A Study on Black Holes Hair Modes and Gravitational Index of Small Black Rings

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SUBHODIP BANDYOPADHYAY

(Enrollment No. : PHYS11201804014)

National Institute of Science Education and Research,

Jatani - 752050

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Name : Dr. Yogesh Kumar Srivastava

Designation : Associate Professor

Department/ Centre : School of Physical Sciences

Name of the CI/ OCC : NISER, Bhubaneswar

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Conferences and Schools:

1. Participated in the *STRINGS 2025* conference at the New York University, Abu Dhabi in January 2025. Here I presented a poster based on my work *Supersymmetric black hole hair and* $AdS_3 \times S^3$.

2. Participated in the *National Strings Meeting 2024* at IIT Ropar in December 2024. Here I gave a talk on my work *Supersymmetric black hole hair and* $AdS_3 \times S^3$.

3. I also participated in the school *Student Talks on Trending Topics in Theory (ST4)* (July 2023) at IIT Mandi. Here, I took a mini lecture series on "Black hole microstate counting" along with Sourav Maji from IIT Bhubaneswar (currently postdoc at HRI Prayagraj).

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Contents

Thesis Summary

String theory has achieved remarkable success in calculating the index or degeneracy associated with certain classes of supersymmetric black holes [15, 16]. In some cases, exact helicity trace indices have been expressed through specific modular forms. A corresponding gravitational approach reproduces these results by evaluating the string path integral in the black hole's near-horizon geometry [18]. This prescription leads to a puzzle: the Breckenridge–Myers–Peet–Vafa (BMPV) black holes [22] in flat space and in Taub–NUT space share the same near-horizon geometry, yet they possess distinct microscopic indices. Thus, a single near-horizon geometry cannot account for the differing microscopic results. This discrepancy is now attributed to black hole hair modes—smooth, normalizable degrees of freedom outside the horizon—many of which have been explicitly identified and constructed [20, 21]. With these hair modes present, reconciling the near-horizon path integral with the exact microscopic results becomes unclear. Their inclusion, however, is essential for a complete correspondence [25, 26]. Conversely, incorporating hair modes from the near-horizon perspective is challenging. Recent advances suggest that evaluating the path integral over the full asymptotically flat spacetime offers a unified framework encompassing both near-horizon and hair contributions.

In our paper [127], which consists the first part of the thesis, we analyze the spectrum of modes in $AdS_3 \times S^3$ and identify those corresponding to known supersymmetric hair modes in full black hole geometries. A key objective is to understand these hair modes from the near-horizon perspective. Another aim is to clarify how these hair deformations relate to other types of AdS_3 deformations, such as null-warped AdS_3. We carry out a detailed analysis of the boundary conditions satisfied by the hair modes in AdS_3, showing that all of them obey non-normalizable boundary conditions. Our findings make it evident that any gravitational computation in the near-horizon region aiming to reproduce the exact microscopic results must necessarily involve non-normalizable modes. We also incorporate the hair modes of the complete black hole solutions within the framework of minimal six-dimensional supergravity coupled to a tensor multiplet. The microscopic expression for the logarithm of the supersymmetric index successfully reproduces the Bekenstein–Hawking entropy of the corresponding supersymmetric black hole in the large-charge limit. Although this agreement is striking, it still leaves several open questions concerning the precise connection between the supersymmetric index and the actual degeneracy of states. In recent years, a framework has been introduced to compute the index directly from the gravitational side using the gravitational path integral [71, 72].

Some supersymmetric elementary string states carrying angular momentum can be interpreted as small black rings within five-dimensional string theory. These black rings are characterized by having a vanishing horizon area. Through the 4D–5D correspondence, such small black rings are related to small, non-rotating black holes in four dimensions.

Recent developments have introduced saddle-point configurations in the gravitational path integral that compute the supersymmetric index for small black holes [82–84]. Building on this, we propose in [128], which is the second part of the thesis, a similar saddle solution tailored to five-dimensional small black rings. The leading contribution comes from a black ring saddle rotating in both independent angular directions with a finite-area horizon. This configuration is described by a three-center Bena–Warner solution and reformulated within the chiral null model, aiding future developments.

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Chapter 1

Introduction

1.1 Black holes in general relativity

Black holes are formed due to gravitational collapse of massive objects having mass several times that of the solar mass. The system collapses continuously due to intense gravity until an "event horizon" is formed. This is a hypothetical surface surrounding black hole which even light can't escape.

In general relativity, black holes are solutions to Einstein's equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{1.1.1}$$

where G_N and Λ are Newton's constant and cosmological constant respectively. For convenience, we work in natural units, i.e, $G_N = 1 = \hbar = k_B = c$. The Einstein equations are highly non-linear and in general, cannot be solved exactly. However they have been solved in many special cases. One such solution is the Schwarzschild metric, which represents a static, spherically symmetric, and uncharged object in vacuum, meaning $T_{\mu\nu} = 0$ and $\Lambda = 0$.

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta^d\phi^2) \tag{1.1.2}$$

In this setup, the variables t and r represent the Schwarzschild time and radial distance, and M is the mass of the gravitational source. This metric effectively describes the space-time outside any spherical body, as long as we are considering the region where $r \geq 2M$. Yet, in most physical situations, the coordinates become problematic at $r = 2M$ because this lies within the matter distribution of stars or planets, making the vacuum solution invalid at that point. For extreme cases like black holes, where all the mass is compressed to a point, the Schwarzschild Solution can still be used at $r = 2M$. In our discussion, we will treat the metric in equation (1.1.2) as specifically describing a black hole, and we will not consider other spherical configurations.

In this black hole context, the metric (1.1.2) has a time component g_{tt} that becomes zero and a radial component g_{rr} that becomes infinite at $r = 2M$. This indicates a coordinate singularity at that radius—not a real physical one, but an artifact of the coordinate system used. Alternative coordinate systems, such as Eddington-Finkelstein or Kruskal coordinates [6], allow us to analyze the interior region of a black hole without this issue.

The hypothetical surface at $r = 2M$ around a Schwarzschild black hole behaves as a one-way membrane through which information can pass only into the interior and nothing can come out classically. This surface is known as the event horizon. The metric (1.1.2) has another singularity at $r = 0$ which is a proper singularity[1] of the geometry and cannot be removed by any choice of coordinates. In general, when a star collapses under gravity, a black hole is formed. Penrose's *Cosmic Censorship Conjecture (CCC)* [2] suggests that singularities resulting from such collapse are always hidden behind an event horizon. This horizon prevents any information from escaping, making the singularity unobservable to distant observers. If no event horizon were present, the singularity would be exposed, forming a "naked singularity," which CCC proposes does not occur. We will see that the event horizon is crucial to understanding many aspects of black hole physics.

Up to this point, black holes have been treated as purely classical objects. However, when quantum effects are included, black holes are predicted to emit radiation and possess a thermal temperature. Through semi-classical analysis[2], Hawking [3] demonstrated that black holes emit radiation like any other hot body, with a temperature known as the *Hawking temperature*, given by

[1] The square of the Riemann tensor $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges at $r \to 0$.

[2] By 'semi-classical computation' we mean quantum field theory effects are considered in a curved geometry.

$$T_H = \frac{\kappa}{2\pi} \tag{1.1.3}$$

where κ represents the surface gravity at the horizon as seen by a distant observer. This surface gravity can be interpreted as the force per unit mass required to hold an object at the horizon from infinity. Besides Hawking's original quantum field-theoretic derivation, other approaches — such as Euclidean techniques (see [4]) — have also reproduced the result. In quantum field theory, using a Wick rotation to imaginary time τ, periodicity T in τ is related to inverse temperature $\beta = \frac{1}{T}$, implying thermal behavior. Based on this idea, performing a Wick rotation $\tau = it$ in the Schwarzschild metric, we obtain the form of the metric 1.1.2 near the horizon as

$$ds_E^2 \approx 16G^2M^2[(\rho^2 d\omega^2 + d\rho^2) + \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{1.1.4}$$

where $\rho = \sqrt{\frac{r}{2GM} - 1}$ and $\omega = \frac{\tau}{4M}$. In these new variables the event horizon lies at $\rho = 0$, which represents the origin of the (ρ, ω)-plane, where ω serves as the angular coordinate and ρ as the radial one. Much like polar coordinates in the plane, the entire Euclidean space can be described using (ρ, ω)-coordinates by making ω periodic with a period 2π, including the point $\rho = 0$. This leads to the conclusion that the Euclidean time τ must also be periodic with a period $P = 8\pi M$ in order to avoid any conical singularity at the center $\rho = 0$. Consequently, for the Schwarzschild black hole, the temperature associated with the event horizon is found to be

$$T_H = \frac{1}{8\pi M}. \tag{1.1.5}$$

The Schwarzschild solution represents the most basic black hole, characterized solely by its mass. However, black holes may also possess electric charge Q in addition to mass M. These configurations arise as solutions to Einstein's equations when coupled with a U(1) gauge field, described by the action

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right). \tag{1.1.6}$$

The field equations derived from this action are

$$G_{\mu\nu} = 2\left(F_{\mu\lambda}F^{\lambda}_{\nu} - \frac{1}{4}F_{\lambda\sigma}F^{\lambda\sigma}g_{\mu\nu}\right), \tag{1.1.7}$$

$$\nabla_{\mu}F^{\mu\nu} = 0, \tag{1.1.8}$$

These are inherently non-linear, reflecting that the solutions cannot be obtained via perturbative expansions around flat spacetime. The spherically symmetric Reissner–Nordström (RN) metric, which solves these equations, is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1.1.9}$$

where $A = \frac{Q}{r}dt$ and $F = dA$ denotes the Maxwell field strength sourced by the black hole. In this case, Q represents the electric charge. The time-time component g_{tt} becomes zero at $r = r_{\pm}$, where

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \tag{1.1.10}$$

So there are two horizons of RN-black hole, outer horizon at $r = r_+$ and the inner horizon at $r = r_-$. The value of mass and charge for which $Q^2 > M^2$ causes naked singularity. Hence it is not allowed according to Cosmic Censorship conjecture. This restricts the values of M and Q to satisfy $M \geq Q$. The black holes that saturate the condition $Q = M$ are called extremal RN black holes, they are stable (soliton-like), are at zero thermal temperature and they don't radiate. For $M > Q$ (non-extremal) they have finite positive temperature and can Hawking radiate. For extremal black holes the inner horizon and the outer horizon coincides and are at $r = r_+ = r_- = M$.

Since stars and planets rotate, it is natural to expect that black holes—formed from their collapse—also possess angular momentum. As a result, solutions to Einstein's equations exist that describe rotating black holes, which are characterized by both mass and angular momentum J. These black holes can be either uncharged, described by the Kerr metric [7], or charged, represented by the Kerr–Newman solution [8]. According to black hole uniqueness theorems, these represent the only stationary and asymptotically flat solutions to Einstein's equations. While non-rotating black holes are static, rotating ones are dynamic, yet still stationary and axisymmetric.

In the 1970s, a profound connection between black hole behavior and classical thermodynamics was established. Notably, the four laws of black hole mechanics, which mirror those of thermodynamics, were formulated by Bardeen, Carter, and Hawking in 1973 [9]. The entropy associated with black holes, known as Bekenstein–Hawking entropy [10], is proportional to the area of the event horizon and is given by

$$S_{\mathrm{BH}} = \frac{A}{4} \tag{1.1.11}$$

where A denotes the area of the event horizon. This formulation is consistent with the black hole area law [11], which states that "the area of a classical black hole's event horizon never decreases." This parallels the second law of thermodynamics, where entropy in a closed system also never diminishes.

From the statistical mechanical point of view, this amount of entropy requires $e^{S_{\mathrm{BH}}}$ numbers of microscopic states of a particular black hole. There are several no-hair theorems [6] according to which for classical black holes associated with usual matter content in $D = 4$ the solutions are uniquely defined in terms of their mass M, electromagnetic charge Q and angular momentum J. Thus construction of $e^{S_{\mathrm{BH}}}$ number of microstates to account for the Bekenstein-Hawking entropy (1.1.11) is not possible in the classical treatment of a black hole. Quantum effects has to be taken into account [16, 17, 19].[3]

[3]However another problem came into picture which is known as the "Universality" problem. It was not

When a black hole emits Hawking radiation, it eventually disappears, leaving only thermal radiation behind. This process transforms a pure state into a mixed one, violating unitarity—a foundational principle in quantum mechanics. Since Hawking radiation reflects only the black hole's mass and charge, it lacks any details about how the black hole originally formed. Once the black hole evaporates entirely, this information is permanently lost. This conundrum is known as the *information loss paradox*. To resolve it, we require a quantum theory of gravity.

String theory is a leading candidate for such a theory. It provides a framework in which problems like black hole microstates and information loss can be studied and, to some extent, addressed. Although this thesis does not go into detail about resolving the information loss paradox, readers interested in further exploration may refer to [29].

1.2 String Theory

This section introduces foundational concepts in string theory. We focus on fundamental strings and branes, with particular emphasis on D-branes. We'll explore how dual descriptions of D-branes lead to the development of AdS/CFT correspondence [30, 31], a crucial tool in understanding black hole microstates. Comprehensive introductions to string theory can be found in [32, 33].

String theory's basic components include higher-dimensional entities like one-dimensional strings and extended objects (branes) of two or more dimensions. Specifically, strings serve as the fundamental degrees of freedom in the perturbative regime, while p-branes arise as non-perturbative objects.[4]

A p-brane has p spatial dimensions—for example, a two-dimensional brane is called a 2-brane or a membrane. Similarly, there are 3-branes, 4-branes, and higher-dimensional analogs. A special class of p-branes, known as D-branes or Dp-branes, allows fundamental strings to have their endpoints attached to them. String vibrations correspond to particles and can pro-

clear why the semi-classical computation (It is QFT on curved spacetime where gravity is not quantized) by Bekenstein-Hawking yields the same entropy as obtained from quantum gravity (gravity is also quantized).

[4]They have masses inversely proportional to the string coupling.

duce both massive and massless states. Among the massless modes of closed strings is the graviton, which mediates the gravitational interaction. String theory is formulated in higher dimensions, and lower-dimensional effective theories can be derived by compactifying the extra dimensions on small geometric spaces.[5] More details about compactification are discussed in Section 1.5.2.

Fundamental strings may be open, with two free ends,[6] or closed, forming loops. These closed strings can have different types of boundary conditions—periodic or anti-periodic—depending on the particular version of string theory being considered.

Originally, string theory included only the bosonic degrees of freedom of strings and was therefore called *bosonic string theory*. For consistency, this theory requires 26 spacetime dimensions. Bosonic strings may be either oriented or unoriented. In this framework, closed strings are required to have periodic boundary conditions. However, all versions of bosonic string theory suffer from the presence of a tachyon in the ground state—a particle with negative mass squared—which signals vacuum instability.

To overcome this instability and to incorporate fermions (which are essential in the Standard Model), supersymmetry was introduced into string theory, leading to the development of *superstring theory*. This version of string theory includes both bosonic and fermionic degrees of freedom and avoids the problems associated with tachyons.

Superstring theory requires ten spacetime dimensions. Supersymmetric closed strings can satisfy either periodic boundary conditions (Ramond sector) or anti-periodic boundary conditions (Neveu–Schwarz sector) along their length.

There are five known consistent superstring theories:

1. **Type-I**

   This theory features $\mathscr{N} = 1$ supersymmetry in ten dimensions. It includes both open and closed strings. The fundamental strings in this theory are unoriented and can be open. In contrast, in type-II superstring theories, fundamental strings are always closed

[5] Typically, these are compact dimensions with radii smaller than the Planck length l_P.

[6] They must satisfy specific boundary conditions, requiring them to lie on particular subspaces.

and oriented.

2. **Type-II A**

   The bosonic field content of the low-energy limit of this theory includes the spacetime metric g_{MN}, the dilaton ϕ, and the NS-NS two-form field B_{MN}, which couples to the fundamental string. It also contains Ramond-Ramond (R-R) fields: the one-form $C^{(1)}$ and the three-form $C^{(3)}$. These R-R fields are associated with D-branes; specifically, $C^{(1)}$ corresponds to a D0-brane and $C^{(3)}$ to a D2-brane. Additionally, there exist electromagnetic duals of these R-R fields.

3. **Type-II B**

   In this theory, the low-energy bosonic field content includes the metric g_{MN}, dilaton ϕ, and the NS-NS two-form field B_{MN}. It also features R-R fields: a zero-form $C^{(0)}$, a two-form $C^{(2)}$, a four-form $C^{(4)}$, and a six-form $C^{(6)}$, along with their respective electromagnetic duals. The $C^{(0)}$ field is linked to a D-instanton that carries charge under the axion χ. The fields $C^{(2)}$, $C^{(4)}$, and $C^{(6)}$ correspond to D1-branes, D3-branes, and D5-branes, respectively.

4. **Heterotic** $SO(32)$

   This class of supersymmetric string theory also has $\mathcal{N} = 1$ supersymmetry in ten dimensions, like the Type-I theory. It possesses Yang-Mills gauge symmetry with the gauge group $SO(32)$.

5. **Heterotic** $E_8 \times E_8$

   In this version, the gauge symmetry is governed by the $E_8 \times E_8$ Lie group.

Among these five superstring theories, Type-I and the heterotic models have $\mathcal{N} = 1$ supersymmetry, while Type-IIA and Type-IIB possess $\mathcal{N} = 2$ supersymmetry in ten dimensions. In the case of Type-IIA, left- and right-moving spinors have opposite chirality, whereas in Type-IIB, both have the same chirality.

Despite their differences, these theories are not entirely independent. They are related through several dualities, such as S-duality and T-duality.

All five string theories are believed to be different low-energy limits of a deeper, unified framework known as *M-theory* [34]. While the complete formulation of M-theory remains unknown (there are various proposals like matrix theory, membrane theory etc.), it is considered to encompass all known string theories via duality relations.

However, it is known that at low energies, M-theory reduces to 11-dimensional supergravity. In this regime, BPS solutions include M2-branes and M5-branes, which have two and five spatial dimensions respectively. When one dimension is compactified, the 11-dimensional supergravity reduces to Type-IIA supergravity. Many Type-IIA objects can thus be understood via M-theory. For instance, D0-branes correspond to Kaluza-Klein (KK) modes along the compact 11th dimension, while D2-branes arise from M2-branes wrapping this compact direction. Likewise, D4-branes originate from M5-branes wrapping one spatial dimension, and D6-branes, being electromagnetic duals of D0-branes, correspond to KK-monopoles in 11 dimensions. The $E_8 \times E_8$ heterotic string theory can also be interpreted within the M-theory framework in eleven dimensions.

At low energies, string theory is effectively described by supergravity. In this thesis, we focus exclusively on this supergravity approximation, considering only massless fields. Specifically, we work with the bosonic part of Type-IIA and Type-IIB supergravity.

The bosonic part of Type-II supergravity includes the spacetime metric $g_{\mu\nu}$, the dilaton ϕ, a NS-NS 2-form field $B_{\mu\nu}$, and R-R fields $C^{(p+1)}$, where p varies depending on the theory. The actions in the string frame for type IIA and IIB supergravities are given by [32, 33]

$$S_{\text{IIA}} = \frac{1}{16\pi G_{10}} \left[\int d^{10}x \sqrt{-g}\, e^{-2\Phi} \left(R + 4\,\partial_M \Phi\, \partial^M \Phi - \tfrac{1}{2}|H_3|^2 \right) \right.$$
$$\left. - \frac{1}{2} \int d^{10}x \sqrt{-g} \left(|F_2|^2 + |\tilde{F}_4|^2 \right) \right] \quad - \frac{1}{16\pi G_{10}} \int B_2 \wedge F_4 \wedge F_4 \qquad (1.2.1)$$

with field strengths

$$H_3 = dB_2, \qquad F_2 = dC_1, \qquad F_4 = dC_3, \qquad \tilde{F}_4 = F_4 - C_1 \wedge H_3. \tag{1.2.2}$$

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{16\pi G_{10}} \Bigg[\int d^{10}x \sqrt{-g}\, e^{-2\Phi} \Big(R + 4\,\partial_M \Phi\, \partial^M \Phi - \tfrac{1}{2}|H_3|^2 \Big) \\ - \frac{1}{2} \int d^{10}x \sqrt{-g} \Big(|F_1|^2 + |\tilde{F}_3|^2 + \tfrac{1}{2}|\tilde{F}_5|^2 \Big) \Bigg] - \frac{1}{16\pi G_{10}} \int C_4 \wedge H_3 \wedge F_3, \end{aligned} \tag{1.2.3}$$

where the field strengths are

$$H_3 = dB_2, \qquad F_1 = dC_0, \qquad F_3 = dC_2, \qquad F_5 = dC_4, \tag{1.2.4}$$

$$\tilde{F}_3 = F_3 - C_0 H_3, \qquad \tilde{F}_5 = F_5 - \tfrac{1}{2} C_2 \wedge H_3 + \tfrac{1}{2} B_2 \wedge F_3, \tag{1.2.5}$$

and the self-duality condition

$$\tilde{F}_5 = *\tilde{F}_5 \tag{1.2.6}$$

is imposed by hand.

The last terms in both the actions are the Chern-Simons terms. These actions represent the low-energy limit of string theory, including only the massless content. Consequently, it describes only closed string modes in ten-dimensional spacetime.

As discussed before, open strings can also appear in the theory. Unlike closed strings which loop back onto themselves, open strings have endpoints. If their ends were not anchored, excitations would violate momentum conservation. Therefore, open string endpoints must lie on a higher-dimensional object. These endpoints obey Dirichlet boundary conditions in the directions perpendicular to the surface. Hence these higher dimensional surfaces are known as D-branes (D stands for Dirichlet) or Dp-branes, p denoting the spatial dimension of the D-brane. D-branes also appear in bosonic string theory however, only in superstring theories some of the D-branes carry charges and are stable. In another way D-branes are solitonic (massive) solutions of non-linear supergravity equations, with masses inversely proportional to coupling, which makes them non-perturbative in nature.

Alongside fundamental strings, D-branes serve as key elements in non-perturbative formulations of string theory.

In the following sections, we aim to provide a concise overview of D-branes and fundamental strings, as they form essential components in the study of black hole microstates. Furthermore, we will discuss how D-branes relate to the AdS/CFT correspondence, which connects gravity in anti-de Sitter (AdS) space with conformal field theory (CFT) defined on its boundary.

1.3 D-branes and AdS/CFT duality

Solutions involving D-branes can be understood as higher-dimensional analogues of the Reissner–Nordström (RN) black hole. The extremal RN black hole metric is given by:

$$ds^2 = -\left(1-\frac{Q}{r}\right)^2 dt^2 + \left(1-\frac{Q}{r}\right)^{-2} dr^2 + r^2 d\Omega^2. \tag{1.3.1}$$

Introducing a coordinate change via $r-Q=R$, we observe that the event horizon moves to $R=0$, allowing us to rewrite the metric as:

$$ds^2 = -\left(1-\frac{Q}{R}\right)^{-2} dt^2 + \left(1-\frac{Q}{R}\right)^2 (dR^2 + R^2 d\Omega^2). \tag{1.3.2}$$

When modeling a point-like source, the function $\left(1-\frac{Q}{R}\right)$ emerges as a harmonic function in the transverse space. Here, the first term of the metric describes the worldvolume geometry of the object, while the second part represents the geometry of the surrounding transverse space. This structure resembles the metric of a 0-brane solution in string theory.

A Dp-brane is a p-dimensional generalization of the extremal 0-brane RN solution, where $p \leq 10$. In alignment with the extremality condition, D-branes also satisfy the BPS (Bogomol'nyi–Prasad–Sommerfield) bound and preserve half of the original 32 supersymmetries of $\mathcal{N}=2$ supersymmetry in ten-dimensional type-II supergravity. When contrasting the motion of a point particle—traced along a 1-dimensional worldline—with that of a string, whose dynamics unfold on a 2-dimensional worldsheet, the corresponding generalization to a Dp-brane involves a $(p+1)$-dimensional worldvolume. The associated spacetime geometry is described in string frame by the metric

$$ds^2 = H_p(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H_p(r)^{1/2}dz^i dz^i, \tag{1.3.3}$$

where $\mu,\nu = 0,1,\ldots,p$ span the D-brane worldvolume, and $z^i z^i$ with $i = p+1,\ldots,9$ cover the directions transverse to the brane. Here, $H_p(r)$ is a harmonic function in the radial direction of the transverse space, given by $r^2 = \sum_{i=p+1}^{9} z_i^2$, and in flat asymptotic regions takes the form

$$H_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p}. \tag{1.3.4}$$

Much like how point particles interact with 1-form gauge fields, Dp-branes couple electrically to $(p+1)$-form Ramond–Ramond (R–R) gauge potentials $C^{(p+1)}$ [35]. The associated field strength is a $(p+2)$-form, denoted $F^{(p+2)}$. The constant L_p in equation (1.2.16) characterizes the brane's charge. For a stack of N Dp-branes, this charge is obtained from the R–R flux in the transverse directions and is expressed as

$$L_p^{7-p} = \frac{(4\pi)^{(5-p)/2}}{\Gamma\left(\frac{7-p}{2}\right)} g_s N \alpha'^{(7-p)/2}, \tag{1.3.5}$$

where g_s is the string coupling, and α' is related to the string length l_s by $\alpha' = l_s^2$, or to the string tension T as $\alpha' = \frac{1}{2\pi T}$.

On the other hand, if the Dp-brane is magnetically charged, it couples to a $(7-p)$-form potential or equivalently to an $(8-p)$-form magnetic field strength (see Box B.1). This offers a gravitational description of D-branes. Alternatively, D-branes can also be described using open strings. Interestingly, the AdS/CFT correspondence emerges naturally by relating these two seemingly different perspectives on D-branes.

1.4 Two different descriptions of D-branes

One way to understand D-branes is through the **(a) closed string or supergravity framework**, where D-branes appear as soliton-like configurations in the low-energy limit of string theory. This supergravity description is reliable when the string coupling is large, i.e., $g_s \gg 1$, and the dynamics are captured by spacetime geometry. In this regime, D-branes source a curved geometry, which in the near-horizon limit becomes Anti-de Sitter (AdS) space—a solution of Einstein's equations with a negative cosmological constant ($\Lambda < 0$).

The alternative **(b) open string or brane picture** treats D-branes as extended objects on which open strings terminate. Both open and closed string excitations can exist near D-branes, but in the weak coupling limit $g_s \ll 1$, the open strings dominate. Open strings describe fluctuations on the D-brane worldvolume, while closed strings correspond to excitations in the full 10D bulk. At low energies, these modes decouple, and in the infrared (IR) limit, the field theory on the D-brane becomes a conformal field theory (CFT) with no massive degrees of freedom.

The **AdS/CFT correspondence** asserts that these two pictures—the closed string/ supergravity view and the open string/brane view—are actually equivalent. This duality is deep and nontrivial. To illustrate this idea concretely, we focus on the simplest case involving D3-branes, which clearly shows how gauge theory and gravity are related. A detailed source for this topic is [40].

1.4.1 D3-branes and AdS_5/CFT_4

D3-branes are BPS solitonic configurations in type-IIB supergravity, extended along three spatial dimensions and evolving along time, forming a 3+1-dimensional worldvolume. With the help of the harmonic function ansatz, one can express the 10D spacetime metric as:

$$ds_{10}^2 = H^{-1/2}(-dt^2 + d\vec{X}^2) + H^{1/2}(dr^2 + r^2 d\Omega_5^2), \tag{1.4.1}$$

where the $(t, \vec{X})$ part of the metric corresponds to the worldvolume of the D3-branes with the isometry group $SO(3,1)$. The remaining part of the metric (1.4.1) corresponds to six transverse spatial directions with isometry $SO(6)$. $X^i, i = 1,2,3$ corresponds to the directions along which the D3-brane extends, $d\Omega_5$ is the five dimensional transverse sphere with radius r.

The harmonic function H associated with this geometry takes the form

$$H = 1 + \frac{L^4}{r^4}, \tag{1.4.2}$$

where the parameter L is determined by the D3-brane's $U(1)$ charge and tension. When considering N identical D3-branes, L is given as

$$L = \left(4\pi g_s N \alpha'^2\right)^{1/4}. \tag{1.4.3}$$

Here, $r = 0$ designates the position of the D3-brane, acting as the source in the geometry. A 4-form gauge potential $C^{(4)}$ resides on the brane's $3+1$ dimensional worldvolume, and it gives rise to a 5-form field strength $F^{(5)} = dC^{(4)}$. Since in ten dimensions the Hodge dual of a 5-form is again a 5-form, the D3-branes are said to be self-dual. The solution (1.4.1) approaches flat spacetime at large distances $r \to \infty$, meaning $H \sim 1$.

In the opposite regime, when $r \ll L$, the function behaves like $H \sim \frac{L^4}{r^4}$, and the metric simplifies to

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + d\vec{X}^2) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2. \tag{1.4.4}$$

As $r \to 0$, the term $\frac{r^2}{L^2}$ vanishes, indicating that the brane's worldvolume becomes infinitely distant. Meanwhile, the radial component $\frac{dr}{r} \sim \log r$, which diverges. This configuration is known as the near-horizon geometry. The metric (1.4.4) can then be split into two components:

$$ds^2 = ds^2_{\mathrm{AdS}_5} + L^2 d\Omega_5^2, \tag{1.4.5}$$

with the substitution $u = \frac{L^2}{r}$, leading to

$$\begin{aligned} ds^2_{\mathrm{AdS}} &= \frac{r^2}{L^2}(-dt^2 + d\vec{X}^2) + \frac{L^2}{r^2}dr^2 \\ &= \frac{L^2}{u^2}(-dt^2 + d\vec{X}^2 + du^2). \end{aligned} \tag{1.4.6}$$

It is the metric for five dimensional AdS-space i.e. AdS_5 with radius L. So the near horizon limit gives $AdS_5 \times S^5$, S^5 having the constant radius L. Supergravity limit is a good approximation when the AdS radius L is much larger than the string length l_s, i.e. small curvature of the AdS-spacetime. We can see from the expression for AdS-radius(1.4.3) that

$$\frac{L}{l_s} = (4\pi g_s N)^{1/4}. \tag{1.4.7}$$

This means that $L \gg l_s$ corresponds to $g_s N \gg 1$, which defines the regime where the supergravity approximation is valid—often referred to as the strong coupling limit. In this limit, the D3-brane configuration is described purely in terms of closed strings propagating in a curved, asymptotically flat background with the self-dual five-form field strength $F^{(5)}$ through S^5. $F^{(5)}$ is field strength for R-R form $C^{(4)}$. No open-string modes or explicit D-brane excitations appear in this description.

Switching to the open-string viewpoint, let us consider a stack of N parallel D3-branes embedded in ten-dimensional spacetime, spanning the directions t, X^1, X^2, X^3. Here, open strings represent small perturbations of the D3-branes. This framework is appropriate for weak string coupling $g_s \ll 1$. In this case, the effective open-string coupling is $g_s N$, and small $g_s N$ corresponds to the weakly coupled regime. At low energies, the dynamics on the D3-branes are governed by a supersymmetric gauge theory localized on their worldvolume.

Open strings stretching along the transverse directions yield scalar fields living on the brane. For D3-branes, there are six such scalars ϕ^i, where $i = 1, \ldots, 6$, corresponding to fluctuations in the transverse dimensions. Each D-brane hosts a $U(1)$ gauge theory for its associated open strings. When the branes coincide, the gauge symmetry is enhanced to $U(N)$. The worldvolume theory becomes a $3+1$-dimensional $SU(N)$[7] super-Yang–Mills theory with $\mathcal{N} = 4$ supersymmetry, which includes 16 conserved supercharges. The low-energy excitations of this theory come from open strings, while closed string modes in the bulk effectively decouple.

AdS/CFT duality [30] implies that the supergravity theory on $AdS_5 \times S^5$ is equivalent to the $\mathcal{N} = 4$, $SU(N)$ super-Yang–Mills theory on the 3+1-dimensional brane world-volume. It's a strong statement, and it has only been proved completely in the large N-limit, which corresponds to the planar limit. The couplings on both the sides of the theory are related to each other as,

$$g_{\mathrm{YM}}^2 = 2\pi g_s, \qquad 2\lambda = \left(\frac{L}{l_s}\right)^4, \tag{1.4.8}$$

where $\lambda = g_{\mathrm{YM}}^2 N$ represents the effective coupling in the large N limit, commonly referred to as the 't Hooft coupling.

In many examples of black holes constructed within string theory, the near-horizon geometry typically takes the form of $AdS_3 \times S^3$. For instance, the D1-D5 system exhibits a near-horizon $AdS_3 \times S^3$ geometry. This connection between the logarithm of microstate counts in the field theory and the Bekenstein-Hawking entropy from the gravitational perspective illustrates the duality between supersymmetric gauge theories on AdS_3 and the conformal field theory (CFT) living on its two-dimensional boundary.

We also come across fundamental strings, also referred to as F1-strings, in the microstate structure of black holes. Hence, it's important to briefly understand these fundamental components in string theory.

[7]If we remove one $U(1)$ degrees of freedom corresponding to the collective mode of N-coincident branes.

1.4.2 Fundamental strings (F1 a.k.a NS1)

These strings serve as the basic elements in perturbative string theory, meaning in the weak coupling regime. By applying the harmonic superposition principle, the configuration of fundamental strings or F1-strings can be expressed through the following metric:[8]

$$ds^2_{\text{string}} = H_1^{-1}[-dt^2 + dy^2] + \sum_{i=1}^{8} dx_i dx_i, \tag{1.4.9}$$

where the string is wound n_1 times along the compact S^1 direction labeled by y. The coordinate y is periodic, $y \sim y + 2\pi R$. The x_i coordinates represent directions transverse to the string. By switching to null coordinates $u = t + y$, $v = t - y$, the above metric (1.4.9) becomes:

$$ds^2_{\text{string}} = -H_1^{-1} du dv + \sum_{i=1}^{8} dx_i dx_i. \tag{1.4.10}$$

H_1 is the harmonic function in the transverse directions given by,

$$H_1 = 1 + \frac{Q_1}{r^6}, \quad \text{where} \quad Q_1 = \frac{g_s \alpha'^3}{V n_1}. \tag{1.4.11}$$

Here, $r = \sqrt{x_i x^i}$ represents the radial distance in the transverse space. This background metric also includes the Kalb-Ramond B-field and a dilaton, which are expressed as:

$$B_{uv} = \frac{1}{2}(e^{2\Phi} - 1), \qquad e^{2\Phi} = H_1^{-1}. \tag{1.4.12}$$

This metric from equation (1.4.9) will be used in deriving the F1-P solution. These F1-P solutions are important building blocks in constructing black hole microstates within string theory.

In the next section, we will briefly review how black holes are constructed in the framework of string theory. Some comprehensive reviews on this topic can be found in references [36] [39].

[8] The subscript 'string' refers to string frame.

1.5 Black holes in string theory

The formulation of black hole microstates in string theory has played a key role in addressing both the black hole entropy problem and the information loss paradox. Since superstring theory operates in ten spacetime dimensions, and standard black holes are observed in four, extra spatial dimensions are typically compactified in such setups. This compactification results in the appearance of new fields when reduced to lower-dimensional effective theories.

Susskind and collaborators [13] proposed a compelling framework for analyzing black holes within string theory. They considered a string in a highly excited state, characterized by a large mass $M \gg \alpha'^{-1/2}$, and assumed a weak string coupling $g_s \ll 1$, so that interactions are negligible and the string behaves as essentially free.

In this regime, the excitation levels of left- and right-moving oscillators satisfy $N_L, N_R \sim \sqrt{\alpha'}M \gg 1$ leading to a vast degeneracy $\mathcal{N}$ of states at fixed mass. The logarithm of this degeneracy yields the microscopic entropy

$$S_{\text{micro}} = \ln[\mathcal{N}] \sim \sqrt{\alpha'}M \tag{1.5.1}$$

The precise coefficient depends on the number of compactified dimensions, since compact directions allow for winding modes that enhance the entropy.

Now, if the string coupling g_s is gradually increased, gravitational effects become significant because the Newton constant scales as $G \sim g_s^2$. For sufficiently large mass M, the system transitions into a black hole configuration. The corresponding Bekenstein-Hawking entropy is given by $S_{\text{BH}} = \frac{A}{4G}$.

For a Schwarzschild black hole in four-dimensional spacetime (i.e., $3+1$ noncompact dimensions), this yields $S_{\text{BH}} \sim M^2$. More generally, in a spacetime with D noncompact dimensions, the entropy scales as

$$S_{\text{BH}} \sim M^{\frac{D-2}{D-3}} \tag{1.5.2}$$

Although S_{micro} and S_{BH} scale differently with mass M, this discrepancy does not render the comparison meaningless. In fact, it is quite significant that the microscopic entropy exhibits any power-law growth with M at all. Specifically, even in the limit of vanishing string coupling $g_s = 0$ the number of accessible microstates increases exponentially with energy. This behavior is captured by

$$S_{\text{micro}} \sim \sqrt{\alpha'} M \tag{1.5.3}$$

Such exponential growth arises from the extended nature of the string. Unlike point particles, which at $g_s = 0$ would simply disperse and fail to form a bound state with high degeneracy, the string can absorb energy M into its internal vibrational modes. This capacity to store energy in a vast number of configurations is what gives rise to the large entropy—even in the absence of gravitational interactions.

1.5.1 BPS States

The mismatch between S_{micro} and S_{BH} arises because the energy spectrum of states shifts as the string coupling changes. Specifically, as we vary the string coupling g_s the energy levels of states are altered, making it invalid to directly compare degeneracies at different values of g_s. This issue can be resolved by focusing on BPS states in string theory. These states carry both mass and charge, satisfying $Q = M$ in suitable units. The mass of a BPS state is determined by its charges and the values of the moduli in the theory, and remains invariant under changes in g_s. Therefore, all BPS states with a given mass M evolve coherently as g_s varies, allowing meaningful comparisons of their degeneracies across different coupling regimes. This leads to the expectation

$$S_{\text{micro}} = S_{\text{BH}} \tag{1.5.4}$$

for BPS configurations. If this equality were not satisfied, it would raise serious concerns about the validity of string theory as a consistent theory of quantum gravity. The entropy S_{BH} is derived from semiclassical thermodynamic arguments, and it is the responsibility of the full quantum theory to reproduce this entropy through a microscopic count of states.

Conversely, if the microscopic and macroscopic entropies do agree, then string theory passes a profound test—demonstrating that it possesses the correct degrees of freedom required for a quantum theory of gravity.

1.5.2 Compactification

String theory naturally exists in higher-dimensional spaces—specifically, ten spacetime dimensions for superstrings. On the other hand, physical black holes exist in a 3+1-dimensional spacetime. To reconcile this, it makes sense to reduce or "compactify" the extra spatial dimensions into small sizes, making them unobservable at low energies or in experiments. When higher-dimensional theories are compactified, they give rise to scalar fields in the effective lower-dimensional theory. These scalar fields come along with the lower-dimensional spacetime metric and other vector fields. This occurs due to the breakdown of the higher-dimensional symmetry groups into lower-dimensional ones. These scalar fields form the moduli space of the lower dimensional theory.

The foundational concepts behind compactification were first introduced by T. Kaluza and O. Klein in 1920 [37]. To understand how this functions, consider the 10-dimensional metric $g_{\mu\nu}$, which is a symmetric 10×10 matrix with indices $\mu, \nu = 0, 1, \ldots, 9$. If we compactify the 9th spatial direction into a small circle, the remaining fields in the resulting 9-dimensional theory are:

$$\text{(metric)}\, g_{ij},\; i, j = 0, 1, \ldots, 8, \qquad \text{(scalar/dilaton)}\, \phi = g_{99}, \qquad \text{(vector field)}\, A_i = g_{i9}. \tag{1.5.5}$$

In addition to the metric components, various form fields also decompose, yielding additional scalar fields in the lower-dimensional setup.

As an example, in case of D1-D5 system in type-IIB string theory compactified on a four-dimensional torus T^4, the moduli space becomes 25-dimensional. This space is described by ten components from the 4×4 block of the full metric, six components from the antisymmetric B-field, the dilaton ϕ, one toroidal component from the 4-form Ramond-Ramond field $C^{(4)}$, six components from the 2-form field $C^{(2)}$, and one scalar from the 0-form $C^{(0)}$. In the near-horizon $AdS_3\times S^3$ limit, this space simplifies to a 20-dimensional manifold. The attractor mechanism implies that, in this limit, the moduli fields are determined solely by specific charges and do not depend on their values at infinity.

1.5.3 1-charge System

To begin, we examine the most elementary BPS states in string theory. For specificity, consider type IIA string theory, which can also be viewed as the dimensional reduction of 11-dimensional M-theory. We compactify one spatial dimension on a circle S^1, parameterized by a coordinate y such that

$$0\leq y<2\pi R \tag{1.5.6}$$

An elementary string, specifically an NS1 brane, can be wrapped around this compact circle. If the string carries no excitations (i.e., no oscillator modes are turned on), it constitutes a BPS state. More generally, we can consider a string wound n_1 times around the circle. For large n_1, this configuration yields a BPS state with substantial mass.

The corresponding supergravity background sourced by such a string takes the form:

$$ds^2_{\text{string}}=H_1^{-1}(-dt^2+dy^2)+\sum_{i=1}^{8}dx^idx^i \tag{1.5.7}$$

where H_1 is a harmonic function associated with the string charge, and x^i denote the transverse spatial directions. The dilaton profile for the NS1 brane solution is given by

$$e^{2\phi}=H_1^{-1},\quad \text{where}\quad H_1=1+\frac{Q_1}{r^6} \tag{1.5.8}$$

Here, ds^2_{string} denotes the 10-dimensional string frame metric, and x^i represent the eight spatial directions transverse to the string. As $r \to 0$, the dilaton $\phi \to -\infty$, indicating that the string coupling vanishes and the physical size of the compact y circle shrinks to zero. Importantly, the geometry lacks a horizon at any finite r. If we interpret $r = 0$ as the location of a horizon, then the area of this surface, measured in the Einstein frame, is zero. Consequently, the Bekenstein-Hawking entropy is

$$S_{\text{BH}} = 0 \tag{1.5.9}$$

This result aligns with the microscopic entropy. Since the NS1 brane is in its oscillator ground state, its degeneracy arises solely from the string's zero modes, which yield 128 bosonic and 128 fermionic states. Thus, the microscopic entropy is

$$S_{\text{micro}} = \ln(256) \tag{1.5.10}$$

This entropy does not scale with the winding number n_1, so in the macroscopic limit $n_1 \to \infty$, we can write at the leading order $S_{\text{micro}} = 0$ which agrees with the vanishing S_{BH}.

To understand why this configuration fails to produce a black hole with finite horizon area, consider the NS1 brane as an M2 brane in M-theory, wrapping the directions x_{11} and y. A brane exerts tension along its worldvolume, causing the cycles it wraps to contract. As a result, the x_{11} circle shrinks to zero size at $r = 0$, which manifests as $\phi \to -\infty$ in the IIA description. Similarly, the y circle also collapses in the M-theory picture.

In contrast, compact directions that are transverse to the brane tend to expand. This expansion occurs because the brane radiates flux, and the energy of this flux is minimized when it spreads over a larger transverse volume.

1.5.4 2-charge System

To prevent the shrinking of the x_{11} direction, we can introduce M5-branes oriented such that they lie transverse to x_{11}. Upon dimensional reduction to type IIA string theory, these M5-

branes become NS5-branes.

Since NS5-branes have five spatial worldvolume directions, we require additional compact dimensions to accommodate them. Therefore, we compactify a four-torus T^4 along with the circle S^1, and wrap the NS5-branes on the product space $T^4 \times S^1$.

We continue to include NS1-branes wrapped along the y direction, but now within this more richly compactified background In the presence of both NS1 and NS5 branes, the radial dependence in the harmonic functions changes. The resulting string frame metric is

$$ds^2_{\text{string}} = H_1^{-1}(-dt^2 + dy^2) + H_5 \sum_{i=1}^{4} dx^i dx^i + \sum_{a=1}^{4} dz^a dz^a \tag{1.5.11}$$

The dilaton profile becomes

$$e^{2\phi} = \frac{H_5}{H_1} \tag{1.5.12}$$

with the harmonic functions defined as

$$H_1 = 1 + \frac{Q_1}{r^2}, \quad H_5 = 1 + \frac{Q_5}{r^2} \tag{1.5.13}$$

Here, the torus T^4 is parametrized by coordinates z^a, where $a = 1, \ldots, 4$, and Q_5 is proportional to n_5, the number of NS5 branes.

As $r \to 0$, the dilaton ϕ approaches a constant value, indicating stabilization of the x_{11} circle. The volume of the T^4 also remains finite at the origin: NS5 branes tend to contract the torus since their worldvolume wraps it, while NS1 branes—being transverse to T^4 induce an expansion. The underlying reason is the same as described at the end of the previous subsection 1.5.3. These competing effects balance out.

1.5.5 3-charge System

To prevent the shrinking of the y circle, we introduce momentum charge P along that direction. If there are n_p units of momentum, the energy associated with these modes is $E_p = \frac{n_p}{R}$ which decreases as the radius R increases. This inverse dependence counteracts the contributions from the NS1 and NS5 branes, whose energies scale linearly with R. The resulting string frame metric becomes

$$\begin{aligned} ds^2_{\text{string}} &= H_1^{-1}(-dt^2+dy^2+K(dt+dy)^2)+H_5\sum_{i=1}^{4}dx^idx^i+\sum_{a=1}^{4}dz^adz^a \\ e^{2\phi} &= \frac{H_5}{H_1} \\ H_1 &= 1+\frac{Q_1}{r^2}, \quad H_5=1+\frac{Q_5}{r^2}, \quad K=\frac{Q_p}{r^2}. \end{aligned} \tag{1.5.14}$$

This geometry features a horizon at $r=0$. To analyze its area, we examine the noncompact spatial directions in polar coordinates near the horizon

$$H_5\sum_{i=1}^{4}dx^idx^i = H_5(dr^2+r^2d\Omega_3^2) \approx Q_5\left(\frac{dr^2}{r^2}+d\Omega_3^2\right) \tag{1.5.15}$$

Thus, the area of the transverse 3-sphere S^3 stabilizes to a constant as $r \to 0$, given by:

$$A^{\text{string}}_{S^3} = (2\pi^2)Q_5^{3/2} \tag{1.5.16}$$

This area will later be converted to the Einstein frame to compute the Bekenstein-Hawking entropy. At the horizon $r \to 0$, the physical length of the y circle in the string frame is:

$$L_y^{\text{string}} = 2\pi R\left(\frac{K}{H_1}\right)^{1/2} = 2\pi R\left(\frac{Q_p}{Q_1}\right)^{1/2} \tag{1.5.17}$$

Let the coordinate volume of the compact T^4, parametrized by z_a, be $(2\pi)^4V$. Then the physical volume of the torus at $r \to 0$ is

$$V^{\text{string}}_{T^4} = (2\pi)^4V \tag{1.5.18}$$

Combining these, the total horizon area in the string frame is

$$A^{\text{string}} = A_{S^3}^{\text{string}} \cdot L_y^{\text{string}} \cdot V_{T^4}^{\text{string}} = (2\pi^2)(2\pi R)((2\pi)^4 V) Q_1^{-1/2} Q_5^{3/2} Q_p^{1/2} \tag{1.5.19}$$

To convert this to the Einstein frame, we use the relation between the 10-dimensional Einstein and string metrics

$$g_{ab}^E = e^{2\phi} g_{ab}^S = \left(\frac{H_1}{H_5}\right)^{1/4} g_{ab}^S \tag{1.5.20}$$

At $r \to 0$, the dilaton stabilizes to $e^{2\phi} = \frac{Q_5}{Q_1}$ Thus, the horizon area in the Einstein frame becomes:

$$A^E = \left(\frac{g_{ab}^E}{g_{ab}^S}\right)^4 A^{\text{string}} = \left(\frac{Q_1}{Q_5}\right) A^{\text{string}} = (2\pi^2)(2\pi R)((2\pi)^4 V)(Q_1 Q_5 Q_p)^{1/2} \tag{1.5.21}$$

This yields a finite horizon area, consistent with a macroscopic black hole solution.

The five-dimensional Newton constant G_5, relevant for the noncompact spacetime directions, is related to the ten-dimensional Newton constant G_{10} by

$$G_5 = \frac{G_{10}}{(2\pi R)(2\pi)^4 V} \tag{1.5.22}$$

Using this relation, the Bekenstein-Hawking entropy becomes

$$S_{\text{Bek}} = \frac{A_E}{4G_{10}} = \frac{(2\pi^2)(2\pi R)(2\pi)^4 V (Q_1 Q_5 Q_p)^{1/2}}{4G_{10}} = \frac{(2\pi^2)(Q_1 Q_5 Q_p)^{1/2}}{4G_5} \tag{1.5.23}$$

Now, we wish to find the expression for the D-brane charges in terms of various quantum numbers and moduli parameters. Dp-branes couple electrically to Ramond–Ramond (RR) gauge potentials through the Wess–Zumino term

$$S_{\text{WZ}} = \mu_p \int_{\mathscr{W}_{p+1}} C_{p+1}, \tag{1.5.24}$$

where $\mathscr{W}_{p+1}$ denotes the brane worldvolume and μ_p is the D-brane charge density.

This coupling can be rewritten as a spacetime integral by introducing a localized current J_{9-p}

$$S_{\mathrm{WZ}} = \mu_p \int C_{p+1} \wedge J_{9-p}. \tag{1.5.25}$$

The bulk kinetic term for the RR field strength $F_{p+2} = dC_{p+1}$ is given by [32, 35, 115]

$$S_{\mathrm{bulk}} = -\frac{1}{4\kappa_{10}^2} \int F_{p+2} \wedge \star F_{p+2}. \tag{1.5.26}$$

Varying the total action with respect to C_{p+1} yields

$$d(\star F_{p+2}) = 2\kappa_{10}^2 \mu_p J_{9-p}. \tag{1.5.27}$$

The normalization of μ_p is fixed by string theory considerations [35]. To start with, supersymmetry implies

$$\mu_p = T_p. \tag{1.5.28}$$

Now, the task at hand will be finding the Dp brane tension T_p. This can be fixed considering open string disk amplitudes and considering T-duality consistency.

Dependence on the string coupling

Disk amplitudes imply

$$T_p \propto \frac{1}{g_s}. \tag{1.5.29}$$

Dimensional analysis

Since the action is dimensionless and $\ell_s^2 = \alpha'$ is the fundamental length scale, one finds

$$[T_p] = (\text{length})^{-(p+1)}, \tag{1.5.30}$$

which implies

$$T_p \propto \frac{1}{g_s(\alpha')^{(p+1)/2}}. \tag{1.5.31}$$

Fixing normalization via T-duality

Under T-duality along a circle of radius R

$$R \to \frac{\alpha'}{R}, \tag{1.5.32}$$

and the tensions satisfy

$$T_{p-1} = 2\pi R T_p. \tag{1.5.33}$$

Assuming

$$T_p = \frac{A_p}{g_s(\alpha')^{(p+1)/2}}, \tag{1.5.34}$$

one obtains the recursion relation

$$A_{p-1} = 2\pi A_p, \tag{1.5.35}$$

which gives

$$A_p = (2\pi)^{-p}. \tag{1.5.36}$$

Thus, putting all the pieces together, we have an expression for D brane tension as

$$T_p = \frac{1}{(2\pi)^p g_s \alpha'^{(p+1)/2}}. \tag{1.5.37}$$

Integrating the equation of motion gives

$$\int \star F_{p+2} = 2\kappa_{10}^2 \mu_p N_p, \tag{1.5.38}$$

so that

$$N_p = \frac{1}{2\kappa_{10}^2 \mu_p} \int \star F_{p+2}. \tag{1.5.39}$$

We are using uppercase to denote the Dp brane charge N_p in order to avoid confusion with momentum charge n_p which will be introduced later on.

Now, we are dealing with Type IIB string theory compactified on $\mathbb{R}^{4,1} \times S^1 \times T^4$.

The brane configuration consists of

- D1-branes wrapped on S^1,
- D5-branes wrapped on $S^1 \times T^4$.

The conserved charges are given by

$$N_5 = \frac{1}{(2\pi)^2\alpha'} \int_{S^3} F_3, \tag{1.5.40}$$

$$N_1 = \frac{1}{(2\pi)^6 g_s \alpha'^3} \int_{S^3 \times T^4} \star F_3. \tag{1.5.41}$$

RR Field Strength for the D1–D5 System

The RR three-form field strength is

$$F_3 = d(H_1^{-1}) \wedge dt \wedge dy + \star_4 dH_5, \tag{1.5.42}$$

where

$$H_1 = 1 + \frac{Q_1}{r^2}, \qquad H_5 = 1 + \frac{Q_5}{r^2}. \tag{1.5.43}$$

Evaluation of the D5 Charge

Using

$$dH_5 = -\frac{2Q_5}{r^3} dr, \qquad \star_4 dr = r^3 d\Omega_3, \tag{1.5.44}$$

we obtain

$$F_3 = 2Q_5 \, d\Omega_3. \tag{1.5.45}$$

Thus

$$\int_{S^3} F_3 = 4\pi^2 Q_5, \tag{1.5.46}$$

which implies

$$Q_5 = g_s \alpha' N_5. \tag{1.5.47}$$

Evaluation of the D1 Charge

At large r

$$\partial_r H_1^{-1} = \frac{2Q_1}{r^3}. \tag{1.5.48}$$

Then

$$\star F_3 \sim \partial_r H_1^{-1}\, r^3 d\Omega_3 \wedge d^4 z. \tag{1.5.49}$$

Using

$$\int_{S^3} d\Omega_3 = 2\pi^2, \qquad \int_{T^4} d^4 z = (2\pi)^4 V, \tag{1.5.50}$$

we obtain

$$Q_1 = \frac{g_s \alpha'^3}{V_4} N_1. \tag{1.5.51}$$

Calculating Momentum Charge

In asymptotically flat spacetimes, conserved charges such as energy and momentum are defined using the ADM formalism. The momentum along a compact direction y is obtained from the asymptotic behavior of the metric component g_{ty}. In ten dimensions, the ADM momentum is given by

$$P_y = \frac{1}{16\pi G_{10}} \int_{\partial\Sigma} dS_i\, \partial_i g_{ty}, \tag{1.5.52}$$

where $\partial\Sigma$ is a closed spatial surface enclosing the source.

For Type IIB compactified on

$$\mathbb{R}^{4,1} \times S^1 \times T^4, \tag{1.5.53}$$

the total number of spatial dimensions is nine. The ADM surface therefore has dimension 8 and is given by

$$\partial\Sigma = S^3 \times S^1 \times T^4. \tag{1.5.54}$$

Since the solution is independent of the internal coordinates, the integral factorizes as

$$\int_{S^3\times S^1\times T^4} = (2\pi R)(2\pi)^4 V \int_{S^3}. \tag{1.5.55}$$

This allows us to express the momentum in terms of the five-dimensional Newton constant

$$P_y = \frac{1}{16\pi G_5}\int_{S^3} dS_i\,\partial_i g_{ty}. \tag{1.5.56}$$

For the D1–D5–P solution, the relevant metric component is

$$g_{ty} = H_1^{-1}K, \tag{1.5.57}$$

where asymptotically

$$H_1 \to 1, \qquad K = \frac{Q_P}{r^2}, \quad \text{as } r \to \infty. \tag{1.5.58}$$

Thus

$$g_{ty} \approx \frac{Q_P}{r^2}. \tag{1.5.59}$$

The radial derivative is

$$\partial_r g_{ty} = -\frac{2Q_P}{r^3}. \tag{1.5.60}$$

The surface element on S^3 is

$$dS_r = r^3 d\Omega_3. \tag{1.5.61}$$

Therefore

$$\int_{S^3} dS_r\,\partial_r g_{ty} = -2Q_P\int_{S^3} d\Omega_3 = -4\pi^2 Q_P. \tag{1.5.62}$$

Substituting into the ADM expression

$$P_y = \frac{\pi Q_P}{4G_5}. \tag{1.5.63}$$

Momentum along the compact circle is quantized

$$P_y = \frac{n_P}{R}. \tag{1.5.64}$$

Equating the two expressions

$$\frac{n_P}{R} = \frac{\pi Q_P}{4G_5}. \tag{1.5.65}$$

The five-dimensional Newton constant is related to the ten-dimensional one by

$$G_5 = \frac{G_{10}}{(2\pi R)(2\pi)^4 V} \quad \text{with} \quad G_{10} = 8\pi^6 g_s^2 \alpha'^4. \tag{1.5.66}$$

This gives $G_5 = \frac{\pi}{4}\frac{g_s^2\alpha'^4}{RV}$. Substituting it into the momentum relation 1.5.65 yields

$$Q_P = \frac{g_s^2\alpha'^4}{R^2 V} n_P. \tag{1.5.67}$$

Now, the ten-dimensional Newton constant is

$$G_{10} = 8\pi^6 g_s^2 \alpha'^4 \tag{1.5.68}$$

Substituting these expressions for charges into the entropy expression 1.5.23 yields [94]

$$S_{\text{Bek}} = 2\pi\sqrt{N_1 N_5 n_p} \tag{1.5.69}$$

This matches the microscopic entropy obtained from counting BPS states in the D1-D5-P system.

It is important to observe that the moduli g_s, V, and R have completely canceled out from the final expression for the entropy. This cancellation is essential for reproducing the entropy through a microscopic calculation.

In the microscopic description, we consider a bound state composed of the quantized charges N_1, N_5, and n_p, and compute the degeneracy of this bound state. Since we are dealing with BPS configurations, the degeneracy is protected and remains independent of the moduli.

This moduli-independence ensures that the microscopic entropy matches the macroscopic result across different regimes of the theory.

Now, let's make a passing comment on the origin of equation 1.5.68. In string theory, we describe the string moving in a background spacetime with a metric $g_{\mu\nu}$, a dilaton Φ and an antisymmetric field $B_{\mu\nu}$. The worldsheet action takes the form of a non-linear sigma model action [32]

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}[h^{ab}g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi(X) + ..] \tag{1.5.70}$$

For the string theory to be consistent, the worldsheet theory have to be conformally invariant. this means, the physics shouldn't change if we rescale the worldsheet coordinates. At the quantum level, this requirement implies that the Renormalization Group(RG) beta functions for the background fields must vanish

$$\beta^g_{\mu\nu} = 0, \quad \beta^B_{\mu\nu} = 0, \quad \beta^\Phi = 0. \tag{1.5.71}$$

At one loop order in α', we get equation of motion

$$\beta^g_{\mu\nu} = \alpha'(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\lambda\sigma}H_\nu^{\lambda\sigma}) + \mathscr{O}(\alpha'^2) = 0. \tag{1.5.72}$$

this equation corresponds to an effective action

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}(R + 4\partial_\mu\Phi\partial^\mu\Phi - ..) \tag{1.5.73}$$

Changing to Einstein frame by $g^E_{\mu\nu} = e^{-\frac{4(\Phi-\Phi_0)}{D-2}} g^S_{\mu\nu}$, and noting that $e^{\Phi_0} = g_s$, the Einstein frame metric looks like

$$S = \frac{1}{2\kappa_{10}^2 g_s^2}\int d^{10}x\sqrt{g_E}(R_E - \frac{1}{2}(\partial\Phi)^2 \ldots) \tag{1.5.74}$$

Comparing with standard Einstein-Hilbert action of the form

$$S_{EH} = \frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-g_E}R_E \tag{1.5.75}$$

we get the relation 1.5.68.

1.6 Counting of states

The idea of counting black hole microstates using configurations of intersecting strings and branes was independently developed by Sen, and also by Strominger and Vafa. A. Sen's analysis in 1995 [38] involved counting the microstates of vibrating fundamental strings (F1-P). In his setup, the 10-dimensional string theory is compactified on $S^1 \times K3$, resulting in a five-dimensional theory. This is related to the heterotic string compactified on a five-torus T^5. Strominger and Vafa performed a similar microstate count for the D1-D5-P system [15]. In these studies, microstates are enumerated in the weak coupling regime, and the resulting entropy—given by the logarithm of the number of states—is shown to agree with the Bekenstein-Hawking entropy of the corresponding black hole in the strong coupling limit.
Supersymmetry ensures that the number of microstates doesn't change with the change of coupling[9].

This outcome not only aids in addressing the black hole entropy problem but also strongly supports the idea of a duality between supergravity in AdS space—which governs the near-horizon geometry of branes—and the conformal field theory (CFT) on the boundary.

[9] To be precise, the object that is computed is an 'index' which remains invariant under the change of moduli

This naturally leads to the question: how do different microstates manifest within a gravitational framework? An effort to visualize this was initiated in 2001 by Mathur and Lunin, aiming to provide a geometric understanding of these microstates [61, 62, 92]. Their proposal remains an active area of research [64–67]. There are also several insightful reviews discussing the fuzzball paradigm and its ongoing developments [94–97].

We present two simple examples of entropy calculation from microscopic point of view.

1.6.1 Entropy for 2-charge NS1-P system

Consider the two-charge system consisting of NS1 branes and momentum P. For this configuration, identifying the bound states is straightforward. First, all windings of the NS1 branes must be joined to form a single, multiply wound string. This "long string" wraps around the compact S^1 a total of n_1 times before reconnecting to itself.

The momentum P must also be bound to this long string. If it were not, it would appear as free, massless excitations of the type IIA theory—such as gravitons or gauge bosons—propagating around the S^1. However, when the momentum is genuinely bound to the NS1, it manifests as traveling waves along the string. Thus, the NS1–P bound state is described by a single multi-wound string carrying left-moving excitations along the S^1.

There are many distinct ways to distribute the same total momentum P across different vibrational modes of the string. This freedom leads to a large degeneracy of states for fixed values of n_1 and n_p.

One approach is to treat the system as an elementary string with winding number n_1, momentum n_p, and purely left-moving excitations at level N_L, with no right-moving modes (i.e., $N_R = 0$). This choice preserves the supersymmetries from the right-moving sector, ensuring the state remains BPS.

The mass formula for such a string state is [94]

$$m^2 = \left(2\pi R n_1 T - \frac{n_p}{R}\right)^2 + 8\pi T N_L = \left(2\pi R n_1 T + \frac{n_p}{R}\right)^2 + 8\pi T N_R \tag{1.6.1}$$

where R is the radius of the compact circle S^1, and $T = \frac{1}{2\pi\alpha'}$ is the tension of the elementary string.

Setting $N_R = 0$, we find

$$N_L = n_1 n_p, \quad m = 2\pi R n_1 T + \frac{n_p}{R} \tag{1.6.2}$$

This corresponds to a threshold bound state of NS1 and P, meaning there is no binding energy. For large n_1 and n_p, the excitation level $N_L \gg 1$, allowing for a rich spectrum of states. The oscillator modes contributing to this level include 8 bosonic degrees of freedom (yielding central charge $c = 8$) and 8 fermionic degrees of freedom (with central charge $c = 4$). Thus, the total central charge of the system is $c = 8 + 4 = 12$.

Using Cardy's formula, the total number of microstates at oscillator level N_L is approximately:

$$\mathcal{N} \sim \exp\left(2\pi\sqrt{\frac{c}{6}N_L}\right) \tag{1.6.3}$$

Substituting $N_L = n_1 n_p$ and the total central charge $c = 12$, we obtain

$$\mathcal{N} \sim \exp\left(2\pi\sqrt{2}\sqrt{n_1 n_p}\right) \tag{1.6.4}$$

Taking the logarithm we compute the microscopic entropy

$$S_{\text{micro}} = \ln\mathcal{N} = 2\pi\sqrt{2}\sqrt{n_1 n_p} \tag{1.6.5}$$

Another counting method is based on counting partitions of an integer. Since we are considering BPS states, increasing the compactification radius R does not affect the number of available states. In this large R limit, the NS1 string can be treated as having small transverse oscillations. Using the Dirac–Born–Infeld (DBI) action in the static gauge, one can describe these oscillations by a quadratic action in their amplitudes. The resulting waves move along the y direction at the speed of light, and each Fourier mode behaves as an independent harmonic oscillator.

The total length of the NS1 string is given by

$$L_T = 2\pi R n_1. \tag{1.6.6}$$

Each Fourier mode with wave number k carries an energy and momentum

$$e_k = p_k = \frac{2\pi k}{L_T}. \tag{1.6.7}$$

The overall momentum on the string is therefore

$$P = \frac{n_p}{R} = \frac{2\pi n_1 n_p}{L_T}. \tag{1.6.8}$$

If we first look at just one transverse direction of vibration, and let m_i be the number of quanta occupying the ith Fourier mode k_i, the total momentum constraint implies

$$\sum_i m_i k_i = n_1 n_p. \tag{1.6.9}$$

The number of ways this can happen equals the number of partitions of the integer $n_1 n_p$. The asymptotic number of such partitions behaves as $\sim \exp\left(2\pi\sqrt{\frac{n_1 n_p}{6}}\right)$. However, the total momentum is distributed among both bosonic and fermionic modes—eight bosonic and eight fermionic in total. The eight fermionic modes effectively contribute as four bosonic ones, giving an overall count equivalent to twelve bosonic modes. Thus, the total degeneracy is obtained by raising the single-mode degeneracy to the 12th power, where we assume that the momentum is distributed democratically between the modes corresponding to the different degrees of freedom, hence each degree of freedom effectively having $\frac{n_1 n_p}{12}$ amount of momentum. This gives

$$\mathcal{N} = \left[\exp\left(2\pi\sqrt{\frac{n_1 n_p}{72}}\right)\right]^{12} = \exp\left(2\pi\sqrt{2n_1 n_p}\right). \tag{1.6.10}$$

1.6.2 Entropy of 3 charge system

Suppose there is only one NS5 brane. Since the NS1 brane extends along the NS5 and is bound to it, we can imagine that the NS1 can oscillate within the plane of the NS5 but cannot move outside it. The momentum P is still carried by waves traveling along the NS1, but now only four transverse directions of vibration are available—those lying inside the NS5 and perpendicular to the NS1. Therefore, the bosonic contribution comes from 4 instead of 8 degrees of freedom. As the three-charge bound state is supersymmetric, there are four fermionic excitation modes as well. Hence, we have

$$c = 4 + 2 = 6. \tag{1.6.11}$$

To generalize to the case $n_5 > 1$, we need to understand why the winding number n_1 effectively becomes $n_1 n_5$ when there are several NS5 branes present. From dualities, we know that

$$NS1(n_1), \quad P(n_p) \longleftrightarrow NS5(n_1), \quad NS1(n_p). \tag{1.6.12}$$

Let us first analyze the NS1-P system. Suppose the NS1 wraps only once around the circle S^1. The n_p units of momentum are distributed among different harmonics, with momenta quantized in multiples of $1/R$. If instead the NS1 wraps $n_1 > 1$ times around S^1, then the total length of the multiwound string is $2\pi R n_1$, and the momentum quantization becomes

$$\Delta p = \frac{1}{n_1 R}. \tag{1.6.13}$$

The total momentum n_p/R must still be an integer multiple of $1/R$, as required for any system living on a circle of radius R. Hence, we have $n_1 n_p$ units of fractional momentum, each of strength Δp, which can be partitioned in different ways among the allowed states of the system.

Now consider the NS5–NS1 system obtained after applying the duality. If there is a single NS5 brane ($n_5 = 1$), then there are n_p NS1 branes bound to it. The various possible states correspond to different ways of partitioning these bound branes. In the NS1-P setup, we can think of counting states by dividing the total winding number n_p in different ways. Essentially, the NS1 strings within the NS5 brane can join together to create "multiwound" strings. Thus, there can be n_p singly wound loops, one string wound n_p times, or any other combination such that the total winding satisfies

$$\sum_i m_i k_i = n_p. \tag{1.6.14}$$

If we consider multiple NS5 branes ($n_1 > 1$), duality implies that the NS1 strings split into $n_1 n_p$ "fractional" strands. These can again be arranged in different combinations, leading to a total number of states corresponding to all possible partitions of $n_1 n_p$:

$$\sum_i m_i k_i = n_1 n_p. \tag{1.6.15}$$

To reproduce the entropy, we count these partitions, where each "multiwound" strand acts as a *component string*. The important property of these component strings is that each possesses four fermionic zero modes from left movers and four from right movers. This follows from a more detailed analysis of the bound states in the *orbifold CFT*.

Quantizing these modes gives four "raising" and four "lowering" operators for both left and right movers. Starting from the ground state (annihilated by all lowering operators), we can choose whether or not to apply each of the four raising operators. This yields $2^4 = 16$ possible ground states per component string. States formed by an even number of raising operators are bosonic, while those with an odd number are fermionic. Hence, each component string with winding k has 8 bosonic and 8 fermionic states.

Counting all possible states of the NS5–NS1 configuration now parallels the counting for the NS1–P system. Partitioning $n_1 n_p$ as above, with 8 bosonic and 8 fermionic states per partition element, the total number of microstates $\mathcal{N}$ is

$$\ln \mathcal{N} = 2\sqrt{2}\pi\sqrt{n_1 n_p}. \tag{1.6.16}$$

Let us revisit the three-charge system under consideration, which consists of n_5 NS5-branes and n_1 NS1-branes. When these branes form a bound state, they collectively behave as an *effective string* with a total winding number given by $n_1 n_5$. This effective string can be decomposed into multiple *component strings*, each characterized by a winding number k_i and appearing with multiplicity m_i. The total winding number is then distributed among these components according to the relation

$$\sum_i m_i k_i = n_1 n_5 \tag{1.6.17}$$

This combinatorial structure reflects the different possible microstates of the system, each corresponding to a distinct way of partitioning the total winding among the component strings. We will focus on a specific subset of states in which all component strings have identical winding numbers k and the same fermionic zero modes. For this case, the total number of component strings is

$$m = \frac{n_1 n_5}{k}. \tag{1.6.18}$$

At one extreme, all strings are singly wound, meaning

$$k = 1, \qquad m = n_1 n_5. \tag{1.6.19}$$

At the opposite extreme, there is just a single component string, so

$$k = n_1 n_5, \qquad m = 1. \tag{1.6.20}$$

Next, we introduce momentum charge P into the system. The NS1–NS5 bound state can take any of the configurations described by equation 1.6.17, and the total momentum n_p can be distributed arbitrarily among the component strings. Each such configuration represents a microstate of the NS1–NS5–P system and contributes to the overall entropy. However, for small n_p, the dominant contribution comes from configurations with fewer component strings (each having larger winding number k).

To illustrate this, consider $n_p = 1$. For the first extreme case ($k = 1$), every component string is singly wound, so there is no fractionalization. We can place a single unit of momentum on any one of these component strings. Since all component strings are identical (each having the same zero modes), exciting one or another gives the same overall state. The resulting state can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{n_1 n_5}}\Big[(\text{component string 1 excited}) + \cdots + (\text{component string } n_1 n_5 \text{ excited})\Big]. \tag{1.6.21}$$

The momentum excitation can occur in 4 bosonic and 4 fermionic modes, yielding 8 possible states for the system.

Now consider the other extreme ($k = n_1 n_5$), where there is only a single component string. Its total winding is $w = n_1 n_5$, so a single unit of momentum corresponds to an excitation at level $n_1 n_5$ on this string. The number of states can then be determined using the partition of this level into harmonic modes, giving

$$\mathcal{N} \sim e^{2\pi\sqrt{\frac{c}{6} n_1 n_5}} = e^{2\pi\sqrt{n_1 n_5}}. \tag{1.6.22}$$

Let us examine the opposite limit where the entire system consists of a single component string. Since this string has a winding number $w = n_1 n_5$, placing just one unit of momentum on it corresponds to an excitation at level $n_1 n_5$. The number of possible states at this excitation level is determined by how the energy is partitioned among different harmonics. This leads to a degeneracy

$$\mathcal{N} \sim e^{2\pi\sqrt{\frac{c}{6}n_1 n_5}} = e^{2\pi\sqrt{n_1 n_5}} \tag{1.6.23}$$

Here, the central charge is taken as $c = 6$, accounting for 4 bosonic and 4 fermionic degrees of freedom. This result yields a significantly larger number of states compared to configurations where the component strings have minimal winding, such as $k_1 = 1$. To compute the leading-order entropy of the NS1–NS5–P system, one can focus on the configuration described by 1.6.20, treating the NS1–NS5 pair as a bound state and neglecting other possible arrangements. The n_p units of momentum are then distributed over this single long string. This effectively creates an excitation level $n_1 n_5 n_p$ and yields an entropy

$$S_{micro} = 2\pi\sqrt{n_1 n_5 n_p} \tag{1.6.24}$$

1.7 The D1-D5 System

For these two-charge configurations, all the fuzzball geometries can be derived from the F1-P geometries by applying appropriate dualities. The total entropy in this case depends on the D1 and D5 charges, Q_1 and Q_5, and is approximately proportional to $\sqrt{Q_1 Q_5}$. Despite this, the corresponding entropies don't match the entropy expected from a classical black hole. Introducing a third charge, such as the momentum P, leads to solutions whose entropy matches the Bekenstein-Hawking value.

In the framework of type-IIB string theory compactified on $T^4 \times S^1$, the D1-D5 system can be modeled effectively. The S^1 direction is labeled by y, and the torus directions by z_α. The configuration includes n_1 D1-branes extended along S^1, and n_5 D5-branes wrapping $T^4 \times S^1$.

The coordinates x_i for $i = 1\dots4$ span the non-compact space. A simplified form of the metric describing this system, using the harmonic superposition principle, is:

$$ds^2 = \frac{1}{\sqrt{H_1 H_5}}(-dt^2 + dy^2) + \sqrt{H_1 H_5}\sum_{i=1}^{4} dx_i dx^i + \sqrt{\frac{H_1}{H_5}}\sum_{\alpha=1}^{4} dz_\alpha dz^\alpha \tag{1.7.1}$$

where H_1 and H_5 are Harmonic functions in the transverse space given by,

$$H_1 = 1 + \frac{Q_1}{r^2}, \qquad H_5 = 1 + \frac{Q_5}{r^2}. \tag{1.7.2}$$

With $r^2 = \sum_i x_i^2$. The integer number of D1, D5, and P branes n_1, n_5, respectively are related to the parameters appearing in the metric as follows,

$$Q_1 = \frac{g_s \alpha'^3}{V} n_1, \quad Q_5 = g_s \alpha' n_5 \tag{1.7.3}$$

where g_s is string coupling, volume of the torus T^4 is $(2\pi)^4 V$ and in terms of string length l_s, $\alpha' = l_s^2$ is the parameter that defines the tension on a fundamental string which is $T = 1/2\pi\alpha'$. The metric (1.7.1) is associated with a 3-form field strength $F^{(3)}$ and the associated dilaton field Φ is given by,

$$e^{2\Phi} = \frac{H_1}{H_5} \tag{1.7.4}$$

The naive D1-D5 metric has a zero sized horizon at $r = 0$ thus it gives vanishing Bekenstein-Hawking entropy. According to fuzzball proposal this metric is a superposition of the actual D1-D5 microstate geometries of a black hole. These microstate geometries are smooth and horizonless and can be obtained by a set of duality maps (S and T dualities) from the momentum carrying fundamental string (F1-P) solution.

[12]This is the string frame metric.

1.7.1 Obtaining the D1-D5 metric

By applying a set of S,T dualities we can go from F1-P system in Type-IIB string theory to D1-D5 system in type-IIB string theory. By this duality transformations the bound state of a fundamental string wrapped n_5 times around the y-circle and carrying n_1 units of momentum, maps to the bound state of a D1-brane wrapped n_1 times around the y-circle and a D5-brane wrapped n_5 times around $T^4 \times S^1$. The set of S and T dualities are given by[13]:

$$\left|\begin{array}{c} P(y) \\ F1(y) \end{array}\right| \xrightarrow{S} \left|\begin{array}{c} P(y) \\ D1(y) \end{array}\right| \xrightarrow{T_{z\alpha}} \left|\begin{array}{c} P(y) \\ D5(y_{z_\alpha}) \end{array}\right| \xrightarrow{S} \left|\begin{array}{c} P(y) \\ NS5(y_{z_\alpha}) \end{array}\right| \xrightarrow{T_{yz_1}} \left|\begin{array}{c} F1(y) \\ NS5(y_{z_\alpha}) \end{array}\right| \xrightarrow{S} \left|\begin{array}{c} D1(y) \\ D5(y_{z_\alpha}) \end{array}\right| \tag{1.7.5}$$

Duality transformation on F1-P solution of heterotic string theory maps to D1-D5 solution of type on Kähler manifold $K3$.

1.7.2 F1-P Solution

The fundamental string solution was previously discussed in Section 1.4.2. To extend this setup, we now introduce n_p units of left-moving momentum along the compact y direction. This is achieved by applying a Lorentz boost along y, which modifies the original metric 1.4.9 into the form

$$ds^2_{\text{string}} = H_1^{-1}\left[-dt^2 + dy^2 + K(dt-dy)^2\right] + \sum_{i=1}^{8} dx^i dx^i \tag{1.7.6}$$

where the function K encodes the momentum contribution and is given by:

$$K = \frac{Q_p}{r^6} \tag{1.7.7}$$

[13]The expression inside the bracket '()' indicates the direction along which the respective brane or string is extended.

Rewriting the metric in terms of null coordinates $u = t - y$, $v = t + y$, we obtain

$$ds^2_{\text{string}} = H_1^{-1}\left[-du\,dv + K\,dv^2\right] + \sum_{i=1}^{8} dx^i dx^i \tag{1.7.8}$$

This boosted geometry describes the F1–P system and features a horizon at $r = 0$, though the horizon area vanishes. The microstate geometries for this configuration were constructed by Dabholkar and Harvey [52]. These solutions are singular at the string source location $r = 0$, but they do not possess a classical horizon. However, when higher-derivative corrections are included, horizons can emerge.

The F1–P solutions are generated using the Garfinkle–Vachaspati transform [50], applied to the fundamental string background described by metric 1.4.9. In null coordinates, this background can be expressed as

$$ds^2 = -e^{2\phi}du\,dv + \sum_{i=1}^{8} dx^i dx^i \tag{1.7.9}$$

The background solution includes components of the NS-NS two-form field and the dilaton, given by:

$$B_{uv} = \frac{1}{2}\left(e^{2\phi} - 1\right), \quad \text{with} \quad e^{2\phi} = 1 + \frac{Q}{r^6} \tag{1.7.10}$$

This geometry possesses translational symmetry along the null directions ∂_u and ∂_v, corresponding to Killing vectors in those directions. To introduce traveling wave deformations into such a background, one employs the Garfinkle–Vachaspati (GV) transformation.

The GV transformation modifies the background metric $g_{\mu\nu}$ as follows:

$$g'_{\mu\nu} = g_{\mu\nu} + e^S k_\mu k_\nu \tag{1.7.11}$$

Here, k_μ is a null Killing vector of the background geometry that is also hypersurface orthogonal

$$\nabla_{[\mu}k_{\nu]} = k_{[\mu}\nabla_{\nu]}S. \tag{1.7.12}$$

satisfying This means the covector k_μ is proportional to the gradient of some scalar function,i.e, $\nabla_\mu S$. The scalar function S satisfies massless scalar field equation in undeformed background

$$\Box\Psi = 0 \tag{1.7.13}$$

This formalism allows one to construct new solutions by embedding traveling wave profiles into backgrounds with null symmetries, such as the F1–P system.
We apply this method to the fundamental string solution 1.7.9. We choose $k^\mu = \partial_\mu$. Lowering the index using undeformed background metric the nonzero component of k_μ becomes

$$k_v = g_{vu}k^u = -\frac{1}{2}e^{-2\phi}. \tag{1.7.14}$$

Applying the Garfinkle–Vachaspati (GV) transformation to the background geometry, the metric is deformed as:

$$g_{\mu\nu} \to g_{\mu\nu} + e^{-2\phi}T'(v,x^i)\,k_\mu k_\nu \tag{1.7.15}$$

where the scalar function used in the GV formalism is $S = -2\phi$. Substituting the explicit form of the null Killing vector k_μ from 1.7.14, the deformation simplifies to

$$g_{vv} \to g_{vv} + e^{-2\phi}T'(v,x^i)\,k_v^2 \tag{1.7.16}$$

This leads to the transformed metric

$$ds^2 = -e^{2\phi}du\,dv + \frac{1}{4}e^{2\phi}T'(v,x^i)\,dv^2 + \sum_{i=1}^{8} dx^i dx^i \tag{1.7.17}$$

Alternatively, by redefining the deformation profile as $T(v,x^i) = -\frac{1}{4}T'(v,x^i)$, the metric can be written as

$$ds^2 = -e^{2\phi}\left(du\,dv - T(v,x^i)\,dv^2\right) + dx^i dx^i \tag{1.7.18}$$

The deformation function $T(v,x^i)$ introduced via the Garfinkle–Vachaspati transformation must satisfy the wave equation in the background geometry $\partial^2 T = 0$ where derivatives are taken with respect to the eight transverse spatial directions x^i. The presence of the dv^2 term in the metric indicates a left-moving traveling wave on the fundamental string background.

To incorporate right-moving excitations, one can similarly choose the null Killing vector $k^v = 1$, allowing symmetric construction of right-moving deformations.

Since $T(v, x^i)$ is harmonic in the transverse space, it can be expanded in terms of spherical harmonics on the 8-dimensional transverse space. Retaining only the physically relevant modes associated with string sources, the function takes the form

$$T(v, \vec{x}) = \vec{f(v)} \cdot \vec{x} + \frac{p(v)}{r^6} \tag{1.7.19}$$

Here, the second term represents a gravitational wave component not localized on the string and is therefore discarded in constructing oscillating string solutions. We retain only the first term.

However, the linear dependence on x^i implies that the deformation does not vanish at spatial infinity. Consequently, the geometry is not asymptotically flat. To restore asymptotic flatness, one must perform a suitable set of diffeomorphisms that reframe the solution 1.7.19 into a physically acceptable form.

To restore asymptotic flatness, we perform the following coordinate transformation

$$v = v' \tag{1.7.20}$$

$$u = u' - 2\dot{\vec{F}} \cdot \vec{x}' + 2\dot{\vec{F}} \cdot \vec{F} - \int^{v'} \dot{\vec{F}}^2 dv \tag{1.7.21}$$

$$\vec{x} = \vec{x}' - \vec{F(v)}. \tag{1.7.22}$$

where the wave profile function $f(v)$ is related by $f(v) = -2\ddot{F}(v)$. This function $F(v)$ characterizes the shape of the traveling wave along the string.

After applying this diffeomorphism, the metric in the new coordinates (u',v',x') becomes

$$ds^2 = -e^{2\phi}du'dv' - (e^{2\phi}-1)F^2dv'^2 + 2(e^{2\phi}-1)\dot{\vec{F}}\cdot d\vec{x}'dv' + d\vec{x}'\cdot d\vec{x}' \qquad (1.7.23)$$

Relabeling the coordinates $(u',v',x') \to (u,v,x)$, the metric simplifies to

$$ds^2 = -e^{2\phi}du\,dv - (e^{2\phi}-1)\dot{F}^2dv^2 + 2(e^{2\phi}-1)\dot{\vec{F}}\cdot d\vec{x}dv + d\vec{x}\cdot d\vec{x} \qquad (1.7.24)$$

The associated NS-NS field components are

$$B_{uv} = \frac{1}{2}(e^{2\phi}-1), \quad B_{vi} = \dot{F}_i(e^{2\phi}-1), \quad e^{-2\phi} = 1 + \frac{Q}{|\vec{x}-\vec{F}(v)|^6} \qquad (1.7.25)$$

The condition $\vec{x} = \vec{F}(v)$ identifies the location of the string in the transverse space, where the vector function $\vec{F}(v) = (F_1(v),F_2(v),F_3(v),F_4(v))$ describes the oscillation profile of the string. On this surface, the dilaton field satisfies:

$$e^{2\phi} = 0 \quad \text{at} \quad \vec{x} = \vec{F}(v) \qquad (1.7.26)$$

To express the geometry compactly, we define the following functions

$$A_i = \frac{Q\dot{F}_i(v)}{|\vec{x}-\vec{F}(v)|^6} \qquad (1.7.27)$$

$$K = \frac{Q\dot{F}^2(v)}{|\vec{x}-\vec{F}(v)|^6} \qquad (1.7.28)$$

$$H^{-1} = 1 + \frac{Q}{|\vec{x}-\vec{F}(v)|^6} \qquad (1.7.29)$$

With these identifications, the metric takes the form

$$ds^2 = H\left(-du\,dv + K\,dv^2 + 2A_i\,dx^i\,dv\right) + \sum_{i=1}^{8} dx^i dx^i \qquad (1.7.30)$$

The field components associated with the F1–P background take the form:

$$B_{uv} = -G_{uv} = H^{-1}, \quad B_{ui} = -G_{ui} = -HA_i, \quad e^{2\Phi} = H^{-1} \tag{1.7.31}$$

This configuration corresponds to the chiral null model, a class of solutions where the functions H and K are harmonic in the transverse space and satisfy linear wave equations. These functions depend on the string's vibration profile $\vec{F}(v)$, and one can construct more general solutions by superposing contributions from multiple strings, each with its own profile. For a system of m strings, each vibrating with a distinct profile $\vec{F}_i(v)$, the harmonic functions are given by

$$H^{-1} = 1 + \sum_m \frac{Q}{|\vec{x} - \vec{F}_m|^2} \tag{1.7.32}$$

$$K = \sum_m \frac{Q\dot{F}_m^2}{|\vec{x} - \vec{F}_m|^2} \tag{1.7.33}$$

$$A_i = \sum_m \frac{Q\dot{F}_{mi}}{|\vec{x} - \vec{F}_m|^2} \tag{1.7.34}$$

To reinterpret this solution in the D1–D5 frame, one typically compactifies the transverse directions on a four-torus T^4. The metric then becomes

$$ds^2 = H\left(-du\,dv + K\,du^2 + 2A_i\,dx^i\,du\right) + \delta_{ij}dx^i dx^j + dz_a dz_a, \quad a = 1, \ldots, 4 \tag{1.7.35}$$

Here, the compact directions z_a have periodicities $z_a \sim z_a + 2\pi R_a$, and the harmonic functions are taken to be uniform over the torus. Assuming the string vibrations occur only in the noncompact directions, the harmonic functions retain the same form

$$H^{-1} = 1 + \sum_m \frac{Q}{|\vec{x} - \vec{F}_m|^2}, \quad K = \sum_m \frac{Q\dot{F}_m^2}{|\vec{x} - \vec{F}_m|^2}, \quad A_i = \sum_m \frac{Q\dot{F}_{mi}}{|\vec{x} - \vec{F}_m|^2} \tag{1.7.36}$$

The one-form A_i, where $i = 1, \ldots,$ can be interpreted as a $U(1)$ gauge potential over the transverse space. Its associated field strength tensor is defined as $F_{ij} = \partial_i A_j - \partial_j A_i$. The functions H^{-1}, K, and F_{ij} are harmonic and satisfy the Laplace equation in the transverse directions

$$\partial^2 H^{-1} = 0, \quad \partial^2 K = 0, \quad \partial^2 F_{ij} = 0$$

In the case of a multiply wound fundamental string wrapped along the compact y direction with a large winding number n_5, the individual strands can be treated as densely packed. This approximation allows the configuration to be modeled as a continuous distribution of sources, simplifying the construction of supergravity solutions.

The harmonic functions for a multiply wound fundamental string can be expressed as integrals along its profile

$$H^{-1} = 1 + \frac{Q}{L}\int_0^L \frac{dv}{|\vec{x}-\vec{F}(v)|^2}, \quad K = \frac{Q}{L}\int_0^L \frac{dv\,\dot{\vec{F}}^2(v)}{|\vec{x}-\vec{F}(v)|^2}, \quad A_i = -\frac{Q}{L}\int_0^L \frac{\dot{F}_i(v)\,dv}{|\vec{x}-\vec{F}(v)|^2} \tag{1.7.37}$$

Here, $L = 2\pi n_5 R$ denotes the total length of the string, wrapped n_5 times around the compact y circle of radius R. This setup yields the locally matched F1–P solution in type IIB supergravity.

We apply a sequence of S- and T-duality transformations given by 1.7.5. First, we start with NS1-P and reach NS5-NS1. The explicit map is

$$\begin{pmatrix} g_s \\ Q_1 \\ R \\ R_6 \\ V \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \frac{1}{g_s} \\ \frac{Q_1}{g_s} \\ \frac{R}{\sqrt{g_s}} \\ \frac{R_6}{\sqrt{g_s}} \\ \frac{V}{g_s^2} \end{pmatrix} \xrightarrow{T_{6789}} \begin{pmatrix} \frac{g_s}{V} \\ \frac{Q_1}{g_s} \\ \frac{R}{\sqrt{g_s}} \\ \frac{\sqrt{g_s}}{R_6} \\ \frac{g_s^2}{V} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \frac{V}{g_s} \\ \frac{Q_1 V}{g_s^2} \\ \frac{R\sqrt{V}}{g_s} \\ \frac{\sqrt{V}}{R_6} \\ V \end{pmatrix} \xrightarrow{T_{56}} \begin{pmatrix} \frac{R_6}{R} \\ \frac{Q_1 V}{g_s^2} \\ \frac{g_s}{R\sqrt{V}} \\ \frac{R_6}{\sqrt{V}} \\ R_6^2 \end{pmatrix} \tag{1.7.38}$$

A final S-duality takes the NS5–NS1 to D5–D1:

$$\begin{pmatrix} \frac{R_6}{R} \\ \frac{Q_1 V}{g_s^2} \\ \frac{g_s}{R\sqrt{V}} \\ \frac{R_6}{\sqrt{V}} \\ R_6^2 \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \frac{R}{R_6} \\ \frac{Q_1 V R}{g_s^2 R_6} \\ \frac{g_s}{\sqrt{R R_6 V}} \\ \sqrt{\frac{R_6 R}{V}} \\ R^2 \end{pmatrix} \equiv \begin{pmatrix} g_s' \\ Q_5' \\ R' \\ R_6' \\ V' \end{pmatrix} \tag{1.7.39}$$

This sequence shows how the NS1 charge Q_1 becomes the D1 charge, and the NS5 charge Q_5 maps to the D5 charge after duality. At each stage, coordinates are chosen to ensure the metric remains asymptotically flat. While T-duality leaves the string-frame metric invariant due to conformal scaling, S-duality requires a rescaling of coordinates to preserve the asymptotic structure.

In the NS1–P frame, the harmonic function sourced by the NS1 branes behaves at large r as

$$H^{-1} \approx 1 + \frac{Q_1}{r^2} \tag{1.7.40}$$

After applying dualities to reach the D1–D5 frame, the harmonic function transforms to

$$H^{-1} \approx 1 + \frac{Q_5'}{r^2} \tag{1.7.41}$$

where $Q_5' = \mu^2 Q_1$ with $\mu^2 = \frac{VR}{g_s^2 R_6}$. We also note

$$Q_5' = \mu^2 Q_1 = \mu^2 \frac{g_s^2 n_1}{V} = g_s' n_1 \tag{1.7.42}$$

This solution maps to the D1–D5 frame, where the metric takes the form

$$ds^2 = \sqrt{\frac{H}{1+K}} \left[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + \sqrt{\frac{1+K}{H}} dx^i dx^i + \sqrt{H(1+K)} dz_a dz_a \tag{1.7.43}$$

The associated fields in this frame are

$$C^{(2)} = \frac{1}{1+K}\left[-dt\wedge B_i dx^i + dy\wedge A_i dx^i\right] + \frac{K}{1+K} dt\wedge dy + C_{ij} dx^i \wedge dx^j \tag{1.7.44}$$

where

$$dB = -\star_4 dA, \quad dC = -\star_4 dH^{-1} \tag{1.7.45}$$

1.8 An example: rotating NS1-P→ rotating D1-D5 system

Start with a NS1-P solution describing a profile for a rotating string

$$F_1 = a\cos\omega v, \quad F_2 = a\sin\omega v, \quad F_3 = 0, \quad F_4 = 0. \tag{1.8.1}$$

where a is a constant. The NS1 profile looks like an uniform helix with rotation on $x1-x2$ plane and the axis of the helix being the S^1 direction y. We excite the lowest harmonic on the string by choosing

$$\omega = \frac{1}{n_1 R} \tag{1.8.2}$$

This makes the NS1 to have only one turn of helix in the covering space. The integrals can be computed by going to the polar coordinates

$$\begin{aligned} x^1 &= \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi}, \quad x^2 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi} \\ x^3 &= \tilde{r}\cos\tilde{\theta}\cos\tilde{\psi}, \quad x^4 = \tilde{r}\cos\tilde{\theta}\cos\tilde{\psi} \end{aligned} \tag{1.8.3}$$

And again, another set of transformations

$$\tilde{r} = \sqrt{r^2 + a^2\sin\theta^2}, \quad \cos\tilde{\theta} = \frac{\cos\theta}{\sqrt{r^2 + a^2\sin\theta^2}} \tag{1.8.4}$$

This gives the harmonic functions

$$H^{-1} = 1 + \frac{Q_1}{r^2 + a^2\cos\theta^2}, \quad K = \frac{a^2}{n_1^2 R^2}\frac{Q_1}{r^2 + a^2\cos\theta^2} \tag{1.8.5}$$

After the sequence of dualities given by 1.7.5, we recast this system into D1-D5 system with harmonic functions

$$H'^{-1} = 1 + \frac{Q_5'}{f}, \quad K' = \mu^2\frac{Q_p}{f} = \frac{Q_1'}{f} \tag{1.8.6}$$

with $f = r^2 + a^2\cos\theta^2$ and Q_1' and Q_5' are D1 and D5 charges respectively. The constant a is given as

$$a = \frac{\sqrt{Q_1' Q_5'}}{R'} \tag{1.8.7}$$

where R' is the final radius of the y circle at the end of the sequence of dualities as given by 1.7.39. Removing the primes and using $\tilde{\psi} \equiv \psi, \tilde{\phi} \equiv \phi$, we write the D1-D5 solution as

$$\begin{aligned} ds^2 &= -\frac{1}{h}(-dt^2 + dy^2) + hf(d\theta^2 + \frac{dr^2}{r^2 + a^2}) - \frac{2a\sqrt{Q_1Q_5}}{hf}(\cos\theta^2 dy d\psi + \sin\theta^2 dt d\phi) \\ &+ h[(r^2 + \frac{a^2 Q_1 Q_5 \cos\theta^2}{h^2 f^2})\cos\theta^2 d\psi^2 + (r^2 + a^2 - \frac{a^2 Q_1 Q_5 \cos\theta^2}{h^2 f^2})\sin\theta^2 d\phi^2] + \sqrt{\frac{Q_1 + f}{Q_5 + f}} dz_a dz_a \\ A_\phi &= -a\sqrt{Q_1Q_5}\frac{\sin\theta^2}{f} \end{aligned} \tag{1.8.8}$$

with $h = \sqrt{(1 + \frac{Q_1}{f})(1 + \frac{Q_5}{f})}$.

1.9 The Geometry

Let's look at the metric from 1.8.8. At large r this metric goes over to flat space. Now, consider the opposite limit $r << (Q_1Q_5)^{1/4}$. Writing $r' = \frac{r}{a}$

$$\begin{aligned} ds^2 &= -(r'^2+1)\frac{a^2dt^2}{\sqrt{Q_1Q_5}} + r'^2\frac{a^2dy^2}{\sqrt{Q_1Q_5}} + \sqrt{Q_1Q_5}\frac{dr'^2}{r'^2+1} \\ &+ \sqrt{Q_1Q_5}[d\theta^2 + \cos\theta^2(d\psi - \frac{ady}{\sqrt{Q_1Q_5}})^2 + \sin\theta^2(d\phi - \frac{adt}{\sqrt{Q_1Q_5}})^2)] \\ &+ \sqrt{\frac{Q_1}{Q_5}}dz_adz_a \end{aligned} \tag{1.9.1}$$

Transforming to new angular coordinates

$$\psi' = \psi - \frac{a}{\sqrt{Q_1Q_5}}y, \quad \phi' = \phi - \frac{a}{\sqrt{Q_1Q_5}}t \tag{1.9.2}$$

the metric 1.9.1 becomes

$$\begin{aligned} ds^2 &= \sqrt{Q_1Q_5}[-(r'^2+1)\frac{dt^2}{R^2} + r'^2\frac{dy^2}{R^2} + \frac{dr'^2}{r'^2+1}] \\ &+ \sqrt{Q_1Q_5}[d\theta^2 + \cos\theta^2 d\psi'^2 + \sin\theta^2 d\phi'^2] + \sqrt{\frac{Q_1}{Q_5}}dz_adz_a \end{aligned} \tag{1.9.3}$$

The geometry corresponds to $\mathrm{AdS}_3 \times S^3 \times T^4$. At large radial distances, the spacetime asymptotically approaches flat space. As one moves inward, the geometry develops a throat-like region that closely resembles the naive supergravity solution. However, instead of ending in a singularity at $r = 0$, the geometry smoothly caps off, indicating the absence of a horizon or singular core.

Different regions of the geometry

The region $r >> a$ corresponds to the outer region of the geometry. Here, the geometry looks asymptotically flat. On the other hand, for $r << \sqrt{Q}$, the geometry locally looks like

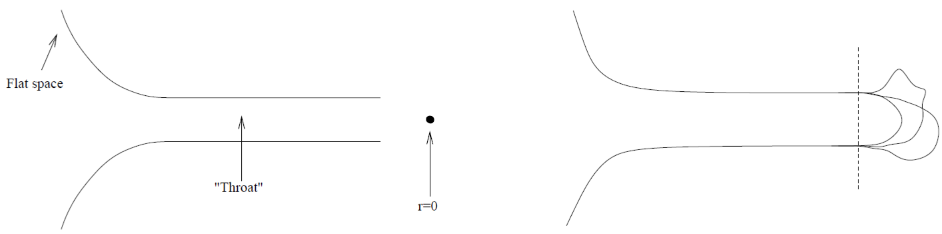

Figure 1.1: The left image shows naive D1-D5 geometry where the right one shows actual fuzzball geometries. The dashed line denotes the location of cap $r \sim a$ where the microstaes differ.

$AdS_3 \times S^3 \times T^4$. For large charge configuration $R_{AdS}^2 = \sqrt{Q_1 Q_5} \sim g_s \alpha' \sqrt{n_1 n_5 \frac{V}{R_y^2}}$, which is macroscopically large in the string units, whereas the profile radius is set by the microscopic vibration scale of the string, often comparable to $\sqrt{\alpha'}$ [94]. This establishes a scale hierarchy $a << R_{AdS}$.

From the AdS_3 metric, $g_{rr} = \frac{\sqrt{Q_1 Q_5}}{r^2} = \frac{R_{AdS}^2}{r^2}$. So, the typical length of the throat is

$$L_{throat} = \int_a^{R_{AdS}} \sqrt{g_{rr}} dr = R_{AdS} ln(R_{AdS}/a) \tag{1.9.4}$$

So, for $R_{AdS} >> a$, i.e, $\sqrt{Q_1 Q_5} >> a$, we have a “long” AdS throat.

We define a parameter $\varepsilon \equiv \frac{\sqrt{Q}}{R_y} = \frac{a}{\sqrt{Q}}$ which is a measure of the shape of the geometry. For $\varepsilon << 1$, the geometry develops a long AdS throat where there can be several units of AdS radius along the radial length of the throat. In the decoupling limit $\varepsilon \to 0$ the outer asymptotically flat spacetime separates from the inner “cap($r \sim a$)+ throat” region of the geometry. The geometry of the cap region depends on the profile function $\vec{F}(v)$, i.e, on the particular microstate.

1.10 Outline of the thesis

The thesis is organised as follows. In chapter 3, I’ll discuss my first work which is on construction of hair modes in $AdS_3 \times S^3$ and relating them to to the hair modes of the BMPV black holes in six dimensions.

Chapter 2 will provide some motivation for that. In chapter five, I discuss my second work, which is on small black rings and gravitational index. The chapters 4 and initial part of 5 will discuss some necessary background material for that.

Chapter 2

Black Holes and Hair

2.1 Macroscopic Entropy

Classically, black hole entropy is purely a horizon dependent quantity. For two derivative theories, black hole entropy is given by Bekenstein-Hawking entropy

$$S_{BH} = \frac{\mathscr{A}}{4} \tag{2.1.1}$$

where $\mathscr{A}$ is the area of the event horizon.

In presence of higher derivative terms in action, this entropy formula changes and the relevant measure is given by the Wald entropy in that case. For two derivative theories, Wald entropy reduces to the Bekenstein-Hawking entropy.

2.2 Index: the microscopic counterpart

While we are interested in counting state of clack holes, this is a very hard problem. Instead, Witten introduced an index [12], which allows one to interpolate between weak and strong coupling due to protections ensured by supersymmetry. The Witten index is defined as

$$\mathscr{I} = Tr[(-1)^F e^{-\beta H}] \tag{2.2.1}$$

where F is the fermion parity operator giving $+1$ while acting on bosonic states and -1 while acting on fermionic states. Since in supersymmetric multiplets bosons and fermions appear as pairs at non-zero energy levels, Witten index gets non-zero contribution only from the ground states, which have zero energy and so, preserve supersymmetry. These states don't need to appear as pairs as acting one state with supersymmetry generator will simply annihilate the state rather than giving some other partner state. Hence, these ground states contribute nonzero values to the index.

$$\mathscr{I} = n_b^0 - n_f^0 \tag{2.2.2}$$

where n_b^0 and n_f^0 denote the numbers of supersymmetric bosonic and fermionic ground states respectively. Another important property of the Witten index is that it is preserved under the change of coupling. Since the multiplet strucure typically doesn't change under the change in coupling, the Witten index remains insensitive to coupling.

In case of extended supersymmetry theories, where some of the supersymmetries are broken, all the states appear in some multiplet and there are no unpaired supersymmetric ground states. In his case, the Witten index vanishes. In that case, we define helicity trace indices. In case of 4n broken supersymmetries in 4d, the relevant helicity trace index is [1, 5, 14]

$$B_{2n} = \frac{1}{2n!} Tr[(-1)^F e^{-\beta H} (2J_3)^n] \tag{2.2.3}$$

where J_3 is the third component of angular momentum under any $SU(2)$ rotation.

In calculating microscopic state counts in case of supersymmetric black holes, we typically count some version of index. Now one might argue that the index theoretically is not the same as entropy as it counts the bosonic and fermionic staes with different weights. But in particular cases of black holes carrying large charges, the index and entropy matches up to very high accuracy.

2.3 The 4D-5D entropy puzzle and its resolution

String theory has been very successful in computing the index/degeneracy for a class of supersymmetric black holes [15, 16]. In some cases, exact counting formulas are known for helicity trace indices in terms of certain modular forms.

A concrete proposal for computing the same from macroscopic(i.e, bulk gravitational theory side) is to do a string path integral in the near horizon geometry of the black hole [18]. An year after this proposal was made, a puzzle regarding this was outlined along with a possible resolution in [20, 21]. The puzzle goes as follows: the Breckenridge-Myers-Peet-Vafa (BMPV) black hole [22] in flat space and in Taub-NUT space have identical near-horizon geometry but different microscopic indices. The same near-horizon geometry cannot account for different microscopic answers.

The difference is attributed to the presence of black hole hair modes: smooth, normalizable, bosonic and fermionic degrees of freedom living outside the horizon. In classical general relativity, there are several no hair theorems that prohibits the existence of such modes. But in string theory, they can exist. Some of these extra degrees of freedom were identified and constructed in [20,21]. These computations were extended to a wide class of other models in [23] where the need to include both the twisted and untwisted sector hair modes was identified and many gaps in the earlier calculations were filled. See also [24].

Chapter 3

Hair Modes on $AdS_3 \times S^3$ and Relation with Black Hole Hair

3.1 Review: black holes, near horizon geometry, and bosonic hair modes

In this section, we review the construction of bosonic hair modes for $J_L = 0$ BMPV black hole in flat space and Taub-NUT space [20, 21, 23].

Black hole in flat space

In 5 dimensions, the spatial rotation group is $SO(4) \simeq SU(2)_L \times SU(2)_R$. The supersymmetric black holes are allowerd to carry only one $SU(2)$ angular momentum, which here we take to be $SU(2)_L$ and denote the corresponding angular momentum as J_L. The rotating,i.e, $J_L \neq 0$ BMPV black hole metric takes the form

$$
\begin{aligned}
ds^2 &= G_{MN} dx^M dx^N && (3.1.1) \\
&= \psi^{-1}(r)\left[du\,dv + (\psi(r)-1)dv^2 - 2\zeta dv\right] + \psi(r) ds^2_{\text{flat}}, && (3.1.2)
\end{aligned}
$$

where

$$
u = x^5 - t, \qquad v = x^5 + t \qquad \text{and} \qquad \zeta = -\frac{\tilde{J}}{8r}(dx^4 + \cos\theta d\phi). \qquad (3.1.3)
$$

The $J_L = 0$ BMPV black hole [22] metric uplifted to six-dimensions takes the form,

$$\begin{aligned} ds^2 &= G_{MN} dx^M dx^N && (3.1.4)\\ &= \psi^{-1}(r)\left[du\,dv + (\psi(r)-1)dv^2\right] + \psi(r) ds^2_{\text{flat}}, && (3.1.5)\end{aligned}$$

Here x^5 is the coordinate on the sixth dimension, a Kaluza-Klein circle denoted S^1 with period $2\pi R_5$. The function $\psi(r)$ appearing in the metric is

$$\psi(r) = 1 + \frac{r_0}{r}, \tag{3.1.6}$$

with r_0 related to the charges carried by the black hole. The coordinate r is the Gibbons-Hawking radial coordinate on 4D flat space introduced as follows. The spherical polar coordinates for four-dimensional Euclidean flat space $(\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{\psi})$, in which the metric takes the form,

$$ds^2_{\text{flat}} = d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \cos^2\tilde{\theta} d\tilde{\phi}^2 + \sin^2\tilde{\theta} d\tilde{\psi}^2), \tag{3.1.7}$$

are related to cartesian coordinates as,

$$w^1 = \tilde{r}\cos\tilde{\theta}\cos\tilde{\phi}, \qquad w^2 = \tilde{r}\cos\tilde{\theta}\sin\tilde{\phi}, \tag{3.1.8}$$

$$w^3 = \tilde{r}\sin\tilde{\theta}\cos\tilde{\psi}, \qquad w^4 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\psi}. \tag{3.1.9}$$

The Gibbons-Hawking coordinates (r, θ, ϕ, x^4) are defined via,

$$\tilde{r} = 2\sqrt{r}, \qquad \tilde{\theta} = \frac{\theta}{2}, \tag{3.1.10}$$

$$\tilde{\phi} = \frac{1}{2}(x^4 + \phi), \qquad \tilde{\psi} = \frac{1}{2}(x^4 - \phi). \tag{3.1.11}$$

In these coordinates flat space metric takes the form

$$ds^2_{\text{flat}} = \frac{1}{r}dr^2 + r(d\theta^2 + \sin^2\theta d\phi^2 + (dx^4 + \cos\theta d\phi)^2). \tag{3.1.12}$$

For details on the identifications of the angular coordinates, we refer the reader to refs. [21,23].

The other fields supporting the solutions are as follows. The six-dimensional dilaton is kept fixed at its constant asymptotic value throughout the spacetime,

$$e^{\Phi} = \lambda. \tag{3.1.13}$$

The self-dual three-form Ramond-Ramond (RR) field $F^{(3)}$ is only other nontrivial field

$$F^{(3)} = \frac{r_0}{\lambda}\left(\varepsilon_3 + \star_6 \varepsilon_3\right), \tag{3.1.14}$$

where $\varepsilon_3 = \sin\theta\, dx^4 \wedge d\theta \wedge d\phi$ and $\varepsilon^{t54r\theta\phi} = +1$.

Black hole in Taub-NUT space

Next we consider $J_L = 0$ BMPV black hole in Taub-NUT space [46,47], following the notation of refs. [21, 23]. The metric of the four dimensional Taub-NUT space in Gibbons-Hawking coordinates is given as a U(1) fibre over an $\mathbb{R}^3$ base space,

$$ds^2_{TN} = \left(\frac{4}{R_4^2} + \frac{1}{r}\right)^{-1}(dx^4 + \cos\theta d\phi)^2 + \left(\frac{4}{R_4^2} + \frac{1}{r}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{3.1.15}$$

Compared to flat space in Gibbons-Hawking coordinates written above in 3.1.12, the only difference is that the $\frac{1}{r}$ factors are now replaced with $\left(\frac{4}{R_4^2} + \frac{1}{r}\right)$. The x^4 coordinate labels the circle $\widetilde{S}^1$ and it is periodic with size $2\pi R_4$ at infinity.

The metric describing the $J_L = 0$ BMPV black hole in Taub-NUT space takes the form,

$$ds^2 = \psi^{-1}(r)[du\,dv + (\psi(r) - 1)dv^2] + \psi(r) ds^2_{TN}. \tag{3.1.16}$$

The dilaton is set to its asymptotic value $e^{\Phi} = \lambda$. The self-dual field strength $F^{(3)}$ supporting this solution is,

$$F^{(3)} = \frac{r_0}{\lambda}\Big[(\varepsilon_3 + \star_6\varepsilon_3)\Big]. \tag{3.1.17}$$

Near-horizon geometry

It is well known that the two black holes discussed above have the identical near-horizon geometry [20, 48]. For the black hole in flat space, we can take the near-horizon limit by taking the limit $r \to 0$ to get

$$ds^2 = \frac{r}{r_0}\left(dudv + \frac{r_0}{r}dv^2\right) + \frac{r_0}{r^2}dr^2 + r_0\left(d\theta^2 + \sin^2\theta d\phi^2 + (dx^4 + \cos\theta d\phi)^2\right). \tag{3.1.18}$$

This metric describes extremal BTZ $\times$ S^3. To bring the metric in a more standard form, we perform the following coordinate transformation

$$\phi = \tilde{\phi} - \tilde{\psi}\,, \qquad \theta = 2\tilde{\theta}\,, \qquad x^4 = \tilde{\phi} + \tilde{\psi}\,, \qquad \frac{r}{r_0} = \frac{\ell^2}{z^2}, \qquad r_0 = \frac{1}{4}\ell^2, \tag{3.1.19}$$

to get

$$ds^2 = \frac{\ell^2}{z^2}\left(du\,dv + dz^2 + \frac{z^2}{\ell^2}dv^2\right) + \ell^2 d\Omega^2_{S^3}, \tag{3.1.20}$$

with

$$d\Omega^2_{S^3} = d\tilde{\theta}^2 + \cos^2\tilde{\theta}d\tilde{\phi}^2 + \sin^2\tilde{\theta}d\tilde{\psi}^2. \tag{3.1.21}$$

Here, ℓ is the AdS_3 length. The boundary of AdS_3 is at $z = 0$. The horizon of the extremal BTZ is at $z \to \infty$.

Metric (3.1.20) can be simplified further by introducing Poincaré coordinates in the AdS_3 part of the geometry. The transformation

$$\bar{v} = e^{\frac{2v}{\ell}}, \qquad \bar{u} = u - \frac{1}{\ell}z^2, \qquad \bar{z} = \sqrt{\frac{2}{\ell}}ze^{\frac{v}{\ell}}. \tag{3.1.22}$$

takes us to the Poincaré coordinates $(\bar{u}, \bar{v}, \bar{z})$.[1] The final metric takes the form

$$ds^2 = \frac{\ell^2}{\bar{z}^2}\left(d\bar{u}\,d\bar{v} + d\bar{z}^2\right) + \ell^2 d\Omega^2_{S^3}. \tag{3.1.23}$$

The periodic identifications of u and v coordinates are not so natural in the Poincaré coordinates $(\bar{u}, \bar{v}, \bar{z})$. The horizon is located at $\bar{z} \to \infty$. For local analyses (such as regularity of the modes), Poincaré coordinates are the most convenient to work with. For identifying modes in AdS_3 with the hair modes of the black holes, (u, v, z) are the most convenient coordinates to work with.

Bosonic hair modes on black hole in flat space

Now we consider the hair modes [21, 23]. The solution-generating technique of Garfinkle and Vachaspati [50, 51] was used to add hair modes to the above systems. Given a space-time metric G_{MN} with null Killing vector k_M that is hypersurface orthogonal, i.e.,

$$\nabla_{[M}k_{N]} = k_{[M}\nabla_{N]}A \tag{3.1.24}$$

for some function A, new exact solutions to the supergravity equations are constructed by the following transform,

$$G'_{MN} = G_{MN} + e^A\, T\, k_M\, k_N, \tag{3.1.25}$$

and matter fields remain unchanged. The transformed metric is a valid solution if T satisfies,

$$\Box T = 0 \qquad \text{and} \qquad k^M \partial_M T = 0, \tag{3.1.26}$$

where $\Box$ is d'Alembertian with respect to the background metric G_{MN}.

[1] This coordinate transformation is a special case of Banados, Chamblin, and Gibbons [49] transformations. They are discussed in more detail in section 3.2.1.

The BMPV black string in six-dimensions possesses such a null Killing vector,

$$k^M \partial_M = \frac{\partial}{\partial u}, \tag{3.1.27}$$

with

$$e^A = \psi. \tag{3.1.28}$$

Application of the Garfinkle-Vachaspati transform gets [23]

$$ds^2 = \psi^{-1} \left[du\,dv + (\psi - 1 + T(v,\vec{w}))\,dv^2\right] + \psi\, ds^2_{\text{flat}}. \tag{3.1.29}$$

We demand regularity at infinity and at the origin. We keep only the terms that cannot be removed by coordinate transformations [52] and choose

$$T(v,\vec{w}) = f_i(v) w^i, \qquad \int_0^{2\pi R_5} f_i(v) dv = 0, \tag{3.1.30}$$

with four arbitrary functions $f_i(v)$. The deformed metric (3.1.29) is not manifestly asymptotically flat, but can be made so by applying a standard change of coordinates [52]. We can take the near-horizon limit of the deformed metric by taking the $r \to 0$ limit. It gives

$$ds^2 = \frac{\ell^2}{z^2}\left(du dv + dz^2 + \left(\frac{z^2}{\ell^2} + T(z)\right) dv^2\right) + \ell^2 d\Omega^2_{S^3}. \tag{3.1.31}$$

where

$$T(z) = f_i(v) w^i(z). \tag{3.1.32}$$

Bosonic hair modes on black hole in Taub-NUT space

Next we recall the hair mode deformations of the BMPV black hole in Taub-NUT space. A class of these deformations is generated by the Garfinkle-Vachaspati transform. The deformed metric takes the form

$$ds^2 = \psi^{-1}(r)[du\,dv + (\psi(r) - 1 + \widetilde{T}(v,x^4,r,\theta,\phi))\,dv^2] + \psi(r)\,ds^2_{TN}, \tag{3.1.33}$$

where now the condition is that $\widetilde{T}(v,x^4,r,\theta,\phi)$ is a harmonic function on four-dimensional Taub-NUT space. For an x^4 independent function, the condition simply reduces to the function $\widetilde{T}$ being harmonic on the three-dimensional transverse space $\mathbb{R}^3$ spanned by (r,θ,ϕ). As in the BMPV case, requiring the deformation to be regular at the origin and at infinity and dropping terms that can be removed by coordinate transformations, we can choose

$$\widetilde{T}(v,\vec{y}) = g_i(v)y^i, \qquad \int_0^{2\pi R_5} g_i(v)dv = 0, \tag{3.1.34}$$

where y^i are cartesian coordinates on $\mathbb{R}^3$ and $g_i(v)$ are three arbitrary functions. As before, we can take the near-horizon limit of the deformed metric by taking the $r \to 0$ limit.

The BMPV black hole in Taub-NUT space admits another class of deformations [21] sourced by anti-self-dual three-forms. To keep things as simple as possible, in this chapter, we consider only one such three-form to be present. These new deformations arise (essentially) because the Taub-NUT space admits a self-dual harmonic form,

$$\omega_{TN} = -\frac{r}{4r+R_4^2}\sin\theta\,d\theta\wedge d\phi + \frac{R_4^2}{(4r+R_4^2)^2}dr\wedge(dx^4+\cos\theta\,d\phi), \tag{3.1.35}$$

where $\varepsilon_{x^4 r\theta\phi} = +\sqrt{\det g_{TN}}$. Using this two form, a six-dimensional anti-self-dual three-form can be constructed as

$$H^{(3)} = h(v)\,dv\wedge\omega_{TN}, \tag{3.1.36}$$

where $h(v)$ is an arbitrary function of v. The metric sourced by such a form field is given by

$$ds^2 = \psi^{-1}(r)[dudv + (\psi(r) - 1 + S(v,r))dv^2] + \psi(r)\,ds^2_{TN}, \tag{3.1.37}$$

with

$$S(v,r) = -\frac{4r}{R_4^2(4r+R_4^2)} h(v)^2. \tag{3.1.38}$$

As the function $S(v,r)$ does not vanish asymptotically, the deformed metric does not look manifestly asymptotically flat. This can be remedied by shifting u coordinate as [21, 23],

$$u \to u + \frac{1}{R_4^2} \int_0^v h(v')^2 dv'. \tag{3.1.39}$$

Once again, we can take the near-horizon limit of the deformed metric by taking the $r \to 0$ limit.

3.2 Garfinkle-Vachaspati hair modes in the near-horizon region

We start by considering Garfinkle-Vachaspati deformations in the near-horizon region (3.1.20). As we reviewed in the previous section, the details of the Garfinkle-Vachaspati deformations are very different for the BMPV black hole in Taub-NUT space vs in flat space. The aim of this section to understand this difference from the near-horizon perspective.

The Garfinkle-Vachaspati deformation (3.1.25) applied to the near-horizon metric (3.1.20) with $k = \partial_u$ results in,

$$ds^2 = \frac{\ell^2}{z^2}\left(dudv + dz^2 + \left(\frac{z^2}{\ell^2} + H\right) dv^2\right) + \ell^2 d\Omega^2_{S^3}, \tag{3.2.1}$$

where $H(v,z,\tilde{\theta},\tilde{\phi},\tilde{\psi})$ is a harmonic function for the six-dimensional metric (3.1.20). Expanding H in terms of the S^3 spherical harmonics $H(v,z,\tilde{\theta},\tilde{\phi},\tilde{\psi}) = H^I(v,z)Y^I(\tilde{\theta},\tilde{\phi},\tilde{\psi})$ we get equations for functions $H^I(v,z)$,

$$z^3 \partial_z \left(\frac{1}{z}\partial_z H^I(v,z)\right) - L(L+2)H^I(v,z) = 0, \tag{3.2.2}$$

where we have used the fact that the spherical harmonics of S^3 satisfies $\nabla^2 Y^I = -L(L+2)Y^I$, with L taking values in non-negative integers $\mathbb{N}$. Scalar spherical harmonics on S^3 are labeled as $I = L, m_+, m_-$ with $|m_\pm| \leq \frac{L}{2}$ and $\frac{L}{2} - m_\pm \in \mathbb{N}$. Explicitly, these spherical harmonics can be written as [56]

$$Y^I(\tilde{\theta}, \tilde{\phi}, \tilde{\psi}) \propto e^{i(S\tilde{\phi}+D\tilde{\psi})}(1-x)^{\frac{S}{2}}(1+x)^{\frac{D}{2}} P^{(S,D)}_{\frac{L-S-D}{2}}(x), \tag{3.2.3}$$

where $P_n^{(a,b)}$ are the Jacobi polynomials and $x = \cos 2\tilde{\theta}, S = m_+ + m_-, D = m_+ - m_-$. We will see below that the black hole hair modes discussed in the previous section correspond to different L modes from the $AdS_3 \times S^3$ perspective. In Poincaré coordinates $(\bar{u}, \bar{v}, \bar{z})$ introduced in (3.1.22), the deformed metric (3.2.1) takes the form

$$ds^2 = \frac{\ell^2}{\bar{z}^2}\left(d\bar{u}d\bar{v} + d\bar{z}^2 + \bar{H}d\bar{v}^2\right) + \ell^2 d\Omega^2_{S^3}, \tag{3.2.4}$$

where the function $\bar{H}(\bar{v}, \bar{z})$ satisfies the same equation as $H(v, z)$

$$\bar{z}^3 \partial_{\bar{z}} \left(\frac{1}{\bar{z}} \partial_{\bar{z}} \bar{H}^I(\bar{v}, \bar{z})\right) - L(L+2)\bar{H}^I(\bar{v}, \bar{z}) = 0. \tag{3.2.5}$$

3.2.1 $L = 0$ modes

With $L = 0$, equation (3.2.5) simply reduces to the wave equation in AdS_3. Since there are no propagating degrees of freedom in three-dimensional AdS_3 gravity, we expect for (3.2.4) either a trivial solution or a solution which is related to $AdS_3 \times S^3$ by a coordinate transformation. Indeed, this expectation is realized. Solving equation (3.2.2), we get (the index I takes the value $(0,0,0)$ and is dropped for simplicity),

$$\bar{H} = a(\bar{v})\bar{z}^2 + b(\bar{v}). \tag{3.2.6}$$

The term corresponding to $b(\bar{v})$ (non-normalisable mode) can be removed by transforming the coordinate $\bar{u}$,

$$\bar{u} \to \bar{u} - \int^{v} b(\bar{v}')d\bar{v}'. \tag{3.2.7}$$

The term corresponding to $a(\bar{v})$ (normalisable mode) can be removed by a transformation of the type discussed by Banados, Chamblin, and Gibbons [49],

$$\bar{v} \to f(\bar{v}')\,, \qquad \bar{u} \to \bar{u}' - \frac{\bar{z}'^2 f''(\bar{v}')}{2f'(\bar{v}')}\,, \qquad \bar{z} \to \bar{z}'\sqrt{f'(\bar{v}')}, \tag{3.2.8}$$

where $f'(\bar{v}'), f''(\bar{v}')$ are the first and the second derivatives of the function $f(\bar{v}')$ with respect to $\bar{v}'$. These transformations change the metric to

$$ds^2 = \frac{\ell^2}{\bar{z}'^2}\left(d\bar{u}'d\bar{v}' + d\bar{z}'^2\right) + \ell^2 d\Omega^2_{S^3}, \tag{3.2.9}$$

which is the original metric (3.1.23) without the deformation in the new coordinates, provided the function $f(\bar{v}')$ is obtained via solving the equation,

$$a(f(\bar{v}'))f'(\bar{v}')^2 - \frac{1}{2}S\{f(\bar{v}'),\bar{v}'\} = 0, \tag{3.2.10}$$

for the given function $a(\bar{v})$. In this equation, $S\{f(x),x\}$ is the Schwarzian derivative of the function $f(x)$,

$$S\{f(x),x\} = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{3.2.11}$$

3.2.2 $L = 1$ modes

For $L = 1$ equation (3.2.2) becomes,

$$z^3\partial_z\left(\frac{1}{z}\partial_z H^I(z,v)\right) - 3H^I(z,v) = 0. \tag{3.2.12}$$

A general solution to this equation is given by

$$H^I = c^I(v)z^3 + \frac{d^I(v)}{z}. \tag{3.2.13}$$

The normalisable modes $c^I(v)z^3$ are not regular at the horizon $z \to \infty$. The non-normalisable modes $\frac{d^I(v)}{z}$ can be regular at the horizon. The index I takes values $(1, m_+, m_-)$ with $m_+ = \pm\frac{1}{2}, m_- = \pm\frac{1}{2}$. Discarding the normalisable modes, we get,

$$H = \frac{1}{z} \sum_{m_+, m_-} \left(d^{1,m_+,m_-}(v) Y^{1,m_+,m_-} \right). \tag{3.2.14}$$

The spherical harmonics can be evaluated as

$$Y^{1,\pm\frac{1}{2},\pm\frac{1}{2}} \propto \sin\tilde{\theta}\, e^{\pm i\tilde{\phi}}, \tag{3.2.15}$$

$$Y^{1,\pm\frac{1}{2},\mp\frac{1}{2}} \propto \cos\tilde{\theta}\, e^{\pm i\tilde{\psi}}. \tag{3.2.16}$$

From (3.1.19) and (3.1.10) we observe that $\frac{1}{z} \sim \tilde{r}$. Thus, we see that H in equation (3.2.14) correspond to $f_i(v)w^i$ with cartesian coordinates defined w^i defined in (3.1.9). The form of the function $f_i(v)w^i$ is precisely the form that correspond to the Garfinkle-Vachaspati deformations for the BMPV black hole in flat space discussed in the previous section. Thus we conclude that the non-normalisable $L = 1$ modes in the near horizon region correspond to the Garfinkle-Vachaspati deformations for the BMPV black hole in flat space.

Is the deformed metric regular at the horizon $z \to \infty$? In terms of the barred coordinates, the deformed near horizon metric takes the form,

$$ds^2 = \frac{\ell^2}{\bar{z}^2}\left(d\bar{u}d\bar{v} + d\bar{z}^2 + \frac{\bar{f}_i(\bar{v})n^i}{\bar{z}} d\bar{v}^2 \right) + \ell^2 d\Omega^2_{S^3}. \tag{3.2.17}$$

Here $n^i = w^i/|w|$ is a 4-dimensional unit vector. The horizon is at $\bar{z} \to \infty$. The functions $\bar{f}_i(\bar{v})$ are related to the functions $f_i(v)$ though they are not identical. With the coordinate transformation

$$\bar{v} = -\frac{1}{V} , \qquad \bar{z} = \frac{1}{VW} , \qquad \bar{u} = U + \frac{1}{VW^2}, \tag{3.2.18}$$

the deformed metric can be brought to the form,

$$ds^2 = \ell^2 \left(W^2 dU dV + \frac{dW^2}{W^2} + \frac{W^3}{V} \bar{f}_i(\bar{v}(V)) n^i dV^2 \right) + \ell^2 d\Omega^2_{S^3}. \tag{3.2.19}$$

The horizon is now located at $V = 0$. The metric appears singular at $V = 0$. However, as earlier, we can ensure that the g_{VV} component vanishes by a shift in the U coordinate

$$U = \widetilde{U} - WG(V, \tilde{\theta}^i), \tag{3.2.20}$$

with

$$G(V, \tilde{\theta}^i) = \int_0^V \frac{n^i f_i(\bar{v}(V'))}{V'} dV', \tag{3.2.21}$$

Here $\tilde{\theta}^i$ collectively denotes $\tilde{\theta}, \tilde{\phi}, \tilde{\psi}$. Note that the function $G(V, \tilde{\theta}^i)$ is such that in the limit $V \to 0$ it vanishes. Shift (3.2.20) generates the following additional terms in the metric,

$$-\ell^2 W^2 G(V, W, \tilde{\theta}^i) dW dV - \ell^2 W^3 \partial_{\tilde{\theta}_i} G(V, \tilde{\theta}^j) dW d\tilde{\theta}^i. \tag{3.2.22}$$

These additional terms all vanish in the $V \to 0$ limit. The resulting metric becomes smooth near $V = 0$. Yhe V derivatives of the metric are not smooth. This leads to divergences in the Riemann tensor. For example, R_{VWVW} diverges as $1/V$ as $V \to 0$. The conclusion is in line with the fact that these modes are also singular on the BMPV black hole horizon in the full black hole geometry [21].

3.2.3 $L = 2$ modes

Next we consider $L = 2$ modes. For these modes, equation (3.2.2) simplifies to

$$z^3 \partial_z \left(\frac{1}{z} \partial_z H^I(z,v) \right) - 8H^I(z,v) = 0. \tag{3.2.23}$$

The general solution to this equation is given by

$$H^I = c^I(v) z^4 + \frac{d^I(v)}{z^2}. \tag{3.2.24}$$

The black hole horizon is at $z \to \infty$ and hence we select the non-normalisable $\frac{d^I(v)}{z^2}$ part of the solution that seems regular at the horizon.[2] The modes with specific values for the index I can be identified with the Garfinkle-Vachaspati hair modes for the black hole in Taub-NUT space. From (3.1.19) and (3.2.3), we observe that x^4 independent modes correspond to $m_+ = 0$. The allowed values of m_- are $m_- = 0, \pm 1$. Thus, the general x^4 independent deformation takes the form,

$$H = \frac{1}{z^2} \left(d^{(2,0,0)}(v) Y^{(2,0,0)} + d^{(2,0,-1)} Y^{(2,0,-1)} + d^{(2,0,1)} Y^{(2,0,1)} \right). \tag{3.2.25}$$

These spherical harmonics can be evaluated as

$$Y^{(2,0,0)} \propto \cos\theta, \qquad Y^{(2,0,1)} \propto \sin\theta \, e^{i\phi}, \qquad Y^{(2,0,-1)} \propto \sin\theta \, e^{-i\phi}. \tag{3.2.26}$$

We see that the function H in (3.2.25) corresponds to $g_i(v) y^i$ for $i = 1,2,3$ and hence to the Garfinkle-Vachaspati hair modes for the black hole in Taub-NUT space (3.1.34). Indeed, by the taking the $r \to 0$ limit of the GV deformed black hole metric in Taub-NUT we obtain $AdS_3 \times S^3$ metric (3.2.1) with H given as in (3.2.25).

[2] In a different context, the normalisable $c^I(v) z^4$ part of the solution was considered in [57], though in a different coordinate system. Ref. [57] concluded that these modes are not regular at the horizon. The aim of ref. [57] was to find solutions that maintain the self-dual orbifold asymptotics and hence they chose the normalisable modes. We are interested in the black hole hair modes. Therefore, to begin with we choose a solution that seems regular at the horizon.

In (U,V,W) coordinates, cf. (3.2.18), the deformed metric takes the form

$$ds^2 = \ell^2 \left(W^2 dU dV + \frac{dW^2}{W^2} + W^4 g_i(\bar{v}(V)) n^i dV^2 \right) + \ell^2 d\Omega^2_{S^3}. \tag{3.2.27}$$

The functions $\bar{g}_i(\bar{v})$ is related to the functions $g_i(v)$ though they are not identical. A key difference compared to the $L=1$ deformed metric (3.2.19) is that the coefficient of dV^2 terms now goes as W^4. The metric appears regular at $V=0$. Though, there is a catch: as $V\to 0$, $\bar{v}$ coordinate changes rapidly from a finite value to infinity. Thus, it is not obvious in these coordinates if metric (3.2.27) is regular or not at $V=0$. To address this, we can ensure that g_{VV} vanishes by a shift in the U coordinate

$$U = \widetilde{U} - W^2 G(V, \tilde{\theta}^i), \tag{3.2.28}$$

with

$$G(V,\tilde{\theta}^i) = \int_0^V n^i g_i(v(V'))dV'. \tag{3.2.29}$$

The shift generates additional terms

$$-2\ell^2 W^3 G(V,\tilde{\theta}^i) dW dV - \ell^2 W^4 \partial_{\tilde{\theta}^i} G(V,\tilde{\theta}^i) dW d\tilde{\theta}^i. \tag{3.2.30}$$

These additional terms all vanish in the $V\to 0$ limit. The resulting metric becomes smooth near $V=0$, however, V derivatives of the metric are not. Particularly, $\partial_V^2 G$ diverges at $V=0$. These divergences, however, do not appear in the Riemann tensor components.

We conclude that the three functions $g_i(v)$ generate smooth deformations of the near-horizon geometry.[3] This conclusion is in line with the analysis of [21] where it was shown that the three Garfinkle-Vachaspati hair modes for the black hole in Taub-NUT space are also regular at the horizon.

[3]We can now qualify the statement in [57] that $L=2$ modes are not regular in the bulk. We can pick $L=2$ solutions which are, in fact, regular in the bulk but are non-normalisable.

3.3 Form field hair modes in the near horizon region

We now consider a deformation of $AdS_3 \times S^3$ due to the anti-self-dual form field H_{MNP}. The relevant perturbation equations were given in [45]. For our analysis the most relevant equation is the anti-self-duality condition for the perturbing form field with mixed components. This equation takes the form [45, eq. (4.18)]

$$H_{ab\mu} + \frac{1}{2}\varepsilon_{\mu}{}^{\nu\rho}\varepsilon_{ab}{}^{c}H_{c\nu\rho} = 0, \tag{3.3.1}$$

where latin indices $a, b, c, \dots$ stand for the sphere coordinates and greek indices $\mu, \nu, \rho, \dots$ for the AdS coordinates. For all six-dimensions we use $M, N, P, \dots$ indices. Let B_{MN} be the two-form for which H_{MNP} is the field strength, i.e., $H_{MNP} = \partial_{[M}B_{NP]}$.

Consider expanding B_{MN} in spherical harmonics. To identify the relevant black hole hair modes, we do not need the most general decomposition. We only need the decomposition of B_{MN} with one sphere coordinate and one AdS coordinate. The components $B_{\mu a}$ can be expanded in terms of spherical harmonics on S^3 as

$$B_{\mu a} = \sum b^I_\mu Y^I_a, \tag{3.3.2}$$

where Y^I_a are the one-form harmonics. It turns out, we do not even need general properties of one-form harmonics. As it will become shortly, we only need one of the simplest one-form harmonic obtained by lowering the Killing vector [56]

$$\partial_{\tilde{\phi}} + \partial_{\tilde{\psi}}. \tag{3.3.3}$$

In components this one-form is

$$\xi = \cos^2\tilde{\theta}\, d\tilde{\phi} + \sin^2\tilde{\theta}\, d\tilde{\psi}. \tag{3.3.4}$$

This one-form satisfies

$$\bar{\varepsilon}_a{}^{bc}\partial_b\xi_c = -2\xi_a, \tag{3.3.5}$$

where $\bar{\varepsilon}_{abc}$ is the epsilon tensor for the unit-sphere. We use the convention $\bar{\varepsilon}_{\tilde{\theta}\tilde{\phi}\tilde{\psi}} = \sin\tilde{\theta}\cos\tilde{\theta}$. Eq. (3.3.2) now simplifies to

$$B_{\mu a} = b_\mu \xi_a. \tag{3.3.6}$$

Computing H_{MNP} for the B-field of the form (3.3.6) and substituting it in (3.3.1), we get

$$\frac{1}{2}\varepsilon_\mu{}^{\nu\rho}(\partial_\nu b_\rho - \partial_\rho b_\nu) = \frac{1}{\ell}\eta b_\mu, \tag{3.3.7}$$

with $\eta = -2$. For later use we keep η general. To solve equation (3.3.7) we make the ansatz that only the b_v component of the one-form b_μ is non-vanishing.[4] Furthermore, we assume that the perturbation is compatible with ∂_u Killing symmetry, i.e., b_v only depends on the v and z coordinates. Substituting this ansatz in equation (3.3.7), we get

$$z\partial_z b_v(v,z) = \eta b_v(v,z), \qquad \text{where} \qquad \varepsilon_{uvz} = \frac{\ell^3}{2z^3}. \tag{3.3.8}$$

This gives as a solution

$$b_v = c(v)z^\eta, \tag{3.3.9}$$

which in turn gives,

$$B_{v\tilde{\phi}} = \frac{c(v)}{z^2}\cos^2\tilde{\theta}, \qquad\qquad B_{v\tilde{\psi}} = \frac{c(v)}{z^2}\sin^2\tilde{\theta}. \tag{3.3.10}$$

As will shortly become clear, this perturbation describes the anti-self-dual form-field perturbations to the black hole in Taub-NUT space from the near horizon region perspective.

To see this, let us recall that the three-form hair perturbation for the black hole in Taub-

[4] Applying ∇^μ on (3.3.7) we see that $\nabla^\mu b_\mu = 0$, where ∇^μ is the covariant derivative compatible with the three-dimensional AdS_3 part of the metric. We can use this condition to set $b_z = 0$. Next, requiring db to be compatible with the k^M Killing symmetry, we have $\pounds_k db = d(i_k db) = 0$. Choosing $i_k db = 0$ gives us $i_k(\star_3 b) = 0$ from (3.3.7), which implies $\star_3(b \wedge k) = 0$. This condition can be used to set $b_u = 0$.

NUT space takes the form (3.1.36). In the near horizon limit $r \to 0$ the H-field becomes,

$$H = -\frac{h(v)r}{R_4^2} \sin\theta \, dv \wedge d\theta \wedge d\phi + \frac{h(v)}{R_4^2} dv \wedge dr \wedge (dx^4 + \cos\theta d\phi), \tag{3.3.11}$$

which in coordinates $(u, v, z, \tilde{\theta}, \tilde{\phi}, \tilde{\psi})$, takes the form

$$H_{v\tilde{\theta}\tilde{\phi}} = -\frac{\ell^4}{R_4^2 z^2} h(v) \sin\tilde{\theta} \cos\tilde{\theta}, \qquad H_{v\tilde{\theta}\tilde{\psi}} = \frac{\ell^4}{R_4^2 z^2} h(v) \sin\tilde{\theta} \cos\tilde{\theta}, \tag{3.3.12}$$

$$H_{vz\tilde{\phi}} = -\frac{\ell^4}{R_4^2 z^3} h(v) \cos^2\tilde{\theta}, \qquad H_{vz\tilde{\psi}} = -\frac{\ell^4}{R_4^2 z^3} h(v) \sin^2\tilde{\theta}. \tag{3.3.13}$$

The B_{MN} sourcing this H_{MNP} can be chosen to be of the form (3.3.10) with $c(v) = -\frac{\ell^4}{2R_4^2} h(v)$.

Now that we have a solution to the anti-self dual form-field, we can consider its back reaction on the metric and solve the corresponding Einstein equation with the stress tensor sourced by the form-field deformation. In our conventions, Einstein equation with anti-self-dual form field as an additional source take the form [21, 23]

$$R_{MN} = F_{MPQ} F_N{}^{PQ} + H_{MPQ} H_N{}^{PQ}. \tag{3.3.14}$$

With the H_{MNP} given in (3.3.12)–(3.3.13) a simple calculation shows that it only sources the vv component of the Einstein equation, and moreover the sphere dependence drops out. We find[5]

$$H^{uPQ} H_{vPQ} = \frac{8\ell^2}{R_4^4} \left(\frac{h(v)^2}{z^2} \right). \tag{3.3.15}$$

To capture the deformation in the metric we make the ansatz,

$$ds^2 = \frac{\ell^2}{z^2} \left(dudv + dz^2 + \left(\frac{z^2}{\ell^2} + S(v,z) \right) dv^2 \right) + \ell^2 d\Omega_{S^3}^2, \tag{3.3.16}$$

where we have introduced the function $S(v,z)$ in the dv^2 term. Only $R^u{}_v$ component of the

[5] It is simplest to write this result in mixed component. We will see shortly that the left hand side of the Einsteins equations take a particularly simple form with mixed indices of this type.

Einstein equation gives a non-trivial equation. We find,

$$z^3\partial_z\left(\frac{1}{z}\partial_z S(v,z)\right)+\frac{8\ell^4}{R_4^4}\left(\frac{h(v)^2}{z^2}\right)=0. \tag{3.3.17}$$

This equation can be readily solved for S to yield,

$$S(v,z)=-\frac{\ell^4}{R_4^4}\frac{h(v)^2}{z^2}. \tag{3.3.18}$$

This is precisely the $r\to 0$ limit of the function S that appears in equation (3.1.38). We conclude that the B-field (3.3.10) together with metric (3.3.16) describes the anti-self-dual form-field perturbations to the black hole in Taub-NUT space from the near horizon region perspective.

We also note that the three-dimensional part of the metric (3.3.16) is of the following special form,

$$g_{\mu\nu}=g_{\mu\nu}^{AdS_3}-4\ell^2R_4^{-4}h(v)^2k_\mu k_\nu, \tag{3.3.19}$$

where $k=\partial_u$.

In (U,V,W) coordinates, cf. (3.2.18), the deformed metric takes the form

$$ds^2=\ell^2\left(W^2dUdV+\frac{dW^2}{W^2}-W^4\bar{h}(\bar{v}(V))^2dV^2\right)+\ell^2d\Omega_{S^3}^2. \tag{3.3.20}$$

The function $\bar{h}(\bar{v})$ is related to the function $h(v)$ though they are not identical. As far as the powers of V and W are concerned, this metric has the same form as (3.2.27). Therefore, the arguments given below (3.2.27) regarding the smoothness of the deformation apply without any change. The form field H_{MNP} is also regular in (U,V,W) coordinates. We conclude that the anti-self-dual form-field gives rise to a smooth deformation of the near-horizon geometry. This conclusion is in line with the analysis of [21] where it was shown that the anti-self-dual form-field hair modes for the black hole in Taub-NUT space are regular at the horizon.

3.4 Fermionic hair modes in the near horizon region

In this section, we make some observations on the non-normalisability of the fermionic hair modes of refs. [20, 21, 23] in $AdS_3 \times S^3$. We find it easiest to present this analysis in six-dimensions. The relevant linearised equations for the gravitino ψ_M^α are

$$\Gamma^{MNP} D_N \Psi_P^\alpha - F^{MNP} \Gamma_N \widehat{\Gamma}^1_{\alpha\beta} \Psi_P^\beta = 0. \tag{3.4.1}$$

Our aim is to solve this equation in the near horizon geometry. We work with the form of the near horizon geometry (3.1.18). Making the ansatz, $\Psi_M^\alpha = 0$, for $M \neq v$, and $\partial_u \Psi_v^\alpha = 0$, and choosing the gauge condition $\Gamma^v \Psi_v^\alpha = 0$, we obtain a class of solutions

$$\Psi_v \propto r^{3/2} \varepsilon(v, \theta, \phi) \quad \text{for} \quad \widehat{\Gamma}^1 \varepsilon = -\varepsilon, \tag{3.4.2}$$

with the angular dependence given as

$$\varepsilon(v,\theta,\phi) = h(v) e^{i\phi/2} \begin{pmatrix} \cos(\theta/2) \\ -\sin(\theta/2) \end{pmatrix}, \quad \text{or} \quad \varepsilon(v,\theta,\phi) = h(v) e^{-i\phi/2} \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}. \tag{3.4.3}$$

The same solutions are obtained by taking the $r \to 0$ limit of the corresponding expressions in [21, 23]. These solutions are regular at the horizon.[6] A key point that we wish to highlight is that in (u, v, z) coordinates, cf. (3.1.19), the gravitino deformation grows near the boundary $z = 0$ of AdS_3 as

$$\Psi_v \propto \frac{1}{z^3} \varepsilon(v, \tilde{\theta}, \tilde{\phi}, \tilde{\psi}). \tag{3.4.4}$$

The gravitino hair modes are non-normalizable in the near-horizon region.

[6] We do not present a smoothness analysis in this chapter. However, it is clear that a straightforward adaptation of the analysis of ref. [21, 23] is applicable.

3.5 Discussion

In this chapter, we discussed the hair modes of the BMPV black hole in flat space and in Taub-NUT space from the near-horizon $AdS_3 \times S^3$ perspective. We found that the non-trivial hair modes involve non-zero angular momentum on the S^3 and are regular at the AdS_3 horizon. Most importantly, all non-trivial hair modes change the asymptotics of the near-horizon geometry. Our results clearly show that these modes "leak-out" from near-horizon region to the asymptotically flat region: they grow towards the AdS boundary before falling off to zero in the asymptotically flat region.

We wish to make a few technical remarks regarding the above analysis.

1. Using the AdS/CFT dictionary, we can identify the dimensions of the CFT operators that correspond to the bulk deformations discussed above. Turning on non-normalisable modes correspond to deformations of the CFT with operators of conformal dimension $\Delta > 2$. For operators dual to scalars of mass m in AdS_3, conformal dimension Δ is

$$\Delta = 1 + \sqrt{1 + m^2 \ell^2}. \tag{3.5.1}$$

The Garfinkle-Vachaspati deformations in $AdS_3 \times S^3$ with $L = 1$ and $L = 2$ can be identified with scalars of masses $m^2 \ell^2 = L(L+2)$. Hence, the smooth hair mode with $L = 2$ corresponds to a deformation of the CFT by an operator of mass dimensions

$$\Delta = 1 + \sqrt{1 + 8} = 4. \tag{3.5.2}$$

The same conclusion can be reached by considering the Garfinkle-Vachaspati deformation as a metric deformation. Since the deformation term grows as z^{-4} near the boundary $z = 0$, it corresponds to a deformation of the CFT by an operator of mass dimensions $\Delta = 4$, see, e.g., comments in [58].

2. We can similarly identify the dimensions of the operators that correspond to the form-

field deformation. Taking one more exterior derivative of equation (3.3.7), we can convert it into the equation for a massive vector field

$$\star_3 d \star_3 db = \ell^{-2}\eta^2 b. \tag{3.5.3}$$

A deformation by a massive vector field of mass $m^2\ell^2 = \eta^2 = 4$ corresponds to deforming the boundary CFT by an operator of dimension $\Delta = 1 + \sqrt{m^2\ell^2} = 3$ [58]. In addition, since the form-field back reacts and generates a non-normalisable metric deformation that grows as z^{-4} near the boundary $z = 0$, we conclude that the CFT deformation is accompanied with an additional deformation by a $\Delta = 4$ operator.

3. In section 3.3, we made the observation that the three-dimensional part of the metric (3.3.16) can be written in a special form (3.3.19). For the case when the function $h(v)$ is a constant, we recognise metric (3.3.19) as the null warped AdS_3 metric (and also as Schrödinger metric with $z = 2$). As is well known, these metrics can be obtained as solutions to three-dimensional Einstein gravity with negative cosmological constant coupled to a massive vector field [59]. As discussed in the previous remark, the metric deformation discussed in section 3.3 can also be thought of as a deformation sourced by a massive vector field (3.5.3). Thus, the fact that the metric (3.3.16) can be written in the null warped AdS_3 form is perhaps not unexpected. Null warped AdS_3 metrics are known to be duals of deformations of the boundary CFT by operators of dimension 4 [60]. This is in line with the observations made above. It will be interesting to identify such operators in the D1-D5 CFT and explore the connection to the null warped AdS_3 further.

In refs. [64–66] various deformations of $AdS_3 \times S^3$ were considered in the context of the fuzzball program. Even though the starting point is the same, there are many differences between our results and their analyses. Firstly, since there are no horizons in the fuzzball cases, these references did not need to consider the regularity at the horizon. The analyses in these references is instead in global AdS. Secondly, the metric perturbations considered in these references do not have direct analogs in our analysis, but roughly speaking a class of modes considered in [64] corresponds to our $L = 0$ Garfinkle-Vachaspati modes.[7]

In ref. [68], it was suggested that fuzzballs should also admit hair modes like the ones that we have considered in this chapter. Specifically, by putting fuzzballs in Taub-NUT space, it was conjectured that there should be fuzzball hair, like the ones obtained by putting the BMPV black hole in Taub-NUT space. We hope to report our progress on constructing such hair on a class of simplest fuzzballs in the future.

[7] See also point 3 on page 26 of ref. [64] where a related observation is made.

Chapter 4

Review of Gravitational Index

4.1 Index vs degeneracy

In string theory, we compute the microscopic degeneracy of BPS black holes. This microscopic degeneracy d_{micro} is often replaced by a proxy element, which is some version of "index". The degeneracy calculated from the macroscopic side S_{macro} is often classically given by general relativity. On the other hand, the appropriate index in case of $4n$ broken supersymmetries is given by

$$B_{2n} = (-1)^n Tr[(-1)^{2h}(2h)^{2n}]/(2n)! \tag{4.1.1}$$

Now, theoretically index will always be less than entropy. But in special cases, e.g, in large charge limit index matches with entropy upto very high accuracy. For detailed discussion, one can refer to [26].

The idea to define gravitational index lets us compare similar quantities, i.e, index from both macroscopic and the microscopic side.

4.2 Grand canonical partition function for $D = 4$ $\mathcal{N} = 2$ ungauged SUGRA

For the sake of concreteness, focus on $\mathcal{N} = 2$ ungauged supergravity in four dimensions. The fields of the theory include a metric $g_{\mu\nu}$, a complex spin-3/2 gravitino Ψ and a $U(1)$ graviphoton A [27]. In natural units, i.e, $G_N = c = \hbar = 1$, the quadratic part of the action looks [27]

$$I[g,\Psi,A] = -\frac{1}{16\pi}\int_M d^4x\sqrt{g}\left[R - \frac{1}{4}F^2 - \frac{i}{2}\bar{\Psi}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho + \dots\right] + I_{bdy}, \tag{4.2.1}$$

dots denote cubic and quartic terms completely fixed by supersymmetry. This action is invariant under $\mathcal{N} = 2$ supersymmetry given by $\delta_\varepsilon\Psi = \hat{\nabla}\varepsilon + \mathscr{O}(\Psi^2)$ with $\hat{\nabla}_\mu \equiv \nabla_\mu + \frac{i}{8}F_{\mu\nu}\Gamma^{\mu\nu}\Gamma_\mu$. The Euclidean path integral of interest in asymptotically flat space with boundary topology $S^1 \times S^2$,

$$ds^2 = d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \mathscr{O}(1/r), \quad r \to \infty, \tag{4.2.2}$$

with the identifications

$$\phi \sim \phi + 2\pi \qquad \text{and} \qquad (\tau,\phi) \sim (\tau+\beta, \phi + i\beta\Omega). \tag{4.2.3}$$

The time circle has asymptotic periodicity $\beta = T^{-1}$ with β being the inverse temperature. the twist parameter Ω denotes angular velocity of the horizon along some direction within S^2. Impose falloff condition $\Psi \sim \mathscr{O}(1/r^2)$ at infinity with periodicity

$$\begin{aligned}\Psi(\tau,r,\theta,\phi) &= -\Psi(\tau+\beta,r,\theta,\phi+i\beta\Omega) \\ &= -e^{\beta\Omega J}\,\Psi(\tau+\beta,r,\theta,\phi),\end{aligned} \tag{4.2.4}$$

where J, the generator of rotations along ϕ has been defined in the second line. To fix the charges, the appropriate choice of boundary term will be [121]

$$I_{\text{bdy}} = -\frac{1}{8\pi}\int_{\partial M}\sqrt{h}K - \frac{1}{4\pi}\int_{\partial M}\sqrt{h}n_a F^{ab}A_b. \tag{4.2.5}$$

with boundary metric h_{ab}. The partition function is defined schematically via gravitational path integral [120]

$$Z_{\text{grav}}(\beta,\Omega,Q) \equiv \int \mathscr{D}g\mathscr{D}\Psi\mathscr{D}A\; e^{-I[g,\Psi,A]}, \tag{4.2.6}$$

For theories with a gravitational dual, we fix Q, Ω and β by imposing the above boundary conditions. The gravitational path integral computes $\mathrm{Tr}_Q\left(e^{-\beta H+\beta\Omega J}\right)$, with Tr_Q is a trace over states with charge Q. This is not a precise relation for theories without an explicit gravitational dual since a quantum mechanical Hilbert space and operators in a boundary theory are not being defined which would give an independent calculation of the right-hand side.

4.3 Classical saddles

Classical saddles consistent with the above boundary conditions are the Kerr-Newman black holes. The Lorentzian metric obtained via analytic continuation $\tau = it$ is given in terms of the parameters (a, M, Q)

$$\begin{aligned} g_{KN\mu\nu}dx^\mu dx^\nu &= -\frac{\Delta}{\rho^2}\left[dt + a\sin^2\theta d\phi\right]^2 + \frac{\rho^2 dr^2}{\Delta} + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(adt + (r^2+a^2)d\phi\right)^2, \\ A_{KN} &= \frac{Q\cos\theta}{\rho^2}(adt + (r^2+a^2)d\phi), \end{aligned} \tag{4.3.1}$$

where $\rho^2 \equiv r^2 + a^2\cos^2\theta$ and $\Delta \equiv r^2 + a^2 - 2Mr + Q^2$. The gravitino solution has $\Psi = 0$. The locations of outer and inner event horizons are at radii

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}. \tag{4.3.2}$$

The solution is given in Boyer–Lindquist coordinates, in which the metric becomes asymptotically flat. In Euclidean signature, the region of interest is the outer horizon r_+ and $r \in [r_+, \infty)$. Smoothness of the Euclidean horizon fixes β and Ω M and a, or equivalently r_+ and a,

$$\Omega = \frac{a}{r_+^2 + a^2}, \qquad T = \beta^{-1} = \frac{r_+^2 - a^2 - Q^2}{4\pi(r_+^2 + a^2)r_+}. \tag{4.3.3}$$

Using the fact that for a classical solution $R = 0$ and F^2 is a total derivative, the classical solution can be calculated [120]

$$I_{\text{classical}}(\beta,\Omega,Q) = \frac{\beta}{2}\frac{a^2+Q^2+r_+^2}{2r_+} + \frac{\beta}{2}\frac{Q^2 r_+}{r_+^2+a^2}, \tag{4.3.4}$$

r_+ and a are implicit functions of β, Q and Ω. Regularization was done by subtracting the contribution from empty flat space.

4.4 Solving for the partition function

The boundary conditions(4.2.4) are invariant under the shift $i\beta\Omega \to i\beta\Omega - 4\pi n$, with $n \in \mathbb{Z}$. The path integral becomes a sum over these different solutions which share the same boundary conditions [122, 123]

$$Z_{\text{grav}}(\beta,\Omega,Q) = \sum_{n\in\mathbb{Z}} e^{-I_{\text{classical}}\left(\beta,\Omega+\frac{4\pi i}{\beta}n,Q\right)} Z_{\text{1-loop}}(\beta,\Omega,Q;n)(1+\dots). \tag{4.4.1}$$

The sum is taken over classical solutions and the first term gives their classical action contribution $e^{-I_{\text{classical}}\left(\beta,\Omega+\frac{4\pi i}{\beta}n,Q\right)}$. The action at low temperatures with fixed real $i\beta\Omega$

$$\begin{aligned} -I_{\text{classical}}\left(\beta,\Omega+\frac{4\pi i}{\beta}n,Q\right) &= -\beta Q + \pi Q^2 + \frac{Q^3(4\pi^2-(i\beta\Omega-4\pi n)^2)}{2\beta} + \dots \\ &\overset{\Omega\to\frac{4\pi i\alpha}{\beta}}{=} -\beta Q + \pi Q^2 + \frac{2\pi^2 Q^3(1-4(\alpha+n)^2)}{\beta} + \dots, \end{aligned} \tag{4.4.2}$$

where $\alpha \equiv -\frac{i\beta\Omega}{4\pi}$. Generically the $n = 0$ term dominates, although, leading quantum effects can possibly change this. The dots denote subleading terms at low temperatures.

The one loop determinant $Z_{\text{1-loop}}(\beta,\Omega,Q;n)$ that captures the fluctuations of metric, graviphoton, and gravitino around the n-th solution. This term is the difference between bosonic Einstein-Maxwell theory and $\mathcal{N} = 2$ ungauged supergravity. The reason being, only metric and graviphoton will constitute Einstein-Maxwell theory. At large charge, Q, low temperature, and small angular velocity (fixed real α), this can be approximated by [122]

$$Z_{1\text{-loop}}(\beta,\Omega,Q;n) \overset{\Omega\to\frac{4\pi i\alpha}{\beta}}{\sim} \beta\frac{(\alpha+n)\cot(\pi\alpha)}{(1-4(\alpha+n)^2)^2}\,Q^{c_{\log}}\left[1+O(Q^{-1})\right]. \tag{4.4.3}$$

The quantity $c_{\log}$ can be calculated by Sen's quantum entropy function [124] and depends on the matter content of the theory. It can be absorbed by a logarithmic correction to the area term in the classical action.

4.5 Defining the index

The Witten index is schematically defined as

$$\text{Index}(\beta,Q) = \text{Tr}_Q\left[(-1)^F e^{-\beta H}\right]. \tag{4.5.1}$$

We explain a naive approach that is misleading. Take the Reissner-Nordström black hole. The metric given in (4.3.1) with $a=0$ and fixed T. In the notation above, this corresponds to computing $Z(\beta,\Omega=0)=\text{Tr}_Q\left(e^{-\beta H}\right)$ [27]. Contractibility of the Euclidean time circle requires the boundary conditions on the fermion to be $\Psi(\tau+\beta,r,\theta,\phi)=-\Psi(\tau,r,\theta,\phi)$. But, presence of the $(-1)^F$ factor in the trace might lead one to naively take the fermions to be periodic around time circle, contradicting its holonomy around the contractible time circle. To cure this we make an observation: $(-1)^F=e^{2\pi iJ}$. Accordingly, we propose to compute the index [72]

$$\begin{aligned}\text{Index}(\beta,Q) &= Z_{\text{grav}}\left(\beta,\Omega=\frac{2\pi i}{\beta},Q\right)\\ &= \sum_{n\geq 0} e^{-I_{\text{classical}}\left(\beta,\frac{2\pi i}{\beta}+\frac{4\pi i}{\beta}n,Q\right)}\,Z_{1\text{-loop}}\left(\beta,\frac{2\pi i}{\beta},Q;n\right)(1+\ldots).\end{aligned} \tag{4.5.2}$$

Instead of changing the boundary conditions of the fermions by hand, this prescription effectively turns on an imaginary angular velocity $\Omega\to\frac{2\pi i}{\beta}+\frac{4\pi i}{\beta}n$. The solutions contributing to the path integral computing the index are become smooth after introducing this rotation. This purely imaginary value of Ω corresponds to real angular velocity in Euclidean space since $\Omega_E = i\Omega$.

The solutions with $\Omega = \pm\left(\frac{2\pi i}{\beta} + \frac{4\pi i}{\beta} n\right)$ can be interchanged by shifting $n \to -1-n$. These are interchangable under parity transformation. Since this is a symmetry of the theory, we restrict ourselves to the positive sign of Ω and to $n \geq 0$ without any loss of generality. This choice of Ω, for any value of n, leads to periodic fermions around the time circle [72]

$$\begin{aligned} \Psi(\tau, r, \theta, \phi) &= -\Psi(\tau+\beta, r, \theta, \phi+2\pi), \\ &= \Psi(\tau+\beta, r, \theta, \phi). \end{aligned} \tag{4.5.3}$$

In the first equality, smoothness of fermions with respect to the contractible circle at the horizon has been used. In the second line we use that the fermion is always antiperiodic around $\phi \sim \phi + 2\pi$. Thus we get a classical configuration that solves the equations of motion and is completely smooth for both the bosonic and the fermionic fields.

4.6 Saddles of the index

The $n = 0$ saddle: Using the expressions from equation (4.3.3), we can solve for the special value of a, say a_* which solves $\beta(a_\star, r_+, Q)\Omega(a_\star, r_+, Q) = 2\pi i$.
This gives $a \to ir_+ \pm iQ$ and we pick $a_\star = ir_+ - iQ$ (since the other solution is not physical). Now, we can use the equation that determines r_+ and insert this value of $a_\star$, yielding

$$\begin{aligned} \Delta(r_+) &= r_+^2 + a_\star^2 - 2Mr_+ + Q^2 = 2r_+(Q-M) = 0, \\ &\Rightarrow \; M = Q. \end{aligned} \tag{4.6.1}$$

This shows that the choice $\Omega \to \Omega_\star = 2\pi i/\beta$ fixes $M = Q$, giving supersymmetric solution. Hence, the Killing spinor equation $\hat{\nabla}_\mu \varepsilon = 0$ is integrable.

Even though these solutions have $M = Q$, their temperature is nonzero. Writing $a = i\mathfrak{a}$ where real $\mathfrak{a} = r_+ - Q$, the expressions for temperature, mass and angular velocity turn out to be [72]

$$T(\mathfrak{a}) = \frac{1}{2\pi}\frac{\mathfrak{a}}{Q(Q+2\mathfrak{a})}, \quad M(\mathfrak{a}) = Q, \quad \Omega(\mathfrak{a}) = 2\pi i T(\mathfrak{a}). \tag{4.6.2}$$

4.7 Finding index from partition function

Rotation group in five dimensions is $SO(4) = SU(2)_L \times SU(2)_R$. A generic black hole in five dimensions carry three $U(1)$ charges, mass M, angular momenta J_ϕ and J_ψ in mutually orthonormal planes with azimuthal angles ψ and ϕ. The procedure for genrating such solutions is outlined in [125]. The third components of $SU(2)_L$ and $SU(2)_R$ angular momenta are related to J_ϕ and J_ψ as

$$J_{3L} = \frac{1}{2}(J_\phi - J_\psi), \qquad J_{3R} = \frac{1}{2}(J_\phi + J_\psi) \tag{4.7.1}$$

Inverse temperature β, chemical potentials $\vec{\mu}$ for the charges $\vec{Q}$ and the angular momenta Ω_L, Ω_R conjugate to J_{3L}, J_{3R} are the conjugate variables. Classical supersymmetric black holes have $J_{3R} = 0$.

The supersymmetry generators decompose into $(2_L, 1_R)$ and $(1_L, 2_R)$ representations in equal numbers. For states preserving four supercharges, we choose the unbroken generators to lie in the $(1_L, 2_R)$ representation, with the remainder broken. The broken generators give rise to fermion zero modes. The relevant index for a supersymmetric black hole breaking $2n$ $SU(2)_L$ invariant supersymmetries, or $(1_L, 2_R)$ supersymmetry [99]

$$N_{BPS}(\vec{Q}, J_{3L}) = e^{S_{BPS}(\vec{Q}, J_{3L})} = Tr_{\vec{Q}, J_{3L}, \vec{k}=0}[(-1)^F (2J_{3R})^n], \quad (-1)^F = e^{2\pi i J_{3R}} \tag{4.7.2}$$

where the trace is over all states with fixed charge vector $\vec{Q}$, angular momentum J_{3L} and zero momentum.

In the absence of the $(2J_{3R})^n$ insertion, the trace over the fermion zero modes in the $(1_L, 2_R)$ sector would vanish. The insertion precisely saturates the n pairs of zero modes, yielding a nonzero result, as in four dimensions. Additional zero modes arise in the $(2_L, 1_R)$ representation.

From the macroscopic side, we calculate the gravitational partition function with the boundary conditions mentioned before. Setting $\Omega_R = -2\pi i/\beta$, the partition function becomes [99]

$$\begin{aligned} Z(\beta, \vec{\mu}, \Omega_L, -2\pi i/\beta) &= Tr[e^{-\beta E - \beta\vec{\mu}.\vec{Q} - \beta\Omega_L J_{3L} + 2\pi i J_{3R}}(2J_{3R})^n] \\ &= Tr[e^{-\beta E - \beta\vec{\mu}.\vec{Q} - \beta\Omega_L J_{3L}}(-1)^F(2J_{3R})^n] \end{aligned} \tag{4.7.3}$$

Assuming the dispersion relation $E = M_{BPS} + \vec{k}^2/2M_{BPS}$, the partition function becomes [99]

$$Z(\beta, \vec{\mu}, \Omega_L, -2\pi i/\beta) = \sum_{\vec{Q}, J_{3L}} e^{S_{BPS} - \beta M_{BPS} - \beta\vec{\mu}.\vec{Q} - \beta\Omega_L J_{3L}} \int d^{n_T}k (\frac{L}{2\pi})^{n_T} e^{-\beta\vec{k}^2/2M_{BPS}} \tag{4.7.4}$$

where $\vec{k}$ is an n_T dimensional vector that is invariant under the action of $e^{\beta\Omega_L J_{3L}}$. Performing the $\vec{k}$ integral and inverting the previous relation we get

$$e^{S_{BPS}} \sim L^{-n_T}(\frac{\beta}{M_{BPS}})^{n_T/2}\beta^{n_V+1}\int d^{n_V}\mu d\Omega_L e^{\beta M_{BPS} + \beta\vec{\mu}.\vec{Q} + \beta\Omega_L J_{3L} + lnZ(\beta, \vec{\mu}, \Omega_L, -2\pi i/\beta)} \tag{4.7.5}$$

lnZ_0, the leading classical result for lnZ, is given by

$$lnZ_0(\beta, \vec{\mu}, \vec{\Omega}) = S_0 - \beta M - \beta\vec{\mu}.\vec{Q} - \beta\vec{\Omega}.\vec{J} \tag{4.7.6}$$

Using this expression of lnZ_0 in (4.7.5) and carrying out the $\vec{k}, \vec{\mu}$ and Ω_L integrals using saddle point approximation one obtains [99]

$$\begin{aligned} e^{S_{BPS}} &\sim L^{-n_T}(\frac{\beta}{M_{BPS}})^{n_T/2}\beta^{n_V+1}(det\frac{\partial^2 lnZ_0}{\partial\mu_i\partial\mu_j})^{-1/2}(\frac{\partial^2 lnZ_0}{\partial\Omega_L^2})^{-1/2} \\ &\times e^{\beta M_{BPS} + \beta\vec{\mu}.\vec{Q} + \beta\Omega_L J_{3L} + lnZ_0(\beta, \vec{\mu}, \Omega_L, -2\pi i/\beta)} \end{aligned} \tag{4.7.7}$$

where at saddle point

$$\frac{1}{\beta}\frac{\partial lnZ_0}{\partial \mu_i} = Q_i, \qquad \frac{1}{\beta}\frac{\partial lnZ_0}{\partial \Omega_L} = J_{3L} \tag{4.7.8}$$

Using various scaling properties of the black hole parameters provided in [99] and setting $D = 5$

$$\begin{aligned} S_{BPS} &\simeq \beta M_{BPS} + \beta\vec{\mu}.\vec{Q} + \beta\Omega_L J_{3L} + lnZ_0 \\ &\quad - n_T lnL - \frac{n_T}{2}ln\lambda - \frac{n_V}{2}ln\lambda - \frac{3}{2}ln\lambda + \delta lnZ \end{aligned} \tag{4.7.9}$$

where δlnZ stands for the logarithmic corrections from integration over massless fields in the gravitational path integral and inclusion of $(2J_{3R})^n$ in Z. Using the condition $\beta\Omega_R = -2\pi i/\beta$ and expression for lnZ_0, we obtain

$$\begin{aligned} S_{BPS} &\simeq \beta M_{BPS} + S_0 - \beta M + 2\pi i J_{3R} \\ &\quad - n_T lnL - \frac{n_T}{2}ln\lambda - \frac{n_V}{2}ln\lambda - \frac{3}{2}ln\lambda + \delta lnZ \end{aligned} \tag{4.7.10}$$

Here M and S_0 are mass and entropy of a classical black hole having temperature β with $\beta\Omega_R = -2\pi i$, charges $\vec{Q}$ and $SU(2)_L$ charge J_{3L} and in general are not the same as those of a zero temperature black hole M_{BPS} and S_{BPS}. The classical result $S^{(0)}_{BPS}$ of S_{BPS} is given as

$$S^{(0)}_{BPS} = \beta M_{BPS} + S_0 - \beta M + 2\pi i J_{3R} \tag{4.7.11}$$

Given this result relating an extremal black hole and its non-extremal index saddle, we can apply this test to find index saddle of supersymmetric small black ring. In the next chapter, we introduce black rings and search for index saddle.

Chapter 5

Black Rings and Gravitational Index

Black hole uniqueness theorems in four spacetime dimensions state that stationary black holes have spherical horizon topology and are completely specified by their conserved charges. However, these results fail in five dimensions. In D=5, the vacuum Einstein equations admit stationary, asymptotically flat solutions with horizon topology $S^1 \times S^2$, known as rotating black rings. These solutions are not uniquely characterized by their conserved charges like mass and angular momenta and the charges do not distinguish black rings from spherical black holes. Charged black ring solutions with analogous features were subsequently constructed. We'll first review neutral black rings and then charged black rings.

5.1 Ring Coordinates

In four spatial dimensions, the rotation group $SO(4)$ contains two commuting $U(1)$ subgroups, allowing for independent rotations in two orthogonal planes. To make this explicit, consider flat four-dimensional space and group the coordinates into two pairs, introducing polar coordinates in each plane

$$x^1 = r_1 \cos\phi, \quad x^2 = r_1 \sin\phi, \quad x^3 = r_2 \cos\psi, \quad x^4 = r_2 \sin\psi. \tag{5.1.1}$$

Rotations in the ϕ and ψ directions correspond to two independent angular momenta, J_ϕ and J_ψ. We focus on ring-like configurations lying in the (x^3, x^4)-plane and rotating along ψ, so that J_ψ is nonzero.

As is often useful in general relativity, it is advantageous to introduce adapted coordinates.

A natural way to do this is by constructing equipotential surfaces associated with a source resembling the black hole of interest. Rather than using a scalar potential sourced by a ring, it is more convenient to work with the equipotential surfaces of a two-form potential $B_{\mu\nu}$. In this picture, the ring is treated as a circular string acting as an electric source for the three-form field strength $H = dB$, which satisfies

$$\partial_\mu \left(\sqrt{-g} H^{\mu\nu\rho} \right) = 0 \tag{5.1.2}$$

away from the source.

Flat four-dimensional space, expressed in these coordinates, takes the form

$$ds^2 = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2. \tag{5.1.3}$$

The solution corresponding to a circular electric source located at $r_1 = 0$, $r_2 = R$, and extending over $0 \leq \psi < 2\pi$ can be obtained using standard methods from classical electrodynamics. The resulting two-form potential is [85]

$$B_{t\psi} = \frac{R}{2\pi} \int_0^{2\pi} d\psi' \frac{r_2 \cos\psi'}{r_1^2 + r_2^2 + R^2 - 2Rr_2 \cos\psi'} = -\frac{1}{2} \left(1 - \frac{R^2 + r_1^2 + r_2^2}{\Sigma} \right), \tag{5.1.4}$$

where

$$\Sigma = \sqrt{(r_1^2 + r_2^2 + R^2)^2 - 4R^2 r_2^2}. \tag{5.1.5}$$

One can also construct the Hodge dual of this field. In five spacetime dimensions the dual of the three-form field strength satisfies $\star H = F = dA$, where A is a one-form potential. Thus, the dual description of electric string is a magnetic monopole. In this case a, the dual description is a circular distribution of monopoles. Surfaces of constant A_ϕ are orthogonal to those of constant $B_{t\psi}$. For the dual field, one finds

$$A_\phi = -\frac{1}{2} \left(1 + \frac{R^2 - r_1^2 - r_2^2}{\Sigma} \right). \tag{5.1.6}$$

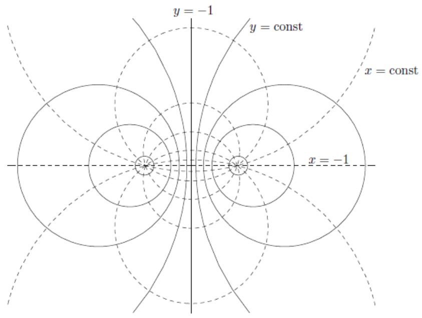

Figure 5.1: Ring coordinates for flat four-dimensional space are illustrated on a section with fixed ϕ and ψ (together with the antipodal section $\phi+\pi, \psi+\pi$). The dashed curves represent surfaces of constant $|x| \in [0,1]$, while the solid curves correspond to surfaces of constant $y \in [-\infty,-1]$. As $y \to -\infty$, these surfaces shrink to zero size, marking the location of the ring of radius R. The disk enclosed by the ring forms the axis of the rotational symmetry generated at $x=+1$.

We now introduce new coordinates y and x, defined so that they correspond to constant values of $B_{t\psi}$ and A_ϕ, respectively. A convenient choice is [76]

$$y = -\frac{R^2+r_1^2+r_2^2}{\Sigma}, \qquad x = \frac{R^2-r_1^2-r_2^2}{\Sigma}, \tag{5.1.7}$$

with inverse relations

$$r_1 = R\frac{\sqrt{1-x^2}}{x-y}, \qquad r_2 = R\frac{\sqrt{y^2-1}}{x-y}. \tag{5.1.8}$$

The allowed coordinate ranges are $-\infty \le y \le -1, \quad -1 \le x \le 1$. Here, $y \to -\infty$ corresponds to the location of the ring source, while spatial infinity is reached as $x \to y \to -1$. The axis of rotation along ψ , which is actually a plane rather than a line, lies at $y=-1$. The axis associated with ϕ, located at $r_1=0$, splits into two regions: $x=1$, corresponding to the disk $r_2 \le R$, and $x=-1$, corresponding to its complement outside the ring, $r_2 \ge R$.

In these coordinates, the flat metric becomes [76]

$$ds^2 = \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right]. \tag{5.1.9}$$

This coordinate system is illustrated in picture 5.1, showing a section at fixed ψ and ϕ (along with the antipodal section for clarity). It is also useful to rewrite the geometry in a form better adapted to the region near the ring. For this, define new coordinates r and θ by

$$r = -\frac{R}{y}, \quad \cos\theta = x, \tag{5.1.10}$$

with ranges $0 \leq r \leq R, \qquad 0 \leq \theta \leq \pi$. In terms of these variables, the flat metric becomes

$$ds^2 = \frac{1}{\left(1+\frac{r\cos\theta}{R}\right)^2}\left[\left(1-\frac{r^2}{R^2}\right)R^2 d\psi^2 + \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{5.1.11}$$

The apparent singularity at $r = R$ is not physical. It simply corresponds to the axis of rotation in the ψ-direction. It is evident that surfaces of constant r (equivalently, constant y) have the topology of $S^2 \times S^1$, with the S^2 parametrized by (θ, ϕ) and the S^1 by ψ. Since the horizons and ergosurfaces of black rings occur at fixed values of y or r, their topology is naturally exhibited in these coordinates.

Σ^{-1} satisfies the Laplace equation for a scalar field sourced by a ring, $\nabla^2\Sigma^{-1} = 0$. Since $\Sigma^{-1} = \frac{x-y}{2R^2}$, surfaces of constant scalar potential generally do not coincide with surfaces of constant x or y, except in the limit of large negative y (i.e., $r \ll R$), where $\Sigma^{-1} \approx -\frac{y}{2R^2} = \frac{1}{2Rr}$.

Although one could introduce coordinates adapted to constant Σ and its gradient flow, this leads to a more complicated form of the black ring solutions. By contrast, the (x, y) coordinates, being naturally adapted to the two-form potential B, are particularly convenient, especially for analyzing configurations with dipole charges, such as supersymmetric black rings.

5.2 Non-extremal two charge black rings

Elvang and Emparan in ref. [86] presented a five-dimensional smooth singly spinning non-extremal black ring with two independent charges. The technique they used for adding the two charges is rather straightforward. It involves the following three steps:

1. We start by adding to a vacuum five-dimensional solution of Einstein equations a flat sixth dimension z. (In the case of ref. [86] the vacuum five-dimensional solution is the neutral Emparan-Reall black ring [73].) A Lorentz boost (with boost parameter α) gives a solution with linear momentum in the z-direction.

2. We next apply T-duality in the z-direction that exchanges the momentum with a fundamental string charge.

3. We apply another boost (with boost parameter β) in the z-direction to get a solution with both charge and momentum. The resulting configuration is a solution to the classical equations of motion of the low-energy NS-NS sector of superstring theory compactified on T^4. Compactifying the z-direction on a circle we get the five-dimensional solution of interest.

The technique can be applied to any vacuum five-dimensional solution. In section 5.2.1, we review the Elvang-Emparan solution. In section 5.2.2, we apply the same technique to the Pomeransky-Sen'kov black ring solution. Later in the paper, in section 5.7.4 we apply the same technique to the five-dimensional Kerr black string. For completeness, the technique is reviewed in appendix D.

5.2.1 Singly spinning charged black ring

We start with a very brief review of the neutral Emparan-Reall black ring [73] to set the notation.

Emparan-Reall black ring

The metric takes the form [76],

$$\begin{aligned} ds^2 &= -\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\psi_{ER}\right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi_{ER}^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi_{ER}^2\right], \end{aligned} \tag{5.2.1}$$

where

$$F(\xi) = 1 + \lambda\,\xi\,, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi)\,, \qquad C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,. \tag{5.2.2}$$

The dimensionless parameters λ and ν take values $0 < \nu \leq \lambda < 1$. The dimension-full parameter R sets the scale of the solution. The coordinate ranges are $-1 \leq x \leq 1$, $-\infty < y < -1$, with asymptotic infinity at $x, y = -1$. Regularity at infinity requires the angular coordinates to have periodicities (to avoid conical singularities at $x = -1$ and $y = -1$ the angular variables must be identified with period),

$$\Delta\psi_{ER} = \Delta\phi_{ER} = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu}. \tag{5.2.3}$$

To avoid also a conical singularity at $x = +1$ we must have

$$\Delta\phi_{ER} = 2\pi\frac{\sqrt{1+\lambda}}{1+\nu}. \tag{5.2.4}$$

This is compatible with (5.2.3) only if we take the parameter λ in terms of ν as

$$\lambda = \lambda_c \equiv \frac{2\nu}{1+\nu^2}. \tag{5.2.5}$$

This condition is often called the balancing condition. The name "balancing condition" originates from the intuition that the angular momentum must be tuned so that the centrifugal force balances the tension and self-attraction of the ring. The event horizon is at $y=-1/\nu$. Once the balancing condition is imposed, $\psi=\psi_{ER}\sqrt{1+\nu^2}$, $\phi=\phi_{ER}\sqrt{1+\nu^2}$ have canonical periodicity 2π.

Now consider the (r,θ) coordinates introduced earlier in 5.1.10, along with a redefinition of parameters $(\nu,\lambda)\to(r_0,\sigma)$

$$\nu=\frac{r_0}{R}, \qquad \lambda=\frac{r_0\cosh^2\sigma}{R}. \tag{5.2.6}$$

In these variables, the metric takes a somewhat less transparent form [76]

$$ds^2=-\frac{\hat{f}}{\hat{g}}\left(dt-r_0\sinh\sigma\cosh\sigma\sqrt{\frac{R+r_0\cosh^2\sigma}{R-r_0\cosh^2\sigma}\frac{r}{R}-1}\,\frac{R}{r\hat{f}}d\psi\right)^2$$
$$+\frac{\hat{g}}{\left(1+\frac{r\cos\theta}{R}\right)^2}\left[\frac{f}{\hat{f}}\left(1-\frac{r^2}{R^2}\right)R^2d\psi^2+\frac{dr^2}{\left(1-\frac{r^2}{R^2}\right)f}+\frac{r^2}{g}d\theta^2+\frac{g}{\hat{g}}r^2\sin^2\theta\,d\phi^2\right], \tag{5.2.7}$$

where

$$f=1-\frac{r_0}{r}, \qquad \hat{f}=1-\frac{r_0\cosh^2\sigma}{r}, \tag{5.2.8}$$

and

$$g=1+\frac{r_0}{R}\cos\theta, \qquad \hat{g}=1+\frac{r_0\cosh^2\sigma}{R}\cos\theta. \tag{5.2.9}$$

Now consider the regime r, r_0, $r_0\cosh^2\sigma\ll R$, where $g\approx\hat{g}\approx 1$, and define $\psi=z/R$. The metric can be written as

$$ds^2=-\hat{f}(dt+\tanh\sigma(1-\hat{f})dz)^2+\frac{f}{\hat{f}}dz^2+\frac{dr^2}{f}+r^2(d\theta^2+\sin^2\theta d\phi^2) \quad . \tag{5.2.10}$$

where the (t,z) part ultimately looks like

$$ds^2_{(t,z)}=-(1-\frac{r_0\cosh^2\sigma}{r})dt^2-\frac{2r_0\sinh\sigma\cosh\sigma}{r}dtdz+(1+\frac{r_0\sinh^2\sigma}{r})dz^2 \tag{5.2.11}$$

We see that in this limit, the metric reduces exactly to that of a boosted black string extended along the z-direction, with boost parameter σ. The horizon sits at $r = r_0$, and requiring regularity fixes the period of ψ to 2π, so that $z \sim z + 2\pi R$. Thus, this limit corresponds to a ring whose radius R is much larger than its thickness r_0, focusing on the region near the ring $r \sim r_0$.

This makes precise the intuitive picture of a black ring as a boosted black string bent into a circle. It also clarifies the meaning of the parameters λ and ν. From their definitions, ν measures the ratio of the horizon S^2 radius r_0 to the ring radius R, so smaller ν corresponds to thinner rings. Meanwhile, λ/ν controls the rotation, and more precisely, $\sqrt{1-(\nu/\lambda)}$ can be identified (approximately) with the local boost velocity $v = \tanh\sigma$.

Elvang-Emparan two-charge black ring

Following the solution generating procedure outlined in appendix D, we get the two charge Elvang-Emparan black ring starting with the neutral Emparan-Reall black ring.[1] The Lagrangian for the theory is reviewed in appendix D. The five-dimensional Einstein frame metric takes the form,

$$\begin{aligned} ds_5^2 &= -\frac{1}{(h_\alpha h_\beta)^{2/3}}\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}c_\alpha c_\beta\, d\psi_{ER}\right)^2 \\ &\quad +(h_\alpha h_\beta)^{1/3}\frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi_{ER}^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi_{ER}^2\right] \end{aligned} \tag{5.2.12}$$

where we have defined

$$h_\alpha(x,y) = 1 + \frac{\lambda(x-y)}{F(x)}s_\alpha^2, \qquad h_\beta(x,y) = 1 + \frac{\lambda(x-y)}{F(x)}s_\beta^2, \tag{5.2.13}$$

[1] Note that our coordinates and parameters are slightly different from Elvang and Emparan [86]. They use coordinates and parameters different from the original black ring of [73], which in turn are different from the more modern notation [76]. Throughout this paper, we use notation of [76] for singly spinning black rings. This also means that we cannot simply take equations from Elvang and Emparan's paper; we had to work out the solution from the start.

and we use the short hand $c_\alpha = \cosh\alpha$, $c_\beta = \cosh\beta$, $s_\alpha = \sinh\alpha$, $s_\beta = \sinh\beta$. The dilaton Φ and the extra scalar χ are given as

$$e^{-2\Phi} = h_\alpha(x,y)\,, \qquad e^{-\frac{\sqrt{3}}{\sqrt{2}}\chi} = \frac{h_\beta(x,y)}{\sqrt{h_\alpha(x,y)}}\,. \tag{5.2.14}$$

The gauge fields are

$$A_t^{(1)} = \frac{(x-y)\lambda}{F(x)h_\beta(x,y)}c_\beta s_\beta\,, \qquad A_{\psi_{ER}}^{(1)} = \frac{CR(1+y)}{F(x)h_\beta(x,y)}c_\alpha s_\beta\,, \tag{5.2.15}$$

$$A_t^{(2)} = \frac{(x-y)\lambda}{F(x)h_\alpha(x,y)}s_\alpha c_\alpha\,, \qquad A_{\psi_{ER}}^{(2)} = \frac{CR(1+y)}{F(x)h_\alpha(x,y)}s_\alpha c_\beta\,. \tag{5.2.16}$$

The antisymmetric tensor field has as non-zero components,

$$B_{\psi_{ER}t} = \frac{CR(1+y)s_\alpha s_\beta}{F(x)h_\alpha(x,y)}\,. \tag{5.2.17}$$

The balancing condition remains the same. The event horizon is at $y = -1/\nu$. The charged solution is smooth everywhere on and outside the horizon.

Removing Dirac-Misner string singularity

In black ring solutions, we are interested in horizon topology $S^1 \times S^2$. The coordinates (x,ϕ) parametrize a sphere S^2 only if ∂_ϕ has fixed points at $x = \pm 1$, i.e, at the two poles of the sphere. But the orbit of ∂_ϕ doesn't close off when ω_ϕ doesn't go to 0 at $x = \pm 1$, indicating the presence of Dirac-Misner strings. Just like the case with Dirac strings in magnetic monopoles, one can try to eliminate the string by covering the manifold with two different coordinate patches, each one regular at each pole. The coordinate transformation where the two patches overlap requires time t to be periodically identified as $\Delta t = \omega_\phi(x=1)\Delta\phi$ [77]. But periodicity in t will lead to the existence of closed timelike curves everywhere outside the horizon.

To get rid of this trouble, we need to put $\omega_\phi(x=1) = 0$. And, from the expression of ω_ϕ in doubly rotating black ring case, we can see that this condition is indeed satified, saving us from a breakdown in global causality.

Physical properties of the Elvang-Emparan black ring

The mass of the black ring and the horizon area are

$$M = \frac{\pi R^2}{4G_N}\frac{\lambda}{1-\nu}(1+\cosh 2\alpha+\cosh 2\beta), \quad (5.2.18)$$

$$\mathscr{A}_{\rm H} = 8\pi^2 R^3 \frac{\nu^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)} c_\alpha c_\beta. \quad (5.2.19)$$

The angular momentum and angular velocity in the ϕ direction vanish. In the ψ direction, we have

$$J_\psi = \frac{\pi R^3}{2G_N}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\nu)^2} c_\alpha c_\beta, \quad (5.2.20)$$

$$\Omega_\psi = \frac{1}{R c_\alpha c_\beta}\sqrt{\frac{\lambda-\nu}{\lambda(1+\lambda)}}. \quad (5.2.21)$$

The temperature of the horizon is

$$T = \frac{1}{4\pi R c_\alpha c_\beta}(1+\nu)\sqrt{\frac{1-\lambda}{\lambda\nu(1+\lambda)}}. \quad (5.2.22)$$

Finally, the U(1) charges of the solution can be defined as,

$$\mathbf{Q}_i = \frac{1}{16\pi G_N}\int_{S^3 \text{ at } \infty} e^{-2\Phi_i} \star_5 F^{(i)}, \quad (5.2.23)$$

where $\Phi_1 = \frac{\sqrt{2}}{\sqrt{3}}\chi$ and $\Phi_2 = \Phi - \frac{1}{\sqrt{6}}\chi$. They take values,

$$\mathbf{Q}_1 = \frac{\pi R^2}{4G_N}\frac{\lambda}{1-\nu}\sinh 2\alpha, \quad (5.2.24)$$

$$\mathbf{Q}_2 = \frac{\pi R^2}{4G_N}\frac{\lambda}{1-\nu}\sinh 2\beta. \quad (5.2.25)$$

The solution also carries a dipole charge fixed in terms of the other parameters of the solution. We will discuss the dipole charge when we consider the supersymmetric limit.

5.2.2 Doubly spinning charged black ring

Neutral doubly spinning black ring was constructed by Pomeransky and Sen'kov [88]. We follow slightly different notation from the one introduced in [88]. This is mainly to facilitate comparison with the black ring solutions discussed in the previous section.

Pomeransky-Sen'kov black ring

Compared to [88], we choose mostly plus signature, and exchange $\phi \leftrightarrow \psi$ to conform to the notation of the previous section. In [88], angles ψ and ϕ have been rescaled to have canonical periodicity 2π. This can be a potential source of confusion. For this reason we have used ψ_{ER} and ϕ_{ER} to denote the Emparan-Reall coordinates in the previous section.

The metric functions take a fairly complicated form in the general case in which the black ring is not in equilibrium, but they simplify significantly when the balancing condition is imposed. In all metrics we write for the doubly spinning black rings, the balancing condition is imposed. Moreover, λ in the solution in [88] has different meaning from the solution above. It roughly corresponds to the parameter ν above. ν in [88] is the new parameter of the doubly spinning black ring. We replace (λ, ν) of [88] with (ν, η), respectively. To summarise,

$$\lambda_{PS} = \nu, \tag{5.2.26}$$

$$\nu_{PS} = \eta. \tag{5.2.27}$$

With this notation, when we take η to zero we recover the balanced Emparan-Reall black ring.

The doubly spinning black ring metric takes the form,

$$
\begin{aligned}
ds^2 &= -\frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\psi^2 - 2\frac{J(x,y)}{H(y,x)}d\psi d\phi + \frac{F(y,x)}{H(y,x)}d\phi^2 \\
&\quad + \frac{2k^2 H(x,y)}{(x-y)^2(1-\eta)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right), \qquad (5.2.28)
\end{aligned}
$$

with

$$
\begin{aligned}
\Omega &= \omega_\phi d\phi + \omega_\psi d\psi \qquad (5.2.29)\\
&= -\frac{2k\nu\sqrt{(1+\eta)^2-\nu^2}}{H(y,x)}\big[(1-x^2)y\sqrt{\eta}\,d\phi \\
&\quad + \frac{1+y}{1-\nu+\eta}\left(1+\nu-\eta+x^2y\eta(1-\nu-\eta)+2\eta x(1-y)\right)d\psi\big], \qquad (5.2.30)
\end{aligned}
$$

and the functions G, H, J, F take the form

$$
\begin{aligned}
G(x) &= (1-x^2)\left(1+\nu x+\eta x^2\right), \qquad (5.2.31)\\
H(x,y) &= 1+\nu^2-\eta^2+2\nu\eta(1-x^2)y+2x\nu(1-y^2\eta^2)+x^2y^2\eta(1-\nu^2-\eta^2), \qquad (5.2.32)\\
J(x,y) &= \frac{2k^2(1-x^2)(1-y^2)\nu\sqrt{\eta}}{(x-y)(1-\eta)^2}\left(1+\nu^2-\eta^2+2(x+y)\nu\eta-xy\eta(1-\nu^2-\eta^2)\right), \qquad (5.2.33)\\
F(x,y) &= \frac{2k^2}{(x-y)^2(1-\eta)^2}\Big[G(x)(1-y^2)\left[\left((1-\eta)^2-\nu^2\right)(1+\eta)+y\nu(1-\nu^2+2\eta-3\eta^2)\right] \\
&\quad + G(y)\big[2\nu^2+x\nu((1-\eta)^2+\nu^2)+x^2\left((1-\eta)^2-\nu^2\right)(1+\eta)+x^3\nu(1-\nu^2-3\eta^2+2\eta^3) \\
&\quad - x^4(1-\eta)\eta(-1+\nu^2+\eta^2)\big]\Big]. \qquad (5.2.34)
\end{aligned}
$$

In order to recover the metric (5.2.1) together with the condition (5.2.5) one must take $\eta \to 0$, identify $R^2 = 2k^2(1+\nu^2)$, and relate $\psi_{ER} = \frac{\psi}{\sqrt{1+\nu^2}}$, $\phi_{ER} = \frac{\phi}{\sqrt{1+\nu^2}}$.

In the above metric, note that the parameter η appears under the square root sign. Therefore, for the metric to be real $\eta \geq 0$. The positivity of the mass of the black ring requires $\nu > 0$ and the finiteness requires $1+\eta-\nu > 0$ (see below). The finiteness of angular momenta requires $\eta < 1$. Furthermore, the parameters ν and η are restricted to $\nu \geq 2\sqrt{\eta}$. This condition is a Kerr-like bound on the rotation of the S^2. To see this, consider the equation for vanishing $G(y)$, which determines the position of the event horizon within the allowed range $-\infty < y < -1$,

$$1+\nu y+\eta y^2=0. \tag{5.2.35}$$

Imposing that the roots of (5.2.35) are real yields the required bound. The regular event horizon is at

$$y_h = \frac{-\nu+\sqrt{\nu^2-4\eta}}{2\eta}. \tag{5.2.36}$$

To summarise, the restrictions on the parameters ν, η are

$$0 \le \eta < 1\,, \qquad 2\sqrt{\eta} \le \nu < 1+\eta\,. \tag{5.2.37}$$

Doubly spinning two-charge black ring

Following the solution generating procedure outlined in appendix D applied to the Pomeransky-Sen'kov black ring, we get the doubly spinning two-charge black ring. This solution was also considered by Hoskisson [90][2]. We have,

$$\begin{aligned} ds^2 &= -\frac{1}{(h_\alpha h_\beta)^{2/3}}\frac{H(y,x)}{H(x,y)}(dt+c_\alpha c_\beta \Omega)^2 + (h_\alpha h_\beta)^{1/3}\frac{2k^2 H(x,y)}{(x-y)^2(1-\eta)^2}\left(\frac{dx^2}{G(x)}-\frac{dy^2}{G(y)}\right) \\ &\quad +(h_\alpha h_\beta)^{1/3}\left[-\frac{F(x,y)}{H(y,x)}d\psi^2 - 2\frac{J(x,y)}{H(y,x)}d\psi d\phi + \frac{F(y,x)}{H(y,x)}d\phi^2\right], \end{aligned} \tag{5.2.38}$$

where we have defined

$$h_\alpha(x,y) = c_\alpha^2 - s_\alpha^2\frac{H(y,x)}{H(x,y)}, \qquad h_\beta(x,y) = c_\beta^2 - s_\beta^2\frac{H(y,x)}{H(x,y)}. \tag{5.2.39}$$

[2] Once again the parameters used there are slightly different from what we use. The ADM mass is not computed correctly in [90].

The dilaton Φ and the extra scalar χ are given as

$$e^{-2\Phi} = h_\alpha(x,y)\,, \qquad e^{-\frac{\sqrt{3}}{\sqrt{2}}\chi} = \frac{h_\beta(x,y)}{\sqrt{h_\alpha(x,y)}}\,. \tag{5.2.40}$$

The gauge fields are

$$A_t^{(1)} = \frac{H(x,y)-H(y,x)}{H(x,y)h_\beta(x,y)} c_\beta s_\beta\,, \qquad A_t^{(2)} = \frac{H(x,y)-H(y,x)}{H(x,y)h_\alpha(x,y)} c_\alpha s_\alpha\,, \tag{5.2.41}$$

$$A_\psi^{(1)} = -\frac{H(y,x)}{H(x,y)h_\beta(x,y)} \omega_\psi c_\alpha s_\beta\,, \qquad A_\psi^{(2)} = -\frac{H(y,x)}{H(x,y)h_\alpha(x,y)} \omega_\psi c_\beta s_\alpha\,, \tag{5.2.42}$$

$$A_\phi^{(1)} = -\frac{H(y,x)}{H(x,y)h_\beta(x,y)} \omega_\phi c_\alpha s_\beta\,, \qquad A_\phi^{(2)} = -\frac{H(y,x)}{H(x,y)h_\alpha(x,y)} \omega_\phi c_\beta s_\alpha\,. \tag{5.2.43}$$

The antisymmetric tensor field has as non-zero components,

$$B_{\psi t} = -\frac{H(y,x)}{H(x,y)h_\alpha(x,y)} \omega_\psi s_\alpha s_\beta\,, \tag{5.2.44}$$

$$B_{\phi t} = -\frac{H(y,x)}{H(x,y)h_\alpha(x,y)} \omega_\phi s_\alpha s_\beta\,. \tag{5.2.45}$$

The event horizon is at $y = y_h$ given in (5.2.36). The charged solution is smooth everywhere on and outside the event horizon.

Physical properties of the doubly spinning two-charge black ring

The physical parameters, mass M, angular momenta J_ψ, J_ϕ, area of the event horizon $\mathscr{A}_H$, angular velocities Ω_ψ, Ω_ϕ, and the temperature of the horizon T for the solution can be computed following, say, [88, 91]. We find,

$$\begin{aligned}
M &= \frac{k^2\pi\nu}{G_N(1-\nu+\eta)}\left(1+\cosh 2\alpha+\cosh 2\beta\right), && (5.2.46)\\
J_\phi &= \frac{4k^3\pi\nu\sqrt{\eta}\sqrt{(1+\eta)^2-\nu^2}}{G_N(1-\eta)^2(1-\nu+\eta)}c_\alpha c_\beta, && (5.2.47)\\
J_\psi &= \frac{2k^3\pi\nu\left(1+\nu-6\eta+\nu\eta+\eta^2\right)\sqrt{(1+\eta)^2-\nu^2}}{G_N(1-\eta)^2(1-\nu+\eta)^2}c_\alpha c_\beta, && (5.2.48)\\
\mathscr{A}_H &= -\frac{32\pi^2k^3(1+\nu+\eta)\nu}{(y_h-1/y_h)(1-\eta)^2}c_\alpha c_\beta, && (5.2.49)\\
\Omega_\phi &= \frac{1}{kc_\alpha c_\beta}\frac{\nu(1+\eta)-(1-\eta)\sqrt{\nu^2-4\eta}}{4\nu\sqrt{\eta}}\sqrt{\frac{1+\eta-\nu}{1+\eta+\nu}}, && (5.2.50)\\
\Omega_\psi &= \frac{1}{2kc_\alpha c_\beta}\sqrt{\frac{1+\eta-\nu}{1+\eta+\nu}}, && (5.2.51)\\
T &= \frac{\sqrt{\nu^2-4\eta}(1-\eta)(y_h^{-1}-y_h)}{8\pi k\nu(1+\nu+\eta)c_\alpha c_\beta}, && (5.2.52)
\end{aligned}$$

where y_h is the location of the horizon given in (5.2.36). The U(1) charges defined via (5.2.23) take values

$$\begin{aligned}
\mathbf{Q}_1 &= \frac{k^2\pi\nu}{G_N(1-\nu+\eta)}\sinh 2\alpha, && (5.2.53)\\
\mathbf{Q}_2 &= \frac{k^2\pi\nu}{G_N(1-\nu+\eta)}\sinh 2\beta. && (5.2.54)
\end{aligned}$$

One can readily check that these expressions go over to the singly spinning case when $\eta\to 0$.

5.3 Dipoles

A circular string sources an electric field $B_{t\psi}$, whose magnetic dual A_ϕ is sourced by a circular distribution of magnetic monopoles [76]. In addition to the charges $\mathbf{Q}_i$ defined in (5.4.4), the topology of a black ring allows to define 'dipole charges' q_i as

$$q_i=\frac{1}{2\pi}\int_{S^2}F^i\,, \qquad (5.3.1)$$

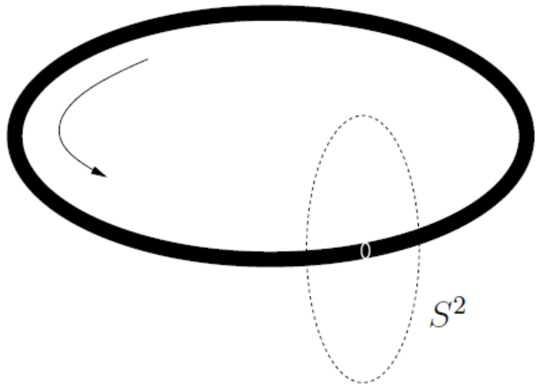

Figure 5.2: The dipole q_i measured from the magnetic flux of F^i across an S^2 enclosing a section of the ring.

by performing the integral on a surface S^2 that links the ring once.

Even if there is a local distribution of charge, the total magnetic charge is zero. It can be attibuted to the fact that conserved charges are calculated by integrating over fluxes at infinity and in this case, no flux reaches infinity.

5.4 5D $U(1)^3$ supergravity down from 11D

The conserved charges and dipole moments of black rings allow for a natural interpretation within string/M-theory. In this framework, the relevant five-dimensional supergravity theories are most conveniently understood as arising from dimensional reduction of eleven-dimensional supergravity, which is the low-energy limit of M-theory.

Eleven-dimensional supergravity with the metric and a 3-form potential A with 4-form field strength $F = dA$ as bosonic fields has action

$$I_{11} = \frac{1}{16\pi G_{11}} \int \left(R_{11} \star_{11} 1 - \frac{1}{2} F \wedge \star_{11} F - \frac{1}{6} F \wedge F \wedge A \right) , \tag{5.4.1}$$

where R_{11} and $\star_{11}$ denote the eleven-dimensional Ricci scalar and Hodge dual respectively. Perform dimensional reduction on T^6 using the ansatz

$$\begin{aligned} ds^2_{11} &= ds^2_5 + Y^1\left(dx_1^2+dx_2^2\right)+Y^2\left(dx_3^2+dx_4^2\right)+Y^3\left(dx_5^2+dx_6^2\right), \\ A &= A^1\wedge dx_1\wedge dx_2 + A^2\wedge dx_3\wedge dx_4 + A^3\wedge dx_5\wedge dx_6 \,. \end{aligned} \tag{5.4.2}$$

to get a five-dimensional supergravity theory.

T^6 is parametrized by the coordinates $\{x^1,x^2,x^3,x^4,x^5,x^6\}$. Assuming that nothing depends on the coordinates x^i parametrizing the T^6, ds_5^2, Y^i and A^i can be regarded as a five-dimensional metric, scalars, and vectors respectively.

We demand the constraint $Y^1Y^2Y^3 = 1$ to make T^6 have a constant volume, making the metric ds_5^2 five-dimensional Einstein-frame metric. The eleven-dimensional action reduces to the action of five-dimensional $U(1)^3$ supergravity

$$I_5 = \frac{1}{16\pi G_5}\int\left(R\star 1 - \mathscr{G}_{ij}dY^i\wedge\star dY^j - \mathscr{G}_{ij}F^i\wedge\star F^j - \frac{1}{6}\mathscr{C}_{ijk}F^i\wedge F^j\wedge A^k\right), \tag{5.4.3}$$

with $\mathscr{G}_{ij} = \frac{1}{2}\mathrm{diag}\left((Y^1)^{-2},(Y^2)^{-2},(Y^3)^{-2}\right)$, $\mathscr{C}_{ijk} = 1$ when (ijk) is a permutation of (123) and 0 otherwise. $F^i = dA^i$ are Maxwell field strengths. Thus, we end up with $\mathscr{N} = 1 D = 5$ minimal supergravity coupled with two $U(1)$ vector multiplets.

Conserved electric charges for asymptotically flat solutions are given by

$$\mathbf{Q}_i = \frac{1}{16\pi G_5}\int_{S^3}(Y^i)^{-2}\star_5 F^i, \tag{5.4.4}$$

where the S^3 is taken at spatial infinity. Following [126] it can be shown that regular solutions of the theory satisfy

$$M \geq |\mathbf{Q}_1| + |\mathbf{Q}_2| + |\mathbf{Q}_3|, \tag{5.4.5}$$

where M is the ADM mass. The equality holds for supersymmetric cases. The form of the eleven-dimensional field strength suggests that the electric charge $\mathbf{Q}_i$ that couples to F^i arises from M2-branes wrapped on the internal T^6, e.g F^1 is sourced by M2-branes wrapping the 12 cycle of T^6 etc. The charges are quantized in terms of the wrapping numbers of the M2-branes as [76]

$$N_i = \left(\frac{4G_5}{\pi}\right)^{1/3} \mathbf{Q}_i \,. \tag{5.4.6}$$

These solutions can be mapped to a U-duality frame that is convenient for microscopic analysis. For example, these can be recast as solutions for a D1-D5 system with momentum along their common direction. To demonstrate this, dimensionally reduce the eleven-dimensional solution above on the x^6 direction to give a solution of ten-dimensional type IIA supergravity. Performing T-dualities along the x^5, x^4, x^3 directions gives a solution of type IIB supergravity with metric

$$ds^2 = -(Y^3)^{1/2} ds_5^2 + (Y^3)^{-3/2} \left(dx + A^3\right)^2 + Y^1 (Y^3)^{1/2} d\mathbf{x}_4^2, \tag{5.4.7}$$

where the T^5 is parametrized by the coordinates $x \equiv x^5$ and $\mathbf{x}_4 \equiv (x^1, x^2, x^3, x^4)$.
The other non-zero IIB fields are

$$e^{2\Phi} = \frac{Y^1}{Y^2}, \qquad F_{(3)} = \left(Y^1\right)^{-2} \star_5 F^1 + F^2 \wedge \left(dx + A^3\right), \tag{5.4.8}$$

where Φ is the dilaton, $F_{(3)}$ the Ramond-Ramond 3-form field strength, and $\star_5$ denotes the Hodge dual with respect to the five-dimensional metric.
Examining the RR 3-form reveals that the electric charges that couple to F^1 and F^2 arise from D5-branes wrapped on T^5 and D1-branes wrapped around the z-circle respectively. The appearance of A^3 in the metric reveals that F^3 is electrically sourced by momentum (P) around the KK x-circle.

5.5 Supersymmetric two charge black ring with single rotation

We use the balancing condition (5.2.5) to replace λ which appears in the singly spinning solution in favour of ν. The BPS limit is obtained by taking $\nu \to 0, \alpha \to \infty, \beta \to \infty$ such that we keep the charges fixed [86]. In this limit, the factor $\frac{\lambda}{1-\nu}$ that features prominently in the mass and charge expressions (5.2.18), (5.2.24)–(5.2.25) goes to 2ν. We define,

$$2\nu \sinh 2\alpha = \frac{Q_1}{R^2}, \qquad 2\nu \sinh 2\beta = \frac{Q_2}{R^2}. \tag{5.5.1}$$

These relations imply,

$$\alpha = \frac{1}{2}\sinh^{-1}\frac{Q_1}{2R^2\nu}, \qquad \beta = \frac{1}{2}\sinh^{-1}\frac{Q_2}{2R^2\nu}, \tag{5.5.2}$$

i.e., α and β are replaced in favour of Q_1 and Q_2. After this replacements, we take the $\nu \to 0$ limit.

In this limit, $F(y) \to 1$, $F(x) \to 1$, and

$$h_\alpha \to 1 + \frac{Q_1}{2R^2}(x-y), \qquad h_\beta \to 1 + \frac{Q_2}{2R^2}(x-y), \tag{5.5.3}$$

and

$$Cc_\alpha c_\beta \to \frac{1}{2\sqrt{2}R^2}\sqrt{Q_1 Q_2}. \tag{5.5.4}$$

As a result, metric (5.2.12) becomes,

$$ds_5^2 = -f^{-2}\left(dt - \frac{1}{2\sqrt{2}R}\sqrt{Q_1 Q_2}(1+y)d\psi\right)^2 + f ds^2_{\text{base}}, \tag{5.5.5}$$

where ds^2_{base} is the four-dimensional flat base space in ring coordinates,

$$ds^2_{\text{base}} = \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right]. \tag{5.5.6}$$

The coordinates ϕ and ψ now have canonical periodicity 2π. The function f takes the form,

$$f^3 = \left(1+\frac{Q_1}{2R^2}(x-y)\right)\left(1+\frac{Q_2}{2R^2}(x-y)\right). \tag{5.5.7}$$

The right hand side of eq. (5.5.7) is a product of two harmonic functions, in the same notation as in ref. [76, section 5.1], with the identification $Q_3 = 0, q_1 = 0, q_2 = 0, q_3 = \frac{1}{\sqrt{2}R}\sqrt{Q_1 Q_2}$. Note that, in this solution the dipole charge q_3 is fixed in terms of Q_1 and Q_2; it is not an independent parameter.

To expand a little more on this point about the harmonic functions, let us write four-dimensional flat space in the coordinates

$$ds^2_{\text{base}} = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2. \tag{5.5.8}$$

The transformation

$$y = -\frac{R^2+r_1^2+r_2^2}{\Sigma}, \qquad x = \frac{R^2-r_1^2-r_2^2}{\Sigma}, \qquad \Sigma = \sqrt{(r_1^2+r_2^2+R^2)^2-4R^2r_2^2}, \tag{5.5.9}$$

with inverse

$$r_1 = R\frac{\sqrt{1-x^2}}{x-y}, \qquad r_2 = R\frac{\sqrt{y^2-1}}{x-y}. \tag{5.5.10}$$

takes us to the ring coordinates (5.5.6). With these transformations at hand, we observe that

$$\frac{1}{2R^2}(x-y) = \frac{1}{\Sigma}. \tag{5.5.11}$$

The function Σ^{-1} solves the Laplace equation on four-dimensional flat space for a ring source at $r_1 = 0$, $r_2 = R$, $0 \le \psi < 2\pi$.

A key point is that the event horizon area of this BPS solution is zero. To see this, we observe that the prefactor

$$\frac{\nu^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)} \tag{5.5.12}$$

in (5.2.19) goes as $\sqrt{2}\nu^2$ in the $\nu \to 0$ limit. Thus, we see that if we keep the mass finite, the area goes to zero. We conclude that upon taking the supersymmetric limit we get a "small" black ring with zero horizon area,

$$S_{\text{BPS}} = 0. \tag{5.5.13}$$

It is well appreciated [86] that the supersymmetric solution described above is dual to a D1-D5 supertube [92, 93, 98]. In the F1-P duality frame (for heterotic set-up), this system has been called a "small" black ring in 5D supergravity [78–80]; it is expected that if we consider higher derivative corrections to the supergravity action a finite area horizon would emerge.

5.6 The complex saddle solution with two rotations

In this section, we propose a complex solution with two rotations that serves as a saddle for computing the index for the small BPS black ring of the previous section.

To set the context, we start with a quick review of the key ideas from [71, 72] following [99]. The rotation group in five dimensions is $SO(4) = SU(2)_L \times SU(2)_R$. A generic black hole in five dimensions rotates in two orthogonal planes with ψ and ϕ as the azimuthal angles. For black rings, motivated by the 4D-5D connection [100, 101], the third components of $SU(2)_L$ and $SU(2)_R$ angular momenta are identified as [78–80],

$$J_{3L} = J_\psi, \tag{5.6.1}$$

$$J_{3R} = J_\phi. \tag{5.6.2}$$

The supersymmetric black rings of the previous section have $J_{3R} = J_\phi = 0$. The relevant index for a supersymmetric black hole that breaks some $SU(2)_L$ invariant supersymmetries is [99][3]

$$e^{S_{\text{BPS}}} = \text{Tr}_{\vec{\mathbf{Q}},J_{3L}}\left[(-1)^F\right], \tag{5.6.3}$$

where the trace is taken over all states carrying fixed charges $\vec{\mathbf{Q}} = (\mathbf{Q}_1, \mathbf{Q}_2)$ and J_{3L}. The trace is taken over all values of $\vec{J}^2_{3L}, J_{3R}, \vec{J}^2_{3R}$.

To compute $e^{S_{\text{BPS}}}$ from the macroscopic side we begin with the gravitational partition function with boundary conditions appropriate to that of an Euclidean black hole with inverse temperature β and angular velocities $\Omega_{L,R}$. Let the entropy of the Euclidean black hole be S_0. Going around the Euclidean time $\tau = it$ gives the periodic identification (see, e.g., [102]),

$$(\tau, \psi, \phi) \equiv (\tau + \beta, \psi - i\beta\,\Omega_\psi, \phi - i\beta\,\Omega_\phi). \tag{5.6.4}$$

The partition function defined via the gravitational path integral in this way computes,

$$Z(\beta, \Omega_\psi, \Omega_\phi, \mu_1, \mu_2) = \text{Tr}\left[e^{-\beta M + \beta\Omega_\psi J_\psi + \beta\Omega_\phi J_\phi + \beta\mu_1\mathbf{Q}_1 + \beta\mu_2\mathbf{Q}_2}\right], \tag{5.6.5}$$

where μ_1 and μ_2 are the chemical potentials for the charges $\mathbf{Q}_1$ and $\mathbf{Q}_2$ respectively and the trace is over all states.[4]

For computing the index [71, 72], we set the chemical potential Ω_ϕ dual to J_ϕ such that,

$$\beta\Omega_\phi = 2\pi i. \tag{5.6.6}$$

The partition function (5.6.5) can then be related to the index $e^{S_{\text{BPS}}}$. In the classical limit, the relation is [99],

$$S_{\text{BPS}} = S_0 - \beta M + \beta M_{\text{BPS}} + 2\pi i J_\phi. \tag{5.6.7}$$

[3]A discussion that also includes fermion zero modes can be found in [99]. For a more detailed discussion on the definition of an appropriate index for small black rings, see section 3 of [81].

[4]Our conventions are such that the first law of black hole mechanics takes a more standard form $TdS = dM - \Omega_i dJ_i - \mu_i d\mathbf{Q}_i$.

Here, M denotes the mass of the black hole solution corresponding to the saddle point that contributes to the index. Although, there is no detailed understanding, in all examples known so far, and on physical grounds we expect,

$$M = M_{\text{BPS}}. \tag{5.6.8}$$

In that case, equation (5.6.9) becomes,

$$S_{\text{BPS}} = S_0 + 2\pi i J_\phi. \tag{5.6.9}$$

This relates the entropy of an extremal (zero temperature BPS) black hole S_{BPS} to that of a non-extremal (non-zero temperature but supersymmetric) black hole S_0. We test both (5.6.8) and (5.6.9) for the small black ring of the previous section.

A key intuition for finding the relevant saddle solutions comes from the following observation: If we identify $y \to -R/r$, $\nu \to 2m/R$ and $\eta \to a^2/R^2$, which are the correct identifications in the infinite radius limit of the doubly spinning black ring to the boosted Kerr black string, then equation (5.2.35) becomes the familiar $r^2 - 2mr + a^2 = 0$ for the Kerr black hole. Refs. [82, 83] identify the saddle solutions for small black holes in four dimensions via the analytic continuation $a \to ia$. This suggests that the analytic continuation we need to consider is to negative values of η. We also replace the parameter k with the parameter R via

$$k = \frac{R}{\sqrt{2(1+\nu^2)}}. \tag{5.6.10}$$

In the $\eta \to 0$ limit, the parameter R matches on the parameter R of the singly spinning solution.

We propose that the saddle solution for the small black ring is obtained by taking the supersymmetric limit[5] $\nu \to 0$ of the doubly spinning two-charge black ring. We take the

[5]Although we call this the supersymmetric limit, one cannot conclude the existence of Killing spinors until the solution is mapped to a Bena-Warner form, which is done later in the paper. With this hindsight we call the $\nu \to 0$ limit the supersymmetric limit. We reserve the phrase "BPS" for referring to the singly spinning supersymmetric black ring.

supersymmetric limit keeping the temperature and charges fixed. We define,

$$\alpha = \frac{1}{2}\sinh^{-1}\frac{Q_1}{2R^2\nu}, \qquad \beta = \frac{1}{2}\sinh^{-1}\frac{Q_2}{2R^2\nu}, \tag{5.6.11}$$

and take $\nu \to 0$. Note that, we use the same relations as we used above for the singly spinning black ring (5.5.2). In particular, we have not introduced any η dependent factors in (5.6.11). This has both advantages and disadvantages. The advantages are that some of the equations below come out simpler. The disadvantage is that the canonically defined charges $\mathbf{Q}_1$, $\mathbf{Q}_2$ take the form

$$\mathbf{Q}_1 = \frac{\pi}{4G_N}\frac{Q_1}{1+\eta}, \qquad \mathbf{Q}_2 = \frac{\pi}{4G_N}\frac{Q_2}{1+\eta}, \tag{5.6.12}$$

with an η dependent factor. The advantages outnumber this (minor) disadvantage, and we use (5.6.11) as we take the supersymmetric limit.

Let us now look at the supersymmetric limit of the ADM mass, cf. (5.2.46). We get,

$$M = \frac{\pi}{4G_N}\left(\frac{Q_1}{1+\eta} + \frac{Q_2}{1+\eta}\right). \tag{5.6.13}$$

Thus,

$$M = \mathbf{Q}_1 + \mathbf{Q}_2, \tag{5.6.14}$$

which confirms (5.6.8), i.e., the saddle solution has the same mass as the BPS black ring in terms of the physical charges $\mathbf{Q}_1$, $\mathbf{Q}_2$. Algebraically speaking, the η dependent factor needs to be taken into account when expressing $\mathbf{Q}_1$, $\mathbf{Q}_2$ in terms of Q_1 and Q_2.

In this limit, the entropy $S_0 = \mathscr{A}_H/(4G_N)$ of the doubly spinning two-charge black ring behaves as, cf. (5.2.49),

$$S_0 = \frac{\pi^2 R\sqrt{-\eta}}{\sqrt{2}G_N(1-\eta)^2}\sqrt{Q_1 Q_2}. \tag{5.6.15}$$

For $-1 < \eta < 0$, this quantity is non-zero and positive. In the same limit J_ϕ becomes, cf. (5.2.47),

$$J_\phi = \frac{\pi R\sqrt{\eta}}{2\sqrt{2}G_N(1-\eta)^2}\sqrt{Q_1Q_2}. \tag{5.6.16}$$

For the branch

$$\sqrt{\eta} = i\sqrt{-\eta}, \tag{5.6.17}$$

in the range $-1 < \eta < 0$, we observe that the expressions are such that

$$S_0 + 2\pi i J_\phi = 0. \tag{5.6.18}$$

This confirms (5.6.9). In particular, J_ϕ is purely imaginary. We conclude that the proposed saddle solution satisfies both (5.6.8) and (5.6.9). In this limit, (5.6.6) is also satisfied.

The angular momentum J_ψ of the saddle solution is given as, cf. (5.2.48),

$$J_\psi = \frac{\pi R(1-6\eta+\eta^2)}{4\sqrt{2}G_N(1-\eta)^2(1+\eta)}\sqrt{Q_1Q_2}. \tag{5.6.19}$$

This expression depends on the parameter η. The inverse temperate β takes the form,

$$\beta = \frac{\pi\sqrt{Q_1Q_2}}{\sqrt{2}R(1-\eta)}. \tag{5.6.20}$$

Apart from the physical charges $\mathbf{Q}_1, \mathbf{Q}_2, J_\psi$, the saddle solution has an extra parameter η which allows us to adjust the size of the Euclidean time circle. Ω_ψ in the supersymmetric limit goes to zero as expected for supersymmetric rotating black holes. Unfortunately, β does not go to infinity as $\eta \to 0$. A similar issue as was pointed out by Chen, Murthy, and Turiaci in [83] (and it was implicit in the analysis of [82]) for four-dimensional small black holes. We do not have anything further to add to this point than what was already said in [83], except that unlike [82, 83] expression (5.6.20) has some non-trivial dependence on the S^2 rotation parameter η. More work is required to fully understand small black holes in the four-dimensional context. The fact that $\beta^{-1} \neq 0$ in general suffices for our considerations in this chapter.

5.7 The nature of the saddle solution

In this section, we present a detailed analysis of the saddle solution. Since the area of the small black ring is zero, the saddle solution is expected to have some singularities. We analyse these singular locations first.

5.7.1 Singular locations

In the supersymmetric limit $\nu \to 0$, the location of the horizon (5.2.36) becomes,

$$y = y_h = -\frac{1}{\sqrt{-\eta}}. \tag{5.7.1}$$

For $\eta < 0$ this is in the physically allowed range of the y coordinate. In the following, it is convenient to take

$$\eta = -b^2, \tag{5.7.2}$$

then

$$y = y_h = -b^{-1}, \tag{5.7.3}$$

is the location of the horizon. In the $\nu \to 0$ limit, the five-dimensional dilaton (5.2.40) takes the form,

$$e^{-2\Phi} = 1 + \frac{Q_1}{2R^2} \frac{(x-y)(1+b^2xy)}{(1-b^2)(1-b^2x^2y^2)}. \tag{5.7.4}$$

At the poles $x = \pm 1$ of the S^2 cross-section of the horizon $y_h = -b^{-1}$ the dilaton $e^{-2\Phi}$ diverges. This in particular means that the solution is singular at the poles of the S^2 cross-section of the horizon. The situation is analogous to the small black holes analysed in ref. [82]. The singular locations are two rings,

$$y = -b^{-1}, \qquad x = \pm 1, \qquad 0 \leq \psi < 2\pi. \tag{5.7.5}$$

We leave an investigation of the geometry near the singular locations for the future. For the small black holes this geometry was studied in detail in [82].[6] Instead, we focus on the nature of the sources that make up the supersymmetric solution.

5.7.2 Four-dimensional base space

In the $\nu \to 0$ limit, functions (5.2.31)–(5.2.34) become,

$$G(x) = (1-x^2)\left(1+\eta x^2\right), \quad H(x,y) = (1-\eta^2)(1+\eta x^2 y^2), \tag{5.7.6}$$

$$J(x,y) = 0, \quad F(x,y) = \frac{R^2}{(x-y)^2}(1+\eta)(1-y^2)(1+\eta x^2)(1+\eta x^2 y^2). \tag{5.7.7}$$

In particular,

$$\frac{H(y,x)}{H(x,y)} = 1. \tag{5.7.8}$$

With this observation, it is clear that the final metric takes the form,

$$ds_5^2 = -f^{-2}\left(dt + \Omega^{(s)}\right)^2 + f ds^2_{\text{base}}, \tag{5.7.9}$$

where

$$ds^2_{\text{base}} = \frac{R^2 H(x,y)}{(x-y)^2(1-\eta)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right) - \frac{F(x,y)}{H(y,x)}d\psi^2 + \frac{F(y,x)}{H(y,x)}d\phi^2, \tag{5.7.10}$$

and

$$\Omega^{(s)} = \omega^{(s)}_\phi d\phi + \omega^{(s)}_\psi d\psi \tag{5.7.11}$$

$$= -\frac{\left(1-x^2\right) y\sqrt{Q_1 Q_2}}{2\sqrt{2}(1-\eta)R\left(1+\eta x^2 y^2\right)} i\sqrt{-\eta}\, d\phi - \frac{(1+y)\sqrt{Q_1 Q_2}\left(1-\eta\left(1-2x+2xy-x^2 y\right)-\eta^2 x^2 y\right)}{2\sqrt{2}R\left(1-\eta^2\right)\left(1+\eta x^2 y^2\right)} d\psi. \tag{5.7.12}$$

[6] For the small black rings (with one rotation) this analysis of done in great detail in [80].

For $-1 < \eta < 0$, we note that $\Omega^{(s)}$ necessarily has an imaginary part. Since it is a priori not clear what regularity conditions must be imposed on a complex metric, we cannot comment on the smoothness of the metric. It is as good as a complex metric as other metrics that have featured in similar discussions [102].

A simple calculation using Mathematica shows that ds^2_{base} is flat space.[7] The following coordinate transformation

$$r_1^2 = \frac{R^2\left(1-x^2\right)\left(1+\eta y^2\right)}{(1-\eta)(x-y)^2}, \tag{5.7.13}$$

$$r_2^2 = \frac{R^2\left(y^2-1\right)\left(1+\eta x^2\right)}{(1-\eta)(x-y)^2}, \tag{5.7.14}$$

brings the base space in a standard form,

$$ds^2 = dr_1^2 + dr_2^2 + r_1^2 d\phi^2 + r_2^2 d\psi^2. \tag{5.7.15}$$

Using $\eta = -b^2$, we observe that singularities (5.7.5) are at

$$y = -b^{-1}, \quad x = +1 \implies r_1 = 0, \quad r_2 = R_+ = \frac{(1-b)}{\sqrt{1+b^2}}R, \tag{5.7.16}$$

$$y = -b^{-1}, \quad x = -1 \implies r_1 = 0, \quad r_2 = R_- = \frac{(1+b)}{\sqrt{1+b^2}}R. \tag{5.7.17}$$

Note that for $b > 0$, $R_+ < R_-$.

Even though the solution is singular and the metric is complex, the area calculation is unambiguous. We note that at the horizon $y = y_h = -b^{-1}$ the size of the ϕ circle vanishes in the four-dimensional base metric, i.e., $r_1 = 0$ from (5.7.13). Using this observation in (5.7.9) and computing the area of the horizon by considering the induced metric on constant t and

[7] The 4D base space (5.7.10) might be described as a flat space metric in ellipsoidal ring coordinates: that is, surfaces of constant y correspond to rings with ellipsoidal (rather than spherical) cross-sections at fixed ψ. We leave a more detailed investigation of this point too for the future. We thank Roberto Emparan for this comment.

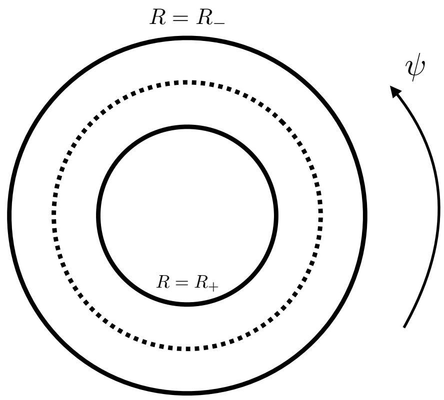

Figure 5.3: *For the non-extremal saddle solution, the harmonic functions are sourced at two rings $r_1 = 0, r_2 = R_\pm, 0 \leq \psi < 2\pi$ on the four-dimensional flat base space. The $\pm$ signs refer to the poles $x = \pm 1$ of the S^2 cross-section of the horizon. Since $x = -1$ lies on the outside, $R_- > R_+$.*

constant y surface, we find

$$(\det g)\Big|_{y=-b^{-1},\ t=\text{constant}}^{(x,\phi,\psi)} = -g_{xx}^{\text{base}} \cdot g_{\psi\psi}^{\text{base}} \cdot \left(\omega_\phi^{(s)}\right)^2. \tag{5.7.18}$$

All factors on the right hand side of this equation are regular. In particular, singular terms all cancel out. Thus the area of the solution is well defined. It is expected that the higher derivative corrections in 5D would smoothen out the singular features of the solution. A detailed analysis of these issues is certainly beyond the scope of this paper.

5.7.3 Solution in terms of 4D harmonic functions

The function f in eq. (5.7.9) is of the form,

$$f^3 = h_1 h_2, \tag{5.7.19}$$

where the multiplicative factors are obtained via

$$h_\alpha \quad \to \quad h_1 = 1 + \frac{Q_1}{2R^2}\frac{(x-y)(1-\eta xy)}{(1+\eta)\,(1+\eta x^2y^2)}, \tag{5.7.20}$$

$$h_\beta \quad \to \quad h_2 = 1 + \frac{Q_2}{2R^2}\frac{(x-y)(1-\eta xy)}{(1+\eta)\,(1+\eta x^2y^2)}, \tag{5.7.21}$$

in the $\nu \to 0$ limit. The vector fields take the following simple form,

$$A_t^{(1)} = 1 - h_2^{-1}, \qquad A_\psi^{(1)} = -h_2^{-1}\Omega_\psi^{(s)}, \qquad A_\phi^{(1)} = -h_2^{-1}\Omega_\phi^{(s)}, \tag{5.7.22}$$

$$A_t^{(2)} = 1 - h_1^{-1}, \qquad A_\psi^{(2)} = -h_1^{-1}\Omega_\psi^{(s)}, \qquad A_\phi^{(2)} = -h_1^{-1}\Omega_\phi^{(s)}. \tag{5.7.23}$$

The antisymmetric tensor field has as non-zero components,

$$B_{\psi t} = -h_1^{-1}\Omega_\psi^{(s)}, \qquad B_{\phi t} = -h_1^{-1}\Omega_\phi^{(s)}. \tag{5.7.24}$$

A simple calculation using Mathematica shows that the function that appears in (5.7.20)–(5.7.21)

$$\frac{(x-y)(1-\eta xy)}{(1+\eta x^2y^2)} \tag{5.7.25}$$

is a hamonic function for the metric (5.7.10). The function f^3 is thus a product of two harmonic functions. Using $\eta = -b^2$, we observe that the harmonic function (5.7.25) can be split as

$$\frac{(x-y)(1+b^2xy)}{(1-b^2x^2y^2)} = \frac{1+b}{2}\left(\frac{x-y}{1-bxy}\right) + \frac{1-b}{2}\left(\frac{x-y}{1+bxy}\right), \tag{5.7.26}$$

such that each of the two terms on the right hand side is harmonic. This split strongly suggests the "splitting-centers" picture of [89]: the harmonic functions of the saddle solution are sourced at two points on a suitable three-dimensional base space. We can make this precise as follows.

For a ring source located at $r_1 = 0, r_2 = R$ in the (r_1, ϕ, r_2, ψ) coordinates, we saw that the relevant harmonic function is Σ^{-1}, cf. (5.5.11), where

$$\Sigma = \sqrt{(r_1^2+r_2^2+R^2)^2-4R^2r_2^2}. \tag{5.7.27}$$

For the situation of interest now, the ring sources are located at $r_2 = R_\pm$ (5.7.16)–(5.7.17). A simple calculation shows that in the (x,ϕ,y,ψ) coordinates with metric (5.7.10), we have,

$$\Sigma_+ = \sqrt{(r_1^2+r_2^2+R_+^2)^2-4R_+^2r_2^2} = \frac{2R^2(1-b)(1+bxy)}{(1+b^2)(x-y)}, \tag{5.7.28}$$

$$\Sigma_- = \sqrt{(r_1^2+r_2^2+R_-^2)^2-4R_-^2r_2^2} = \frac{2R^2(1+b)(1-bxy)}{(1+b^2)(x-y)}. \tag{5.7.29}$$

Thus indeed,

$$h_1 = 1+\frac{Q_1}{2(1+b^2)}\left(\frac{1-b}{1+b}\frac{1}{\Sigma_+}+\frac{1+b}{1-b}\frac{1}{\Sigma_-}\right), \tag{5.7.30}$$

$$h_2 = 1+\frac{Q_2}{2(1+b^2)}\left(\frac{1-b}{1+b}\frac{1}{\Sigma_+}+\frac{1+b}{1-b}\frac{1}{\Sigma_-}\right), \tag{5.7.31}$$

i.e., the harmonic functions $h_{1,2}$ are sourced at the rings $R_\pm$. These rings are shown in figure 5.3. It turns out we can go one step further. The full saddle solution can be written in the Bena-Warner form [103, 104] with *three* centers. This is shown in section 5.8.

5.7.4 Infinite radius limit

In the infinite radius limit, the black ring saddle solution becomes the saddle solution for a four-dimensional small black hole uplifted to five dimensions with momentum added along the fifth-dimension. To see this, consider the boosted Kerr black string in five dimensions,

$$\begin{aligned} ds^2 &= -\left(1-\frac{2mr\cosh^2\sigma}{\Sigma}\right)dt^2+\frac{2mr\sinh(2\sigma)}{\Sigma}dtdw+\left(1+\frac{2mr\sinh^2\sigma}{\Sigma}\right)dw^2 \\ &\quad +\frac{\Sigma}{\Delta}dr^2+\Sigma d\theta^2+\frac{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 \\ &\quad -\frac{4mr\cosh\sigma}{\Sigma}a\sin^2\theta dtd\phi-\frac{4mr\sinh\sigma}{\Sigma}a\sin^2\theta dwd\phi, \end{aligned} \tag{5.7.32}$$

where

$$\Delta = r^2 + a^2 - 2mr, \qquad \Sigma = r^2 + a^2\cos^2\theta, \tag{5.7.33}$$

and where σ is the boost parameter, m the mass parameter, and a is the angular momentum parameters of the 4D Kerr solution. This metric solves vacuum Einstein equations in five-dimensions. Next, consider applying boost-T-duality-boost to add two charges to this black string. In the resulting metric, consider taking the supersymmetric limit as,

$$m \to 0, \qquad \alpha, \beta \to \infty, \tag{5.7.34}$$

such that the charges

$$\widetilde{Q}_1 = 4m\sinh 2\alpha, \qquad \widetilde{Q}_2 = 4m\sinh 2\beta, \tag{5.7.35}$$

are kept finite. Finally, we replace the rotation parameter as $a = ic$ and set $\sinh\sigma = 1$. We get,

$$ds^2 = -\frac{1}{(\widetilde{h}_1\widetilde{h}_2)^{\frac{2}{3}}}\left(dt + \frac{(2\widetilde{Q}_1\widetilde{Q}_2)^{\frac{1}{2}}r}{4(r^2 - c^2\cos^2\theta)}\left(ic\sin^2\theta d\phi - dw\right)\right)^2 + (\widetilde{h}_1\widetilde{h}_2)^{\frac{1}{3}}ds^2_{\text{base}}, \tag{5.7.36}$$

where now,

$$ds^2_{\text{base}} = \frac{r^2 - c^2\cos^2\theta}{r^2 - c^2}dr^2 + (r^2 - c^2\cos^2\theta)d\theta^2 + (r^2 - c^2)\sin^2\theta d\phi^2 + dw^2, \tag{5.7.37}$$

and

$$\widetilde{h}_1 = 1 + \frac{\widetilde{Q}_1 r}{2(r^2 - c^2\cos^2\theta)}, \qquad \widetilde{h}_2 = 1 + \frac{\widetilde{Q}_2 r}{2(r^2 - c^2\cos^2\theta)}. \tag{5.7.38}$$

These configurations serve as saddles for computing the gravitational index for a four-dimensional small black hole uplifted to five dimensions with (a specific value of) momentum added along the fifth-dimension.

The black ring saddle metric (5.7.9) becomes (5.7.36), upon changing

$$x = \cos\theta, \qquad y = -\frac{R}{r}, \qquad b = \frac{c}{R}, \qquad Q_{1,2} = R\widetilde{Q}_{1,2}, \qquad \psi = -\frac{w}{R}. \tag{5.7.39}$$

in the $R \to \infty$ limit. The other fields also match.

5.8 Saddle solution in the Bena-Warner form

Using results from [105], we can write the harmonic functions (5.7.30)–(5.7.31) in a form such that they are sourced at two points on a three-dimensional base space as suggested in [89]. We do the following series of coordinate transformations. First,

$$r_1 = \rho\cos\Theta, \qquad r_2 = \rho\sin\Theta, \qquad \phi = \frac{1}{2}(\phi_1 + \phi_2) \qquad \psi = \frac{1}{2}(\phi_1 - \phi_2), \tag{5.8.1}$$

then,

$$\Theta = \frac{1}{2}\theta, \qquad\qquad \rho = 2\sqrt{r}, \tag{5.8.2}$$

and finally,

$$x_1 = r\sin\theta\cos\phi_2, \qquad x_2 = r\sin\theta\sin\phi_2, \qquad x_3 = r\cos\theta. \tag{5.8.3}$$

The four-dimensional flat base space can be written as

$$ds^2_{\text{base}} = r(d\phi_1 + \cos\theta d\phi_2)^2 + \frac{1}{r}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi_2^2), \tag{5.8.4}$$

In these coordinates,

$$\frac{1}{4}\Sigma_+ = |\vec{x}-\vec{x}_+| = \sqrt{x_1^2+x_2^2+\left(x_3+\frac{R_+^2}{4}\right)^2}, \tag{5.8.5}$$

$$\frac{1}{4}\Sigma_- = |\vec{x}-\vec{x}_-| = \sqrt{x_1^2+x_2^2+\left(x_3+\frac{R_-^2}{4}\right)^2}, \tag{5.8.6}$$

where

$$\vec{x}_\pm = \left(0,0,-\frac{R_\pm^2}{4}\right). \tag{5.8.7}$$

In the form (5.8.4), the three-dimensional base in flat space. Now, we can easily trace through these coordinate transformations and write the full solution in the Bena-Warner form [103, 104]. For the Anupam-Chowdhury-Sen saddle solutions [99] this was done in [106, 107]. We refer the reader to [107] for a concise review of the Bena-Warner form and its relation to a standard $N=2, D=4$ supergravity notation.

For the black ring saddles, the eight Bena-Warner functions take the following form. The function V is simply,

$$V = \frac{1}{|\vec{x}|}, \tag{5.8.8}$$

which corresponds to the center at $\vec{x}=0$. This is expected, as the four-dimensional base space is flat space. The remaining functions are all either constants (zero or one) or have sources at $\vec{x}=\vec{x}_\pm$. Thus clearly, the $\vec{x}=0$ center is smooth. The three magnetic functions are

$$K^1 = 0, \tag{5.8.9}$$

$$K^2 = 0, \tag{5.8.10}$$

$$K^3 = \frac{k_+^3}{|\vec{x}-\vec{x}_+|}+\frac{k_-^3}{|\vec{x}-\vec{x}_-|}, \tag{5.8.11}$$

with

$$k_+^3 = \frac{\sqrt{Q_1 Q_2}}{4\sqrt{2}(1+b^2)R}\left(\frac{1-b}{1+b}+i\right), \tag{5.8.12}$$

$$k_-^3 = \frac{\sqrt{Q_1 Q_2}}{4\sqrt{2}(1+b^2)R}\left(\frac{1+b}{1-b}-i\right). \tag{5.8.13}$$

Note that these coefficients have imaginary parts. The three electric functions are

$$L_1 = h_1 = 1 + \frac{Q_1}{8(1+b^2)}\left(\frac{1-b}{1+b}\frac{1}{|\vec{x}-\vec{x}_+|} + \frac{1+b}{1-b}\frac{1}{|\vec{x}-\vec{x}_-|}\right), \tag{5.8.14}$$

$$L_2 = h_2 = 1 + \frac{Q_2}{8(1+b^2)}\left(\frac{1-b}{1+b}\frac{1}{|\vec{x}-\vec{x}_+|} + \frac{1+b}{1-b}\frac{1}{|\vec{x}-\vec{x}_-|}\right), \tag{5.8.15}$$

$$L_3 = 1, \tag{5.8.16}$$

and finally the function M is,

$$M = m_0 + \frac{m_+}{|\vec{x}-\vec{x}_+|} + \frac{m_-}{|\vec{x}-\vec{x}_-|}, \tag{5.8.17}$$

where

$$m_0 = -\frac{\sqrt{Q_1 Q_2}}{4\sqrt{2}(1-b^2)R}, \tag{5.8.18}$$

$$m_+ = \frac{R\sqrt{Q_1 Q_2}(1-b)^2}{32\sqrt{2}(1+b)\left(1+b^2\right)^2}\left(1-b-i(1+b)\right), \tag{5.8.19}$$

$$m_- = \frac{R\sqrt{Q_1 Q_2}(1+b)^2}{32\sqrt{2}(1-b)\left(1+b^2\right)^2}\left(1+b+i(1-b)\right). \tag{5.8.20}$$

The coefficients of the function M also have imaginary parts.

Several comments are in order here.

1. The magnetic function K^3 is dipolar: it is dipolar in the sense that it captures the dipole charge and also dipolar in the sense that the charges at the two centers are complex conjugate of each other.

2. The electric functions are real. However, the charges are split between the $\vec{x}_{\pm}$ centers in a non-trivial (un-equal) way. The sum of charges at the two centers equals the total charge.

3. The coefficients at the centers for the function M are fairly cumbersome. This is essentially due to the cumbersome nature of the Pomeransky-Sen'kov solution. The coefficients at the centers are not complex conjugate of each other.

It will be good to understand if these coefficients can be understood from the new form of attraction point of view [108]. Since the full solution has three centers, the formalism of [108] needs to be adapted somewhat to address this question. A further analysis of the solution, including the nature of the singularities at the centers, and its relation to other solutions discussed in the literature[8] is left for the future.

In the $b \to 0$ limit, the eight harmonic functions simply reduce to the eight harmonic functions for the two-charge black ring, $V = 1/|\vec{x}|, K^1 = 0, K^2 = 0, L_3 = 1$,

$$M = -\frac{\sqrt{Q_1 Q_2}}{4\sqrt{2}R} + \frac{R\sqrt{Q_1 Q_2}}{16\sqrt{2}|\vec{x}-\vec{x}_0|}, \qquad K^3 = \frac{\sqrt{Q_1 Q_2}}{2\sqrt{2}R|\vec{x}-\vec{x}_0|}, \tag{5.8.21}$$

$$L_1 = 1 + \frac{Q_1}{4|\vec{x}-\vec{x}_0|}, \qquad L_2 = 1 + \frac{Q_2}{4|\vec{x}-\vec{x}_0|}, \tag{5.8.22}$$

where $\vec{x}_0 = \left(0, 0, -\frac{1}{4}R^2\right)$.

[8]Perhaps to the bubbling supertubes [103] via the so-called spectral flow transformations [109] together with some analytic continuation.

Chapter 6

Conclusions and future directions

This thesis consists of two parts. In the first part, we study "hair modes", i.e, deformation modes in case $AdS_3 \times S^3$ geometry in the context of six dimensional minimal supergravity coupled to a tensor multiplet. This part is based on our published paper [127]. In the second part, we discuss about our second published work [128], which is on gravitational index of small black rings.

Let's talk about [127] first. The 4D-5D connection allows us to view the same near horizon geometry as part of a 4D black hole or a 5D black hole. A much studied example of this phenomenon is the BMPV black hole uplifted to 6D with flat base space versus Taub-NUT base space. This uplift is done by adding a compact circle, compactifying along which gives a 5D black hole with flat base and 4D black hole with Taub-NUT base space. These black holes have identical near horizon $AdS_3 \times S^3$ geometry. In this chapter, we study modes in $AdS_3 \times S^3$ and identify those that correspond to supersymmetric hair modes in the full black hole spacetimes. We show that these modes satisfy non-normalisable boundary conditions in AdS_3. The non-normalisable boundary conditions are different for different hair modes and for different asymptotic completion. We also discuss how the supersymmetric hair modes on BMPV black holes fit into the classification of supersymmetric solutions of 6D supergravity.

This work leads to some ideas which we'd like to explore in future. In ref. [68], it was suggested that fuzzballs should also admit hair modes like the ones that we have considered in this chapter. Specifically, by putting fuzzballs in Taub-NUT space, it was conjectured that there should be fuzzball hair, like the ones obtained by putting the BMPV black hole in Taub-NUT space. We hope to report our progress on constructing such hair on a class of simplest fuzzballs in the future.

Now, let's give a brief outline of [128]. Certain supersymmetric elementary string states with angular momentum can be viewed as small black rings in a five-dimensional string theory. These black rings have zero area event horizon. The 4D-5D connection relates these small rings to small black holes without angular momentum in one less dimension. Recent works have proposed saddle solutions that compute the supersymmetric index for small black holes using gravitational path integral. Index saddles for small black holes are now reasonably well understood [82–84]. It is natural to ask if those calculations can be extended to the case of small black rings? In this chapter, we propose an analogous saddle solution for a five-dimensional small black ring. The dominant contribution comes from a black ring saddle that rotates in both independent planes in five dimensions and has a finite area event horizon. We also write the saddle solution as a three center Bena-Warner solution.

We considered the supersymmetric two charge black ring of Elvang and Emparan [86] and found a saddle that contributes to the index for this black ring. Our main reason for working with the Elvang-Emparan solution is the technical simplicity of the corresponding saddle solution, which rotates in both planes. It is certainly more interesting to work with a supersymmetric black ring with one independent dipole charge. However, working with dipole black rings with both rotations is fairly cumbersome. This is because adding dipole charges [110] on neutral black rings is not straightforward; for example, these charges cannot be added using boost-duality based solution generating techniques.

A versatile method is, however, known that adds a single dipole charge [111] on neutral black rings. This method was used in, for example, ref. [112], to present solutions with both rotations and a dipole charge. Given the cumbersome nature of the metrics not much has been done using these solutions. However, we believe that these solutions with the addition of two charges can be used to construct index saddle solutions for five-dimensional small black rings of [77–80].

In the last few months, there has been significant progress in understanding the nature of the index saddles for a variety of black holes in five dimensions [89, 107, 114]. Certain ideas were discussed in [89] for constructing saddles for supersymmetric black rings. They suggested considering uplift of 4D multi-center supersymmetric configurations to 5D and searching for saddles corresponding to black rings in this parameter space. At present it is not clear to us how practical this method is. On the one hand, for the small black ring of Elvang and Emparan, which in our opinion is the simplest supersymmetric black ring set-up, we could not have guessed the three-center solution in the form we reported in this chapter. On the other hand, the non-extremal solutions with two rotations, three independent dipole, and three independent electric charges are prohibitively cumbersome to work with. Perhaps a midway path is most likely to achieve the most success. Taking inspiration from the analysis we have performed, it is perhaps possible to find saddles for finite entropy supersymmetric black rings by considering uplift of 4D multi-center supersymmetric configurations. We hope to make progress on this question in our future work.

Appendices

Appendix A

String Dualities

There exist several kinds of dualities in physics that connect two seemingly unrelated theories. In this section, we aim to briefly summarize some of the key dualities found in string theory.

Electromagnetic duality

In string theory, Dp-branes are known to carry $U(1)$ charges, which may correspond to both electric and magnetic types. When dealing with a point particle, its worldline can interact with a 1-form gauge field A_μ, contributing a term to the action of the form

$$Q \int A_\mu, \tag{B.1.1}$$

where Q denotes the Maxwell charge of the point particle. Likewise, when considering a p-brane, its worldvolume couples to a $(p+1)$-form gauge field $A_{(p+1)}$, and the associated electric field strength is expressed as

$$F_{(p+2)} = dA_{(p+1)}. \tag{B.1.2}$$

The magnetic dual of the field strength tensor is given by the Hodge star operation as,

$$*F_{(p+2)} = \tilde{F}_{D-(p+2)}. \tag{B.1.3}$$

S-duality

This duality is a symmetry of type-IIB string theory, since under S-duality type-IIB string theory maps into itself. Also, under S-duality type-I superstring theory can be mapped to $SO(32)$ heterotic string theory. In type-IIB theory, two configurations are considered dual if their fields transform under S-duality in the following way:

$$g^{(\mathrm{E})}_{\mu\nu} \to g^{(\mathrm{E})}_{\mu\nu}, \quad \Phi \leftrightarrow -\Phi, \quad B_{\mu\nu} \leftrightarrow C_{\mu\nu}, \tag{B.2.4}$$

Here, $g^{(\mathrm{E})}_{\mu\nu}$ is the Einstein-frame metric, Φ is the dilaton, and $B_{\mu\nu}$ and $C_{\mu\nu}$ are the two-form fields. This symmetry represents a strong-weak coupling duality because the string coupling g_s, determined by the asymptotic dilaton value Φ_0, satisfies:

$$g_s = e^{\Phi_0} \tag{B.2.5}$$

As a result, when one theory has a coupling g_s, its S-dual has a coupling $1/g_s$. Since the NS-NS field $B_{\mu\nu}$ and the RR field $C_{\mu\nu}$ are exchanged, S-duality effectively swaps fundamental strings and D-branes. For instance, a fundamental string $F1$ aligned along the z-direction becomes a D1-brane along the same direction under S-duality:

$$F1(z) \leftrightarrow D1(z) \tag{B.2.6}$$

T-duality

This is a duality between type-IIA superstring theory and type-IIB superstring theory. When T-duality is applied along a compact spatial direction of radius R, the closed strings in one theory correspond to closed strings in a dual theory where the radius is replaced by α'/R. Likewise, for a compact space of volume V, T-duality transforms it to one with volume α'^4/V.

The transformation of background fields under T-duality follows specific prescriptions known as the Buscher rules. These describe how the fields change in the low-energy limit of string theory. For T-duality applied along the z-direction, the transformations at leading order for the background fields are given by:

$$G''_{zz} = \frac{1}{G_{zz}}, \tag{B.3.8}$$

$$G''_{\mu z} = \frac{B_{\mu z}}{G_{zz}}, \tag{B.3.9}$$

$$G''_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu z}G_{\nu z} - B_{\mu z}B_{\nu z}}{G_{zz}}, \tag{B.3.10}$$

$$B''_{\mu z} = \frac{G_{\mu z}}{G_{zz}}, \tag{B.3.11}$$

$$B''_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu z}G_{\nu z} - G_{\mu z}B_{\nu z}}{G_{zz}}, \tag{B.3.12}$$

$$e^{2\phi'} = \frac{e^{2\phi}}{G_{zz}}, \tag{B.3.13}$$

$$C^{(n)}_{\mu_1\ldots\mu_{n-1}\alpha z} = C^{(n-1)}_{\mu_1\ldots\mu_{n-1}\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu_1\ldots\mu_{n-2}|z|}G_{\mu_{n-1}]z}}{G_{zz}}, \tag{B.3.14}$$

$$C^{(n)}_{\mu_1\ldots\mu_{n-1}\alpha\beta} = C^{(n+1)}_{\mu_1\ldots\mu_{n-1}\alpha\beta z} + nC^{(n-1)}_{[\mu_1\ldots\mu_{n-2}|z|}B_{\mu_{n-1}]\beta} + n(n-1)\frac{C^{(n-1)}_{[\mu_1\ldots\mu_{n-2}|z|}B_{\mu_{n-1}]z}G_{\beta z}}{G_{zz}}. \tag{B.3.15}$$

Appendix B

Hair deformations using six dimensional supergravity

The set-up we are interested in is six-dimensional minimal supergravity coupled to one tensor multiplet. The bosonic equations of motion for this set-up are given as [69]

$$\begin{aligned} R_{MN} &= 2\nabla_M\phi\nabla_N\phi + e^{2\sqrt{2}\phi}\left(G_{MPQ}G_N{}^{PQ} - \frac{1}{6}g_{MN}G_{PQR}G^{PQR}\right), && \text{(B.0.1)} \\ dG &= 0, && \text{(B.0.2)} \\ d\star_6(Ge^{2\sqrt{2}\phi}) &= 0, && \text{(B.0.3)} \\ \nabla^2\phi &= \frac{1}{3\sqrt{2}}e^{2\sqrt{2}\phi}G_{PQR}G^{PQR}. && \text{(B.0.4)} \end{aligned}$$

For the constant dilaton case $G^2 = G_{PQR}G^{PQR}$ must vanish and we have

$$\begin{aligned} dG &= 0, && \text{(B.0.5)} \\ d\star_6 G &= 0, && \text{(B.0.6)} \\ R_{MN} &= G_{MPQ}G_N{}^{PQ}. && \text{(B.0.7)} \end{aligned}$$

Breaking G_{MNP} into self-dual F_{MNP} and anti self-dual H_{MNP} parts, we get equation (3.3.14).

Bena, Giusto, Shigemori, and Warner [70] wrote a linear system of equations for supersymmetric solutions to the above set-up (B.0.1)–(B.0.4). The metric ansatz considered in their work is of the form

$$ds^2 = -2H^{-1}(dv+\beta)\left(du+\omega+\frac{1}{2}\mathscr{F}(dv+\beta)\right)+Hds_4^2, \tag{B.0.8}$$

where H and $\mathscr{F}$ are functions and β, ω are one-forms on the four dimensional base with metric $ds_4^2 = h_{mn}dx^m dx^n$. A priori, the functions H, $\mathscr{F}$, the one-forms β, ω, and the metric h_{mn} all depend on v, x^m but not on u. We consider the case when the base metric is hyper-Kähler. Moreover, we restrict to the cases where the one-form β and the base metric are taken to be independent of the v coordinate. Under these assumptions, the exterior derivative of the one-form β becomes self-dual on the base space

$$\star_4 \widetilde{d}\beta = \widetilde{d}\beta, \tag{B.0.9}$$

where $\widetilde{d}$ is the exterior derivative restricted to the base space. Ultimately, we are interested in the constant dilaton case, however, to make connection with [70] it is better to keep ϕ non-vanishing for now. Furthermore, we will be only interested in the cases when the hyper-Kähler base is either flat space or Taub-NUT space. These assumptions already make the linear system of Bena, Giusto, Shigemori and Warner manageable. For the cases we are interested in a further drastic simplification occurs, namely

$$\beta = 0. \tag{B.0.10}$$

This is sometimes called a "trivial fibration" in the literature. This makes the linear system very manageable.

Under these assumptions, the tensor gauge field G_{MNP} takes the form,

$$G = d\left[-\frac{1}{2}Z_1^{-1}(du+\omega)\wedge dv\right]+\widehat{G}_1, \tag{B.0.11}$$

$$e^{2\sqrt{2}\phi}\star_6 G = d\left[-\frac{1}{2}Z_2^{-1}(du+\omega)\wedge dv\right]+\widehat{G}_2, \tag{B.0.12}$$

where[1]

$$\widehat{G}_1 = \frac{1}{2}\star_4 (DZ_2) + dv \wedge \Theta_1, \tag{B.0.13}$$
$$\widehat{G}_2 = \frac{1}{2}\star_4 (DZ_1) + dv \wedge \Theta_2, \tag{B.0.14}$$

and where for our simplified set-up $D\Phi = \widetilde{d}\Phi - \beta \wedge \dot{\Phi} = \widetilde{d}\Phi$. Θ_1 and Θ_2 are two two-forms on the base space. Z_1 and Z_2 are two functions on the base space. Physically speaking, Z_1 and Z_2 are the electric potentials and Θ_1 and Θ_2 are the magnetic two-forms. The BPS conditions imply that the electric potentials Z_1 and Z_2 and related to the function H that appears in the metric (B.0.8) as

$$Z_1 = He^{\sqrt{2}\phi}, \qquad Z_2 = He^{-\sqrt{2}\phi}. \tag{B.0.15}$$

For the hyper-Kähler base space cases, the magnetic two-forms on the base are self-dual

$$\star_4\Theta_1 = \Theta_1, \qquad \star_4\Theta_2 = \Theta_2. \tag{B.0.16}$$

The "first layer" of the BPS equations determines (Z_1, Z_2) and (Θ_1, Θ_2). The equations are,

$$D\star_4 DZ_1 = 0, \qquad D\star_4 DZ_2 = 0, \tag{B.0.17}$$

and

$$\widetilde{d}\Theta_2 = \frac{1}{2}\star_4 D\dot{Z}_1, \qquad \widetilde{d}\Theta_1 = \frac{1}{2}\star_4 D\dot{Z}_2 \tag{B.0.18}$$

where over-dots are the Lie derivative with respect to v.

The "second layer" of the BPS equations determines the function $\mathscr{F}$ and the one-form

[1] Our Hodge star conventions are the same as [70].

ω. The one-form ω carries information about the angular momentum of the solution and the function $\mathscr{F}$ about the left-moving momentum. The equations are,

$$D\omega + \star_4 D\omega = 2Z_1\Theta_1 + 2Z_2\Theta_2, \tag{B.0.19}$$

$$\star_4 D \star_4 \left(\dot{\omega} - \frac{1}{2}D\mathscr{F}\right) = -2\star_4 \Theta_1 \wedge \Theta_2 + Z_1\partial_v^2 Z_2 + Z_2\partial_v^2 Z_1 + \partial_v Z_1 \partial_v Z_2. \tag{B.0.20}$$

Underlying black hole: BMPV

We can embed $J_L = 0$ BMPV black hole in this formalism rather easily. The black holes correspond to no v-dependence and no magnetic fluxes $\Theta_1 = \Theta_2 = 0$. Furthermore, for our set-up, the dilaton ϕ is set to its constant asymptotic value. We can adjust this value such that $\phi = 0$, i.e., $Z_1 = Z_2 = H$. For the non-rotating black hole the one form $\omega = 0$. Equation (B.0.17) then implies H is harmonic; equation (B.0.20) tells us that $\mathscr{F}$ is harmonic. In Gibbons-Hawking coordinates (3.1.12) the choice $H = \psi(r)$ and $\mathscr{F} = \psi(r) - 1$, cf. (3.1.6), gives us the $J_L = 0$ BMPV black hole. We can check that $2G = d(\psi^{-1} du \wedge dv) + \star_4 DH$ matches with $F^{(3)} = r_0(\star_6 \varepsilon_3 + \varepsilon_3)$, cf. (3.1.14), with $\varepsilon_3 = \sin\theta dx^4 \wedge d\theta \wedge d\phi$.

The embedding of $J_L \neq 0$ BMPV black hole additionally requires introducing a v-independent one form ω such that

$$D\omega + \star_4 D\omega = 0. \tag{B.0.21}$$

As a result (B.0.19) is satisfied. The functions $\mathscr{F}$ and H remain the same. The one-form ω enters the three-form G. It is straightforward to see that the one-form ζ of [23] satisfies (B.0.21) and gives the correct three-form G.

Underlying black hole: BMPV in Taub-NUT

We take the base metric $ds_4^2 = ds_{TN}^2$, cf. (3.1.15). As in the previous paragraph we take $\phi = 0$, i.e., $Z_1 = Z_2 = H$. Equation (B.0.17) implies H is harmonic on the Taub-NUT base. Equation (B.0.20) tells us that $\mathscr{F}$ is harmonic on the Taub-NUT base. In Gibbons-Hawking coordinates (3.1.15) the choice $H = \psi(r)$ and $\mathscr{F} = \psi(r) - 1$, cf. (3.1.6), gives us the $J_L = 0$

BMPV black hole in Taub-NUT. We can check that $2G = d(\psi^{-1}du \wedge dv) + \star_4 DH$ matches with $F^{(3)} = r_0(\star_6 \varepsilon_3 + \varepsilon_3)$, cf. (3.1.17), with $\varepsilon_3 = \sin\theta dx^4 \wedge d\theta \wedge d\phi$. The embedding of the $J_L \neq 0$ BMPV black hole in Taub-NUT is exactly the same as discussed above except that $\star_4$ in (B.0.21) is now with respect to the Taub-NUT metric (3.1.15).

Garfinkle-Vachaspati hair modes

In both the above cases, equation (B.0.20) tells us that $\mathscr{F}$ is harmonic on the base space. Thus, apart from $\psi - 1$ we have the freedom of adding a harmonic function to $\mathscr{F}$. This freedom is precisely the same as applying the Garfinkle-Vachaspati transform.

Form-field hair modes

To turn on the anti-self-dual field deformations on the above discussed black hole in Taub-NUT, we take in addition

$$\Theta_1 = h(v)\omega_{TN} = -\Theta_2, \tag{B.0.22}$$

where ω_{TN} is introduced in (3.1.35). Since ω_{TN} is self-dual, equations (B.0.16) are satisfied. Since ω_{TN} is closed, i.e., $d\omega_{TN} = \tilde{d}\omega_{TN} = 0$, equations (B.0.18) imply that we can continue to choose H to be v-independent harmonic function. Then, equation (B.0.19) reduces to

$$D\omega + \star_4 D\omega = d\omega + \star_4 d\omega = 2Z_1\Theta_1 + 2Z_2\Theta_2 = 0. \tag{B.0.23}$$

Thus, $\omega = 0$ continues to be the solution for the non-rotation case. For rotating case $\omega = \zeta$ continues to be the solution. The only non-trivial change is in $\mathscr{F}$. Equation (B.0.20) simplifies to

$$\star_4 d \star_4 d\mathscr{F} = 4 \star_4 \Theta_1 \wedge \Theta_2 = -4h(v)^2 \star_4 \omega_{TN} \wedge \omega_{TN}. \tag{B.0.24}$$

Solution to this equation for $\mathscr{F}$ is $S(v,r)$ as given in (3.1.38).

Appendix C

Schematics of 4D-5D connetion

We start by considering a black hole in 5D flat spacetime. The spatial infinity is S^3 and the rotation group is $SO(4) \simeq SU(2)_L \times SU(2)_R$. So a non-rotating 5D black hole (like the 5D Schwarzschild or BMPV with zero angular momentum) has the full $SO(4)$ rotational symmetry, both the $SU(2)$'s act.

A rotating 5D black hole (say, BMPV) has angular momenta (J_L, J_R) under $SU(2)_L \times SU(2)_R$.

For BPS configurations like BMPV, typically $J_L = 0$ and $J_R \neq 0$. That means only one of the two $SU(2)$'s, say $SU(2)_L$ is preserved.

So the rotational symmetry reduces to $SO(4) \longrightarrow SU(2)_L \times U(1)_R \simeq SO(3) \times U(1)$. Physically $SO(3)$ corresponds to rotations in the three noncompact spatial directions orthogonal to the rotation plane, $U(1)$ corresponds to the rotation itself,i.e. the isometry generated by the angular momentum. Hence, a rotating 5D black hole has only $SO(3)$ symmetry (plus the rotational $U(1)$).

Now consider the symmetry of the Taub-NUT space. It itself has $SO(3)_{base} \times U(1)_\psi$ symmetry. Placing the black hole at the center, where the ψ circle shrinks to zero, it locally looks like the rotational direction of the black hole itself. The ψ circle stabilizes to a constant radius asymptotically. So, we can identify the angular momentum J_ψ near the center as the momentum P_ψ around the compact ψ direction.

$$\text{5D spin around } \psi \,(\text{rotation near center}) \longleftrightarrow \text{4D momentum along } \psi \,(\text{translation at infinity}).$$

Now, performing KK reduction along the asymptotic ψ circle gives a non-rotating 4D black hole at the center with the momentum along asymptotic ψ circle or J_ψ of the 5D black hole at the center interpreted as an KK $U(1)$ field charge.

Now, let's talk more precisely in the language of string theory. Consider a 3-charge rotating black hole in 5 dimensions obtained by toroidal compactification of type IIB theory. the system can be interpreted as a D1-D5-P bound stae. It can preserve 8 of total 32 supersymmetries, i.e, provide a $\frac{1}{4}$ BPS configuration. It has an entropy

$$S = \frac{A}{4\pi} = \sqrt{Q_1 Q_5 n - J^2} \tag{C.0.1}$$

where $J = J_{12} = J_{34}$, Q_1 is the number of D1 branes wrapped around y^1 circle, Q_5 gives the number of D5 branes on $y^1, \cdots, y^5$ torus and n being the momentum along the y^1 circle. Through a set of dualities, and adding an M theory circle y^6, this system can be mapped to a system with n D2 branes along y^1 and y^6 and another two sets of D2 branes. To sum up, we have 3 sets of M2 branes along 3 orthogonal tori.

Now consider a 4D non-rotating BH with Q_1, Q_2, Q_3, Q_4 number of D2,D2,D2 and D6 branes respectively with P_1 D0 branes added. This 5-charge system has entropy

$$S = 2\pi \sqrt{Q_1 Q_2 Q_3 Q_4 - \frac{1}{4} P_1^2 Q_1^2} \tag{C.0.2}$$

If we identify D0 and D6 branes as electrically and magnetically charged respectively, then we perform an electric-magnetic duality transformation and rename $Q_1 = P_0$ and $P_1 = Q_0$. Then, the entropy takes the form

$$S = 2\pi \sqrt{P_0 Q_2 Q_3 Q_4 - \frac{1}{4} P_0^2 Q_0^2} \tag{C.0.3}$$

Now, we see that this agrees with C.0.1 for $P_0 = 1$ and the identification $J = Q_0/2$.

Setting $P_0 = 1$ in case of 4d black hole means that there is only one D6 brane. D6 brane in type IIA theory is the higher dimensional analog of a KK monopole. Thus, from M theory point of view, the 4 dimensions transverse to the D6 form Taub-NUT geometry

$$ds_{TN}^2 = (1 + \frac{R}{2r})(dr^2 + r^2 d\Omega_2^2) + (1 + \frac{R}{2r})^{-1}(dy + r \sin\theta/2^2 d\phi)^2 \tag{C.0.4}$$

This geometry can be pictured as a cigar-shaped space, where the D6-brane is localized near the tip. Away from the tip, the geometry resembles $\mathbb{R}^3 \times S^1$, where the circle corresponds to the M-theory direction whose radius in type IIA units is $R = g_s \ell_s$. The presence of Q_0 D0-branes implies that there are Q_0 units of momentum along this M-theory circle. Near the tip, however, the geometry would appear nonsingular and approximately $\mathbb{R}^4$ if no other branes were involved—but the additional branes modify this structure in a nontrivial way.

In the strong-coupling limit of this four-dimensional setup, the M-theory circle expands, and the Taub–NUT geometry becomes asymptotically flat $\mathbb{R}^4$ at large distances. Close to the origin, one finds a 5d black hole. The Q_0 units of momentum along the M-theory circle is now realised as angular momentum $J_{34} = J_{12} = Q_0/2$ around the origin, corresponding to what was previously the tip of the cigar. Thus, in this limit, the system describes a five-dimensional black hole carrying M2-brane charges Q_1, Q_2, Q_3 and total angular momentum $J = J_{34} = J_{12}$. As the entropy is independent of the string coupling and hence of the M-theory circle's size, it remains identical for both the four- and five-dimensional black holes, consistent with the result obtained earlier.

Appendix D

Generating two charge solutions

We are interested in solutions to the classical equations of motion obtained from the action of the low energy NS-NS sector of the superstring theory compactified on $T^4 \times S^1$. It is convenient to start in six dimensions. A suitable truncation is to six-dimensional metric, antisymmetric two-form field B_{MN}, and dilaton Φ. The six-dimensional action in string frame takes the form,

$$S_{6S} = \frac{1}{16\pi G_6} \int d^6x \sqrt{-G^{(S)}} e^{-2\Phi} \left[R^{(S)} + 4(\nabla\Phi)^2 - \frac{1}{12} H_{MNP} H^{MNP} \right], \tag{D.0.1}$$

where $H = dB$. We can go to Einstein frame using,

$$G^{(E)}_{MN} = e^{-\Phi} G^{(S)}_{MN}. \tag{D.0.2}$$

The Einstein frame action reads,

$$S_{6E} = \frac{1}{16\pi G_6} \int d^6x \sqrt{-G^{(E)}} \left[R^{(E)} - (\nabla\Phi)^2 - \frac{1}{12} e^{-2\Phi} H_{MNP} H^{MNP} \right]. \tag{D.0.3}$$

A Kaluza-Klein reduction on a circle gives the theory of interest in five dimensions. Using the following ansatz for the metric,

$$ds^2_{6E} = e^{\frac{1}{\sqrt{6}}\chi} ds^2_{5E} + e^{-\frac{\sqrt{3}}{\sqrt{2}}\chi} (dz + A^{(1)})^2, \tag{D.0.4}$$

we go to the five-dimensional Einstein frame. The ansatz for the reduction of the NS-NS two-form $B(x,z)$ field is

$$B(x,z) = B(x) + A^{(2)}(x) \wedge dz, \tag{D.0.5}$$

where $B(x)$ is a two form in five dimensions and $A^{(2)}(x)$ is a one form. We get the following five dimensional Einstein frame action upon dimensional reduction,

$$S = \frac{1}{16\pi G_N} \int d^5x \sqrt{-g}\, \mathscr{L}, \tag{D.0.6}$$

where

$$\mathscr{L} = R - \frac{1}{2}(\nabla\chi)^2 - (\nabla\Phi)^2 - \frac{1}{12} e^{-\frac{\sqrt{2}}{\sqrt{3}}\chi - 2\Phi} H^2 - \frac{1}{4} e^{-\frac{2\sqrt{2}}{\sqrt{3}}\chi} \left(F^{(1)}\right)^2 - \frac{1}{4} e^{\frac{\sqrt{2}}{\sqrt{3}}\chi - 2\Phi} \left(F^{(2)}\right)^2, \tag{D.0.7}$$

with the field strengths defined as

$$H = dB - dA^{(2)} \wedge A^{(1)}\ , \tag{D.0.8}$$

and $F^{(1)} = dA^{(1)}$, $F^{(2)} = dA^{(2)}$.

To construct charged solutions of interest to theory (D.0.6), we start with a general stationary metric that solves vacuum Einstein equations in five dimensions and generate a two charge metric using boosts and T-duality. We take the seed metric of the general form,

$$ds_5^2 = g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + g_{\psi\psi}d\psi^2 + g_{tt}dt^2 + 2g_{\phi\psi}d\phi d\psi + 2g_{\phi t}d\phi dt + 2g_{\psi t}dt d\psi. \tag{D.0.9}$$

The five coordinates are (r,θ,ϕ,ψ,t). Metric (D.0.9) is diagonal in the (r,θ) space and has the most general form in the (ϕ,ψ,t) space, with $\partial_t, \partial_\phi, \partial_\psi$ as three commuting Killing vectors. For black rings, r is to thought of as y and θ as x. All seed metrics that we use will be of this form. We can interpret metric (D.0.9) as a string frame metric to ten-dimensional string theory by adding five flat directions z, z_1, z_2, z_3, z_4 with all other NS-NS and R-R fields set to zero. We have,

$$ds^2 = ds_5^2 + dz^2 + \sum_{i=1}^{4} dz_i dz_i, \tag{D.0.10}$$

$$e^{2\Phi} = 1, \qquad B = 0. \tag{D.0.11}$$

Since nothing happens in the z_i directions, we ignore them in what follows and concentrate on the six-dimensional metric and fields. To add charges, we proceed in three steps:

1. We first perform a boost along the z-direction,

$$t = t' \cosh\alpha + z' \sinh\alpha, \tag{D.0.12}$$

$$z = z' \cosh\alpha + t' \sinh\alpha. \tag{D.0.13}$$

2. Next we perform a T-duality along the z' direction using the following rules (see e.g., [115]). For ease of notation let us call $z' = s$. We have, the transformed fields as,

$$G'_{ss} = \frac{1}{G_{ss}}, \qquad e^{2\Phi'} = \frac{e^{2\Phi}}{G_{ss}}, \tag{D.0.14}$$

$$G'_{\mu s} = \frac{B_{\mu s}}{G_{ss}}, \qquad B'_{\mu s} = \frac{G_{\mu s}}{G_{ss}}, \tag{D.0.15}$$

$$G'_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu s}G_{\nu s} - B_{\mu s}B_{\nu s}}{G_{ss}}, \qquad B'_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu s}G_{\nu s} - G_{\mu s}B_{\nu s}}{G_{ss}}. \tag{D.0.16}$$

3. Finally, we perform another boost in the z' direction with a different boost parameter,

$$t' = t'' \cosh\beta + z'' \sinh\beta, \tag{D.0.17}$$

$$z' = z'' \cosh\beta + t'' \sinh\beta. \tag{D.0.18}$$

At the end of these steps, we get the desired fields in 6D in the string frame. Dimensionally reducing to 5D using the reduction procedure discussed above, we get the final 5D fields. Dropping the primes, we have the final Einstein frame metric $g^{E\,5D}_{\mu\nu}$ in terms of the seed metric

$g_{\mu\nu}$ as,

$$
\begin{aligned}
g^{E\,5D}_{rr} &= H g_{rr}, && \text{(D.0.19)}\\
g^{E\,5D}_{\theta\theta} &= H g_{\theta\theta}, && \text{(D.0.20)}
\end{aligned}
$$

$$
\begin{aligned}
g^{E\,5D}_{tt} &= H^{-2} g_{tt}, && \text{(D.0.21)}\\
g^{E\,5D}_{t\phi} &= H^{-2} c_\alpha c_\beta g_{t\phi}, && \text{(D.0.22)}\\
g^{E\,5D}_{t\psi} &= H^{-2} c_\alpha c_\beta g_{t\psi}, && \text{(D.0.23)}
\end{aligned}
$$

$$
\begin{aligned}
g^{E\,5D}_{\psi\psi} &= H^{-2}\left(c_\alpha^2 c_\beta^2 g_{\psi\psi} + (c_\alpha^2 s_\beta^2 + s_\alpha^2 c_\beta^2 + s_\alpha^2 s_\beta^2 g_{tt})(g_{\psi\psi} g_{tt} - g_{\psi t}^2)\right), && \text{(D.0.24)}\\
g^{E\,5D}_{\phi\phi} &= H^{-2}\left(c_\alpha^2 c_\beta^2 g_{\phi\phi} + (c_\alpha^2 s_\beta^2 + s_\alpha^2 c_\beta^2 + s_\alpha^2 s_\beta^2 g_{tt})(g_{\phi\phi} g_{tt} - g_{\phi t}^2)\right), && \text{(D.0.25)}\\
g^{E\,5D}_{\phi\psi} &= H^{-2}\left(c_\alpha^2 c_\beta^2 g_{\psi\phi} + (c_\alpha^2 s_\beta^2 + s_\alpha^2 c_\beta^2 + s_\alpha^2 s_\beta^2 g_{tt})(g_{\psi\phi} g_{tt} - g_{\phi t} g_{\psi t})\right), && \text{(D.0.26)}
\end{aligned}
$$

where we use the short hand notation $c_\alpha = \cosh\alpha, c_\beta = \cosh\beta, s_\alpha = \sinh\alpha, s_\beta = \sinh\beta$ and we have also defined,

$$
H = h_\alpha^{\frac{1}{3}} h_\beta^{\frac{1}{3}}, \qquad h_\alpha = c_\alpha^2 + s_\alpha^2 g_{tt}, \qquad h_\beta = c_\beta^2 + s_\beta^2 g_{tt}. \tag{D.0.27}
$$

The five-dimensional dilaton and the scalar χ take the form,

$$
e^{-2\Phi} = h_\alpha, \qquad e^{-\frac{\sqrt{3}}{\sqrt{2}}\chi} = \frac{h_\beta}{\sqrt{h_\alpha}}. \tag{D.0.28}
$$

The two vectors $A^{(1)}$ and $A^{(2)}$ take the form,

$$A_t^{(1)} = \frac{(1+g_{tt})}{c_\beta^2 + g_{tt} s_\beta^2} c_\beta s_\beta, \qquad A_t^{(2)} = \frac{(1+g_{tt})}{c_\alpha^2 + g_{tt} s_\alpha^2} s_\alpha c_\alpha, \tag{D.0.29}$$

$$A_\psi^{(1)} = \frac{g_{t\psi}}{c_\beta^2 + g_{tt} s_\beta^2} c_\alpha s_\beta, \qquad A_\psi^{(2)} = \frac{g_{t\psi}}{c_\alpha^2 + g_{tt} s_\alpha^2} s_\alpha c_\beta, \tag{D.0.30}$$

$$A_\phi^{(1)} = \frac{g_{t\phi}}{c_\beta^2 + g_{tt} s_\beta^2} c_\alpha s_\beta, \qquad A_\phi^{(2)} = \frac{g_{t\phi}}{c_\alpha^2 + g_{tt} s_\alpha^2} s_\alpha c_\beta. \tag{D.0.31}$$

Finally, the five-dimensional B-field has non-zero components as,

$$B_{\phi t} = \frac{g_{t\phi} s_\alpha s_\beta}{c_\alpha^2 + g_{tt} s_\alpha^2}, \tag{D.0.32}$$

$$B_{\psi t} = \frac{g_{t\psi} s_\alpha s_\beta}{c_\alpha^2 + g_{tt} s_\alpha^2}. \tag{D.0.33}$$

A five-dimensional duality relation: We can convert the two-form potential B into a one form potential $A^{(3)}$ and interpret the solution as a solution of $\mathrm{U}(1)^3$ supergravity. The duality relation is, see for example [116][1],

$$\exp\left[-\sqrt{\frac{2}{3}}\chi - 2\Phi\right] \star_5 H = -dA^{(3)}. \tag{D.0.34}$$

Our conventions for the orientation with the black ring coordinates is $\varepsilon_{tyx\phi\psi} = +\sqrt{-\det g}$. For the Hodge star operation we always use the Polchinski conventions [117].

Area computation: To compute the area of a black hole we need the determinant of the five-dimensional Einstein frame metric on the section of constant t and constant r. For singly spinning cases, with spin in the ψ direction, $g_{t\phi} = 0$ and $g_{\phi\psi} = 0$. The determinant expression

[1] In [116] the sign of the 5D Chern-Simons term was taken to be $+$, whereas in the Bena-Warner literature it is taken to be $-$, hence the sign difference.

simplifies and we get,

$$
\begin{aligned}
(\det g^{E\,5D})_{\theta\phi\psi} &= c_\alpha^2 c_\beta^2 g_{\theta\theta} g_{\phi\phi} g_{\psi\psi} \\
&\quad + (c_\alpha^2 s_\beta^2 + s_\alpha^2 c_\beta^2) g_{\theta\theta} g_{\phi\phi} \left(g_{\psi\psi} g_{tt} - g_{t\psi}^2 \right) \\
&\quad + s_\alpha^2 s_\beta^2 g_{\theta\theta} g_{\phi\phi} g_{tt} \left(g_{\psi\psi} g_{tt} - g_{t\psi}^2 \right). \qquad \text{(D.0.35)}
\end{aligned}
$$

For the general situation define,

$$
g_3 = (\det g)_{\phi\psi t} = \begin{pmatrix} g_{\phi\phi} & g_{\phi\psi} & g_{\phi t} \\ g_{\phi\psi} & g_{\psi\psi} & g_{\psi t} \\ g_{\phi t} & g_{\psi t} & g_{tt} \end{pmatrix}. \qquad \text{(D.0.36)}
$$

We have,

$$
(\det g^{E\,5D})_{\theta\phi\psi} = c_\alpha^2 c_\beta^2 g_{\theta\theta} (g_{\phi\phi} g_{\psi\psi} - g_{\phi\psi}^2) + (c_\alpha^2 s_\beta^2 + s_\alpha^2 c_\beta^2) g_{\theta\theta} g_3 + s_\alpha^2 s_\beta^2 g_{\theta\theta} g_{tt} g_3 \qquad \text{(D.0.37)}
$$

Appendix E

Saddle solution in the chiral null model form

Taking motivation from [92], in this appendix we write the saddle solution uplifted to six-dimensions in the chiral null model form. Since all supersymmetric F1-P solutions can be written in this form, it is perhaps not a surprise that the saddle solution can be written in this form. Nonetheless, exhibiting this is interesting. It gives us confidence that saddle solutions for the higher dimensional supersymmetric small black rings [80] can be explored following this line of thought.

Recall that the five-dimensional vector fields and the B-field for the saddle solution take the simplified form given in (5.7.22)–(5.7.23) and (5.7.24). Using these expressions, we uplift the solution to six-dimensions and express it in string frame. The string frame metric reads,

$$
\begin{aligned}
ds^2_{6S} &= e^{\Phi} ds^2_{6E} = \frac{1}{\sqrt{h_1}} ds^2_{6E} && \text{(E.0.1)} \\
&= -\frac{1}{h_1 h_2}\left(dt + \Omega^{(s)}\right)^2 + ds^2_{\text{flat}} + \frac{h_2}{h_1}(dz + A^{(1)})^2. && \text{(E.0.2)}
\end{aligned}
$$

From the 5D B-field, we get the the following 6D B-field,

$$
B = B_{\psi t} d\psi \wedge dt + B_{\phi t} d\phi \wedge dt + A^{(2)} \wedge dz. \tag{E.0.3}
$$

We can connect this form of the string frame metric to the chiral null model metrics [118, 119]. A standard form of the chiral null model is [92],

$$
ds^2 = H\left(-dudv + Kdv^2 + 2\mathscr{A}_i dx^i dv\right) + dx_i dx_i, \tag{E.0.4}
$$

with other fields given as,

$$e^{-2\Phi} = H^{-1}, \qquad B_{uv} = -\frac{1}{2}H, \qquad B_{vi} = -H\mathscr{A}_i. \tag{E.0.5}$$

We can regard $\mathscr{A}_i$ as a vector field on the four-dimensional flat base space $dx_i dx_i$ and we construct $\mathscr{F}_{ij} = \partial_i \mathscr{A}_j - \partial_j \mathscr{A}_i$. Then, the functions in the chiral null model must satisfy,

$$\partial^2 H^{-1} = 0, \qquad \partial^2 K = 0, \qquad \partial_i \mathscr{F}^{ij} = 0. \tag{E.0.6}$$

In these equations, ∂^2 is the Laplacian in the x_i Cartesian coordinates.

From the Bena-Warner analysis of section 5.8, we conclude that the one form $\Omega^{(s)}$ on the four-dimensional flat base space satisfies,

$$d\left[d\Omega^{(s)} + \star_4 d\Omega^{(s)}\right] = 0 \implies d\star_4 d\left(\Omega^{(s)}\right) = 0. \tag{E.0.7}$$

In particular, if we identify, $\mathscr{A} = -\Omega^{(s)}$, the chiral null model equations for $\mathscr{A}$ are satisfied. With the identifications $H = h_1^{-1}, K = h_2 - 1, u = t - z$, and $v = t + z$, the string frame metric for the saddle solution readily matches the chiral null model form. The other fields also match.

Appendix F

Constructing small black rings in d-dimensional supergravity

We discuss the construction of small black ring solutions within heterotic string theory, focusing on configurations with d noncompact spacetime dimensions that describe a rotating fundamental string. We follow the approach given by [80]. A related solution, viewed through a different U-duality frame, was previously obtained in [85] as a supergravity supertube. The setup involves heterotic string theory compactified on the space $\mathbb{R}_t \times \mathbb{R}^{d-1} \times S^1 \times T^{9-d}, \quad$ with $4 \leq d \leq 9$. The coordinates are labeled as follows: t for time, $x = (x^1, \ldots, x^{d-1})$ for the noncompact spatial directions, and x^d for the compact circle S^1, whose radius is denoted by R_d. Our goal is to analyze the geometry of a small black ring localized in the noncompact d-dimensional spacetime $\mathbb{R}_t \times \mathbb{R}^{d-1}$. To do this, we treat the solution as arising from a $(d+1)$-dimensional effective theory, obtained by reducing the ten-dimensional heterotic string theory on the internal torus T^{9-d}. This approach requires expressing the solution in terms of $(d+1)$-dimensional fields and assuming that it is independent of the compact torus coordinates achieved by smearing the ten-dimensional solution over T^{9-d}. Earlier works [52, 53] constructed a broad class of supergravity solutions describing a fundamental string carrying arbitrary left-moving traveling waves, with profiles of the form $x = F(t - x^d), \quad$ where $F = (F_1, \ldots, F_{d-1})$ are arbitrary functions. In [92] (see also [94]), the case $d = 5$ was studied in detail, where the fundamental string wraps the compact x^d direction $-w$ times (with $w \gg 1$) and carries $n \gg 1$ units of momentum along x^d. It was argued that in this regime, the appropriate supergravity description is obtained by smearing the solution of [52] along the x^d direction. Compactifying this smeared solution on x^d yields a five-dimensional geometry, where the fundamental string has an arbitrary transverse profile $\vec{x} = \vec{F}(v)$ in the noncompact $\mathbb{R}^4$ space.

The construction presented in [92] can be generalized to an arbitrary number of dimensions d. Specifically, one obtains a solution of the $(d+1)$-dimensional supergravity equations describing a fundamental string wound w times ($w \gg 1$) around the compact direction x^d, carrying n units of momentum ($n \gg 1$) along the same direction, and having an arbitrary shape in the non-compact transverse space $\mathbb{R}^{d-1}$. The string profile in the transverse space is denoted by $\vec{x} = \vec{F}(v), \quad 0 \le v \le L$, and the string-frame metric, dilaton, and B-field components are given by [80]

$$\begin{aligned} ds^2_{\text{str},,d+1} &= f_f^{-1}\left[-(dt - A_i, dx^i)^2 + (dx^d - A_i, dx^i)^2 + (f_p - 1)(dt - dx^d)^2\right] + dx^i dx^i, \\ e^{2\Phi_{d+1}} &= g^2 f_f^{-1}, \qquad B_{td} = -(f_f^{-1} - 1), \qquad B_{ti} = -B_{di} = f_f^{-1} A_i. \end{aligned} \tag{F.0.1}$$

Here $i = 1, 2, \ldots, d-1$. The harmonic functions and gauge potentials associated with the string are [80]

$$\begin{aligned} f_f(\vec{x}) &= 1 + \frac{Q_f}{L}\int_0^L \frac{dv}{|\vec{x} - \vec{F}(v)|^{d-3}}, \quad f_p(\vec{x}) = 1 + \frac{Q_f}{L}\int_0^L \frac{|\dot{\vec{F}}(v)|^2, dv}{|\vec{x} - \vec{F}(v)|^{d-3}}, \\ A_i(\vec{x}) &= -\frac{Q_f}{L}\int_0^L \frac{\dot{F}_i(v), dv}{|\vec{x} - \vec{F}(v)|^{d-3}}. \end{aligned} \tag{F.0.2}$$

The dot denotes differentiation with respect to v, and the total length of the string is $L = 2\pi w R_d$.[1] The total momentum parameter is defined as

$$Q_p \equiv \frac{Q_f}{L}\int_0^L |\dot{\vec{F}}(v)|^2 dv. \tag{F.0.3}$$

At large distances ($|\vec{x}| \to \infty$), the functions $(f_f - 1)$ and $(f_p - 1)$ fall off respectively as $Q_f/|\vec{x}|^{d-3}$ and $Q_p/|\vec{x}|^{d-3}$. By computing the fluxes of the gauge fields $G_{d\mu}$ and $B_{d\mu}$ at infinity, one obtains the relations between the parameters Q_f, Q_p and the quantized winding and

[1] The relation to the harmonic functions in [92, 94] is $f_f^{-1} = H, f_p = K + 1$.

momentum numbers w, n as [54]

$$Q_f = \frac{16\pi G_d R_d}{(d-3),\Omega_{d-2},\alpha'} w, \qquad Q_p = \frac{16\pi G_d}{(d-3),\Omega_{d-2},R_d} n \tag{F.0.4}$$

where Ω_D is the area of the unit sphere S^D, and G_d is the d-dimensional Newton constant,

$$16\pi G_d = \frac{16\pi G_{d+1}}{2\pi R_d} = \frac{(2\pi)^{d-3} g_s^2 \alpha'^{(d-1)/2}}{R_d}. \tag{F.0.5}$$

Studying the asymptotic form of the metric and results from [55] the angular momentum of the solution in the $x^i - x^j$ plane is given by

$$J_{ij} = \frac{(d-3)\Omega_{d-2}}{16\pi G_d} \frac{Q_f}{L} \int_0^L (F_i \dot{F}_j - F_j \dot{F}_i) dv \tag{F.0.6}$$

Before analyzing the small black ring configuration, it is useful to first examine a simpler case of a circular fundamental string profile. The shape of the string is chosen as

$$\vec{F} = \vec{F}^{(0)}, \qquad \left\{ F_1^{(0)} + iF_2^{(0)} = Re^{i\omega v},\ F_3^{(0)} = \cdots = F_{d-1}^{(0)} = 0, \right. \tag{F.0.7}$$

which describes a circular loop of radius R lying in the x^1-x^2 plane and rotating with frequency $\omega = \frac{2\pi Q}{L} = \frac{Q}{wR_d}$. This configuration describes a fundamental string winding Q times around a ring of radius R in the x^1-x^2 plane. To express the geometry conveniently, we introduce a set of coordinates $(s, \psi, w\xi^a)$ defined as

$$x^1 = s\cos\psi, \qquad x^2 = s\sin\psi, \qquad x^3 = w\xi^1, \quad x^4 = w\xi^2, \ \ldots, \ x^{d-1} = w\xi^{d-3} \tag{F.0.8}$$

with the constraint $\sum_{a=1}^{d-3} (\xi^a)^2 = 1$. These coordinates naturally adapt to the circular symmetry of the ring. Using these coordinates, the harmonic functions from equationF.0.2 can be evaluated as

$$f_f = 1 + Q_f(s^2+w^2+R^2)^{-\frac{d-3}{2}}{}_2F_1\left(\frac{d-3}{4},\frac{d-1}{4};1;\frac{4R^2s^2}{(s^2+w^2+R^2)^2}\right),$$
$$f_p = 1 + Q_p(s^2+w^2+R^2)^{-\frac{d-3}{2}}2F_1\left(\frac{d-1}{4},\frac{d+1}{4};1;\frac{4R^2s^2}{(s^2+w^2+R^2)^2}\right),$$
$$A_\psi = -\frac{(d-3)}{2},qR^2s^2(s^2+w^2+R^2)^{-\frac{d-1}{2}}{}_2F_1\left(\frac{d-1}{4},\frac{d+1}{4};2;\frac{4R^2s^2}{(s^2+w^2+R^2)^2}\right) \tag{F.0.9}$$

where we have defined

$$q \equiv Q_f\omega. \tag{F.0.10}$$

Here, ${}_2F_1(\alpha,\beta;\gamma;z)$ denotes the hypergeometric function. For odd values of d, these functions simplify to rational expressions, whereas for even d they involve elliptic integrals. Using equation F.0.3, the momentum parameter becomes

$$Q_p = Q_fR^2\omega^2. \tag{F.0.11}$$

Combining this with the definitions in equations F.0.10 and F.0.7, one finds

$$q = \frac{16\pi G_d}{(d-3)\Omega_{d-2}\alpha'}Q. \tag{F.0.12}$$

Finally, from equation F.0.6, it follows that the configuration carries angular momentum

$$J = \frac{QR^2}{\alpha'}. \tag{F.0.13}$$

Using F.0.4, F.0.11, F.0.13, one finds

$$JQ = nw \tag{F.0.14}$$

which implies saturation of the Regge bound by the circular configuration.

To construct the small black ring geometry, we begin by introducing a small deformation to the circular string profile. This is done by adding a fluctuation to the original configuration

$$\vec{F} = \vec{F}^{(0)} + \delta\vec{F}, \tag{F.0.15}$$

where $\delta\vec{F}$ represents small oscillations around the background $\vec{F}^{(0)}$. The detailed form of $\delta\vec{F}$ is unimportant as long as it satisfies certain conditions to be discussed later. As a simple example, we take

$$\delta F_1 + i\delta F_2 = ae^{i(\nu v+b)}, \tag{F.0.16}$$

which describes a fluctuation with amplitude a and phase b. Eventually, we consider the limit

$$\frac{a}{R} \to 0, \qquad \frac{\nu}{\omega} \to \infty, \qquad a\nu = \text{fixed}. \tag{F.0.17}$$

This corresponds to a very small-amplitude ($a \ll R$), high-frequency ($\nu \gg \omega$) perturbation. Due to the first condition in (F.0.17), the profile $\vec{F}$ appearing in the denominators of F.0.2 can be approximated by $\vec{F}^{(0)}$. Expanding the distance factor in the harmonic functions yields

$$\begin{aligned} |\vec{x} - \vec{F}^{(0)}|^{-(d-3)} &= \left[s^2 + w^2 + R^2 - 2sR\cos(\omega v - \psi)\right]^{-\frac{d-3}{2}} \\ &= (s^2 + w^2 + R^2)^{-\frac{d-3}{2}} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma\left(\frac{d-3}{2} + k\right)}{\Gamma\left(\frac{d-3}{2}\right)} \left[\frac{2sR\cos(\omega v - \psi)}{s^2 + w^2 + R^2}\right]^k. \end{aligned} \tag{F.0.18}$$

On the other hand, for the numerator term in f_p, we find

$$|\dot{\vec{F}}|^2 = |\dot{\vec{F}}^{(0)} + \delta\dot{\vec{F}}|^2 = R^2\omega^2 + a^2\nu^2 + 2Ra\omega\nu\cos\left[(\omega - \nu)v - b\right]. \tag{F.0.19}$$

When multiplying (F.0.18) and (F.0.19) and integrating over v, only those terms survive for which the oscillatory factors cancel. This occurs when $\nu/\omega = 1$, but in the limit $\nu/\omega \to \infty$ of (F.0.17), such terms average out. Hence, $\delta\vec{F}$ does not contribute to the potentials A_i in F.0.2 only $\vec{F}^{(0)}$ contributes. Therefore, the sole effect of introducing the perturbation (F.0.16) is to modify F.0.11 as

$$Q_p = Q_f(R^2\omega^2 + a^2\nu^2). \qquad \text{(F.0.20)}$$

This gives the supergravity black ring solution. It can be realised noting that for this solution

$$JQ < nw \qquad \text{(F.0.21)}$$

Even if we consider more general types of fluctuations than those in equation F.0.16 by taking linear combinations of several modes along all x^i directions $(1 \leq i \leq (d-1))$—the previous conclusions still hold as long as each mode satisfies the condition F.0.17. The only difference is that the $a^2\nu^2$ term in F.0.20 must now be replaced by a sum of contributions from all the different modes.

The fact that the resulting solution F.0.9 does not depend on the detailed form of the fluctuation $\delta\vec{F}$ shows that this supergravity small black ring captures all the possible microstates. The entropy of these microstates is given by equation $S_{micro} = 4\pi\sqrt{nw - JQ}$. Although the small black ring solution was originally written in terms of the coordinates $(s, \psi, w, \vec{\xi})$ in F.0.9, it is more useful to switch to a new coordinate system $(y, \psi, x, \vec{\xi})$, defined as [80] in order to make connection with the present literature

$$s = \frac{\sqrt{y^2-1}}{x-y}R, \qquad w = \frac{\sqrt{1-x^2}}{x-y}R, \qquad -1 \leq x \leq 1, \quad -\infty < y \leq -1. \qquad \text{(F.0.22)}$$

In this new coordinate system, the harmonic functions from F.0.9 take the form

$$\begin{aligned}
f_f &= 1 + \frac{Q_f}{R^{d-3}} \frac{(x-y)^{(d-3)/2}}{(-2y)^{(d-3)/2}}\, {}_2F_1\left(\frac{d-3}{4}, \frac{d-1}{4}; 1; 1-\frac{1}{y^2}\right), \\
f_p &= 1 + \frac{Q_p}{R^{d-3}} \frac{(x-y)^{(d-3)/2}}{(-2y)^{(d-3)/2}}\, {}_2F_1\left(\frac{d-3}{4}, \frac{d-1}{4}; 1; 1-\frac{1}{y^2}\right), \\
A_\psi &= -\left(\frac{d-3}{2}\right) \frac{q}{R^{d-5}} \frac{(y^2-1)(x-y)^{(d-5)/2}}{(-2y)^{(d-1)/2}}\, {}_2F_1\left(\frac{d-1}{4}, \frac{d+1}{4}; 2; 1-\frac{1}{y^2}\right). \qquad \text{(F.0.23)}
\end{aligned}$$

Finally, the flat $(d-1)$-dimensional metric can be expressed as

$$d\vec{x}_{d-1}^2 = \frac{R^2}{(x-y)^2}\left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\Omega_{d-4}^2\right]. \tag{F.0.24}$$

These coordinates are called the “ring coordinates” and we have used them extensively in our work in black rings. For a good introduction to these coordinates one may refer to [76].

Coherent state nature of the string profile

The classical circular string configuration can be viewed as a coherent quantum state in the Fock space of string oscillators. We now describe this correspondence in detail.

Classical profile

Consider the classical circular profile used in constructing the small black ring:

$$F_1 + iF_2 = Re^{i\omega v}, \tag{F.0.25}$$

where $v = \tau - \sigma$ is the left-moving worldsheet coordinate. This describes a purely left-moving traveling wave of radius R in the (x_1, x_2) plane.

Left-moving mode expansion

In light-cone or conformal gauge, the left-moving part of the string coordinate field is expanded as

$$X^i(v) = x^i + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha_m^i e^{-imv} \tag{F.0.26}$$

where the oscillator modes obey

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n,0}. \tag{F.0.27}$$

The right-moving coordinates $X^i(u)$ are absent in the BPS configuration we consider, so the solution is purely left-moving.

Classical mode and its quantized version

The above circular profile corresponds to a vibration in one transverse complex direction, i.e.

$$F_1 + iF_2 = Re^{inv}, \tag{F.0.28}$$

which has a single Fourier mode of wavenumber n. This can be matched to the oscillator α^i_{-n} in the mode expansion (F.0.26).

Thus, in the quantum description, the circular string corresponds to a state with only one non-zero left-moving mode excited.

Definition of the coherent state

We define a coherent state built from this oscillator mode as

$$||F\rangle\rangle = \exp\left[\frac{1}{n}\sqrt{\frac{2}{\alpha'}}C_i\alpha^i_{-n}\right]|0\rangle \tag{F.0.29}$$

where C_i is a complex polarization vector specifying the amplitude and direction of the excitation.

This state satisfies

$$\alpha^i_n||F\rangle\rangle = \sqrt{\frac{2n}{\alpha'}}C^i||F\rangle\rangle, \qquad \alpha^i_{-n}||F\rangle\rangle = 0, \tag{F.0.30}$$

showing that it is an eigenstate of the annihilation operator, i.e. a standard coherent state.

Expectation value of the embedding field

Using (F.0.26), the expectation value of the coordinate operator in this state is

$$\langle\langle F|X^i(v)|F\rangle\rangle = i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\langle\langle F|\alpha_m^i|F\rangle\rangle e^{-imv} \quad = i\sqrt{\frac{\alpha'}{2}}\frac{1}{n}\langle\langle F|\alpha_n^i|F\rangle\rangle e^{-inv} = C^i e^{-inv}. \tag{F.0.31}$$

Thus the expectation value of $X^i(v)$ reproduces the classical left-moving wave:

$$\langle X_1 + iX_2\rangle = Re^{inv} \tag{F.0.32}$$

where we identify $C_1 + iC_2 = R/2$.

Chirality and absence of the conjugate mode

Note that the expectation value contains only a term proportional to e^{-inv}, with no complex conjugate piece. This reflects the chiral nature of the state: we have quantized only the left-moving sector of the string. In a general configuration that included both left- and right-movers, one would have $X^i(\tau,\sigma) = X_L^i(v) + X_R^i(u)$, and a real classical profile would involve both e^{-inv} and e^{inu} terms. In contrast, the BPS circular string (and the black ring it sources) is purely left-moving, so only one of these chiral sectors is excited.

Physical interpretation

The coherent state (F.0.29) represents a macroscopic traveling wave on the fundamental string, corresponding to a single vibration mode with fixed amplitude and phase. The expectation value of the embedding field reproduces the classical profile used in supergravity.

More general superpositions of such modes yield multi-frequency coherent states describing more complicated string shapes. The ensemble of all such excitations with fixed total winding and momentum gives rise to the entropy $S \sim 4\pi\sqrt{nw - JQ}$.

On the "blackness" of small black rings

Naively, from the definition of the harmonic functions F.0.2 it might seem that the solutions are singular at the location of the string profile $\vec{F}^0(v)$. But, it has been shown these solutions are smooth [94] that these configuratons are smooth in general. So, the configuration is a ring, just not a "black" one. Once one adds the pertubation F.0.16, the picture changes. Microstates appear and hence the classical solution develops a horizon, giving its "blackness". Though in two derivative theories, they have zero area (and hence, the name "small" black rings), it has been shown by scaling analysis in [80] that higher derivative corrections generate a nonzero area, producing

$$S_{BH} = C\sqrt{nw - JQ} \tag{F.0.33}$$

for some constant factor C.

The microscopic picture

For the pure circular profile

$$F_1 + iF_2 = Re^{inv}, \tag{F.0.34}$$

the string carries a single harmonic of vibration with mode number n. Hence only one set of oscillators, α^1_{-n} and α^2_{-n} (combined into a complex mode), are excited, while all others vanish. The total excitation level is

$$N = \sum_{m>0} mN_m, \qquad N_m = \sum_i \alpha^i_{-m}\alpha^i_m. \tag{F.0.35}$$

For the circular profile, only N_n is nonzero

$$N = nN_n. \tag{F.0.36}$$

Thus, all left-moving momentum (or energy) is concentrated in a single mode, making this configuration a classical or coherent state. Introducing small perturbations excites additional modes α^i_{-m} , distributing the total excitation among different frequencies

$$N = \sum_{m>0} mN_m. \tag{F.0.37}$$

This distribution gives rise to a large degeneracy of microstates, with entropy

$$S_{micro} \sim 4\pi\sqrt{nw - JQ}. \tag{F.0.38}$$

this can be seen considering heterotics strings with momentum n, winding w and angular momentum J. All the right movers are placed at their respective ground states for the BPS configuration. The left moving sector consists of 24 bosons. the left moving oscillator occupation number satisfies the constraint

$$N_L = nw \tag{F.0.39}$$

When small, high-frequency vibrations are added on top of the circular profile, they are represented by oscillator modes with much higher mode numbers,

$$m = \nu w \frac{R_d}{\alpha'}, \tag{F.0.40}$$

whose total contribution to the left-moving excitation level is denoted by N_{vib}. Since the total level is fixed by the momentum–winding quantum numbers,

$$N_L = nw, \tag{F.0.41}$$

we can decompose it as

$$N_L = N_J + N_{\text{vib}} \quad \Rightarrow \quad nw = JQ + N_{\text{vib}}. \tag{F.0.42}$$

Thus, the internal vibrations carry the remaining excitation number

$$N_{\text{vib}} = nw - JQ. \tag{F.0.43}$$

Now, the entropy follows from Cardy's formula $S = 2\pi\sqrt{\frac{c}{6}N}$ with $c = 24$ for 24 bosons and $N = N_{\text{vib}}$.

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