

Thesis Summary

String theory has achieved remarkable success in calculating the index or degeneracy associated with certain classes of supersymmetric black holes [15, 16]. In some cases, exact helicity trace indices have been expressed through specific modular forms. A corresponding gravitational approach reproduces these results by evaluating the string path integral in the black hole’s near-horizon geometry [18]. This prescription leads to a puzzle: the Breckenridge–Myers–Peet–Vafa (BMPV) black holes [22] in flat space and in Taub–NUT space share the same near-horizon geometry, yet they possess distinct microscopic indices. Thus, a single near-horizon geometry cannot account for the differing microscopic results. This discrepancy is now attributed to black hole hair modes—smooth, normalizable degrees of freedom outside the horizon—many of which have been explicitly identified and constructed [20, 21]. With these hair modes present, reconciling the near-horizon path integral with the exact microscopic results becomes unclear. Their inclusion, however, is essential for a complete correspondence [25, 26]. Conversely, incorporating hair modes from the near-horizon perspective is challenging. Recent advances suggest that evaluating the path integral over the full asymptotically flat spacetime offers a unified framework encompassing both near-horizon and hair contributions.

In our paper [127], which consists the first part of the thesis, we analyze the spectrum of modes in $AdS_3 \times S^3$ and identify those corresponding to known supersymmetric hair modes in full black hole geometries. A key objective is to understand these hair modes from the near-horizon perspective. Another aim is to clarify how these hair deformations relate to other types of AdS_3 deformations, such as null-warped AdS_3 . We carry out a detailed analysis of the boundary conditions satisfied by the hair modes in AdS_3 , showing that all of them obey non-normalizable boundary conditions. Our findings make it evident that any gravitational computation in the near-horizon region aiming to reproduce the exact microscopic results must necessarily involve non-normalizable modes. We also incorporate the hair modes of the complete black hole solutions within the framework of minimal six-dimensional supergravity

coupled to a tensor multiplet. The microscopic expression for the logarithm of the supersymmetric index successfully reproduces the Bekenstein–Hawking entropy of the corresponding supersymmetric black hole in the large-charge limit. Although this agreement is striking, it still leaves several open questions concerning the precise connection between the supersymmetric index and the actual degeneracy of states. In recent years, a framework has been introduced to compute the index directly from the gravitational side using the gravitational path integral [71,72].

Some supersymmetric elementary string states carrying angular momentum can be interpreted as small black rings within five-dimensional string theory. These black rings are characterized by having a vanishing horizon area. Through the 4D–5D correspondence, such small black rings are related to small, non-rotating black holes in four dimensions.

Recent developments have introduced saddle-point configurations in the gravitational path integral that compute the supersymmetric index for small black holes [82–84]. Building on this, we propose in [128], which is the second part of the thesis, a similar saddle solution tailored to five-dimensional small black rings. The leading contribution comes from a black ring saddle rotating in both independent angular directions with a finite-area horizon. This configuration is described by a three-center Bena–Warner solution and reformulated within the chiral null model, aiding future developments.