetilde{r}_j(x', -x_n), \ (x', x_n) \in \Omega,$$

and, as in the Schrödinger case, observe that

$$\|r_j\|_{L^2(\Omega)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{a} \quad \text{and} \quad \|r_j^*\|_{L^2(\Omega)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{a},$$
$$\|r_j\|_{L^2(\Omega)} \leq |B_R|^{\frac{1}{2}} \quad \text{and} \quad \|r_j^*\|_{L^2(\Omega)} \leq |B_R|^{\frac{1}{2}}.$$

Next, we recall from (6.4.28) and (6.4.29) that $\widetilde{u}_2$ and $\widetilde{v}$ satisfy the PDEs

$$P\widetilde{u}_2 = f_1 \quad \text{and} \quad P\widetilde{v} = f_2 \ \text{ in } B_R, \ \text{ respectively},$$

where

$$P = \Delta^2 \ \text{ and } \ f_1 = (k^2 - ikb)^2\widetilde{u}_2 - \widetilde{q}_{e,2}\widetilde{u}_2, \ f_2 = (k^2 + ikb)^2\widetilde{v} - \widetilde{q}_{e,1}\widetilde{v}.$$

Then, using the interior regularity for elliptic operators (see Theorem 2.3.3) in B_R, we can conclude that

$$\|\widetilde{u}_2\|_{H^4(\widetilde{\Omega})} \leq C\|f_1\|_{L^2(V)} + C\|\widetilde{u}_2\|_{L^2(V)} \text{ and } \|\widetilde{v}\|_{H^4(\widetilde{\Omega})} \leq C\|f_2\|_{L^2(V)} + C\|\widetilde{v}\|_{L^2(V)},$$

for any V such that $\widetilde{\Omega} \subset\subset V \subset\subset B_R$. From the above estimates, we obtain

$$\|\widetilde{u}_2\|_{H^4(\widetilde{\Omega})} \leq C\|f_1\|_{L^2(B_R)} + C\|\widetilde{u}_2\|_{L^2(B_R)} \text{ and } \|\widetilde{v}\|_{H^4(\widetilde{\Omega})} \leq C\|f_2\|_{L^2(B_R)} + C\|\widetilde{v}\|_{L^2(B_R)}. \tag{6.4.39}$$

Now, for f_1 and f_2, as in the Schrödinger case, we have the bounds

$$\|f_1\|_{L^2(B_R)} \leq C(k^2 + kb)^2\|\widetilde{u}_2\|_{L^2(B_R)}, \tag{6.4.40}$$

and

$$\|f_2\|_{L^2(B_R)} \leq C(k^2 + kb)^2\|\widetilde{v}\|_{L^2(B_R)}, \tag{6.4.41}$$

for some constant C that depends on M and Ω.

Also, from the above discussion, we note that u_2 and v defined in (6.4.38) are elements of $H^4(\Omega)$.

6.5 Stability estimate for the Schrödinger case

Proof of theorem 6.2.1

Let $u_1 \in H^1(\Omega)$ be a solution to the problem

$$\begin{cases} -\Delta u_1 - (k^2 - ikb)u_1 + q_1 u_1 = 0 & \text{in } \Omega, \\ u_1 = u_2 & \text{on } \partial\Omega, \end{cases}$$

where u_2 is as defined in (6.4.21). Then $u := u_1 - u_2 \in H^1(\Omega)$ satisfies

$$\begin{cases} -\Delta u - (k^2 - ikb)u + q_1 u = -(q_1 - q_2)u_2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the function u defined above and v as defined in (6.4.21) in the integral identity (6.3.4), we get

$$\int_\Omega q u_2 \bar{v}\, dx = \int_\Gamma \partial_\nu u\, \bar{v}\, dS = \int_\Gamma (\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2})(f)\, \bar{v}\, dS, \tag{6.5.1}$$

where $u_1 = u_2 = f$ on Γ and the last equality follows from the definition of the Dirichlet-to-Neumann map $\mathcal{N}^S_q$. Now, taking the absolute value of both sides and using the trace theorem, we obtain:

$$\begin{aligned} \left|\int_\Omega q u_2 \bar{v}\, dx\right| &= \left|\int_\Gamma (\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2})(f)\, \bar{v}\, dS\right| \\ &\leq \|(\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2})(f)\|_{H^{-\frac{1}{2}}(\Gamma)} \|v\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} \\ &\leq C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* \|f\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* \|u_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \|v\|_{H^1(\Omega)} \\ &\leq C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* \|u_2\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned} \tag{6.5.2}$$

Now, observe that

$$\|u_2\|_{H^1(\Omega)} \leq 2\|\widetilde{u}_2\|_{H^1(\widetilde{\Omega})} \quad \text{and} \quad \|v\|_{H^1(\Omega)} \leq 2\|\widetilde{v}\|_{H^1(\widetilde{\Omega})}.$$

Combining these estimates with those in (6.4.23), (6.4.24), (6.4.25) and the fact that $k \geq 1$, we get

$$\|u_2\|_{H^1(\Omega)} \leq C(k^2 + kb)\|\widetilde{u}_2\|_{L^2(B_R)} \quad \text{and} \quad \|v\|_{H^1(\Omega)} \leq C(k^2 + kb)\|\widetilde{v}\|_{L^2(B_R)}. \tag{6.5.3}$$

Next, we estimate the bounds for $\|\widetilde{u}_2\|_{L^2(B_R)}$ using the bounds in (6.4.5) and (6.4.20) as follows:

$$\begin{aligned} \|\widetilde{u}_2\|^2_{L^2(B_R)} &= \int_{B_R} |\widetilde{u}_2|^2 dx = \int_{B_R} \left|e^{i\eta_2\cdot x}(1 + \widetilde{r}_2(x))\right|^2 dx \leq \int_{B_R} e^{2|\mathrm{Im}\,(\eta_2)||x|}|1 + \widetilde{r}_2(x)|^2 dx \\ &\leq 2e^{2(b+a)R} \int_{B_R} (1 + |\widetilde{r}_2|^2) dx = 2e^{2(b+a)R}(|B_R| + \|\widetilde{r}_2\|^2_{L^2(B_R)}) \\ &\leq 2e^{2(b+a)R}(|B_R| + |B_R|) \leq Ce^{2(b+a)R}. \end{aligned}$$

Similarly, for $\|\widetilde{v}\|_{L^2(B_R)}$, we have

$$\|\widetilde{v}\|^2_{L^2(B_R)} \leq Ce^{2(b+a)R}.$$

The estimates above, when combined with those in (6.5.3), lead to

$$\|u_2\|_{H^1(\Omega)} \leq C(k^2 + kb)e^{(b+a)R} \quad \text{and} \quad \|v\|_{H^1(\Omega)} \leq C(k^2 + kb)e^{(b+a)R}.$$

Consequently, the estimate (6.5.2) becomes

$$\left| \int_\Omega qu_2\bar{v}\, dx \right| \leq C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_*(k^2 + kb)^2 e^{2(b+a)R}. \tag{6.5.4}$$

Next, by substituting the CGO type solutions u_2 and v defined in (6.4.21) into the left-hand side of the integral identity (6.5.1), we get

$$\begin{aligned}
\int_\Omega qu_2\bar{v}\, dx &= \int_\Omega q \left(e^{i\eta_2\cdot x}(1 + r_2) - e^{i\eta_2\cdot x^*}(1 + r_2^*)\right) \\
&\qquad \left(e^{-i\overline{\eta_1}\cdot x}(1 + \overline{r_1}) - e^{-i\overline{\eta_1}\cdot x^*}(1 + \overline{r_1^*})\right) dx \\
&= \int_\Omega q \left(e^{i(\eta_2 - \overline{\eta_1})\cdot x}(1 + r_2)(1 + \overline{r_1}) + e^{i(\eta_2 - \overline{\eta_1})\cdot x^*}(1 + r_2^*)(1 + \overline{r_1^*})\right) dx \\
&\quad - \int_\Omega q \left(e^{i[\eta_2\cdot x - \overline{\eta_1}\cdot x^*]}(1 + r_2)(1 + \overline{r_1^*}) + e^{i[\eta_2\cdot x^* - \overline{\eta_1}\cdot x]}(1 + r_2^*)(1 + \overline{r_1})\right) dx \\
&= \int_\Omega q \left(e^{i(\eta_2 - \overline{\eta_1})\cdot x} + e^{i(\eta_2 - \overline{\eta_1})\cdot x^*}\right) dx - \int_\Omega q \left(e^{i[\eta_2\cdot x - \overline{\eta_1}\cdot x^*]} + e^{i[\eta_2\cdot x^* - \overline{\eta_1}\cdot x]}\right) dx \\
&\qquad + \int_\Omega q\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*)dx,
\end{aligned}$$

where $x^* := (x', -x_n)$ and

$$\begin{aligned}
\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*) &= e^{i(\eta_2 - \overline{\eta_1})\cdot x}(r_2 + \overline{r_1} + r_2\overline{r_1}) + e^{i(\eta_2 - \overline{\eta_1})\cdot x^*}(r_2^* + \overline{r_1^*} + r_2^*\overline{r_1^*}) \\
&\quad - e^{i[\eta_2\cdot x - \overline{\eta_1}\cdot x^*]}(r_2 + \overline{r_1^*} + r_2\overline{r_1^*}) - e^{i[\eta_2\cdot x^* - \overline{\eta_1}\cdot x]}(r_2^* + \overline{r_1} + r_2^*\overline{r_1}) \\
&:= w_1 + w_2 + w_3 + w_4.
\end{aligned} \tag{6.5.5}$$

Now, we will calculate the terms $\eta_2 \cdot x - \overline{\eta_1} \cdot x^*$ and $\eta_2 \cdot x^* - \overline{\eta_1} \cdot x$ for $x \in \Omega$.

For the first term, we obtain

$$\begin{aligned}
&\eta_2 \cdot x - \overline{\eta_1} \cdot x^* \\
&\quad = \left(-\frac{\xi'}{2} - A\beta' - i(B\beta' + a\alpha'), -\frac{\xi_n}{2} - A\beta_n - i(B\beta_n + a\alpha_n)\right) \cdot (x', x_n) \\
&\qquad\quad - \left(\frac{\xi'}{2} - A\beta' - i(B\beta' + a\alpha'), \frac{\xi_n}{2} - A\beta_n - i(B\beta_n + a\alpha_n)\right) \cdot (x', -x_n) \\
&\quad = (-\xi', -2A\beta_n - 2iB\beta_n - 2ia\alpha_n) \cdot (x', x_n) \\
&\quad = \left(-\xi', -2(A + iB)\frac{|\xi'|}{|\xi|}\right) \cdot (x', x_n) \quad \left(\text{since } \beta_n = \frac{|\xi'|}{|\xi|} \text{ and } \alpha_n = 0\right) \\
&\quad = -\xi_+ \cdot x, \quad \text{where } \xi_+ := \left(\xi', 2(A + iB)\frac{|\xi'|}{|\xi|}\right).
\end{aligned} \tag{6.5.6}$$

Similarly, for the second term, we have

$$\eta_2 \cdot x^* - \overline{\eta_1} \cdot x = -\xi_- \cdot x, \quad \text{where } \xi_- := \left(\xi', -2(A + iB)\frac{|\xi'|}{|\xi|}\right). \tag{6.5.7}$$

These two terms, along with $\eta_2 - \overline{\eta_1} = -\xi$, gives

$$\begin{aligned}
\int_\Omega q u_2 \bar{v}\, dx = \int_\Omega q\left(e^{-i\xi\cdot x} + e^{-i\xi\cdot x^*}\right) dx - \int_\Omega q\left(e^{-i\xi_+\cdot x} + e^{-i\xi_-\cdot x}\right) dx \\
+ \int_\Omega q\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*) dx.
\end{aligned} \tag{6.5.8}$$

Next, we will calculate the first two terms on the right-hand side of the above equation in terms of the Fourier transform.

For the first term, we have

$$\begin{aligned}
\int_\Omega q\left(e^{-i\xi\cdot x} + e^{-i\xi\cdot x^*}\right) dx &= \int_\Omega q(x', x_n) e^{-i\xi\cdot x} dx + \int_{\Omega^*} q(x', -x_n) e^{-i\xi\cdot x} dx \\
&= \int_{\widetilde{\Omega}} q_e(x) e^{-i\xi\cdot x} dx = \int_{\mathbb{R}^n} \widetilde{q}_e(x) e^{-i\xi\cdot x} dx = \mathcal{F}[\widetilde{q}_e](\xi).
\end{aligned}$$

Note that

$$\xi_+ = \left(\xi', 2(A + iB)\frac{|\xi'|}{|\xi|}\right) = \underbrace{\left(\xi', 2A\frac{|\xi'|}{|\xi|}\right)}_{\xi_+^{\mathrm{Re}}} + i\underbrace{\left(0, 2B\frac{|\xi'|}{|\xi|}\right)}_{\xi_+^{\mathrm{Im}}},$$

$$\xi_- = \left(\xi', -2(A+iB)\frac{|\xi'|}{|\xi|}\right) = \underbrace{\left(\xi', -2A\frac{|\xi'|}{|\xi|}\right)}_{\xi_-^{\mathrm{Re}}} + i\underbrace{\left(0, -2B\frac{|\xi'|}{|\xi|}\right)}_{\xi_-^{\mathrm{Im}}},$$

and

$$|\mathrm{Im}\,(\xi_\pm)| = 2|B|\frac{|\xi'|}{|\xi|} \leq 2b. \tag{6.5.9}$$

Using the lower bound for A (see Appendix I), we see that

$$|\xi_\pm^{\mathrm{Re}}|^2 = |\xi'|^2 + 4A^2\frac{|\xi'|^2}{|\xi|^2} \geq |\xi'|^2 + 4\left(k^2 + a^2 - \frac{|\xi|^2}{4}\right)\frac{|\xi'|^2}{|\xi|^2} = 4\left(k^2+a^2\right)\frac{|\xi'|^2}{|\xi|^2}. \tag{6.5.10}$$

Also,

$$\int_\Omega q(x)e^{-i\xi_+\cdot x}\,dx = \int_\Omega q(x)e^{\xi_+^{\mathrm{Im}}\cdot x}e^{-i\xi_+^{\mathrm{Re}}\cdot x}\,dx = \int_\Omega q(x)e^{2B\frac{|\xi'|}{|\xi|}x_n}e^{-i\xi_+^{\mathrm{Re}}\cdot x}\,dx,$$

and

$$\int_\Omega q(x)e^{-i\xi_-\cdot x}\,dx = \int_\Omega q(x)e^{\xi_-^{\mathrm{Im}}\cdot x}e^{-i\xi_-^{\mathrm{Re}}\cdot x}\,dx = \int_\Omega q(x)e^{-2B\frac{|\xi'|}{|\xi|}x_n}e^{-i\xi_-^{\mathrm{Re}}\cdot x}\,dx.$$

As in the previous chapter, consider the functions $g_+, g_- : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$g_+(x,\eta) := e^{2B\frac{|\eta'|}{|\eta|}x_n},\; x\in\Omega,\; \eta\in\mathbb{R}^n\setminus\{0\},$$
$$g_-(x,\eta) := e^{-2B\frac{|\eta'|}{|\eta|}x_n},\; x\in\Omega,\; \eta\in\mathbb{R}^n\setminus\{0\}.$$

We note that $\forall\eta\in\mathbb{R}^n\setminus\{0\}$, $g_+(\cdot,\eta), g_-(\cdot,\eta)\in H^s(\Omega)\cap L^\infty(\Omega)$ (see Appendix 5.4) and

$$\|g_+(\cdot,\eta)\|_{H^s(\Omega)} \leq Ce^{4bR}, \quad \|g_+(\cdot,\eta)\|_{L^\infty(\Omega)} \leq e^{2bR},$$
$$\|g_-(\cdot,\eta)\|_{H^s(\Omega)} \leq Ce^{4bR}, \quad \|g_-(\cdot,\eta)\|_{L^\infty(\Omega)} \leq e^{2bR},$$

where the bounds are independent of η.

The choice of the special coordinates (discussed in Section 6.4) helps us in getting rid of the parameter a from the imaginary parts of $\xi_\pm$. Otherwise, we would have the above bounds multiplied by a term of the form e^{Ca}, which would have significantly hampered the decay estimates that we obtain below.

Next, as in the previous chapter, we consider the functions $q_\pm : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$q_\pm(x,\eta) := q(x)g_\pm(x,\eta), \; x \in \Omega, \; \eta \in \mathbb{R}^n \setminus \{0\}.$$

Then, as observed in Appendix 5.4, $q_\pm(\cdot,\eta) \in H^s(\Omega), \; \forall \eta \in \mathbb{R}^n \setminus \{0\}$ and

$$\|q_\pm(\cdot,\eta)\|_{H^s(\Omega)} \le Ce^{4bR}, \; \forall \eta \in \mathbb{R}^n \setminus \{0\}.$$

Let $(q_\pm(\cdot,\eta))_z$ denote the zero extensions of $q_\pm(\cdot,\eta)$ from Ω to $\mathbb{R}^n$. Then

$$\|(q_\pm(\cdot,\eta))_z\|_{H^s(\mathbb{R}^n)} \le C\|q_\pm(\cdot,\eta)\|_{H^s(\Omega)} \le Ce^{4bR}. \tag{6.5.11}$$

By Lemma 2.3.4, there exists a constant $C > 0$ and for any $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that $\forall p \in \mathbb{R}^n$ and $\tau \in (0,1)$, we have

$$|\mathcal{F}[(q_\pm(\cdot,\eta))_z](p)| \le \frac{C_N}{(1+\tau|p|)^N}\|(q_\pm(\cdot,\eta))_z\|_{H^s(\mathbb{R}^n)} + C\tau^s\|(q_\pm(\cdot,\eta))_z\|_{H^s(\mathbb{R}^n)}. \tag{6.5.12}$$

Using (6.5.11) in (6.5.12), we observe that $\forall \eta \in \mathbb{R}^n \setminus \{0\}$ and $\forall p \in \mathbb{R}^n$,

$$|\mathcal{F}[(q_\pm(\cdot,\eta))_z](p)| \le \frac{Ce^{4bR}}{(1+\tau|p|)^N} + Ce^{4bR}\tau^s. \tag{6.5.13}$$

Note that the constant C depends only on N, s, Ω and M.

The second term in (6.5.8) can be written as

$$\begin{aligned}\int_\Omega q\left(e^{-i\xi_+\cdot x} + e^{-i\xi_-\cdot x}\right)dx &= \int_{\mathbb{R}^n}(q_+(\cdot,\xi))_z(x)e^{-i\xi_+^{\mathrm{Re}}\cdot x}dx + \int_{\mathbb{R}^n}(q_-(\cdot,\xi))_z(x)e^{-i\xi_-^{\mathrm{Re}}\cdot x}dx \\ &= \mathcal{F}\left[(q_+(\cdot,\xi))_z\right](\xi_+^{\mathrm{Re}}) + \mathcal{F}\left[(q_-(\cdot,\xi))_z\right](\xi_-^{\mathrm{Re}}).\end{aligned}$$

Therefore, from (6.5.8), we have

$$\begin{aligned}|\mathcal{F}[\widetilde{q}_e](\xi)| \le \left|\int_\Omega qu_2\bar{v}\,dx\right| + |\mathcal{F}\left[(q_+(\cdot,\xi))_z\right](\xi_+^{\mathrm{Re}})| + |\mathcal{F}\left[(q_-(\cdot,\xi))_z\right](\xi_-^{\mathrm{Re}})| \\ + \left|\int_\Omega q\mathcal{G}(x,r_1,r_2,r_1^*,r_2^*)dx\right|.\end{aligned} \tag{6.5.14}$$

Next, we will calculate the bounds for the last three terms on the right-hand side of the above estimate, as we already have the bound for the first term above from (6.5.4).

For the second and third terms of the right-hand side of the estimate (6.5.14), using (6.5.13) with $\eta = \xi$ and $p = \xi_{\pm}^{\mathrm{Re}}$, we obtain

$$|\mathcal{F}\left[(q_{\pm}(\cdot,\xi))_z\right](\xi_{\pm}^{\mathrm{Re}})| \leq \frac{Ce^{4bR}}{(1+\tau|\xi_{\pm}^{\mathrm{Re}}|)^N} + Ce^{4bR}\tau^s.$$

Now, using (6.5.10), we see that

$$|\mathcal{F}\left[(q_{\pm}(\cdot,\xi))_z\right](\xi_{\pm}^{\mathrm{Re}})| \leq \frac{Ce^{4bR}}{\left(1+2\tau(k^2+a^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}\right)^N} + Ce^{4bR}\tau^s,$$

where the constant C depends only on N, s, Ω and M.

For the last term of the right-hand side of the estimate (6.5.14), using (6.5.5), (6.5.6), (6.5.7), (6.5.9) and the relation $\eta_2 - \overline{\eta_1} = -\xi$, we have

$$|\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*)| \leq |w_1| + |w_2| + |w_3| + |w_4|, \tag{6.5.15}$$

where

$$\begin{aligned}
|w_1| &\leq \left|e^{-i\xi\cdot x}\right| |r_2 + \overline{r_1} + r_2\overline{r_1}| \leq |r_2| + |r_1| + |r_2||r_1|,\\
|w_2| &\leq \left|e^{-i\xi\cdot x^*}\right| |r_2^* + \overline{r_1^*} + r_2^*\overline{r_1^*}| \leq |r_2^*| + |r_1^*| + |r_2^*||r_1^*|,\\
|w_3| &\leq \left|e^{-i\xi_+\cdot x}\right| |r_2 + \overline{r_1^*} + r_2\overline{r_1^*}| \leq e^{2bR}(|r_2| + |r_1^*| + |r_2||r_1^*|),\\
|w_4| &\leq \left|e^{-i\xi_-\cdot x}\right| |r_2^* + \overline{r_1} + r_2^*\overline{r_1}| \leq e^{2bR}(|r_2^*| + |r_1| + |r_2^*||r_1|).
\end{aligned} \tag{6.5.16}$$

Integrating both sides in (6.5.16) and applying the Cauchy-Schwarz inequality along with the bounds from (6.4.22), we get

$$\begin{aligned}
\int_\Omega |w_j(x)|dx &\leq |\Omega|^{\frac{1}{2}}\frac{C_0M|B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}} + |\Omega|^{\frac{1}{2}}\frac{C_0M|B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}} + |B_R|^{\frac{1}{2}}\frac{C_0M|B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}}\\
&\leq 3|B_R|^{\frac{1}{2}}\frac{C_0M|B_R|^{\frac{1}{2}}}{(k^2+a^2)^{\frac{1}{2}}} = \frac{3C_0M|B_R|}{(k^2+a^2)^{\frac{1}{2}}} \quad \text{for } j = 1, 2,
\end{aligned}$$

$$\text{and} \quad \int_\Omega |w_j(x)|dx \leq e^{2Rb}\frac{3C_0M|B_R|}{(k^2+a^2)^{\frac{1}{2}}} \quad \text{for } j = 3, 4.$$

Thus, using the fact that $e^{2bR} \geq 1$, the last term of the right-hand side of (6.5.14) becomes

$$\begin{aligned}\left|\int_\Omega q\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*)dx\right| &\leq \int_\Omega |q|\,|\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*)|dx \\ &\leq \|q\|_{L^\infty(\Omega)}\int_\Omega (|w_1| + |w_2| + |w_3| + |w_4|)dx \\ &\leq 4M\left[e^{2Rb}\frac{3C_0M|B_R|}{(k^2+a^2)^{\frac{1}{2}}}\right] \leq C\frac{e^{2Rb}}{(k^2+a^2)^{\frac{1}{2}}},\end{aligned}$$

where the constant C depends only on C_0, M and Ω (note that the constant C depends on B_R as well but B_R in turn depends on Ω only).

Consequently, by applying these bounds to estimate (6.5.14) along with the bound in (6.5.4), we obtain

$$\begin{aligned}|\mathcal{F}[\widetilde{q}_e](\xi)| \leq C\|\mathcal{N}_{q_1}^S - \mathcal{N}_{q_2}^S\|_*(k^2+kb)^2e^{2(b+a)R} + \frac{Ce^{4bR}}{\left(1 + 2\tau(k^2+a^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}\right)^N} \\ + Ce^{4bR}\tau^s + \frac{Ce^{2Rb}}{(k^2+a^2)^{\frac{1}{2}}},\end{aligned} \tag{6.5.17}$$

for $0 < |\xi|^2 \leq 3(k^2+a^2)$ with $|\xi'| > 0$, $s \in (0, \frac{1}{2})$, $\tau \in (0,1)$ and $N \in \mathbb{N}$. Here the constant C depends only on C_0 (which depends only on n and Ω), M, s and Ω.

Next, we estimate the H^{-1} norm of $\widetilde{q}_e$ using (6.5.17). First, for $\rho > 1$, we define

$$\mathcal{K}_\rho := \{(\xi', \xi_n) \in \mathbb{R}^n \;\;:\;\; 0 < |\xi'| < \rho \text{ and } |\xi_n| < \rho\}\,.$$

Then, for $\xi \in \mathcal{K}_\rho$, using the estimate (6.5.17), we have

$$\begin{aligned}|\mathcal{F}[\widetilde{q}_e](\xi)| \leq C\|\mathcal{N}_{q_1}^S - \mathcal{N}_{q_2}^S\|_*(k^2+kb)^2e^{2(b+a)R} + \frac{Ce^{4bR}}{\left(1 + \frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^N} \\ + Ce^{4bR}\tau^s + \frac{Ce^{2Rb}}{(k^2+a^2)^{\frac{1}{2}}},\end{aligned}$$

since $|\xi| < (\rho^2+\rho^2)^{\frac{1}{2}} = \sqrt{2}\rho < 2\rho$.

Now, we can express the H^{-1} norm of $\widetilde{q}_e$ as follows:

$$\|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)}^2 = \int_{\mathcal{K}_\rho}\frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2}d\xi + \int_{\mathbb{R}^n\setminus\mathcal{K}_\rho}\frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2}d\xi. \tag{6.5.18}$$

By applying Parseval's identity, the second integral above can be estimated as follows:

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathcal{K}_\rho}\frac{\left|\mathcal{F}[\widetilde{q}_e](\xi)\right|^2}{1+|\xi|^2}d\xi &\le \int_{\mathbb{R}^n\setminus\mathcal{K}_\rho}\frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{|\xi|^2}d\xi \le \int_{\mathbb{R}^n\setminus\mathcal{K}_\rho}\frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{\rho^2}d\xi \\ &\le \frac{\|\mathcal{F}[\widetilde{q}_e]\|^2_{L^2(\mathbb{R}^n)}}{\rho^2} \le \frac{C}{\rho^2},\end{aligned} \tag{6.5.19}$$

where C depends on M and Ω. For the first integral in (6.5.18), we have

$$\begin{aligned}\int_{\mathcal{K}_\rho}\frac{\left|\mathcal{F}[\widetilde{q}_e](\xi)\right|^2}{1+|\xi|^2}d\xi &\le \int_{\mathcal{K}_\rho}\left|\mathcal{F}[\widetilde{q}_e](\xi)\right|^2 d\xi \\ &= \int_{\mathcal{K}_\rho}\Bigg|C\|\mathcal{N}^S_{q_1}-\mathcal{N}^S_{q_2}\|_*(k^2+kb)^2e^{2(b+a)R}+\frac{Ce^{2Rb}}{(k^2+a^2)^{\frac12}} \\ &\qquad + Ce^{4bR}\tau^s+\frac{Ce^{4bR}}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac12}|\xi'|\right)^N}\Bigg|^2 d\xi \\ &\le C\|\mathcal{N}^S_{q_1}-\mathcal{N}^S_{q_2}\|^2_*(k^2+kb)^4e^{4(b+a)R}\left(\int_{\mathcal{K}_\rho}d\xi\right)+\frac{Ce^{4Rb}}{(k^2+a^2)}\left(\int_{\mathcal{K}_\rho}d\xi\right) \\ &\qquad + Ce^{8bR}\tau^{2s}\left(\int_{\mathcal{K}_\rho}d\xi\right)+Ce^{8bR}\int_{\mathcal{K}_\rho}\frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac12}|\xi'|\right)^{2N}}.\end{aligned} \tag{6.5.20}$$

We next estimate the integrals in the above estimate. For the first integral, we note that

$$\int_{\mathcal{K}_\rho}d\xi=\int_{-\rho}^{\rho}\int_{|\xi'|<\rho}d\xi' d\xi_n=2\rho\int_0^\rho\int_{|\theta|=1}r^{n-2}d\theta dr=2\rho\frac{\rho^{n-1}}{n-1}\int_{|\theta|=1}d\theta\le C\rho^n. \tag{6.5.21}$$

Now, using the fact that $\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac12}|\xi'|\right)^{2N}>\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac12}|\xi'|\right)^N$ for $N\in\mathbb{N}$, the second integral can be estimated as follows:

$$\begin{aligned}\int_{\mathcal{K}_\rho}\frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac12}|\xi'|\right)^{2N}} &\le \int_{\mathcal{K}_\rho}\frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac12}|\xi'|\right)^{N}} \\ &= \int_{-\rho}^{\rho}\int_{|\xi'|<\rho}\frac{d\xi' d\xi_n}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac12}|\xi'|\right)^{N}}\end{aligned}$$

$$= 2\rho \int_0^\rho \int_{|\theta|=1} \frac{r^{n-2} d\theta dr}{\left(1 + \frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}} r\right)^N}$$
$$= 2\rho \left(\int_{|\theta|=1} d\theta\right) \int_0^\rho \frac{r^{n-2} dr}{\left(1 + \frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}} r\right)^N}$$
$$\le C\rho \int_0^\rho \frac{r^{n-2} dr}{\left(1 + \frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}} r\right)^N}.$$

Denote $y := \frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}} r$ and choose $N > n-1 > 1$. Then, we have

$$\int_0^\rho \frac{r^{n-2} dr}{\left(1 + \frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}} r\right)^N} \le \int_0^\infty \frac{y^{n-2}}{(1+y)^N} \left(\frac{\rho}{\tau(k^2+a^2)^{\frac{1}{2}}}\right)^{n-2} \left(\frac{\rho}{\tau(k^2+a^2)^{\frac{1}{2}}}\right) dy$$
$$= \frac{\rho^{n-1}}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}} \int_0^\infty \frac{y^{n-2}}{(1+y)^N} dy \le \frac{C\rho^{n-1}}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}},$$

where

$$\int_0^\infty \frac{y^{n-2}}{(1+y)^N} dy = \int_0^d \frac{y^{n-2}}{(1+y)^N} dy + \int_d^\infty \frac{y^{n-2}}{(1+y)^N} dy \quad \text{(for some } d > 0\text{)}$$
$$\le \int_0^d y^{n-2} dy + \int_d^\infty \frac{dy}{y^{N-n+2}} < \infty.$$

Therefore

$$\int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1 + \frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}} |\xi'|\right)^{2N}} \le \frac{C\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}. \tag{6.5.22}$$

Using the estimates (6.5.21) and (6.5.22) in (6.5.20), we obtain

$$\int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \le C\|\mathcal{N}_{q_1}^S - \mathcal{N}_{q_2}^S\|_*^2 (k^2+kb)^4 e^{4(b+a)R} \rho^n + \frac{Ce^{4Rb}}{(k^2+a^2)} \rho^n$$
$$+ Ce^{8bR} \tau^{2s} \rho^n + \frac{Ce^{8bR} \rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}. \tag{6.5.23}$$

Now, using (6.5.19), (6.5.23) and the fact that $e^{8Rb} \ge 1$ in (6.5.18), we get

$$\|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)}^2 \le C\|\mathcal{N}_{q_1}^S - \mathcal{N}_{q_2}^S\|_*^2 (k^2+kb)^4 e^{4(b+a)R} \rho^n + \frac{Ce^{8Rb} \rho^n}{(k^2+a^2)}$$
$$+ Ce^{8Rb} \tau^{2s} \rho^n + \frac{Ce^{8Rb} \rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}} + \frac{Ce^{8Rb}}{\rho^2}. \tag{6.5.24}$$

Next, we choose $\tau \in (0,1)$ and $\rho > 0$ in such a way that the last four terms in the right-hand side of the above estimate can be combined together.

First, we choose $\tau^2 = (k^2+a^2)^{-\frac{n-1}{2s+n-1}}$. Then, given that $k, a \geq 1$, we find that

$$\tau \in (0,1) \quad \text{and} \quad \tau^{2s} = \frac{1}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}.$$

Consequently, we obtain

$$e^{8Rb}\tau^{2s}\rho^n + \frac{e^{8Rb}\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}} = 2\frac{e^{8Rb}\rho^n}{(k^2+a^2)^{\frac{s(n-1)}{2s+n-1}}}.$$

Now, observe the fact that

$$\frac{s(n-1)}{2s+n-1} \leq \frac{s(n-1)}{n-1} = s < \frac{1}{2}.$$

Using this, along with the fact that $k^2+a^2 > 1$, we have

$$\frac{e^{8Rb}\rho^n}{(k^2+a^2)} + \frac{2e^{8Rb}\rho^n}{(k^2+a^2)^{\frac{s(n-1)}{2s+n-1}}} \leq \frac{e^{8Rb}\rho^n}{(k^2+a^2)^{\frac{1}{2}}} + \frac{2e^{8Rb}\rho^n}{(k^2+a^2)^{\frac{s(n-1)}{2s+n-1}}} \leq 3\frac{e^{8Rb}\rho^n}{(k^2+a^2)^{\frac{s(n-1)}{2s+n-1}}}.$$

So far, we have combined the three terms into

$$\frac{Ce^{8Rb}\rho^n}{(k^2+a^2)^{\frac{s(n-1)}{2s+n-1}}}.$$

Finally, we choose $\rho = (k^2+a^2)^{\frac{s(n-1)}{(n+2)(2s+n-1)}}$. Then, we get

$$\rho > 1 \quad \text{and} \quad \frac{\rho^n}{(k^2+a^2)^{\frac{s(n-1)}{2s+n-1}}} = \frac{1}{\rho^2}$$

using the facts that $k, a \geq 1$. With these choices, the estimate (6.5.24) can be rewritten as

$$\begin{aligned}\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\|(\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2})\|^2_*(k^2+kb)^4 e^{4(b+a)R}(k^2+a^2)^{\frac{s(n-1)n}{(n+2)(2s+n-1)}} \\ + \frac{Ce^{8Rb}}{(k^2+a^2)^{\frac{2s(n-1)}{(n+2)(2s+n-1)}}}.\end{aligned}$$

Note that we have derived this estimate under the assumption that if $\xi \in \mathcal{K}_\rho$, then the condition $|\xi|^2 \leq 3(k^2+a^2)$ is satisfied. Next we check that this is indeed true. To see this, observe that

$$\begin{aligned} &\frac{s(n-1)n}{(n+2)(2s+n-1)} \le \frac{s(n-1)n}{n(n-1)} = s < \frac{1}{2} \\ \text{and} \quad &\frac{2s(n-1)}{(n+2)(2s+n-1)} \le \frac{2s(n-1)}{2(n-1)} = s < \frac{1}{2}. \end{aligned} \tag{6.5.25}$$

Now, suppose that $\xi \in \mathcal{K}_\rho$, that is, $0 < |\xi'| < \rho,\ |\xi_n| < \rho$. Then

$$\frac{|\xi|^2}{3(k^2+a^2)} \le \frac{2\rho^2}{3(k^2+a^2)} = \frac{2}{3}\frac{(k^2+a^2)^{\frac{2s(n-1)}{(n+2)(2s+n-1)}}}{(k^2+a^2)} = \frac{2}{3}\frac{1}{(k^2+a^2)^{1-\frac{2s(n-1)}{(n+2)(2s+n-1)}}} < 1,$$

for $s \in (0, \frac{1}{2})$ and $n \ge 3$. Hence, we have $|\xi|^2 \le 3(k^2+a^2)$.

Let us denote $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)}$. Using (6.5.25) along with the fact $k, a \ge 1$, we obtain

$$\begin{aligned} \|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\le C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|^2_*(k^2+kb)^4 e^{4(b+a)R}(k^2+a^2)^{\frac{1}{2}} + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\le C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|^2_*(k^2+kb)^4 e^{4(b+a)R}(k^2a^2+k^2a^2)^{\frac{1}{2}} + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\le C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|^2_*(k^2+kb)^4 e^{4(b+a)R}ka + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\le C\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|^2_*(k^2+kb)^4 e^{4(b+a)R}(k^2+kb)e^{Ra} + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\le Ce^{8Rb}\left[(k^2+kb)^5\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|^2_* e^{5Ra} + \frac{1}{(k^2+a^2)^\sigma}\right], \end{aligned}$$

where the second last line follows from the facts that $k \le k^2+kb$ and $a \le e^{Ra}$.

Next, we choose $a > 1$ suitably to obtain the stability estimate.

Let $a = \frac{E}{5R}$. Then the previous estimate becomes

$$\begin{aligned} \|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} &\le Ce^{4Rb}\left[(k^2+kb)^5\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|^2_* e^{5R\frac{E}{5R}} + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}} \\ &= C = e^{4Rb}\left[(k^2+kb)^5\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|^2_* e^{E} + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}} \\ &= Ce^{4Rb}\left[(k^2+kb)^5\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}}. \end{aligned} \tag{6.5.26}$$

However, we need to ensure that with our choice of a, the conditions $a > 1$ and $(k^2+a^2)^{\frac{1}{2}} > C_0 M$ for all $k \ge 1$ are satisfied. This leads to the following condition on $\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_*$.

Let

$$\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* < \delta := \frac{1}{e^{5Rc}}, \quad \text{where} \;\; c = 1 + C_0 M.$$

Then

$$e^{5R(1+C_0M)} < \frac{1}{\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_*} \iff 5R(1+C_0M) < \log(\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_*^{-1}) = E$$
$$\implies 1 + C_0M < \frac{E}{5R} = a < (k^2+a^2)^{\frac{1}{2}}, \; \forall k \geq 1.$$

Hence, our choice of a satisfies both the required conditions.

From (6.5.26), we obtain the estimate

$$\|q\|_{H^{-1}(\Omega)} \leq \|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} \leq Ce^{4Rb}\left[(k^2+kb)^5\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}}.$$

The case when $\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* \geq \frac{1}{e^{5Rc}}$ easily follows from the following fact:

$$\|q\|_{H^{-1}(\Omega)} \leq C\|q\|_{L^\infty(\Omega)} \leq \frac{CM}{\delta^{\frac{1}{2}}}\delta^{\frac{1}{2}} \leq \frac{CM}{\delta^{\frac{1}{2}}}\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_*^{\frac{1}{2}}$$
$$\leq Ce^{4Rb}\left[(k^2+kb)^5\|\mathcal{N}^S_{q_1} - \mathcal{N}^S_{q_2}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}},$$

where we denote $\delta := \frac{1}{e^{5Rc}}$ and the constant C in the last inequality depends only on n, s, Ω and M. Thus the stability estimate (6.2.1) follows.

6.6 Stability estimate for the biharmonic case

Proof of theorem 6.2.4

Let u_1 be a solution to the problem

$$\begin{cases} \Delta^2 u_1 \; - \; (k^2 - ikb)^2 u_1 \; + \; q_1 u_1 & = 0 \;\; \text{in } \Omega, \\ \qquad\qquad (u_1, \Delta u_1) & = (u_2, \Delta u_2) \;\; \text{on } \partial\Omega, \end{cases}$$

where u_2 is the CGO type solution defined in (6.4.38). Then $u := u_1 - u_2 \in H^4(\Omega)$ satisfies

$$\begin{cases} \Delta^2 u \; - \; (k^2 - ikb)^2 u \; + \; q_1 u & = -(q_1 - q_2)u_2 \;\; \text{in } \Omega, \\ \qquad\qquad (u, \Delta u) & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Denote $q = q_2 - q_1$. Then, using the function u defined above and the function v defined in (6.4.38) in the integral identity (6.3.7), we obtain

$$\begin{aligned}
\left|\int_\Omega q u_2 \bar{v}\, dx\right| &= \left|\int_\Gamma \partial_\nu (\Delta u)\, \bar{v}\, dS + \int_\Gamma \partial_\nu u \left(\overline{\Delta v}\right)\, dS\right| \\
&\le \|\partial_\nu(\Delta u_1 - \Delta u_2)\|_{L^2(\Gamma)} \|v\|_{L^2(\partial\Omega)} + \|\partial_\nu(u_1 - u_2)\|_{L^2(\Gamma)} \|\Delta v\|_{L^2(\partial\Omega)} \\
&\le C\left(\|\partial_\nu u_1 - \partial_\nu u_2\|_{L^2(\Gamma)} \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\partial_\nu(\Delta u_1) - \partial_\nu(\Delta u_2)\|_{L^2(\Gamma)} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \\
&\le C\left(\|\partial_\nu u_1 - \partial_\nu u_2\|_{L^2(\Gamma)} + \|\partial_\nu(\Delta u_1) - \partial_\nu(\Delta u_2)\|_{L^2(\Gamma)}\right) \\
&\qquad \left(\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Delta v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right),
\end{aligned}$$

where in the first inequality we have used the fact that $v = 0 = \Delta v$ on Γ_0. By applying the trace theorem and the definition of the Dirichlet-to-Neumann map $\mathcal{N}_q^B$, it follows that

$$\begin{aligned}
\left|\int_\Omega q u_2 \bar{v}\, dx\right| &\le C\left(\|\partial_\nu u_1 - \partial_\nu u_2\|_{H^{\frac{5}{2}}(\Gamma)} + \|\partial_\nu(\Delta u_1) - \partial_\nu(\Delta u_2)\|_{H^{\frac{1}{2}}(\Gamma)}\right) \\
&\qquad \left(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}\right) \\
&\le C\|(\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B)(f,g)\|_{H^{\frac{5}{2},\frac{1}{2}}(\Gamma)} \|v\|_{H^4(\Omega)} \\
&\le C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_* \|(f,g)\|_{\widetilde{H}^{\frac{7}{2},\frac{3}{2}}(\Gamma)} \|v\|_{H^4(\Omega)} \\
&\le C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_* \left(\|u_2\|_{\widetilde{H}^{\frac{7}{2}}(\Gamma)} + \|\Delta u_2\|_{\widetilde{H}^{\frac{3}{2}}(\Gamma)}\right) \|v\|_{H^4(\Omega)} \\
&= C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_* (\|u_2\|_{H^{\frac{7}{2}}(\partial\Omega)} + \|\Delta u_2\|_{H^{\frac{3}{2}}(\partial\Omega)}) \|v\|_{H^4(\Omega)} \\
&\le C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_* \left(\|u_2\|_{H^4(\Omega)} + \|\Delta u_2\|_{H^2(\Omega)}\right) \|v\|_{H^4(\Omega)} \\
&\le C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_* \|u_2\|_{H^4(\Omega)} \|v\|_{H^4(\Omega)},
\end{aligned} \tag{6.6.1}$$

where $(u_1, \Delta u_1) = (u_2, \Delta u_2) = (f, g)$ on Γ. Now, observe that

$$\|u_2\|_{H^4(\Omega)} \le 2\|\widetilde{u}_2\|_{H^4(\widetilde{\Omega})} \quad \text{and} \quad \|v\|_{H^4(\Omega)} \le 2\|\widetilde{v}\|_{H^4(\widetilde{\Omega})}.$$

These estimates, combined with those in (6.4.39), (6.4.40), (6.4.41) and the fact that $k \ge 1$, we get

$$\|u_2\|_{H^4(\Omega)} \le C(k^2 + kb)^2 \|\widetilde{u}_2\|_{L^2(B_R)} \quad \text{and} \quad \|v\|_{H^4(\Omega)} \le C(k^2 + kb)^2 \|\widetilde{v}\|_{L^2(B_R)}. \tag{6.6.2}$$

Next, using (6.4.37), we observe that

$$\|\widetilde{u}_2\|^2_{L^2(B_R)} \leq Ce^{2(b+a)R}, \ \|\widetilde{v}\|^2_{L^2(B_R)} \leq Ce^{2(b+a)R}.$$

The estimates above, when combined with those in (6.6.2), lead to

$$\|u_2\|_{H^4(\Omega)} \leq C(k^2+kb)^2 e^{(b+a)R} \quad \text{and} \quad \|v\|_{H^4(\Omega)} \leq C(k^2+kb)^2 e^{(b+a)R}.$$

Consequently, the estimate (6.6.1) becomes

$$\left|\int_\Omega qu_2\bar{v}\,dx\right| \leq C\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_*(k^2+kb)^4 e^{2(b+a)R}. \tag{6.6.3}$$

Now, by substituting u_2 and v and proceeding as in the Schrödinger case, we observe that

$$\begin{aligned}\int_\Omega qu_2\bar{v}\,dx = \int_\Omega q\left(e^{-i\xi\cdot x} + e^{-i\xi\cdot x^*}\right)dx - \int_\Omega q\left(e^{-i\xi_+\cdot x} + e^{-i\xi_-\cdot x}\right)dx \\ + \int_\Omega q\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*)dx,\end{aligned} \tag{6.6.4}$$

where $\mathcal{G}$ is as defined in (6.5.5). Recall that the first two terms in the right-hand side of the previous identity can be written as

$$\int_\Omega q\left(e^{-i\xi\cdot x} + e^{-i\xi\cdot x^*}\right)dx = \mathcal{F}[\widetilde{q}_e](\xi),$$

and

$$\int_\Omega q\left(e^{-i\xi_+\cdot x} + e^{-i\xi_-\cdot x}\right)dx = \mathcal{F}\left[(q_+(\cdot,\xi))_z\right](\xi_+^{\mathrm{Re}}) + \mathcal{F}\left[(q_-(\cdot,\xi))_z\right](\xi_-^{\mathrm{Re}}).$$

Consequently, from (6.6.4), we have

$$\begin{aligned}|\mathcal{F}[\widetilde{q}_e](\xi)| \leq \left|\int_\Omega qu_2\bar{v}\,dx\right| + \left|\mathcal{F}\left[(q_+(\cdot,\xi))_z\right](\xi_+^{\mathrm{Re}})\right| + \left|\mathcal{F}\left[(q_-(\cdot,\xi))_z\right](\xi_-^{\mathrm{Re}})\right| \\ + \left|\int_\Omega q\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*)dx\right|.\end{aligned} \tag{6.6.5}$$

Next, we will calculate the bounds for the last three terms on the right-hand side of the above estimate, as we already have the bound for the first term in (6.6.3).

Recall that

$$|\mathrm{Im}\,(\xi_\pm)| \leq 2b, \quad |\xi_\pm^{\mathrm{Re}}|^2 \geq 4\left(k^2+a^2\right)\frac{|\xi'|^2}{|\xi|^2}. \tag{6.6.6}$$

Proceeding as in the Schrödinger case, and applying the quantitative version of the Riemann-Lebesgue lemma together with the estimates in (6.6.6), we obtain

$$|\mathcal{F}\left[(q_{\pm}(\cdot,\xi))_z\right](\xi_{\pm}^{\mathrm{Re}})| \leq \frac{Ce^{4bR}}{\left(1+2\tau(k^2+a^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}\right)^N} + Ce^{4bR}\tau^s, \tag{6.6.7}$$

where the constant C depends only on N, s, Ω and M.

For the fourth term of the right-hand side of the estimate (6.6.5), we again proceed as in the Schrödinger case. Recall that $\mathcal{G}$ can be estimated as in (6.5.15)-(6.5.16), and then proceeding exactly as in that case, we get

$$\begin{aligned}\int_\Omega |w_j(x)|dx &\leq |\Omega|^{\frac{1}{2}}\frac{C_0M|B_R|^{\frac{1}{2}}}{a} + |\Omega|^{\frac{1}{2}}\frac{C_0M|B_R|^{\frac{1}{2}}}{a} + |B_R|^{\frac{1}{2}}\frac{C_0M|B_R|^{\frac{1}{2}}}{a}\\ &\leq \frac{3C_0M|B_R|}{a} \quad \text{for } j=1,2,\\ \text{and} \quad \int_\Omega |w_j(x)|dx &\leq e^{2Rb}\frac{3C_0M|B_R|}{a} \quad \text{for } j=3,4.\end{aligned}$$

Using this and the fact that $e^{2bR} \geq 1$, we note that the fourth term of the right-hand side of (6.6.5) can be estimated as

$$\begin{aligned}\left|\int_\Omega q\mathcal{G}(x,r_1,r_2,r_1^*,r_2^*)dx\right| &\leq \int_\Omega |q|\;|\mathcal{G}(x,r_1,r_2,r_1^*,r_2^*)|dx\\ &\leq \|q\|_{L^\infty(\Omega)}\int_\Omega (|w_1|+|w_2|+|w_3|+|w_4|)dx\\ &\leq 4M\left[e^{2Rb}\frac{3C_0M|B_R|}{a}\right] \leq C\frac{e^{2Rb}}{a},\end{aligned} \tag{6.6.8}$$

where the constant C depends only on C_0, M and Ω.

Let us further assume that $a > k$. Then

$$\sqrt{2}a > (k^2+a^2)^{\frac{1}{2}} \implies \frac{1}{a} < \frac{\sqrt{2}}{(k^2+a^2)^{\frac{1}{2}}}.$$

Using this in (6.6.8), we see that

$$\left|\int_\Omega q\mathcal{G}(x,r_1,r_2,r_1^*,r_2^*)dx\right| \leq C\frac{e^{2Rb}}{(k^2+a^2)^{\frac{1}{2}}}. \tag{6.6.9}$$

Using (6.6.3), (6.6.7) and (6.6.9) in (6.6.5), we obtain

$$|\mathcal{F}[\widetilde{q}_e](\xi)| \leq C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_*(k^2+kb)^4 e^{2(b+a)R} + \frac{Ce^{4bR}}{\left(1+2\tau(k^2+a^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}\right)^N} + Ce^{4bR}\tau^s + \frac{Ce^{2Rb}}{(k^2+a^2)^{\frac{1}{2}}}, \tag{6.6.10}$$

for $0 < |\xi|^2 \leq 3(k^2+a^2)$ with $|\xi'| > 0$, $s \in (0, \frac{1}{2})$, $\tau \in (0,1)$ and $N \in \mathbb{N}$.

Next, we find suitable bounds for the H^{-1} norm of $\widetilde{q}_e$ using the previous estimate. The steps are similar to the Schrödinger case.

For $\rho > 1$, we consider the set

$$\mathcal{K}_\rho := \{(\xi', \xi_n) \in \mathbb{R}^n \ : \ 0 < |\xi'| < \rho \text{ and } |\xi_n| < \rho\}.$$

Then, for $\xi \in \mathcal{K}_\rho$, using the estimate (6.6.10) and the fact that $|\xi| < 2\rho$, we see that

$$|\mathcal{F}[\widetilde{q}_e](\xi)| \leq C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_*(k^2+kb)^4 e^{2(b+a)R} + \frac{Ce^{4bR}}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^N} + Ce^{4bR}\tau^s + \frac{Ce^{2Rb}}{(k^2+a^2)^{\frac{1}{2}}}.$$

Now

$$\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi + \int_{\mathbb{R}^n \setminus \mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \leq \int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi + \frac{C}{\rho^2}. \tag{6.6.11}$$

For the first integral on the right-hand side of (6.6.11), we observe that

$$\int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \leq C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_*^2(k^2+kb)^8 e^{4(b+a)R}\left(\int_{\mathcal{K}_\rho} d\xi\right) + \frac{Ce^{4Rb}}{(k^2+a^2)}\left(\int_{\mathcal{K}_\rho} d\xi\right) + Ce^{8bR}\tau^{2s}\left(\int_{\mathcal{K}_\rho} d\xi\right) + Ce^{8bR}\int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{2N}}.$$

Proceeding as in the Schrödinger case, choosing $N > n-1$, we obtain

$$\int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \leq C\|\mathcal{N}_{q_1}^B - \mathcal{N}_{q_2}^B\|_*^2(k^2+kb)^8 e^{4(b+a)R}\rho^n + \frac{Ce^{4Rb}}{(k^2+a^2)}\rho^n + Ce^{8bR}\tau^{2s}\rho^n + \frac{Ce^{8bR}\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}. \tag{6.6.12}$$

Using (6.6.12) and the fact that $e^{8Rb} \geq 1$ in (6.6.11), we get

$$\begin{aligned} \|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|^2_*(k^2+kb)^8 e^{4(b+a)R}\rho^n + \frac{Ce^{8Rb}\rho^n}{(k^2+a^2)} \\ + Ce^{8Rb}\tau^{2s}\rho^n + \frac{Ce^{8Rb}\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}} + \frac{Ce^{8Rb}}{\rho^2}. \end{aligned} \tag{6.6.13}$$

Next, as in the Schrödinger case, we choose τ such that $\tau^2 = (k^2+a^2)^{-\frac{n-1}{2s+n-1}}$ and

$$\rho = (k^2+a^2)^{\frac{s(n-1)}{(n+2)(2s+n-1)}}.$$

The suitability of these choices follows exactly as in the Schrödinger case. Using these choices in (6.6.13), we see that

$$\begin{aligned} \|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|^2_*(k^2+kb)^8 e^{4(b+a)R}(k^2+a^2)^{\frac{s(n-1)n}{(n+2)(2s+n-1)}} \\ + \frac{Ce^{8Rb}}{(k^2+a^2)^{\frac{2s(n-1)}{(n+2)(2s+n-1)}}}. \end{aligned}$$

Let $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)}$. Then using (6.5.25) and the fact that $k, a \geq 1$, we obtain

$$\begin{aligned} \|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|^2_* k^8(k+b)^8 e^{4(b+a)R}(k^2+a^2)^{\frac{1}{2}} + \frac{Ce^{8Rb}}{(k^2+a^2)^\sigma} \\ &\leq Ce^{12Rb}\left[e^{17Rk}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|^2_* e^{5Ra} + \frac{1}{(k^2+a^2)^\sigma}\right]. \end{aligned} \tag{6.6.14}$$

Next, we choose $a = k + C_0Mk + \frac{E}{5R}$. Note that this choice of a fulfills the required conditions $a > k \geq 1$ and $a > C_0M$ for all $k \geq 1$.

Using this choice of a in (6.6.14), we see that for $\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_* < \delta$, where $\delta := \frac{1}{e}$,

$$\begin{aligned} &\|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} \\ &\leq Ce^{6Rb}\left[e^{17Rk}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|^2_* e^{5R(k+C_0Mk+\frac{E}{5R})} + \frac{1}{(k^2+\left(k+C_0Mk+\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}} \\ &= Ce^{6Rb}\left[e^{17Rk}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|^2_* e^{5Rk(1+C_0M)}e^E + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}} \\ &\leq Ce^{6Rb}\left[e^{CRk}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}} \end{aligned}$$

and hence

$$\|q\|_{H^{-1}(\Omega)} \leq Ce^{6Rb}\left[e^{CRk}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_* + \frac{1}{(k^2 + \left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}}.$$

The case when $\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_* \geq \delta$ can be easily derived as follows:

$$\begin{aligned}\|q\|_{H^{-1}(\Omega)} \leq C\|q\|_{L^\infty(\Omega)} \leq \frac{CM}{\delta^{\frac{1}{2}}}\delta^{\frac{1}{2}} \leq \frac{CM}{\delta^{\frac{1}{2}}}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_*^{\frac{1}{2}} \\ \leq Ce^{6Rb}\left[e^{CRk}\|\mathcal{N}^B_{q_1} - \mathcal{N}^B_{q_2}\|_* + \frac{1}{(k^2 + \left(\frac{E}{5R}\right)^2)^\sigma}\right]^{\frac{1}{2}},\end{aligned}$$

where the constant C depends only on n, s, Ω and M. Thus the stability estimate (6.2.2) follows.

6.7 Appendix

Appendix I

In this appendix, we recall, from [31], a few results related to $\eta_j,\ j = 1, 2$, that we have used in the calculations.

First, for η_1, we have

$$\begin{aligned}&\eta_1 \cdot \eta_1 \\ &= \Big(\frac{\xi}{2}\Big)\cdot\Big(\frac{\xi}{2}\Big) + \Big(\frac{\xi}{2}\Big)\cdot\Big(-A\beta\Big) + \Big(\frac{\xi}{2}\Big)\cdot\Big(iB\beta\Big) + \Big(\frac{\xi}{2}\Big)\cdot\Big(ia\alpha\Big) \\ &\quad + \Big(-A\beta\Big)\cdot\Big(\frac{\xi}{2}\Big) + \Big(-A\beta\Big)\cdot\Big(-A\beta\Big) + \Big(-A\beta\Big)\cdot\Big(iB\beta\Big) + \Big(-A\beta\Big)\cdot\Big(ia\alpha\Big) \\ &\quad + \Big(iB\beta\Big)\cdot\Big(\frac{\xi}{2}\Big) + \Big(iB\beta\Big)\cdot\Big(-A\beta\Big) + \Big(iB\beta\Big)\cdot\Big(iB\beta\Big) + \Big(iB\beta\Big)\cdot\Big(ia\alpha\Big) \\ &\quad + \Big(ia\alpha\Big)\cdot\Big(\frac{\xi}{2}\Big) + \Big(ia\alpha\Big)\cdot\Big(-A\beta\Big) + \Big(ia\alpha\Big)\cdot\Big(iB\beta\Big) + \Big(ia\alpha\Big)\cdot\Big(ia\alpha\Big) \\ &= \frac{|\xi|^2}{4} + (A^2 - B^2)|\beta|^2 - 2iAB|\beta|^2 - a^2|\alpha|^2 \qquad (\text{since } \alpha\cdot\beta = \alpha\cdot\xi = \beta\cdot\xi = 0) \\ &= \frac{|\xi|^2}{4} + k^2 + a^2 - \frac{|\xi|^2}{4} + ikb - a^2 \qquad (\text{since } |\alpha| = |\beta| = 1) \\ &= k^2 + ikb,\end{aligned}$$

where we have used the facts that $A^2 - B^2 = k^2 + a^2 - \frac{|\xi|^2}{4}$ and $2AB = -kb$, which can be obtained by squaring $A + iB$. Similarly, for η_2, we get

$$
\begin{aligned}
&\eta_2 \cdot \eta_2 \\
&= \Big(-\frac{\xi}{2}\Big)\cdot\Big(-\frac{\xi}{2}\Big) + \Big(-\frac{\xi}{2}\Big)\cdot\Big(-A\beta\Big) + \Big(-\frac{\xi}{2}\Big)\cdot\Big(-iB\beta\Big) + \Big(-\frac{\xi}{2}\Big)\cdot\Big(-ia\alpha\Big) \\
&\qquad + \Big(-A\beta\Big)\cdot\Big(-\frac{\xi}{2}\Big) + \Big(-A\beta\Big)\cdot\Big(-A\beta\Big) \\
&\qquad\qquad + \Big(-A\beta\Big)\cdot\Big(-iB\beta\Big) + \Big(-A\beta\Big)\cdot\Big(-ia\alpha\Big) \\
&\qquad + \Big(-iB\beta\Big)\cdot\Big(-\frac{\xi}{2}\Big) + \Big(-iB\beta\Big)\cdot\Big(-A\beta\Big) \\
&\qquad\qquad + \Big(-iB\beta\Big)\cdot\Big(-iB\beta\Big) + \Big(-iB\beta\Big)\cdot\Big(-ia\alpha\Big) \\
&\qquad + \Big(-ia\alpha\Big)\cdot\Big(-\frac{\xi}{2}\Big) + \Big(-ia\alpha\Big)\cdot\Big(-A\beta\Big) \\
&\qquad\qquad + \Big(-ia\alpha\Big)\cdot\Big(-iB\beta\Big) + \Big(-ia\alpha\Big)\cdot\Big(-ia\alpha\Big) \\
&= \frac{|\xi|^2}{4} + (A^2 - B^2)|\beta|^2 + 2iAB|\beta|^2 - a^2|\alpha|^2 \quad \text{(since } \alpha\cdot\beta = \alpha\cdot\xi = \beta\cdot\xi = 0) \\
&= \frac{|\xi|^2}{4} + k^2 + a^2 - \frac{|\xi|^2}{4} - ikb - a^2 \qquad \text{(since } |\alpha| = |\beta| = 1) \\
&= k^2 - ikb.
\end{aligned}
$$

Also, for $j = 1, 2$,

$$
|\eta_j|^2 = \frac{|\xi|^2}{4} + A^2|\beta|^2 + B^2|\beta|^2 + a^2|\alpha|^2 = \frac{|\xi|^2}{4} + A^2 + B^2 + a^2,
$$

where

$$
\begin{aligned}
A^2 + B^2 = |A + iB|^2 &= \left|\Big(k^2 + a^2 - \frac{|\xi|^2}{4} - ikb\Big)^{\frac{1}{2}}\right|^2 = \left|k^2 + a^2 - \frac{|\xi|^2}{4} - ikb\right| \\
&= \left(\Big(k^2 + a^2 - \frac{|\xi|^2}{4}\Big)^2 + k^2b^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Next, we observe the following bound for $\operatorname{Im}(\eta_j)$ when $|\xi|^2 \le 3(k^2 + a^2)$.

Recall that

$$
\operatorname{Im}(\eta_1) = B\beta + a\alpha, \quad \text{and} \quad \operatorname{Im}(\eta_2) = -B\beta - a\alpha,
$$

and hence $|\mathrm{Im}\,(\eta_j)|^2 = B^2 + a^2,\ j = 1, 2$. Additionally, using the facts that $A > 0$, and $A^2 - B^2 = k^2 + a^2 - \frac{|\xi|^2}{4}$, $2AB = -kb$, we obtain

$$\begin{aligned}
|B| = \frac{kb}{2A} &= \frac{kb}{\sqrt{2}\sqrt{A^2 - B^2 + \sqrt{(A^2-B^2)^2 + 4A^2B^2}}} \\
&= \frac{kb}{\sqrt{2}\sqrt{k^2 + a^2 - \frac{|\xi|^2}{4} + \sqrt{(k^2 + a^2 - \frac{|\xi|^2}{4})^2 + k^2b^2}}} \\
&\leq \frac{kb}{\sqrt{2}\sqrt{\frac{1}{4}(k^2+a^2) + \sqrt{(\frac{1}{4}(k^2+a^2))^2}}} \\
&= \frac{kb}{\sqrt{2}\sqrt{\frac{1}{4}(k^2+a^2) + \frac{1}{4}(k^2+a^2)}} = \frac{kb}{\sqrt{2}\sqrt{\frac{1}{2}(k^2+a^2)}} \\
&= \frac{kb}{\sqrt{k^2+a^2}} < \frac{kb}{k} = b,
\end{aligned}$$

where we have used the fact that $|\xi|^2 \leq 3(k^2+a^2)$ in the second line of the above calculation. Hence, we have

$$|\mathrm{Im}\,(\eta_j)|^2 \leq b^2 + a^2 \leq (b+a)^2,$$

and therefore,

$$|\mathrm{Im}\,(\eta_j)| \leq b + a,\ j = 1, 2.$$

Also,

$$\begin{aligned}
A^2 &= \frac{1}{2}\left[\left(\left(k^2 + a^2 - \frac{|\xi|^2}{4}\right)^2 + k^2b^2\right)^{\frac{1}{2}} + \left(k^2 + a^2 - \frac{|\xi|^2}{4}\right)\right] \\
&\geq \frac{1}{2}\cdot 2\left(k^2 + a^2 - \frac{|\xi|^2}{4}\right) = \left(k^2 + a^2 - \frac{|\xi|^2}{4}\right).
\end{aligned}$$

Appendix II

In this appendix, we derive the equation satisfied by the remainder term in the CGO solution for the biharmonic problem. We note that $\widetilde{w}(x) = e^{i\eta\cdot x}(1 + \widetilde{r}(x))$ satisfies

$$\Delta^2\widetilde{w} - (\eta\cdot\eta)^2\widetilde{w} + q\widetilde{w} = 0 \quad \text{in } B_R$$

if and only if $\widetilde{r}$ satisfies

$$e^{-i\eta\cdot x}(\Delta^2 - (\eta\cdot\eta)^2 + q)(e^{i\eta\cdot x}(1+\widetilde{r})) = 0 \quad \text{in } B_R.$$

Now

$$\begin{aligned}
&e^{-i\eta\cdot x}\Delta^2(e^{i\eta\cdot x}(1+\widetilde{r}))\\
&\quad= e^{-i\eta\cdot x}\Delta\left(\Delta(e^{i\eta\cdot x}(1+\widetilde{r}))\right)\\
&\quad= e^{-i\eta\cdot x}\Delta\left(\Delta(e^{i\eta\cdot x})(1+\widetilde{r}) + 2\nabla(e^{i\eta\cdot x})\cdot\nabla(1+\widetilde{r}) + e^{i\eta\cdot x}\Delta(1+\widetilde{r})\right)\\
&\quad= e^{-i\eta\cdot x}\Delta\left(-(\eta\cdot\eta)e^{i\eta\cdot x}(1+\widetilde{r}) + 2i(\eta\cdot\nabla\widetilde{r})e^{i\eta\cdot x} + e^{i\eta\cdot x}\Delta\widetilde{r}\right)\\
&\quad= e^{-i\eta\cdot x}\left(-(\eta\cdot\eta)\Delta(e^{i\eta\cdot x}(1+\widetilde{r})) + 2i\Delta((\eta\cdot\nabla\widetilde{r})e^{i\eta\cdot x}) + \Delta(e^{i\eta\cdot x}\Delta\widetilde{r})\right)\\
&\quad= -e^{-i\eta\cdot x}(\eta\cdot\eta)\left(-(\eta\cdot\eta)e^{i\eta\cdot x}(1+\widetilde{r}) + 2i(\eta\cdot\nabla\widetilde{r})e^{i\eta\cdot x} + e^{i\eta\cdot x}\Delta\widetilde{r}\right)\\
&\qquad + 2ie^{-i\eta\cdot x}\left(-(\eta\cdot\eta)(\eta\cdot\nabla\widetilde{r})e^{i\eta\cdot x} + 2i\eta\cdot\nabla(\eta\cdot\nabla\widetilde{r})e^{i\eta\cdot x} + \Delta(\eta\cdot\nabla\widetilde{r})e^{i\eta\cdot x}\right)\\
&\qquad + e^{-i\eta\cdot x}\left(-(\eta\cdot\eta)e^{i\eta\cdot x}\Delta\widetilde{r} + 2ie^{i\eta\cdot x}(\eta\cdot\nabla(\Delta\widetilde{r})) + e^{i\eta\cdot x}\Delta^2\widetilde{r}\right)\\
&\quad= (\eta\cdot\eta)^2(1+\widetilde{r}) - 2i(\eta\cdot\eta)(\eta\cdot\nabla\widetilde{r}) - (\eta\cdot\eta)\Delta\widetilde{r} - 2i(\eta\cdot\eta)(\eta\cdot\nabla\widetilde{r})\\
&\qquad - 4\eta\cdot\nabla(\eta\cdot\nabla\widetilde{r}) + 2i(\eta\cdot\nabla(\Delta\widetilde{r})) - (\eta\cdot\eta)\Delta\widetilde{r} + 2i(\eta\cdot\nabla(\Delta\widetilde{r})) + \Delta^2\widetilde{r}\\
&\quad= \Delta^2\widetilde{r} - 4i(\eta\cdot\eta)(\eta\cdot\nabla\widetilde{r}) - 2(\eta\cdot\eta)\Delta\widetilde{r} - 4\eta\cdot\nabla(\eta\cdot\nabla\widetilde{r})\\
&\qquad + 4i(\eta\cdot\nabla(\Delta\widetilde{r})) + (\eta\cdot\eta)^2(1+\widetilde{r}).
\end{aligned}$$

Therefore, $\widetilde{r}$ satisfies

$$\begin{aligned}
0 &= e^{-i\eta\cdot x}\Delta^2(e^{i\eta\cdot x}(1+\widetilde{r})) - (\eta\cdot\eta)^2(1+\widetilde{r}) + q(1+\widetilde{r})\\
&= \Delta^2\widetilde{r} - 4i(\eta\cdot\eta)(\eta\cdot\nabla\widetilde{r}) - 2(\eta\cdot\eta)\Delta\widetilde{r} - 4\eta\cdot\nabla(\eta\cdot\nabla\widetilde{r}) + 4i(\eta\cdot\nabla(\Delta\widetilde{r})) + q(1+\widetilde{r}),
\end{aligned}$$

and hence

$$\begin{aligned}
\Delta^2\widetilde{r} - 4i(\eta\cdot\eta)(\eta\cdot\nabla\widetilde{r}) - 2(\eta\cdot\eta)\Delta\widetilde{r} - 4\eta\cdot\nabla(\eta\cdot\nabla\widetilde{r}) &+ 4i(\eta\cdot\nabla(\Delta\widetilde{r}))\\
&= -q(1+\widetilde{r}) \quad \text{in } B_R.
\end{aligned}$$

We can rewrite this as

$$\left[(\Delta + 2i\eta\cdot\nabla)^2 - 2(\eta\cdot\eta)\left(\Delta + 2i\eta\cdot\nabla\right)\right]\widetilde{r} = -q(1+\widetilde{r}) \text{ in } B_R.$$

Consider the linear differential operator

$$P \equiv (\Delta + 2i\eta \cdot \nabla)^2 - 2(\eta \cdot \eta)(\Delta + 2i\eta \cdot \nabla)$$

and the corresponding symbol

$$P(\xi^*) = \left(|\xi^*|^2 + 2(\eta \cdot \xi^*)\right)^2 + 2(\eta \cdot \eta)\left(|\xi^*|^2 + 2(\eta \cdot \xi^*)\right), \ \xi^* \in \mathbb{R}^n.$$

Recall the notation

$$\widetilde{P}(\xi^*) = \left(\sum_{|\alpha| \geq 0} |\partial_\xi^\alpha P(\xi^*)|^2\right)^{\frac{1}{2}}, \ \ \xi^* \in \mathbb{R}^n.$$

Next, we note that

$$\begin{aligned}
P(\xi^*) &= \big(|\xi^*|^2 + 2(\eta \cdot \xi^*)\big)^2 + 2(\eta \cdot \eta)\big(|\xi^*|^2 + 2(\eta \cdot \xi^*)\big) \\
&= (|\xi^*|^2 + 2(\eta \cdot \xi^*) + 2(\eta \cdot \eta))(|\xi^*|^2 + 2(\eta \cdot \xi^*)), \\
\frac{\partial P}{\partial \xi_j}(\xi^*) &= 2(|\xi^*|^2 + 2(\eta \cdot \xi^*))(2\xi_j^* + 2\eta_j) + 2(\eta \cdot \eta)(2\xi_j^* + 2\eta_j) \\
&= 4(|\xi^*|^2 + 2(\eta \cdot \xi^*) + (\eta \cdot \eta))(\xi_j^* + \eta_j), \\
\frac{\partial^2 P}{\partial \xi_j^2}(\xi^*) &= 4(2\xi_j^* + 2\eta_j)(\xi_j^* + \eta_j) + 4(|\xi^*|^2 + 2(\eta \cdot \xi^*) + (\eta \cdot \eta))(1) \\
&= 8(\xi_j^* + \eta_j)^2 + 4(|\xi^*|^2 + 2(\eta \cdot \xi^*) + (\eta \cdot \eta)), \\
\frac{\partial^3 P}{\partial \xi_j^3}(\xi^*) &= 16(\xi_j^* + \eta_j)(1) + 4(2\xi_j^* + 2\eta_j) = 24(\xi_j^* + \eta_j).
\end{aligned}$$

Using these, we see that

$$\begin{aligned}
\left(\widetilde{P}(\xi^*)\right)^2 &= \sum_{|\alpha| \geq 0} |\partial_\xi^\alpha P(\xi^*)|^2 \geq \sum_{j=1}^n \left|\frac{\partial^3 P}{\partial \xi_j^3}(\xi^*)\right|^2 = 24 \sum_{j=1}^n |\xi_j^* + \eta_j|^2 \\
&= 24 \sum_{j=1}^n \left((\xi_j^* + \mathrm{Re}\,(\eta_j))^2 + (\mathrm{Im}\,(\eta_j))^2\right) \\
&\geq 24 \sum_{j=1}^n (\mathrm{Im}\,(\eta_j))^2 \geq 4|\mathrm{Im}\,(\eta)|^2,
\end{aligned}$$

and hence

$$\widetilde{P}(\xi^*) \geq 2|\mathrm{Im}\,(\eta)|.$$

Chapter 7

Inverse polyharmonic problem with attenuation

In this chapter, based on the work [11], we study high-frequency stability estimates for the determination of the zeroth order perturbation of the polyharmonic operator with constant attenuation from the partial Dirichlet-to-Neumann map in domains satisfying the flatness condition $(\mathcal{A})$. Our results extend the results obtained in the previous chapter (see also [14]) for the Schrödinger equation and the biharmonic operator to the polyharmonic case.

7.1 Introduction

Let us consider the following boundary value problem for the polyharmonic operator posed in a bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 3:$

$$\begin{cases} (-\Delta)^m u \ - \ (k^2 - ikb)^m u \ + \ qu &= 0 \quad \text{in } \Omega, \\ \qquad\qquad (u, \Delta u, ..., \Delta^{m-1} u) &= (f_1, f_2, ..., f_m) \ \text{ on } \partial\Omega. \end{cases} \tag{7.1.1}$$

Here m is a positive integer greater than or equal to 3, the attenuation b is assumed to be a positive constant. The potential q is assumed to be real-valued and belongs to the space $L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$. The constant k denotes the frequency and is assumed to be greater than or equal to 1. The Navier boundary data $(f_1, f_2, ..., f_m)$ is assumed to be in the space

$$H^{2m-\frac{1}{2}}(\partial\Omega) \times H^{2m-\frac{5}{2}}(\partial\Omega) \times \cdots \times H^{\frac{3}{2}}(\partial\Omega) =: \prod_{j=1}^{m} H^{2j-\frac{1}{2}}(\partial\Omega).$$

We assume that the domain Ω satisfies the flatness condition $(\mathcal{A})$. The supports of the boundary data f_j $(1 \leq j \leq m)$ are assumed to be contained in the open subset $\Gamma := \partial\Omega \setminus \Gamma_0$

of the boundary and the boundary measurements $\partial_\nu(\Delta^j u),\ 0 \le j \le m-1$ are available on Γ only. The flat part Γ_0 is, therefore, assumed to be inaccessible in this setup.

In this chapter, we address the question of stability of the recovery of the potential (zeroth order perturbation) q from the knowledge of the boundary data and boundary measurements (encoded in the partial Dirichlet-to-Neumann map) on Γ. We study the dependence of the stability estimate on the frequency k. Following the works [13,14,33], we only assume that the potential $q \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$. To the best of our knowledge, the work [3] is the only one, prior to our work in this chapter, that discusses stability estimates for the polyharmonic case. Our strategy closely follows that in the previous chapter (see also [14]). The major difficulty in this direction was in proving the existence of the complex geometric optics (CGO) type solutions for the polyharmonic case.

The plan of the chapter is as follows. In Section 7.2, we discuss our main result on the stability estimate. In Section 7.3, we discuss the construction of complex geometric optics (CGO) type solutions and the derivation of the stability estimate. In the appendix, we discuss some auxiliary results related to the construction of the CGO type solutions.

7.2 Statement of the main result

As in the previous chapters, we work with the Sobolev spaces $\widetilde{H}^s(\Gamma)$ and $H^s(\Gamma)$.

Owing to the fact that the imaginary part of $(k^2 - ikb)^m$ is non-zero (and hence, not a part of the spectrum of $(-\Delta)^m + q$), it follows that there exists a unique solution to (7.1.1) when the boundary condition $(f_1, f_2, ..., f_m) \in \prod_{j=1}^m \widetilde{H}^{2j-\frac{1}{2}}(\Gamma)$.

The corresponding partial Dirichlet-to-Neumann (D-N) map is defined as

$$\mathcal{N}_q^{\mathbb{P}} : \prod_{j=1}^{m} \widetilde{H}^{2j-\frac{1}{2}}(\Gamma) \to \prod_{j=0}^{m-1} H^{2j+\frac{1}{2}}(\Gamma) \ \text{ such that} \qquad (7.2.1)$$

$$(f_1, f_2, ..., f_m) \mapsto \left(\partial_\nu u\Big|_\Gamma, \partial_\nu(\Delta u)\Big|_\Gamma, ..., \partial_\nu(\Delta^{m-1} u)\Big|_\Gamma\right).$$

For the partial D-N map $\mathcal{N}_q^{\mathbb{P}}$, we consider the operator norm defined by

$$\|\mathcal{N}_q^{\mathbb{P}}\|_* := \sup\{\|\mathcal{N}_q^{\mathbb{P}}(f_1, f_2, ..., f_m)\|_{H^{2m-\frac{3}{2},...,\frac{1}{2}}(\Gamma)} \ : \ \|(f_1, f_2, ..., f_m)\|_{\widetilde{H}^{2m-\frac{1}{2},...,\frac{3}{2}}(\Gamma)} = 1\}.$$

The following is our stability result for the recovery of the potential q from the partial D-N map $\mathcal{N}_q^{\mathbb{P}}$.

Theorem 7.2.1. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$ satisfying the flatness condition $(\mathcal{A})$ described earlier and $\Omega \subset\subset B(0,R)$ for some $R > 1$. Let b be a positive constant and suppose that the frequency $k \geq 1$. Further, suppose that the potentials $q_1, q_2 \in L^\infty(\Omega) \cap H^s(\Omega),\ 0 < s < \frac{1}{2}$ satisfy*

$$\|q_j\|_{L^\infty(\Omega)} + \|q_j\|_{H^s(\Omega)} \leq M,\ j = 1, 2,$$

for some $M > 0$. Then there exists a constant $C > 0$, depending only on n, s, Ω and M such that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq Ce^{2(m+1)Rb} \left[e^{CRk} \|\mathcal{N}_{q_1}^{\mathbb{P}} - \mathcal{N}_{q_2}^{\mathbb{P}}\|_* + \frac{1}{(k^2 + \left(\frac{E}{5R}\right)^2)^\sigma} \right]^{\frac{1}{2}}, \tag{7.2.2}$$

where $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)} < \frac{1}{2}$ and $E := |\log \|\mathcal{N}_{q_1}^{\mathbb{P}} - \mathcal{N}_{q_2}^{\mathbb{P}}\|_*|$.

Remark 7.2.2. The stability estimate (7.2.2) tends to suggest an improvement in stability with growing frequency but a qualitative justification of the fact seems to be difficult due to the presence of the exponentially growing factor e^{CRk} in the Lipschitz part.

7.3 CGO type solutions and the stability estimate

In this section, we discuss a proof of our main result Theorem 7.2.1. As a first step, we construct appropriate CGO type solutions which together with the quantitative Riemann-Lebesgue lemma help us in the estimation of the Fourier transform of $q_1 - q_2$.

7.3.1 CGO type solutions

We denote $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as $x = (x', x_n)$, where $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Now, for a given $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, with $\xi' \neq 0$, we choose unit vectors α and β as in the previous chapter.

Let $a > 1$ be a parameter to be chosen later. For $\xi \in \mathbb{R}^n$ satisfying $|\xi|^2 \leq 3(k^2 + a^2)$, we denote the principal square root of $k^2 + a^2 - \frac{|\xi|^2}{4} - ikb$ by

$$X + iY := \left(k^2 + a^2 - \frac{|\xi|^2}{4} - ikb\right)^{\frac{1}{2}}.$$

The condition $|\xi|^2 \leq 3(k^2 + a^2)$ guarantees that $X > 0$. Using the vectors α, β, we define

$$\zeta_1 := \frac{\xi}{2} - X\beta + i(Y\beta + a\alpha), \quad \zeta_2 := -\frac{\xi}{2} - X\beta - i(Y\beta + a\alpha).$$

Then

$$\zeta_2 - \overline{\zeta_1} = -\xi, \quad \zeta_1 \cdot \zeta_1 = k^2 + ikb, \; \zeta_2 \cdot \zeta_2 = k^2 - ikb, \tag{7.3.1}$$

$$|\zeta_j|^2 = \frac{|\xi|^2}{4} + X^2 + Y^2 + a^2, \quad j = 1, 2.$$

Note that

$$X^2 + Y^2 = |X + iY|^2 = \left(\left(k^2 + a^2 - \frac{|\xi|^2}{4}\right)^2 + k^2 b^2\right)^{\frac{1}{2}}, \tag{7.3.2}$$

$|\mathrm{Im}\,(\zeta_j)|^2 = Y^2 + a^2,\ j = 1, 2$, and $|Y| \leq b$ (see [14], [31]). Therefore

$$|\mathrm{Im}\,(\zeta_j)|^2 \leq b^2 + a^2 \leq (b + a)^2 \quad \text{for } j = 1, 2. \tag{7.3.3}$$

Next, we extend the potentials $q_j,\ j = 1, 2$ from Ω to $\mathbb{R}^n$ by using a suitable reflection across Γ_0 as follows.

Let $\widetilde{\Omega} := \Omega \cup \Omega^*$, where $\Omega^* := \{(x', x_n) \in \mathbb{R}^n : (x', -x_n) \in \Omega\}$ is the reflection of Ω by $\{x_n = 0\}$ and let $q_{z,j}$ denote the extension of q_j to $\mathbb{R}^n$ by zero. Since $s < \frac{1}{2}$ (see [1]), the zero extension $q_{z,j} \in H^s(\mathbb{R}^n)$.

Next, let us denote the reflection of $q_{z,j}$ by the plane $x_n = 0$ by $q_{re,j}$. In other words,

$$q_{re,j}(x', x_n) = q_{z,j}(x', -x_n) \quad \text{for } (x', x_n) \in \mathbb{R}^n.$$

Then $q_{re,j} \in H^s(\mathbb{R}^n)$ and

$$\|q_{z,j}\|_{H^s(\mathbb{R}^n)} \le C\|q_j\|_{H^s(\Omega)} \le CM, \quad \|q_{re,j}\|_{H^s(\mathbb{R}^n)} \le C\|q_{z,j}\|_{H^s(\mathbb{R}^n)} \le CM,$$

$$\|q_{z,j}\|_{L^\infty(\mathbb{R}^n)} = \|q_j\|_{L^\infty(\Omega)} \le M, \quad \|q_{re,j}\|_{L^\infty(\mathbb{R}^n)} = \|q_{z,j}\|_{L^\infty(\mathbb{R}^n)} \le M.$$

The constant C above appears due to the boundedness of the reflection and the extension operators (see [1]). In what follows, C will denote a generic constant.

Let $\widetilde{q}_{e,j} := q_{z,j} + q_{re,j}$. Then, we have $\widetilde{q}_{e,j} \in H^s(\mathbb{R}^n)$ and

$$\|\widetilde{q}_{e,j}\|_{H^s(\mathbb{R}^n)} \le \|q_{z,j}\|_{H^s(\mathbb{R}^n)} + \|q_{re,j}\|_{H^s(\mathbb{R}^n)} \le CM,$$

$$\|\widetilde{q}_{e,j}\|_{L^\infty(\mathbb{R}^n)} \le \|q_{z,j}\|_{L^\infty(\mathbb{R}^n)} + \|q_{re,j}\|_{L^\infty(\mathbb{R}^n)} \le CM.$$

Let $q_{e,j}$ denote the restriction of $\widetilde{q}_{e,j}$ to the domain $\widetilde{\Omega}$. Then, for $j = 1, 2$, $q_{e,j} \in H^s(\widetilde{\Omega})$ and satisfy

$$q_{e,j}(x) := \begin{cases} q_j(x', x_n) & \text{if } (x', x_n) \in \Omega, \\ q_j(x', -x_n) & \text{if } (x', x_n) \in \Omega^*. \end{cases}$$

Next, we choose $R > 1$ such that $\widetilde{\Omega} \subset\subset B_R := B(0, R)$. Then the restriction $\widetilde{q}_{e,j}|_{B_R} \in H^s(B_R)$. Also

$$\begin{aligned} &\|q_{e,j}\|_{L^\infty(\widetilde{\Omega})} \le M, \ \|\widetilde{q}_{e,j}\|_{L^\infty(B_R)} \le \|\widetilde{q}_{e,j}\|_{L^\infty(\mathbb{R}^n)} \le CM, \\ &\|q_{e,j}\|_{H^s(\widetilde{\Omega})} \le CM, \ \|\widetilde{q}_{e,j}\|_{H^s(B_R)} \le CM. \end{aligned} \tag{7.3.4}$$

Denoting $q := q_1 - q_2$, $q_z := q_{z,1} - q_{z,2}$, $\widetilde{q}_e = \widetilde{q}_{e,1} - \widetilde{q}_{e,2}$ and $q_e := q_{e,1} - q_{e,2}$, we see that

$$\begin{aligned} &\|q\|_{L^\infty(\Omega)} \le CM, \ \|q_z\|_{L^\infty(\mathbb{R}^n)} \le CM, \ \|q_e\|_{L^\infty(\widetilde{\Omega})} \le CM, \ \|\widetilde{q}_e\|_{L^\infty(B_R)} \le CM, \\ &\|q\|_{H^s(\Omega)} \le CM, \ \|q_z\|_{H^s(\mathbb{R}^n)} \le CM, \ \|q_e\|_{H^s(\widetilde{\Omega})} \le CM, \ \|\widetilde{q}_e\|_{H^s(B_R)} \le CM. \end{aligned} \tag{7.3.5}$$

Next, let us consider the equations

$$(-\Delta)^m \widetilde{u}_2 - (k^2 - ikb)^m \widetilde{u}_2 + \widetilde{q}_{e,2} \widetilde{u}_2 = 0 \quad \text{in } B_R, \tag{7.3.6}$$

and

$$(-\Delta)^m \widetilde{v} - (k^2 + ikb)^m \widetilde{v} + \widetilde{q}_{e,1} \widetilde{v} = 0 \quad \text{in} \ B_R. \tag{7.3.7}$$

We observe (see Appendix 7.4) that the CGO solutions

$$\widetilde{u}_2(x) = e^{i\zeta_2 \cdot x}(1 + \widetilde{r}_2(x)) \quad \text{and} \quad \widetilde{v}(x) = e^{i\zeta_1 \cdot x}(1 + \widetilde{r}_1(x))$$

are distributional solutions to (7.3.6) and (7.3.7) respectively if and only if the remainder term $\widetilde{r}_j,\ j = 1, 2$, satisfy

$$\left[\sum_{l=0}^{m-1} (-1)^{m-l}\ {}^m C_l (\zeta_j \cdot \zeta_j)^l (\Delta + 2i\zeta_j \cdot \nabla)^{m-l} \right] \widetilde{r}_j = -\widetilde{q}_{e,j}(1 + \tilde{r}_j). \tag{7.3.8}$$

The corresponding linear differential operators

$$P_j \equiv \sum_{l=0}^{m-1} (-1)^{m-l}\ {}^m C_l (\zeta_j \cdot \zeta_j)^l (\Delta + 2i\zeta_j \cdot \nabla)^{m-l},\ j = 1, 2,$$

with the symbols

$$P_j(\xi^*) = \sum_{l=0}^{m-1} {}^m C_l (\zeta_j \cdot \zeta_j)^l (|\xi^*|^2 + 2\,\zeta_j \cdot \xi^*)^{m-l},\ \xi^* \in \mathbb{R}^n,$$

satisfy the estimates

$$\widetilde{P}_j(\xi^*) \geq 2|\text{Im}\,(\eta_j)| \geq 2a > 1. \tag{7.3.9}$$

By Theorem 2.3.2, there exists a bounded linear operator E_j on $L^2(B_R)$ such that

$$P_j E_j f = f, \quad \forall f \in L^2(B_R). \tag{7.3.10}$$

Additionally, for any linear partial differential operator Q with constant coefficients, we have

$$\|Q E_j g\|_{L^2(B_R)} \leq C_0 \sup_{\xi^* \in \mathbb{R}^n} \left| \frac{\widetilde{Q}(\xi^*)}{\widetilde{P}_j(\xi^*)} \right| \|g\|_{L^2(B_R)},\ \forall g \in L^2(B_R). \tag{7.3.11}$$

The positive constant C_0 above depends only on n, B_R and the order of P_j (which is $2m$ here).

The existence of CGO solutions to (7.3.6) and (7.3.7) follows, as discussed below, as an application of the Browder fixed-point theorem (Theorem 2.3.1).

Lemma 7.3.1. *Let $a > 1$ be such that*

$$C_0 M \leq a. \tag{7.3.12}$$

Then, the functions

$$\widetilde{u}_2(x) = e^{i\eta_2 \cdot x}(1 + \widetilde{r}_2(x)) \quad \textit{and} \quad \widetilde{v}(x) = e^{i\eta_1 \cdot x}(1 + \widetilde{r}_1(x))$$

are distributional solutions to (7.3.6) *and* (7.3.7) *respectively with*

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq \frac{C_0 M |B_R|^{\frac{1}{2}}}{a}, \quad j = 1, 2. \tag{7.3.13}$$

□

The proof of the lemma closely follows the argument in the previous chapter and we therefore omit the details. From (7.3.12) and (7.3.13), we see that $\widetilde{r}_j$ additionally satisfies the estimate

$$\|\widetilde{r}_j\|_{L^2(B_R)} \leq |B_R|^{\frac{1}{2}}.$$

Now, the functions $\widetilde{u}_2$ and $\widetilde{v}$ satisfy the PDEs

$$P\widetilde{u}_2 = f_1 \quad \text{and} \quad P\widetilde{v} = f_2 \ \text{ in } B_R,$$

where

$$P = (-\Delta)^m \ \text{ and } \ f_1 = (k^2 - ikb)^m \widetilde{u}_2 - \widetilde{q}_{e,2}\widetilde{u}_2, \ f_2 = (k^2 + ikb)^m \widetilde{v} - \widetilde{q}_{e,1}\widetilde{v}.$$

From the interior regularity of P (see Theorem 2.3.3) in B_R, we infer that

$$\|\widetilde{u}_2\|_{H^{2m}(\widetilde{\Omega})} \leq C\|f_1\|_{L^2(V)} + C\|\widetilde{u}_2\|_{L^2(V)},$$
$$\|\widetilde{v}\|_{H^{2m}(\widetilde{\Omega})} \leq C\|f_2\|_{L^2(V)} + C\|\widetilde{v}\|_{L^2(V)},$$

for any V such that $\widetilde{\Omega} \subset\subset V \subset\subset B_R$, and hence

$$\|\widetilde{u}_2\|_{H^{2m}(\widetilde{\Omega})} \leq C\|f_1\|_{L^2(B_R)} + C\|\widetilde{u}_2\|_{L^2(B_R)}, \quad \|\widetilde{v}\|_{H^{2m}(\widetilde{\Omega})} \leq C\|f_2\|_{L^2(B_R)} + C\|\widetilde{v}\|_{L^2(B_R)}.$$

The L^2 norms of f_1 and f_2 satisfy the estimates (see [14])

$$\|f_1\|_{L^2(B_R)} \le C(k^2+kb)^m \|\widetilde{u}_2\|_{L^2(B_R)},$$
$$\|f_2\|_{L^2(B_R)} \le C(k^2+kb)^m \|\widetilde{v}\|_{L^2(B_R)}.$$

Therefore, using the fact that $k^2+kb \ge 1$, we see that

$$\|\widetilde{u}_2\|_{H^{2m}(\widetilde{\Omega})} \le C(k^2+kb)^m \|\widetilde{u}_2\|_{L^2(B_R)}, \quad \|\widetilde{v}\|_{H^{2m}(\widetilde{\Omega})} \le C(k^2+kb)^m \|\widetilde{v}\|_{L^2(B_R)}. \tag{7.3.14}$$

The constant C above depends only on M and Ω. The restrictions r_j and r_j^* of $\widetilde{r}_j$ to Ω and Ω^*, defined by

$$r_j := \widetilde{r}_j|_\Omega \ \text{ and } \ r_j^*(x', x_n) := \widetilde{r}_j(x', -x_n), \ (x', x_n) \in \Omega,$$

satisfy the estimates

$$\begin{aligned} \|r_j\|_{L^2(\Omega)} \le \frac{C_0 M |B_R|^{\frac{1}{2}}}{a}, \quad \|r_j^*\|_{L^2(\Omega)} \le \frac{C_0 M |B_R|^{\frac{1}{2}}}{a}, \\ \|r_j\|_{L^2(\Omega)} \le |B_R|^{\frac{1}{2}}, \quad \|r_j^*\|_{L^2(\Omega)} \le |B_R|^{\frac{1}{2}}. \end{aligned} \tag{7.3.15}$$

We now define

$$\begin{aligned} u_2(x', x_n) &:= e^{i\zeta_2\cdot(x',x_n)}(1 + r_2(x', x_n)) - e^{i\zeta_2\cdot(x',-x_n)}(1 + r_2^*(x', x_n)), \\ v(x', x_n) &:= e^{i\zeta_1\cdot(x',x_n)}(1 + r_1(x', x_n)) - e^{i\zeta_1\cdot(x',-x_n)}(1 + r_1^*(x', x_n)), \ (x', x_n) \in \Omega. \end{aligned} \tag{7.3.16}$$

It is easy to see that $u_2, v \in H^{2m}(\Omega)$ satisfy

$$\begin{cases} (-\Delta)^m u_2 - (k^2 - ikb)^m u_2 + q_2 u_2 &= 0 \quad \text{in } \Omega, \\ (u_2, \Delta u_2, ..., \Delta^{m-1} u_2) &= 0 \text{ on } \Gamma_0, \end{cases}$$

and

$$\begin{cases} (-\Delta)^m v - (k^2 + ikb)^m v + q_1 v &= 0 \quad \text{in } \Omega, \\ (v, \Delta v, ..., \Delta^{m-1} v) &= 0 \quad \text{on } \Gamma_0, \end{cases} \tag{7.3.17}$$

respectively. We shall henceforth refer to them as CGO type solutions.

7.3.2 Integral identity

We next discuss an integral identity that will be used in the derivation of the stability estimate.

Lemma 7.3.2. *Let $u_2, v \in H^{2m}(\Omega)$ be solutions of* (7.3.17) *and suppose that u_1 satisfy*

$$\begin{cases} (-\Delta)^m u_1 - (k^2 - ikb)^m u_1 + q_1 u_1 &= 0 \quad \text{in } \Omega, \\ (u_1, \Delta u_1, ..., \Delta^{m-1} u_1) &= (u_2, \Delta u_2, ..., \Delta^{m-1} u_2) \text{ on } \partial\Omega. \end{cases}$$

Then, we have the integral identity

$$\int_\Omega (q_1 - q_2) u_2 \bar{v} \, dx = \sum_{j=0}^{m-1} \int_\Gamma \left[\partial_\nu((-\Delta)^j u)\right] \left[(-\Delta)^{m-1-j} \bar{v}\right] \, dS, \tag{7.3.18}$$

where $u := u_1 - u_2$ is the unique solution to the problem

$$\begin{cases} (-\Delta)^m u - (k^2 - ikb)^m u + q_1 u &= -(q_1 - q_2) u_2 \quad \text{in } \Omega, \\ (u, \Delta u, ..., \Delta^{m-1} u) &= 0 \text{ on } \partial\Omega. \end{cases} \tag{7.3.19}$$

Proof. From (7.3.19), we see that

$$-\int_\Omega (q_1 - q_2) u_2 \bar{v} \, dx = \int_\Omega (-\Delta)^m u \, \bar{v} \, dx - \int_\Omega (k^2 - ikb)^m u \, \bar{v} \, dx + \int_\Omega q_1 u \bar{v} \, dx.$$

Using Green's identity and the boundary conditions satisfied by u and v, we get

$$\begin{aligned} -\int_\Omega &(q_1 - q_2) u_2 \bar{v} \, dx \\ &= \int_\Omega (-\Delta)^{m-1} u (-\Delta \bar{v}) \, dx - \int_\Gamma \partial_\nu((-\Delta)^{m-1} u) \bar{v} \, dS \\ &\qquad - \int_\Omega (k^2 - ikb)^m u \bar{v} \, dx + \int_\Omega q_1 u \bar{v} \, dx \\ &= \int_\Omega (-\Delta)^{m-2} u (-\Delta)^2 \bar{v} \, dx - \int_\Gamma \partial_\nu((-\Delta)^{m-2} u)(-\Delta \bar{v}) \, dS \\ &\qquad - \int_\Gamma \partial_\nu((-\Delta)^{m-1} u) \bar{v} \, dS - \int_\Omega (k^2 - ikb)^m u \bar{v} \, dx + \int_\Omega q_1 u \bar{v} \, dx \end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&= \int_\Omega (-\Delta u)(-\Delta)^{m-1}\bar{v}\,dx - \int_\Omega (k^2 - ikb)^m u\bar{v}\,dx\\
&\qquad + \int_\Omega q_1 u\bar{v}\,dx - \sum_{j=1}^{m-1}\int_\Gamma [\partial_\nu((-\Delta)^j u)][(-\Delta)^{m-1-j}\bar{v}]\,dS\\
&= \int_\Omega u(-\Delta)^m\bar{v}\,dx - \int_\Omega (k^2 - ikb)^m u\bar{v}\,dx + \int_\Omega q_1 u\bar{v}\,dx\\
&\quad - \int_\Gamma \partial_\nu u(-\Delta)^{m-1}\bar{v}\,dS - \sum_{j=1}^{m-1}\int_\Gamma [\partial_\nu((-\Delta)^j u)][(-\Delta)^{m-1-j}\bar{v}]\,dS.
\end{aligned}
$$

The identity (7.3.18) follows from the identity above by using the fact that v satisfies (7.3.17). □

7.3.3 Derivation of the stability estimate.

Next, we discuss the derivation of the stability estimate. Using (7.3.16) in the left hand side of the integral identity (7.3.18), we have

$$
\begin{aligned}
\int_\Omega q u_2\bar{v}\,dx = \int_\Omega q\left(e^{i(\zeta_2-\overline{\zeta_1})\cdot x} + e^{i(\zeta_2-\overline{\zeta_1})\cdot x^*}\right)dx - \int_\Omega q\left(e^{i[\zeta_2\cdot x-\overline{\zeta_1}\cdot x^*]} + e^{i[\zeta_2\cdot x^*-\overline{\zeta_1}\cdot x]}\right)dx\\
+ \int_\Omega q\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*)dx,
\end{aligned}
\tag{7.3.20}
$$

where $x^* := (x', -x_n)$ and

$$
\begin{aligned}
\mathcal{G}(x, r_1, r_2, r_1^*, r_2^*) = e^{i(\zeta_2-\overline{\zeta_1})\cdot x}(r_2 + \overline{r_1} + r_2\overline{r_1}) + e^{i(\zeta_2-\overline{\zeta_1})\cdot x^*}(r_2^* + \overline{r_1^*} + r_2^*\overline{r_1^*})\\
- e^{i[\zeta_2\cdot x-\overline{\zeta_1}\cdot x^*]}(r_2 + \overline{r_1^*} + r_2\overline{r_1^*}) - e^{i[\zeta_2\cdot x^*-\overline{\zeta_1}\cdot x]}(r_2^* + \overline{r_1} + r_2^*\overline{r_1}).
\end{aligned}
$$

Let us denote the four terms in the right-hand side above by w_1, w_2, w_3 and w_4. It is easy to see that for $x \in \Omega$,

$$
\zeta_2\cdot x - \overline{\zeta_1}\cdot x^* = -\xi_+\cdot x, \quad \text{where } \xi_+ := \left(\xi', 2(X+iY)\frac{|\xi'|}{|\xi|}\right), \text{ and} \tag{7.3.21}
$$

$$
\zeta_2\cdot x^* - \overline{\zeta_1}\cdot x = -\xi_-\cdot x, \quad \text{where } \xi_- := \left(\xi', -2(X+iY)\frac{|\xi'|}{|\xi|}\right). \tag{7.3.22}
$$

Using (7.3.21) and (7.3.22) in (7.3.20), along with the fact that $\zeta_2 - \overline{\zeta_1} = -\xi$, we get

$$\begin{aligned}\int_\Omega q u_2 \bar{v}\, dx = \int_\Omega q \left(e^{-i\xi\cdot x} + e^{-i\xi\cdot x^*}\right) dx - \int_\Omega q \left(e^{-i\xi_+\cdot x} + e^{-i\xi_-\cdot x}\right) dx \\ + \int_\Omega q \mathcal{G}(x, r_1, r_2, r_1^*, r_2^*) dx.\end{aligned} \tag{7.3.23}$$

Note that

$$\int_\Omega q \left(e^{-i\xi\cdot x} + e^{-i\xi\cdot(x', -x_n)}\right) dx = \mathcal{F}[\widetilde{q}_e](\xi). \tag{7.3.24}$$

To estimate the terms involving ξ_+ and ξ_-, we proceed as in chapters 5 and 6. We note that the real part ξ_+^{Re} and the imaginary part ξ_+^{Im} of ξ_+ are given by

$$\xi_+^{\mathrm{Re}} = \left(\xi', 2X\frac{|\xi'|}{|\xi|}\right), \quad \xi_+^{\mathrm{Im}} = \left(0, 2Y\frac{|\xi'|}{|\xi|}\right).$$

Similarly, the real and the imaginary part of ξ_- are given by

$$\xi_-^{\mathrm{Re}} = \left(\xi', -2X\frac{|\xi'|}{|\xi|}\right), \quad \xi_-^{\mathrm{Im}} = \left(0, -2Y\frac{|\xi'|}{|\xi|}\right).$$

Also (see [14]),

$$|\mathrm{Im}\,(\xi_\pm)| \le 2b, \qquad |\xi_\pm^{\mathrm{Re}}|^2 \ge 4(k^2 + a^2)\frac{|\xi'|^2}{|\xi|^2} > 0. \tag{7.3.25}$$

Let us consider the functions $q_\pm : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$q_\pm(x, \eta) := q(x) e^{\pm 2Y \frac{|\eta'|}{|\eta|} x_n}, \; x \in \Omega, \; \eta \in \mathbb{R}^n \setminus \{0\}.$$

Proceeding as in chapter 5, we see that $q_\pm(\cdot, \eta) \in H^s(\Omega), \; \forall \eta \in \mathbb{R}^n \setminus \{0\}$ and

$$\|q_\pm(\cdot, \eta)\|_{H^s(\Omega)} \le Ce^{4Rb}, \; \forall \eta \in \mathbb{R}^n \setminus \{0\}.$$

Since $s < \frac{1}{2}$, the zero extension $(q_\pm(\cdot, \eta))_z$ of $q_\pm(\cdot, \eta)$ from Ω to $\mathbb{R}^n$ belongs to $H^s(\mathbb{R}^n)$ and

$$\|(q_\pm(\cdot, \eta))_z\|_{H^s(\mathbb{R}^n)} \le Ce^{4Rb}. \tag{7.3.26}$$

Using Lemma 2.3.4 (the quantitative Riemann-Lebesgue lemma) and (7.3.26), we see that there exists a constant $C > 0$ and for every $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that $\forall p \in \mathbb{R}^n$ and $\tau \in (0,1)$, we have

$$|\mathcal{F}\left[(q_\pm(\cdot,\eta))_z\right](p)| \leq \frac{Ce^{4Rb}}{(1+\tau|p|)^N} + Ce^{4Rb}\tau^s, \tag{7.3.27}$$

where the constant C depends only on N, s, Ω and M. Note that

$$\int_\Omega q(x)e^{-i\xi_+\cdot x}\,dx = \int_\Omega q(x)e^{2Y\frac{|\xi'|}{|\xi|}x_n}e^{-i\xi_+^{\text{Re}}\cdot x}\,dx = \mathcal{F}\left[(q_+(\cdot,\xi))_z\right](\xi_+^{\text{Re}}),$$
$$\int_\Omega q(x)e^{-i\xi_-\cdot x}\,dx = \int_\Omega q(x)e^{-2Y\frac{|\xi'|}{|\xi|}x_n}e^{-i\xi_-^{\text{Re}}\cdot x}\,dx = \mathcal{F}\left[(q_-(\cdot,\xi))_z\right](\xi_-^{\text{Re}}).$$

Therefore, from (7.3.25) and (7.3.27), we have

$$|\mathcal{F}\left[(q_\pm(\cdot,\xi))_z\right](\xi_\pm^{\text{Re}})| \leq \frac{Ce^{4Rb}}{\left(1+2\tau(k^2+a^2)^{\frac{1}{2}}\frac{|\xi'|}{|\xi|}\right)^N} + Ce^{4Rb}\tau^s, \tag{7.3.28}$$

where the constant C depends only on N, s, Ω and M. The last term in (7.3.23) can be estimated (following [14]) using (7.3.15) and (7.3.25) to obtain

$$\left|\int_\Omega q\mathcal{G}(x,r_1,r_2,r_1^*,r_2^*)dx\right| \leq \frac{Ce^{2Rb}}{a}. \tag{7.3.29}$$

Additionally, let us assume that the parameter a is chosen so that $a > k$. Then using the fact that $\sqrt{2}a > (k^2+a^2)^{\frac{1}{2}}$, from (7.3.29), we have

$$\left|\int_\Omega q\mathcal{G}(x,r_1,r_2,r_1^*,r_2^*)dx\right| \leq C\frac{e^{2Rb}}{(k^2+a^2)^{\frac{1}{2}}}.$$

To estimate the first term in the identity (7.3.23), we proceed as follows. Using Cauchy-Schwarz inequality, from (7.3.18), we observe that

$$\begin{aligned}\left|\int_\Omega qu_2\bar{v}\,dx\right| &\leq \sum_{j=0}^{m-1}\|\partial_\nu(\Delta^j u)\|_{L^2(\Gamma)}\|\Delta^{m-1-j}v\|_{L^2(\Gamma)}\\ &\leq \left(\sum_{j=0}^{m-1}\|\partial_\nu(\Delta^j u)\|_{L^2(\Gamma)}\right)\left(\sum_{j=0}^{m-1}\|\Delta^{m-1-j}v\|_{L^2(\partial\Omega)}\right)\\ &\leq \left(\sum_{j=0}^{m-1}\|\partial_\nu(\Delta^{m-1-j}u)\|_{H^{2j+\frac{1}{2}}(\Gamma)}\right)\left(\sum_{j=0}^{m-1}\|\Delta^j v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right).\end{aligned}$$

From this, using the trace theorem and the definition of the partial D-N map, we have the estimate

$$\left|\int_\Omega q u_2 \bar{v}\, dx\right| \le C\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* \|u_2\|_{H^{2m}(\Omega)} \|v\|_{H^{2m}(\Omega)}. \tag{7.3.30}$$

The H^{2m} norms of u_2 and v can be estimated from (7.3.14) by noting the following bounds for $\widetilde{u}_2$ and $\widetilde{v}$:

$$\|\widetilde{u}_2\|^2_{L^2(B_R)} \le \int_{B_R} e^{2|\mathrm{Im}\,(\zeta_2)||x|} |1 + \widetilde{r}_2(x)|^2 dx \le Ce^{2(b+a)R},$$
$$\|v\|^2_{L^2(B_R)} \le Ce^{2(b+a)R} \text{ (follows similarly as that in the case of } \widetilde{u}_2).$$

Using these, we obtain

$$\|u_2\|_{H^{2m}(\Omega)} \le C(k^2 + kb)^m e^{(b+a)R} \quad \text{and} \quad \|v\|_{H^{2m}(\Omega)} \le C(k^2 + kb)^m e^{(b+a)R}.$$

Therefore, from (7.3.30), we see that

$$\left|\int_\Omega q u_2 \bar{v}\, dx\right| \le C(k^2 + kb)^{2m} \|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* e^{2(b+a)R}. \tag{7.3.31}$$

Using (7.3.24), (7.3.28) and (7.3.31) in (7.3.23), we have

$$\begin{aligned} |\mathcal{F}[\widetilde{q}_e](\xi)| \le C\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* (k^2 + kb)^{2m} e^{2(b+a)R} &+ \frac{Ce^{4Rb}}{\left(1 + 2\tau(k^2 + a^2)^{\frac{1}{2}} \frac{|\xi'|}{|\xi|}\right)^N} \\ &+ Ce^{4Rb}\tau^s + \frac{Ce^{2Rb}}{(k^2 + a^2)^{\frac{1}{2}}}, \end{aligned} \tag{7.3.32}$$

for $0 < |\xi|^2 \le 3(k^2 + a^2)$ with $|\xi'| > 0$, $s \in (0, \frac{1}{2})$, $\tau \in (0, 1)$ and $N \in \mathbb{N}$.

We can now estimate the H^{-1} norm of $\widetilde{q}_e$ by proceeding in the following manner. For $\rho > 1$ (which we will choose later), we consider the subset $\mathcal{K}_\rho$ of $\mathbb{R}^n$ defined by

$$\mathcal{K}_\rho := \{(\xi', \xi_n) \in \mathbb{R}^n \ : \ 0 < |\xi'| < \rho \text{ and } |\xi_n| < \rho\}.$$

Later, we shall choose ρ so that $\xi \in \mathcal{K}_\rho$ implies that $|\xi|^2 \le 3(k^2 + a^2)$. We note that if $\xi \in \mathcal{K}_\rho$, then (since $|\xi| < 2\rho$) from (7.3.32), we have

$$\begin{aligned} |\mathcal{F}[\widetilde{q}_e](\xi)| \le C\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* (k^2 + kb)^{2m} e^{2(b+a)R} &+ \frac{Ce^{4Rb}}{\left(1 + \frac{\tau}{\rho}(k^2 + a^2)^{\frac{1}{2}} |\xi'|\right)^N} \\ &+ Ce^{4Rb}\tau^s + \frac{Ce^{2Rb}}{(k^2 + a^2)^{\frac{1}{2}}}. \end{aligned} \tag{7.3.33}$$

Using Parseval's identity, we see that

$$\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi + \int_{\mathbb{R}^n\setminus\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \le \int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi + \frac{C}{\rho^2}. \tag{7.3.34}$$

To estimate the integral over K_ρ in (7.3.34), we use (7.3.33) and the fact that $\int_{\mathcal{K}_\rho} d\xi \le C\rho^n$ to obtain

$$\int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \le \int_{\mathcal{K}_\rho} |\mathcal{F}[\widetilde{q}_e](\xi)|^2\, d\xi \le C\|\mathcal{N}^{\mathbb{P}}_{q1} - \mathcal{N}^{\mathbb{P}}_{q2}\|^2_*(k^2+kb)^{4m} e^{4(b+a)R}\rho^n$$
$$+ Ce^{8Rb}\tau^{2s}\rho^n + \frac{Ce^{4Rb}}{(k^2+a^2)}\rho^n + Ce^{8Rb}\int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{2N}}.$$

Choosing $N > n-1 > 1$, we see that (see [14])

$$\int_{\mathcal{K}_\rho} \frac{d\xi}{\left(1+\frac{\tau}{\rho}(k^2+a^2)^{\frac{1}{2}}|\xi'|\right)^{2N}} \le \frac{C\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}}.$$

Therefore

$$\int_{\mathcal{K}_\rho} \frac{|\mathcal{F}[\widetilde{q}_e](\xi)|^2}{1+|\xi|^2} d\xi \le C\|\mathcal{N}^{\mathbb{P}}_{q1} - \mathcal{N}^{\mathbb{P}}_{q2}\|^2_*(k^2+kb)^{4m} e^{4(b+a)R}\rho^n + Ce^{8Rb}\tau^{2s}\rho^n$$
$$+ \frac{Ce^{4Rb}}{(k^2+a^2)}\rho^n + \frac{Ce^{8Rb}\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}},$$

and hence, using the fact that $e^{8Rb} \ge 1$, we have

$$\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \le C\|\mathcal{N}^{\mathbb{P}}_{q1} - \mathcal{N}^{\mathbb{P}}_{q2}\|^2_*(k^2+kb)^{4m} e^{4(b+a)R}\rho^n + Ce^{8Rb}\tau^{2s}\rho^n$$
$$+ \frac{Ce^{8Rb}\rho^n}{(k^2+a^2)} + \frac{Ce^{8Rb}\rho^n}{\tau^{n-1}(k^2+a^2)^{\frac{n-1}{2}}} + \frac{Ce^{8Rb}}{\rho^2}. \tag{7.3.35}$$

Next, we choose τ such that $\tau^2 = (k^2+a^2)^{-\frac{n-1}{2s+n-1}}$ and $\rho = (k^2+a^2)^{\frac{s(n-1)}{(n+2)(2s+n-1)}}$. Then, proceeding as in [14], from (7.3.35), we see that

$$\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} \le C\|\mathcal{N}^{\mathbb{P}}_{q1} - \mathcal{N}^{\mathbb{P}}_{q2}\|^2_*(k^2+kb)^{4m} e^{4(b+a)R}(k^2+a^2)^{\frac{s(n-1)n}{(n+2)(2s+n-1)}}$$
$$+ \frac{Ce^{8Rb}}{(k^2+a^2)^{\frac{2s(n-1)}{(n+2)(2s+n-1)}}}.$$

Let $\sigma := \frac{2s(n-1)}{(n+2)(2s+n-1)}$. Then, using the fact that $\frac{s(n-1)n}{(n+2)(2s+n-1)} < \frac{1}{2}$, we obtain

$$\begin{aligned}\|\widetilde{q}_e\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|^2_*(k^2+kb)^{4m}e^{4(b+a)R}(k^2+a^2)^{\frac{1}{2}} + \frac{Ce^{8Rb}}{(k^2+a^2)^{\sigma}} \\ &\leq Ce^{(4m+4)Rb}\left[e^{(8m+1)Rk}\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|^2_* e^{5Ra} + \frac{1}{(k^2+a^2)^{\sigma}}\right].\end{aligned}$$

Finally, we choose $a = k + C_0Mk + \frac{E}{5R}$ and observe that for $\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* < \delta$, where $\delta := \frac{1}{e}$,

$$\begin{aligned}\|\widetilde{q}_e\|_{H^{-1}(\mathbb{R}^n)} &\leq Ce^{2(m+1)Rb}\Bigg[e^{(8m+1)Rk}\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|^2_* e^{5R(k+C_0Mk+\frac{E}{5R})} \\ &\qquad + \frac{1}{(k^2+\left(k+C_0Mk+\frac{E}{5R}\right)^2)^{\sigma}}\Bigg]^{\frac{1}{2}} \\ &= Ce^{2(m+1)Rb}\left[e^{(8m+1)Rk}\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|^2_* e^{5Rk(1+C_0M)}e^{E} + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}} \\ &\leq Ce^{2(m+1)Rb}\left[e^{CRk}\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\|q\|_{H^{-1}(\Omega)} \leq Ce^{2(m+1)Rb}\left[e^{CRk}\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}}.$$

The case when $\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* \geq \delta$ follows from the observation that

$$\begin{aligned}\|q\|_{H^{-1}(\Omega)} \leq C\|q\|_{L^\infty(\Omega)} &\leq \frac{CM}{\delta^{\frac{1}{2}}}\delta^{\frac{1}{2}} \leq \frac{CM}{\delta^{\frac{1}{2}}}\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|^{\frac{1}{2}}_* \\ &\leq Ce^{2(m+1)Rb}\left[e^{CRk}\|\mathcal{N}^{\mathbb{P}}_{q_1} - \mathcal{N}^{\mathbb{P}}_{q_2}\|_* + \frac{1}{(k^2+\left(\frac{E}{5R}\right)^2)^{\sigma}}\right]^{\frac{1}{2}},\end{aligned}$$

where the constant C depends only on n, s, Ω and M.

7.4 Appendix

In this appendix, we provide derivation of the equation (7.3.8) satisfied by the remainder term r of a CGO solution of the form $u(x) = e^{i\zeta\cdot x}(1 + r(x))$ and also discuss a proof of the estimate (7.3.9). To begin with, we note that

$$\begin{aligned}-\Delta[e^{i\zeta\cdot x}(1+r)] &= -\Delta(e^{i\zeta\cdot x})\,(1+r) - 2\nabla(e^{i\zeta\cdot x})\cdot\nabla(1+r) - e^{i\zeta\cdot x}\Delta(1+r)\\ &= [-(\Delta+2i\zeta\cdot\nabla)r + (\zeta\cdot\zeta)(1+r)]e^{i\zeta\cdot x}.\end{aligned} \tag{7.4.1}$$

$$\begin{aligned}(-\Delta)^2[e^{i\zeta\cdot x}(1+r)] &= -\Delta\left([-(\Delta+2i\zeta\cdot\nabla)r + (\zeta\cdot\zeta)(1+r)]e^{i\zeta\cdot x}\right)\\
&= \Delta[(\Delta+2i\zeta\cdot\nabla)r]e^{i\zeta\cdot x} + 2\nabla(e^{i\zeta\cdot x})\cdot\nabla[(\Delta+2i\zeta\cdot\nabla)r]\\
&\quad + \Delta(e^{i\zeta\cdot x})[(\Delta+2i\zeta\cdot\nabla)r] + (\zeta\cdot\zeta)\left(-\Delta[e^{i\zeta\cdot x}(1+r)]\right)\\
&= [\big(\Delta^2 + 2i\zeta\cdot\nabla(\Delta) + 2(i\zeta)\cdot\nabla(\Delta) + 2(i\zeta)\cdot\nabla(2i\zeta\cdot\nabla)\\
&\qquad - (\zeta\cdot\zeta)(\Delta+2i\zeta\cdot\nabla)\big)r]e^{i\zeta\cdot x}\\
&\qquad + (\zeta\cdot\zeta)[-(\Delta+2i\zeta\cdot\nabla)r + (\zeta\cdot\zeta)(1+r)]e^{i\zeta\cdot x}\\
&= [\big(\Delta^2 + 4i\zeta\cdot\nabla(\Delta) - 4(\zeta\cdot\nabla)^2\big)r]e^{i\zeta\cdot x} - (\zeta\cdot\zeta)[2(\Delta+2i\zeta\cdot\nabla)r]e^{i\zeta\cdot x}\\
&\qquad + (\zeta\cdot\zeta)^2(1+r)e^{i\zeta\cdot x}\\
&= [\big((\Delta+2i\zeta\cdot\nabla)^2 - 2(\zeta\cdot\zeta)(\Delta+2i\zeta\cdot\nabla)\big)r + (\zeta\cdot\zeta)^2(1+r)]e^{i\zeta\cdot x}\\
&= \left[\sum_{l=0}^{1}(-1)^{2-l}\,{}^2C_l(\zeta\cdot\zeta)^l(\Delta+2i\zeta\cdot\nabla)^{2-l}r\right]e^{i\zeta\cdot x} + (\zeta\cdot\zeta)^2(1+r)e^{i\zeta\cdot x}.\end{aligned} \tag{7.4.2}$$

In general, we have

$$\begin{aligned}(-\Delta)^m[e^{i\zeta\cdot x}(1+r)] &= \left[\sum_{l=0}^{m-1}(-1)^{m-l}\,{}^mC_l(\zeta\cdot\zeta)^l(\Delta+2i\zeta\cdot\nabla)^{m-l}r\right]e^{i\zeta\cdot x}\\ &\qquad + (\zeta\cdot\zeta)^m(1+r)e^{i\zeta\cdot x}.\end{aligned} \tag{7.4.3}$$

To see this, we use mathematical induction as follows. For $m=1$ and $m=2$, that the above statement is true follows from (7.4.1) and (7.4.2).

Now, assume that the statement (7.4.3) is true for $m \le n$. Then, for $m = n$, we have

$$\begin{aligned}(-\Delta)^n[e^{i\zeta\cdot x}(1+r)] &= \left[\sum_{l=0}^{n-1}(-1)^{n-l}\,{}^nC_l(\zeta\cdot\zeta)^l(\Delta+2i\zeta\cdot\nabla)^{n-l}r\right]e^{i\zeta\cdot x}\\ &\qquad + (\zeta\cdot\zeta)^n(1+r)e^{i\zeta\cdot x}.\end{aligned}$$

Using this, we see that

$$
\begin{aligned}
&(-\Delta)^{n+1}[e^{i\zeta\cdot x}(1+r)] \\
&\quad = (-\Delta)\left(\left[\sum_{l=0}^{n-1}(-1)^{n-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l(\Delta+2i\zeta\cdot\nabla)^{n-l}r\right]e^{i\zeta\cdot x}\right) \\
&\qquad\qquad\qquad + (-\Delta)\left[(\zeta\cdot\zeta)^n(1+r)e^{i\zeta\cdot x}\right] \\
&\quad = \sum_{l=0}^{n-1}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l\,\Delta\left[((\Delta+2i\zeta\cdot\nabla)^{n-l}r)e^{i\zeta\cdot x}\right] \\
&\qquad - (\zeta\cdot\zeta)^n\left[(\Delta+2i\zeta\cdot\nabla)r-(\zeta\cdot\zeta)(1+r)\right]e^{i\zeta\cdot x} \\
&\quad = \sum_{l=0}^{n-1}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l\Big[\big(\Delta(\Delta+2i\zeta\cdot\nabla)^{n-l}+2i\zeta\cdot\nabla(\Delta+2i\zeta\cdot\nabla)^{n-l} \\
&\qquad - (\zeta\cdot\zeta)(\Delta+2i\zeta\cdot\nabla)^{n-l}\big)r\Big]e^{i\zeta\cdot x} - (\zeta\cdot\zeta)^n\left[(\Delta+2i\zeta\cdot\nabla)r-(\zeta\cdot\zeta)(1+r)\right]e^{i\zeta\cdot x} \\
&\quad = \sum_{l=0}^{n-1}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l\Big[\big([\Delta^2+4i\zeta\cdot\nabla(\Delta)-4(\zeta\cdot\nabla)^2](\Delta+2i\zeta\cdot\nabla)^{(n-1)-l} \\
&\qquad - (\zeta\cdot\zeta)(\Delta+2i\zeta\cdot\nabla)^{n-l}\big)r\Big]e^{i\zeta\cdot x} - (\zeta\cdot\zeta)^n\left[(\Delta+2i\zeta\cdot\nabla)r-(\zeta\cdot\zeta)(1+r)\right]e^{i\zeta\cdot x} \\
&\quad = \sum_{l=0}^{n-1}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l\Big[\big((\Delta+2i\zeta\cdot\nabla)^2(\Delta+2i\zeta\cdot\nabla)^{(n-1)-l}\big)r\Big]e^{i\zeta\cdot x} \\
&\qquad - \sum_{l=0}^{n-1}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l\Big[(\zeta\cdot\zeta)(\Delta+2i\zeta\cdot\nabla)^{n-l}r\Big]e^{i\zeta\cdot x} \\
&\qquad - (\zeta\cdot\zeta)^n\left[(\Delta+2i\zeta\cdot\nabla)r-(\zeta\cdot\zeta)(1+r)\right]e^{i\zeta\cdot x} \\
&\quad = \sum_{l=0}^{n-1}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l\Big[(\Delta+2i\zeta\cdot\nabla)^{(n+1)-l}r\Big]e^{i\zeta\cdot x} \\
&\qquad - \sum_{l=0}^{n-2}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^{l+1}\Big[(\Delta+2i\zeta\cdot\nabla)^{n-l}r\Big]e^{i\zeta\cdot x} \\
&\qquad - (-1)^2\,{}^{n}C_{n-1}(\zeta\cdot\zeta)^n\big((\Delta+2i\zeta\cdot\nabla)r\big)e^{i\zeta\cdot x} \\
&\qquad - (\zeta\cdot\zeta)^n\left[(\Delta+2i\zeta\cdot\nabla)r-(\zeta\cdot\zeta)(1+r)\right]e^{i\zeta\cdot x} \\
&\quad = (-1)^{n+1}\,{}^{n}C_0(\zeta\cdot\zeta)^0\Big[(\Delta+2i\zeta\cdot\nabla)^{n+1}r\Big]e^{i\zeta\cdot x} \\
&\qquad + \sum_{l=1}^{n-1}(-1)^{(n+1)-l}\,{}^{n}C_l(\zeta\cdot\zeta)^l\Big[(\Delta+2i\zeta\cdot\nabla)^{(n+1)-l}r\Big]e^{i\zeta\cdot x} \\
&\qquad - \sum_{l=1}^{n-1}(-1)^{(n+2)-l}\,{}^{n}C_{l-1}(\zeta\cdot\zeta)^l\Big[(\Delta+2i\zeta\cdot\nabla)^{(n+1)-l}r\Big]e^{i\zeta\cdot x} \\
&\qquad - \left[{}^{n}C_{n-1}+1\right](\zeta\cdot\zeta)^n\big((\Delta+2i\zeta\cdot\nabla)r\big)e^{i\zeta\cdot x} + (\zeta\cdot\zeta)^{n+1}(1+r)e^{i\zeta\cdot x}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{n+1}\, {}^{n+1}C_0 (\zeta\cdot\zeta)^0 \big[(\Delta + 2i\zeta\cdot\nabla)^{n+1} r\big] e^{i\zeta\cdot x} \\
&\qquad + \sum_{l=1}^{n-1} (-1)^{(n+1)-l}\, \big[{}^{n}C_l + {}^{n}C_{l-1}\big](\zeta\cdot\zeta)^l \big[(\Delta + 2i\zeta\cdot\nabla)^{(n+1)-l} r\big] e^{i\zeta\cdot x} \\
&\qquad\quad - \ (n+1)(\zeta\cdot\zeta)^n \big((\Delta + 2i\zeta\cdot\nabla) r\big) e^{i\zeta\cdot x} + (\zeta\cdot\zeta)^{n+1}(1+r) e^{i\zeta\cdot x} \\
&= (-1)^{n+1}\, {}^{n+1}C_0 (\zeta\cdot\zeta)^0 \big[(\Delta + 2i\zeta\cdot\nabla)^{(n+1)-0} r\big] e^{i\zeta\cdot x} \\
&\quad + \sum_{l=1}^{n-1} (-1)^{(n+1)-l}\, {}^{n+1}C_l (\zeta\cdot\zeta)^l \big[(\Delta + 2i\zeta\cdot\nabla)^{(n+1)-l} r\big] e^{i\zeta\cdot x} \\
&\quad + (-1)^{(n+1)-n}\, {}^{n+1}C_n (\zeta\cdot\zeta)^n \big[(\Delta + 2i\zeta\cdot\nabla)^{(n+1)-n} r\big] e^{i\zeta\cdot x} + (\zeta\cdot\zeta)^{n+1}(1+r) e^{i\zeta\cdot x} \\
&= \sum_{l=0}^{n} (-1)^{(n+1)-l}\, {}^{n+1}C_l (\zeta\cdot\zeta)^l \big[(\Delta + 2i\zeta\cdot\nabla)^{(n+1)-l} r\big] e^{i\zeta\cdot x} + (\zeta\cdot\zeta)^{n+1}(1+r) e^{i\zeta\cdot x}
\end{aligned}
$$

and hence, (7.4.3) holds true for $m = n+1$.

Now, using (7.4.3), we see that

$$
\begin{aligned}
0 &= e^{-i\zeta\cdot x}(-\Delta)^m [e^{i\zeta\cdot x}(1+r)] - (\zeta\cdot\zeta)^m (1+r) + q(1+r) \\
&= \left[\sum_{l=0}^{m-1} (-1)^{m-l}\, {}^{m}C_l (\zeta\cdot\zeta)^l (\Delta + 2i\zeta\cdot\nabla)^{m-l} r\right] + (\zeta\cdot\zeta)^m (1+r) \\
&\qquad\qquad - (\zeta\cdot\zeta)^m (1+r) + q(1+r) \\
&= \sum_{l=0}^{m-1} (-1)^{m-l}\, {}^{m}C_l (\zeta\cdot\zeta)^l (\Delta + 2i\zeta\cdot\nabla)^{m-l} r + q(1+r),
\end{aligned}
$$

and therefore r satisfies the equation

$$
\left[\sum_{l=0}^{m-1} (-1)^{m-l}\, {}^{m}C_l (\zeta\cdot\zeta)^l (\Delta + 2i\zeta\cdot\nabla)^{m-l}\right] r = -q(1+r),
$$

which proves (7.3.8).

Next, we consider the linear differential operator

$$
P \equiv \sum_{l=0}^{m-1} (-1)^{m-l}\, {}^{m}C_l (\zeta\cdot\zeta)^l (\Delta + 2i\zeta\cdot\nabla)^{m-l}
$$

and the corresponding symbol

$$
P(\xi^*) = \sum_{l=0}^{m-1} {}^{m}C_l \ (\zeta\cdot\zeta)^l (|\xi^*|^2 + 2\zeta\cdot\xi^*)^{m-l}, \ \xi^* \in \mathbb{R}^n.
$$

We recall that

$$\widetilde{P}(\xi^*) = \left(\sum_{|\alpha|\geq 0} |\partial_\xi^\alpha P(\xi^*)|^2 \right)^{\frac{1}{2}}, \quad \xi^* \in \mathbb{R}^n.$$

To derive the estimate (7.3.9), we will only use the derivatives of P order $2m-1$. For $\xi^* \in \mathbb{R}^n$, we see that

$$\begin{aligned}
\frac{\partial^{2m-1}P}{\partial\xi_j^{2m-1}}(\xi^*) &= \sum_{l=0}^{m-1} {}^mC_l\,(\zeta\cdot\zeta)^l \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}[(|\xi|^2+2\zeta\cdot\xi)^{m-l}](\xi^*)\\
&= {}^mC_0\,(\zeta\cdot\zeta)^0 \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}[(|\xi|^2+2\zeta\cdot\xi)^{m-0}](\xi^*)\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}[(|\xi|^2+2\zeta\cdot\xi)^{m}](\xi^*).
\end{aligned}$$

Using binomial expansion, we have

$$\begin{aligned}
\frac{\partial^{2m-1}P}{\partial\xi_j^{2m-1}}(\xi^*) &= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}[(|\xi|^2+2\zeta\cdot\xi)^{m}](\xi^*)\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[\sum_{q=0}^{m} {}^mC_q|\xi|^{2q}(2\zeta\cdot\xi)^{m-q}\right](\xi^*)\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}[{}^mC_m|\xi|^{2m}(2\zeta\cdot\xi)^0 + {}^mC_{m-1}|\xi|^{2m-2}(2\zeta\cdot\xi)^1](\xi^*)\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[|\xi|^{2m}\right](\xi^*) + 2m\frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[|\xi|^{2m-2}(\zeta\cdot\xi)\right](\xi^*).
\end{aligned}$$

Next, using the general Leibniz rule, we see that

$$\begin{aligned}
&\frac{\partial^{2m-1}P}{\partial\xi_j^{2m-1}}(\xi^*)\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[|\xi|^{2m}\right](\xi^*) + 2m\sum_{t=0}^{2m-1} {}^{2m-1}C_t \frac{\partial^{2m-1-t}}{\partial\xi_j^{2m-1-t}}\left[|\xi|^{2m-2}\right](\xi^*)\frac{\partial^t}{\partial\xi_j^t}\left[\zeta\cdot\xi\right](\xi^*)\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[|\xi|^{2m}\right](\xi^*) + 2m\ {}^{2m-1}C_0 \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[|\xi|^{2m-2}\right](\xi^*)\,(\zeta\cdot\xi^*)\\
&\quad + 2m\ {}^{2m-1}C_1 \frac{\partial^{2m-2}}{\partial\xi_j^{2m-2}}\left[|\xi|^{2m-2}\right](\xi^*)\frac{\partial}{\partial\xi_j}\left[\zeta\cdot\xi\right](\xi^*)\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[|\xi|^{2m}\right](\xi^*) + 2m\,(0)\,(\zeta\cdot\xi^*) + 2m(2m-1)\frac{\partial^{2m-2}}{\partial\xi_j^{2m-2}}\left[|\xi|^{2m-2}\right](\xi^*)\,\zeta_j\\
&= \frac{\partial^{2m-1}}{\partial\xi_j^{2m-1}}\left[|\xi|^{2m}\right](\xi^*) + 2m(2m-1)\zeta_j\frac{\partial^{2m-2}}{\partial\xi_j^{2m-2}}\left[|\xi|^{2m-2}\right](\xi^*).
\end{aligned}$$

Again, using binomial expansion, we observe that

$$\begin{aligned}
&\frac{\partial^{2m-2}}{\partial \xi_j^{2m-2}} \left[|\xi|^{2m-2}\right](\xi^*) \\
&\qquad = \frac{\partial^{2m-2}}{\partial \xi_j^{2m-2}} \left[(\xi_1^2 + \ldots + \xi_n^2)^{m-1}\right](\xi^*) \\
&\qquad = \frac{\partial^{2m-2}}{\partial \xi_j^{2m-2}} \left[\sum_{q=0}^{m-1} \left(\sum_{i=1,\ i\neq j}^{n} \xi_i^2\right)^{m-1-q} \xi_j^{2q}\right](\xi^*) \\
&\qquad = \frac{\partial^{2m-2}}{\partial \xi_j^{2m-2}} \left[\left(\sum_{i=1,\ i\neq j}^{n} \xi_i^2\right)^{0} \xi_j^{2m-2}\right](\xi^*) = \frac{\partial^{2m-2}}{\partial \xi_j^{2m-2}} \left[\xi_j^{2m-2}\right](\xi^*) \\
&\qquad = (2m-2)!\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^{2m-1} P}{\partial \xi_j^{2m-1}}(\xi^*) &= \frac{\partial^{2m-1}}{\partial \xi_j^{2m-1}} \left[|\xi|^{2m}\right](\xi^*) + 2m(2m-1)(2m-2)!\, \zeta_j \\
&= \frac{\partial^{2m-1}}{\partial \xi_j^{2m-1}} \left[|\xi|^{2m}\right](\xi^*) + (2m)!\, \zeta_j\ .
\end{aligned}$$

Using this, for $m \geq 2$, we see that

$$\begin{aligned}
\left(\widetilde{P}(\xi^*)\right)^2 &= \sum_{|\alpha|\geq 0} |\partial_\xi^\alpha P(\xi^*)|^2 \geq \sum_{j=1}^{n} \left|\frac{\partial^{2m-1} P}{\partial \xi_j^{2m-1}}(\xi^*)\right|^2 \\
&= \sum_{j=1}^{n} \left|\frac{\partial^{2m-1}}{\partial \xi_j^{2m-1}} \left[|\xi|^{2m}\right](\xi^*) + (2m)!\, \zeta_j\right|^2 \\
&= \sum_{j=1}^{n} \left[\left(\frac{\partial^{2m-1}}{\partial \xi_j^{2m-1}} \left[|\xi|^{2m}\right](\xi^*) + (2m)!\, \mathrm{Re}\,(\zeta_j)\right)^2 + ((2m)!\, \mathrm{Im}\,(\zeta_j))^2\right] \\
&\geq \sum_{j=1}^{n} ((2m)!\, \mathrm{Im}\,(\zeta_j))^2 \geq 4|\mathrm{Im}\,(\zeta)|^2,
\end{aligned}$$

whence (7.3.9) follows.

Conclusion

In this thesis, we have studied high-frequency stability estimates for the recovery of the zeroth order perturbation of the Schrödinger equation, the biharmonic operator, and the polyharmonic operator with constant attenuation, from the linearized and the nonlinear partial Dirichlet-to-Neumann (D-N) map in domains satisfying the condition $(\mathcal{A})$.

For the Schrödinger case, in both the linearized and the nonlinear problems, the stability estimates we obtained exhibit a polynomial dependence on the frequency in the Lipschitz part. In this scenario, we observe an improvement in the stability estimate from logarithmic type to Hölder type with an appropriate choice of the frequency k.

For the biharmonic and polyharmonic cases, polynomial dependence on the frequency is obtained only in the linearized problem. In the nonlinear problem, this dependence becomes exponential. The following table summarizes the resulting frequency dependence in all the cases considered here.

Stability estimate		
Operator	**linearized DN map**	**DN map**
Schrödinger	Polynomial growth in k	Polynomial growth in k
Biharmonic	Polynomial growth in k	Exponential growth in k
Polyharmonic	Polynomial growth in k	Exponential growth in k

Here, we analyzed the question of stability for the above operators in the presence of constant attenuation. It will be interesting to examine similar questions for non-constant attenuation, under similar assumptions on the domain. It will also be worthwhile to see if similar stability estimates can be derived in the case of more general partial data problems.

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