Relativistic dissipative hydrodynamics with extended relaxation time approximation

By

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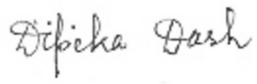

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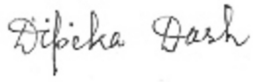

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DEDICATIONS

To

My Supervisor and Family

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ABSTRACT

The development of a new framework for the derivation of order-by-order hydrodynamics from the Boltzmann equation becomes essential as the widely used Anderson-Witting formalism leads to violations of fundamental conservation laws when the relaxation-time depends on particle energy or when considered in a hydrodynamic frame other than the Landau frame. To address these issues, we propose a generalized framework that allows for a consistent derivation of relativistic dissipative hydrodynamics from the Boltzmann equation while considering an energy-dependent relaxation time that extends the Anderson-Witting relaxation-time approximation. In this new framework, we ensure the compatibility of the derived hydrodynamic equations with fundamental conservation laws. Specifically, we focus on deriving first-order hydrodynamic equations in the Landau frame. An important aspect of the present framework is its treatment of transport coefficients, including shear and bulk viscosity, as well as charge and heat diffusion currents. We find that these transport coefficients exhibit corrections due to the energy dependence of the relaxation-time, contrasting with the predictions obtained from the Anderson-Witting approximation of the collision term. To further analyze the system, we introduce a parametrized relaxation-time and study the ratio of the transport coefficients. Through this analysis, we identify several interesting scaling features that shed light on the behavior of the fluid in different regimes and under various physical conditions.

We employ the extended relaxation time approximation (ERTA) with a particle-energy dependent relaxation time to derive second-order viscous hydrodynamics from the Boltzmann equation for a massless particle system. The resulting transport coefficients are sensitive to the energy dependence of the relaxation time, significantly affecting the fluid's evolution. By studying the (0+1)-dimensional expansion with Bjorken symmetry, we investigate the fixed point structure inherent in the hydrodynamic equations. Employing a power law parametrization for the energy dependence, we successfully reproduce the stable free-streaming fixed point for a specific power. The impact of the energy-dependent relaxation time on isotropization and thermalization processes in an expanding plasma is discussed. This study enhances our understanding of relativistic hydrodynamics and its role in describing the behavior of high-energy systems.

Contents

Summary

Heavy-ion collisions have proven to be invaluable experiments for gaining insights into the properties of strongly interacting hot quark-gluon plasma, a state of matter described by quantum chromodynamics (QCD). The matter produced in these collisions exhibits fluid-like properties, and as a result, the theory of relativistic dissipative hydrodynamics has been remarkably successful in describing many of its characteristics with unprecedented accuracy.

However, a key challenge arises when considering the collision term in hydrodynamics, which depends on the energy of the particles involved. This energy dependence reflects the microscopic nature of the system but introduces a complexity. Specifically, the inclusion of this microscopic dependency in the traditional hydrodynamics framework may lead to violations of the fundamental laws of energy conservation.

To address this challenge and provide a more comprehensive and sustainable framework, a new approach has been proposed. This novel framework extends dissipative hydrodynamics to incorporate an energy-dependent collision term. Importantly, this new framework ensures that no fundamental laws of hydrodynamics are violated, making it a robust and well-defined framework to accommodate all phenomena of hydrodynamics, particularly those involving momentum-dependent parameters.

In the initial paper, the authors discuss this new framework and present its derivation up to first-order dissipative hydrodynamics. This extension allows for the exploration of various new and intriguing behaviors of transport coefficients in a generalized system. However, it is noted that the first-order hydrodynamics derived from this framework was found to be acausal.

To address the causality issue, the study is extended to second-order hydrodynamics. In this context, causality is restored. The authors employ a simplification by considering a massless Maxwell-Boltzmann system to perform second-order dissipative hydrodynamics. This choice allows for a more tractable analysis while still capturing essential features of the system's behavior.

The study also successfully explores the thermalization, isotropization, and fixed points

in the context of free streaming and demonstrates their correlation with momentum-dependent parameters. This comprehensive approach not only addresses causality concerns but also provides a deeper understanding of the behavior of strongly interacting matter in extreme conditions, offering valuable insights for high-energy physics research.

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Chapter 1

Introduction

1.1 History

The most challenging and delicate task of science is to understand the fundamental nature of matter and the universe. By colliding particles at extremely high energies, scientists can probe the underlying structure of matter and the fundamental forces that govern its behavior. This has led to many important discoveries, such as the Higgs boson, which explains how particles acquire mass. Another reason to study high-energy physics is to investigate the origins of the universe. By recreating the conditions that existed in the early universe, scientists can gain insights into how the universe evolved and how structures like galaxies and stars formed. So many scientists have worked in predicting the nature by various theoretical models and experimental studies.

Currently, high-energy nuclear physics encourages the study of heavy ion nuclei collision under extremely hot and dense conditions[1]. At a temperature million times higher than the core of the sun and density quadrillion times denser than water, the thermodynamics nature and transport properties of nuclear matter are studied widely with extensive research interest in recent works. Earlier, protons and neutrons were assumed to be fundamental particles which interact with each other through strong force by exchange of force carriers named pions [2]. Later it was found that pions, protons and neutrons are not fundamental particles. However, those particles have an inner substructure. In 1968, at the Stanford Linear Accelerator Center (SLAC), California, USA, deep inelastic scattering experiments were performed and concluded that a proton is made up of quarks, anti-quarks and gluons, collectively known as partons by Feynman[3, 4, 5]. Quantum Chromodynamics (QCD) is

the fundamental strong interactions theory of elementary particles, quarks and gluons[6]. Similar to the photon in electromagnetic interaction(QED), gluons play the role of mediators in the strong force. Particles (electron, muons, neutrino) carry the electric charge in QED as QCD particles(quarks) carry color charge (color) of three different types[7, 8, 9]. This property makes the interaction of gluons different from photon interaction.

Furthermore, QCD shows two interesting properties (1) Confinement and (2) Asymptotic freedom[10, 11, 12]. Confinement is a phenomenon of color charge or quark particles, which can't be isolated, and hence a bare quark is not possible to observe. Colorless bound states of quarks are known as hadrons and can be observed experimentally within the QCD framework. Asymptotic freedom is an interesting property of QCD which tells us that the interaction between quarks and gluons gets asymptotically weaker at high energy or low distance, which implies the free particle nature of quarks at high energy. Physicists expected the system of quarks and gluons to be a weakly coupled system to which perturbation theory should be applicable. At low energy, due to QCD confinement, the quarks are confined to form hadrons as mesons (quark anti-quark pair) or baryons (three quarks). As the temperature increases, eventually, the bound states break up to form a weakly coupled deconfined quark-gluon state. During the transition between two configurations, a phase transition must have occurred in which a hadronic degree of freedom vanishes, and quark-gluon degree of freedom for a new state of matter is manifested directly over a certain volume. This new form of matter is called quark-gluon plasma (QGP) which can be achieved at high temperatures and low particle density[13, 14].

1.2 QCD-phase diagram

In normal matter, the atoms that make up our bodies and everything around us are made up of quarks bound together by gluons. However, at very high temperatures and densities,

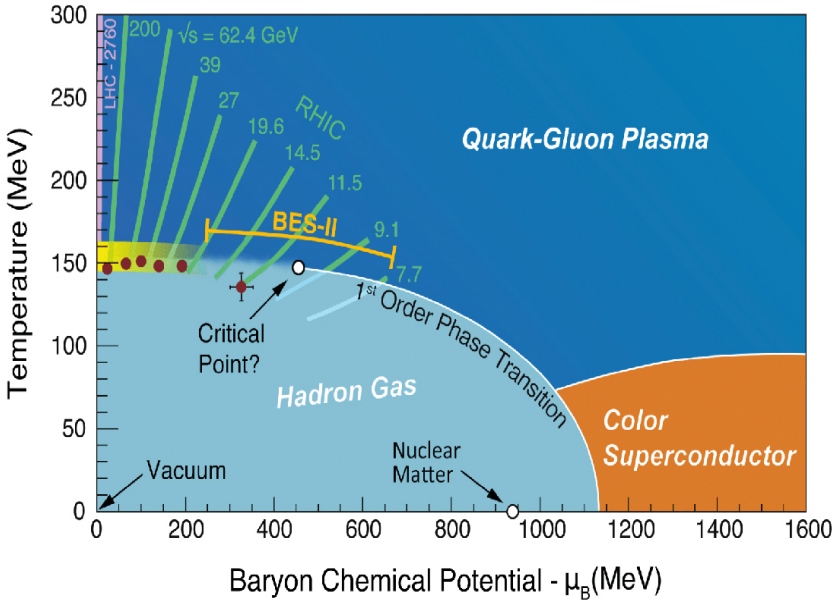

Figure 1.1: Schematic QCD phase diagram as a function of temperature and chemical potential [15].

the strong force that binds quarks and gluons together becomes weaker, and the quarks and gluons are no longer confined within individual protons and neutrons. Instead, they form a hot, dense and extremely energetic soup-like mixture of particles known as QGP.

The existense of QGP state in heavy-ion collisions is a result of the extreme conditions created by the collisions, which generate temperatures and densities that are several orders of magnitude greater than those found in ordinary matter. The resulting QGP is a unique state of matter being achieved in the laboratory for the study of the strong force and the behavior of matter at the highest energies and densities. QGP at high temperature and low density is assumed to have existence in the early universe just after the (few microseconds) Big Bang. Some other forms of QGP at low temperature and high density may exist at the inner core of a neutron star or massive compact star whose estimated density may reach ten times higher than the ordinary nuclei. After the Big Bang, the QGP state is created, and subsequent cooling and rapid expansion of the universe causes the quark-gluon confinement to produce hadrons. In the phase diagram Fig.1.1, at high temperatures and low chemical

potential, the hot QGP is expected during the Big Bang [15]. The high density and low-temperature regime of the phase diagram represents strongly degenerate quark matter and possibly in the superconducting phase. The phase diagram of QCD can be understood as a function of temperature and baryon doping. Baryon doping can be defined as the excess of quarks over anti-quarks, which is parameterized by the quantity chemical potential for baryon number μ_B. Moving in a transverse direction from the left region of the phase diagram shows the QGP formation with varying energy ($\sqrt{S}_{NN}$) of heavy ion collisions. There is a phase transition line in Fig.1.1, which separates the confined and deconfined phases of QCD [16]. High-density and low-temperature deconfinement phase transitions are of the first order [17], and the phase transition at low density with a high-temperature regime is mostly a crossover, as anticipated from QCD lattice simulation. In the late 1990s, progressing in classical work in the 2000s, the first principle of lattice QCD calculation clears that the transition of hot QCD matter to hadrons proceeded through a continuous crossover, not exactly a first-order phase transition [18]. So any fluctuation longer than the fm-scale of hot QCD matter will not be addressed. That means it has no significant imprint for a microsecond-old universe to be detected in any way today. There is a critical point between the phase transition, which has grabbed a lot of attention in current work and investigation.

It is not possible to implement a single theoretical framework for the whole QCD phase structure because of the non-perturbative nature of QCD. Therefore, each portion of the phase diagram will be explored with different approaches and approximations. The low energy part of the hadronic phase can be studied with chiral perturbation theory [19], where the chiral symmetry is spontaneously broken. QCD perturbation method can be used for QGP at very high-temperature regions of a phase diagram. Since the cosmological observation of primordial hot QCD matter is not possible nowadays, our central goal is to recreate the droplets of Big-Bang matter in the laboratory, where we can learn about material properties

and the phase diagram of QCD can be investigated.

Ultra-relativistic collision of heavy ions has shown substantial and significant success in reproducing a few trillion degrees hot matter, which is a liquid [20]. However, at these temperatures, the matter is not made of hadrons. Instead, it was expected to be made of weakly coupled plasma state of quark and gluons. Later it was found by scientists that the state reproduced by heavy ion collision is a strongly coupled perfect fluid. This ambiguity of the QGP state still needs to be understood properly. Next section, the study of the relativistic heavy in collisions and its result for the current time will be discussed.

1.3 Relativistic heavy ion collision

Multiple attempts have been made to produce QGP in the laboratory since the mid-1980s. Using ultra-relativistic heavy nuclei energies and advanced detection techniques of the modern era, good progress is achieved in this field [21]. Thermalization and hydrodynamics flow have shown good agreement to be fitted as QGP theory with the experimental data collection. But still, the detectors are not efficient in observing the QGP state directly; some indirect signatures are strengthening the QGP existence with doubts and controversial results. In the era 1995-2003, both the *European Organization for Nuclear Research (CERN)* and *Brookhaven National Laboratory (BNL)* are the second set of experiments for QGP production. At that time, CERN beam energy of heavy ion nuclei reached up to 11GeV(per nucleon), creating a new strongly coupled matter with thousands of particles per event and energy density greater than 1GeV/fm^3. The data was able to interpret QGP formation as close to perfect fluid rather than weakly interacting gas as expected.

Again to investigate more on this with higher beam energy and advanced detection techniques *Relativistic Heavy Ion Collider(RHIC)* in BNL started taking data [23, 24, 25, 26, 27, 28, 29, 30] by colliding Au+Au ions with approximately 20 times higher value than the beam

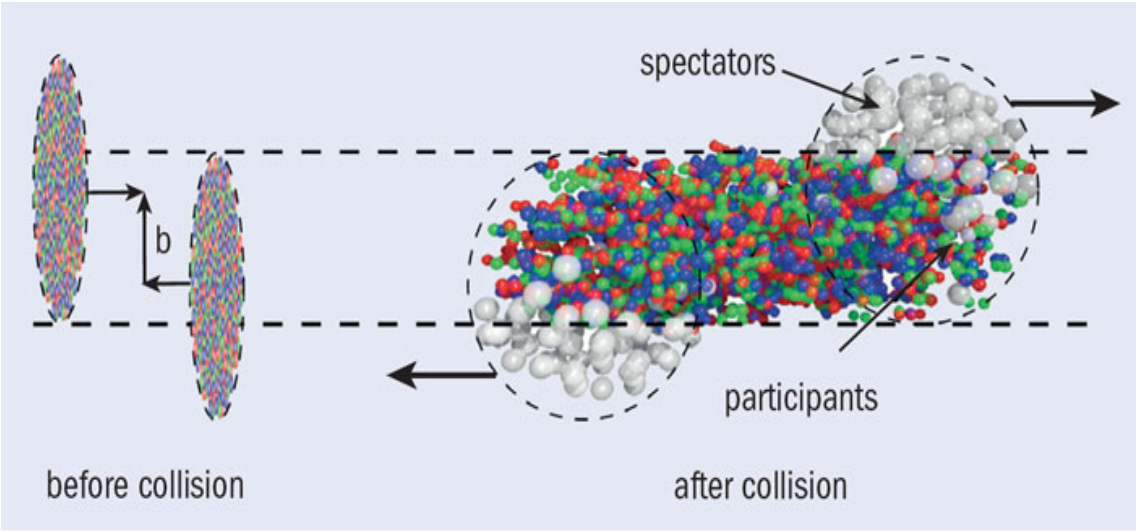

Figure 1.2: Left: Just before the collision, two-disc size heavy ions with impact parameter b. Right: The spectator and participant nucleons just after the collision [22].

energy of *Super Proton Synchrotron(LHC)* in CERN in the summer of 2000 [31, 32, 33, 34]. At this level, it was cleared that QGP is not a gaseous plasma but rather a new state of matter which expands collectively following the laws of relativistic fluid dynamics and maintains the local thermal equilibrium throughout its evolution. This experiment successfully ensure the creation of QGP by exceeding the critical temperature ($T_c \approx 155MeV$) required for it and getting evidence for a large degree of thermalization ($\leq 1fm/c$) after nuclear impact. Next, to determine the quantitative thermodynamics and transport properties precisely, on-going experiments with even higher collision energy ($\sqrt{S}_{NN} = 2.76TeV$) were conducted at *Large Hadron Collider (LHC)* at CERN starting in 2010. At LHC, apart from heavy ions (A-A) collision, proton-nucleus (p-A) collision and proton-proton (p-p) collisions are also performed. Earlier it wass observed that QGP state is produced during A-A collision only, but recent studies have also shown that QGP state can exist even at high multiplicity of p-A and p-p collisions but with less probability. So A-A collision is assumed to be the preferable source for QGP state [35].

A head-on collision is an idealized case and most of the collision will be non-central

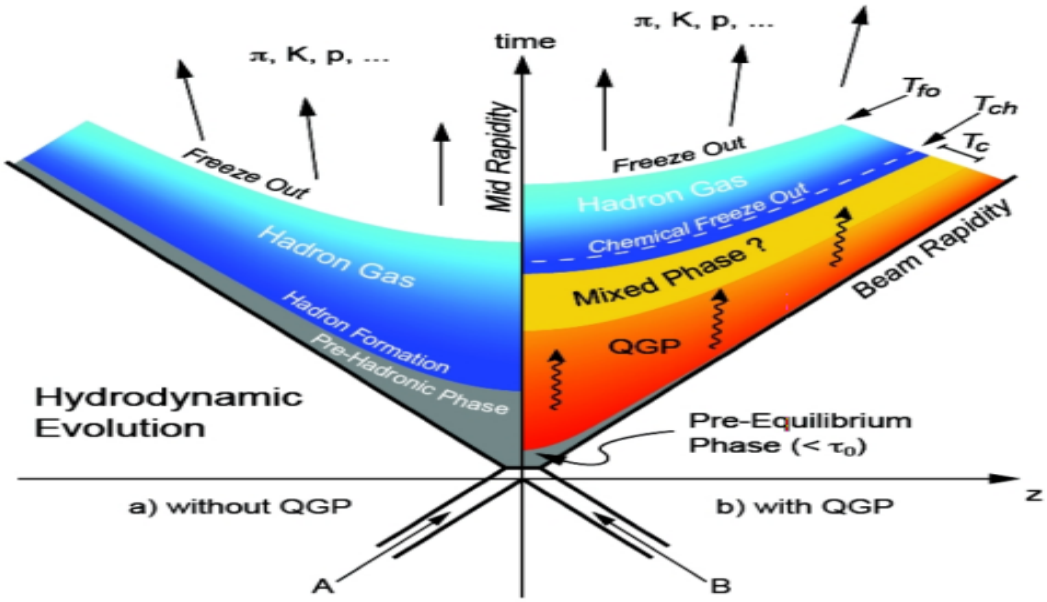

Figure 1.3: Space time evolution of a system just after the heavy ion collisions. Different stages of heavy ions collision a) without and b) with QGP evolution [36]

in nature. Due to relativity, spherical ions takes the Lorentz contracted disk shape in the direction perpendicular to the beam direction. The intersection part of the two discs is governed by an impact parameter which is defined as the distance between two centres of colliding nuclei as shown in the figure Fig: 1.2. From the number of produced particles, the centrality parameter of the collision can also be determined. The nucleons which don't take part in a collision and pass each other without colliding are known as spectators. The highest number of particles are produced in events having zero impact parameter. That means for peripheral collision there will be the lowest number of particles and correspondingly impact parameter measures the number of particles produced during an event.

Let us consider Fig. 1.3, which represents the different stages of heavy ion collision in space-time co-ordinate [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48]. Various signatures for QGP production has been proposed and studied in details [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59]. It is now well established that due to interactions between the constituents being sufficiently strong, a proximity to local thermal equilibrium is achieved after a typi-

cal time scale of $\tau_0 \equiv 1fm$. The subsequent evolution of the fireball can then be described by relativistic dissipative hydrodynamics [44, 60, 61, 62]. Fluid dynamics and thermodynamics inspired models, such as blast wave model and thermal model, have been applied extensively to study the spectra and flow observables [63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78]. In a fluid dynamics-based approach, detailed knowledge of microscopic dynamics is not necessary, but understanding the equation of state that links pressure, energy density, and baryon density is essential. Similarly, a comprehensive grasp of transport coefficients, such as viscosity, is crucial in understanding the evolution of the fireball.

The application of viscous hydrodynamics to high-energy heavy-ion collisions has garnered significant attention, particularly following the surprising estimation of a remarkably low value of the ratio of shear viscosity to entropy density, η/s, from the analysis of elliptic flow data [79]. In the realm of hydrodynamics, η/s plays a central role. Essentially, η/s serves as a natural, dimensionless measure of the impact of shear viscosity on a relativistic fluid and is often referred to as the "specific viscosity." It is interesting to note that, within the strong coupling limit of a wide range of holographic theories, this ratio η/s is found to be $1/4\pi$ [80, 81, 82]. Kovtun, Son, and Starinets (KSS) conjectured this strong coupling result to be the absolute lower bound for all fluids [83]. Indeed, the estimated η/s of QGP was remarkably close to the KSS bound, prompting the assertion that the matter produced at RHIC represented the most perfect fluid ever observed. An accurate determination of η/s is essential for understanding the characteristics of QCD matter, and it currently remains a subject of intense research; for further information, please refer to [60] and its associated references.

1.4 Relativistic fluid dynamics

To understand the microscopic dynamics of a system of a large number of interacting particles, a mathematical formulation is required, which is a non-trivial task for theoretical physics. However, at a large distance and time scales, an effective macroscopic description is provided considering the relevant degrees of freedom at these scales. Mostly the microscopic variables fluctuate rapidly in space and time. Hence their average result from the microscopic scales will be reflected on a macroscopic level. Very small changes in average values, due to the rapid fluctuations, have a negligible contribution to macroscopic dynamics. On the other side, slowly varying variables, such as the conserved quantities, have an important role in the effective mathematical description of the system.

Relativistic hydrodynamics is a theoretical framework used to describe the long wavelength and low-frequency collective phenomenon effectively in a large number of interacting particle systems. The state of a relativistic system can be determined by some slowly varying local field of macroscopic quantities like energy-momentum tensor and charge currents. The fundamental assumption of hydrodynamics is that the system must be limited for those close to the local thermal equilibrium state. A continuous system having each infinitesimal volume element near thermodynamics equilibrium throughout its evolution is defined as fluid [84]. Alternately, we can describe each point of the space by means of an infinitesimal volume called a fluid element, assuming homogeneity of matter. That means, for a non-disipative fluid, spatial gradients are neglected in a fluid element and it can be described by a finite number of macroscopic thermodynamics variables. This leads to the fact that each fluid element must be small enough compared to the macroscopic distance scale to guarantee the continuum limit and simultaneously large enough relative to the microscopic distance scale, to ensure the thermodynamics equilibrium assumption. If the scale of microscopic and macroscopic length are defined sufficiently far apart, the co-existence of both

the thermodynamics limit (infinite volume) and continuity limit (zero volume) within the fluid elements can be understood clearly [84].

So here, each fluid element can be described by slowly varying macroscopic observables like temperature, fluid velocity, and chemical potential, where "slowly" means the hydrodynamics macroscopic quantities variation in space and time is much smaller compared to the characteristics of the microscopic length scale of the system. To understand this, an effective expansion parameter known as the Knudsen number is defined as the ratio of microscopic scale to macroscopic scale in hydrodynamics [85, 86]. Typical ratios of length-scales, such as Knudsen number and Reynolds number, are important in understanding the regime of applicability of relativistic hydrodynamics [87, 88, 89, 90, 91, 92, 93, 94, 95].

Three types of typical time scales can be defined to govern the dynamical evolution of the system for a dilute medium with well-defined quasiparticles [85]. Mean collision time (τ_0) is the shortest one, which is defined as $\tau_0 = r_0/\bar{v}$, where r_0 represents the interaction length and $\bar{v}$ is the average velocity of the particles. The second time scale, $\tau_{mfp} = l_{mfp}/\bar{v}$, is the interval between two successive collisions, where l_{mfp} is the mean free path of the particles. The third macroscopic time scale, τ_{loc}, occurs in addition to these two microscopic time scales. It is the typical time for the achievement of local thermal equilibrium in distinct subsystems, each of which is macroscopically tiny but yet comprises a significant number of particles. The corresponding length scale can be denoted as l_{loc} and it establishes the smallest fluid element size at which local equilibration may occur. Lastly, a characteristic time τ_{glo} can be added to allow for the formation of complete or global thermal equilibrium. This time scale is determined by the system's size, it is unaffected by the specifics of microphysics [96]. In short, we can relate the time scale as follows

$$\tau_{glo} >> \tau_{loc} >> \tau_{mfp} >> \tau_0.$$

The ideas of the collisional time scale and the mean-free path are not clearly defined

in strongly coupled systems, and there is only one microscopic interaction time scale, τ_{mic} where $\tau_{mic} << \tau_{loc} << \tau_{glo}$. The time scale within τ_0 and τ_{loc} refers to the kinetic stage and is described using the statistical distribution function. For the sufficiently larger time which lies within τ_{loc} and τ_{glo}, the system approaches the local thermodynamic equilibrium. The corresponding local macroscopic quantities are enough to describe a system, whereas the simplified description of the system corresponds to the hydrodynamic stage.

A macroscopic energy-momentum exchange between adjacent subsystems of the size $l \geq l_{loc}$ takes place in this stage to facilitate subsequent equilibration processes. The microscopic interactions result in the local equilibrium in each fluid element. So, for systems with strong interactions, the local equilibration time scale (τ_{loc}) is smaller. By using macroscopic dissipative fluxes, the hydrodynamic regime's irreversible processes are characterized. These fluxes develop when the system reacts to regional differences or anisotropies in thermodynamic forces and mitigate the system toward its global thermal equilibrium. These dissipative fluxes may be thought of as linear functions of the thermodynamic forces for a slight departure from the global equilibrium.

Landau-Lifshitz [97] and Eckart [98] provided a relativistic model of dissipative hydrodynamics based on the linear response theory. Yet, these first-order theories and formulations of relativistic fluid dynamics are incompatible with the core ideas of special relativity. The parabolic set of equations that result from the linear relationships between the dissipative fluxes and thermodynamic forces contain collective modes propagating at arbitrarily high speeds as a solution [99, 100, 101]. These first-order theories not only exhibits acausality but is also unstable [99, 102].

The exploration of the quark-gluon plasma (QGP), formed in ultra-relativistic heavy-ion collisions, has been a subject of extensive study within the framework of relativistic fluid dynamics. Initially, the simplest version, ideal hydrodynamics, is often employed [103, 104]. This approach neglects viscous effects and assumes that local equilibrium is perfectly

maintained throughout the expansion of the QGP. On a microscopic level, this assumption implies that the scattering time of the constituent particles is significantly shorter than the macroscopic expansion time. In essence, ideal hydrodynamics assumes that the mean free path of particles is much smaller than the size of the system. However, since all fluids inherently exhibit dissipative behavior due to the quantum mechanical uncertainty principle [105], ideal fluid results serve primarily as a reference point for understanding the system's dynamics, particularly when dissipative effects become substantial.

1.5 Structure of thesis

In Chapter 1, we embark on a historical journey through the realm of particle physics, providing insight into the emergence of the Quark-Gluon Plasma (QGP) state. We delve into the intriguing question of where in the universe we can encounter this exotic state of matter. Moreover, we meticulously examine the experimental evidence that substantiates the existence of the QGP state, with a particular focus on indirect signatures that lend support to its presence. Subsequently, we elucidate the theoretical framework that elegantly aligns with the characteristics of this plasma state.

Chapter 2 embarks on a comprehensive exploration of relativistic fluid dynamics, offering a nuanced perspective from a phenomenological standpoint. The chapter's progression is as follows:
We commence by systematically deriving the equations governing the motion of an ideal relativistic fluid. This foundational section serves as a springboard for our exploration. Next, we introduce the concept of dissipation into our analysis, adopting a phenomenological approach. This step allows us to account for the effects of viscosity and other dissipative phenomena within the framework of relativistic fluid dynamics. The chapter then advances to the derivation of the equations comprising the relativistic Navier-Stokes theory. This

is accomplished by invoking the second law of thermodynamics, a fundamental principle that plays a pivotal role in understanding the behavior of fluids. Subsequently, we extend our examination to the Israel-Stewart theory, an advanced framework that builds upon the foundations of relativistic Navier-Stokes theory. This section provides insights into the more intricate aspects of relativistic dissipative fluid dynamics. Our exploration takes a brief detour to touch upon relativistic kinetic theory, offering a bridge between the microscopic and macroscopic perspectives. We elucidate how various hydrodynamic quantities can be expressed in terms of the single-particle phase-space distribution function, a concept fundamental to the understanding of relativistic fluids. Finally, Chapter 2 culminates with a thought-provoking discussion on the evolution of the phase-space distribution function as governed by the Boltzmann equation. This multifaceted equation plays a central role in our understanding of how particles within a fluid interact and evolve over time. In essence, Chapter 2 serves as a comprehensive journey through the theoretical underpinnings of relativistic fluid dynamics, providing readers with a solid foundation for the subsequent discussions and analyses presented in this work.

Moving on to Chapter 3, we shed light on the primary motivations that underpin the selection of the specific research undertaken in this work.

Chapter 4 unfolds a crucial phase of our study, where we delve into the development of a novel framework essential for our research objectives. Within this chapter, we embark on a rigorous exploration of the Navier-Stokes equation tailored for a generalized statistical system. Our exposition goes beyond conventional boundaries as we explicitly elucidate the distinctive characteristics of this equation when applied to a spectrum of systems, including both massless and massive particles, as well as those carrying electric charge. Notably, we unveil intriguing and novel features associated with transport coefficients, especially in scenarios where the relaxation time parameter is extended beyond the customary bounds. Chapter 4, in essence, serves as the cornerstone of our study, where we lay the ground-

work for the subsequent analyses and discussions. It marks a pivotal juncture in our quest to deepen our understanding of the complex behavior of various physical systems under diverse conditions.

Chapter 5 represents a significant expansion of our proposed framework, where we elevate our analysis to the second order. In doing so, we achieve a successful restoration of causality, a critical achievement in our study. Within this chapter, we meticulously derive the second-order transport coefficients, particularly focusing on systems characterized by masslessness and following the Maxwell-Boltzmann statistics. As part of our investigations, we employ the concept of Bjorken flow to elucidate the behavior of the system. This framework allows us to shed light on key phenomena, including isotropization and thermalization. Furthermore, we explore the profound implications of energy dependence on free-streaming fixed points, providing valuable insights into the intricate dynamics at play in our chosen systems. Chapter 5 marks a pivotal juncture in our research journey as we delve into the complexities of second-order effects and their impact on the behavior of physical systems. This extension of our framework is a testament to the depth and scope of our study as we strive to unravel the intriguing intricacies of these phenomena.

In the culminating Chapter 6, we embark on a comprehensive recapitulation of our research findings and insights. This chapter serves as a platform for summarizing the key results that have emerged from our rigorous investigation. Additionally, we look ahead and delve into the future perspectives that beckon for further exploration and study. These discussions encompass the potential avenues for extending and building upon the research presented in this work, offering a glimpse into the evolving landscape of this field. In essence, Chapter 6 serves as both a reflection on the journey we have undertaken in this study and a launching pad for the exciting possibilities and directions that lie ahead in the realm of our research.

1.6 Notation and convention

Some of the most common notations used in our thesis are listed below.

$$\begin{aligned} T &\rightarrow \text{Temperature of the fluid} \\ \mu &\rightarrow \text{Chemical potenitial of the fluid} \\ n &\rightarrow \text{particle density} \\ \mathcal{E} &\rightarrow \text{energy density} \\ \mathcal{P} &\rightarrow \text{isotropic pressure} \\ s &\rightarrow \text{entropy density} \end{aligned} \tag{1.1}$$

$\hbar = c = k_\beta$=1 with $\hbar = \frac{h}{2\pi}$ where h is the Planck constant, k_β the Boltzmann constant, and c is the speed of light. As per Minkowskian coordinates $g_{\mu\nu} = diag(+1, -1, -1, -1), x^\mu = (t, x, y, z), p \cdot q \equiv p_\mu q^\mu$. Tensors without indices will correspond to lorentz scalar. Fluid four velocity is denoted by u^μ and the Lorentz contraction factor by γ.The projector onto the space orthogonal to u^μ is defined as $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ where $\Delta^{\mu\nu}$ satisfies the condition $\Delta^{\mu\nu} u_\mu = \Delta^{\mu\nu} u_\mu = 0$ with trace $\Delta^\mu_\mu = 3$. The partial derivative ∂^μ can be decomposed as $\partial^\mu = \nabla^\mu + u^\mu D$ where $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ and $D \equiv u^\mu \partial_\mu$ and $\dot{f} = Df$

$$\begin{aligned} A_{(\mu}B_{\nu)} &\equiv \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu), \\ A_{[\mu}B_{\nu]} &\equiv \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu), \\ A_{<\mu}B_{\nu>} &\equiv \Delta^{\alpha\beta}_{\mu\nu} A_\alpha B_\beta \\ \Delta^{\alpha\beta}_{\mu\nu} &\equiv \frac{1}{2}\left(\Delta^\alpha_\mu \Delta^\beta_\nu + \Delta^\alpha_\nu \Delta^\beta_\mu - \frac{2}{3}\Delta^{\alpha\beta}\Delta_{\mu\nu}\right) \\ \Delta^{\alpha\beta}_{\mu\nu}\Delta_{\alpha\beta} &= \Delta^{\alpha\beta}_{\mu\nu}\Delta^{\mu\nu} = 0 \end{aligned} \tag{1.2}$$

The commonly used local fluid rest frame variables in dissipative viscous hydro dynamics are expressed in terms of energy momentum tensor $T^{\mu\nu}$ and charge four current N^{μ} and entropy four current S^{μ} as follows using the above notation

$$
\begin{aligned}
n &\equiv u_{\mu}N^{\mu} \quad \text{net charge density;} \\
n^{\mu} &\equiv \Delta^{\mu}_{\nu}N^{\nu} \quad \text{net flow of charge;} \\
\epsilon &\equiv u_{\mu}T^{\mu\nu}u_{\nu} \quad \text{energy density} \\
P + \Pi &\equiv \frac{-1}{3}\Delta_{\mu\nu}T^{\mu\nu} \quad \text{P:thermal pressure,}\Pi\text{:bulk pressure;} \\
h &\equiv \frac{(\epsilon + P)}{n} \quad \text{enthalpy;}
\end{aligned}
$$

$$
\begin{aligned}
\pi^{\mu\nu} &= T^{<\mu\nu>} \quad \text{shear stress tensor;} \\
h^{\mu} &\equiv u_{\nu}T^{\nu\lambda}\Delta^{\mu}_{\lambda} \quad \text{energy flow;} \\
q^{\mu} &\equiv h^{\mu} - hn^{\mu} \quad \text{heat flow;} \\
s &\equiv u_{\mu}S^{\mu} \quad \text{entropy density;} \\
\Phi^{\mu} &\equiv \Delta^{\mu}_{\nu}S^{\nu} \quad \text{entropy flux;}
\end{aligned}
$$

$$
\begin{aligned}
c_s^2 &\equiv \left(\frac{dP}{d\epsilon}\right)_{\frac{s}{n}} \quad \text{adiabatic speed of sound squared.} \\
\theta &\equiv \partial \cdot u \quad \text{expansion rate,} \\
\sigma^{\mu\nu} &\equiv \nabla^{<\mu}u^{\nu>} = \frac{1}{2}(\nabla^{\mu}u^{\nu} + \nabla^{\nu}u^{\mu}) - \frac{1}{3}\nabla^{\mu\nu}\partial_{\alpha}u^{\alpha} \quad \text{velocity stress tensor,} \\
\omega^{\mu\nu} &\equiv \nabla^{[\mu}u^{\nu]} \quad \text{vorticity tensor} \\
\nabla^{\mu}u^{\nu} &\equiv \left(\sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{3}\Delta^{\mu\nu}\theta\right)
\end{aligned} \tag{1.3}
$$

Chapter 2

Relativistic hydrodynamics

The simplicity and generality of relativistic fluid dynamics are its most enticing features. It is straightforward in the sense that the system's whole information is encoded in its thermodynamic and transport characteristics, namely in its equation of state and transport coefficients. Because it is based only on the premise that the system would remain relatively near to its local thermodynamic equilibrium throughout its evolution, fluid dynamics is also universal. Although the notion of closeness to local equilibrium is extremely strong, it prevents us from making any more assumptions about how the particles and fields interact, whether the underlying processes are classical or quantum, etc. In this chapter, we review the fundamentals of thermodynamics in this chapter and discuss relativistic fluid dynamics from a phenomenological standpoint. Moreover, the key elements of kinetic theory in relation to fluid dynamics will be explored. The next Chapters will need to use the ideas from this Chapter in order to construct dissipative hydrodynamic equations for use in high-energy heavy-ion physics.

We will arrange the chapters as follows: In Sec 2.1, We provide an overview of thermodynamics' fundamental rules and develop the thermodynamic relationships that will be employed later on in this thesis. Section 2.2 includes a brief idea about relativistic fluid dynamics for the ideal case. We obtain the equations of motion and the overall form of the conserved currents in an ideal fluid. In Sec. 2.3 We introduce dissipation in fluid dynamics, go over the fundamentals of dissipative fluid dynamics, and then use the second law of thermodynamics to develop a covariant version of Navier-Stokes theory. We go through the causality and instability of the Navier-Stokes theory in the relativistic domain.

In addition, we discuss Israel-Stewart's theory and demonstrate how causal fluid dynamical equations may be derived from the second law of thermodynamics. The relativistic kinetic theory is discussed in Sec. 2.5, where fluid dynamical currents are expressed in terms of single-particle phase-space distribution functions. We also go over the fundamentals of the relativistic Boltzmann equation and get how it affects the distribution function evolution. First-order and second-order theories are obtained using the relativistic kinetic theory successfully.

2.1 Thermodynamics properties

Thermodynamics has no assumptions about the small-scale or microscopic structure; it is an empirical explanation of the macroscopic or large-scale characteristics of matter. Its laws may be derived from statistical mechanics and are solely concerned with the average behavior of a very large number of tiny components. A broad collection of tiny variables, such as the system's volume (V), total energy (E), entropy (S), and number of particles (N), can be used to define a thermodynamic system. Four phenomenological principles serve as the foundation for thermodynamics and describe how these quantities are connected to one another and change over time [106, 107, 108].

The Zeroth law establishes the concept of temperature by saying that the entire system is in equilibrium if all conceivable pairs of systems are in thermal equilibrium with one another. The first law is a straightforward declaration of energy conservation. Similar to the Zeroth law, the second law introduces a brand-new idea called entropy, which must always rise in order for an isolated system to have the propensity to move toward equilibrium. It is impossible to attain absolute zero temperature in a limited number of steps, according to the third law.

Since heat exchange is not a state variable, it is preferable to define it in terms of a state

variable entropy using the relation $\delta Q = T\delta S$. Here temperature T is a proportionality constant. We can write because of the energy's extensibility. The first law of thermodynamics' mathematical formulation in terms of thermodynamic state variables is provided by,

$$dE = TdS - PdV + \mu dN$$

where E, S, V, and N, the system's internal energy, volume, and particle number, are the extensive variables that may be represented in terms of the intensive variables T, P, and, or temperature, pressure, and chemical potential, respectively. The intensive quantities P, T and μ can be defined as

$$\left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{1}{T}\,, \left(\frac{\partial S}{\partial V}\right)_{N,E} = \frac{P}{T}\,, \left(\frac{\partial S}{\partial N}\right)_{E,V} = -\frac{\mu}{T}$$

Mathematically, entropy is defined as an additive and extensive function of the state variables.

$$S(\lambda V, \lambda E, \lambda N) = \lambda S(V, E, N)$$

Differentiating both sides of the above relation with respect to λ, we get

$$S = V\left(\frac{\partial S}{\partial \lambda V}\right)_{\lambda N,\lambda E} + E\left(\frac{\partial S}{\partial \lambda E}\right)_{\lambda N,\lambda V} + N\left(\frac{\partial S}{\partial \lambda N}\right)_{\lambda E,\lambda V} \tag{2.1}$$

Setting the value of λ to be 1, we obtain the Euler's relation

$$E = -PV + TS + \mu N \tag{2.2}$$

Using both the Euler's relation (2.1) and with first law of thermodynamics (2.2), we obtain the famous Gibbs-Duhem relation

$$VdP = SdT + Nd\mu \tag{2.3}$$

The intensive variables, energy density, entropy density, and number density can be defined as $\mathcal{E} \equiv E/V$, $s \equiv S/V$, and $n \equiv N/V$ respectively. Now the Euler's and Gibbs-Duhem

relations are reduced to

$$\mathcal{E} = -P + Ts + \mu N \tag{2.4}$$

$$dP = sdT + nd\mu \tag{2.5}$$

Using (2.4) and differentiating (2.5), the analogous form of the first law of thermodynamics can be obtained

$$d\mathcal{E} = Tds + \mu dn \implies ds = \frac{1}{T}d\mathcal{E} - \frac{\mu}{T}dn \tag{2.6}$$

The equilibrium state of a system is characterized as a steady state where the extensive and intensive variables of the system remain constant. In accordance with the second law of thermodynamics, which governs the behavior of isolated thermodynamic systems, the entropy of such a system must either increase or remain unchanged. Therefore, in the case of a thermodynamic system in equilibrium, the entropy, being an extensive variable, remains constant. Conversely, for a system that deviates from equilibrium, the entropy always exhibits an increasing trend. This fundamental concept plays a pivotal role in this chapter, as it serves to impose constraints and derive the equations of motion for a dissipative fluid. With this concise overview of the fundamental principles of thermodynamics, we refer interested readers to [108] for a more comprehensive examination. Subsequently, the following section introduces and presents the derivation of the equations of relativistic ideal fluid dynamics.

2.2 Relativistic ideal hydrodynamics

An ideal fluid is characterized by the underlying assumption of local thermal equilibrium, wherein every constituent element of the fluid exists in a state of exact thermodynamic equilibrium [97, 109]. This implies that at each space-time coordinate of the fluid denoted as $x \equiv x_\mu$, and it is possible to assign specific thermodynamic properties, namely the

temperature T(x), the chemical potential $\mu(x)$, and a collective four-velocity field($u_\mu(x)$).

$$u_\mu \equiv \frac{dx_\mu}{d\tau} \tag{2.7}$$

where $d\tau$ represents the proper time increment given by

$$d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - d\vec{x}^2 = dt^2(1 - \vec{v}^2) \tag{2.8}$$

Here $\vec{v} = \frac{d\vec{x}}{dt}$. This leads to

$$u^\mu(x) = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(\vec{v}) \begin{bmatrix} 1 \\ \vec{v} \end{bmatrix} \tag{2.9}$$

The primary fluid-dynamical variables, commonly denoted as T, μ, and u_μ, play a crucial role in characterizing the state of a fluid. The complete specification of a fluid's state requires the densities and currents associated with conserved quantities, such as energy, momentum, and (net) particle number. In the case of a relativistic fluid, the state variables are represented by the energy-momentum tensor, $T_{\mu\nu}$, and the (net) particle four-current, N_μ.

To derive the general expressions for these currents in an ideal fluid, we first establish the local rest frame (LRF) of the fluid. In the LRF, the fluid velocity $\vec{v}$ is zero, and the energy-momentum tensor $T^{\mu\nu}_{LRF}$, the (net) particle four-current N^μ_{LRF} and the entropy four-current S^μ_{LRF} should exhibit the characteristic features of a system in static equilibrium.

In other words, within the local rest frame, the energy flow (T^{i0}_{LRF}) vanishes, and the pressure(P) is isotropic ($T^{ij}_{LRF} = \delta^{ij}P$) with zero net particle or entropy flow ($N^\mu_{LRF} = 0$ and $S^\mu_{LRF} = 0$).

Consequently, in this frame, the energy-momentum tensor, particle four-current, and entropy four-current assume the following simple forms:

$$T^{\mu\nu}_{LRF} = \begin{bmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, N^\mu_{LRF} = \begin{bmatrix} n \\ 0 \\ 0 \\ 0 \end{bmatrix}, S^\mu_{LRF} = \begin{bmatrix} s \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.10}$$

For an ideal relativistic fluid, the energy-momentum tensor $T^{\mu\nu}_{(0)}$, the (net) particle four-current $N^{\mu}_{(0)}$ and the entropy four-current $S^{\mu}_{(0)}$ can be expressed in terms of the hydrodynamic tensor degrees of freedom, namely the fluid velocity vector u^{μ} and the metric tensor $g_{\mu\nu}$.

To ensure that $T^{\mu\nu}_{(0)}$ is symmetric and transforms as a tensor and that $N^{\mu}_{(0)}$ and $S^{\mu}_{(0)}$ transform as vectors under Lorentz transformations, the most general form allowed is given by the following expressions:

$$T^{\mu\nu}_{(0)} = a_1 u^{\mu} u^{\nu} + a_2 g^{\mu\nu}, \quad N^{\mu}_{(0)} = a_3 u^{\mu}, S^{\mu}_{(0)} = a_4 u^{\mu} \tag{2.11}$$

In the local rest frame (LRF), $\vec{v} = 0$, so $u^{\mu} = (1, \vec{0})$. Now in LRF, (2.11) takes the form

$$T^{\mu\nu}_{(0)LRF} = \begin{bmatrix} a_1 + a_2 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 \\ 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & -a_2 \end{bmatrix}, N^{\mu}_{(0)LRF} = \begin{bmatrix} a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, S^{\mu}_{(0)LRF} = \begin{bmatrix} a_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.12}$$

If we will compare the above expression with the equation (2.10)

$$a_1 = \mathcal{E} + P, \quad a_2 = -P, \quad a_3 = n, \quad a_4 = s \tag{2.13}$$

For an ideal fluid, the conserved currents will be expressed as

$$T^{\mu\nu}_{(0)} = \mathcal{E} u^{\mu} u^{\nu} - P\Delta^{\mu\nu}, \quad N^{\mu}_{(0)} = n u^{\mu}, \quad S^{\mu}_{(0)} = s u^{\mu} \tag{2.14}$$

Here $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu} u^{\nu}$ is called a projection operator onto the 3D, which is also orthogonal to u^{μ}. A few properties of an orthogonal projector are listed below.

$$\Delta^{\mu\nu} u_{\nu} = u_{\mu} \Delta^{\mu\nu} = 0, \quad \Delta^{\mu}_{\rho} \Delta^{\rho\nu} = \Delta^{\mu\nu}, \quad \Delta^{\mu}_{\mu} = 3 \tag{2.15}$$

The dynamical description of an ideal fluid is derived by applying the conservation laws of energy, momentum, and (net) particle number. Mathematically, these conservation laws are expressed through the four divergences of the energy-momentum tensor and the particle four-current, resulting in the following equations:

$$\partial_{\mu} T^{\mu\nu}_{(0)} = 0, \quad \partial_{\mu} N^{\mu}_{(0)} = 0 \tag{2.16}$$

In the context of Lorentz transformations, the partial derivative ∂_μ behaves as a covariant vector. To further analyze the derivative ∂_μ, it can be decomposed into components parallel and perpendicular to the four-velocity u^μ using the projection operator $\Delta_{\mu\nu}$. This decomposition allows for projecting the derivative along and orthogonal to u^μ.

$$D \equiv u^\mu \partial_\mu, \nabla_\mu \equiv \Delta^\rho_\mu \partial_\rho, \implies \partial_\mu = u_\mu D + \nabla_\mu \tag{2.17}$$

By projecting the energy-momentum conservation equation along and orthogonal to u^μ, along with the conservation law for particle number, we obtain the equations of motion for ideal fluid dynamics. These equations govern the behaviour of the ideal fluid and can be derived as follows:

$$u_\mu \partial_\nu T^{\mu\nu}_{(0)} = 0 \qquad \implies D\mathcal{E} + (\mathcal{E} + P)\theta = 0, \tag{2.18}$$

$$\Delta^\alpha_\mu \partial_\nu T^{\mu\nu}_{(0)} = 0 \qquad \implies (\mathcal{E} + P) Du^\alpha - \nabla^\alpha P = 0, \tag{2.19}$$

$$\partial_\mu N^\mu_{(0)} = 0 \qquad \implies Dn + n\theta = 0 \tag{2.20}$$

The quantity $\theta = \partial_\mu u^\mu$ represents the expansion rate of the fluid. It is important to note that an ideal fluid is described by four fields: $\mathcal{E}$, P, n, and u_μ, which correspond to six independent degrees of freedom. However, the conservation laws provide only five equations of motion. To close this system of equations, the equation of state of the fluid, P = P(n, $\mathcal{E}$), relating the pressure to other thermodynamic variables, must be specified.

The existence of an equation of state is ensured by assuming local thermal equilibrium, which guarantees that the equations of ideal fluid dynamics are always closed. In other words, the assumption of local thermal equilibrium allows for the determination of the pressure as a function of the energy density and particle number density, completing the set of equations needed to describe the behaviour of an ideal fluid.

In the following, we will express the equilibrium thermodynamic relations derived in Section 2.1 (equations (2.4),(2.5) and (2.6)) in a covariant form [101, 110]. To facilitate this, we introduce the notations:

$$\beta \equiv \frac{1}{T} \quad \alpha \equiv \frac{\mu}{T} \quad \beta^\mu \equiv \frac{\beta u^\mu}{T}$$

In the introduced notations, the covariant versions of Euler's relation (Eq.(2.4))) and the Gibbs-Duhem relation (Eq. (2.5)) can be expressed as follows, respectively:

$$S^\mu_{(0)} = P\beta^\mu + \beta_\nu T^{\mu\nu}_{(0)} - \alpha N^\mu_{(0)}, \tag{2.21}$$

$$d(P\beta^\mu) = N^\mu_{(0)} d\alpha - T^{\mu\nu}_{(0)} d\beta_\nu. \tag{2.22}$$

The first law of thermodynamics Eq.(2.6) can be derived using the above relations

$$dS^\mu_{(0)} = \beta_\nu dT^{\mu\nu}_{(0)} - \alpha dN^\mu_{(0)}.$$

To derive the expression for the entropy four-current divergence, we start with the first law of thermodynamics (Eq. (2.2)) and get that:

$$\partial_\mu S^\mu_{(0)} = \beta_\mu \partial_\nu T^{\mu\nu}_{(0)} - \alpha \partial_\mu N^\mu_{(0)}$$

After using the conservation laws for energy-momentum and net particle number, the expression for the entropy four-current divergence simplifies, leading to the conservation of entropy, $\partial_\mu S^\mu_{(0)} = 0$. It is crucial to emphasize that within equilibrium thermodynamics, the conservation of entropy is a natural consequence of energy-momentum and particle number conservation, together with the first law of thermodynamics.

The equation of motion of the entropy density can be obtained to be

$$\partial_\mu S^\mu_{(0)} = 0 \implies Ds + s\theta = 0.$$

It is observed that the rate of the equation of the entropy density behaves similarly to that of particle number. So for ideal hydrodynamics, entropy density to number density ratio (s/n) is a constant of motion.

2.3 Relativistic dissipative hydrodynamics

Dissipative effects in a fluid arise from irreversible thermodynamic processes that occur during the fluid's motion. Unlike in ideal hydrodynamics, where every fluid element is in equilibrium with the entire fluid, dissipative fluids exhibit deviations from global equilibrium. Dissipative effects arise as a consequence of deviations from local thermodynamic equilibrium. It's important to note that, in practice, achieving perfect local thermodynamic equilibrium is an idealized condition that is never strictly realized. This inherent non-ideality is a fundamental characteristic of all fluids and is fundamentally rooted in the uncertainty principle [105] from quantum mechanics. Consequently, even in seemingly well-behaved systems, there are always subtle dissipative phenomena at play due to the probabilistic nature of particle interactions. To approach equilibrium, each fluid element exchanges heat with its surroundings, and due to relative motion, it can also dissipate energy through friction. It is essential to incorporate all these processes to achieve a comprehensive and realistic description of a relativistic fluid. By considering these dissipative mechanisms, we can obtain a more accurate and comprehensive understanding of the behaviour and evolution of the fluid system.

The earliest covariant formulations of dissipative fluid dynamics were independently developed by Eckart in 1940 [98] and later by Landau and Lifshitz in 1959 [97]. These theories, collectively known as first-order theories (or theories of the first order of gradients), were based on a covariant generalization of the well-established non-relativistic Navier-Stokes theory, which had already proven successful in describing a wide range of dissipative fluid phenomena.

The Navier-Stokes theory was effectively utilized to describe various non-relativistic fluids, including weakly coupled gases like air and strongly coupled fluids like water. Given its remarkable success in the non-relativistic context, researchers saw a relativistic general-

ization of the Navier-Stokes theory as the most promising approach to describe relativistic dissipative fluids. Thus, the formulations of Eckart and Landau-Lifshitz provided crucial groundwork for understanding the behaviour of dissipative fluids in a covariant manner, and they laid the foundation for further developments in relativistic dissipative hydrodynamics. These first-order theories are valuable tools for studying a wide range of phenomena, from relativistic heavy-ion collisions to the dynamics of astrophysical systems.

The formulation of relativistic dissipative hydrodynamics posed challenges as the straightforward relativistic generalization of Navier-Stokes theory was found to be intrinsically unstable [102, 99]. This instability arises due to the inherent acausal behavior of the theory [111, 112]. In the relativistic generalization of Navier-Stokes theory, signals can propagate with infinite speed in the medium, which is problematic. In non-relativistic theories, this feature is not a concern and can be overlooked. However, in relativistic systems, where causality is a fundamental physical property that must be preserved, this acausal behavior leads to equations of motion that are intrinsically unstable.

Despite the instability issues, it is valuable to review the first-order theories of relativistic dissipative hydrodynamics, as they represent an essential initial step in understanding the basic features of relativistic dissipative fluid dynamics. They provide insights into certain aspects of the dynamics of relativistic fluids, even though they may not be fully accurate in all situations. Subsequent developments in relativistic hydrodynamics have led to more sophisticated and stable higher-order theories, which better capture the complex behavior of relativistic dissipative fluids while preserving causality. Similar to ideal fluids, the fundamental equations governing the motion of dissipative fluids are derived from the conservation laws of energy-momentum and (net) particle number, as mentioned below.

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0$$

where we have defined

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \tau^{\mu\nu} = \mathcal{E}u^\mu u^\nu - P\Delta^{\mu\nu} + \tau^{\mu\nu}, \quad N^\mu = N^\mu_{(0)} + n^\mu = nu^\mu + n^\mu.$$

In the case of dissipative fluids, the energy-momentum tensor is no longer diagonal and isotropic in the local rest frame. The presence of dissipation introduces additional terms that account for the viscous stresses and heat flux. These additional terms are collectively represented by the dissipative tensor $\tau_{\mu\nu}$. Moreover, due to the presence of diffusion processes, the particle flow is expected to appear in the local rest frame of the fluid element. To incorporate these effects, an additional dissipative current n_μ, representing the particle number flow, is introduced.

Consequently, the complete set of equations for dissipative fluids includes the previously derived ideal currents $T^{(0)}_{\mu\nu}$ and $N^{(0)}_\mu$, and the additional dissipative currents $\tau_{\mu\nu}$ and n_μ, which provide a more comprehensive description of the fluid's behavior, accounting for dissipative effects. Solving this extended set of equations allows us to study the complex dynamics of dissipative fluids and understand the transport phenomena occurring in such systems. Indeed, in order to satisfy angular momentum conservation, the dissipative tensor $\tau_{\mu\nu}$ is required to be symmetric ($\tau_{\mu\nu} = \tau_{\nu\mu}$). This symmetry ensures that the conservation of angular momentum is obeyed in fluid dynamics, which is a fundamental principle in physics [113, 114].

The primary objective in the study of dissipative fluids is to find the dynamical or constitutive equations that govern the behavior of the dissipative currents $\tau_{\mu\nu}$ and n_μ. These equations, known as the constitutive relations, establish the relationship between the dissipative currents and the relevant thermodynamic variables and gradients. They play a crucial role in dissipative hydrodynamics as they determine how the dissipative effects, such as viscosity and heat conduction, depend on the local thermodynamic state and the gradients of the fluid properties. By obtaining appropriate constitutive equations, one can fully describe

the dynamics of dissipative fluids and analyze how dissipation affects the evolution and transport processes in the system. The study of these constitutive equations and their properties is a significant area of research in dissipative fluid dynamics, and it allows for a deeper understanding of the complex behavior of real-world fluid systems.

The subsequent critical aspect involves exploring relativistic fluids with finite dissipation. While the theory of ideal hydrodynamics characterizes a fluid in an equilibrium state, dissipative hydrodynamics deals with fluids that are not in equilibrium. Describing such out-of-equilibrium fluids gives rise to certain subtle ambiguities. Due to the system's departure from equilibrium, the physical quantities lose their conventional meaning, leading to challenges in their interpretation. This issue will be addressed and examined in the following discussion.

2.3.1 Frame and matching conditions in hydrodynamics

The introduction of dissipative currents in a fluid alters its equilibrium characteristics, making the definition of equilibrium variables ill-defined. In the presence of dissipation, the fluid can no longer be considered to be in local thermodynamic equilibrium at each spacetime point. This poses a challenge in defining the thermodynamic variables in dissipative fluids. To address this issue, in a dissipative fluid, the thermodynamic variables can only be defined in terms of an artificial equilibrium state. This artificial equilibrium state is constructed in such a way that the thermodynamic relations are still valid as if the fluid were in local thermodynamic equilibrium. This approach allows us to extend the concepts of thermodynamics to dissipative fluids while maintaining consistency with the fundamental laws of thermodynamics.

The first step in constructing such an equilibrium state is to define the total energy density ($\mathcal{E}$) and particle number density (n) in the local rest frame of the fluid element. These quantities serve as reference values that provide a basis for defining thermodynamic vari-

ables in the presence of dissipation. The subsequent steps involve determining the constitutive equations that relate the dissipative currents to the gradients of the fluid properties, thus establishing a complete description of the fluid's behavior in a dissipative regime. This can be assured by imposing the matching conditions [110]

$$\mathcal{E} \equiv u_\mu u_\nu T^{\mu\nu} \quad , \quad n \equiv u_\mu N^\mu. \tag{2.23}$$

These matching conditions lead to the following constraints on dissipative currents

$$u_\mu u_\nu \tau^{\mu\nu} = 0 \quad , \quad u_\mu n^\mu = 0. \tag{2.24}$$

Once the total energy density ($\mathcal{E}$) and particle number density (n) are defined, an artificial equilibrium state can be constructed using the equation of state. The equation of state relates the thermodynamic quantities, such as entropy density (s), pressure (P), temperature (T), chemical potential (μ), and others, to the energy density and particle number density in equilibrium conditions.

It is crucial to emphasize that while the energy and particle densities are physically defined and have a clear physical meaning, all the other thermodynamic quantities (e.g. s, P, T, μ, etc.) are defined only in terms of this artificial equilibrium state. In the presence of dissipation, these quantities no longer have their usual physical interpretation.

In dissipative fluids, the thermodynamic variables are effective quantities that account for the influence of dissipation and do not necessarily retain their usual meanings in the context of local thermodynamic equilibrium. Nevertheless, they are still valuable in describing the fluid's behavior and serve as essential parameters for understanding the dissipative effects and transport processes occurring in the system.

To make further progress, a convenient step is to decompose $\tau_{\mu\nu}$ into its irreducible components. This decomposition involves expressing $\tau_{\mu\nu}$ as a sum of a scalar, a four-vector, and a traceless and symmetric second-rank tensor. Moreover, this tensor decomposition

must satisfy the matching or orthogonality condition, as described by Eq.(2.24), that $\tau_{\mu\nu}$ fulfils. To achieve this tensor decomposition while ensuring orthogonality with respect to the fluid four-velocity u_μ, we introduce another projection operator - the double symmetric, traceless projector orthogonal to u_μ.

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\nu}_{\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right) \tag{2.25}$$

satisfying the following properties,

$$\Delta^{\mu\nu}{}_{\alpha\beta} = \Delta_{\alpha\beta}{}^{\mu\nu}, \quad \Delta^{\mu\nu}_{\rho\sigma}\Delta^{\rho\sigma}_{\alpha\beta} = \Delta^{\mu\nu}_{\alpha\beta}, \quad u_\mu\Delta^{\mu\nu}_{\alpha\beta} = g_{\mu\nu}\Delta^{\mu\nu}_{\alpha\beta} = 0, \quad \Delta^{\mu\nu}_{\mu\nu} = 5 \tag{2.26}$$

The terms within the parentheses of the above equation denote the symmetrization of the Lorentz indices($A^{(\mu\nu)} \equiv (A^{\mu\nu} + A^{\nu\mu})/2$). The irreducible form of dissipative current $\tau_{\mu\nu}$ can be tensor decomposed using u^μ, $\Delta^{\mu\nu}$ and $\Delta^{\mu\nu}_{\alpha\beta}$ as

$$\tau^{\mu\nu} \equiv -\Pi\Delta^{\mu\nu} + 2u^{(\mu}h^{\nu)} + \pi^{\mu\nu} \tag{2.27}$$

where we have defined

$$\Pi \equiv -\frac{1}{3}\Delta_{\alpha\beta}\tau^{\alpha\beta}, \quad h^\mu \equiv \Delta^{\mu}_{\alpha}u_\beta\tau^{\alpha\beta}, \quad \pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta}\tau^{\alpha\beta} \tag{2.28}$$

The scalar Π represents the bulk viscous pressure, the four-vector h^μ corresponds to the energy-diffusion four-current, and the second-rank tensor $\pi^{\mu\nu}$ represents the shear-stress tensor.

The projection operators Δ^{μ}_{α} and $\Delta^{\mu\nu}_{\alpha\beta}$ play a vital role in the tensor decomposition. They ensure that the vector h^μ and the tensor $\pi^{\mu\nu}$ are orthogonal to the fluid four-velocity u^μ, and they also guarantee that the shear-stress tensor $\pi^{\mu\nu}$ is traceless.

With these definitions in place, we can express all the irreducible hydrodynamic fields in

terms of the net particle four-current N^μ and the energy-momentum tensor $T^{\mu\nu}$ as follows:

$$\begin{aligned} \mathcal{E} = u_\alpha u_\beta T^{\alpha\beta}, \quad n = u_\alpha N^\alpha, \quad \Pi = -P - \frac{1}{3}\Delta_{\alpha\beta}T^{\alpha\beta} \\ h^\mu = u_\alpha T^{\langle\mu\rangle\alpha}, \quad n^\mu = N^{\langle\mu\rangle}, \quad \pi^{\mu\nu} = T^{\langle\mu\nu\rangle}, \end{aligned} \tag{2.29}$$

Here the angular bracket notations mean $A^{\langle\mu\rangle} \equiv \Delta^\mu_\alpha A^\alpha$ and $B^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} B^{\alpha\beta}$.

Indeed, the energy-momentum tensor $T^{\mu\nu}$ is a symmetric second-rank tensor with ten independent components, and the net particle four-current N_μ is a four-vector with four independent components. Altogether, they contribute to a total of fourteen independent components. In the tensor decompositions, the vector h^μ and the net particle four-current n^μ are orthogonal to the fluid four-velocity u^μ, allowing each of them to have only three independent components. The shear-stress tensor $\pi^{\mu\nu}$ is symmetric, traceless, and orthogonal to u_μ, resulting in five independent components. When we combine the independent components of u_μ, $\mathcal{E}$, n, and Π, which have a total of six independent components (note that pressure P is related to $\mathcal{E}$ via the equation of state), we count a total of seventeen independent components, which is three more than expected. The discrepancy arises because, up to this point, the velocity field u_μ was introduced as a generally normalized four-vector without specifying its components explicitly. To reduce the number of independent components to the correct value, the velocity field u_μ needs to be defined. This specification will correctly constrain the number of independent components, allowing us to describe the dynamics of the dissipative fluid appropriately.

The use of the term 'hydrodynamic frame' in the context of relativistic hydrodynamics can indeed be a source of confusion, especially for those new to the field. In this context, 'frame' doesn't refer to Lorentz frames, as it does in special relativity. Instead, it pertains to different definitions of the fluid four-velocity (u_μ), and these frames cannot be simply related through Lorentz transformations. In the process of formulating the theory of dissipative fluid dynamics, the next crucial step is to fix the fluid velocity u_μ. In the case of

ideal fluids, the local rest frame was well-defined as the frame in which there is no net energy and particle flow simultaneously. However, in the context of dissipative fluids, this definition is no longer valid due to the presence of energy and particle diffusion. From a mathematical perspective, fluid velocity can be defined in various ways. However, from a physical perspective, physically meaningful choices for defining fluid velocity:

- Landau-Lifshitz frame: In this frame, the fluid velocity u_μ is defined to be the normalized eigenvector of the energy-momentum tensor $T^{\mu\nu}$ corresponding to the maximum eigenvalue. This choice aligns the fluid velocity with the direction of energy flow and is particularly relevant for describing systems with predominantly dissipative effects, such as in relativistic heavy-ion collisions [97].

$$u_\nu T^{\mu\nu} = \mathcal{E} u^\mu, \quad u^\mu = \frac{u_\nu T^{\mu\nu}}{u_\alpha u_\beta T^{\alpha\beta}} \implies h^\mu = 0 \tag{2.30}$$

- Eckart frame: In the Eckart frame, the fluid velocity u_μ is defined such that the particle four-current N_μ is parallel to u_μ. This choice ensures that the fluid velocity coincides with the direction of particle flow and is more suitable for systems with strong particle number fluctuations [115].

$$N^\mu = n u^\mu, \quad u^\mu = \frac{N^\mu}{u_\alpha N^\alpha} \implies n^\mu = 0 \tag{2.31}$$

The selection of the appropriate frame depends on the specific physical system and the dominant dissipative effects. Both choices have their advantages and are commonly used in the study of dissipative fluid dynamics to analyze and understand various phenomena in different contexts.

The Landau-Lifshitz frame offers several advantages over the Eckart frame in the context of dissipative fluid dynamics:

- Causality preservation: In the Landau-Lifshitz frame, the fluid velocity u_μ is defined as the normalized eigenvector of the energy-momentum tensor $T^{\mu\nu}$ corresponding to

the maximum eigenvalue. This choice ensures that the velocity u_μ is always causal, i.e., it does not exceed the speed of light. On the other hand, in the Eckart frame, the fluid velocity is not guaranteed to be causal, which can lead to acausal behavior in certain situations.

- Simplicity in description: The Landau-Lifshitz frame is mathematically simpler to describe and work with due to its basis in the eigenvalues of the energy-momentum tensor. This simplicity makes it easier to analyze and interpret the equations of motion and transport coefficients in relativistic dissipative fluid dynamics.

- Suitable for systems with dominant energy dissipation: The Landau-Lifshitz frame is particularly relevant for describing systems where energy dissipation is the dominant dissipative effect. This includes scenarios like relativistic heavy-ion collisions, where energy transport is a crucial aspect. The Landau-Lifshitz frame provides a natural choice for studying dissipative phenomena associated with strong energy gradients.

- Consistency with non-relativistic limits: In the non-relativistic limit, the Landau-Lifshitz frame smoothly reduces to the familiar Newtonian hydrodynamics, ensuring consistency with non-relativistic fluid dynamics [116].

- Applicability to high-energy physics: The Landau-Lifshitz frame is commonly used in high-energy physics, such as in the study of quark-gluon plasma, where relativistic effects and energy dissipation are significant.

While the Eckart frame also has its applications in specific scenarios, the Landau-Lifshitz frame's causal nature, simplicity, and relevance to systems with dominant energy dissipation make it more preferable and widely used in relativistic dissipative fluid dynamics, especially in high-energy physics and relativistic heavy-ion collisions.

In the context of dissipative hydrodynamics, particularly in the Landau frame, the relation $q_\mu = -\frac{(\mathcal{E}+P)}{n} n_\mu$ takes on a specific significance [117]. The Landau frame is a hydrodynamic frame where the heat flow q_μ is set to zero. In this frame, the relation simplifies as follows:

$$q_\mu = -\frac{(\mathcal{E}+P)}{n} n_\mu = 0$$

In the Landau frame, the heat flow is forced to be zero, meaning there is no net flow of thermal energy in the fluid. This simplification is particularly useful when analyzing systems that are very close to thermal equilibrium. The absence of a heat flow simplifies the equations of motion and can aid in gaining insights into the behavior of the fluid near equilibrium. Landau frame essentially states that, in this frame, the heat flow is entirely determined by the energy density ($\mathcal{E}$), pressure (P), four-velocity (u^μ), and particle number density (n). When these quantities are such that the heat flow is set to zero, it reflects a specific state of the fluid where thermal equilibrium or near-equilibrium conditions prevail.

Therefore, the conserved currents in the Landau frame can be written as

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \quad N^\mu = n u^\mu + n^\mu \tag{2.32}$$

Similar to ideal fluids, the energy-momentum conservation equation can be projected along and orthogonal to u^μ and are known as equations of motions for dissipative fluids.

$$u_\mu \partial_\nu T^{\mu\nu} = 0 \implies \dot{\mathcal{E}}(\mathcal{E} + P + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0 \tag{2.33}$$

$$\Delta^\alpha_\mu \partial_\nu T^{\mu\nu} = 0 \implies (\mathcal{E} + P + \Pi)\dot{u}^\alpha - \nabla^\alpha (P + \Pi) + \Delta^\alpha_\mu \partial_\nu \pi^{\mu\nu} = 0 \tag{2.34}$$

$$\partial_\mu N^\mu = 0 \implies \dot{n} + n\theta + \partial_\mu n^\mu = 0 \tag{2.35}$$

where we have defined $\dot{A} \equiv DA = u^\mu \partial_\mu A$ is known as the comoving derivative, $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ is a space like derivative operator being orthogonal to u^μ and the shear tensor is defined as $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu} u^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta$

While there are a total of fourteen independent components from the energy-momentum tensor $T^{\mu\nu}$ and net particle four-current N^{μ}, the equations (Eq (2.33),(2.34),(2.35)) only provide five equations. To obtain the complete set of equations for dissipative fluid dynamics, it is essential to derive an additional nine equations of motion that will close the system.

These additional equations of motion correspond to finding the closed dynamical or constitutive relations satisfied by the dissipative tensors Π (bulk viscous pressure), n^{μ} (energy-diffusion four-current), and $\pi^{\mu\nu}$ (shear-stress tensor). The constitutive relations determine how these dissipative tensors depend on the gradients of the fluid properties and the thermodynamic variables.

2.3.2 First order Navier-Stokes theory

The entropy doesn't obey conservation laws in the presence of dissipative currents, which means $\partial_{\mu}S^{\mu} \neq 0$. To account for this, we must determine the form of the entropy four-current for dissipative fluids, which is not known beforehand and requires careful consideration. For dissipative fluids, we must extend the expression of equation (2.21) to incorporate additional terms that account for the entropy production due to dissipation. These additional terms arise from the dissipative processes and the gradients of fluid properties, and they play a crucial role in capturing the non-equilibrium behavior of the fluid.

$$S^{\mu} = P\beta^{\mu} + \beta_{\nu}T^{\mu\nu} - \alpha N^{\mu} \tag{2.36}$$

The extension of the entropy four-current for dissipative fluids remains valid because it was based on constructing an artificial equilibrium state using matching conditions to satisfy the thermodynamic relations as if in equilibrium. This crucial step was proposed by Eckart, Landau, and Lifshitz as a fundamental approach in deriving the relativistic Navier-Stokes theory [97, 98]. By constructing the artificial equilibrium state, we ensure that the thermodynamic relations still hold despite the presence of dissipative effects.

The next step is to calculate the entropy generation, $\partial_\mu S^\mu$, in dissipative fluids. To achieve this, we substitute the form of the energy-momentum tensor $T^{\mu\nu}$ and the particle four-current N^μ for dissipative fluids, as given in Eq. (2.32), into Eq. (2.36). Then, by taking the divergence and using the equations (Eqs (2.33),(2.34),(2.35)) that govern the dissipative currents, we obtain the expression for the entropy generation as follows.

$$\partial_\mu S^\mu = -\beta\Pi\theta - n^\mu \nabla_\mu \alpha + \beta\pi^{\mu\nu}\sigma_{\mu\nu} \tag{2.37}$$

This expression accounts for how the dissipative effects contribute to the change in entropy over spacetime. The relativistic Navier-Stokes theory is derived by applying the second law of thermodynamics to each fluid element. This involves the requirement that the entropy production $\partial_\mu S^\mu$ must always be positive, in accordance with the principle of irreversibility.

$$-\beta\Pi\theta - n^\mu \nabla_\mu \alpha + \beta\pi^{\mu\nu}\sigma_{\mu\nu} \geq 0 \tag{2.38}$$

If one will assume that the bulk viscous pressure (Π), the shear stress tensor ($\pi^{\mu\nu}$) and the particle-diffusion current (n^μ) are linearly proportional to θ, $\sigma^{\mu\nu}$ and $\nabla^\mu\alpha$ respectively, the above inequality will hold for all possible fluid configurations. This implies to

$$\Pi = -\zeta\theta, \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, n^\mu = \kappa\nabla^\mu\alpha \tag{2.39}$$

The proportionality coefficients ζ, η and κ correspond to the bulk viscosity, particle diffusion, and shear viscosity, respectively. These coefficients quantify the strength of the dissipative effects and play a crucial role in characterizing the behavior of dissipative fluids. When we substitute the expressions for the dissipative currents from the constitutive relations into the equation for entropy production, we observe that the source term for entropy production becomes a quadratic function of the dissipative currents.

$$\partial_\mu S^\mu = \frac{\beta}{\zeta}\Pi^2 - \frac{1}{\kappa}n^\mu n_\mu + \frac{\beta}{2\eta}\pi^{\mu\nu}\pi_{\mu\nu} \tag{2.40}$$

The quadratic nature of the source term signifies that entropy production is directly related to the interactions between different dissipative effects. The coefficients ζ, η and κ determine the relative contributions of bulk viscosity, particle diffusion, and shear viscosity to the overall entropy generation in the fluid. In the Eq (2.40) n^μ is orthogonal to u^μ which is time like. So n^μ is a space-like and will give the relation $n^\mu n_\mu < 0$. The symmetric tensor $\pi^{\mu\nu}$, representing the dissipative pressure tensor, has important properties. In the local rest frame of the fluid, $\pi^{0\mu} = \pi^{\mu 0} = 0$, meaning that there are no components of the dissipative pressure tensor along the time direction. This property arises due to the dissipative effects being purely spatial in the fluid's local rest frame. Furthermore, since $\pi^{\mu\nu}$ is a symmetric tensor, the trace of the square of $\pi^{\mu\nu}$, denoted as $\pi^{\mu\nu}\pi_{\mu\nu}$, will always be positive. Therefore, as long as the coefficients ζ,η and κ are non-negative (i.e. $\zeta \geq 0$, $\eta \geq 0$ and $\kappa \geq 0$), the entropy production $\partial_\mu S^\mu$ will always be positive. The constitutive relations for the dissipative quantities (given in Eq. (2.39)), along with the equations (2.33)-(2.34) governing the dissipative currents, collectively form the relativistic Navier-Stokes equations. These equations provide a comprehensive description of the behavior of dissipative fluids in the relativistic regime.

The relativistic Navier-Stokes theory, as obtained by Landau and Lifshitz [97], and independently by Eckart [98], suffers from acausality and instability. The acausality arises from the constitutive relations for the dissipative currents, where linear relationships between dissipative currents and gradients of the primary fluid-dynamical variables lead to instantaneous effects. This instantaneous response is incompatible with relativistic causality, ultimately causing the theory to be unstable.

To address this issue, various theories have been developed to incorporate dissipative effects in fluid dynamics while respecting causality and stability. Some of these alternative theories include the Grad-Israel-Stewart theory [101, 118, 101], the divergence-type theory [100, 119], extended irreversible thermodynamics [120], Carter's theory [121], and the

Öttinger-Grmela theory [122], among others.

Among these, Israel and Stewart's formulation of causal relativistic dissipative fluid dynamics has gained significant popularity and widespread use. This approach introduces additional dynamical equations for the evolution of dissipative currents that involve time derivatives of the dissipative quantities. By doing so, it allows for finite propagation speeds of dissipative effects and ensures that causality is preserved in the theory. Israel and Stewart's theory has proven to be a successful and robust framework for describing relativistic dissipative fluids, particularly in scenarios such as relativistic heavy-ion collisions and the early universe, where dissipative effects play a crucial role. Their formulation provides a stable and physically consistent description of dissipative fluid dynamics in a relativistic setting, making it a preferred choice for many researchers in the field.

2.3.3 Causal Israel-Stewart theory

The relativistic generalization of fluid dynamics has indeed been a non-trivial task, particularly in addressing the issues of causality and stability. Until recently, the relativistic versions of Navier-Stokes theory were believed to suffer from causality and stability problems, mainly due to the parabolic nature of the system's dispersion relation. This parabolic dispersion relation implies that perturbations can propagate infinitely fast, leading to acausal behavior and instability.

To overcome these challenges, second-order hydrodynamics theories were formulated, which have been found to be causal due to the hyperbolic nature of the dispersion relation. This hyperbolic nature ensures that perturbations propagate at finite speeds, preserving causality and stability in the theory. The pursuit of a consistent and reliable relativistic hydrodynamics theory remains an active area of research. Various approaches and formulations have been proposed, leading to multiple causal theories of relativistic hydrodynamics. Among them, the first and most widely known is the Muller-Israel-Stewart (MIS)

theory, which has been influential in the field [123, 124, 125]. In recent times, there has been growing interest in a class of first-order causal theories, which have gained popularity [126, 127]. These theories offer alternative approaches to tackle the challenges of relativistic hydrodynamics and have shown promise in providing a comprehensive understanding of the dynamics of relativistic fluids.

The Navier-Stokes theory, which suffers from causality and stability issues, can be extended to a causal theory of relativistic fluid dynamics by introducing a relaxation time parameter. The incorporation of this relaxation time introduces a time delay between the generation of gradients and the corresponding forces, allowing for a more physically realistic and causal description of dissipative effects. This relaxation process is described by the Maxwell-Cattaneo law [128]. In the context of relativistic fluid dynamics, Muller, Israel and Stewart (MIS) proposed a theory based on the formulation of the entropy current. They developed a systematic approach to include dissipative effects in a causal manner. By assuming the entropy current to have an algebraic form, they derived a consistent and physically motivated description of relativistic dissipative fluid dynamics. To account for dissipative effects up to second-order terms, the entropy current can be written as a function of the primary fluid-dynamical variables and their gradients. The inclusion of second-order terms is important to capture more complex and realistic dissipative behavior that arises in relativistic fluids. Following this, the second-order entropy current can be written as

$$S^{\mu} = su^{\mu} - \alpha n^{\mu} - \beta(\alpha_0 \Pi \Delta^{\mu\nu} + \alpha_1 \pi^{\mu\nu}) n_{\nu} - (\beta_0 \Pi^{\mu} + \beta_1 n_{\nu} n^{\nu} + \beta_2 \pi_{\alpha\beta} \pi^{\alpha\beta}) \beta^{\mu} \quad (2.41)$$

Implementing the requirement of positive entropy production using the above relation, we

can obtain the functional dependencies of the dissipative currents as shown in ref[].

$$\begin{aligned}
\dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\frac{1}{\beta_0}\Big[\theta + \beta_{\Pi\Pi}\Pi\theta + \alpha_0\nabla_\mu n^\mu + \psi\alpha_n\Pi^n n_\mu \dot{u}^\mu + \psi\alpha\Pi_n n_\mu\nabla^\mu\alpha\Big] \\
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= \frac{1}{\beta_2}\Big[\sigma^{\mu\nu} - \beta_{\pi\pi}\pi^{\mu\nu}\theta - \alpha_1\nabla^{\langle\mu}n^{\nu\rangle} - \chi\alpha_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha - \chi\alpha_{n\pi}n^{\langle\mu}\dot{u}^{\nu\rangle}\Big] \\
\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} &= \frac{1}{\beta_1}\Big[T\nabla^\mu\alpha - \beta_{nn}n^\mu\theta + \alpha_0\nabla^\mu\Pi + \alpha_1\Delta^\mu_\gamma\nabla_\mu\pi^{\gamma\nu} + \tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}^{\langle\mu\rangle}u \\
&\quad + \tilde{\psi}\alpha_{\Pi n}\Pi\nabla^\mu\alpha \;\; + \tilde{\chi}\alpha_{\pi n}\pi^\mu_\nu\nabla^\nu\alpha + \tilde{\chi}\alpha_{n\pi}\pi^\mu_\nu\dot{u}^\nu\Big]
\end{aligned} \tag{2.42}$$

The above equations for dissipative quantities are called relaxation type equations where the relaxation time are defined as

$$\tau_\Pi = \zeta\beta_0, \quad \tau_\pi = 2\eta\beta_2, \quad \tau_n \equiv \lambda\beta_1 = \kappa\beta_1/T \tag{2.43}$$

Since the relaxation times are positive, the Taylor expansion coefficients β_0,β_1 and β_2 must all be greater than zero.

One of the most important aspects of the IS theory is the introduction of relaxation times associated with the dissipative currents. These relaxation times represent the characteristic time scales within which the dissipative currents respond to the hydrodynamic gradients. Unlike in the relativistic Navier-Stokes theory, where this response is instantaneous, the inclusion of relaxation times in the IS theory restores causality and makes the dissipative currents independent dynamical variables satisfying partial differential equations rather than constitutive relations. This introduction of relaxation processes in the IS theory is crucial for achieving a causal and stable relativistic hydrodynamics framework. It ensures that the dissipative effects propagate at finite speeds, respecting the principles of causality and preventing unphysical instantaneous responses or acausal behavior. This significant advantage comes with the introduction of five new parameters, namely β_0, β_1, β_2, α_0 and α_1, in the theory. These parameters are associated with relaxation times and cannot be determined solely within the framework of thermodynamics.

Additional information from microscopic theories, such as kinetic theory, is necessary to determine these coefficients. Kinetic theory provides a more fundamental description of the behavior of particles in a fluid and their interactions, allowing for the determination of these relaxation times and coefficients. The need for microscopic input from kinetic theory is a common feature in the study of dissipative fluid dynamics. By combining the macroscopic framework of hydrodynamics with the microscopic details of particle interactions, researchers can obtain a complete understanding of the evolution of dissipative fluids and determine the coefficients required for a fully specified evolution equation. In the subsequent section, the basics of relativistic kinetic theory and the Boltzmann transport equation are typically explored to gain insights into the microscopic dynamics and the determination of the relaxation times and coefficients in the IS theory.

2.4 Relativistic kinetic theory

The kinetic theory offers a statistical framework that connects the macroscopic properties of a many-body system to the interactions among its constituent particles and the external constraints acting on the system. This framework allows us to express macroscopic quantities in terms of a single-particle phase-space distribution function, which describes the probability of finding a particle in a specific phase-space region. In the context of relativistic dissipative hydrodynamics, the various formulations presented in my thesis are derived within the framework of relativistic kinetic theory. This approach is particularly well-suited for studying systems where relativistic effects are significant, such as high-energy physics or astrophysical scenarios. By considering the statistical behavior of particles in the fluid and accounting for collisions and interactions, the kinetic theory allows us to derive the constitutive equations governing the dissipative currents, such as the shear viscosity, bulk viscosity, and particle diffusion. The main steps involved in using kinetic theory for de-

riving dissipative hydrodynamics involve Boltzmann's transport equation and its various approximations, such as the Chapman-Enskog expansion, Grad's moment method, or the Grad-Israel-Stewart (MIS) formalism. These approaches allow us to relate the moments of the phase-space distribution function to the macroscopic fluid variables, such as density, velocity, and temperature [129].

In relativistic kinetic theory, a system of relativistic particles is described by a large number of particles, each with rest mass m, momentum $\vec{p}$, and energy p_0. The energy-momentum relation in relativity is given by $p^0 = ((\vec{p})^2 + m^2)^{(1/2)}$. To characterize the statistical behavior of this system, we introduce the single particle distribution function f(x, p), which provides the distribution of the four-momentum $p = (p^0, \vec{p})$ at each point in spacetime (x) within the system. Specifically, $f(x,p)\Delta^3 x \Delta^3 p$ gives the average number of particles at a given time t in the volume element $\Delta^3 x$ around the point $\vec{x}$ with momenta in the range $(\vec{p}, \vec{p} + \Delta p)$.

It is important to note that this definition of the single-particle phase-space distribution function f(x, p) assumes two key conditions:

- The number of particles contained in the volume element $\Delta^3 x$ is large: This statistical approach is valid when dealing with a large number of particles, enabling us to treat the system using statistical methods.

- The volume element $\Delta^3 x$ is small compared to the macroscopic point of view: The distribution function f(x, p) allows us to study the behavior of particles at a fine-grained level, where the spatial and momentum variations are considered at a microscopic scale.

By employing this single-particle distribution function, relativistic kinetic theory enables us to explore the statistical behavior of relativistic systems with many particles and derive

macroscopic properties, such as fluid densities, velocities, and temperatures, from the microscopic dynamics of individual particles.

Let's define particle density $n(x)$ for a non-uniform system such that $n(x)\Delta^3 x$ will provide the average number of particles in a volume $\Delta^3 x$ at $(\vec{x}, t)$. Similarly, we can define particle flow $\vec{j}(x)$ as the particle current. Particle density and particle flow can be expressed in terms of $f(x, p)$ as follows.

$$n(x) = \int d^3p f(x,p), \quad \vec{j}(x) = \int d^3p \vec{v} f(x,p) \tag{2.44}$$

Here $\vec{v} = \vec{p}/p^0$ is the particle velocity. A four-vector field called four-flow $N^\mu = (n, \vec{j})$ can be written combining the two local quantities, particle density and particle flow. So four-vector field $N^\mu(x)$ in terms of $f(x, p)$ will be expressed as

$$N^\mu(x) = \int \frac{d^3p}{p^0} p^\mu f(x,p) \tag{2.45}$$

It is to be noted that the quantity $\frac{d^3p}{p^0}$ is a Lorentz invariant quantity, so $f(x, p)$ must be a scalar to satisfy the transformation of N^μ as a four-vector.

Also, the average energy density and energy flow can be written in terms of the distribution function as

$$T^{00}(x) = \int d^3p\, p^0 f(x,p), \quad T^{0,i}(x) = \int d^3p\, p^0 v^i f(x,p) \tag{2.46}$$

The average value of particle momenta p^i is defined as momentum density, and momentum flow or pressure tensor represents the flow in direction j of momentum in the direction i. That means we have

$$T^{i0}(x) = \int d^3p\, p^i f(x,p), \quad T^{ij}(x) = \int d^3p\, p^i v^j f(x,p) \tag{2.47}$$

Using $v^i = p^i/p^0$ these in a compact covariant form we obtain the energy-momentum tensor to be

$$T^{\mu\nu}(x) = \int \frac{d^3p}{p^0}\, p^\mu p^\nu f(x,p) \tag{2.48}$$

The above expression of the energy-momentum tensor corresponds to the second moment of the distribution function for which $T^{\mu\nu}$ is a symmetric quantity.

For small deviations from equilibrium, $f(x,p)$ can be written as $f = f_0 + \delta f$. Where f_0 is defined as the equilibrium distribution function.

$$f_0(x,p) = \frac{1}{exp(\beta u \cdot p - \alpha) + r} \tag{2.49}$$

$(u \cdot p)$ is a scalar product defined as $u \cdot p \equiv u_\mu p^\mu$ and $r = 0, 1$ and -1 for Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein statistics respectively. For $f(x,p) = f_0(x,p)$,the equation (2.45) and (2.48) will reduce to ideal particle four-flow and energy momentum tensor $N^\mu_{(0)}$ and $T^{\mu\nu}_{(0)}$ accordingy. Therefore the dissipative quantities (the bulk viscous pressure Π, the shear stress tensor $\pi^{\mu\nu}$ and the particle diffusion current n^μ) can be written as

$$\Pi = -\frac{1}{3}\Delta_{\mu\nu}\int \frac{d^3p}{p^0}\, p^\mu p^\nu \delta f, \quad \pi^{\alpha\beta} = \Delta^{\alpha\beta}_{\mu\nu}\int \frac{d^3p}{p^0}\, p^\mu p^\nu \delta f, \quad n^\mu = \Delta^{\mu\nu}\int \frac{d^3p}{p^0}\, p_\nu \delta f \tag{2.50}$$

We are referring to the Boltzmann equation, which plays a central role in describing the evolution of the phase-space distribution function in kinetic theory. The Boltzmann equation provides a fundamental and reliable description of the microscopic dynamics of a system of particles undergoing $2 \Leftrightarrow 2$ elastic collisions. The Boltzmann equation for the single-particle distribution function $f(x,p)$ in the presence of $2 \Leftrightarrow 2$ elastic collisions is typically given by:

$$p^\mu \partial_\mu f = \mathcal{C}[f] = \frac{1}{2}\int dp' dk dk' W_{pp'\to kk'}(f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} - f_p f_{p'} \tilde{f}_k \tilde{f}_{k'}) \tag{2.51}$$

$f(x,p)$ is the single-particle phase-space distribution function, representing the probability density of finding a particle with four-momentum p at the spacetime point x, t represents time, x_μ is the spacetime coordinates, and p^μ are the components of the four-momentum $p = (p_0, \vec{p})$. $p^\mu \partial f(x,p)/\partial x_\mu$ represents the advection term, accounting for the transport of particles in the phase space due to their motion. $\mathcal{C}[f]$ is the collision term, representing the

effect of particle interactions through $2 \Leftrightarrow 2$ elastic collisions. The collision term describes how the distribution function changes due to these collisions, redistributing particles in the phase space. The first and second terms within the integral of equation (2.51) corresponds to the $kk' \to pp'$ and $pp' \to kk'$, respectively. Solving the Boltzmann equation allows one to track the time evolution of the distribution function $f(x, p)$ and, in turn, understand how the microscopic interactions influence the macroscopic behavior of the fluid. By examining the moments of the distribution function, such as the energy-momentum tensor and particle four-current, one can derive the evolution equations for the macroscopic quantities, including the dissipative currents.

It is worth noting that the Boltzmann equation is a highly complex partial integrodifferential equation, and its direct solution is often challenging. Various approximation techniques, such as the Chapman-Enskog expansion, Grad's moment method, or the Grad-Israel-Stewart formalism, are used to simplify the equation and derive the constitutive relations for the dissipative currents, which eventually lead to the causal theories of relativistic dissipative hydrodynamics [130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 138, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 117, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192].

The Chapman-Enskog method is an important technique used in relativistic dissipative hydrodynamics to derive constitutive relations for the dissipative currents from the Boltzmann equation. It is a systematic expansion method based on the smallness of gradients and deviations from local equilibrium [152]. The Chapman-Enskog method simplifies the Boltzmann equation by performing a systematic expansion of the distribution function and its gradients in terms of Knudsen number (Kn), which is a dimensionless parameter characterizing the ratio of the mean free path of particles to the macroscopic length scale of the system. The Knudsen number measures the degree of deviation from local equilibrium. The

expansion is typically done as follows: The distribution function is expanded as a power series in Kn:

$$f(x,p) = f_0(x,p) + Knf_1(x,p) + Kn^2 f_2(x,p) + ...,$$

where $f_0(x,p)$ represents the local equilibrium distribution function, and $f_1(x,p)$, $f_2(x,p)$, etc., are the first-order, second-order, and higher-order corrections due to deviations from equilibrium. Similarly, the hydrodynamic quantities (such as energy-momentum tensor and particle four-current) are expanded as power series in Kn. The Boltzmann equation is then solved order by order in Kn, starting from the lowest order. At each order, the collision term in the Boltzmann equation is expanded in Kn, and the resulting equations are matched with the corresponding hydrodynamic equations. The constitutive relations for the dissipative currents are obtained by expressing the dissipative corrections $(f_1, f_2, etc.)$ in terms of the hydrodynamic variables and their gradients.

Boltzmann equation

In the context of kinetic theory, the evolution of the single-particle phase-space distribution function f(x,p)f(x, p)f(x,p) is described by the Boltzmann equation. The Boltzmann equation contains terms involving the collision kernel, denoted as $\mathcal{C}[f, \bar{f}]$ and $\bar{\mathcal{C}}[f, \bar{f}]$. The collision kernel represents the interactions and collisions between particles and is a crucial term of the Boltzmann equation.

$$p^\mu \partial_\mu f = \mathcal{C}\left[f, \bar{f}\right], \qquad p^\mu \partial_\mu \bar{f} = \bar{\mathcal{C}}\left[f, \bar{f}\right], \tag{2.52}$$

For systems that are close to thermal equilibrium, Anderson and Witting proposed an approximation for the collision kernel[193]. This approximation, often referred to as the Relaxation Time Approximation (RTA), is formulated as follows:

$$\mathcal{C}[f, \bar{f}] = -\frac{(u\cdot p)}{\tau_{\mathrm{R}}(x)}(f - f_{\mathrm{eq}}), \quad \bar{\mathcal{C}}[f, \bar{f}] = -\frac{(u\cdot p)}{\bar{\tau}_{\mathrm{R}}(x)}(\bar{f} - \bar{f}_{\mathrm{eq}}), \tag{2.53}$$

This approximation assumes that the relaxation time scale τ_{R} is independent of the energies of the particles and anti-particles. In other words, it simplifies the collision term by assuming that particles tend to relax towards thermal equilibrium on a characteristic timescale that is the same for all particles, regardless of their energy. Conservation of energy-momentum tensor and net-charge current, defined in Eqs. (5.1) and (4.2), leads to

$$\partial_\mu T^{\mu\nu} = \partial_\mu \int dP p^\mu p^\nu \left(f + \bar{f}\right) = 0 \tag{2.54}$$

$$\partial_\mu N^\mu = \partial_\mu \int dP\, p^\mu \left(f - \bar{f}\right) = 0. \tag{2.55}$$

This imposes the following conditions on the collision term in the RTA approximation defined in Eq. (4.16):

$$u_\mu \int dP\, p^\mu p^\nu \left[\frac{f}{\tau_{\mathrm{R}}} + \frac{\bar{f}}{\bar{\tau}_{\mathrm{R}}}\right] = u_\mu \int dP\, p^\mu p^\nu \left[\frac{f_{\mathrm{eq}}}{\tau_{\mathrm{R}}} + \frac{\bar{f}_{\mathrm{eq}}}{\bar{\tau}_{\mathrm{R}}}\right], \tag{2.56}$$

$$u_\mu \int dP\, p^\mu \left[\frac{f}{\tau_{\mathrm{R}}} - \frac{\bar{f}}{\bar{\tau}_{\mathrm{R}}}\right] = u_\mu \int dP\, p^\mu \left[\frac{f_{\mathrm{eq}}}{\tau_{\mathrm{R}}} - \frac{\bar{f}_{\mathrm{eq}}}{\bar{\tau}_{\mathrm{R}}}\right]. \tag{2.57}$$

In the realm of relativistic hydrodynamics, the Relaxation Time Approximation (RTA) has proven invaluable, particularly when relaxation times for particles and antiparticles are energy-independent $\tau_{\mathrm{R}} = \bar{\tau}_{\mathrm{R}}$ [177]. In such cases, the RTA aligns seamlessly with both the Landau frame condition, which stipulates $T^{\mu\nu}u_\nu = \mathcal{E}u^\mu$, and the matching conditions, where $\mathcal{E}$ and n match their equilibrium values ($\mathcal{E}_0$ and n_0). However, it becomes evident that when relaxation times depend on particle energy or momentum, the standard RTA no longer satisfies the conservation equations (typically encompassing energy-momentum and particle number conservation) even within the Landau frame. Recognizing this limitation, we embark on a journey to extend the RTA framework, tailoring it to gracefully accommodate scenarios with energy-dependent relaxation times. This extension holds the promise of ensuring that the conservation equations remain steadfast, even as relaxation times take on energy-dependent characteristics.

Chapter 3

Motivation for ERTA

In previous studies, relaxation time (τ_R) has typically been treated as independent of particle momentum. However, there are two compelling reasons to explore the possibility of making τ_R momentum-dependent, as outlined below[84]:

- Microscopic Realism: Realistic systems often exhibit a dependence of the relaxation time on the momentum or energy of the particles involved. This dependence arises from the specifics of the microscopic interactions taking place within the system. By considering a momentum-dependent relaxation time, we aim to capture the underlying physics more faithfully and bring our theoretical framework closer to the actual behavior of particles in the system.

- Improved Theoretical Consistency: The use of an energy-independent relaxation time in the macroscopic sector can lead to a contradiction with the microscopic picture where the mean free time is, in general, momentum dependent. In contrast, a momentum-dependent τ_R offers a path to achieve greater theoretical consistency. It allows us to develop a framework in which the assumptions made at the microscopic level align more cohesively with the macroscopic conservation laws, leading to a more robust and realistic model.

By embracing a momentum-dependent relaxation time, we aim to enhance the accuracy and applicability of our theoretical descriptions, aligning them more closely with the behavior of real-world systems. This, in turn, can lead to deeper insights and a better understanding of the physical phenomena under investigation.

3.1 Coupling of QCD

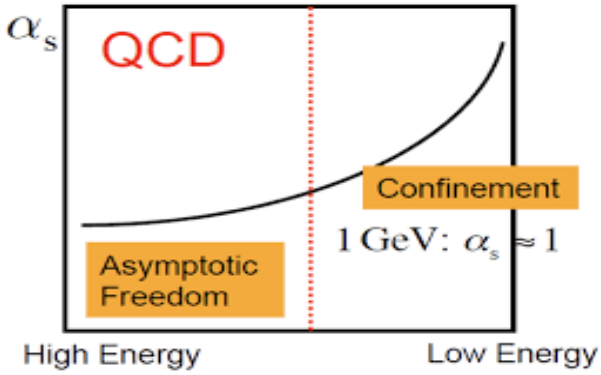

Figure 3.1: Asymptotic nature of QCD coupling constant with respect to energy [194].

In Figure (3.1), which illustrates the relationship between the running coupling constant and energy in Quantum Chromodynamics (QCD), a notable trend is evident. As energy increases, the QCD coupling becomes weaker, while at lower energies, it strengthens. This behavior of QCD coupling leads to the phenomena of "confinement" and "asymptotic freedom". The implication of this behavior on relaxation time can be understood by considering the relationship between relaxation time and the cross-sectional area (σ). A larger cross-sectional area corresponds to a shorter mean free path (λ_{mfp}) for particles. Importantly, the cross-sectional area is directly proportional to the coupling constant (α_s). As a result, there exists a direct connection between the relaxation time and the energy of the particles. This relationship forms a key aspect of our investigation.

$$\tau_{\text{R}}\,(\text{Relaxation time}) \propto \frac{1}{\sigma} \propto \frac{1}{\alpha_s} \propto E(p) \tag{3.1}$$

3.2 Different power of momentum dependence

In the table below, we present the momentum dependence of relaxation time (τ_{R}) across various scenarios, each defined by distinct ansatz and physical processes. In the table, τ_{R} is

Model	Physics	Formula
Relaxation time, ($\tau_{\mathrm{R}} \propto p$)	Relaxation time grows with particle momentum	$\chi(p) \propto p^2$
Relaxation time, ($\tau_{\mathrm{R}} = \text{cons.}$)	Relaxation time independent of momentum	$\chi(p) \propto p$
Scalar theory	Randomizing collisions which happen rarely	$\chi(p) \propto p^2$
QCD Soft Scattering	Soft $q \sim gT$ collisions lead to a random walk of hard particles	$\chi(p) \propto p^2$
QCD Hard Scattering	Hard $q \sim \sqrt{pT}$ collisions lead to a random walk of hard particles	$\chi(p) \propto \frac{p^2}{\log(p/T)}$
QCD radiation energy (E-loss)	Radiative energy controls the approach to equi- librium. In the LPM regime $\hat{q}$ controls the radiation rate.	$\chi(p) \propto \frac{p^{3/2}}{\alpha_s \sqrt{q}}$

Table 3.1: Summary of the functional dependence of the departure from equilibrium on the theory and approximation considered [195].

referred as τ_p for simplicity. Here the parameter $\chi(p)$ is related with τ_{R} with the relation:

$$\tau_{\mathrm{R}} = \chi(p) * (\frac{T}{p}).$$

In kinetic theory description, two key ansatz are primarily encountered: the quadratic ansatz, where equilibration time is proportional to energy (p), and the linear ansatz, where equilibration time remains independent of energy [195]. Ohter possible scenario are:

- **Scalar Theory ($\lambda\phi^4$):** In this scenario, quadratic ansatz is described where the cross-section decreases as $\frac{1}{p}$. Consequently, particles with higher energy exhibit slower equilibration as they encounter a more transparent medium.

- **Soft gT Collisions:** In the context of soft scattering approximation, a fascinating phenomenon unfolds. These soft collisions give rise to an intriguing momentum diffusion effect because of 'dragging' hard gluons through the dense QCD medium resulting quadratic ansatz.

- **Logarithmic Correction Ansatz:** In this case, momentum diffusion scales logarithmically with particle energy.

- **QED (Quantum Electrodynamics):** Here, bremsstrahlung dominates energy loss compared to ionization. Consequently, the penetration depth of an electromagnetic shower scales as $\tau_{\mathrm{R}} \propto p$.

- **QCD (Quantum Chromodynamics):** In the analogous QCD scenario, high-energy particles mainly lose energy through inelastic gluon radiation. The relaxation time scales as $\tau_{\mathrm{R}} \propto E^{\frac{1}{2}}$ due to the Landau-Pomeranchuk-Migdal (LPM) suppression effect. This effect leads to the rate of inelastic energy loss scaling as $\frac{dE}{dt} \propto E^{\frac{1}{2}}$ with incident energy.

To address these complexities, multiple models have proposed modifications to the Anderson-Witting (AW) relaxation time approximation (RTA) in order to ensure conservation laws with momentum dependence relaxation time [143, 196, 197]. We have introduced an alternative approach that retains the AW form of collision term but with extended form of RTA (ERTA) while satisfying conservation equations at different gradient orders of the distribution function expansion [183, 190]. We have formulated transport coefficients for dissipative quantities up to the first order in a generalized case [183]. We further extend this derivation to the second order under specific conditions [190].

3.3 Violation of energy-momentum conservation

In this subsection, we show that the Boltzmann equation with momentum dependent relaxation time leads to violation of energy-momentum conservation. The Boltzmann equation in the relaxation-time approximation is given by

$$p^{\mu}\partial_{\mu}f = -\frac{(u\cdot p)}{\tau_{\mathrm{R}}(x,p)}\delta f. \tag{3.2}$$

For energy-momentum conservation, we have

$$\partial_\mu T^{\mu\nu} = 0 \Rightarrow \partial_\mu \int dP\, p^\mu p^\nu f = 0 \Rightarrow \int dP\, p^\nu \left(p^\mu \partial_\mu f\right) = 0. \tag{3.3}$$

Using Eq. (3.2) in the above equation, we find that the the following condition should be satisfied

$$\int dP\, p^\nu \left[-\frac{(u\cdot p)}{\tau_{\rm R}(x,p)}\delta f\right] = 0. \tag{3.4}$$

This is essentially the constraint that the first moment of the collision kernel should vanish.

In the case of momentum independent $\tau_{\rm R}$,

$$-\int dP\, p^\nu \frac{(u\cdot p)}{\tau_{\rm R}(x)}\delta f = -\frac{u_\mu}{\tau_{\rm R}(x)}\int dP\, p^\mu p^\nu \delta f = -\frac{1}{\tau_{\rm R}(x)} u_\mu \delta T^{\mu\nu} = 0, \tag{3.5}$$

where $\delta T^{\mu\nu}$ is the out-of-equilibrium correction to the energy-momentum tensor. The last identity in the above equation follows from the Landau frame condition. Therefore, we conclude that RTA with momentum independent relaxation time conserves energy-momentum with the Landau frame definition for fluid velocity.

For the case of momentum dependent $\tau_{\rm R}$, we have

$$-\int dP\, p^\nu \frac{(u\cdot p)}{\tau_{\rm R}(x,p)}\delta f = -u_\mu \int dP \frac{1}{\tau_{\rm R}(x,p)} p^\mu p^\nu \delta f. \tag{3.6}$$

We find that the above expression does not vanish for general $\tau_{\rm R}(x,p)$ with the Landau frame definition of the fluid velocity. Therefore we conclude that a naive extension of the RTA, to include momentum dependence relaxation time, leads to violation of energy-momentum conservation.

Chapter 4

First order dissipative hydrodynamics using ERTA

4.1 Introduction

Relativistic Boltzmann equation is a transport equation governing the space-time evolution of the single particle phase-space distribution function, and is capable of accurately describing the collective dynamics of the system. However, the Boltzmann equation is difficult to solve directly as the collision term depends on the integral of the product of distribution functions, making it a complicated integro-differential equation. Many approximation for simplifying the collision term in the linearized regime have been proposed over several decades [198, 199, 200, 193]. The Anderson and Witting relaxation time approximation (RTA) of the collision term [193] enormously simplifies the Boltzmann equation by assuming that the effect of collisions are to exponentially drive the system towards local equilibrium, controlled by a single parameter–'relaxation' time. This approximation has been used quite successfully in several fields of physics. It has been employed to formulate relativistic dissipative hydrodynamics and derive transport coefficients [132, 133, 134, 135, 136, 137, 138, 151, 147, 155, 165, 171], and has also been recently used to study the domain of applicability of hydrodynamics and investigate the hydrodynamization phenomena [201, 202, 203, 204, 205, 206, 87, 207, 208, 209, 210, 178, 211, 212, 213].

The usual formulation of hydrodynamics from RTA assumes the relaxation-time to be independent of particle energy (or momentum), and one is restricted to work in Landau frame to ensure macroscopic conservation laws. However, the collision time scale typi-

cally depends on the microscopic interactions for any realistic system [195, 214, 215, 143, 216]. Promoting the relaxation-time to be particle energy-dependent or employing a general matching condition leads to violation of microscopic conservation laws, which has generated much interest towards a consistent approximation of the collision term which satisfies microscopic and macroscopic conservation laws [217, 218, 197, 196, 219]. Recently, a modification of the RTA was proposed [196] to ensure microscopic conservation of particle number and energy momentum for energy-dependent relaxation-time irrespective of choice of hydrodynamic frames. In the present work, we consider a different resolution to the problem [143]. By defining the equilibrium distribution function appearing in the RTA approximation in the "thermodynamic frame" [220, 221, 222], we develop a framework for consistent derivation of order-by-order hydrodynamics where conservation laws hold at each order in the gradient expansion. Within this framework, we derive first-order hydrodynamic equations in Landau frame with an energy-dependent relaxation-time and show that the transport coefficients, such as shear and bulk viscosity as well as charge and heat diffusion currents, have corrections due to the energy dependence of relaxation-time. We further study the ratios of these transport coefficients using a power law parametrization for the energy dependence of the relaxation-time and find several new and interesting scaling features that we report here.

4.2 Relativistic hydrodynamics

Conserved energy momentum tensor and net-charge current can be expressed in terms of the single particle phase–space distribution function:

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu \left(f + \bar{f}\right) = \mathcal{E} u^\mu u^\nu - (\mathcal{P} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \tag{4.1}$$

$$N^\mu = \int dP\, p^\mu \left(f - \bar{f}\right) = n u^\mu + n^\mu, \tag{4.2}$$

where $dP = g\, d\mathbf{p}/[(2\pi)^3 p_0]$ is the invariant momentum-space integration measure with g being the degeneracy factor and p^μ is the particle four momenta. Here f and $\bar{f}$ are the phase-space distribution functions for particles and anti-particles respectively. In the tensor decomposition, $\mathcal{E}$, $\mathcal{P}$, and n are the energy density, equilibrium pressure, and the net number density. The projection operator $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is orthogonal to the hydrodynamic four-velocity u^μ defined in the Landau frame: $T^{\mu\nu}u_\nu = \mathcal{E}u^\mu$. We work with the Minkowskian metric tensor $g^{\mu\nu} \equiv \text{diag}(+,-,-,-)$.

The energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$, and particle four-current conservation, $\partial_\mu N^\mu = 0$, yields the fundamental evolution equations for $\mathcal{E}$, u^μ and n, as

$$\dot{\mathcal{E}} + (\mathcal{E} + \mathcal{P} + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{4.3}$$

$$(\mathcal{E} + \mathcal{P} + \Pi)\dot{u}^\alpha - \nabla^\alpha(\mathcal{P} + \Pi) + \Delta^\alpha_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{4.4}$$

$$\dot{n} + n\theta + \partial_\mu n^\mu = 0. \tag{4.5}$$

Here we use the standard notation $\dot{A} = u^\mu \partial_\mu A$ for co-moving derivatives, $\nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu$ for space-like derivatives, $\theta \equiv \partial_\mu u^\mu$ for the expansion scalar and, $\sigma^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\theta\Delta^{\mu\nu}$ for the velocity stress tensor.

In order to calculate the thermodynamic quantities corresponding to a system of single species of relativistic particles, we consider the equilibrium phase-space distribution functions for particles, f_{eq}, and anti-particles, $\bar{f}_{\text{eq}}$, given as:

$$f_{\text{eq}} \equiv \frac{1}{e^{\beta(u\cdot p)-\alpha} + a}, \qquad \bar{f}_{\text{eq}} \equiv \frac{1}{e^{\beta(u\cdot p)+\alpha} + a}\,, \tag{4.6}$$

where $u\cdot p \equiv u_\mu p^\mu$, $\beta \equiv 1/T$ is the inverse temperature and $\alpha \equiv \mu/T$ is the ratio of the chemical potential to temperature. Also, a represents the species of particles; $a = -1, 0, 1$ for Bose-Einstein, Boltzmann, and Fermi-Dirac gas, respectively.

The temperature, T, and chemical potential, μ, of a non-equilibrium system are auxiliary quantities and are defined using the matching conditions $\mathcal{E} = \mathcal{E}_0$ and $n = n_0$, where $\mathcal{E}_0$ and

n_0 are the energy density and the net number density in equilibrium:

$$\mathcal{E}_0 \equiv u_\mu u_\nu \int dP\, p^\mu p^\nu \left(f_{\rm eq} + \bar{f}_{\rm eq}\right) = I^+_{2,0}\,, \tag{4.7}$$

$$n_0 \equiv u_\mu \int dP\, p^\mu \left(f_{\rm eq} - \bar{f}_{\rm eq}\right) = I^-_{1,0}\,. \tag{4.8}$$

Here, the thermodynamic integrals $I^\pm_{n,q}$ are defined as

$$I^\pm_{n,q} \equiv \frac{1}{(2q+1)!!}\int {\rm dP}\,(u\cdot p)^{n-2q}\big(\Delta_{\alpha\beta}\,p^\alpha p^\beta\big)^q\big(f_{\rm eq} \pm \bar{f}_{\rm eq}\big)\,. \tag{4.9}$$

It is easy to see that the equilibrium pressure and entropy density are

$$\mathcal{P} \equiv -\frac{1}{3}\Delta_{\mu\nu}\int dP\, p^\mu p^\nu \left(f_{\rm eq} + \bar{f}_{\rm eq}\right) = -I^+_{2,1}\,, \tag{4.10}$$

$$s_0 \equiv \frac{\mathcal{E}_0 + \mathcal{P} - \mu\, n_0}{T} = \frac{1}{T}\left(I^+_{2,0} - I^+_{2,1} - \mu\, I^-_{1,0}\right). \tag{4.11}$$

Using the expressions for these thermodynamic quantities, the speed of sound squared can be obtained from $c_s^2 \equiv d\mathcal{P}/d\mathcal{E}\big|_{s/n}$.

For a system close to local thermodynamic equilibrium, the non-equilibrium phase-space distribution function can be written as $f = f_{\rm eq} + \delta f$, where $|\delta f| \ll f$ is out-of-equilibrium correction to the distribution function. Using Eqs. (5.1) and (4.2), the bulk viscous pressure, Π, the shear stress tensor, $\pi^{\mu\nu}$ and the dissipative charge diffusion, n^μ, are expressed in terms of δf as

$$\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\,(\delta f + \delta\bar{f}), \tag{4.12}$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dP\, p^\alpha p^\beta\,(\delta f + \delta\bar{f}), \tag{4.13}$$

$$n^\mu = \Delta^\mu_\nu \int dP\, p^\nu(\delta f - \delta\bar{f}), \tag{4.14}$$

where $\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$ is a traceless symmetric projection operator orthogonal to u^μ as well as $\Delta^{\mu\nu}$. In order to derive the expressions for the above dissipative quantities in terms of hydrodynamic gradients and calculate the associated transport coefficients, we require the form of δf and $\delta\bar{f}$. In the next section, we obtain the first-order

correction to $f_{\rm eq}$ by extending the Anderson-Witting RTA approximation of the collision term to incorporate energy dependence of relaxation-time.

4.3 Boltzmann equation

The evolution of single-particle phase-space distribution function $f(x,p)$ within the framework of kinetic theory is governed by the Boltzmann equation:

$$p^\mu \partial_\mu f = \mathcal{C}\left[f,\bar{f}\right], \qquad p^\mu \partial_\mu \bar{f} = \bar{\mathcal{C}}\left[f,\bar{f}\right], \tag{4.15}$$

where $\mathcal{C}[f,\bar{f}]$ and $\bar{\mathcal{C}}[f,\bar{f}]$ are the so called collision kernel which is the most nontrivial part of Boltzmann equation. For a system close to equilibrium, Anderson and Witting proposed an approximation for the collision kernel,

$$\mathcal{C}[f,\bar{f}] = -\frac{(u\cdot p)}{\tau_{\rm R}(x)}(f-f_{\rm eq}), \quad \bar{\mathcal{C}}[f,\bar{f}] = -\frac{(u\cdot p)}{\bar{\tau}_{\rm R}(x)}(\bar{f}-\bar{f}_{\rm eq}), \tag{4.16}$$

where $\tau_{\rm R}$ and $\bar{\tau}_{\rm R}$ are relaxation time scales for particles and anti-particles, respectively. The above model, which we refer to as RTA in the following, assumes $\tau_{\rm R}$ and $\bar{\tau}_{\rm R}$ to be independent of particle energies. Conservation of energy-momentum tensor and net-charge current, defined in Eqs. (5.1) and (4.2), leads to

$$\partial_\mu T^{\mu\nu} = \partial_\mu \int dP p^\mu p^\nu \left(f+\bar{f}\right) = 0 \tag{4.17}$$

$$\partial_\mu N^\mu = \partial_\mu \int dP\, p^\mu \left(f-\bar{f}\right) = 0. \tag{4.18}$$

This imposes the following conditions on the collision term in the RTA approximation defined in Eq. (4.16):

$$u_\mu \int dP\, p^\mu p^\nu \left[\frac{f}{\tau_{\rm R}} + \frac{\bar{f}}{\bar{\tau}_{\rm R}}\right] = u_\mu \int dP\, p^\mu p^\nu \left[\frac{f_{\rm eq}}{\tau_{\rm R}} + \frac{\bar{f}_{\rm eq}}{\bar{\tau}_{\rm R}}\right], \tag{4.19}$$

$$u_\mu \int dP\, p^\mu \left[\frac{f}{\tau_{\rm R}} - \frac{\bar{f}}{\bar{\tau}_{\rm R}}\right] = u_\mu \int dP\, p^\mu \left[\frac{f_{\rm eq}}{\tau_{\rm R}} - \frac{\bar{f}_{\rm eq}}{\bar{\tau}_{\rm R}}\right]. \tag{4.20}$$

For particle energy independent relaxation times, the above two conditions are consistent with the Landau frame condition, $T^{\mu\nu}u_\nu = \mathcal{E}u^\mu$, and matching conditions, $\mathcal{E} = \mathcal{E}_0$ and $n = n_0$, provided that the relaxation time scales for particles and antiparticles are equal, i.e., $\tau_{\rm R} = \bar{\tau}_{\rm R}$ [177]. In the following we will assume this to be true and assign only one relaxation timescale, $\tau_{\rm R}$, for the system under consideration. However, it can be easily seen that conservation equations (4.19) and (4.20) are not satisfied even in the Landau frame when the relaxation time $\tau_{\rm R}$ becomes particle energy (or momentum) dependent. Therefore the form of usual RTA is rather restrictive and in what follows, we extend the RTA such that the conservation equations are satisfied for energy-dependent relaxation time.

4.3.1 Extended Relaxation time approximation

We consider the following extension of the RTA [143]:

$$p^\mu \partial_\mu f = -\frac{(u\cdot p)}{\tau_{\rm R}(x,p)}\left(f - f^*_{\rm eq}\right), \tag{4.21}$$

with $f \to \bar{f}$ and $f^*_{\rm eq} \to \bar{f}^*_{\rm eq}$ for anti-paticles. Here, $f^*_{\rm eq}$ and $\bar{f}^*_{\rm eq}$ are the local equilibrium distribution functions upon which the system relaxes with a time scale $\tau_{\rm R}(x,p)$,

$$f^*_{\rm eq} \equiv \frac{1}{e^{\beta^*(u^*\cdot p)-\alpha^*}+a}, \quad \bar{f}^*_{\rm eq} \equiv \frac{1}{e^{\beta^*(u^*\cdot p)+\alpha^*}+a}, \tag{4.22}$$

where $\beta^* \equiv 1/T^*$ and $\alpha^* \equiv \mu^*/T^*$. Note that the above equilibrium distribution functions reduces to the Maxwell-Boltzmann/Fermi-Dirac/Bose-Einstein form in the local rest frame of u^*_μ with temperature T^* and chemical potential μ^*. In this sense, we interpret the local rest frame of u^*_μ to be the 'thermodynamic frame' [220, 221, 222] with T^* and μ^* being the corresponding thermodynamic quantities.

At this juncture, we would like to clarify the interpretation and differences between hydrodynamic frame and thermodynamic frame. Firstly, we emphasize that hydrodynamic "frames" (for instance the two well known Landau and Eckart frames or any other definition

of u^μ), should not be confused with Lorentz frames, i.e., they are not related to each other by Lorentz transformations. Instead, these definitions of u^μ specifies the local fluid rest frame. Moreover, for a non-equilibrium system, temperature and chemical potential are auxiliary fields which we define using the matching conditions. Therefore the local equilibrium distribution function, defined in Eq. (4.6), is of the Maxwell-Boltzmann/Fermi-Dirac/Bose-Einstein form in the fluid rest frame, $u^\mu = (1, 0, 0, 0)$, with auxiliary fields T and μ. On the kinetic theory side, where we consider the Boltzmann equation in ERTA [Eq. (4.21)], we propose that the distribution function relaxes to a local equilibrium distribution, Eq. (4.22), which takes the form of Maxwell-Boltzmann/Fermi-Dirac/Bose-Einstein distributions in the rest frame of u^*_μ. In this local rest frame definition, which we term as the thermodynamic frame, the distribution corresponds to temperature T^* and chemical potential μ^*. We insist that the local rest frame of u^μ (defined in the present work as the Landau frame) on the hydrodynamics side need not coincide with that of u^*_μ on the kinetic theory side. In the following, we will relate these two frames, and the corresponding thermodynamic quantities.

4.3.2 Relation between two frames

In order to obtain the out-of-equilibrium correction to the distribution function, we employ Chapman-Enskog like expansion to iteratively solve the ERTA Boltzmann equation (4.21), i.e., $f = f_{\text{eq}} + \delta f_{(1)} + \delta f_{(2)} + \dots$. Here $\delta f_{(i)}$ represents i^{th} order gradient correction to the distribution function. Since we are interested in deriving hydrodynamic equations, the expansion is done about the hydrodynamic equilibrium. The first order gradient correction, $\delta f_{(1)}$ obtained using Eq. (4.21),

$$\delta f_{(1)} = \delta f_* - \frac{\tau_{\text{R}}(x,p)}{(u\cdot p)}\, p^\mu \partial_\mu f_{\text{eq}}, \tag{4.23}$$

and similarly for $\delta\bar{f}_{(1)}$. Here, $\delta f_* \equiv f^*_{\rm eq} - f_{\rm eq}$ adds a further gradient correction to $\delta f_{(1)}$ arising from the difference between the definition of hydrodynamic and thermodynamic frame variables. It is clear that these 'frames' should coincide in equilibrium (as the collision term vanishes in equilibrium), and therefore δf_* vanishes in equilibrium.

Towards determining δf_*, we start by relating the auxiliary hydrodynamic variables u^μ, T and μ to the corresponding variables on the kinetic theory side, u^*_μ, T^* and μ^*,

$$u^*_\mu \equiv u_\mu + \delta u_\mu, \qquad T^* \equiv T + \delta T, \qquad \mu^* \equiv \mu + \delta\mu\,. \tag{4.24}$$

Since δu_μ, δT, and $\delta\mu$ are out of equilibrium corrections, they are at least first order in gradients[1]. To obtain δf_*, we Taylor expand $f^*_{\rm eq}$ about u^μ, T and μ:

$$\begin{aligned}\delta f_* &= \left.\frac{\partial f^*_{\rm eq}}{\partial u^\mu_*}\right|_{(u^\mu,T,\mu)} \delta u^\mu \;+\; \left.\frac{\partial f^*_{\rm eq}}{\partial T_*}\right|_{(u^\mu,T,\mu)} \delta T + \left.\frac{\partial f^*_{\rm eq}}{\partial \mu_*}\right|_{(u^\mu,T,\mu)} \delta\mu \;+\; \mathcal{O}\big(\delta^2\big) \\ &= \left[-\frac{p_\mu \delta u^\mu}{T} + \frac{(u\cdot p-\mu)\delta T}{T^2} + \frac{\delta\mu}{T}\right] f_{\rm eq}\tilde{f}_{\rm eq} + \mathcal{O}\big(\delta^2\big)\,,\end{aligned} \tag{4.25}$$

where $\tilde{f}_{\rm eq} \equiv 1 - a f_{\rm eq}$. For anti-particles, we replace the chemical potential $\mu \to -\mu$ and $f_{\rm eq} \to \bar{f}_{\rm eq}$. In deriving, we have ignored terms of $\mathcal{O}(\delta^2)$ as they are of higher order.

4.3.3 Out-of-equilibrium correction to the distribution function

We state some useful relations and identities which will be used later to obtain the first-order correction, $\delta f_{(1)}$. Using the hydrodynamic equations (5.2)-(4.5) and Eqs. (5.5), (4.8), we obtain

$$\dot{\alpha} = \chi_a\,\theta + \mathcal{O}(\delta^2)\,, \quad \dot{\beta} = \chi_b\,\beta\,\theta + \mathcal{O}(\delta^2)\,, \tag{4.26}$$

$$\nabla^\mu\beta = \frac{n}{\mathcal{E}+\mathcal{P}}(\nabla^\mu\alpha) - \beta\,\dot{u}^\mu + \mathcal{O}(\delta^2)\,, \tag{4.27}$$

[1] Since both u_μ and u^*_μ are time like four-velocities, their normalization leads to, $u\cdot u = u^*\cdot u^* = 1, \implies u\cdot\delta u = \mathcal{O}\big(\delta^2\big)$. Hence the term $u\cdot\delta u$ is second-order in gradients and we shall ignore it in the present derivation.

where we have kept all terms till first-order in gradients. The dimensionless quantities χ_a and χ_b are defined as

$$\chi_a \equiv \frac{(\mathcal{E}+\mathcal{P})J^-_{2,0} - nJ^+_{3,0}}{J^+_{3,0}J^+_{1,0} - J^-_{2,0}J^-_{2,0}}, \quad \chi_b \equiv \frac{(\mathcal{E}+\mathcal{P})J^+_{1,0} - nJ^-_{2,0}}{\beta(J^+_{3,0}J^+_{1,0} - J^-_{2,0}J^-_{2,0})}. \tag{4.28}$$

The $J^{\pm}_{n,q}$ integrals appearing in the above expressions are defined similar to Eq. (4.9), but with $(f_{\rm eq}\tilde{f}_{\rm eq} \pm \bar{f}_{\rm eq}\tilde{\bar{f}}_{\rm eq})$ in the integrand instead of $(f_{\rm eq} \pm \bar{f}_{\rm eq})$.

Using Eqs. (4.25), (4.26) and (4.27), the first-order gradient correction to the distribution function, Eq. (4.23), simplifies to

$$\begin{aligned}\delta f_{(1)} = &\left[-\beta\, p\cdot\delta u + \beta^2\,(u\cdot p - \mu)\,\delta T + \beta\,\delta\mu\right] f_{\rm eq}\,\tilde{f}_{\rm eq} \\ &+ \tau_{\rm R}(x,p)\left[\left(\beta(u\cdot p)\,(\chi_b - 1/3) + \frac{\beta\, m^2}{3(u\cdot p)} - \chi_a\right)\theta\right. \\ &\left. + \frac{\beta}{u\cdot p}\,p^\mu p^\nu \sigma_{\mu\nu} + \left(\frac{n}{\mathcal{E}+\mathcal{P}} - \frac{1}{u\cdot p}\right) p^\mu \nabla_\mu \alpha\right] f_{\rm eq}\,\tilde{f}_{\rm eq}.\end{aligned} \tag{4.29}$$

To evaluate $\delta\bar{f}_{(1)}$ for anti-particles, one needs to replace $\mu \to -\mu$, $\alpha \to -\alpha$, $\chi_a \to -\chi_a$ and $f_{\rm eq}\,\tilde{f}_{\rm eq} \to \bar{f}_{\rm eq}\,\tilde{\bar{f}}_{\rm eq}$ in above expression.

4.4 Hydrodynamic frame and matching conditions

We note that $\delta f_{(1)}$ depends on δT, δu^μ and $\delta\mu$. Imposing Landau frame conditions: $u_\nu T^{\mu\nu} = \mathcal{E}u^\mu$ and the matching conditions: $\mathcal{E} = \mathcal{E}_0$ and $n = n_0$ at first-order in gradients, i.e., with $f = f_1 \equiv f_{\rm eq} + \delta f_{(1)}$, we obtain (as in Appendix 6.1)

$$\delta u^\mu = \mathcal{C}_1 \frac{(\nabla^\mu \alpha)}{T}\,, \qquad \delta T = \mathcal{C}_2\,\theta\,, \qquad \delta\mu = \mathcal{C}_3\,\theta\,, \tag{4.30}$$

where we have introduced the dimensionless variables

$$\mathcal{C}_1 \equiv \frac{T}{\mathcal{E}+\mathcal{P}}\left[K^-_{2,1} - \left(\frac{n}{\mathcal{E}+\mathcal{P}}\right)K^+_{3,1}\right], \tag{4.31}$$

$$\mathcal{C}_2 \equiv T^2\left(\frac{J^-_{2,0}\,y^-_{2,0} - J^+_{1,0}\,y^+_{3,0}}{J^+_{3,0}J^+_{1,0} - J^-_{2,0}J^-_{2,0}}\right), \tag{4.32}$$

$$\mathcal{C}_3 \equiv T\left[\frac{\left(J^-_{2,0} - \mu\,J^+_{1,0}\right)y^+_{3,0} - \left(J^+_{3,0} - \mu\,J^-_{2,0}\right)y^-_{2,0}}{J^+_{3,0}J^+_{1,0} - J^-_{2,0}J^-_{2,0}}\right]. \tag{4.33}$$

In the above equations, $K^{\pm}_{n,q}$ are defined similar to Eq. (4.9) but with $\Big[(f_{\rm eq}\tilde{f}_{\rm eq} \pm \bar{f}_{\rm eq}\tilde{\bar{f}}_{\rm eq})\tau_{\rm R}(x,p)\Big]$ in the integrand instead of $(f_{\rm eq} \pm \bar{f}_{\rm eq})$. The coefficients $y^{\pm}_{n,q}$ are defined as

$$y^{\pm}_{n,q} \equiv \beta\left(\chi_b - \frac{1}{3}\right) K^{\pm}_{n,q} + \frac{m^2}{3T} K^{\pm}_{(n-2),q} - \chi_a K^{\mp}_{(n-1),q}\,. \tag{4.34}$$

When the relaxation time is particle energy independent, the integrals $K^{\pm}_{n,q} \to \tau_{\rm R}(x) J^{\pm}_{n,q}$ and $y^{-}_{2,0}\,, y^{+}_{3,0} \to 0$ leading to vanishing of the coefficients $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$. Thus δu^{μ}, δT and $\delta\mu$ vanishes and ERTA reduces to the usual Anderson-Witting RTA.

As already mentioned, the quantities δu^{μ}, δT and $\delta\mu$ relate the hydrodynamic and thermodynamic frames, and to express these in terms of hydrodynamic fields requires one to impose the hydrodynamic frame and matching condition at each order in gradients. Further, the derivation of $\delta f_{(1)}$ requires the lower order evolution of hydrodynamic fields, Eqs. (4.26) and (4.27), obtained from conservation equations $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu N^{\mu} = 0$ *in the same hydrodynamic frame*. Therefore obtaining $\delta f_{(1)}$ by iteratively solving the ERTA Boltzmann equation, Eq. (4.21), and fixing δT, δu^{μ} and $\delta\mu$ using the same frame and matching conditions ensures macroscopic conservation by construction.

4.5 First order transport coefficients

Equation (4.29) together with Eq. (4.30) and Eqs. (4.31)-(4.33) completely specifies the first-order correction to the equilibrium distribution function, $\delta f_{(1)}$. With this, the relativistic Navier-Stokes expression for dissipative quantities using the definitions given in Eqs. (4.12)-(4.14) is obtained to be,

$$\pi^{\mu\nu} = 2\,\eta\,\sigma^{\mu\nu}, \qquad \Pi = -\zeta\,\theta, \qquad n^{\mu} = \kappa_n \nabla^{\mu}\alpha. \tag{4.35}$$

where the transport coefficients are given by,

$$\eta = \frac{K^{+}_{3,2}}{T}\,, \tag{4.36}$$

$$\zeta = -\mathcal{C}_2\left(\frac{\mathcal{E}+\mathcal{P}-\mu\, n}{T}\right) - \mathcal{C}_3\, n + y^{+}_{3,1}\,, \tag{4.37}$$

$$\kappa_n = \mathcal{C}_1\,\frac{n}{T} + \left(\frac{n}{\mathcal{E}+\mathcal{P}}\right) K^{-}_{2,1} - K^{+}_{1,1}\,. \tag{4.38}$$

The form of the relaxation time $\tau_R(x,p)$ has not yet been specified. In the next Section, we study the behaviour of transport coefficients using a power-law parametrization for the energy dependence of the relaxation time [195, 214, 215, 143, 216],

$$\tau_{\rm R}(x,p) = \tau_{\rm eq}(x)\,\left(\frac{u\cdot p}{T}\right)^{\ell}\,. \tag{4.39}$$

Here $\tau_{\rm eq}(x)$ represents the energy independent part of relaxation time and ℓ is a constant. In this case, the integrals $K^{\pm}_{n,q}$ can be expressed as: $K^{\pm}_{n,q} = \left(\tau_{\rm eq}/T^{\ell}\right)\, J^{\pm}_{(n+\ell),q}$.

4.6 Results and discussions

In this section, we study the behaviour of first-order transport coefficients, obtained in the previous section, in three different cases:

1. A system of chargeless and massless particles. In this case, the bulk viscous pressure and charge flux vanishes and dissipation in the system occurs purely from shear stress tensor. Mathematically this amounts to setting $m = \mu = 0$.

2. A system of chargeless massive particles. In this case, the charge flux vanishes and dissipation in the system is due to shear stress tensor and bulk viscous pressure. This is equivalent to setting $\mu = 0$ and $m \neq 0$.

3. A system of charged massless particles. In this case, the bulk viscous pressure vanishes and dissipation in the system is due to non-zero shear stress tensor and charge flux. Mathematically this translates to setting $m = 0$ and $\mu \neq 0$.

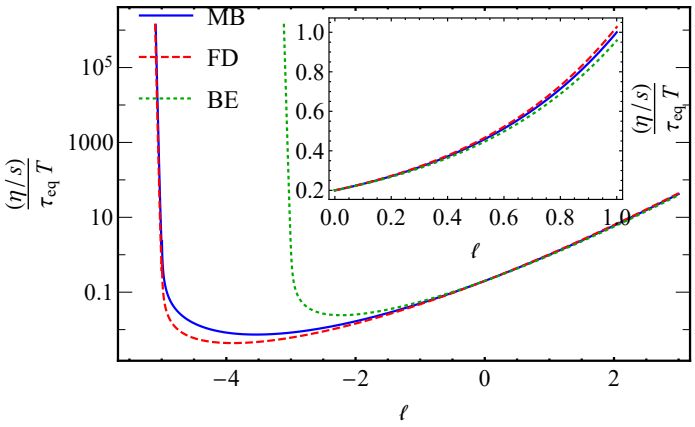

Figure 4.1: Dependence of $\eta/(s\tau_{\rm eq}T)$ on l for Maxwell-Boltzmann (MB), Fermi-Dirac (FD) and Bose-Einstein (BE) distributions. Inset shows the same within the range $0 \leq \ell \leq 1$, relevant for QCD system.

In each of these three cases, we study the dependence of the transport coefficients on thermodynamic parameters ($\alpha,\ \beta$), the momentum dependence of the relaxation time (ℓ) and statistics of the equilibrium distribution (a). In the last two cases, we also study the ratio of transport coefficients, and their limiting behaviours.

System of chargeless and massless particles

A system of massless ($m = 0$) and chargeless ($\mu = 0$) particles has dissipation purely due to shear stress tensor. For such a system, the only relevant transport coefficient is the coefficient of shear viscosity. Note that in this case, with $\mu = 0$, we have $I^-_{n,q} = J^-_{n,q} = 0$, and the coefficients $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ vanishes. The coefficient of shear stress given by . (4.36) reduces to

$$\eta = \frac{\tau_{\rm eq}\, g\, T^4\, \Gamma(5+\ell)}{15\,\pi^2}\left[\frac{{\rm Li}_{4+\ell}(-a)}{-a}\right], \tag{4.40}$$

where g is the degeneracy factor, $\Gamma(x)$ is the Euler Gamma function and

$${\rm Li}_n(x) \equiv \sum_{k=1}^{\infty}(x^k/k^n)$$

is the polylogarithm (also referred to as Jonquiére's function). Note that the quantity inside the square brackets on the right-hand side of the above equation is positive definite for all

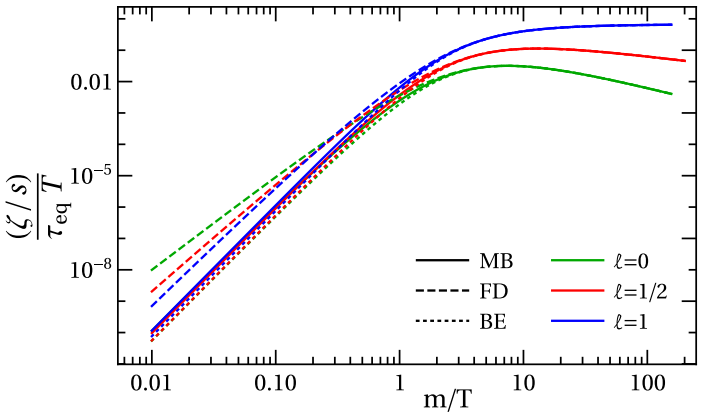

Figure 4.2: Variation of $\zeta/(s\tau_{\rm eq}T)$ with m/T for Maxwell-Boltzmann (MB), Fermi-Dirac (FD) and Bose-Einstein (BE) distributions for three different values of ℓ.

ℓ and is equal to 1, $\zeta(4+\ell)$ and $(1-2^{-3-\ell})\zeta(4+\ell)$ for $a=0$, -1, and 1, respectively. Here, $a=0$, -1, and 1 correspond to Maxwell-Boltzmann (MB), Bose-Einstein (BE), and Fermi-Dirac (FD) statistics, respectively, and $\zeta(s)\equiv\sum_{k=1}^{\infty}(1/k^s)$ is the Riemann zeta function.

To study the relative importance of shear viscosity, it is instructive to consider its dimensionless ratio with entropy density, η/s. For first-order dissipative hydrodynamics, the out-of-equilibrium entropy density is the same as that in equilibrium, $s=s_0$, where s_0 is given in Eq. (5.7). For a system of chargeless and massless particles, the only energy scale is the temperature, and therefore $\tau_{\rm eq}\sim 1/T$. For this case, $\eta/(s\tau_{\rm eq}T)$ is just a function of ℓ and, using Eqs. (5.7) and (4.40), is given by

$$\frac{\eta}{s\tau_{\rm eq}T}=\frac{\Gamma(5+\ell)}{120}\left[\frac{{\rm Li}_{4+\ell}(-a)}{{\rm Li}_4(-a)}\right]. \tag{4.41}$$

Here, we observe that for $\ell=0$, the scaled shear viscosity coefficient, $\eta/(s\tau_{\rm eq}T)$, is independent of the form of the equilibrium distribution. This may be attributed to the fact that in Fig. 4.1 (described below), the three curves for MB, FD and BE, either share a common tangent or cross each other at $\ell=0$. We observe that the latter is true in the present case.

In Fig. 4.1, we show the variation of $\eta/(s\tau_{\rm eq}T)$ with ℓ, the parameter which governs the momentum dependence of the relaxation time, for MB, FD and BE distributions. We

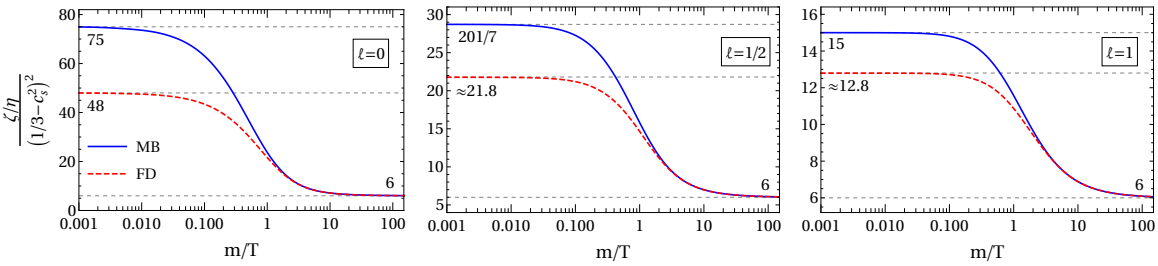

Figure 4.3: Variation of $\frac{\zeta/\eta}{(1/3-c_s^2)^2}$ with m/T for Maxwell-Boltzmann (MB) and Fermi-Dirac (FD) statistics, for $l = 0,\ 1/2$ and 1.

observe a minimum in $\eta/(s\tau_{\rm eq}T)$ for negative values of ℓ which occurs at different values of ℓ for different statistics; $\ell \simeq -3.5, -3.9$ and -2.2 for MB, FD and BE. respectively. In order to understand this, we first observe that the parameter ℓ controls the momentum dependence of the inter-particle interaction strength. Therefore the minima in $\eta/(s\tau_{\rm eq}T)$ may be attributed to the fact that depending on the sign of ℓ, the interaction strength increases or decreases with particle momentum; leading to a change in average momentum transfer. A more detailed analysis of this feature is left for future work. In the inset, we focus on the range $0 \leq \ell \leq 1$ which is relevant for QCD medium [195]. We observe that the specific viscosity increases with increasing ℓ having negligible dependence on the form of equilibrium statistics.

System of chargeless massive particles

For a system consisting of chargeless ($\mu = 0$) but massive ($m \neq 0$) particles, the dissipation is due to shear stress tensor as well as bulk viscous pressure. In this case $\mathcal{C}_1, \mathcal{C}_3$ vanishes irrespective of ℓ and $I^-_{n,q} = J^-_{n,q} = 0$. The transport coefficients η and ζ in Eqs. (4.36) and (4.37), reduces to

$$\eta = \frac{\tau_{\rm eq}}{T^{\ell+1}} J^+_{3+\ell,2}\,, \tag{4.42}$$

$$\zeta = -\mathcal{C}_2 \left(\frac{\mathcal{E}+\mathcal{P}}{T}\right) + \frac{\tau_{\rm eq}}{T^{\ell+1}} \left(\frac{5}{3} J^+_{3+\ell,2} + c_s^2 J^+_{3+\ell,1}\right). \tag{4.43}$$

where the coefficient $\mathcal{C}_2$ and the squared speed of sound, c_s^2, is given by

$$\mathcal{C}_2 = -\frac{\tau_{\rm eq}}{T^{\ell-1}}\left(\frac{c_s^2 J^+_{3+\ell,0} + J^+_{3+\ell,1}}{J^+_{3,0}}\right), \tag{4.44}$$

$$c_s^2 = \chi_b = \frac{\mathcal{E}+\mathcal{P}}{\beta\, J^+_{3,0}} = \frac{\mathcal{E}+\mathcal{P}}{3(\mathcal{E}+\mathcal{P}) + z^2 T J^+_{1,0}}\,. \tag{4.45}$$

Here $z \equiv m/T$ is defined as the ratio of particle mass and temperature. Note that $\mathcal{C}_2$ vanishes when the relaxation time is momentum independent, i.e., for $\ell = 0$. The above expression for c_s^2 agrees with Ref. [151] and [157] for classical and quantum statistics, respectively.

In Fig. 4.2, we show the variation of $\zeta/(s\tau_{\rm eq}T)$ with m/T for Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac distributions for three different values of ℓ: 0, $1/2$ and 1. We see a non-monotonous dependence on m/T for all distributions and all values of ℓ considered here. This is expected and is in qualitative agreement with results obtained in Refs. [197, 196]. We see that the present formulation with ERTA does not lead to negative values for bulk viscosity, and therefore no violation of second law of thermodynamics. In the following, we study the limiting behavior of the ratio of bulk viscosity to shear viscosity, ζ/η.

At this point, we would like to emphasize that in finite temperature QCD, the breaking of conformality arises from strong interactions. Therefore the origin of bulk viscosity lies in the conformality breaking due to strong interaction. Within the kinetic theory picture, the conformality breaking is achieved by incorporating a finite mass of the medium constituents. Nevertheless, one can make a connection between these two pictures by noting that the strong interaction leads to an effective thermal mass of the medium constituents [141]]. This thermal mass is of the order of the temperature of the medium, and therefore non-negligible. Hence, including a finite mass to break conformality in the kinetic theory picture is a reflection of conformality breaking due to strong interaction.

Small m/T behavior: It is instructive to study the scaling behavior of the ratio of viscous coefficients, ζ/η, with the conformality measure, $1/3 - c_s^2$. In order to study this scaling

behavior, we first consider small departure from conformality, i.e., small-z expansion. For the conformality measure, this expansion leads to

$$\frac{1}{3}-c_s^2=\begin{cases}\frac{z^2}{36}+\mathcal{O}(z^3) & \text{MB}\,,\\ \frac{5z^2}{21\pi^2}+\mathcal{O}(z^3) & \text{FD}\,,\\ \frac{5z^2}{12\pi^2}+\mathcal{O}(z^3) & \text{BE}\,.\end{cases} \tag{4.46}$$

For MB and FD statistics, the quantity ζ/η in small-z limit has the leading behavior as

$$\frac{\zeta}{\eta}=\Gamma\left(\frac{1}{3}-c_s^2\right)^2\,, \tag{4.47}$$

where $\Gamma\equiv\lim\limits_{z\to0}\frac{\zeta/\eta}{\left(\frac{1}{3}-c_s^2\right)^2}$. We first consider the case with MB and FD distribution. The BE case will be considered separately.

For classical MB statistics, we obtain

$$\Gamma_{\rm MB}=\frac{15\left(\ell^3+6\ell^2-13\ell+30\right)}{(\ell+1)(\ell+2)(\ell+3)}, \tag{4.48}$$

which results in $\Gamma_{\rm MB}=75,\ 201/7$ and 15 for $\ell=0,\ 1/2$ and 1, respectively. For Fermi-Dirac statistics, we get analytical expressions only for integer l. We obtain $\Gamma_{\rm FD}=48$ for $\ell=0$ and $15+\frac{14\left[7\pi^4\log(2)-45\pi^2\zeta(3)\right]}{375\zeta(5)}\approx12.8$ for $\ell=1$. For non-integer ℓ, the value of this coefficient is obtained numerically and we get $\Gamma_{\rm FD}\simeq21.8$ for $\ell=1/2$. In Fig. 4.4, we show the evolution of Γ with ℓ. We observe a non-monotonic behavior of Γ as a function ℓ with a minimum at $\ell\approx2.5$. We also see that $\Gamma\to15$ for large ℓ irrespective of equilibrium

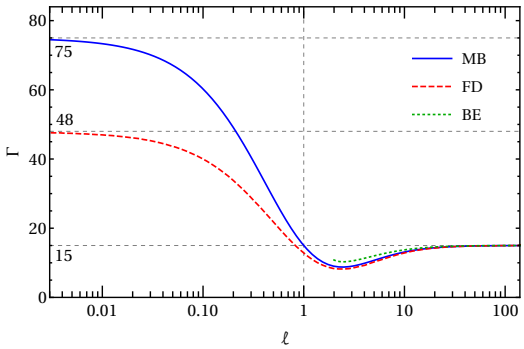

Figure 4.4: Behavior of Γ, defined below Eq. (4.47), with ℓ for Maxwell-Boltzmann (MB), Fermi-Dirac (FD) and Bose-Einstein (BE) equilibrium statistics.

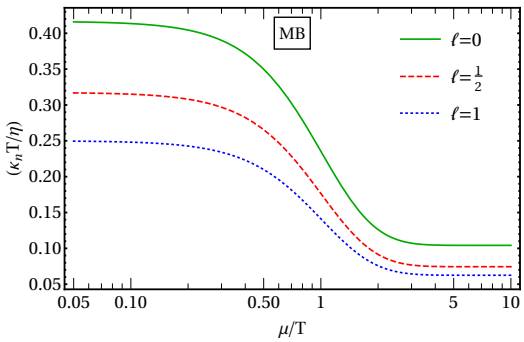

Figure 4.5: The ratio of charge conductivity to shear viscosity multiplied with temperature as a function of μ/T for the case of Maxwell-Boltzmann statistics for $\ell = 0,\ 1/2$ and 1.

statistics and converges to the MB case. To understand this, we note that the integral coefficients, I_{nq} and J_{nq} for quantum statistics can be written as a sum over Boltzmann factor, in the following form [157]

$$I_{n+l,q} = \frac{gT^{n+l+2}z^{n+l+2}}{2\pi^2(2q+1)!!}(-1)^q\sum_{r=1}^{\infty}(-a)^{r-1}\int_0^{\infty}d\theta \times (\cosh\theta)^{n+l-2q}(\sinh\theta)^{2q+2}\ \exp(-rz\cosh\theta), \tag{4.49}$$

$$J_{n+l,q} = \frac{gT^{n+l+2}z^{n+l+2}}{2\pi^2(2q+1)!!}(-1)^q\sum_{r=1}^{\infty}r(-a)^{r-1}\int_0^{\infty}d\theta \times (\cosh\theta)^{n+l-2q}(\sinh\theta)^{2q+2}\ \exp(-rz\cosh\theta). \tag{4.50}$$

We find that, in above equations, for $\ell \gg 1$ and $n + \ell \gg q$, the leading term, i.e., $r = 1$ dominates and the other terms in the summation can be neglected. In this case, the integral coefficients, $I_{n+\ell,q}$ and $J_{n+\ell,q}$ become equal to that obtained using classical MB statistics. Note that for large ℓ ($\ell \gtrsim 2$), we find that a system with BE equilibrium statistics also follows the scaling relation of Eq. (4.47). At this point, it is worth mentioning that the scaling behavior in Eq. (4.47) with $\Gamma = 15$ was obtained by Weinberg for a system of radiation interacting with matter [223, 224].

The case of BE statistics for small values of ℓ is more subtle because, in the massless limit, the soft momenta governs the behavior of ζ/η. We follow the procedure outlined in Ref. [141] to evaluate the thermodynamic integrals. For $\ell = 0$, the leading term in small-z

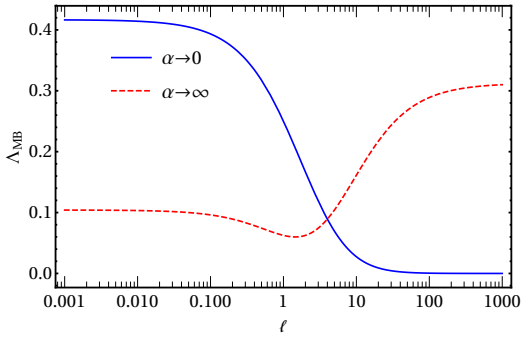

Figure 4.6: The scaling coefficient $\Lambda_{\rm MB}$ defined in Eq. (4.57) as a function of ℓ in the case of Maxwell-Boltzmann statistics for $\mu/T \to 0$ and $\mu/T \to \infty$.

expansion leads to the scaling relation

$$\frac{\zeta}{\eta} = \frac{3\sqrt{15}}{2}\left(\frac{1}{3} - c_s^2\right)^{3/2} . \tag{4.51}$$

As mentioned earlier, for $\ell \gtrsim 2$, we find that a system with BE equilibrium statistics follows the scaling relation of Eq. (4.47), and the value of $\Gamma_{\rm BE}$ for quadratic scaling is plotted in Fig. 4.4. The scaling behavior of ζ/η in the small-z regime for $0 < \ell < 2$ is more involved and is left for future work.

Large m/T behavior: For constitutive particles having large mass compared to the temperature (non-relativistic regime), expansion of c_s^2 in powers of $1/z$ leads to

$$\frac{1}{3} - c_s^2 = \frac{1}{3} - \frac{1}{z} + \frac{3}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) , \tag{4.52}$$

which is independent of statistics. This is not surprising because for large mass, $a = \pm 1$ in Eqs. (4.6) and (4.22), which is relevant for quantum statistics, can be ignored and hence the thermodynamics is dominated by classical physics. Therefore the large mass limit is both non-relativistic and classical limit. Moreover, we see that for very large mass, $\mathcal{P}/\mathcal{E} \to 0$ as $m/T \to \infty$ and hence c_s^2 vanishes.

Further the quantity ζ/η in this limit has the expansion

$$\frac{\zeta}{\eta} = \frac{2}{3} - \frac{4}{z} + \frac{26 + (\ell - 6)\ell}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) . \tag{4.53}$$

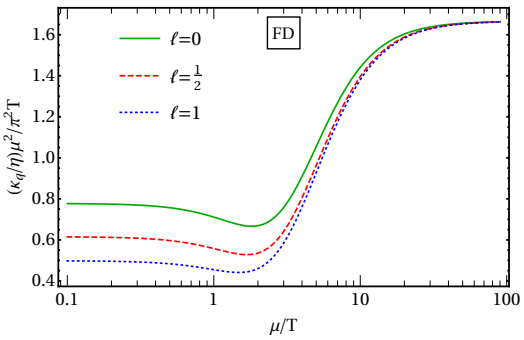

Figure 4.7: The ratio of heat conductivity to shear viscosity multiplied with the factor $\mu^2/\pi^2 T$ as a function of μ/T for the case of Fermi-Dirac statistics for $\ell = 0,\ 1/2$ and 1.

Considering the leading terms in Eqs. (4.52) and (4.53), we have

$$\frac{\zeta}{\eta} = 2\left(\frac{1}{3} - c_s^2\right), \tag{4.54}$$

which corresponds to the lower bound for the ratio in 3-space dimensions [225, 226]. This scaling behavior is valid in the extreme non-relativistic limit for $c_s^2 \to 0$. Therefore, one may conclude that the extreme non-relativistic limit leads to the maximum value of the lower bound on ζ/η. This result is independent of statistics as discussed earlier. Further, the leading order term in ζ/η, is independent of 'ℓ' as well. This may be attributed to the fact that in this extreme non-relativistic limit, the system is essentially 'free-streaming' as $\mathcal{P}/\mathcal{E} \to 0$. Thus, there is no dependence on the collision term $\mathcal{C}[f]$, and hence on 'ℓ'. Therefore at large m/T, $(\zeta/\eta)/(1/3 - c_s^2)^2 = (2/3)/(1/3)^2 = 6$ irrespective of statistics and 'ℓ'.

System of charged massless particles

For a system of massless ($m = 0$) particles with conserved charges ($\mu \neq 0$), the dissipation is due to shear stress tensor and dissipative charge current. The expressions for shear viscosity, η, and charge conductivity, κ_n, are given in Eqs. (4.36) and (4.38), respectively.

The coefficient of thermal conductivity κ_q can be written as [227, 117]

$$\kappa_q = \kappa_n \left(\frac{\mathcal{E} + \mathcal{P}}{nT} \right)^2 . \tag{4.55}$$

It is interesting to note that for Maxwell-Boltzmann statistics, the ratio of the transport coefficients κ_n/η scales as $1/T$ in both small and large μ/T limit. On the other hand, for Fermi-Dirac statistics, the ratio κ_q/η scales as T/μ^2 in both small and large μ/T limit. Calculation of charge/heat conductivity at finite chemical potential is not possible for Bose-Einstein statistics due to the phenomena of Bose-Einstein condensation. In the following, we consider MB and FD statistics for analysis of charge and heat conduction.

For MB statistics, the ratio κ_n/η in both small and large μ/T limit scales as

$$\frac{\kappa_n}{\eta} = \Lambda_{\rm MB} \frac{1}{T}, \tag{4.56}$$

where the coefficient $\Lambda_{\rm MB}$ is a function of ℓ and is given by

$$\Lambda_{\rm MB} = \begin{cases} \frac{5}{(4+\ell)(3+\ell)} & \text{for } \alpha \to 0\,, \\ \frac{5(\ell^2-\ell+4)}{16\,(4+\ell)(3+\ell)} & \text{for } \alpha \to \infty\,. \end{cases} \tag{4.57}$$

In Fig. 4.5, we plot the ratio $\kappa_n T/\eta$ as a function of μ/T for the case of MB statistics for $\ell = 0,\ 1/2$ and 1. We see that this ratio tends to constant value in the limit of both small and large μ/T. We also observe that this ratio is lower for increasing values of ℓ. This is more evident in Fig. 4.6 where we plot the coefficient $\Lambda_{\rm MB}$ as a function of ℓ for $\alpha \to 0$ and $\alpha \to \infty$. We see that in case of $\alpha \to \infty$, for both very small and very large ℓ, $\Lambda_{\rm MB}$ saturates to a constant value indicating that there is no ℓ-dependence in these limits. For $\alpha \to 0$, we find a monotonous decrease in $\Lambda_{\rm MB}$ which approaches zero for large ℓ, as is also evident from Eq. (4.57).

Similarly, for FD statistics, the ratio κ_q/η in small and large μ/T limit scales as

$$\frac{\kappa_q}{\eta} = \Lambda_{\rm FD} \frac{\pi^2 T}{\mu^2}, \tag{4.58}$$

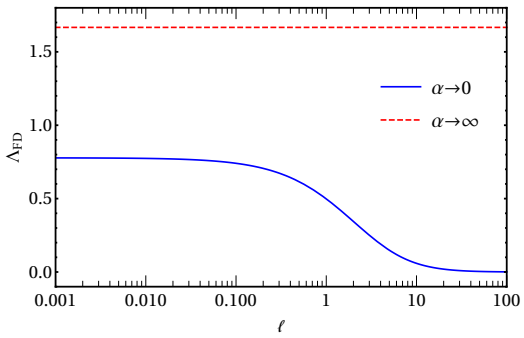

Figure 4.8: The scaling coefficient $\Lambda_{\rm MB}$ defined in Eq. (4.59) as a function of l in the case of Fermi-Dirac statistics for $\mu/T \to 0$ and $\mu/T \to \infty$.

where the coefficient $\Lambda_{\rm FD}$ is again a function of l and is given by

$$\Lambda_{\rm FD} = \begin{cases} \frac{196\,\pi^2\left(2^{1+\ell}-1\right)\zeta(2+\ell)}{45\left(2^{3+\ell}-1\right)(4+\ell)(3+\ell)\zeta(4+\ell)} & \text{for } \alpha \to 0\,, \\ \frac{5}{3} & \text{for } \alpha \to \infty\,. \end{cases} \tag{4.59}$$

In Fig. 4.7, we plot the ratio $(\mu^2\kappa_q)/(\pi^2\eta T)$ as a function of μ/T for the case of FD statistics for $\ell = 0,\ 1/2$ and 1. We see that this ratio tends to constant value in the limit of both small and large μ/T. We see that for large μ/T, the curves for different ℓ coincides indicating that this ratio becomes ℓ-independent. We also observe that this ratio is lower for increasing values of ℓ. This is more evident in Fig. 4.8 where we plot the coefficient $\Lambda_{\rm FD}$ as a function of ℓ for $\alpha \to 0$ and $\alpha \to \infty$. In case of $\alpha \to \infty$, we obtain $\Lambda_{\rm FD} = 5/3$ for all values of ℓ. For $\alpha \to 0$, we find a monotonous decrease in $\Lambda_{\rm FD}$ which approaches zero for large ℓ, as is also evident from Eq. (4.59).

Chapter 5

Second order dissipative hydrodynamics using ERTA

5.1 Introduction

The relativistic Boltzmann equation is a transport equation that governs the space-time evolution of the single-particle phase-space distribution function. It is capable of accurately describing the collective dynamics of the system in the limit of small mean free path and therefore has been employed extensively to formulate the theory of relativistic hydrodynamics [132, 133, 134, 135, 137, 201, 228]. However, solving the Boltzmann equation directly is challenging due to the complicated integro-differential nature of the collision term, which involves the integral of the product of distribution functions. Over several decades, various approximations have been proposed to simplify the collision term in the linearized regime. In 1969, following earlier works by Bhatnagar-Gross-Krook [198] and Welander [199], Marle introduced a relaxation time approximation for non-relativistic systems [200]. However, Marle's version was not applicable to massless particles and was ill-defined in the relativistic limit. Anderson and Witting resolved these issues by generalizing Marle's model to the relativistic regime, qualitatively recovering the results obtained using Grad's method of moments in the relativistic limit [193]. These models, introduced by Marle and Anderson-Witting, incorporate a collision time scale known as the relaxation time. The Anderson-Witting model requires the relaxation time to be independent of particle momenta, making it straightforward to apply in the formulation of relativistic dissipative hydrodynamics.

The Anderson-Witting model achieves enormous simplification by approximating that

collisions drive the system towards local equilibrium exponentially without explicitly describing the interaction mechanism of the microscopic constituents. This approximation provides a highly accurate description of the collective dynamics for systems close to equilibrium. In the following, we will refer to the Anderson-Witting model as the relaxation-time approximation (RTA). Despite its simplistic nature, RTA and its variations have proven to be immensely useful and have been extensively employed in formulating relativistic dissipative hydrodynamics as well as in deriving transport coefficients [136, 138, 151, 165, 171, 155, 177, 229, 230, 231, 188]. Recently, it has also been applied to study the domain of applicability of hydrodynamics [205, 206, 87, 207, 232, 208, 233, 209, 210, 178, 211, 234, 212, 213, 186, 235, 236, 237]. This simple model appears to capture effective microscopic interactions across a wide range of theories.

When deriving dissipative hydrodynamic equations from kinetic theory using the RTA approximation, it is typically assumed that the relaxation time is independent of particle energy (or momentum). Additionally, one is constrained to work in the Landau frame to ensure the preservation of macroscopic conservation laws. However, in realistic systems, the collision time scale generally depends on the microscopic interactions [195, 214, 215, 216]. Introducing an energy-dependent relaxation time leads to a violation of microscopic conservation laws in the Landau frame. As a result, there has been considerable interest in developing a consistent formulation of relativistic dissipative hydrodynamics with an energy-dependent relaxation-time approximation for the Boltzmann equation that satisfies both microscopic and macroscopic conservation laws [143, 197, 196, 183].

Relativistic viscous hydrodynamics-based multistage dynamical models have demonstrated success in accurately describing a broad spectrum of soft hadronic observables in heavy-ion collisions [238, 239, 240]. The hydrodynamics stage of the evolution encompasses the deconfined quarks and gluons regime at high temperatures, the phase transition, and the hadron gas phase [60]. The dynamical properties of the evolving non-equilibrium

nuclear matter are governed by a set of transport coefficients, such as the shear and bulk viscosities [79, 241, 226, 242, 243]. These transport coefficients play a crucial role in explaining the hadronic observables in heavy-ion collisions. Thus, a major goal of heavy-ion phenomenology is to extract the temperature dependence of these transport coefficients for the evolving nuclear matter, and considerable efforts have been made to determine these coefficients from various aspects. Most phenomenological studies adopt parameterized forms for the shear and bulk viscosities [244, 245, 246, 247]. Recent studies have employed Bayesian methods to obtain these parameters and have provided bounds on the transport coefficients [248, 249, 250, 251, 252, 253]. However, since these parameterizations do not stem from microscopic considerations, the predictability of such models is limited. Additionally, the second-order transport coefficients utilized in these hydrodynamic models are obtained for specific interactions, and, as a result, they may fail to accurately capture the system's behavior during its evolution.

In our recent work [183], we presented a framework for the consistent derivation of relativistic dissipative hydrodynamics from the Boltzmann equation, incorporating a particle energy-dependent relaxation time by extending the Anderson-Witting relaxation-time approximation[1]. Within this extended RTA (ERTA) framework, we derived the first-order hydrodynamic equations and demonstrated that the hydrodynamic transport coefficients can exhibit significant variations with the energy dependence of the relaxation time. Notably, the ERTA framework allows for the adjustment of interaction characteristics by tuning the energy dependence of the relaxation time, enabling a partial description of the transition from deconfined quark-gluon plasma at high temperatures to a weakly interacting gas of hadrons at lower temperatures. While the formulation presented in Ref. [183] successfully incorporates an energy-dependent relaxation time into the RTA, it still suffers from the well-known issue of acausality in first-order relativistic hydrodynamics within the Lan-

[1] In a recent work [254], the transseries structure of ERTA was explored.

dau frame [255, 256, 257, 222, 258, 259, 260, 219, 261]. Consequently, there is a need for a second-order theory that addresses this issue [130, 131], allowing for its application in heavy-ion collision simulations.

In the present study, we employ the ERTA framework to derive second-order hydrodynamic equations in the Landau frame for a conformal system without conserved charges, incorporating an energy-dependent relaxation time. The second-order transport coefficients are found to be sensitive to the energy dependence of the relaxation time. We focus on a boost-invariant flow in (0+1) dimensions and investigate the fixed point structure of the hydrodynamic equations. Our analysis reveals that the location of the free-streaming fixed points is influenced by the energy dependence of the relaxation time. By employing a power law parametrization to describe this energy dependence, we successfully reproduce the stable free-streaming fixed point for a specific power of the energy dependence. Furthermore, we explore the impact of the energy-dependent relaxation time on the processes of isotropization and thermalization of a boost invariant expanding plasma.

This paper is organized as follows: In Sec. 5.2 we review the basic hydrodynamic equations for a conformal, chargeless fluid. Sec. 5.3 briefly summarizes the results of Ref. [183] and outlines the steps necessary to derive second-order hydrodynamic equations, which we present in Sec. 5.4. Appendix 6.2 contains the derivation of the results stated in Sec. 5.5. In Sec. 5.8, we consider Bjorken flow and study the effect of the energy dependence of the relaxation time on systems' thermalization. We summarize our results in Sec. 5.9.

5.2 Overview

The energy-momentum tensor for a system of massless particles with no net conserved charge can be expressed in terms of the single-particle phase–space distribution function, $f(x,p)$, as

$$T^{\mu\nu} = \int \mathrm{dP}\, p^\mu p^\nu f = \mathcal{E} u^\mu u^\nu - \mathcal{P}\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{5.1}$$

where dP $= d^3\vec{p}/\left[(2\pi)^3 E_p\right]$ is the invariant momentum-space integration measure with E_p representing the particle energy which is equal to the magnitude of the particle three-momenta for massless particles, $E_p = |\vec{p}|$. The projection operator $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is orthogonal to the hydrodynamic four-velocity u^μ defined in the Landau frame: $u_\nu T^{\mu\nu} = \mathcal{E} u^\mu$, where $\mathcal{E}$ is the energy density. In the above equation, $\mathcal{P}$ is the thermodynamic pressure and $\pi^{\mu\nu}$ is the shear viscous stress. We work with the Minkowskian metric tensor $g^{\mu\nu} \equiv$ diag$(+,-,-,-)$.

The energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ yields the fundamental evolution equations for $\mathcal{E}$ and u^μ as,

$$\dot{\mathcal{E}} + (\mathcal{E} + \mathcal{P})\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{5.2}$$

$$(\mathcal{E} + \mathcal{P})\dot{u}^\alpha - \nabla^\alpha \mathcal{P} + \Delta^\alpha_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{5.3}$$

$$\tag{5.4}$$

Here we use the standard notation $\dot{A} = u^\mu \partial_\mu A$ for co-moving derivatives, $\theta \equiv \partial_\mu u^\mu$ for the expansion scalar, $\sigma^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\theta\Delta^{\mu\nu}$ for the velocity stress tensor, and $\nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu$ for space-like derivatives.

We consider the equilibrium momentum distribution function to have the Maxwell-Boltzmann distribution in the local rest frame of the fluid, $f_{\text{eq}} = \exp\left[-(u\cdot p)/T\right]$. The equilibrium energy density then takes the form,

$$\mathcal{E}_0 = u_\mu u_\nu \int \text{dP}\, p^\mu p^\nu f_{\text{eq}} = \frac{3T^4}{\pi^2}. \tag{5.5}$$

For an out-of-equilibrium system, the temperature T is an auxiliary quantity which we define using the matching condition $\mathcal{E} \equiv \mathcal{E}_0$. Also, the thermodynamic pressure and entropy density are given by,

$$\mathcal{P} = -\frac{1}{3}\Delta_{\mu\nu} \int \text{dP}\, p^\mu p^\nu f_{\text{eq}} = \frac{T^4}{\pi^2}\,, \tag{5.6}$$

$$\mathcal{S} = \frac{\mathcal{E} + \mathcal{P}}{T} = \frac{4T^3}{\pi^2}\,. \tag{5.7}$$

The evolution of temperature is obtained from the hydrodynamic equations of motion (5.2) and (5.3),

$$\dot{\beta} = \frac{\beta\theta}{3} - \frac{\beta}{3(\mathcal{E}+\mathcal{P})}\pi^{\mu\nu}\sigma_{\mu\nu}\,, \tag{5.8}$$

$$\nabla^{\mu}\beta = -\beta\dot{u}^{\mu} - \frac{\beta}{\mathcal{E}+\mathcal{P}}\Delta^{\mu}_{\alpha}\partial_{\nu}\pi^{\alpha\nu}\,, \tag{5.9}$$

where $\beta = 1/T$.

The non-equilibrium phase-space distribution function can be written as $f = f_{\rm eq} + \Delta f$, where Δf represents the out-of-equilibrium correction to the distribution function. Using Eq. (5.1) the shear stress tensor $\pi^{\mu\nu}$ can be expressed in terms of Δf as

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int \mathrm{dP}\, p^{\alpha}p^{\beta}\,\Delta f\,, \tag{5.10}$$

where $\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\nu}_{\alpha}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$ is a doubly symmetric and traceless projection operator orthogonal to u^{μ} as well as $\Delta^{\mu\nu}$. The evolution of the shear stress tensor depends on the evolution of the distribution function. In this work, we consider the evolution of the distribution function to be governed by the Boltzmann equation with the collision term, $\mathfrak{C}[f]$, in the Extended Relaxation Time Approximation (ERTA) [143, 183],

$$p^{\mu}\partial_{\mu}f = \mathfrak{C}[f] = -\frac{(u\cdot p)}{\tau_{\rm R}(x,p)}\left(f - f^{*}_{\rm eq}\right)\,, \tag{5.11}$$

where the relaxation time, $\tau_{\rm R}(x,p)$, may depend on the particle momenta. The equilibrium distribution function is considered to be of the Maxwell-Boltzmann form in the 'thermodynamic frame', $f^{*}_{\rm eq} = \exp\left[-\left(u^{*}\cdot p\right)/T^{*}\right]$. Here the thermodynamic frame is defined to be the local rest frame of a time-like four vector u^{*}_{μ} which need not necessarily correspond to the hydrodynamic four-velocity u_{μ}, and T^{*} is the temperature in the local rest frame of u^{*}_{μ} (see Ref. [183] for a detailed discussion).

We briefly review the derivation of first-order shear stress from the above kinetic equation in the next section.

5.3 First-order hydrodynamics

We employ Chapman-Enskog-like expansion about hydrodynamic equilibrium[2] to iteratively solve the ERTA Boltzmann equation (5.11),

$$f = f_{\rm eq} + \delta f_{(1)} + \delta f_{(2)} + \delta f_{(3)} + \cdots \tag{5.12}$$

Here $\delta f_{(i)}$ represents the ith order gradient correction to the hydrodynamic equilibrium distribution function. The correction to the distribution function to the first order is

$$\delta f_{(1)} = \delta f^*_{(1)} + \frac{\tau_{\rm R}}{T}\frac{p^\mu p^\nu}{u\cdot p}\sigma_{\mu\nu} f_{\rm eq}\,, \tag{5.13}$$

where we have replaced the derivatives of temperature with the derivatives of fluid velocity using Eqs. (5.8,5.9) consistently keeping terms till first order in gradients, and have defined $\delta f^* \equiv f^*_{\rm eq} - f_{\rm eq}$. Defining $T^* \equiv T + \delta T$ and $u^\mu_* \equiv u^\mu + \delta u^\mu$, we obtain the first-order correction $\delta f^*_{(1)}$ by Taylor expanding $f^*_{\rm eq}$ about u^μ and T,

$$\delta f^*_{(1)} = \left(-\frac{(\delta u\cdot p)}{T} + \frac{(u\cdot p)\delta T}{T^2}\right) f_{\rm eq}\,. \tag{5.14}$$

Using equations (5.13) and (5.14), the quantities δu^μ and δT are obtained by imposing the Landau frame conditions, $u_\nu T^{\mu\nu} = \mathcal{E} u^\mu$, and the matching condition, $\mathcal{E} = \mathcal{E}_0$. We find that these quantities vanish for a system of massless and chargeless particles at first-order in gradients, and the resulting first-order correction is given by

$$\delta f_{(1)} = \frac{\tau_{\rm R}}{T}\frac{p^\mu p^\nu}{u\cdot p}\sigma_{\mu\nu} f_{\rm eq}\,. \tag{5.15}$$

It can be easily checked that the microscopic conservation of energy-momentum at first order holds by taking the first momentum-moment of the Boltzmann equation (5.11) with

[2] We shall refer to $f_{\rm eq} = \exp\left[-(u\cdot p)/T\right]$ as the hydrodynamic equilibrium distribution function with u^μ being the fluid four-velocity and T the local fluid temperature in the local rest frame of u^μ.

$f \mapsto f_{(1)} = f_{\rm eq} + \delta f_{(1)}$,

$$\partial_\mu \int {\rm dP}\, p^\mu p^\nu f_{(1)} = -\int {\rm dP}\, \frac{u\cdot p}{\tau_{\rm R}} p^\nu \left(f_{(1)} - f^*_{\rm eq}\right)$$
$$\implies \partial_\mu T^{\mu\nu}_{(1)} = -\frac{\sigma_{\alpha\beta}}{T}\int {\rm dP}\, p^\nu p^\alpha p^\beta f_{\rm eq} = 0\,. \tag{5.16}$$

Using $\delta f_{(1)}$ obtained in Eq. (5.15), the expression of shear stress tensor from the definition (5.10) is obtained to be [183],

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \tag{5.17}$$

where $\eta = K_{3,2}/T$ is the coefficient of shear viscosity. We have defined the integrals

$$K_{n,q} \equiv \frac{1}{(2q+1)!!}\int {\rm dP}\, \tau_{\rm R}(x,p)(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}\,. \tag{5.18}$$

We will now derive the second-order constitutive relation (and evolution equation) for the shear stress tensor in the next section.

5.4 Second-order hydrodynamics

The nonequilibrium correction to the distribution function till second order can be written as

$$f = f_{\rm eq} + \delta f_{(1)} + \delta f_{(2)} + \mathcal{O}(\delta^3) = f_{\rm eq} + \Delta f_{(2)} + \mathcal{O}(\delta^3) \tag{5.19}$$

where we define $\Delta f_{(2)} \equiv \delta f_{(1)} + \delta f_{(2)}$ representing the non-equilibrium correction till second order. Using the kinetic equation (5.11), and employing the Chapman-Enskog expansion, we obtain $\Delta f_{(2)}$ as,

$$\Delta f_{(2)} = \Delta f^*_{(2)} - \frac{\tau_{\rm R}}{u\cdot p}p^\mu\partial_\mu \delta f^*_{(1)} - \frac{\tau_{\rm R}}{u\cdot p}p^\mu\partial_\mu f_{\rm eq} + \frac{\tau_{\rm R}}{u\cdot p}p^\mu p^\nu\partial_\mu\left(\frac{\tau_{\rm R}}{u\cdot p}\partial_\nu f_{\rm eq}\right). \tag{5.20}$$

Here $\delta f^*_{(1)}$ is out-of-equilibrium correction at first order. As discussed in the previous section, the first-order contribution of δu^μ and δT vanishes, and therefore they have contributions starting from second-order. Keeping terms till second-order in gradients, the first

term on the right-hand side (r.h.s.) of the above equation is,

$$\Delta f^*_{(2)} = \left(-\frac{(\delta u \cdot p)}{T} + \frac{(u \cdot p)\delta T}{T^2} \right) f_{\rm eq}. \tag{5.21}$$

The second term on r.h.s. of Eq. (5.20),

$$-\frac{\tau_{\rm R}}{u \cdot p} p^\mu \partial_\mu \delta f^*_{(1)} = \mathcal{O}(\delta^3), \tag{5.22}$$

has correction starting from third-order in gradients because it involves derivatives of δu^μ and δT, which are at least second-order. The third term on the r.h.s. simplifies to,

$$-\frac{\tau_{\rm R}}{u \cdot p} p^\mu \partial_\mu f_{\rm eq} = \frac{\tau_{\rm R}}{T} \left[\frac{p^\mu p^\nu}{u \cdot p} \sigma_{\mu\nu} - \frac{4}{3} \frac{u \cdot p}{\mathcal{E} + \mathcal{P}} \pi^{\mu\nu} \sigma_{\mu\nu} - \frac{1}{\mathcal{E} + \mathcal{P}} \left(p^\mu \nabla_\nu \pi^\nu_\mu - p^\mu \pi^\nu_\mu \dot{u}_\nu \right) \right] f_{\rm eq}\,. \tag{5.23}$$

In deriving, we have kept all terms till second order when replacing derivatives of temperature with derivatives of fluid velocity using Eqs. (5.8) and (5.9). The last term on the r.h.s. of Eq. (5.20) is given by,

$$\begin{aligned} \frac{\tau_{\rm R}}{u \cdot p} p^\mu p^\nu \partial_\mu \left(\frac{\tau_{\rm R}}{u \cdot p} \partial_\nu f_{\rm eq} \right) = &- \frac{\tau_{\rm R}}{T} \left[\dot{\tau}_{\rm R} \frac{p^\mu p^\nu}{u \cdot p} \sigma_{\mu\nu} + (\nabla_\alpha \tau_{\rm R}) \frac{p^\alpha p^\mu p^\nu}{(u \cdot p)^2} \sigma_{\mu\nu} \right] f_{\rm eq} \\ &- \frac{\tau_{\rm R}^2}{T} \left[\frac{2\theta}{3} \frac{p^\mu p^\nu}{u \cdot p} \sigma_{\mu\nu} + \frac{p^\mu p^\nu}{u \cdot p} \dot{\sigma}_{\mu\nu} + \frac{p^\alpha p^\mu p^\nu}{(u \cdot p)^2} (\nabla_\alpha \sigma_{\mu\nu}) \right. \\ &\left. - 2 \frac{p^\alpha p^\mu p^\nu}{(u \cdot p)^2} \sigma_{\mu\nu} \dot{u}_\alpha - \left(\frac{1}{T} + \frac{1}{u \cdot p} \right) \frac{(p^\mu p^\nu \sigma_{\mu\nu})^2}{(u \cdot p)^2} \right] f_{\rm eq} \end{aligned} \tag{5.24}$$

Therefore, the complete non-equilibrium correction till second order from Eqs. (5.20)-(5.24) is given by,

$$\begin{aligned} \Delta f_{(2)} = &\left[\frac{(u \cdot p)\delta T}{T^2} - \frac{(\delta u \cdot p)}{T} - \frac{\tau_{\rm R}}{T} \left\{ (\dot{\tau}_{\rm R} - 1) \frac{p^\mu p^\nu}{u \cdot p} \sigma_{\mu\nu} + (\nabla_\alpha \tau_{\rm R}) \frac{p^\mu p^\nu p^\alpha}{(u \cdot p)^2} \sigma_{\mu\nu} \right. \right. \\ &\left. + \frac{1}{\mathcal{E} + \mathcal{P}} \left(\frac{4}{3} (u \cdot p) \pi^{\mu\nu} \sigma_{\mu\nu} + p^\mu \nabla_\nu \pi^\nu_\mu - p^\mu \pi^\nu_\mu \dot{u}_\nu \right) \right\} \\ &- \frac{\tau_{\rm R}^2}{T} \left\{ \frac{2\, p^\mu p^\nu}{3(u \cdot p)} \theta \sigma_{\mu\nu} + \frac{p^\mu p^\nu}{u \cdot p} \dot{\sigma}_{\mu\nu} + \frac{p^\alpha p^\mu p^\nu}{(u \cdot p)^2} (\nabla_\alpha \sigma_{\mu\nu}) - 2 \frac{p^\alpha p^\mu p^\nu}{(u \cdot p)^2} \sigma_{\mu\nu} \dot{u}_\alpha \right. \\ &\left. \left. - \left(\frac{1}{T} + \frac{1}{u \cdot p} \right) \frac{(p^\mu p^\nu \sigma_{\mu\nu})^2}{(u \cdot p)^2} \right\} \right] f_{\rm eq} \end{aligned} \tag{5.25}$$

5.5 Imposing Landau frame conditions

We note that the second-order correction to the equilibrium distribution function given by Eq. (5.25) has the undetermined quantities δu^μ and δT. We determine these by imposing the Landau frame condition ($u_\mu T^{\mu\nu} = \mathcal{E} u^\nu$) and matching condition ($\mathcal{E} = \mathcal{E}_0$) with $f \mapsto f_{(2)} = f_{\rm eq} + \Delta f_{(2)}$ (see Appendix 6.2 for derivation),

$$\delta u^\mu = \frac{5K_{3,2}}{T(\mathcal{E}+\mathcal{P})^2}\left(\pi^{\mu\nu}\dot{u}_\nu - \nabla_\nu \pi^{\mu\nu} - \pi^{\alpha\beta}\sigma_{\alpha\beta}u^\mu\right) + \frac{2L_{3,2}}{T(\mathcal{E}+\mathcal{P})}\left(2\sigma^{\mu\nu}\dot{u}_\nu + \nabla_\nu\sigma^{\mu\nu} + \sigma^{\alpha\beta}\sigma_{\alpha\beta}u^\mu\right), \tag{5.26}$$

$$\delta T = \frac{5}{3}\frac{K_{3,2}}{(\mathcal{E}+\mathcal{P})^2}\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{\mathcal{E}+\mathcal{P}}\left(L_{3,2} - \frac{L_{4,2}}{3T}\right)\sigma^{\mu\nu}\sigma_{\mu\nu}\,. \tag{5.27}$$

The $L_{n,q}$ integrals appearing in the above expressions are defined as,

$$L_{n,q} \equiv \frac{1}{(2q+1)!!}\int dP\, \tau_{\rm R}^2(x,p)(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}\,. \tag{5.28}$$

In the derivation, we have used the relation between the integrals,

$$X_{n,q} = -\left(\frac{1}{2q+1}\right)X_{n,q-1}\,, \tag{5.29}$$

which holds for all integrals defined in this article. We note that when the relaxation time does not depend on particle energy, the ERTA approximation of the collision term reduces to the Anderson-Witting RTA approximation, and consequently δu^μ and δT vanishes (see Appendix 6.2).

5.6 Verification of microscopic conservation

To verify microscopic energy-momentum conservation up to the second order, we show that the first momentum-moment of the collision kernel is at least third-order in gradients. To this end, we consider the first moment of the collision kernel in the Boltzmann equation (5.11) and substitute $f \mapsto f_{(2)} = f_{\rm eq} + \Delta f_{(2)}$, where $\Delta f_{(2)}$ is given by Eq. (5.25),

$$\int dP\, p^\nu \mathfrak{C}[f] = -\int dP\, \frac{(u\cdot p)}{\tau_{\rm R}} p^\nu \left(\Delta f_{(2)} - \Delta f^*_{(2)}\right). \tag{5.30}$$

Using the expression of $\Delta f_{(2)} - \Delta f^*_{(2)}$ from Eq. (5.20),

$$\int dP\, p^\nu \mathfrak{C}[f] = \int dP\, p^\mu p^\nu \partial_\mu f_{\text{eq}} - \int dP\, p^\nu p^\alpha p^\beta \partial_\alpha \left(\frac{\tau_{\text{R}}}{u\cdot p}\partial_\beta f_{\text{eq}}\right). \tag{5.31}$$

The first term in the right-hand-side of the above equations is simplified as

$$\int dP\, p^\mu p^\nu \partial_\mu f_{\text{eq}} = \dot{u}_\mu \pi^{\mu\nu} - \nabla_\mu \pi^{\mu\nu}. \tag{5.32}$$

Similarly, the second term is simplified as

$$\begin{aligned}\int dP\, p^\nu p^\alpha p^\beta \partial_\alpha \left(\frac{\tau_{\text{R}}}{u\cdot p}\partial_\beta f_{\text{eq}}\right) &= \partial_\alpha \int dP\, \frac{\tau_{\text{R}}}{u\cdot p} p^\nu p^\alpha p^\beta \partial_\beta f_{\text{eq}} = \partial_\mu \left(-2\frac{K_{3,2}}{T}\sigma^{\mu\nu}\right) + \mathcal{O}(\delta^3) \\ &= \dot{u}_\mu \pi^{\mu\nu} - \nabla_\mu \pi^{\mu\nu} + \mathcal{O}(\delta^3).\end{aligned} \tag{5.33}$$

In the last step, we have used the first-order constitutive relation (5.17). Using Eqs. (5.32) and (5.33) in Eq. (5.31), we obtain

$$\int dP p^\nu \mathfrak{C}[f] = \mathcal{O}(\delta^3). \tag{5.34}$$

This demonstrates the preservation of microscopic energy-momentum conservation up to the second order. It is noteworthy that δu^μ and δT did not appear in the equations during the verification of microscopic conservation. This outcome is specific to the case of massless and chargeless particles and does not happen in general. The contribution from these quantities becomes essential to ensure the conservation of energy-momentum and net current in systems involving massive or charged particles.

5.7 Shear stress till second-order

The expression for shear stress tensor till second order in terms of the hydrodynamic fields is obtained by integrating $\Delta f_{(2)}$ in definition (5.10),

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - 2\eta\tau_\pi \left(\dot{\sigma}^{\langle\mu\nu\rangle} + \frac{1}{3}\sigma^{\mu\nu}\theta\right) - \frac{4}{7}\frac{L_{4,2}}{T^2}\sigma_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \frac{4L_{3,2}}{T}\sigma_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\,, \tag{5.35}$$

where $\eta = K_{3,2}/T$ is the first order transport coefficient and we have defined $\tau_\pi \equiv L_{3,2}/K_{3,2}$. The equation presented above is consistent with the one derived in Ref. [262] under the assumption of conformal symmetry. It is worth noting that the above equation retains its conformal invariance regardless of the specific functional dependence of the relaxation time on the particle energy.

One can rewrite Eq. (5.35) as a relaxation-type equation for the evolution of shear stress tensor by replacing $\sigma^{\mu\nu} \to \pi^{\mu\nu}/(2K_{3,2}/T)$[3],

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \mathcal{C}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}\,, \tag{5.36}$$

where $\beta_\pi \equiv \eta/\tau_\pi = \frac{(K_{3,2})^2}{TL_{3,2}}$ and $\mathcal{C} \equiv \frac{2}{7}\frac{L_{4,2}}{TL_{3,2}}$. It is straightforward to verify that when the relaxation time is independent of particle energies, $\tau_\pi \to \tau_{\rm R}$, $\beta_\pi \to (\mathcal{E}+\mathcal{P})/5$, and $\mathcal{C} \to 10/7$, which agrees with the previous results [136, 138]. It is interesting to note that there is one new integral $K_{3,2}$ (corresponding to η) in first-order, and two new integrals, $L_{3,2}$ and $L_{4,2}$ (corresponding to τ_π and $\mathcal{C}$, respectively), in the second order.

As an illustration, we shall consider the following parametrization of the relaxation time [195, 214, 215, 143, 216],

$$\tau_{\rm R}(x,p) = \tau_{\rm eq}(x)\left(\frac{u\cdot p}{T}\right)^{\ell}, \tag{5.37}$$

where $\tau_{\rm eq}(x)$ represents the particle energy-independent part of relaxation time and scales as $1/T$ for conformal systems. We consider $\tau_{\rm eq}(x) = \kappa/T$, where κ is a dimensionless

[3] In deriving, we used the relation

$$\dot{\sigma}^{\langle\mu\nu\rangle} = \frac{\dot{\pi}^{\langle\mu\nu\rangle}}{2\eta} - \left(\frac{TK_{3,2} + Q_{3,2} - K_{4,2}}{3\eta T^2}\right)\sigma^{\mu\nu}\theta\,,$$

where the $Q_{n,q}$ integral is defined as

$$Q_{n,q} \equiv \frac{1}{(2q+1)!!}\int \lceil \mathcal{P}\frac{\partial \tau_{\rm R}}{\partial\beta}(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q f_{\rm eq}.$$

Further, we used the relation, $Q_{n,q} = K_{n+1,q} - (n+1)TK_{n,q}$, and expressed $Q_{n,q}$ in terms of $K_{n,q}$ integral.

constant. Note that the exponents ℓ may depend on the space-time coordinates. With this parametrization, the coefficient of shear viscosity is obtained to be [183],

$$\eta = \frac{K_{3,2}}{T} = \frac{4\kappa T^3}{5\pi^2}\left[\frac{\Gamma(5+\ell)}{24}\right], \text{ iff } \ell > -5. \tag{5.38}$$

Also, the coefficients τ_π, β_π, and $\mathcal{C}$ appearing in Eq. (5.36) can be determined analytically to have the form,

$$\beta_\pi \equiv \frac{(K_{3,2})^2}{TL_{3,2}} = \frac{4T^4}{5\pi^2}\left[\frac{\Gamma(\ell+5)^2}{24\Gamma(2\ell+5)}\right], \tau_\pi \equiv \frac{L_{3,2}}{K_{3,2}} = \frac{\kappa}{T}\left[\frac{\Gamma(2\ell+5)}{\Gamma(\ell+5)}\right], \mathcal{C} \equiv \frac{2}{7}\frac{L_{4,2}}{TL_{3,2}} = \frac{10+4\ell}{7}, \tag{5.39}$$

with the condition $\ell > -5/2$. The above results will be employed in the next section to study the evolution of a plasma undergoing boost-invariant expansion.

5.8 Results

Bjorken flow

We shall now study the hydrodynamic equation obtained for a fluid undergoing Bjorken expansion [263]. Bjorken symmetries enforce translational and rotational symmetry in the transverse (x, y) plane, boost invariance along the z (longitudinal) direction, and reflection symmetry $z \to -z$. The symmetries are manifest in Milne coordinate system (τ, x, y, η_s), where $\tau = \sqrt{t^2 - z^2}$ is the proper time and $\eta_s = \tanh^{-1}(z/t)$ the space-time rapidity. In these coordinates the fluid appears to be static, $u^\mu = (1, 0, 0, 0)$, irrotational ($\omega^{\mu\nu} = 0$) and unaccelerated ($\dot{u}^\mu = 0$), but has a non-zero local expansion rate, $\theta = 1/\tau$. Symmetries further constrain the shear tensor to be diagonal and space-like in Milne coordinates, leaving only one independent component which we take to be the $\eta_s\eta_s$ component: $\pi^{xx} = \pi^{yy} = -\tau^2\pi^{\eta_s\eta_s}/2 \equiv \pi/2$.

The hydrodynamic equations for evolution of energy density (5.2) and the shear tensor

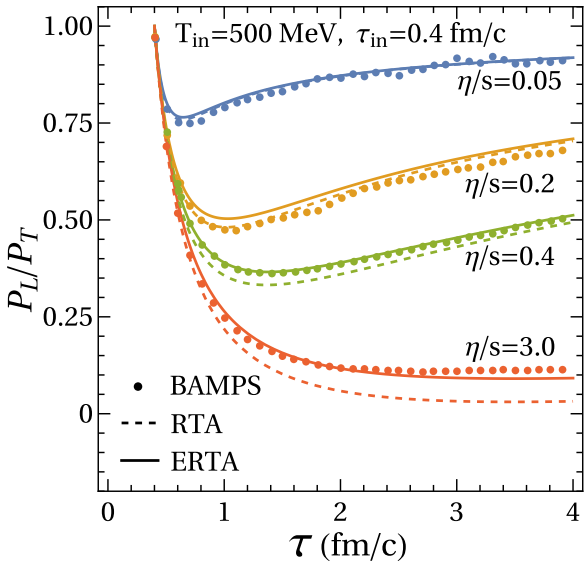

Figure 5.1: Evolution of P_L/P_T in BAMPS (black dots) compared with hydrodynamic evolution using transport coefficients obtained from the RTA approximation (dashed curves) and ERTA approximation with $\ell = 1/2$.

(5.36) in Milne coordinates takes the form (Appendix 6.3),

$$\frac{d\mathcal{E}}{d\tau} = -\frac{1}{\tau}\left(\mathcal{E} + \mathcal{P} - \pi\right), \tag{5.40}$$

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \frac{4}{3}\frac{\beta_\pi}{\tau} - \left(\frac{4+\mathcal{C}}{3}\right)\frac{\pi}{\tau}. \tag{5.41}$$

The above equations can be transformed into an equation for the quantity [206, 233, 234],

$$g \equiv \frac{\tau}{\mathcal{E}}\frac{\partial \mathcal{E}}{\partial \tau} = \frac{\pi}{\mathcal{E}} - \frac{4}{3}. \tag{5.42}$$

When the energy density exhibits power law behavior, g corresponds to the exponent of that specific power law (i.e., if $\mathcal{E} \sim \tau^a$, then $g = a$). Equations (5.40) and (5.41) can be written as a non-linear, first-order, differential equation in g as,

$$-\mathcal{B}(g) = g^2 + \left(\frac{8+\mathcal{C}}{3}\right)g + \frac{4}{3}\left(\frac{4}{3} + \frac{\mathcal{C}}{3} - \frac{\beta_\pi}{\mathcal{E}}\right) + \frac{\tau}{\tau_\pi}\left(g + \frac{4}{3}\right), \tag{5.43}$$

where we have defined $\mathcal{B}(g) \equiv \tau(\mathrm{d}g/\mathrm{d}\tau)$. Note that $\mathcal{C}$ and $\beta_\pi/\mathcal{E}$ are dimensionless.

The hydrodynamic regime is reached when the scattering rate exceeds the expansion rate, i.e., $\tau_\pi \ll \tau$. The last term in the above equation is dominant in this regime, and the

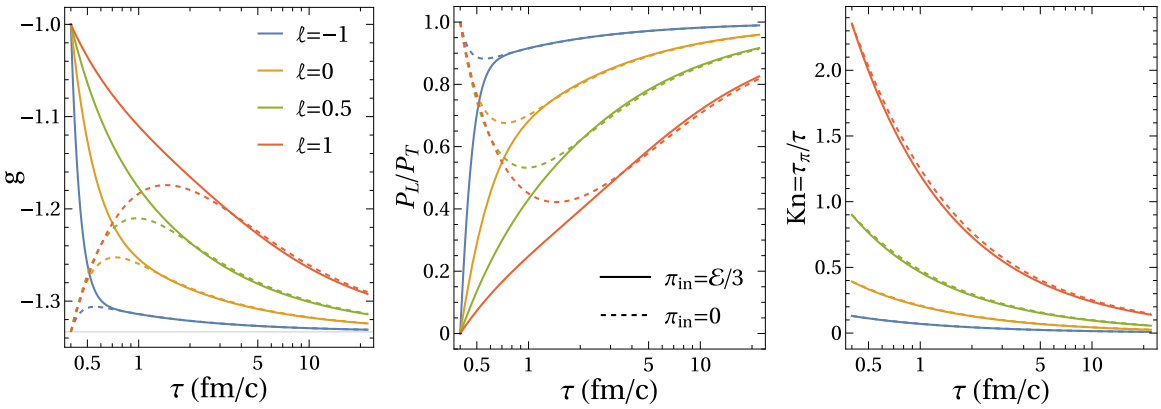

Figure 5.2: Time evolution of the quantity g, pressure anisotropy P_L/P_T, and Knudsen number τ_π/τ. Solid and dashed curves correspond to different initial shear stresses, $\pi = \mathcal{E}/3, 0$, respectively. The blue, orange, green, and red curves correspond to values of $\ell = -1, 0, 1/2, 1$, respectively. As can be seen in all three panels, isotropization and thermalization are delayed as the value of ℓ is increased.

hydrodynamic fixed point g_* is given by:

$$g_* = -\frac{4}{3}\,. \tag{5.44}$$

In the collisionless regime, the expansion rate far exceeds the scattering rate ($\tau_\pi \gg \tau$), and the function $\mathcal{B}(g)$ in Eq. (5.43) is dominated by the terms that do not depend on τ/τ_π. The zeros of this function correspond to the free-streaming fixed points,

$$g_{\rm fp} = -\left(\frac{8+\mathcal{C}}{6}\right) \pm \sqrt{\frac{4\beta_\pi}{3\mathcal{E}} + \frac{\mathcal{C}^2}{36}}\,, \tag{5.45}$$

with the positive root corresponding to the free-streaming stable fixed point. For a plasma undergoing Bjorken expansion, it has been shown that the stable free-streaming fixed point of the exact kinetic solution corresponds to vanishing longitudinal pressure, or $g = -1$ [212, 213, 233][4]. Using the parametrization (5.37) for the relaxation time and the corresponding values of the transport coefficients given in Eqs. (5.38) and (5.39), we observe that the value of the stable fixed point in the exact kinetic equation ($g = -1$) can be recovered from Eq. (5.45) for $\ell \approx 0.763$. Extending the domain of Israel-Stewart-type hydrodynamic

[4] Although this was shown for the RTA Boltzmann equation, it holds true even for the ERTA case since the collision term vanishes in free-streaming.

theories requires the hydrodynamic equations to accurately capture the location of the stable free-streaming fixed point, as emphasized in Ref. [186]. Therefore, the evolution Eq. (5.43), or analogously Eqs. (5.40) and (5.41), is expected to provide a good description of the underlying weakly coupled microscopic theory with $\ell \approx 0.763$, even in far-off-equilibrium regimes. It is worth mentioning that this value of ℓ is not arbitrary; many microscopic theories lie in the range $\ell = [0, 1]$ [195].

To illustrate the impact of the ERTA framework, we show the comparison of the second-order hydrodynamic equations obtained from the RTA approximation ($\ell = 0$) with those derived from the ERTA approximation setting $\ell = 0.5$ and compare them with BAMPS results [264, 265, 169] in Fig 5.1. The initial temperature is set to be 500 MeV at an initial time of 0.4 fm/c with a vanishing initial shear stress. Further, we fix the values of κ appearing in Eqs. (5.38) and (5.39) such that η/s is set to different values as mentioned in the figure[5]. As can be seen from the figure, the solid curves representing the ERTA approximation with $\ell = 0.5$ are in an overall better agreement with the BAMPS solution than the dashed curves [197][6].

In Figure 5.2, we present the evolution of three quantities: g, the pressure anisotropy $P_L/P_T = (P - \pi)/(P + \pi/2)$, and the Knudsen number τ_π/τ. The initial temperature is set to be 500 MeV at an initial time of 0.4 fm/c. Additionally, we consider the parameter κ appearing in Eqs. (5.38) and (5.39) to have the value $5/(4\pi)$[7]. In all three panels, the solid curves represent cases where P_L is initialized at 0, corresponding to $\pi = \mathcal{E}/3$, while the dashed curves are initialized with a vanishing initial shear stress, $\pi = 0$. The blue, orange, green, and red curves correspond to different values of $\ell = -1, 0, 1/2, 1$, respectively. The

[5]For RTA approximation ($\ell = 0$), $\kappa = \{1/4, 1, 2, 15\}$ for $\eta/s = \{0.05, 0.2, 0.4, 3.0\}$, respectively. Similarly for ERTA approximation with $\ell = 0.5$, $\kappa = \{0.11, 0.46, 0.92, 6.9\}$ for $\eta/s = \{0.05, 0.2, 0.4, 3.0\}$, respectively.

[6]We note that the second-order hydrodynamic equations obtained from the RTA approximation (dashed curve) perform better than the one obtained from ERTA approximation (solid curve) for $\eta/s = 0.2$.

[7]The value $\kappa = 5/(4\pi)$ implies $\eta/s = 1/(4\pi)$ when the relaxation time is independent of the particle energies ($\ell = 0$), i.e. when ERTA reduces to Anderson-Witting RTA.

left panel displays the evolution of the quantity g, with the gray dashed line representing the hydrodynamic fixed point g_*. It can be observed from the systematic trend of blue, orange, green, and red curves that the system remains out of equilibrium for a longer duration as the values of ℓ are increased. This feature is also visible in the middle panel where the evolution of pressure anisotropy, P_L/P_T is shown – approach to $P_L/P_T = 1$ is delayed for the orange, green, and red curves compared to the blue curve, indicating a slower isotropization. Also, in the left and middle panels we observe that the solid and dashed curves, representing different initial shear stress, overlap earlier for smaller values of ℓ. Interestingly, the evolution of Knudsen number shown in the right panel is not strongly dependent on the initial values of shear stress but has a strong dependence on the strength of the momentum-dependence of the relaxation time i.e. on ℓ; the solid and dashed curves largely overlap during the entire evolution. It is worth noting that increasing the value of ℓ enhances the initial gradient strength (as τ_π increases), and smaller values of ℓ drive the system towards thermalization at a faster rate, which is evident from the middle panel.

In Figure 5.3, we show the evolution of the temperature normalized with the ideal temperature evolution, $T_{\text{id}} = T_{\text{in}}(\tau_{\text{in}}/\tau)^{1/3}$. It is observed that at a given time, the fluid maintains a higher temperature when the initial shear stress has a large positive value (solid curves), in contrast to when the initial shear stress is vanishing (dashed curves). An interesting observation is that increasing values of ℓ also lead to higher temperatures of the medium, as indicated by the trend of the differently colored curves. This may be understood from the right panel of Fig. 5.2, where we observe that an increase in the values of ℓ results in a larger Knudsen number. Consequently, this leads to increased dissipation, resulting in a slower fall of temperature compared to ideal evolution. Moreover, the interplay between the initial conditions for shear stress and the various medium interactions (characterized by different values of ℓ) is intriguing, and can provide insights towards constraining the initial conditions for hydrodynamic simulation of heavy-ion collisions. Further, the various curves

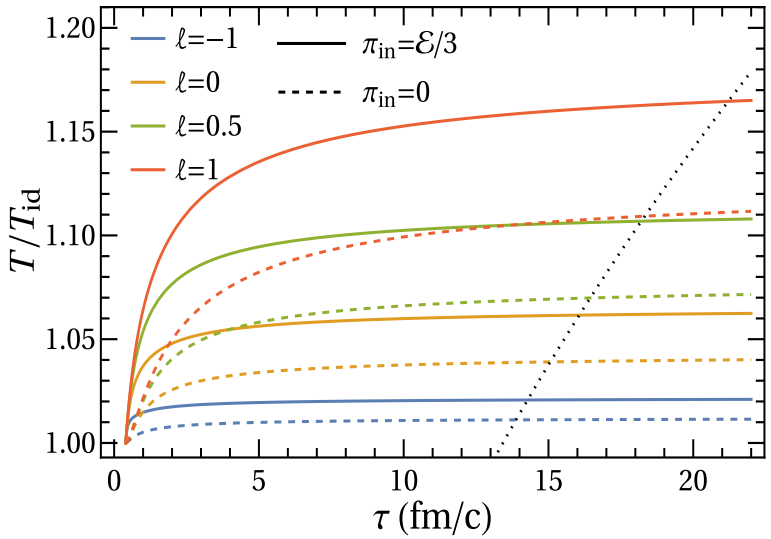

Figure 5.3: Time evolution of temperature normalized with ideal temperature evolution. The initial conditions and legends are the same as in Fig. 5.2. The black dotted curve represents a temperature surface of 155 MeV.

in Fig. 5.3 crossing the temperature surface of 155 MeV (represented by the black dotted curve) at different proper times suggests that a constant temperature particlization surface can be reached at different times with varying anisotropies. This can be seen more clearly in Fig. 5.4, where the evolution of the pressure anisotropy P_L/P_T with τ/τ_π is shown. The evolution of the curves is stopped when the temperature of the plasma reaches 155 MeV during the expansion (at times when the different curves cross the black dotted curve in Fig. 5.3). In Fig. 5.4, we see that the pressure anisotropy across the different colored curves differs significantly. We also note that the evolution of P_L/P_T for different curves in the near-equilibrium regime ($\tau \gtrsim 5\tau_\pi$) is nearly the same, but differs substantially in the far-off-equilibrium regime.

5.9 Summary and outlook

To summarize, we have derived relativistic second-order hydrodynamics from the Boltzmann equation using the extended relaxation time approximation for the collision kernel,

incorporating an energy-dependent relaxation time. The transport coefficients are shown to explicitly depend on the microscopic relaxation rate. We investigated the fixed point structure of the hydrodynamic equations for a plasma undergoing Bjorken flow and showed that the location of the free-streaming fixed points depends on the energy dependence of the relaxation time. Additionally, we employed a power law parametrization to describe the energy dependence of the relaxation time and examined its impact on the thermalization process of the expanding plasma. We demonstrated that the plasma's approach to equilibrium is affected by the relaxation time's dependence on different powers of energy; the plasma remains in the out-off-equilibrium regime and at a higher temperature for longer duration as larger positive values of ℓ are considered.

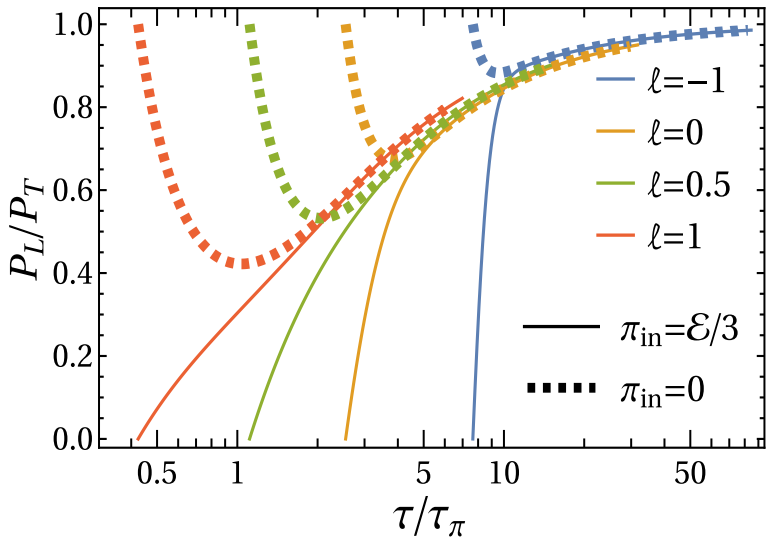

Figure 5.4: Evolution of pressure anisotropy P_L/P_T with τ/τ_π. The evolution of different curves is stopped at proper times when the temperature evolution reaches 155 MeV.

While the derivation in the present article is done for a conformal system without conserved charges, it can be extended for non-conformal systems with conserved charges and quantum statistics by following the steps outlined in the article. It is also desired to have typical relaxation rates for the energy dependence of the relaxation time across different stages of the evolution of the nuclear matter formed in heavy-ion collisions. Such param-

eterization of the relaxation time can have parameters which may depend, for example, on the temperature of the medium[8]. Incorporating these rates will make the full hydrodynamic equations with the associated transport coefficients more suitable for a (3+1)-dimensional hydrodynamic simulation. It should be noted that the functional form of the first-order transport coefficients, such as η, is determined within the framework. Furthermore, such an analysis may also provide insights into the form of distribution function at particlization. These aspects will be investigated in future studies.

[8] It remains to be explored if some of the essential features of a strongly coupled fluid can be captured in this framework by parameterizing the relaxation time.

Chapter 6

Summary and future outlook

In our initial objective, we sought to incorporate energy/momentum-dependent, Extended Relaxation Time Approximation (ERTA) into the Boltzmann equation. However, when we attempted a direct integration of ERTA into the Anderson-Witting Relaxation Time Approximation (AW RTA) form of the Boltzmann equation, a significant problem arose: the violation of macroscopic conservation laws. To address this challenge, we adopted a novel frame concept, which was instrumental in reestablishing both macroscopic and microscopic conservation principles. This alternative frame not only successfully preserved the integrity of conservation laws but also provided a deeper understanding of the behavior of hydrodynamic quantities. Furthermore, our approach uncovered previously unexplored and intriguing aspects of the system, particularly in relation to the energy-dependent Relaxation Time Approximation (RTA). These new insights offer a fresh perspective on how energy-dependent processes influence the behavior of the system at both macroscopic and microscopic scales.

The work reported in the present thesis begins by deriving the first-order dissipative hydrodynamic equations from kinetic theory. These equations describe how various properties like temperature, density, and flow velocity evolve in a system when subjected to gradients. They are essential for understanding the behavior of matter under extreme conditions, as seen in high-energy physics. To derive these equations, an 'extended' RTA collision kernel in the Boltzmann equation is employed. The Boltzmann equation is a fundamental equation in statistical mechanics that describes the behavior of particles in a gas. The 'extension' likely includes additional terms or considerations to account for the microscopic complexity

of the system being studied. In the context of hydrodynamics, relaxation time represents the timescale over which a system returns to equilibrium after being perturbed. By making this relaxation time energy-dependent, the model becomes more accurate in describing systems where particles have varying energies. Transport coefficients are parameters that quantify how a system responds to gradients or forces, such as viscosity and thermal conductivity play a crucial role in understanding the transport properties of a fluid. It is then highlighted that the energy-dependent relaxation time has a direct impact on these transport coefficients. This means that as the energy of particles changes, the transport properties of the system also change. This is an important consideration when modelling systems with varying particle energies, as is often the case in high-energy physics. To further explore this energy dependence, the text mentions the use of a power law parametrization. This is a mathematical representation that allows us to parametrize the relaxation time variation with energy in a systematic way. By employing such a parametrization, we can better understand and quantify the energy-dependent effects on the system. Analysis of the transport coefficient ratios can reveal important insights into the system's behavior. These ratios were examined under different equilibrium statistics, which likely means they considered various initial conditions or configurations of the system. The text concludes by noting that this analysis led to the discovery of new and interesting scaling features. These scaling features suggest that certain properties or relationships within the system exhibit consistent behavior across different scales or conditions.

In the subsequent part of the thesis, we present the successful derivation of relativistic second-order hydrodynamic equations. In our previous research, we successfully derived the Navier-Stokes equation. However, it was noted that the Navier-Stokes equation, in its initial form, exhibited acausal behavior. To address this issue and bring about a more physically meaningful representation, we extended our analysis to second order. This extension was pivotal in restoring causality within the framework of dissipative hydrodynamics. We

explored the fixed point structure of the hydrodynamic equations in the context of a plasma undergoing Bjorken flow. We found that the location of these fixed points is influenced by the energy dependence of the relaxation time. This implies that the system's behavior at equilibrium is affected by the relaxation time's energy dependence. To further understand the energy dependence of the relaxation time, a power law parametrization was employed. This mathematical representation allows for a systematic description of how the relaxation time varies with energy. We also examined how the energy dependence of the relaxation time affects the thermalization process of the expanding plasma. We discovered that the approach to equilibrium of the plasma is influenced by the relaxation time's dependence on different powers of energy. Notably, the study demonstrated that the plasma's approach to equilibrium differs depending on the choice of the power (ℓ) in the energy dependence of the relaxation time. Larger positive values of ℓ lead to the plasma remaining in an out-of-equilibrium state for a longer duration and at a higher temperature.

By considering the energy dependence of relaxation times and its impact on transport coefficients and the thermalization process, the study sheds light on how these complex systems evolve and approach equilibrium. The future outlook for research in this area could involve several exciting avenues and advancements:

- **Formulation for more general system:** Expanding the framework to encompass systems with massive and charged particles introduces an additional layer of complexity. Investigating second-order gradients of parameters in such systems could yield valuable insights into how these factors influence the behavior of matter in high-energy physics contexts. The incorporation of energy-dependent Relaxation Time Approximation (RTA) with different statistics is an exciting avenue for future study. Examining how various statistical distributions affect the system's dynamics and transport properties can enhance our understanding of equilibrium and non-

equilibrium behavior in diverse physical systems.

- **Higher-Order Hydrodynamics:** Building upon the success in deriving first-order and second-order dissipative hydrodynamic equations, researchers may explore even higher-order hydrodynamics to capture more intricate details of the system's behavior. This could lead to more accurate descriptions of extreme conditions in high-energy physics.

- **Refined Energy Dependence:** Further refinement of the energy-dependent relaxation time model could yield deeper insights. It will be interesting to investigate alternative forms of energy dependence or consider more complex energy distributions to better represent realistic scenarios.

- **Extended Parameter Space:** Expanding the analysis to cover a wider parameter space could provide a comprehensive understanding of how different factors interact. This could involve variations in temperature, energy, and other relevant parameters to explore the system's behavior more comprehensively.

- **Experimental Validation:** It is important to consider the simulation of relativistic heavy ion collision by employing the hydrodynamic formulation presented in this thesis. A 3+1 dimensional simulation will help us to compare our predictions with the experimental results and therefore constrain the momentum dependence of the relaxation time.

- **Applications Beyond High-Energy Physics:** Investigating potential applications of these models beyond high-energy physics could be valuable. For example, the ERTA framework may find use in condensed matter systems where momentum dependence of the relaxation time needs to be taken into account.

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Appendix A

6.1 Derivations of ERTA delta quantities with generalised system

The frame condition $u_\nu T^{\mu\nu} = \mathcal{E}_0 u^\mu$ implies

$$
\begin{aligned}
& u_\mu \int dP\, p^\mu p^\nu (f_{\text{eq}} + \delta f_{(1)}) = u^\nu \int dP\, (u\cdot p)^2 f_{\text{eq}} \\
\Longrightarrow\; & u_\mu \int dP\, p^\mu p^\nu \delta f_{(1)} = u^\nu u_\alpha u_\beta \int dP\, p^\alpha p^\beta f_{\text{eq}} - u_\mu \int dP\, p^\mu p^\nu f_{\text{eq}} = u^\nu u_\alpha u_\beta\, I^{\alpha\beta}_{(0)} - u_\mu\, I^{\mu\nu}_{(0)} \\
\Longrightarrow\; & u_\mu \int dP\, p^\mu p^\nu \delta f_{(1)} = 0\,. \qquad (6.1)
\end{aligned}
$$

We note that this equation holds for all orders, i.e., for $\delta f_{(1)} \to \delta f_{(1)} + \cdots + \delta f_{(n)}$. In fact, all these conditions holds for all orders.

The above frame condition when particles and anti-particles are considered gives

$$
u^\nu \left[-\frac{(u\cdot\delta u)}{T} J^+_{3,0} + \frac{\delta T}{T^2}\left(J^+_{3,0} - \mu J^-_{2,0}\right) + \frac{\delta\mu}{T} J^-_{2,0} \right] - \frac{\delta u_\alpha}{T}\Delta^{\nu\alpha} J^+_{3,1} + u^\nu\, y^+_{3,0}\,\theta + x_1 \nabla^\nu \alpha = 0\,, \qquad (6.2)
$$

where

$$
y^+_{3,0} \equiv \frac{1}{T}\left(\chi_b - \frac{1}{3}\right) K^+_{3,0} + \frac{m^2}{3T} K^+_{1,0} - \chi_a\, K^-_{2,0}\,, \qquad x_1 \equiv \left(\frac{n}{\mathcal{E}+\mathcal{P}}\right) K^+_{3,1} - K^-_{2,1}\,. \qquad (6.3)
$$

Now, the term

$$
\frac{\delta u_\alpha}{T}\Delta^{\nu\alpha} J^+_{3,1} = \frac{\delta u_\alpha}{T}\left(g^{\nu\alpha} - u^\nu u^\alpha\right) J^+_{3,1} = \frac{J^+_{3,1}}{T}\left(\delta u^\nu - (u\cdot\delta u) u^\nu\right) \qquad (6.4)
$$

Using this and ignoring terms like $(u \cdot \delta u)$ (as they are of higher order), Eq. (6.2) simplifies to

$$\delta u^\nu = \frac{u^\nu}{J^+_{3,1}} \left[\frac{\delta T}{T} \left(J^+_{3,0} - \mu J^-_{2,0} \right) + \delta\mu \, J^-_{2,0} + y^+_{3,0} \, T \, \theta \right] + \frac{T \, x_1}{J^+_{3,1}} (\nabla^\nu \alpha) \, . \tag{6.5}$$

We also have the matching condition $u_\mu \, u_\nu \, T^{\mu\nu} = \mathcal{E}_0$. This gives

$$\frac{\delta T}{T} \left(J^+_{3,0} - \mu J^-_{2,0} \right) + \delta\mu \, J^-_{2,0} + y^+_{3,0} \, T \, \theta = 0 \, . \tag{6.6}$$

Using this in Eq. (6.5), we get

$$\begin{aligned} \delta u^\nu &= \frac{T \, x_1}{J^+_{3,1}} (\nabla^\nu \alpha) = \mathcal{C}_1 \frac{(\nabla^\nu \alpha)}{T} \, , \\ \text{where} \qquad \mathcal{C}_1 &\equiv \frac{T^2 \, x_1}{J^+_{3,1}} = -\frac{T \, x_1}{\mathcal{E} + \mathcal{P}} = \frac{T}{\mathcal{E} + \mathcal{P}} \left[K^-_{2,1} - \left(\frac{n}{\mathcal{E} + \mathcal{P}} \right) K^+_{3,1} \right] . \end{aligned} \tag{6.7}$$

Now, imposing the condition $u_\mu N^\mu = n_0$, we obtain

$$\frac{\delta T}{T} \left(J^-_{2,0} - \mu J^+_{1,0} \right) + \delta\mu \, J^+_{1,0} + y^-_{2,0} \, T \, \theta = 0 \, , \tag{6.8}$$

where

$$y^-_{2,0} \equiv \frac{1}{T} \left(\chi_b - \frac{1}{3} \right) K^-_{2,0} + \frac{m^2}{3T} K^-_{0,0} - \chi_a \, K^+_{1,0} \, . \tag{6.9}$$

Solving Eqs. (6.6 , 6.8) simultaneously, we obtain

$$\delta T = \mathcal{C}_2 \, \theta \, , \quad \text{where} \quad \mathcal{C}_2 \equiv \frac{T^2 \left(J^-_{2,0} \, y^-_{2,0} - J^+_{1,0} \, y^+_{3,0} \right)}{J^+_{3,0} J^+_{1,0} - J^-_{2,0} J^-_{2,0}} \, , \tag{6.10}$$

$$\delta\mu = \mathcal{C}_3 \, \theta \, , \quad \text{where} \quad \mathcal{C}_3 \equiv \frac{T \left[\left(J^-_{2,0} - \mu \, J^+_{1,0} \right) y^+_{3,0} - \left(J^+_{3,0} - \mu \, J^-_{2,0} \right) y^-_{2,0} \right]}{J^+_{3,0} J^+_{1,0} - J^-_{2,0} J^-_{2,0}} \, . \tag{6.11}$$

6.2 Second order derivation of δu^μ and δT for massless and chargeless MB system

In this Appendix, we obtain δu^μ and δT by imposing Landau frame and matching conditions. The Landau Frame condition, $u_\mu T^{\mu\nu} = \mathcal{E} u^\nu$, with the matching $\mathcal{E} = \mathcal{E}_0$, for a non-equilibrium distribution $f = f_{\rm eq} + \delta f$ can be written as

$$\begin{aligned} u_\mu \int \mathrm{dP}\, p^\mu p^\nu (f_{\rm eq} + \delta f) &= u^\nu \int \mathrm{dP}\, (u\cdot p)^2 f_{\rm eq} \\ \implies u_\mu \int \mathrm{dP}\, p^\mu p^\nu \delta f &= u^\nu u_\alpha u_\beta I^{\alpha\beta} - u_\mu I^{\mu\nu} = 0. \end{aligned} \tag{6.12}$$

where $I^{\alpha\beta} = \int \mathrm{dP}\, p^\alpha p^\beta f_{\rm eq}$. Replacing $\delta f \mapsto \delta f_{(2)}$ obtained in Eq. (5.25), and performing the integrals in the local rest frame of u^μ, the above equation reduces to

$$\begin{aligned} &I_{3,1}\delta u^\mu - I_{3,0}\frac{\delta T}{T}u^\mu + \frac{5K_{3,2}}{(\mathcal{E}+\mathcal{P})}\left(\pi^{\mu\nu}\dot{u}_\nu - \nabla_\nu \pi^{\mu\nu}\right) + \left(6L_{3,2} + 2M_{4,2} - 2\frac{N_{3,2}}{T}\right)\sigma^{\mu\nu}\dot{u}_\nu \\ &+ 2L_{3,2}\nabla_\nu \sigma^{\mu\nu} + \left(10L_{3,2} - 2\frac{L_{4,2}}{T} + 2M_{4,2}\right)\sigma^{\alpha\beta}\sigma_{\alpha\beta}u^\mu = 0. \end{aligned} \tag{6.13}$$

Note that the term $\delta u \cdot u = (\delta u \cdot \delta u)/2 \sim \mathcal{O}(\delta^4)$, since δu^μ is at least second order (see discussion in Section 5.4), and has been ignored in the derivation. Further, we have defined the thermodynamic integrals,

$$I_{n,q} \equiv \frac{1}{(2q+1)!!}\int \mathrm{dP}(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}, \tag{6.14}$$

$$M_{n,q} \equiv \frac{1}{(2q+1)!!}\int \mathrm{dP}\tau_{\rm R}\frac{\partial \tau_{\rm R}}{\partial(u\cdot p)}(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q f_{\rm eq}, \tag{6.15}$$

$$N_{n,q} \equiv \frac{1}{(2q+1)!!}\int \mathrm{dP}\,\tau_{\rm R}\frac{\partial \tau_{\rm R}}{\partial \beta}(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}. \tag{6.16}$$

We note that $M_{n,q}$ and $N_{n,q}$ integrals can be expressed in terms of the $L_{n,q}$ integrals through the relations,

$$M_{n,q} = \frac{1}{2T}L_{n,q} - \frac{n+1}{2}L_{n-1,q}\,, \quad \text{iff } n > -1\,, 2N_{n,q} \quad = L_{n+1,q} - nTL_{n,q}\,. \tag{6.17}$$

Using these relations, Eq. (6.13) simplifies to,

$$I_{3,1}\delta u^\mu - I_{3,0}\frac{\delta T}{T}u^\mu + \frac{5K_{3,2}}{(\mathcal{E}+\mathcal{P})}\left(\pi^{\mu\nu}\dot{u}_\nu - \nabla_\nu\pi^{\mu\nu}\right) + 2L_{3,2}\times\left(2\sigma^{\mu\nu}\dot{u}_\nu + \nabla_\nu\sigma^{\mu\nu}\right) + \left(5L_{3,2} - \frac{L_{4,2}}{T}\right)\sigma^{\alpha\beta}\sigma_{\alpha\beta}u^\mu = 0. \tag{6.18}$$

Similarly, using the matching condition:

$$u_\mu u_\nu T^{\mu\nu} = \mathcal{E}_0 \implies u_\mu u_\nu \int \mathrm{dP}\, p^\mu p^\nu \delta f = 0\,, \tag{6.19}$$

and replacing $\delta f \mapsto \delta f_{(2)}$ obtained in Eq. (5.25), we obtain

$$-I_{3,0}\frac{\delta T}{T} + \frac{5K_{3,2}}{\mathcal{E}+\mathcal{P}}\pi^{\mu\nu}\sigma_{\mu\nu} + \left(3L_{3,2} - \frac{L_{4,2}}{T}\right)\sigma^{\mu\nu}\sigma_{\mu\nu} = 0\,. \tag{6.20}$$

Noting that $I_{3,0} = 3T(\mathcal{E}+\mathcal{P})$ and solving for δT, we obtain

$$\delta T = \frac{5}{3}\frac{K_{3,2}}{(\mathcal{E}+\mathcal{P})^2}\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{\mathcal{E}+\mathcal{P}}\left(L_{3,2} - \frac{L_{4,2}}{3T}\right)\sigma^{\mu\nu}\sigma_{\mu\nu}\,. \tag{6.21}$$

The expression for δu^ν is obtained by using Eq. (6.20) in Eq. (6.18),

$$\delta u^\mu = \frac{5K_{3,2}}{T(\mathcal{E}+\mathcal{P})^2}\left(\pi^{\mu\nu}\dot{u}_\nu - \nabla_\nu\pi^{\mu\nu} - \pi^{\alpha\beta}\sigma_{\alpha\beta}u^\mu\right) + \frac{2L_{3,2}}{T(\mathcal{E}+\mathcal{P})}\left(2\sigma^{\mu\nu}\dot{u}_\nu + \nabla_\nu\sigma^{\mu\nu} + \sigma^{\alpha\beta}\sigma_{\alpha\beta}u^\mu\right), \tag{6.22}$$

where we have used the relation $I_{3,1} = -T(\mathcal{E}+\mathcal{P})$.

In the case when the relaxation time is independent of the particle energies, the integrals $K_{n,q} \to \tau_\mathrm{R}\, I_{n,q}$ and $L_{n,q} \to \tau_\mathrm{R}\, I_{n,q} = \tau_\mathrm{R} K_{n,q}$. Using these, and noting that $\tau_\mathrm{R} = 5\eta/(\mathcal{E}+\mathcal{P})$, Eq. (6.22) simplifies to

$$\begin{aligned}\delta u^\mu &= \frac{5K_{3,2}}{T(\mathcal{E}+\mathcal{P})^2}\left(\pi^{\mu\nu}\dot{u}_\nu - \nabla_\nu\pi^{\mu\nu} - \pi^{\alpha\beta}\sigma_{\alpha\beta}u^\mu\right) + \frac{5K_{3,2}}{T(\mathcal{E}+\mathcal{P})^2}2\eta\left(2\sigma^{\mu\nu}\dot{u}_\nu + \nabla_\nu\sigma^{\mu\nu} + \sigma^{\alpha\beta}\sigma_{\alpha\beta}u^\mu\right)\\ &= \frac{5K_{3,2}}{T(\mathcal{E}+\mathcal{P})^2}\left(3\pi^{\mu\nu}\dot{u}_\nu - \nabla_\nu\pi^{\mu\nu} + 2\eta\nabla_\nu\sigma^{\mu\nu}\right)\\ &= 0\,.\end{aligned} \tag{6.23}$$

In transitioning to the second equality, we employed the first-order relation $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$. Furthermore, we used the relation $\nabla_\nu \eta = -3T\eta\nabla_\nu\beta = 3\eta\dot{u}_\nu$ in the last equality.

Similarly, Eq. (6.21) reduces to

$$\begin{aligned}\delta T &= \frac{1}{\mathcal{E}+\mathcal{P}}\left[\frac{5}{3}\frac{K_{3,2}}{(\mathcal{E}+\mathcal{P})}\pi^{\mu\nu}\sigma_{\mu\nu} + \tau_{\rm R}^2\left(I_{3,2} - \frac{5}{3}I_{3,2}\right)\sigma^{\mu\nu}\sigma_{\mu\nu}\right] \\ &= \frac{1}{\mathcal{E}+\mathcal{P}}\left[\frac{5}{3}\frac{K_{3,2}}{(\mathcal{E}+\mathcal{P})}\pi^{\mu\nu}\sigma_{\mu\nu} - \frac{5\eta}{(\mathcal{E}+\mathcal{P})}\frac{2}{3}K_{3,2}\sigma^{\mu\nu}\sigma_{\mu\nu}\right] \\ &= 0\,. \end{aligned} \tag{6.24}$$

The fact that δu^μ and δT vanish when the relaxation time is independent of particle energy is anticipated since ERTA reduces to the Anderson-Witting RTA, thus providing a consistency validation.

6.2.1 Relation among integrals

For relativistic massless and chargeless systems $(u \cdot p)$ will be the same as energy(E) or momentum (p) term. Let's define it here,

$$\begin{aligned} K_{n,q} &\equiv \frac{1}{(2q+1)!!}\int {\rm dP}\,\tau_{\rm R}(x,p)(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}, \\ L_{n,q} &\equiv \frac{1}{(2q+1)!!}\int {\rm dP}\,\tau_{\rm R}^2(x,p)(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}, \\ M_{n,q} &\equiv \frac{1}{(2q+1)!!}\int {\rm dP}\frac{\tau_{\rm R}\partial\tau_{\rm R}}{\partial(u\cdot p)}(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}, \\ Q_{n,q} &\equiv \frac{1}{(2q+1)!!}\int {\rm dP}\frac{\partial\tau_{\rm R}}{\partial\beta}(u\cdot p)^{n-2q}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q\, f_{\rm eq}. \end{aligned} \tag{6.25}$$

Relation 1 Reduced form of any integrals We already know that $I_{n,q} = \frac{-1}{2q+1}I_{n,q-1}$ for massless and charge less case. So we can easily verify that $K_{n,q}, L_{n,q} and M_{n,q}$ also satisfy the above formula successfully. So in general we can write

$$X_{n,q} = \frac{-1}{(2q+1)}X_{n,q-1}$$

, where $X_{n,q}$ can be $K_{n,q}, L_{n,q} and M_{n,q}$

Relation 2 Reduced form of $M_{n,q}$

$$\begin{aligned}
L_{n,q} =& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp\tau_{\mathrm{R}}^2(u\cdot p)^{n-2q-1}p^{2q+2}\exp^{-\beta p}\\
=& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp\,\underbrace{\tau_{\mathrm{R}}^2 p^{n+1}}_{u}\underbrace{exp^{-\beta p}}_{v}\\
=& \frac{(-1)^q}{2\pi^2(2q+1)!!}\left[\left(\tau_{\mathrm{R}}^2 p^{n+1}\frac{\exp^{-\beta p}}{(-\beta)}\right)_0^{\infty} - \int 2\frac{\partial\tau_{\mathrm{R}}}{\partial p}\tau_{\mathrm{R}}p^{n+1}\frac{\exp^{-\beta p}}{(-\beta)}dp + \int (n+1)p^n\tau_{\mathrm{R}}^2\frac{\exp^{-\beta p}}{(-\beta)}dp\right]\\
=& \frac{2}{\beta}M_{n,q} + \frac{(n+1)}{\beta}L_{n-1,q}\\
\Rightarrow\ M_{n,q} =& \frac{\beta}{2}L_{n,q} - \frac{(n+1)}{2}L_{n-1,q}
\end{aligned} \tag{6.26}$$

Relation 3 The time derivative of $K_{3,2}$ can be derived as follows.

$$\begin{aligned}
K_{3,2} &= \int dP\tau_{\mathrm{R}}(u\cdot p)^{3-4}(p^2)^2\exp^{-\beta(u\cdot p)}\\
&= \int \frac{p^2dp}{p(2\pi)^3}\frac{\tau_{\mathrm{R}}}{p}p^4\exp^{-\beta p}\\
&= \int \frac{dp}{(2\pi)^3}p^4\tau_{\mathrm{R}}\exp^{(-\beta p)}\\
D(K_{3,2}) &= \int \frac{dp}{(2\pi)^3}p^4 D(\tau_{\mathrm{R}}\exp^{(-\beta p)})\\
&= \int \frac{dp}{(2\pi)^3}p^4\left(\frac{\partial\tau_{\mathrm{R}}}{\partial\beta}\dot{\beta} + \frac{\partial\tau_{\mathrm{R}}}{\partial(u\cdot p)}p^\nu\dot{u}_\nu - \tau_{\mathrm{R}}\dot{\beta}(u\cdot p) + \tau_{\mathrm{R}}\beta\dot{u}_\nu p^\nu\right)\exp\{-\beta(u\cdot p)\}\\
&= \int \frac{dp}{(2\pi)^3}p^4\frac{\partial\tau_{\mathrm{R}}}{\partial\beta}\frac{\beta\theta}{3} + 0 - \int \frac{dp}{(2\pi)^3}p^4\tau_{\mathrm{R}}\frac{\beta\theta}{3}(u\cdot p) + 0 \qquad \text{(by doing inverse decomposing)}\\
&= \frac{\beta\theta}{3}Q_{3,2} - \frac{\beta\theta}{3}K_{4,2}\\
D(K_{3,2}) &= \frac{\beta\theta}{3}(Q_{3,2} - K_{4,2})
\end{aligned} \tag{6.27}$$

Relation 4 Time derivative of $\sigma^{\alpha\beta}$

$$
\begin{aligned}
D(\pi^{\alpha\beta}) &= D(2\beta K_{3,2}\sigma^{\alpha\beta}) \\
\dot{\pi}^{\alpha\beta} &= 2\dot{\beta}K_{3,2}\sigma^{\alpha\beta} + 2\beta\dot{K}_{3,2}\sigma^{\alpha\beta} + 2\beta K_{3,2}\dot{\sigma}^{\alpha\beta} \\
\implies 2\beta K_{3,2}\dot{\sigma}^{\alpha\beta} &= \dot{\pi}^{\alpha\beta} - 2\frac{\beta\theta}{3}K_{3,2}\sigma^{\alpha\beta} - 2\beta(Q_{3,2} - K_{4,2})\frac{\beta\theta}{3}\sigma^{\alpha\beta} \\
\implies \dot{\sigma}^{\alpha\beta} &= \frac{\dot{\pi}^{\alpha\beta}}{2\beta K_{3,2}} - \frac{\theta}{3}\sigma^{\alpha\beta} - (\frac{Q_{3,2}}{K_{3,2}} - \frac{K_{4,2}}{K_{3,2}})\frac{\beta\theta}{3}\sigma^{\alpha\beta}
\end{aligned} \tag{6.28}
$$

Relation 5

$$
\begin{aligned}
L_{n,q} =& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp\tau_{\mathrm{R}}^2(u\cdot p)^{n-2q-1}p^{2q+2}\exp^{-\beta p} \\
=& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp\tau_{\mathrm{R}}^2 p^{n+1}exp^{-\beta p} \\
\frac{\partial L_{n\,q}}{\partial\beta} =& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp2\tau_{\mathrm{R}}\frac{\partial\tau_{\mathrm{R}}}{\partial\beta}p^{n+1}\exp^{-\beta p} - \int dp\tau_{\mathrm{R}}^2 p^{n+1}(p)\exp^{-\beta p} \\
=& 2N_{n\,q} - L_{n+1\,q} \\
\implies 2N_{n\,q} =& L_{n+1\,q} + \frac{\partial L_{n\,q}}{\partial\beta}
\end{aligned} \tag{6.29}
$$

If $L_{n\,q} = C_{n\,q}T^n = C_{n\,q}\beta^{-n}$

$$
\frac{\partial L_{n\,q}}{\partial\beta} = C_{n,q}(-n)\beta^{-n-1} = C_{n\,q}\frac{(-n)}{\beta}\beta^{-n} = \frac{(-n)}{\beta}L_{n\,q} = -nTL_{n\,q} \tag{6.30}
$$

Now we can write

$$
2N_{n\,q} = L_{n+1\,q} - nTL_{n\,q} \tag{6.31}
$$

Relation 6

$$
\begin{aligned}
K_{n,q} =& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp\tau_{\mathrm{R}}(u\cdot p)^{n-2q-1}p^{2q+2}\exp^{-\beta p} \\
=& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp\tau_{\mathrm{R}}p^{n+1}exp^{-\beta p} \\
\frac{\partial K_{n\,q}}{\partial\beta} =& \frac{(-1)^q}{2\pi^2(2q+1)!!}\int dp\frac{\partial\tau_{\mathrm{R}}}{\partial\beta}p^{n+1}\exp^{-\beta p} - \int dp\tau_{\mathrm{R}}p^{n+1}(p)\exp^{-\beta p} \\
=& Q_{n\,q} - K_{n+1\,q} \\
\implies Q_{n\,q} =& K_{n+1\,q} + \frac{\partial K_{n\,q}}{\partial\beta}
\end{aligned} \tag{6.32}
$$

if $K_{n,q} = C_{n,q}T^{n+1}$

$$\frac{\partial K_{n\,q}}{\partial \beta} = \frac{\partial}{\partial \beta}(C_{n\,q}T^{n+1}) = \frac{\partial}{\partial \beta}(C_{n\,q}\beta^{-(n+1)}) = -(n+1)C_{n\,q}\beta^{-(n+2)} = -(n+1)TK_{n\,q}$$

Now we can write

$$Q_{n\,q} = K_{n+1\,q} - (n+1)TK_{n\,q} \tag{6.33}$$

Relation 7

$$\begin{aligned}
K_{nq} &= \frac{1}{(2q+1)!!}\int_p \tau_{\mathrm{R}}\,(u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\,f_{\mathrm{eq}} \\
\Longrightarrow\ (\partial_\gamma K_{nq}) &= \frac{1}{(2q+1)!!}\int_p (\partial_\gamma \tau_{\mathrm{R}})\,(u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\,f_{\mathrm{eq}} \\
&+ \frac{1}{(2q+1)!!}\int_p \tau_{\mathrm{R}}\,(n-2q)\,(u\cdot p)^{n-2q-1}\,p^\phi\,(\partial_\gamma u_\phi)\,(p\cdot\Delta\cdot p)^q\,f_{\mathrm{eq}} \\
&+ \frac{1}{(2q+1)!!}\int_p \tau_{\mathrm{R}}\,(u\cdot p)^{n-2q}\,q\,(p\cdot\Delta\cdot p)^{q-1}\,p^\alpha p^\beta\,(\partial_\gamma \Delta_{\alpha\beta})\,f_{\mathrm{eq}} \\
&+ \frac{1}{(2q+1)!!}\int_p \tau_{\mathrm{R}}\,(u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\,(\partial_\gamma f_{\mathrm{eq}}) \\
&= \frac{1}{(2q+1)!!}\int_p \left[\left(\frac{\partial \tau_{\mathrm{R}}}{\partial\,(u\cdot p)}\right)p^\varphi\,(\partial_\gamma u_\varphi) + \left(\frac{\partial \tau_{\mathrm{R}}}{\partial \beta}\right)(\partial_\gamma \beta)\right](u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\,f_{\mathrm{eq}} + 0 \\
&+ 0 - \frac{1}{(2q+1)!!}\int_p \tau_{\mathrm{R}}\,(u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\left[(\partial_\gamma\beta)\,(u\cdot p) + \beta\,(\partial_\gamma u_\varphi)\,p^\varphi\right]f_{\mathrm{eq}} \\
&= \frac{1}{(2q+1)!!}\left[0 + (\partial_\gamma\beta)\int_p\left(\frac{\partial\tau_{\mathrm{R}}}{\partial\beta}\right)\right](u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\,f_{\mathrm{eq}} \\
&- \frac{1}{(2q+1)!!}\left[\int_p \tau_{\mathrm{R}}\,(u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\,(\partial_\gamma\beta)\,(u\cdot p) + 0\right]f_{\mathrm{eq}} \\
&= (\partial_\gamma\beta)\,Q_{nq} - (\partial_\gamma\beta)\,K_{n+1,q} \ \Longrightarrow\ \boxed{(\partial_\gamma K_{nq}) = (\partial_\gamma\beta)\,(Q_{nq} - K_{n+1,q})}
\end{aligned} \tag{6.34}$$

Relation 8 In similar way we can get that

$$\begin{aligned}
L_{nq} &= \frac{1}{(2q+1)!!}\int_p \tau_{\mathrm{R}}^2\,(u\cdot p)^{n-2q}\,(p\cdot\Delta\cdot p)^q\,f_{\mathrm{eq}} \\
&\Longrightarrow\ \boxed{(\partial_\gamma L_{nq}) = (\partial_\gamma\beta)\,(2N_{nq} - L_{n+1,q})}
\end{aligned} \tag{6.35}$$

6.3 Bjorken flow

The concept of "boost invariance" proposed by Bjorken in the context of high-energy physics and relativistic hydrodynamics is a significant simplification used to describe the behavior of matter created in high-energy collisions, such as those in heavy-ion collisions at particle accelerators like the Large Hadron Collider (LHC). Bjorken's idea of "boost invariance" is the assertion that at a longitudinal distance z away from the point of collision and time t after the collision, the matter created in the collision should be moving with a velocity $v_z = \frac{z}{t}$. This concept assumes that transverse dynamics (v_x and v_y) can be neglected. To mathematically describe this concept, Milne coordinates are introduced. These coordinates are based on proper time ($\tau = \sqrt{t^2 - z^2}$)and spacetime rapidity $\eta = \tanh^{-1}(z/t)$. Using Milne coordinates, the relationships between t,z,τ and η are expressed as $t = \tau \cosh \eta$, and $z = \tau \sinh \eta$.

In Milne coordinates, Bjorken's boost invariance can be expressed as follows for the fluid velocity components:

$$u^t = \frac{t}{\tau}\,,\; u^z = \frac{z}{\tau}\,,\; u^\eta = -\frac{u^t \sinh \eta}{\cosh \eta} + \frac{u^z}{\tau}$$

This formulation essentially states that the fluid velocity components depend on τ but are independent of the spacetime rapidity η. Consequently, the energy density ($\mathcal{E}$), pressure (P), fluid velocity (u^μ), and shear stress tensor ($\pi^{\mu\nu}$) are all independent of η and remain unchanged when performing a Lorentz boost. While this boost invariance leads to a highly simplified model where hydrodynamic properties depend only on proper time τ, the system's dynamics are not entirely trivial. This is because, in Milne coordinates, the metric (the spacetime metric tensor) is given by $g^{\mu\nu} = \text{diag}(1, -1, -1, -\tau^2)$, and it is no longer

coordinate-invariant. This non-trivial metric introduces complexities in the description of spacetime in this frame. Few physical quantities can be expressed in Milne coordinate as follows [152].

$$\begin{aligned}
\theta = \frac{1}{\tau},\ \pi^{\eta\eta} = -\frac{\pi}{\tau^2},\ \pi^{xx} = \pi^{yy} = \frac{\pi}{2} \\
\sigma^{\mu\nu} = \frac{1}{\tau}[g^{\mu}_{\eta} g^{\nu\eta} - \frac{1}{3}(g^{\mu\nu} - u^{\mu}u^{\nu})] \\
\sigma^{\eta\eta} = -\frac{2}{3\tau^3},\ \dot{\pi}^{\langle\eta\eta\rangle} = -\frac{1}{\tau^2}\frac{\partial\pi}{\partial\tau},\ \pi^{\angle\eta}_{\gamma}\sigma^{\eta\rangle\gamma} = -\frac{\pi}{3\tau^3} \\
-\tau_{\pi\pi}\pi^{\angle\eta}_{\gamma}\sigma^{\eta\rangle\gamma} = (\frac{\tau_{\pi\pi}}{3} + \delta_{\pi\pi})\frac{\pi}{\tau^3},\ \pi^{\alpha\beta}\sigma_{\alpha\beta} = \frac{\pi}{\tau}
\end{aligned} \tag{6.36}$$

Now we can derive other relations using the above parameters in the Milne coordinate.

- $$\begin{aligned}
T\tau_{\pi} = b = \text{constant} \implies \frac{dT}{d\tau}\tau_{\pi} + \frac{d\tau_{\pi}}{d\tau}T = 0 \\
\frac{dT}{d\tau} = -\frac{b}{\tau_{\pi}^2}(\frac{d\tau_{\pi}}{d\tau})
\end{aligned} \tag{6.37}$$

- $$\dot{\mathcal{E}} + (\mathcal{E} + \mathcal{P})\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0 \implies \frac{d\mathcal{E}}{d\tau} = -\frac{1}{\tau}(\mathcal{E} + \mathcal{P}) + \frac{\pi}{\tau} \tag{6.38}$$

$$\frac{d\mathcal{E}}{d\tau} = \frac{-1}{\tau}(\mathcal{E} + \mathcal{P} - \pi) \tag{6.39}$$

- $$\begin{aligned}
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = 2\beta_{\pi}\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + \underbrace{2\pi^{\langle\mu}_{\gamma}\omega^{\nu\rangle\gamma}}_{0} - \mathcal{C}\pi^{\langle\mu}_{\gamma}\sigma^{\nu\rangle\gamma} \\
\implies \frac{-1}{\tau^2}\frac{d\pi}{d\tau} + \frac{-\pi}{\tau_{\pi}\tau^2} = 2\beta_{\pi}(\frac{-2}{3\tau^3}) - \frac{4\ell + 10}{7}(\frac{-\pi}{3\tau^3}) + \frac{4}{3}\frac{\pi}{\tau^3} \\
\implies \frac{d\pi}{d\tau} = \frac{-\pi}{\tau_{\pi}} + \frac{1}{\tau}(\frac{4}{3}\beta_{\pi} - (\lambda + \frac{4}{3})\pi) \quad \text{where } \lambda = \frac{4\ell + 10}{21}
\end{aligned} \tag{6.40}$$