Some problems on nearly holomorphic modular forms

By

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

- *A converse theorem for quasimodular forms*, M. Charan, J. Meher, K. D. Shankhadhar and R. K. Singh, Forum Math. **34** (2022), no. 2, 547–564.

- *On adjoint of higher order Serre derivatives*, M. Charan, Res. Number Theory **9** (2023), no. 9, Paper No. 30, 8 pp.

Others

- *L-series of weakly holomorphic quasimodular forms and converse theorem*, M. Charan, submitted.

- *On nearly holomorphic Poincaré series*, M. Charan and J. Meher, submitted.

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DEDICATIONS

To my respected teacher

Anjan Kumar Chakrabarty

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ABSTRACT

In this thesis, we discuss four problems in the theory of modular forms. The first problem deals with the various properties of nearly holomorphic Poincaré series, such as the Fourier expansions, the holomorphic projections, etc. We also obtain some limiting properties of certain Fourier coefficients involving nearly holomorphic Poincaré series.

The second problem is about computing the adjoints of higher order Serre derivative maps. We give a formula for the adjoints of the higher order Serre derivative maps with respect to the Petersson inner product in terms of special values of certain shifted Dirichlet series attached to modular forms. As an application, we obtain some identities involving Fourier coefficients of some specific cusp forms and special values of certain shifted Dirichlet series.

The third problem is on the analytic properties of L-series associated to quasimodular forms. We obtain an analogue of Weil's converse theorem for quasimodular forms.

In the fourth problem, we define L-series associated to weakly holomorphic quasimodular forms of level N and study the analytic properties of these L-series twisted by Dirichlet characters. We also establish a converse theorem for weakly holomorphic quasimodular forms which is an analogue of Weil's converse theorem for weakly holomorphic quasimodular forms.

Contents

Summary

We discuss various properties of nearly holomorphic Poincaré series, such as the Fourier expansions, the holomorphic projections, etc. We also obtain some limiting properties of certain Fourier coefficients involving nearly holomorphic Poincaré series. We obtain adjoint maps of higher order Serre derivative maps. As an application, we obtain some identities involving Fourier coefficients of some specific cusp forms and special values of certain shifted Dirichlet series.

We study analytic properties of L-series associated to quasimodular forms. We also obtain a converse theorem for quasimodular forms. We discuss L-series associated to weakly holomorphic quasimodular forms and prove a converse theorem for weakly holomorphic quasimodular forms.

Notations

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ respectively the sets of natural numbers, integers, rational numbers, real numbers and complex numbers. We denote by $\mathbb{R}_+$ the set of all positive real numbers. We denote by $\mathbb{H}$ the complex upper half-plane. For $a, b \in \mathbb{Z}$ we write $a|b$ if b is divisible by a. For $a, b \in \mathbb{Z}$, the notation $a \pmod b$ means that a varies over a complete set of residue classes modulo b and the symbol (a, b) denotes the greatest common divisor of a and b. For $z \in \mathbb{C}$, we denote by $\mathrm{Re}(z)$ the real part of z and by $\mathrm{Im}(z)$ the imaginary part of z. We use $e(z) = e^{2\pi i z}$ with $i = \sqrt{-1}$. We also use $q := e^{2\pi i z}$. For a square matrix γ, we write $\det(\gamma)$ and $\mathrm{tr}(\gamma)$ respectively for the determinant and the trace of the matrix γ respectively. We denote by $C^\infty(\mathbb{H})$ the set of all real analytic functions on $\mathbb{H}$.

Chapter 1

Preliminaries

In this chapter we introduce basic definitions and some basic results of different automorphic forms.

1.1 Modular forms

The group

$$GL_2^+(\mathbb{Q}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Q}, ad - bc > 0 \right\}$$

acts on the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$, by the fractional linear transformation as follows. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$, the action of γ on z is defined by

$$\gamma z := \frac{az + b}{cz + d}. \tag{1.1}$$

For any integer k and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, the slash operator on a function $f : \mathbb{H} \to \mathbb{C}$ is defined by

$$(f|_k\gamma)(z) := (\det(\gamma))^{k/2}(cz + d)^{-k} f(\gamma z). \tag{1.2}$$

The full modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

For a positive integer N, we define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod{N} \right\}.$$

Definition 1.1.1 (Modular form). *Let k and N be positive integers. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. A modular form of weight k, level N and character χ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that*

(1) *for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,*

$$f|_k\gamma = \chi(d) f;$$

(2) *for all $\gamma \in SL_2(\mathbb{Z})$,*

$$f|_k\gamma(z) = \sum_{n=0}^{\infty} a_\gamma(n) q^{n/h},$$

where $h|N$.

Moreover, if the constant terms $a_\gamma(0)$ are zero for all $\gamma \in SL_2(\mathbb{Z})$, then f is said to be a cusp form. We denote the space of all modular forms and the subspace of all cusp forms of weight k, level N and character χ respectively, by $M_k(N, \chi)$ and $S_k(N, \chi)$. We simply write M_k and S_k if $N = 1$ and χ is trivial. The followings are some basic examples of modular forms on $SL_2(\mathbb{Z})$.

Example 1.1.2. *Let k be an even integer greater than 2. The normalized Eisenstein series E_k of weight k on $SL_2(\mathbb{Z})$ is defined by*

$$E_k(z) := \frac{1}{2} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus (0,0) \\ (m,n)=1}} \frac{1}{(mz+n)^k}.$$

It is well known that E_k is a modular form of weight k on $SL_2(\mathbb{Z})$ with Fourier expansion given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.3}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k's are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Remark 1.1.3. *If $k = 2$, the Eisenstein series E_2 is given by*

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n. \tag{1.4}$$

It is well known that E_2 is not a modular form. It satisfies the transformation property

$$(cz + d)^{-2} E_2 \left(\frac{az + b}{cz + d} \right) = E_2(z) + \frac{6}{\pi i} \frac{c}{cz + d} \tag{1.5}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$.

Example 1.1.4. *The Ramanujan delta function is defined by*

$$\Delta(z) := \frac{1}{1728} (E_4(z)^3 - E_6(z)^2).$$

It is a cusp form of weight 12 on $SL_2(\mathbb{Z})$ with Fourier expansion

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $\tau(n)$ is the Ramanujan tau function.

Definition 1.1.5 (Weakly holomorphic modular form). *Let k be an integer and let N be a positive integer. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. A weakly holomorphic modular form of weight k, level N and character χ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that*

(1) *for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,*

$$f|_k \gamma = \chi(d) f;$$

(2) *for all* $\gamma \in SL_2(\mathbb{Z})$,

$$f|_k\gamma(z) = \sum_{n\geq -n_0}^{\infty} a_\gamma(n)q^{n/h}$$

for some $n_0 \in \mathbb{N}$ *and* $h|N$.

Moreover, if $a_\gamma(0)$ are zero for all $\gamma \in SL_2(\mathbb{Z})$, then f is said to be a weakly holomorphic cusp form. We denote the space of all weakly holomorphic modular forms and the subspace of all weakly holomorphic cusp forms of weight k, level N and character χ respectively, by $M_k^!(N,\chi)$ and $S_k^!(N,\chi)$. For a positive integer N, the Fricke involution W_N is defined by

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

The following result is well known [23, Lemma 4.3.2].

Lemma 1.1.6. *If* $f \in M_k(N,\chi)$*, then* $f|_kW_N \in M_k(N,\overline{\chi})$ *and if* $f \in S_k(N,\chi)$*, then* $f|_kW_N \in S_k(N,\overline{\chi})$.

Definition 1.1.7 (Petersson inner product). *Let* $f, g \in M_k(N,\chi)$ *be such that at least one of them is a cusp form. Writing* $z = x+iy$*, the Petersson inner product of* f *and* g *is defined by*

$$\langle f, g\rangle := \frac{1}{\mu_\Gamma}\int\limits_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2}, \tag{1.6}$$

where $\Gamma_0(N)\setminus\mathbb{H}$ *is a fundamental domain,* $\frac{dxdy}{y^2}$ *is a invariant measure under the action of* $SL_2(\mathbb{Z})$ *on* $\mathbb{H}$ *and* μ_Γ *denotes the index of* $\Gamma_0(N)$ *in* $SL_2(\mathbb{Z})$.

It is well-known that $S_k(N,\chi)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product. The following familiar result tells about the growth of the Fourier coefficients of a modular form. The first statement can be easily obtained [10, Theorem 9.2.1] and the second is due to P. Deligne [11].

Proposition 1.1.8. *If* $f = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N,\chi)$*, then for any* $\epsilon > 0$*, we have* $a(n) \ll n^{k-1+\epsilon}$*. If* $f \in S_k(N,\chi)$*, then* $a(n) \ll n^{\frac{k-1}{2}+\epsilon}$.

1.1.1 Hecke operators

For any positive integer n, let

$$X_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{Z}_{\geq 0}, ad = n, 0 \leq b < d \right\}.$$

Definition 1.1.9. *For a positive integer k, the n-th Hecke operator T_n on a function $f : \mathbb{H} \to \mathbb{C}$ is defined by*

$$T_n f = n^{\frac{k}{2}-1} \sum_{\rho \in X_n} f|_k \rho.$$

The above expression means that for any function $f : \mathbb{H} \to \mathbb{C}$, we have

$$(T_n f)(z) = n^{\frac{k}{2}-1} \sum_{ad=n} \sum_{b \bmod d} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \leq b \leq d} f\left(\frac{az+b}{d}\right) \tag{1.7}$$

The following result is well known [10, Proposition 10.2.3].

Theorem 1.1.10. *Let n be a positive integer. If $f \in M_k$, then $T_n f \in M_k$. Also if $f \in S_k$, then $T_n f \in S_k$.*

1.1.2 Poincaré series

Beside Eisenstein series, a very important class of modular forms is constructed via the method of averaging. For any non-negative integer m, the m-th Poincaré series of weight k on $SL_2(\mathbb{Z})$ is defined by

$$P_{m,k}(z) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} e(mz)|_k \gamma, \tag{1.8}$$

where $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$ is the stabilizer of the cusp ∞ for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. It is well known that for $k \geq 4$, the series in (1.8) converges absolutely and uniformly on any compact subset of $\mathbb{H}$ and it is a modular form of weight k on $SL_2(\mathbb{Z})$. In particular, for $m = 0$, $P_{0,k}(z) = E_k(z)$. Also, for $m \geq 1$, $P_{m,k}(z)$ is a cusp form of weight k on $SL_2(\mathbb{Z})$. To state the Fourier series expansion of Poincaré series, we need the definitions of Kloosterman sum and J-Bessel function. For integers n, m, c, the Kloosterman sum is given by

$$K(n,m;c) = \sum_{\substack{r \bmod c \\ (r,c)=1}} e^{2\pi i(nr+mr^{-1})/c}, \tag{1.9}$$

where r^{-1} denotes the inverse of r modulo c and for a non-negative integer ℓ and a real number x, the J-Bessel function of index ℓ is given by

$$J_\ell(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{x}{2})^{2n+\ell}}{n!(\ell+n)!}. \tag{1.10}$$

We have the following theorem [10, Theorem 8.4.5.].

Theorem 1.1.11. *For any integer $m \geq 1$, the Fourier expansion of $P_{m,k}$ is given by,*

$$P_{m,k}(z) = \sum_{n=1}^{\infty} a_{m,k}(n)q^n,$$

where

$$a_{m,k}(n) = \delta_{n,m} + (-1)^{\frac{k}{2}} 2\pi \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} J_{k-1}\left(\frac{4\sqrt{nm}}{c}\right) \tag{1.11}$$

and $\delta_{n,m}$ is the Kronecker symbol.

The following theorem [10, Theorem 8.2.3.] is an important property of Poincaré series.

Theorem 1.1.12. *If $f(z) = \sum\limits_{n=1}^{\infty} a(n)q^n \in S_k$, then for any integer $m \geq 1$, we have*

$$\langle f, P_{m,k} \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a(m). \tag{1.12}$$

By the above theorem, for any $f \in S_k$, we can write

$$f(z) = \sum_{n=1}^{\infty} \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \langle f, P_{n,k} \rangle q^n. \tag{1.13}$$

Using the relation (1.13) and the non-degeneracy of the Petersson inner product, one obtains that the series $P_{m,k}$, $m \geq 1$ span the space S_k. The next result [10, Proposition 10.3.19] is about the action of Hecke operators on Poincaré series.

Proposition 1.1.13. *Let m, n be positive integers and let T_n be the n-th Hecke operator of weight k on $SL_2(\mathbb{Z})$. Then*

$$T_n P_{m,k} = \sum_{d|(m,n)} \left(\frac{n}{d}\right)^{k-1} P_{\frac{mn}{d^2},k}.$$

1.2 Nearly holomorphic modular forms

Definition 1.2.1 (Nearly holomorphic modular form). *Let k, N be positive integers and let p be a non-negative integer. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. A function $F : \mathbb{H} \to \mathbb{C}$ is called a nearly holomorphic modular form of weight k, depth p, level N and character χ if the following conditions hold.*

(1) *There exist holomorphic functions $f_0, f_1, \cdots, f_p$ (called the component functions of f) on $\mathbb{H}$ with $f_p \neq 0$ such that*

$$F(z) = \sum_{j=0}^{p} f_j (2iy)^{-j}.$$

(2) *For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $F|_k\gamma = \chi(d)F$.*

(3) *For each $j \in \{0, 1, \ldots, p\}$, there exists $\alpha_j > 0$ such that*

$$f_j(z) = O(((1 + |z|^2) / \operatorname{Im}(z))^{\alpha_j}) \ \text{ as } \operatorname{Im}(z) \to \infty \ \text{ and } \ \operatorname{Im}(z) \to 0.$$

In this case, we say that f_j is polynomially bounded.

We denote by $M^{\mathrm{nh}}_{k,p}(N,\chi)$ the set of all nearly holomorphic modular forms of weight k, depth p, level N and character χ and we denote by $M^{\mathrm{nh}}_{k,\leq p}(N,\chi)$ the space of all nearly holomorphic modular forms of weight k, depth $\leq p$, level N and character χ. We denote by $M^{\mathrm{nh}}_{k,p}$ the set of all nearly holomorphic modular forms of weight k, depth p on $SL_2(\mathbb{Z})$ (i.e. $N=1$ and χ is trivial) and we denote the space all nearly holomorphic modular forms of weight k, depth $\leq p$ on $SL_2(\mathbb{Z})$ by $M^{\mathrm{nh}}_{k,\leq p}$. We also denote by $M^{\mathrm{nh}}_{k} = \cup_p M^{\mathrm{nh}}_{k,\leq p}$ the space of all nearly holomorphic modular forms of weight k on $SL_2(\mathbb{Z})$.

Definition 1.2.2. *The Maass-Shimura operator R_k on $f \in M^{\mathrm{nh}}_k$ is defined by*

$$R_k f(z) = \frac{1}{2\pi i}\left(\frac{k}{2i\,\mathrm{Im}(z)} + \frac{\partial}{\partial z}\right) f(z).$$

The operator R_k takes M^{nh}_k into M^{nh}_{k+2}. Thus it is called Maass-raising operator. We write $R^m_k := R_{k+2m-2} \circ \cdots \circ R_{k+2} \circ R_k$ with R^0_k =id and $R^1_k = R_k$, where id is the identity map. The following lemma gives an explicit formula of R^m_k.

Lemma 1.2.3. *For $k, \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$ and $f \in C^\infty(\mathbb{H})$, we have*

$$R^m_k f = \frac{1}{(2\pi i)^m} \sum_{l=0}^{m} \binom{m}{l} \frac{(k+m-1)!}{(k+l-1)!} \frac{1}{(2iy)^{m-l}} \frac{\partial^l f}{\partial z^l}.$$

Proof. Using induction on m, one obtains the required formula. □

We state the following decomposition theorem of the space of nearly holomorphic modular forms [30, Lemma 7.8].

Theorem 1.2.4. *Let $k \geq 2$ be even. If $f \in M^{\mathrm{nh}}_{k,\leq p}$ and $p < k/2$, then*

$$M^{\mathrm{nh}}_{k,\leq p} = \bigoplus_{r=0}^{p} R^r_{k-2r} M_{k-2r},$$

and if $p \geq k/2$, then

$$M^{\mathrm{nh}}_{k,\leq p} = \bigoplus_{r=0}^{\frac{k}{2}-1} R^r_{k-2r} M_{k-2r} \oplus \mathbb{C} R_2^{\frac{k}{2}-1} E^*_2,$$

where $E_2^*(z) := E_2(z) - \frac{3}{\pi \operatorname{Im}(z)}$ *is a nearly holomorphic modular form of weight 2 and depth 1 on* $SL_2(\mathbb{Z})$.

Following Shimura [30, pp. 32], we define the slowly increasing and rapidly decreasing functions in M_k^{nh}. Shimura has defined slowly increasing and rapidly decreasing functions in a broader space than M_k^{nh}. Here we define those in M_k^{nh}.

Definition 1.2.5. *Let* $f \in M_k^{\mathrm{nh}}$. *Then* f *is called a*

- **slowly increasing function** *if for all* $\alpha \in SL_2(\mathbb{Q})$, *there exist positive constants* A, B *and* c *depending on* f *and* α *such that*

$$|\operatorname{Im}(\alpha z)^{k/2} f(\alpha z)| < Ay^c \text{ if } y = \operatorname{Im}(z) > B;$$

- **rapidly decreasing function** *if for all* $\alpha \in SL_2(\mathbb{Q})$ *and a positive real number* c, *there exist positive constants* A *and* B *depending on* f, α *and* c *such that*

$$|\operatorname{Im}(\alpha z)^{k/4} f(\alpha z)| < Ay^{-c} \text{ if } y = \operatorname{Im}(z) > B.$$

Remark 1.2.6. *If* $f \in M_k$, *then* f *is a slowly increasing function. In addition, if* $f \in S_k$ *then* f *is a rapidly decreasing function. From the above definitions we observe that the product of a rapidly decreasing function with any nearly holomorphic modular form results in a rapidly decreasing function.*

Let $S_{k,\leq p}^{\mathrm{nh}}$ be the subspace of $M_{k,\leq p}^{\mathrm{nh}}$ consisting of rapidly decreasing functions. Using the property of rapidly decreasing functions and Theorem 1.2.4, we obtain the following result.

Proposition 1.2.7. *Let* $k \geq 2$ *be even integer and let* p *be any non-negative integer. Then*

$$S_{k,\leq p}^{\mathrm{nh}} = \bigoplus_{r=0}^{p} R_{k-2r}^{r} S_{k-2r}.$$

Definition 1.2.8 (Nearly weakly holomorphic modular form). *Let k, N, p be integers with $N \geq 1$ and $p \geq 0$. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. A function $F : \mathbb{H} \to \mathbb{C}$ is called a nearly weakly holomorphic modular form of weight k, depth p, level N and character χ if the following conditions hold.*

(1) *There exist holomorphic functions $f_0, f_1, \cdots, f_p$ (called the component functions of f) on $\mathbb{H}$ with $f_p \neq 0$ such that*

$$F(z) = \sum_{j=0}^{p} f_j (2iy)^{-j}.$$

(2) *For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $F|_k \gamma = \chi(d) F$.*

(3) *For each $j \in \{0, 1, \ldots, p\}$, there exist positive constants c_j, ϵ_j and a polynomial $P_j(z) \in \mathbb{C}[e^{-2\pi i z}]$ such that $f_j(z) = O(e^{c_j/y})$ as $y \to 0$ and $f_j(z) - P_j(z) = O(e^{-\epsilon_j y})$ as $y \to \infty$.*

The set of nearly weakly holomorphic modular forms of weight k, depth p, level N and character χ is denoted by $M_{k,p}^{\mathrm{nh},!}(N, \chi)$. If χ is the trivial character, then we denote the corresponding set by $M_{k,p}^{\mathrm{nh},!}(N)$.

1.3 Quasimodular forms

The Eisenstein series E_2 and the derivatives of modular forms are not modular forms. But they play important roles in the theory of modular forms. They are quasimodular forms. In 1995, M. Kaneko and D. Zagier [16] introduced the notion of a quasimodular form.

Definition 1.3.1 (Quasimodular form). *Let k, N be positive integers and let p be a non-negative integer. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) =$*

$(-1)^k$. *A holomorphic function* $f : \mathbb{H} \to \mathbb{C}$ *is called a quasimodular form of weight* k*, depth* p*, level* N *and character* χ *if the following conditions hold.*

(1) *There exist holomorphic functions* $f_0, f_1, \cdots, f_p$ *(called the component functions of* f*) on* $\mathbb{H}$ *with* $f_p \neq 0$ *such that for any* $\gamma \in \Gamma_0(N)$*, we have*

$$f|_k\gamma\,(z) = \chi(\gamma)\sum_{j=0}^{p} f_j(z)\left(\frac{c}{cz+d}\right)^j. \tag{1.14}$$

(2) *For each* $j \in \{0, 1, \ldots, p\}$*,* f_j *is polynomially bounded.*

The set of all holomorphic quasimodular forms of weight k, depth p, level N and character χ is denoted by $M^{\mathrm{qm}}_{k,p}(N, \chi)$. If χ is the trivial character, then we denote the corresponding set by $M^{\mathrm{qm}}_{k,p}(N)$.

Definition 1.3.2 (Weakly holomorphic quasimodular form). *Let* k, N, p *be integers with* $N \geq 1$ *and* $p \geq 0$*. Let* χ *be a Dirichlet character modulo* N *satisfying* $\chi(-1) = (-1)^k$*. A holomorphic function* $f : \mathbb{H} \to \mathbb{C}$ *is called a* weakly holomorphic quasimodular form *of weight* k*, depth* p*, level* N *and character* χ *if the following conditions hold.*

(1) *There exist holomorphic functions* $f_0, \cdots, f_p$ *(called the* component functions *of* f*) on* $\mathbb{H}$ *with* $f_p \neq 0$ *such that for any* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*, we have*

$$f|_k\gamma\,(z) = \chi(\gamma)\sum_{j=0}^{p} f_j(z)\left(\frac{c}{cz+d}\right)^j.$$

(2) *For each* $j \in \{0, 1, \ldots, p\}$*, there exist positive constants* c_j*,* ϵ_j *and a polynomial* $P_j(z) \in \mathbb{C}[e^{-2\pi i z}]$ *such that* $f_j(z) = O(e^{c_j/y})$ *as* $y \to 0$ *and* $f_j(z) - P_j(z) = O(e^{-\epsilon_j y})$ *as* $y \to \infty$*.*

The set of all weakly holomorphic quasimodular forms of weight k, depth p, level N and character χ is denoted by $M_{k,p}^{\mathrm{qm},!}(N,\chi)$. If χ is the trivial character, then we denote the corresponding set by $M_{k,p}^{\mathrm{qm},!}(N)$.

Chapter 2

Nearly Holomorphic Poincaré Series

2.1 Introduction

Poincaré series give a large class of cusp forms and they form a basis of the vector space of all cusp forms of fixed weight on $SL_2(\mathbb{Z})$. Also they have many applications in the theory of automorphic forms.

In this chapter, we study nearly holomorphic Poincaré series on the full modular group. More precisely, we discuss the Fourier expansions, holomorphic projections etc. of nearly holomorphic Poincaré series. We also discuss some limiting properties of certain Fourier coefficients involving nearly holomorphic Poincaré series. The results of this chapter are contained in [7].

2.1.1 Nearly holomorphic Poincaré series

Let $k \geq 4$ be an even integer and let m and p be integers with $m \geq 1$ and $0 \leq p < k/2 - 1$. The m-th nearly holomorphic Poincaré series of weight k and index p on $SL_2(\mathbb{Z})$ is defined by

$$P_{m,k}^{p}(z) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} (y^{-p} e(mz))|_k \gamma. \tag{2.1}$$

Theorem 2.1.1. *The Poincaré series $P_{m,k}^{p}$ is a nearly holomorphic modular form of weight k and depth $\leq p$ on $SL_2(\mathbb{Z})$.*

Proof. Since for $m \geq 1$ and $\gamma \in SL_2(\mathbb{Z})$,

$$\left|\frac{y^{-p}|j(\gamma,z)|^{2p}}{j(\gamma,z)^k}e(m\gamma z)\right| \leq \frac{y^{-p}}{|j(\gamma,z)|^{k-2p}},$$

the series in the right hand side of (2.1) is absolutely convergent for all $k \geq 4$ and $0 \leq p < k/2$. Now we show that the Poincaré series $P^p_{m,k}$ is a polynomial in $1/y$ of degree $\leq p$ whose coefficients are holomorphic functions on $\mathbb{H}$. The Maass lowering operator L on a function $f : \mathbb{H} \to \mathbb{C}$ is defined by

$$L = -2iy^2\frac{\partial}{\partial \bar{z}}.$$

With respect to the slash operator (1.2), L satisfies the intertwining property

$$L(f|_k\gamma) = (Lf)|_{k-2}\gamma \tag{2.2}$$

for any $k \in \mathbb{Z}$ and $\gamma \in SL_2(\mathbb{R})$. It is easily observed that a smooth function f on $\mathbb{H}$ is a polynomial in $1/y$ of degree $\leq p$ whose coefficients are holomorphic functions on $\mathbb{H}$ (nearly holomorphic) if and only if $L^{p+1}f = 0$. Since L satisfies the intertwining property (2.2) and $L^{p+1}(y^{-p}e(mz)) = 0$, the Poincaré series $P^p_{m,k}$ is a polynomial in $1/y$ of degree $\leq p$ whose coefficients are holomorphic functions on $\mathbb{H}$. Also the Poincaré series $P^p_{m,k}$ satisfies the modularity relation, i.e. $P^p_{m,k}|_k\gamma = P^p_{m,k}$ for all $\gamma \in SL_2(\mathbb{Z})$. Therefore $P^p_{m,k}$ is a nearly holomorphic modular form of weight k and depth $\leq p$ on $SL_2(\mathbb{Z})$. □

The following result [7, Proposition 3.1] gives a relation between holomorphic and nearly holomorphic Poincaré series via Maass-Shimura operator.

Proposition 2.1.2. *For integers m, k, p with $m \geq 1$, $k \geq 4$ even and $p \geq 0$, we have*

$$R^p_k P_{m,k}(z) = \frac{1}{(-4\pi)^p}\sum_{r=0}^{p}\binom{p}{r}\frac{\Gamma(k+p)}{\Gamma(k+r)}(-4\pi m)^r P^{p-r}_{m,k+2p}(z). \tag{2.3}$$

Proof. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, let $\alpha(\gamma, z) = \frac{c}{cz+d}$.
Then we have

$$|j(\gamma, z)|^2 = j(\gamma, z)^2(1 - 2iy\alpha(\gamma, z)). \tag{2.4}$$

Now, by using (2.4), we have

$$\frac{y^{-p}|j(\gamma, z)|^{2p}}{j(\gamma, z)^k} = j(\gamma, z)^{2p-k}y^{-p}(1 - 2iy\alpha(\gamma, z))^p.$$

Simplifying the last term of the above equality, we get

$$\frac{y^{-p}|j(\gamma, z)|^{2p}}{j(\gamma, z)^k} = j(\gamma, z)^{2p-k}(2i)^p\left(\frac{1}{2iy} - \alpha(\gamma, z)\right)^p. \tag{2.5}$$

For any non-negative integer l and any holomorphic function f on $\mathbb{H}$, repeatedly differentiating the identity

$$f|_k\gamma(z) = j(\gamma, z)^{-k}f(\gamma z)$$

yields

$$\frac{d^l}{dz^l}(f|_k\gamma(z)) = \sum_{r=0}^{l}(1)^{l-r}\frac{l!}{r!}\binom{k+l-1}{l-r}\frac{\alpha(\gamma, z)^{l-r}}{j(\gamma, z)^{k+2r}}f^{(r)}(\gamma z) \tag{2.6}$$

for all $\gamma \in SL_2(\mathbb{Z})$. Now using Lemma 1.2.3, we get

$$\begin{aligned} R_k^p P_{m,k}(z) &= \frac{1}{(2\pi i)^p}\sum_{l=0}^{p}\binom{p}{l}\frac{\Gamma(k+p)}{\Gamma(k+l)}\frac{1}{(2iy)^{p-l}}\frac{d^l}{dz^l}\left(\sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} e(mz)|_k\gamma\right) \\ &= \frac{1}{(2\pi i)^p}\sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})}\sum_{l=0}^{p}\binom{p}{l}\frac{\Gamma(k+p)}{\Gamma(k+l)}\frac{1}{(2iy)^{p-l}}\frac{d^l}{dz^l}\Big(e(mz)|_k\gamma\Big). \end{aligned}$$

Using (2.6) in the above expression, we obtain

$$\begin{aligned} R_k^p P_{m,k}(z) = \frac{1}{(2\pi i)^p}\sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})}\sum_{l=0}^{p}\sum_{r=0}^{l}&\frac{(-1)^{l-r}p!}{(p-l)!r!(l-r)!}\frac{(k+p-1)!}{(k+r-1)!}\frac{(2\pi im)^r\alpha(\gamma, z)^{l-r}}{j(\gamma, z)^{k+2r}} \\ &\times\frac{1}{(2iy)^{p-l}}e(m\gamma z). \end{aligned}$$

Simplifying the last two sums of the right hand side of the above expression, we obtain

$$\begin{aligned} R_k^p P_{m,k}(z) = \frac{1}{(2\pi i)^p} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} e(m\gamma z) \sum_{r=0}^{p} \frac{p!(k+p-1)!(2\pi i m)^r}{r!(k+r-1)! j(\gamma, z)^{k+2r}} \\ \times \sum_{l=r}^{p} \frac{(1)^{l-r} \alpha(\gamma, z)^{l-r}}{(p-l)!(l-r)!} \frac{1}{(2iy)^{p-l}}. \end{aligned}$$

Replacing l by $l+r$ in the last sum of the above expression, we deduce that

$$\begin{aligned} R_k^p P_{m,k}(z) &= \frac{1}{(2\pi i)^p} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} e(m\gamma z) \sum_{r=0}^{p} \frac{p!\Gamma(k+p)(2\pi i m)^r}{r!\Gamma(k+r) j(\gamma, z)^{k+2r}} \\ &\quad \times \sum_{l=0}^{p-r} (-1)^l \frac{\alpha(\gamma, z)^l}{l!(p-r-l)!} \frac{1}{(2iy)^{p-r-l}} \\ &= \frac{1}{(2\pi i)^p} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma(k+p)}{\Gamma(k+r)} (2\pi i m)^r \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \left(\frac{1}{2iy} - \alpha(\gamma, z) \right)^{p-r} \\ &\quad \times \frac{e(m\gamma z)}{j(\gamma, z)^{k+2p-2(p-r)}}. \end{aligned}$$

By using (2.5) in the above equality, we obtain

$$\begin{aligned} R_k^p P_{m,k}(z) &= \frac{1}{(2\pi i)^p} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma(k+p)}{\Gamma(k+r)} \frac{(2\pi i m)^r}{(2i)^{p-r}} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \frac{y^{-(p-r)} |j(\gamma, z)|^{2(p-r)} e(m\gamma z)}{j(\gamma, z)^{k-2p}} \\ &= \frac{1}{(-4\pi)^p} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma(k+p)}{\Gamma(k+r)} (-4\pi m)^r \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} (y^{-(p-r)} e(mz))|_{k+2p}\gamma \\ &= \frac{1}{(-4\pi)^p} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma(k+p)}{\Gamma(k+r)} (-4\pi m)^r P_{m,k+2p}^{p-r}(z). \end{aligned}$$

□

Replacing k by $k-2p$ in (2.3), we obtain

$$R_{k-2p}^p P_{m,k-2p}(z) = \frac{1}{(-4\pi)^p} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma(k-p)}{\Gamma(k-2p+r)} (-4\pi m)^r P_{m,k}^{p-r}(z) \tag{2.7}$$

for $0 \le p < \frac{k}{2} - 1$. Solving the system of linear equations (2.7), we obtain $P_{m,k}^p$ as a linear combination of $R_{k-2r}^r P_{m,k-2r}$, $0 \le r \le p$. Therefore from Proposition 1.2.7, we get $P_{m,k}^p \in S_{k,\le p}^{\mathrm{nh}}$ for $0 \le p < \frac{k}{2} - 1$. As an application of Proposition 2.1.2, we have the following result [7, Proposition 3.2].

Proposition 2.1.3. *Suppose that $k \geq 4$ even and $0 \leq p < \frac{k}{2} - 1$. Then the space $S^{\mathrm{nh}}_{k,\leq p}$ is spanned by the Poincaré series $P^r_{m,k}$ $(m \geq 1,\ r = 0, \ldots, p)$.*

Proof. It is well known that the holomorphic Poincaré series $P_{m,k}$ $(m \geq 1)$ span the space S_k. Now the result follows from Proposition 2.1.2 and Proposition 1.2.7. □

2.2 Fourier expansions

In this subsection we compute the Fourier expansion of nearly holomorphic Poincaré series. To establish the Fourier expansion of $P^p_{m,k}$, we need the following lemmas.

Lemma 2.2.1. *If $y > 0$, $k \geq 2$, p and m are integers with $0 \leq p < k/2 - 1$, then we have*

$$\int_{-\infty}^{\infty} \frac{|x+iy|^{2p}}{(x+iy)^k} e^{-2\pi i m x} dx = \begin{cases} 0 & \text{if } m \leq 0, \\ \sum\limits_{l=0}^{p} \binom{p}{l} (-4\pi y)^{p-l} \frac{(-2\pi i)^{k-2p}}{(k-p-l-1)!} m^{k-p-l-1} e^{-2\pi m y} & \text{if } m > 0. \end{cases}$$

To prove the above lemma we recall the following result from [10, Corollary 3.1.19].

Lemma 2.2.2. *Let y be a positive real number and let k, m be integers with $k \geq 2$. Then, we have*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(x+iy)^k} dx = \begin{cases} 0 & \text{if } m \leq 0, \\ \frac{(-2\pi i)^k}{(k-1)!} m^{k-1} e^{-2\pi m y} & \text{if } m > 0. \end{cases}$$

Proof of Lemma 2.2.1. We have

$$\int_{-\infty}^{\infty} \frac{|x+iy|^{2p}}{(x+iy)^k} e^{-2\pi i m x} dx = \int_{-\infty}^{\infty} \frac{(x-iy)^p}{(x+iy)^{k-p}} e^{-2\pi i m x} dx = \int_{-\infty}^{\infty} \frac{(x+iy-2iy)^p}{(x+iy)^{k-p}} e^{-2\pi i m x} dx.$$

Expanding $(x+iy-2iy)^p$, we obtain

$$\int_{-\infty}^{\infty} \frac{|x+iy|^{2p}}{(x+iy)^k} e^{-2\pi i m x} dx = \sum_{l=0}^{p} \binom{p}{l} (-2iy)^{p-l} \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(x+iy)^{k-p-l}} dx$$

Now by using Lemma 2.2.2, we get the result. □

Lemma 2.2.3. *Let A, B and y be real numbers with $A \geq 0$ and $y > 0$. Let k, p be integers such that $k \geq 2$, $0 \leq p < k/2 - 1$. Set*

$$I_k(A, B; y) = \int_{-\infty}^{\infty} \frac{|x+iy|^{2p}}{(x+iy)^k} e^{-2\pi i(A/(x+iy)+B(x+iy))} dx.$$

Then we have

$$I_k(A, B; y) = \begin{cases} i^{-(k-2p)} 2\pi (B/A)^{\frac{k-2p-1}{2}} & \\ \times \sum_{l=0}^{p} \binom{p}{l} (-2y\sqrt{B/A})^{p-l} J_{k-p-l-1}(4\pi\sqrt{AB}) & \text{if } A > 0 \text{ and } B > 0, \\ (-2\pi i)^{k-2p} B^{k-2p-1} & \\ \times \sum_{l=0}^{p} \binom{p}{l} (-4\pi y B)^{p-l}/(k-p-l-1)! & \text{if } A = 0 \text{ and } B > 0, \\ 0 & \text{if } B \leq 0. \end{cases}$$

Proof. Using the series expansion of $e^{-2\pi i A/(x+iy)}$, we get

$$I_k(A, B; y) = \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(-2\pi i A)^j}{j!} \frac{|x+iy|^{2p}}{(x+iy)^{k+j}} e^{-2\pi i B(x+iy)} dx. \tag{2.8}$$

By the uniform convergence, we get

$$I_k(A, B; y) = \sum_{j=0}^{\infty} \frac{(-2\pi i A)^j}{j!} \int_{-\infty}^{\infty} \frac{|x+iy|^{2p}}{(x+iy)^{k+j}} e^{-2\pi i B(x+iy)} dx. \tag{2.9}$$

By Lemma 2.2.1, we have

$$\int_{-\infty}^{\infty} \frac{|x+iy|^{2p}}{(x+iy)^{k+j}} e^{-2\pi i B(x+iy)} dx = \begin{cases} 0 & \text{if } B \leq 0, \\ \sum_{l=0}^{p} \binom{p}{l} (-4\pi y)^{p-l} \frac{(-2\pi i)^{k+j-2p}}{(k+j-p-l-1)!} B^{k+j-p-l-1} & \text{if } B > 0. \end{cases} \tag{2.10}$$

Using (2.10) in (2.9), we get $I_k(A, B; y) = 0$ if $B \leq 0$ and

$$I_k(A, B; y) = \sum_{l=0}^{p} \binom{p}{l} (-4\pi y)^{p-l} (-2\pi i)^{k-2p} B^{k-p-l-1} \sum_{j=0}^{\infty} \frac{(-4\pi^2 AB)^j}{j!(k-p-l+j-1)!} \quad \text{if } B > 0. \tag{2.11}$$

Therefore for $A = 0$ and $B > 0$, we get

$$I_k(A, B; y) = (-2\pi i)^{k-2p} B^{k-2p-1} \sum_{l=0}^{p} \binom{p}{l} \frac{(-4\pi y B)^{p-l}}{(k-p-l-1)!}.$$

For $A > 0$ and $B > 0$, using (1.10) in (2.11), we obtain

$$I_k(A, B; y) = \sum_{l=0}^{p} \binom{p}{l} (-4\pi y)^{p-l} (-2\pi i)^{k-2p} B^{k-p-l-1} (2\pi (AB)^{1/2})^{1-k+p+l} J_{k-p-l-1}(4\pi (AB)^{1/2}).$$

$\square$

We have the following result [7, Theorem 1.4], which is analogous to Theorem 1.1.11 for nearly holomorphic Poincaré series.

Theorem 2.2.4. *For integers $m \geq 1$ and $0 \leq p < \frac{k}{2} - 1$, we have the Fourier expansion*

$$P^p_{m,k}(z) = \sum_{n=1}^{\infty} a^p_{m,k}(n,y) q^n,$$

where

$$a^p_{m,k}(n,y) = y^{-p}\left[\delta_{n,m} + (-1)^{\frac{k-2p}{2}} 2\pi \left(\frac{n}{m}\right)^{\frac{k-2p-1}{2}} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} \right.$$
$$\left. \sum_{l=0}^{p} \binom{p}{l} \left(-2yc\sqrt{\frac{n}{m}}\right)^{p-l} J_{k-p-l-1}\left(\frac{4\sqrt{nm}}{c}\right)\right], \quad (2.12)$$

where $K(n,m;c)$ is the Kloosterman sum defined in (1.9)*, $J_{k-p-l-1}$ is the J-Bessel function defined in* (1.10)*and $\delta_{n,m}$ is the Kronecker symbol.*

Proof. Using the definition of nearly holomorphic Poincaré series (2.1) and the bijection between $\Gamma_\infty \backslash \Gamma$ and $\{(c,d) \in \mathbb{Z}^2 \colon c \geq 0, (c,d) = 1 \text{ and } d = 1 \text{ if } c = 0\}$, we can write

$$P^p_{m,k}(z) = y^{-p} e^{2\pi i m z} + \sum_{\substack{c,d \in \mathbb{Z} \\ c \geq 1,\ (c,\ d)=1}} \frac{y^{-p}|cz+d|^{2p}}{(cz+d)^k} e\left(m\frac{az+b}{cz+d}\right). \quad (2.13)$$

So, we have

$$P^p_{m,k}(z) = y^{-p} e^{2\pi i m z} + y^{-p} \sum_{c=1}^{\infty} \sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1}} |cz+d|^{2p} (cz+d)^{-k} e^{2\pi i m (az+b)/(cz+d)} \quad (2.14)$$

Putting $d = r + nc$ when $c \neq 0$ in the above equation and noting that if $a_0 r - b_0 c = 1$, then $a_0d-(b_0+na_0)c=1$, we obtain

$$\begin{aligned}P^p_{m,k}(z) &= y^{-p}e^{2\pi imz}+y^{-p}\sum_{c=1}^{\infty}\sum_{\substack{r \bmod c\\(r,c)=1}}\sum_{n\in\mathbb{Z}}|cz+r+nc|^{2p}(cz+r+nc)^{-k}e^{2\pi im(az+b)/(cz+r+nc)}\\ &= y^{-p}e^{2\pi imz}+y^{-p}\sum_{c=1}^{\infty}c^{2p-k}\sum_{\substack{r \bmod c\\(r,c)=1}}\sum_{n\in\mathbb{Z}}|z+r/c+n|^{2p}(z+r/c+n)^{-k}\\ &\qquad\qquad \times e^{2\pi i(m/c)(a_0(z+n)+(a_0r-1)/c)/(z+r/c+n)}.\end{aligned} \tag{2.15}$$

Putting

$$S(c,r)=\sum_{n\in\mathbb{Z}}|z+r/c+n|^{2p}(z+r/c+n)^{-k}e^{2\pi i(m/c)(r^{-1}(z+n)+(r^{-1}r-1)/c)/(z+r/c+n)},$$

where $r^{-1}(=a_0)$ denotes an inverse of r modulo c, (2.15) can be written as:

$$P^p_{m,k}(z)=y^{-p}e^{2\pi imz}+y^{-p}\sum_{c=1}^{\infty}c^{2p-k}\sum_{\substack{r \bmod c\\(r,c)=1}}S(c,r). \tag{2.16}$$

For fixed y, we set $f(x)=|x+iy|^{2p}(x+iy)^{-k}e^{-2\pi iA/(x+iy)}$ with $A=m/c^2$. It is clear that $S_{c,r}=e^{2\pi i(m/c)r^{-1}}\sum_{n\in\mathbb{Z}}f(x+r/c+n)$. Now by Poisson summation formula, we get

$$e^{-2\pi i(m/c)r^{-1}}S(c,r)=\sum_{n\in\mathbb{Z}}f(x+r/c+n)=\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi in(x+r/c)}=\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi inr/c}e^{2\pi inx}, \tag{2.17}$$

where

$$\hat{f}(n)=\int_{-\infty}^{\infty}\frac{|x+iy|^{2p}}{(x+iy)^k}e^{-2\pi i(A/(x+iy)+nx)}dx$$

is the Fourier transform of f evaluated at n. By Lemma 2.2.3, we have

$$\hat{f}(n)=\begin{cases}0 & \text{if } n=0,\\ e^{-2\pi ny}i^{-(k-2p)}2\pi((n/m)c^2)^{\frac{k-2p-1}{2}} & \\ \times\sum_{l=0}^{p}\binom{p}{l}\left(-2yc\sqrt{\frac{n}{m}}\right)^{p-l}J_{k-p-l-1}\left(4\pi\frac{\sqrt{mn}}{c}\right) & \text{if } n>0, m>0.\end{cases} \tag{2.18}$$

Using (2.18) in (2.17), we obtain

$$\begin{aligned}S(c,r) &= e^{2\pi i(m/c)r^{-1}}\sum_{n=1}^{\infty}e^{2\pi inr/c}e^{2\pi inx}e^{-2\pi ny}i^{-(k-2p)}2\pi((n/m)c^2)^{\frac{k-2p-1}{2}}\\ &\qquad\times\sum_{l=0}^{p}\binom{p}{l}\left(-2yc\sqrt{\frac{n}{m}}\right)^{p-l}J_{k-p-l-1}\left(4\pi\frac{\sqrt{mn}}{c}\right).\end{aligned}$$

Hence

$$\begin{aligned}
P^p_{m,k}(z) &= y^{-p}e^{2\pi imz} + y^{-p}\sum_{c=1}^{\infty} c^{2p-k} \sum_{\substack{r \bmod c\\ (r,c)=1}} e^{2\pi i(m/c)r^{-1}} \sum_{n=1}^{\infty} e^{2\pi inr/c}e^{2\pi inx}e^{-2\pi ny}i^{-(k-2p)} \\
&\qquad \times 2\pi((n/m)c^2)^{\frac{k-2p-1}{2}} \sum_{l=0}^{p}\binom{p}{l}\left(-2yc\sqrt{\frac{n}{m}}\right)^{p-l} J_{k-p-l-1}\left(4\pi\frac{\sqrt{mn}}{c}\right) \\
&= y^{-p}e^{2\pi imz} + y^{-p}(-1)^{\frac{k-2p}{2}}2\pi\sum_{n=1}^{\infty}\left(\frac{n}{m}\right)^{\frac{k-2p-1}{2}} e^{2\pi nz}\sum_{c=1}^{\infty}c^{-1}\sum_{l=0}^{p}\binom{p}{l}\left(-2yc\sqrt{\frac{n}{m}}\right)^{p-l} \\
&\qquad \times J_{k-p-l-1}\left(4\pi\frac{\sqrt{mn}}{c}\right)\sum_{\substack{r \bmod c\\ (r,c)=1}} e^{2\pi i(nr+mr^{-1})/c} \\
&= y^{-p}q^m + y^{-p}(-1)^{\frac{k-2p}{2}}2\pi\sum_{n=1}^{\infty}\left[\left(\frac{n}{m}\right)^{\frac{k-2p-1}{2}}\sum_{c=1}^{\infty}\frac{K(n,m;c)}{c}\right. \\
&\qquad \left.\sum_{l=0}^{p}\binom{p}{l}\left(-2yc\sqrt{\frac{n}{m}}\right)^{p-l} J_{k-p-l-1}\left(4\pi\frac{\sqrt{mn}}{c}\right)\right]q^n.
\end{aligned}$$

□

2.2.1 Hecke operators and Petersson inner product

For $f \in M_k^{\mathrm{nh}}$ the action of the n-th Hecke operator on f is defined by

$$(T_nf)(z) = n^{\frac{k}{2}-1}\sum_{\rho\in Xn} f|_k\rho, \tag{2.19}$$

where

$$X_n = \left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} \mid a,b,d \in \mathbb{Z}_{\geq 0}, ad = n, 0 \leq b < d\right\}. \tag{2.20}$$

For each integer $n \geq 1$, T_n maps M_k^{nh} to M_k^{nh} and S_k^{nh} to S_k^{nh}. We have the following result [7, Theorem 1.10], which is analogous to Proposition 1.1.13 for nearly holomorphic Poincaré series.

Theorem 2.2.5. *Let m,n,p be integers with $m,n \geq 1$ and $0 \leq p \leq \frac{k}{2}$ and let T_n be the n-th Hecke operator. Then*

$$T_nP^p_{m,k} = \sum_{d|(m,n)}\left(\frac{n}{d}\right)^{k-p-1} d^pP^p_{\frac{mn}{d^2},k}.$$

Proof. From (2.19) we have

$$(T_n P^p_{m,k})(z) = n^{\frac{k}{2}-1} \sum_{\rho \in Xn} \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (y^{-p} e(mz))|_k \gamma \right) |_k \rho.$$

Using a similar idea as in the proof of [10, Theorem 4.4.4.], we get

$$(T_n P^p_{m,k})(z) = n^{\frac{k}{2}-1} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \left(\sum_{\rho \in Xn} (y^{-p} e(mz))|_k \rho \right) |_k \gamma.$$

Now by (2.20), we get

$$T_n P^p_{m,k}(z) = n^{\frac{k}{2}-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(\sum_{ad=n} \sum_{b \pmod d} (y^{-p} e(mz))|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) |_k \gamma.$$

Simplifying the term in the right hand side of the above identity, we get

$$\begin{aligned} T_n P^p_{m,k}(z) &= n^{k-p-1} \sum_{\substack{ad=n \\ d|m}} d^{2p-k+1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(y^{-p} e\left(\frac{mn}{d^2} z \right) \right) |_k \gamma \\ &= \sum_{d|(m,n)} \left(\frac{n}{d} \right)^{k-p-1} d^p P^p_{\frac{mn}{d^2},k}. \end{aligned}$$

□

If f and g are two nearly holomorphic cusp forms of weight k on $SL_2(\mathbb{Z})$, then the Petersson inner product of f and g is defined by

$$\langle f, g \rangle := \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} f(z) \overline{g(z)} y^k \frac{dxdy}{y^2}.$$

We have the following result [7, Theorem 1.7], which is analogous to Theorem 1.1.12 for nearly holomorphic Poincaré series.

Theorem 2.2.6. *Let m, p and q be integers such that $m \geq 1$ and $0 \leq p, q < \frac{k}{2} - 1$. If $f \in M^{\mathrm{nh}}_{k,q}$ is a rapidly decreasing function, then we have*

$$\langle f, P^p_{m,k} \rangle = \sum_{l=0}^{q} \frac{\Gamma(k-p-l-1)}{(4\pi m)^{k-p-l-1}} a_l(m),$$

where $a_l(m)$ is the m-th Fourier coefficient of the l-th component of f.

Proof. We have

$$\langle f, P^p_{m,k}\rangle = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(z) \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} y^{-p}|j_\gamma(z)|^{2p}\overline{j_\gamma(z)^{-k}}\ \overline{e(m\gamma z)}\frac{dxdy}{y^2}.$$

Interchanging the sum and the integral and using the identity

$$\operatorname{Im}(\gamma z) = \frac{\operatorname{Im}(z)}{|j_\gamma(z)|^2},$$

we obtain

$$\langle f, P^p_{m,k}\rangle = \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} \operatorname{Im}(\gamma z)^{k-p} f(\gamma z)\overline{e(m\gamma z)}\frac{dxdy}{y^2}.$$

By the change of variable $z \mapsto \gamma^{-1}z$ in the above expression and using the Rankin's unfolding argument, we obtain

$$\begin{aligned}\langle f, P^p_{m,k}\rangle &= \int_{\Gamma_\infty\backslash\mathbb{H}} y^{k-p} f(z)\overline{e(mz)}\frac{dxdy}{y^2}\\ &= \int_0^\infty\int_0^1 y^{k-p-2} f(x+iy)e^{-2\pi imx}e^{-2\pi my}dxdy.\end{aligned}$$

Putting $f(z) = \sum\limits_{l=0}^{q}\left(\sum\limits_{n=1}^{\infty} a_l(n)e(nz)\right)y^{-l}$ in the above expression, we obtain

$$\begin{aligned}\langle f, P^p_{m,k}\rangle &= \sum_{l=0}^{q}\sum_{n=1}^{\infty} a_l(n)\int_0^\infty\int_0^1 y^{k-p-l-2}e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}dxdy\\ &= \sum_{l=0}^{q} a_l(m)\int_0^\infty y^{k-p-l-2}e^{-2\pi(n+m)y}dxdy\\ &= \sum_{l=0}^{q}\frac{\Gamma(k-p-l-1)}{(4\pi m)^{k-p-l-1}}a_l(m).\end{aligned}$$

This proves the result. ☐

Remark 2.2.7. *From the above theorem, we get the system of linear equations*

$$\langle f, P^p_{m,k}\rangle = \sum_{l=0}^{q}\frac{\Gamma(k-p-l-1)}{(4\pi m)^{k-p-l-1}}a_l(m) \text{ for } p = 0,1,\ldots,q. \tag{2.21}$$

After solving the above system of linear equations, we can get a result like (1.13)*. But it is very hard to solve the system of linear equations* (2.21) *for arbitrary non-negative* *integer q. For $q = 0$ we get (1.13) and for $q = 1$ we get*

$$f(z) = \sum_{n=1}^{\infty}\sum_{t=0}^{1}\sum_{l=0}^{1} \frac{(-1)^{t+l}\Gamma(k+t+l-3)y^{-t}}{\Gamma(k-2)\Gamma(k-3)} \langle f, P_{n,k}^{l}\rangle q^n \tag{2.22}$$

for all weight $k \geq 4$, depth 1 rapidly decreasing nearly holomorphic forms f on $SL_2(\mathbb{Z})$.

As a consequence of the above theorem, we obtain the following result [7, Corollary 1.8].

Corollary 2.2.8. *For integers m, n, p, q with $m, n \geq 1$ and $0 \leq p, q < \frac{k}{2} - 1$, we have*

$$\langle P_{m,k}^{p}, P_{n,k}^{q}\rangle = \frac{\Gamma(k-p-q-1)(4\pi)^{p+q}m^p n^q}{(4\pi\sqrt{mn})^{k-1}} \Bigg[\delta_{m,n} + (-1)^{\frac{k-2p}{2}} 2\pi \sum_{c=1}^{\infty} \frac{K(m,n;c)}{c} \sum_{l=0}^{p} \binom{p}{l} \left(\frac{-c}{2\pi\sqrt{mn}}\right)^{p-l} \frac{\Gamma(k-q-l-1)}{\Gamma(k-p-q-1)} J_{k-p-l-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\Bigg].$$

Proof. By Theorem 2.2.6 and (2.12), we get

$$\begin{aligned}
\langle P_{m,k}^{p}, P_{n,k}^{q}\rangle &= \frac{\Gamma(k-p-q-1)}{(4\pi n)^{k-p-q-1}} \Bigg[\delta_{n,m} + (-1)^{\frac{k-2p}{2}} 2\pi \left(\frac{n}{m}\right)^{\frac{k-2p-1}{2}} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} \\
&\quad J_{k-2p-1}\left(\frac{4\sqrt{nm}}{c}\right)\Bigg] + (-1)^{\frac{k-2p}{2}} 2\pi \left(\frac{n}{m}\right)^{\frac{k-2p-1}{2}} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} \sum_{l=0}^{p-1} \binom{p}{l} \\
&\quad \left(-2c\sqrt{\frac{n}{m}}\right)^{p-l} \frac{\Gamma(k-q-l-1)}{(4\pi n)^{k-q-l-1}} J_{k-p-l-1}\left(\frac{4\sqrt{nm}}{c}\right) \\
&= \frac{\Gamma(k-p-q-1)}{(4\pi n)^{k-p-q-1}} \Bigg[\delta_{n,m} + (-1)^{\frac{k-2p}{2}} 2\pi \left(\frac{n}{m}\right)^{\frac{k-2p-1}{2}} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} \sum_{l=0}^{p} \binom{p}{l} \\
&\quad \left(\frac{-c}{2\pi\sqrt{mn}}\right)^{p-l} \frac{\Gamma(k-q-l-1)}{\Gamma(k-p-q-1)} J_{k-p-l-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\Bigg] \\
&= \frac{\Gamma(k-p-q-1)(4\pi)^{p+q}m^p n^q}{(4\pi\sqrt{mn})^{k-1}} \Bigg[\delta_{m,n} + (-1)^{\frac{k-2p}{2}} 2\pi \sum_{c=1}^{\infty} \frac{K(m,n;c)}{c} \sum_{l=0}^{p} \binom{p}{l} \\
&\quad \left(\frac{-c}{2\pi\sqrt{mn}}\right)^{p-l} \frac{\Gamma(k-q-l-1)}{\Gamma(k-p-q-1)} J_{k-p-l-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\Bigg].
\end{aligned}$$

$\square$

When $p = q$ and $m = n$, we obtain the following result [7, Corollary 1.9].

Corollary 2.2.9. *For integers m, p with $m \geq 1$ and $0 \leq p < \frac{k}{2} - 1$, we have*

$$\langle P^p_{m,k}, P^p_{m,k} \rangle = \frac{\Gamma(k-2p-1)}{(4\pi m)^{k-2p-1}} \Bigg[1 + (-1)^{\frac{k-2p}{2}} 2\pi \sum_{c=1}^{\infty} \frac{K(m,m;c)}{c} \sum_{l=0}^{p} \binom{p}{l} \left(\frac{-c}{2\pi m}\right)^{p-l} \frac{\Gamma(k-p-l-1)}{\Gamma(k-2p-1)} J_{k-p-l-1}\left(\frac{4\pi m}{c}\right) \Bigg]. \tag{2.23}$$

2.3 Holomorphic projection

Nearly holomorphic modular forms are not holomorphic but are C^∞ functions. It is natural to find their projections on the space of holomorphic cusp forms. In [31], J. Sturm gives a complete description of holomorphic projection. The following theorem is a special case of [31, Theorem 1].

Theorem 2.3.1. *Let $k \geq 4$ be an even integer and let*

$$f(x+iy) = \sum_{n=0}^{\infty} a(n,y) q^n \in M_k^{\mathrm{nh}}$$

be a rapidly decreasing function. Let

$$b(n) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \int_0^\infty a(n,y) e^{-4\pi n y} y^{k-2} dy. \tag{2.24}$$

Then $h(z) = \sum_{n=1}^{\infty} b(n) q^n$ is a cusp form of weight k on $SL_2(\mathbb{Z})$. Moreover $\langle g, f \rangle = \langle g, h \rangle$ for all cusp form g of weight k on $SL_2(\mathbb{Z})$.

The function h is called the holomorphic projection of f. As an application of Theorem 2.3.1, we have the following theorem [7, Theorem 1.6].

Theorem 2.3.2. *For integers $m \geq 1$ and $0 \leq p < \frac{k}{2} - 1$, the holomorphic projection of $P^p_{m,k}(z)$ is $(4\pi m)^p \frac{\Gamma(k-p-1)}{\Gamma(k-1)} P_{m,k}(z)$.*

To prove the above theorem, we need the following result on J-Bessel function.

Lemma 2.3.3. *For non-negative integers k and p with $k-2p \geq 0$, we have*

$$J_k(x) = \sum_{l=0}^{p} \binom{p}{l} (-1)^l (\frac{2}{x})^{p-l} \frac{\Gamma(k-l)}{\Gamma(k-p)} J_{k-p-l}(x).$$

Proof. It is well known that the J-Bessel function satisfies the recurrence relation [33, §2.12]

$$J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x).$$

The proof of the lemma follows by using the above identity and induction on p. □

Proof of Theorem 2.3.2. Let $h(z) = \sum_{n=0}^{\infty} b(n) q^n$ be the holomorphic projection of $P^p_{m,k}$. Then by using (2.12) in (2.24), we have

$$b(n) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \int_0^\infty y^{-p} \Bigg[\delta_{n,m} + (-1)^{(k-2p)/2} 2\pi \left(\frac{n}{m}\right)^{(k-2p-1)/2} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} \\ \sum_{l=0}^{p} \binom{p}{l} \left(-2yc\sqrt{\frac{n}{m}}\right)^{p-l} J_{k-p-l-1}\left(\frac{4\sqrt{nm}}{c}\right) \Bigg] e^{-4\pi n y} y^{k-2} dy.$$

Using the definition of the gamma function and simplifying the above integral, we obtain

$$b(n) = \frac{(4\pi m)^p \Gamma(k-p-1)}{\Gamma(k-1)} \Bigg[\delta_{n,m} + (-1)^{k/2} 2\pi \left(\frac{n}{m}\right)^{(k-1)/2} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} \\ \sum_{l=0}^{p} \binom{p}{l} (-1)^l \left(\frac{c}{2\pi\sqrt{nm}}\right)^{p-l} \frac{\Gamma(k-l-1)}{\Gamma(k-p-1)} J_{k-p-l-1}\left(\frac{4\sqrt{nm}}{c}\right) \Bigg].$$

Now by using Lemma 2.3.3, we obtain

$$b(n) = \frac{(4\pi m)^p \Gamma(k-p-1)}{\Gamma(k-1)} \left[\delta_{n,m} + (-1)^{k/2} 2\pi \left(\frac{n}{m}\right)^{(k-1)/2} \sum_{c=1}^{\infty} \frac{K(n,m;c)}{c} J_{k-1}\left(\frac{4\sqrt{nm}}{c}\right) \right] \\ = \frac{(4\pi m)^p \Gamma(k-p-1)}{\Gamma(k-1)} p_{m,k}(n),$$

where $p_{m,k}(n)$ be the n-th Fourier coefficient of the Poincaré series $P_{m,k}$. □

Remark 2.3.4. *Theorem 2.3.2 asserts that the holomorphic projection of the m-th nearly holomorphic Poincaré series is nothing but some scalar multiple of the holomorphic Poincaré series of the same weight.*

2.4 Non-vanishing of nearly holomorphic Poincaré series

The non-vanishing of holomorphic Poincaré series is an interesting problem. It is a conjecture that none of these holomorphic Poincaré series vanish. R. A. Rankin [27] proved the following result [27, Theorem 1].

Theorem 2.4.1. *There exist positive constants k_0 and B, with $B > 4\log 2$ such that for all even integers $k \geq k_0$ and all positive integers m with*

$$m \leq k^2 \exp\left(\frac{-B\log k}{\log\log k}\right),$$

the holomorphic Poincaré series $P_{m,k}(z)$ does not vanish identically.

Later, C. J. Mozzochi [24] and J. Lehner [22] generalized Rankin's result for Poincaré series on $\Gamma_0(N)$ and arbitrary Fuchsian group respectively. Moreover, the non-vanishing of the Poincaré series is related to the famous conjecture of Lehmer [21], which asserts that $\tau(n) \neq 0$, for all $n \geq 1$, where τ is the Ramanujan τ-function. It is natural to obtain the non-vanishing property of nearly holomorphic Poincaré series. We have the following theorem [7, Theorem 1.12] which is a generalization of Theorem 2.4.1.

Theorem 2.4.2. *For a fixed non-negative integer p there exist positive constants k_0 and B, where $B > 4\log 2$, such that for all even integers $k \geq k_0 + 2p$ and all positive integers*

$$m \leq (k-2p)^2 \exp\left(\frac{-B\log(k-2p)}{\log\log(k-2p)}\right),$$

the nearly holomorphic Poincaré series $P^p_{m,k}(z)$ does not vanish identically.

Proof. It is clear that if $f(z) = \sum_{l=0}^{p} f_l(z)y^{-l}$ is a nearly holomorphic modular form, then

$$f \equiv 0 \text{ iff } f_l \equiv 0 \text{ for all } \; l = 0, \ldots, p. \tag{2.25}$$

Using (2.5) in (2.1), we can write $P^p_{m,k}(z) = \sum_{l=0}^{p} f^l_{m,k}(z)y^{-l}$, where

$$f^l_{m,k}(z) = \binom{p}{l}(-2i)^{p-l} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \alpha(\gamma, z)^{p-l} j(\gamma, z)^{-k+2p} e(m\gamma z) \tag{2.26}$$

is the l-th component of $P^p_{m,k}(z)$. Also it is observed that

$$f^p_{m,k}(z) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} j(\gamma, z)^{-k+2p} e(m\gamma z)$$

is the m-th holomorphic Poincaré series of weight $k-2p$. Now the result easily follows from Theorem 2.4.1. □

Remark 2.4.3. *We see that the leading coefficient of $P^p_{m,k}$(as a polynomial of $1/y$) is $P_{m,k-2p}$. In Theorem 2.1.1, we proved that $P^p_{m,k}$ is a nearly holomorphic Poincaré series of weight k and depth $\leq p$ on $SL_2(\mathbb{Z})$. But if we assume the conjecture that none of the holomorphic Poincaré series vanish, then $P^p_{m,k}$ is a nearly holomorphic Poincaré series of weight k and depth p on $SL_2(\mathbb{Z})$.*

In [18, Proposition 1], E. Kowalski et al. obtained the following orthogonality properties of the Fourier coefficients of holomorphic Poincaré series.

Proposition 2.4.4. *If $a_{m,k}(n)$ is the n-th Fourier coefficient of the holomorphic Poincaré series $P_{m,k}(z)$, then for fixed positive integers m and n, we have*

$$\lim_{k \to \infty} a_{m,k}(n) = \delta(m, n).$$

In [18], the authors gave a simple proof of the above proposition. Using a similar method, we obtain the following orthogonality relation of the Fourier coefficients of nearly holomorphic Poincaré series [7, Proposition 1.14].

Proposition 2.4.5. *If $a^l_{m,k}(n)$ is the n-th Fourier coefficient of $f^l_{m,k}$, the l-th component of the nearly holomorphic Poincaré series $P^p_{m,k}(z)$, then for fixed non-negative integers p, m and n with $m, n \geq 1$, we have*

$$\lim_{k\to\infty} a^l_{m,k}(n) = \delta(m,n)\delta(p,l) \textit{ for } 0 \leq l \leq p.$$

Proof. If $l = p$, Proposition 2.4.4 gives the result. If $0 \leq l < p$, we shall prove that

$$\lim_{k\to\infty} a^l_{m,k}(n) = 0 \text{ for any } m, n \geq 1.$$

Since $a^l_{m,k}(n)$ is the n-th Fourier coefficient of $f^l_{m,k}$, we have

$$a^l_{m,k}(n) = \int_U f^l_{m,k}(z)e(-nz)dz, \tag{2.27}$$

where U is a suitable horizontal interval of length one in $\mathbb{H}$. We choose

$$U = \{x + iy_0 : |x| \leq 1/2\}$$

for some fixed $y_0 > 1$. By taking the limit as $k \to \infty$ on both sides of (2.27), we get

$$\lim_{k\to\infty} a^l_{m,k}(n) = \lim_{k\to\infty} \int_U f^l_{m,k}(z)e(-nz)dz. \tag{2.28}$$

We show that for all $z \in U$, $f^l_{m,k}(z) \to 0$ as $k \to \infty$. From (2.26), we have

$$f^l_{m,k}(z) = \binom{p}{l}(-2i)^{p-l} \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} \alpha(\gamma,z)^{p-l} j(\gamma,z)^{-k+2p} e(m\gamma z). \tag{2.29}$$

By taking the limit as $k \to \infty$ on both sides of (2.29), we get

$$\lim_{k\to\infty} f^l_{m,k}(z) = \binom{p}{l}(-2i)^{p-l} \lim_{k\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} \alpha(\gamma,z)^{p-l} j(\gamma,z)^{-k+2p} e(m\gamma z). \tag{2.30}$$

Since $m \geq 1$ and $\gamma z \in \mathbb{H}$ for $z \in \mathbb{H}$ and $\gamma \in SL_2(\mathbb{Z})$, we have

$$|\alpha(\gamma,z)^{p-l} j(\gamma,z)^{-k+2p} e(m\gamma z)| \leq |\alpha(\gamma,z)|^{p-l} |j(\gamma,z)|^{-k+2p}. \tag{2.31}$$

But for $z \in U$and $c, d \in \mathbb{Z}$, we have

$$|cz+d|^2 = (cx+d)^2 + c^2 y_0^2 \geq c^2 y_0^2 > c^2 \text{ as } y_0 > 1. \tag{2.32}$$

Therefore

$$|\alpha(\gamma, z)| = \Big|\frac{c}{cz+d}\Big| < 1 \tag{2.33}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Using (2.33) in (2.31), we deduce that

$$|\alpha(\gamma, z)^{p-l} j(\gamma, z)^{-k+2p} e(m\gamma z)| < |cz+d|^{-k+2p} \tag{2.34}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Now we show that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash SL_2(\mathbb{Z})$, we have

$$\alpha(\gamma, z)^{p-l} j(\gamma, z)^{-k+2p} e(m\gamma z) \to 0 \ \text{ as } k \to \infty.$$

If $c = 0$, $\alpha(\gamma, z)^{p-l} j(\gamma, z)^{-k+2p} e(m\gamma z) = 0$. If $c \neq 0$, then $c^2 y_0^2 > 1$. Hence from (2.32) and (2.34), we obtain

$$|\alpha(\gamma, z)^{p-l} j(\gamma, z)^{-k+2p} e(m\gamma z)| \leq \frac{1}{(c^2 y_0^2)^{\frac{k-2p}{2}}} \to 0 \ \text{ as } k \to \infty.$$

Now from (2.34), we deduce that

$$|\alpha(\gamma, z)^{p-l} j(\gamma, z)^{-k+2p} e(m\gamma z)| \leq |cz+d|^{-4} \tag{2.35}$$

for $k \geq 2p+4$ and $\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})$. Therefore by applying the dominated convergence theorem on the right hand side of (2.30), we obtain

$$\lim_{k \to \infty} f^l_{m,k}(z) = 0$$

for all $z \in U$. Using (2.34) in (2.29), we obtain

$$|f^l_{m,k}(z)| \leq \binom{p}{l} 2^{p-l} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} |cz+d|^{-4}$$

for $k \geq 2p+4$. Since U is compact, by applying dominated convergence theorem on the right hand side of (2.28), we obtain

$$\lim_{k\to\infty} a^{l}_{m,k}(n) = 0.$$

□

As a consequence of Proposition 2.4.5, we obtain the following result [7, corollary 1.15].

Corollary 2.4.6. *For fixed non-negative integers m and p with $m \geq 1$, there exists a sufficiently large positive even integer k such that the nearly holomorphic Poincaré series $P^{p}_{m,k}(z)$ does not vanish identically.*

Proof. From Proposition 2.4.5, we have

$$\lim_{k\to\infty} a^{p}_{m,k}(m) = 1.$$

Hence, there exists a positive integer k_0 such that for all $k > k_0$, $a^{p}_{m,k}(m) \neq 0$. Equivalently, $f^{p}_{m,k}(z)$ (the p-th component of $P^{p}_{m,k}(z)$) is not the zero function. Now using (2.25), we get the result. □

Chapter 3

The adjoints of higher order Serre derivative maps

3.1 Introduction

In this chapter, we compute adjoints of higher order Serre derivative maps with respect to the Petersson scalar product. As an application, we obtain some identities involving Fourier coefficients of some cusp forms and special values of certain shifted Dirichlet series.

In [17], using methods of linear algebra and properties Poincaré series, Kohnen constructed explicitly the adjoint map of the product map (product by a fixed cusp form) on the space of cusp forms S_k for a fixed weight k. Following Kohnen's method, several authors obtained adjoints of various linear maps on the space of cusp forms. The Serre derivative of $f \in M_k$ is defined by

$$\vartheta_k f := Df - \frac{k}{12} E_2 f,$$

where E_2 is defined as in (1.4) and $D := \frac{1}{2\pi i}\frac{d}{dz}$ is the differential operator. It is well known that ϑ_k maps M_k to M_{k+2} [4, Section 5.1]. It preserves the space of cusp forms also. Using the theory of nearly holomorphic modular forms, Kumar [19] constructed the adjoint of the Serre derivative map with respect to the Petersson scalar product. In this chapter, we consider the higher order Serre derivative maps (see subsection

3.2.1 for the precise definition) and find the adjoints of these maps with respect to the Petersson inner product. Our method is different from the method of Kumar. we do not use the theory of nearly holomorphic modular forms to prove our results. The results of this chapter are contained in [6].

3.2 Preliminaries

3.2.1 Higher order Serre derivatives

Let f be a modular form of weight k on $SL_2(\mathbb{Z})$. For any even integer $k \geq 4$, the higher order Serre derivatives are defined as the following [4, pp. 55].
Define

$$\vartheta_k^{[0]} f = f \text{ and } \vartheta_k^{[1]} f = \vartheta_k f = Df - \frac{k}{12} E_2 f.$$

For $n \geq 1$, define

$$\vartheta_k^{[n+1]} f = \vartheta_{k+2n}(\vartheta_k^{[n]} f) - \frac{n(k+n-1)}{144} E_4 \vartheta_k^{[n-1]} f. \tag{3.1}$$

(In particular, $\vartheta_k^{[n]}$ is not simply the n-th iterate of ϑ_k). These functions are given in the closed form by

$$\vartheta_k^{[n]} f(z) = \sum_{r=0}^{n} \binom{n}{r} \frac{(k+n-1)!}{(k+n-r-1)!} (-E_2(z)/12)^r D^{n-r} f(z). \tag{3.2}$$

This closed form can be obtained by using induction on n. We call $\vartheta_k^{[n]}$ the n-th order Serre derivative.

Theorem 3.2.1. *Let n be a non-negative integer. If $f \in M_k$, then $\vartheta_k^{[n]} f \in M_{k+2n}$. Also if $f \in S_k$, then $\vartheta_k^{[n]} f \in S_{k+2n}$.*

Proof. The proof of the theorem follows by using induction on n. □

3.2.2 Poincaré series associated to any q series

Let $\phi(q) = \sum_{n=0}^{\infty} \alpha(n) q^n$ be any q-series on the upper half-plane $\mathbb{H}$, where $\alpha(n)$ grow sufficiently slow for all $n \geq 0$. Following [35], we define the Poincaré series associated to ϕ by

$$\mathbb{P}_k(\phi)(z) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \phi|_k \gamma(z). \tag{3.3}$$

In [35, Section 3], B. Williams proved that the series represented by $\mathbb{P}_k(\phi)$ converges absolutely and uniformly on any compact subset of $\mathbb{H}$ if the coefficients of ϕ satisfy the bound $\alpha(n) = O(n^{k/2-3/2+\epsilon})$ for some $\epsilon > 0$. It is clear that $\mathbb{P}_k(\phi)|_k\gamma = \mathbb{P}_k(\phi)$ for all $\gamma \in SL_2(\mathbb{Z})$. Hence $\mathbb{P}_k(\phi)$ is a modular form of weight k on $SL_2(\mathbb{Z})$. Also if $\alpha(0) = 0$, then $\mathbb{P}_k(\phi)$ is a cusp form of weight k on $SL_2(\mathbb{Z})$. In particular, for $m \geq 1$, $\mathbb{P}_k(q^m)$ is the m-th classical Poincaré series defined in (1.8). Now we recall the following result [35, Theorem 4], which will play an important role in finding the adjoint of $\vartheta_k^{[n]}$.

Theorem 3.2.2. *For any non-negative integers n, m and a positive even integer k with $k \geq 2n+2$, set*

$$\phi(z) = q^m \sum_{r=0}^{n} \binom{n}{r} \frac{(k+n-1)!}{(k+n-r-1)!} (-E_2(z)/12)^r m^{n-r}. \tag{3.4}$$

Then

$$\vartheta^{[n]} \mathbb{P}_k(q^m) = \mathbb{P}_{k+2m}(\phi).$$

3.3 Main Theorem

From Theorem 3.2.1 we know that for any $n \geq 0$, $\vartheta_k^{[n]}$ is a linear map from S_k to S_{k+2n}. Thus the adjoint $\vartheta_k^{[n]*}$ of $\vartheta_k^{[n]}$ is a linear map from S_{k+2n} to S_k satisfying

$$\langle \vartheta_k^{[n]*} f, g \rangle = \langle f, \vartheta_k^{[n]} g \rangle \text{ for all } f \in S_{k+2n} \text{ and } g \in S_k. \tag{3.5}$$

For $n = 0$, $\vartheta_k^{[0]}$ is the identity map and its adjoint is itself. In the following result [6, Theorem 1.4], we obtain the Fourier expansion of $\vartheta_k^{[n]*} f$ for $f \in S_{k+2n}$ and $n \geq 1$.

Theorem 3.3.1. *For a positive integer n and a positive even integer k with $k \geq 2n+2$, the image of $f(z) = \sum_{\ell=1}^{\infty} a(\ell)q^{\ell} \in S_{k+2n}$ under $\vartheta_k^{[n]*}$ is given by*

$$\vartheta_k^{[n]*} f(z) = \frac{\Gamma(k+2n-1)}{\Gamma(k-1)(4\pi)^{2n}} \sum_{m=1}^{\infty} m^{k-1} \left(\sum_{t=0}^{\infty} \frac{a(t+m)\mathcal{E}_{k,n}^{m}(t)}{(t+m)^{k+2n-1}} \right) q^m,$$

where

$$\mathcal{E}_{k,n}^{m}(t) = \sum_{r=0}^{n} \binom{n}{r} \frac{(k+n-1)!}{(k+n+r-1)!} \left(\frac{-1}{12}\right)^r m^{n-r} \varepsilon_r(t) \tag{3.6}$$

and $\varepsilon_r(t)$ is the t-th Fourier coefficient of E_2^r.

We need the following lemma to prove the above theorem.

Lemma 3.3.2. *Let n be a positive integer and let k be a positive even integer with $k \geq 2n+2$. Then for any $f \in S_{k+2n}$, the series*

$$\sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \int_{SL_2(\mathbb{Z})\backslash \mathbb{H}} \left| \, f(z)\overline{\phi|_k\gamma} \; y^{k+2n} \, \right| \; \frac{dxdy}{y^2}$$

converges, where ϕ is as given in (3.4).

Proof. Let $f(z) = \sum_{\ell=1}^{\infty} a(\ell)q^{\ell} \in S_{k+2n}$. By Proposition 1.1.8, we have $a(\ell) = O(\ell^{(k+2n-1)/2+\epsilon})$ for any $\epsilon > 0$. Also $\varepsilon_r(\ell) = O(\ell^{2r-1+\epsilon})$ for any $\epsilon > 0$, where $\varepsilon_r(\ell)$ is the ℓ-th Fourier coefficient of E_2^r. Now using the change of variable $z \mapsto \gamma^{-1}z$, Rankin's unfolding argument and substituting the expression for ϕ from (3.4), we obtain

$$\begin{aligned} &\sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \int_{SL_2(\mathbb{Z})\backslash \mathbb{H}} \left| f(z)\overline{\phi|_k\gamma} \; y^{k+2n} \right| \; \frac{dxdy}{y^2} \\ &= \sum_{r=0}^{n} \binom{n}{r} \frac{(k+n-1)!}{(k+n-r-1)!} \left(\frac{1}{12}\right)^r m^{n-r} \int_{\Gamma_\infty \backslash \mathbb{H}} \left| f(z)\overline{E_2^r(z)q^m} y^{k+2n} \right| \; \frac{dxdy}{y^2}. \end{aligned}$$

Using the Fourier expansions of f and E_2^r, we deduce that

$$\int_{\Gamma_\infty \backslash \mathbb{H}} \left| f(z)\overline{E_2^r(z)q^m} y^{k+2n} \right| \; \frac{dxdy}{y^2} = \int_{\Gamma_\infty \backslash \mathbb{H}} \left| \sum_{\ell=1}^{\infty} \sum_{t=0}^{\infty} a(\ell)\varepsilon_r(t) q^{(\ell+t+m)} y^{k+2n} \right| \; \frac{dxdy}{y^2}.$$

Using the triangle inequality and the estimates for the Fourier coefficients $a(\ell)$ and $\epsilon_r(t)$, we deduce that the right hand side of the above expression is less than or equal to

$$\int_{\Gamma_\infty\backslash\mathbb{H}}\sum_{\ell=1}^{\infty}\sum_{t=0}^{\infty}(\ell+t+m)^{k/2+3n-3/2+2\epsilon}e^{-2\pi(\ell+t+m)y}y^{k+2n}\,\frac{dxdy}{y^2}, \tag{3.7}$$

where C is a positive constant. We have

$$\begin{aligned}
&\sum_{\ell=1}^{\infty}\sum_{t=0}^{\infty}\int_{\Gamma_\infty\backslash\mathbb{H}}(\ell+t+m)^{k/2+3n-3/2+2\epsilon}e^{-2\pi(\ell+t+m)y}y^{k+2n}\,\frac{dxdy}{y^2}\\
&=\sum_{\ell=1}^{\infty}\sum_{t=0}^{\infty}(\ell+t+m)^{k/2+3n-3/2+2\epsilon}\int_{y=0}^{\infty}\int_{x=0}^{1}e^{-2\pi(\ell+t+m)y}y^{k+2n-2}dxdy\\
&=\sum_{\ell=1}^{\infty}\sum_{t=0}^{\infty}(\ell+t+m)^{k/2+3n-3/2+2\epsilon}\frac{\Gamma(k+2n-1)}{(\ell+t+m)^{k+2n-1}}.
\end{aligned}$$

The condition $k\geq 2n+2$ ensures that the above summation is convergent. Therefore by Fubini's theorem, we obtain that the integration in (3.7) is convergent. □

Proof of Theorem 3.3.1. Let $\vartheta_k^{[n]*}f(z)=\sum\limits_{m=1}^{\infty}b(m)q^m$. By Lemma 1.1.12, we have

$$b(m)=\frac{(4\pi m)^{k-1}}{\Gamma(k-1)}\langle\vartheta_k^{[n]*}f,\ \mathbb{P}_k(q^m)\rangle.$$

Using (3.5), we get

$$\langle\vartheta_k^{[n]*}f,\ \mathbb{P}_k(q^m)\rangle=\langle f,\ \vartheta_k^{[n]}\mathbb{P}_k(q^m)\rangle.$$

Now by Theorem 3.2.2, we obtain

$$\langle f,\ \vartheta_k^{[n]}\mathbb{P}_k(q^m)\rangle=\langle f,\ \mathbb{P}_{k+2n}(\phi)\rangle.$$

From (3.3) and using the definition of Petersson inner product, we obtain

$$\langle f,\mathbb{P}_{k+2n}(\phi)\rangle=\int_{SL_2(\mathbb{Z})\backslash\mathbb{H}}f(z)\overline{\sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})}\phi(z)\mid_k\gamma}\;y^{k+2n}\,\frac{dxdy}{y^2}. \tag{3.8}$$

By Lemma 3.3.2, we can interchange the summation and the integral in the right hand side of (3.8). Using Rankin's unfolding argument and substituting the expression for

ϕ from (3.4), the right hand side of (3.8) becomes

$$\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{\phi(z)}y^{k+2n}\ \frac{dxdy}{y^2} = \sum_{r=0}^{n}\binom{n}{r}\frac{(k+n-1)!}{(k+n-r-1)!}\left(\frac{-1}{12}\right)^r m^{n-r} \times \int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{E_2^r(z)q^m}y^{k+2n}\ \frac{dxdy}{y^2}. \tag{3.9}$$

Using the Fourier expansions of f and E_2^r, we obtain

$$\begin{aligned}&\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{E_2^r(z)q^m}y^{k+2n}\ \frac{dxdy}{y^2}\\ &= \int_{y=0}^{\infty}\int_{x=0}^{1}\sum_{\ell=1}^{\infty}\sum_{t=0}^{\infty} a(\ell)\varepsilon_r(t)e^{2\pi i(\ell-t-m)x}e^{-2\pi(\ell+t+m)y}y^{k+2n-2}dxdy.\end{aligned} \tag{3.10}$$

Using a similar technique as in the proof of Lemma 3.3.2, we can interchange the integrals and the summations in the right hand side of (3.10). Then we obtain

$$\begin{aligned}&\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{E_2^r(z)q^m}y^{k+2n}\ \frac{dxdy}{y^2}\\ &= \sum_{\ell=1}^{\infty}\sum_{t=0}^{\infty} a(\ell)\varepsilon_r(t)\int_{y=0}^{\infty}\int_{x=0}^{1} e^{2\pi i(\ell-t-m)x}e^{-2\pi(\ell+t+m)y}y^{k+2n-2}dxdy\\ &= \sum_{t=0}^{\infty} a(t+m)\varepsilon_r(t)\int_{y=0}^{\infty} e^{-4\pi(t+m)y}y^{k+2n-2}dy\\ &= \frac{\Gamma(k+2n-1)}{(4\pi)^{k+2n-1}}\sum_{t=0}^{\infty}\frac{a(t+m)\varepsilon_r(t)}{(t+m)^{k+2n-1}}.\end{aligned}$$

Therefore

$$\langle \vartheta_k^{[n]*}f,\ \mathbb{P}_k(q^m)\rangle = \frac{\Gamma(k+2n-1)}{(4\pi)^{k+2n-1}}\sum_{t\geq 0}\frac{a(t+m)\mathcal{E}_{k,n}^m(t)}{(t+m)^{k+2n-1}}.$$

□

3.4 Applications

We apply Theorem 3.3.1 in some particular cases and find some identities involving special values of certain shifted Dirichlet series. For $k = 12, 16, 18$, we denote the unique normalized cusp forms of S_k by Δ_k with Fourier expansion $\Delta_k(z) = \sum_{n=1}^{\infty}\tau_k(n)q^n$. Note that $\Delta_{12}(z) = \Delta(z)$, whose Fourier coefficients are $\tau(n)$, Ramanujan's tau functions.

Taking $n=2$ and $k=8$ in Theorem 3.3.1, we obtain $\vartheta_8^{[2]*}(\Delta) \in S_8 = \{0\}$. Now using Theorem 3.3.1, we get

$$\frac{\Gamma(11)}{\Gamma(7)(4\pi)^4} \sum_{m=1}^{\infty} m^7 \left(\sum_{t=0}^{\infty} \frac{\tau(t+m)\mathcal{E}_{8,2}^m(t)}{(t+m)^{11}} \right) q^m = 0.$$

This implies that, for all $m \geq 1$

$$\frac{\tau(m)\mathcal{E}_{8,2}^m(0)}{m^{11}} + \sum_{t=1}^{\infty} \frac{\tau(t+m)\mathcal{E}_{8,2}^m(t)}{(t+m)^{11}} = 0. \tag{3.11}$$

If $\varepsilon_r(t)$ is the t-th Fourier coefficient of E_2^r, then we have $\varepsilon_r(0) = 1$ for all $r \geq 0$. and for any $t \geq 1$, we have $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = -24\sigma_1(t)$. Using the identity $E_2^2 = E_4 + 12DE_2$, we obtain that for any $t \geq 1$, $\varepsilon_2(t) = 240\sigma_3(t) - 288t\sigma_1(t)$. Now from (3.6), we have

$$\mathcal{E}_{8,2}^m(0) = \sum_{r=0}^{2} \binom{2}{r} \frac{9!}{(9-r)!} \left(\frac{-1}{12}\right)^r m^{2-r}\varepsilon_r(0) = m^2 - \frac{3}{2}m + \frac{1}{2}. \tag{3.12}$$

And for any $t \geq 1$, we have

$$\mathcal{E}_{8,2}^m(t) = \sum_{r=0}^{2} \binom{2}{r} \frac{9!}{(9-r)!} \left(\frac{-1}{12}\right)^r m^{2-r}\varepsilon_r(t) = (36m - 144t)\sigma_1(t) + 120\sigma_3(t). \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), we obtain

$$\tau(m) = -\frac{24m^{11}}{2m^2 - 3m + 1} \sum_{t=1}^{\infty} \frac{(3m - 12t)\sigma_1(t) + 10\sigma_3(t)}{(t+m)^{11}} \tau(t+m) \text{ for all } m \geq 1.$$

This identity was obtained by B. Williams [35, Example 11] also.

Similarly, taking $n = 2, k = 14$ in Theorem 3.3.1, using Theorem 3.3.1 and the fact that $S_{14} = \{0\}$, we obtain

$$\tau_{18}(m) = -\frac{240m^{17}}{24m^2 - 60m + 135} \sum_{t=1}^{\infty} \frac{(6m - 42t)\sigma_1(t) + 35\sigma_3(t)}{(t+m)^{17}} \tau_{18}(t+m) \text{ for all } m \geq 1.$$

Similarly, taking $n=3$ and $k=10$ in Theorem 3.3.1, we obtain $\vartheta_8^{[3]*}(\Delta) \in S_{10} = \{0\}$. Now using Theorem 3.3.1, we get

$$\frac{\Gamma(15)}{\Gamma(9)(4\pi)^6} \sum_{m=1}^{\infty} m^{13} \left(\sum_{t=0}^{\infty} \frac{\tau_{16}(t+m)\mathcal{E}_{10,3}^m(t)}{(t+m)^{15}} \right) q^m = 0.$$

This implies that for all $m \geq 1$, we have

$$\frac{\tau_{16}(m)\mathcal{E}_{10,3}^{m}(0)}{m^{15}} + \sum_{t=1}^{\infty} \frac{\tau_{16}(t+m)\mathcal{E}_{10,3}^{m}(t)}{(t+m)^{15}} = 0. \tag{3.14}$$

Using the identity $E_2^3 = E_6 + 9DE_4 + 72D^2E_2$, we obtain that for any $t \geq 1$, $\varepsilon_3(t) = -504\sigma_5(t) + 2160t\sigma_3(t) - 1728t^2\sigma_1(t)$. Now from (3.6), we have

$$\mathcal{E}_{10,3}^{m}(0) = \sum_{r=0}^{3} \binom{3}{r} \frac{12!}{(12-r)!} \left(\frac{-1}{12}\right)^{r} m^{3-r}\varepsilon_r(0) = m^3 - 3m^2 + \frac{11}{4}m - \frac{55}{72}. \tag{3.15}$$

For any $t \geq 1$, we have

$$\begin{aligned} \mathcal{E}_{10,3}^{m}(t) &= \sum_{r=0}^{3} \binom{3}{r} \frac{12!}{(12-r)!} \left(\frac{-1}{12}\right)^{r} m^{3-r}\varepsilon_r(t) \\ &= (72m^2 - 792mt + 1320t^2)\sigma_1(t) + (440m - 1650t)\sigma_3(t) + 385\sigma_5(t). \end{aligned} \tag{3.16}$$

Using (3.15) and (3.16) in (3.14), we obtain

$$\begin{aligned} \tau_{16}(m) = &-\frac{72m^{15}}{72m^3 - 216m^2 + 198m - 55} \\ &\times \sum_{t=1}^{\infty} \frac{(72m^2 - 792mt + 1320t^2)\sigma_1(t) + (440m - 1650t)\sigma_3(t) + 385\sigma_5(t)}{(t+m)^{15}} \tau_{16}(t+m) \end{aligned}$$

for all $m \geq 1$.

Chapter 4

Converse theorem for quasimodular forms

4.1 Introduction

In this chapter, we consider twisted Dirichlet series attached to quasimodular forms, study their analytic properties, and prove an analogue of Weil's converse theorem for quasimodular forms. We also give some applications of our results to certain q-series and sign changes of the Fourier coefficients of quasimodular forms.

A converse theorem in the theory of automorphic forms establishes a correspondence between the functions that satisfy certain transformation properties, on one hand, and Dirichlet series satisfying certain analytic properties, on the other hand. For example, the well-known Hecke's converse theorem [13] establishes an equivalence between modular forms on $SL_2(\mathbb{Z})$ and Dirichlet series satisfying a certain functional equation, meromorphic continuation, and boundedness in the vertical strips. A very significant and useful generalization of Hecke's converse theorem to congruence subgroups $\Gamma_0(N)$ was done by Weil [34] which illustrates the meaning of a converse theorem more closely in our context. The converse theorem for GL_n automorphic representations was achieved in the works of several authors in papers [9], [14] and [15].

Any quasimodular form of level N (see subsection 4.2.1 for definition) has a Fourier expansion and hence we can associate a Dirichlet series to it. In [20], Lagos considered Dirichlet series attached to quasimodular forms of depth 1 on $SL_2(\mathbb{Z})$ and generalized Hecke's converse theorem to his settings. This converse theorem has been generalized to quasimodular forms of any depth for the group $SL_2(\mathbb{Z})$ in [1]. In this chapter, we consider twisted Dirichlet series associated to quasimodular forms of level N by Dirichlet characters. We investigate analytic properties of these twisted Dirichlet series and establish an analogue of Weil's converse theorem for quasimodular forms. Then we discuss two applications of our results. The first one discusses the quasimodularity of a certain q-series considered by Ramanujan [26], and the second one establishes the occurrence of infinitely many sign changes for the Fourier coefficients of certain quasimodular forms. The results of this chapter are contained in [8].

4.2 Preliminaries

4.2.1 Quasimodular forms

Let k, N be positive integers and let p be a non-negative integer. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. Let $f \in M_{k,p}^{\mathrm{qm}}(N, \chi)$ with components $f_0, f_1, \ldots, f_p$. We also denote f by $\vec{f} = (f_0, f_1, \ldots, f_p)$. By the transformation property (1.14) for the identity matrix, we get that $f_0 = f$. Moreover, the following result [8, Proposition 2.2] shows that each component f_j of f is again a quasimodular form of weight $k - 2j$ and depth $p - j$. The proof of this result is similar to the proof of [29, Proposition 3.3]. Therefore we omit the proof here.

Proposition 4.2.1. *Let $f \in M_{k,p}^{\mathrm{qm}}(N, \chi)$ with components $f_0, f_1, \ldots, f_p$. Then for every $0 \leq j \leq p$, we have*

$$f_j|_{k-2j}\gamma(z) = \chi(\gamma) \sum_{v=0}^{p-j} \binom{j+v}{v} f_{j+v}(z) \left(X(\gamma)(z)\right)^v \quad \textit{for all } \gamma \in \Gamma_0(N).$$

Remark 4.2.2. *From Proposition 4.2.1, we see that f_p is a modular form of weight $k-2p$, level N and character χ. Since there are no non-zero modular forms of negative weight, we have $p \leq k/2$. Moreover, if $p = k/2$ then f_p is a constant, and hence χ has to be trivial.*

Now, we state two lemmas [8, Lemma 2.4, 2.5] which are useful to establish the desired Fourier expansions for all the components of a quasimodular form and estimates for their Fourier coefficients.

Lemma 4.2.3. *Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a holomorphic function given by the Fourier expansion*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^n,$$

where $a_n \in \mathbb{C}$. Then the following two statements are equivalent.

(1) *The function f is polynomially bounded.*

(2) *We have $a_n = 0$ for $n < 0$. Moreover, $f(z) - a_0 = O\left(e^{-2\pi \mathrm{Im}(z)}\right)$ as $\mathrm{Im}(z) \to \infty$, and $f(z) = O(\mathrm{Im}(z)^{-\sigma})$ as $\mathrm{Im}(z) \to 0$, for some $\sigma > 0$, uniformly in $\mathrm{Re}(z)$.*

Proof. For a proof, see [10, Lemma 5.1.11, Corollary 5.1.17]. □

Lemma 4.2.4. *Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a holomorphic function given by the Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n,$$

where $a(n) \in \mathbb{C}$. Suppose that $f(z) = O(\mathrm{Im}(z)^{-\nu})$ uniformly in $\mathrm{Re}(z)$ as $\mathrm{Im}(z) \to 0$ for some $\nu > 0$. Then $a(n) = O(n^{\nu})$ for all $n \geq 1$.

Proof. For a proof, see [23, Corollary 2.1.6]. □

Now we apply the above two lemmas to get the following proposition [8, Proposition 2.6].

Proposition 4.2.5. *Let $f \in M^{\mathrm{qm}}_{k,p}(N,\chi)$ with components $f_0, f_1, \ldots, f_p$. Then each f_j has a Fourier expansion of the form*

$$f_j(z) = \sum_{n=0}^{\infty} a_j(n) q^n,$$

where $a_j(n) \in \mathbb{C}$ with $a_j(n) = O(n^{\nu_j})$, for some $\nu_j > 0$.

Proof. By Proposition 4.2.1, each f_j is a quasimodular form. Therefore we have $f_j(z+1) = f_j(z)$. This gives us a Fourier expansion

$$f_j(z) = \sum_{n\in\mathbb{Z}} a_j(n) q^n,$$

which converges absolutely and uniformly on any compact subset of $\mathbb{H}$. Since f_j is polynomially bounded, by Lemma 4.2.3 and Lemma 4.2.4 together, we have that $a_j(n) = 0$ for $n < 0$ and $a_j(n) = O(n^{\nu_j})$ for some $\nu_j > 0$. □

We finish this subsection by stating a result [8, Lemma 2.7] which will be useful in the proof of Theorem 4.4.7. For a proof of this lemma, see [23, Lemma 4.3.3].

Lemma 4.2.6. *For a sequence $(a(n))_{n\geq 0}$ of complex numbers, let*

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n.$$

If $a(n) = O(n^{\nu})$ for some $\nu > 0$ then the above series defining $f(z)$ converges absolutely and uniformly on any compact subset of $\mathbb{H}$ and hence $f(z)$ is holomorphic on $\mathbb{H}$. Moreover, $f(z) - a(0) = O\left(e^{-2\pi \mathrm{Im}(z)}\right)$ as $\mathrm{Im}(z) \to \infty$ and $f(z) = O(\mathrm{Im}(z)^{-\nu-1})$ as $\mathrm{Im}(z) \to 0$ uniformly on $\mathrm{Re}(z)$.

4.2.2 Nearly holomorphic modular forms

In this subsection, we briefly review some results on nearly holomorphic modular forms and their relations with quasimodular forms.

If $F \in M_{k,p}^{\mathrm{nh}}(N,\chi)$, then we write

$$F(z) = \sum_{0\le j\le p} f_j(z)(2iy)^{-j}, \quad z = x+iy,$$

for some holomorphic functions f_j on $\mathbb{H}$ which are polynomially bounded. The following result [8, Proposition 2.9] gives a relation between nearly holomorphic modular forms and quasimodular forms.

Proposition 4.2.7. *Let $f_0, f_1, \dots, f_p$ be polynomially bounded holomorphic functions on $\mathbb{H}$. Define the function $F : \mathbb{H} \longrightarrow \mathbb{C}$ by*

$$F(z) = \sum_{0\le j\le p} f_j(z)(2iy)^{-j}, \quad z = x+iy.$$

Then the following two statements are equivalent.

(1) *The function $F \in M_{k,p}^{\mathrm{nh}}(N,\chi)$.*

(2) *The function $f_0 \in M_{k,p}^{\mathrm{qm}}(N,\chi)$ with components $f_0, f_1, \dots, f_p$.*

Proof. For a proof, see [10, Theorem 5.1.22]. □

Unfortunately, the image of a quasimodular form of level N under the usual Fricke involution operator W_N is not a quasimodular form. This adds difficulty in getting the functional equation for the attached Dirichlet series. Therefore we modify the operator W_N appropriately with the help of Proposition 4.2.7 to overcome this difficulty. First, let us discuss the behavior of a nearly holomorphic modular form under the action of the Fricke involution. Let

$$F(z) = \sum_{0\le m\le p} f_m(z)(2iy)^{-m} \in M_{k,p}^{\mathrm{nh}}(N,\chi).$$

For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$, we have

$$F|_k\gamma = \sum_{0\le m\le p} (\det\gamma)^{k/2} f_m(\gamma z) j(\gamma,z)^{m-k}(j(\gamma,z) - 2icy)^m (\det\gamma)^{-m}(2iy)^{-m}.$$

Simplifying, we obtain

$$F|_k\gamma = \sum_{0\le\ell\le p}\left(\sum_{\ell\le m\le p}\binom{m}{\ell}(\det\gamma)^{k/2-m}f_m(\gamma z)j(\gamma,z)^{m+\ell-k}(-c)^{m-\ell}\right)(2iy)^{-\ell}.$$

In particular, for $\gamma = W_N := \left(\begin{smallmatrix}0 & -1\\ N & 0\end{smallmatrix}\right)$, we have

$$F|_k W_N = \sum_{0\le\ell\le p}\widetilde{f_\ell}(z)(2iy)^{-\ell}, \tag{4.1}$$

where

$$\widetilde{f_\ell}(z) = \sum_{\ell\le m\le p}\binom{m}{\ell}(-1)^{m-\ell}N^{k/2-\ell}f_m\left(-\frac{1}{Nz}\right)(Nz)^{m+\ell-k}. \tag{4.2}$$

By the transformation property of F with respect to the group $\Gamma_0(N)$ and (4.2), we obtain the following [8, Lemma 2.10].

Lemma 4.2.8. *If $F \in M^{\mathrm{nh}}_{k,p}(N,\chi)$, then $F|_k W_N \in M^{\mathrm{nh}}_{k,p}(N,\overline{\chi})$.*

In the view of Proposition 4.2.7 and Lemma 4.2.8, we define the operator $\widetilde{W_N}$.

Definition 4.2.9. *Let $f \in M^{\mathrm{qm}}_{k,p}(N,\chi)$ with components $f_0, f_1, \ldots, f_p$. If $\vec{f} = (f_0, f_1, \ldots, f_p)$, then the action of $\widetilde{W_N}$ on $\vec{f}$ is defined by*

$$\vec{f}|_k\widetilde{W_N} = (\widetilde{f_0}, \widetilde{f_1}, \ldots, \widetilde{f_p}),$$

where

$$\widetilde{f_\ell}(z) = \sum_{\ell\le j\le p}\binom{j}{\ell}(-1)^{j-\ell}N^{k/2-\ell}(Nz)^{j+\ell-k}f_j\left(-\frac{1}{Nz}\right), \quad 0\le\ell\le p. \tag{4.3}$$

We have the following result [8, Proposition 2.12].

Proposition 4.2.10. *If $\vec{f} = (f_0, f_1, \ldots, f_p) \in M^{\mathrm{qm}}_{k,p}(N,\chi)$ then*

$$\vec{f}|_k\widetilde{W_N} = (\widetilde{f_0}, \widetilde{f_1}, \ldots, \widetilde{f_p}) \in M^{\mathrm{qm}}_{k,p}(N,\overline{\chi}),$$

where $\widetilde{f_\ell}$ is defined by (4.3). Moreover, for $0\le\ell\le p$, we have

$$f_\ell(z) = i^{2k}\sum_{\ell\le j\le p}\binom{j}{\ell}(-1)^{j-\ell}N^{k/2-\ell}(Nz)^{j+\ell-k}\widetilde{f_j}\left(-\frac{1}{Nz}\right), \tag{4.4}$$

and

$$\vec{f}|_k\widetilde{W}_N|_k\widetilde{W}_N = (-1)^k\vec{f}.$$

Proof. Since each component of $\vec{f}$ is polynomially bounded, from (4.3) it is clear that each component of $\vec{f}|_k\widetilde{W}_N$ is also polynomially bounded. By Proposition 4.2.7, we deduce that

$$F(z) := \sum_{0\le j\le p} f_j(z)(2iy)^{-j} \in M_{k,p}^{\mathrm{nh}}(N,\chi).$$

Now by Lemma 4.2.8 and Proposition 4.2.7, we deduce that

$$\widetilde{f} := \widetilde{f_0} \in M_{k,p}^{\mathrm{qm}}(N,\overline{\chi})$$

with components $\widetilde{f_0}, \widetilde{f_1}, \ldots, \widetilde{f_p}$. Since $W_N^2 = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix}$, we have

$$F|_k W_N^2 = (-1)^k F.$$

Therefore applying (4.1) twice, we get (4.4). □

4.2.3 Some analytic results

In this subsection, we recall Phragmén-Lindelöf theorem and the Stirling's estimate of gamma function which are useful to prove Theorem 4.3.1. See [23, Lemma 4.3.4, (3.2.8)] for more details.

Theorem 4.2.11 (Phragmén-Lindelöf)**.** *For two real numbers ν_1, ν_2 with $\nu_1 < \nu_2$, put*

$$A = \{s \in \mathbb{C} : \nu_1 \le \mathrm{Re}(s) \le \nu_2\}.$$

Let ϕ be a holomorphic function on a domain containing A satisfying

$$|\phi(s)| = O(e^{|\mathrm{Im}(s)|^\delta}) \ \ (|\mathrm{Im}(s)| \to \infty),$$

uniformly on A with $\delta > 0$. For a real number b, if

$$|\phi(s)| = O(\mathrm{Im}(s)^b) \ \ (|\mathrm{Im}(s)| \to \infty), \ \textit{on } \mathrm{Re}(s) = \nu_1 \textit{ and } \mathrm{Re}(s) = \nu_2,$$

then

$$|\phi(s)| = O(\mathrm{Im}(s)^b) \ \ (|\mathrm{Im}(s)| \to \infty), \quad \textit{uniformly on } A.$$

Stirling's estimate: For a complex number $s = \sigma + i\tau$,

$$\Gamma(s) \sim \sqrt{2\pi}\tau^{\sigma-1/2}e^{-\pi|\tau|/2} \ \text{ as } \ |\tau| \to \infty \tag{4.5}$$

uniformly on any vertical strip.

4.3 Hecke's converse theorem for quasimodular forms

For a holomorphic function

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n$$

on $\mathbb{H}$ with $a(n) = O(n^\nu)$, we put

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Since $a(n) = O(n^\nu)$, $L(f, s)$ converges absolutely and uniformly on any compact subset of $\mathrm{Re}(s) > 1 + \nu$. Therefore it is holomorphic on $\mathrm{Re}(s) > 1 + \nu$. We call $L(f, s)$ the Dirichlet series associated to f. For $N \geq 1$, we put

$$\Lambda_N(f, s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s)L(f, s).$$

Now we prove the following result [8, Theorem 3.1] which is an analogue of Hecke's converse theorem for quasimodular forms of level N. This also generalizes the main result of [1] to level N.

Theorem 4.3.1. *Let k, N be positive integers and let p be a non-negative integer with $p \leq \frac{k}{2}$. For each integer $0 \leq j \leq p$, let $(a_j(n))_{n\geq 0}$ and $(b_j(n))_{n\geq 0}$ be a pair of sequences of complex numbers. Assume that there exists a real number $\nu > 0$ such that $a_j(n)$ and $b_j(n)$ are bounded by $O(n^\nu)$. Put*

$$f_j(z) := \sum_{n=0}^{\infty} a_j(n)q^n, \quad g_j(z) := \sum_{n=0}^{\infty} b_j(n)q^n, \quad 0 \leq j \leq p. \tag{4.6}$$

Let $\vec{f} = (f_0, f_1, \dots, f_p)$ and $\vec{g} = (g_0, g_1, \dots, g_p)$. Assume also that f_p and g_p are non-zero constant functions if $p = k/2$. Then the following two statements are equivalent.

(1) $\vec{f}\,|_k\widetilde{W}_N = \vec{g}$.

(2) *For each $j \in \{0, 1, \dots, p\}$, the completed Dirichlet series $\Lambda_N(f_j, s)$ and $\Lambda_N(g_j, s)$ admit meromorphic continuations to the whole s-plane and they satisfy the following functional equations*

$$\Lambda_N(f_j, s) = \sum_{m=0}^{p-j} i^{k-2j-m} N^{\frac{m}{2}} \binom{j+m}{m} \Lambda_N(g_{j+m}, k-2j-m-s). \tag{4.7}$$

Moreover, for each $j \in \{0, 1, \dots, p\}$, the functions

$$\Lambda_N(f_j, s) + \frac{a_j(0)}{s} + \sum_{m=0}^{p-j} \binom{j+m}{m} \frac{i^{k-2j-m} N^{\frac{m}{2}} b_{j+m}(0)}{k-2j-m-s}$$
$$\Lambda_N(g_j, s) + \frac{b_j(0)}{s} + \sum_{m=0}^{p-j} \binom{j+m}{m} \frac{i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0)}{k-2j-m-s}$$

are holomorphic on the whole s-plane and bounded on any vertical strip.

Proof. Let us first prove (1) $\implies$ (2). If $p = k/2$, then $\Lambda_N(f_p, s)$ and $\Lambda_N(g_p, s)$ are identically zero. Therefore the claimed analytic properties for $j = p$ trivially follow by the fact that $f_p = g_p = a_p(0) = b_p(0)$.

If $p = k/2$ and $j \in \{0, 1, \dots, p-1\}$ or $0 \le p < k/2$ and $j \in \{0, 1, \dots, p\}$, then for $\mathrm{Re}(s) > \nu + 1$, we have

$$\begin{aligned}\Lambda_N(f_j, s) &= \int_0^\infty \left(f_j\left(\frac{it}{\sqrt{N}}\right) - a_j(0) \right) t^s \frac{dt}{t} \\ &= \int_0^1 f_j\left(\frac{i}{\sqrt{N}t}\right) t^{-s} \frac{dt}{t} + \int_1^\infty f_j\left(\frac{i}{\sqrt{N}t}\right) t^{-s} \frac{dt}{t}.\end{aligned}$$

With the change of variable $t \mapsto 1/t$ in the first integral, we obtain

$$\begin{aligned}\Lambda_N(f_j, s) &= \int_1^\infty f_j\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t} + \int_1^\infty f_j\left(\frac{i}{\sqrt{N}t}\right) t^{-s} \frac{dt}{t} \\ &= \int_1^\infty \left(f_j\left(\frac{it}{\sqrt{N}}\right) - a_j(0) \right) t^s \frac{dt}{t} + \int_1^\infty f_j\left(\frac{i}{\sqrt{N}t}\right) t^{-s} \frac{dt}{t} - \frac{a_j(0)}{s}.\end{aligned}$$

Since $\vec{f}\,|_k\widetilde{W}_N=\vec{g}$, using (4.4) we obtain

$$\begin{aligned}\Lambda_N(f_j,s)&=\int_1^\infty\left(f_j\left(\frac{it}{\sqrt{N}}\right)-a_j(0)\right)t^s\frac{dt}{t}\\&\quad+\sum_{j\le l\le p}\binom{l}{j}(-1)^{l-j}N^{\frac{l-j}{2}}i^{l+j+k}\int_1^\infty g_l\left(\frac{it}{\sqrt{N}}\right)t^{k-l-j-s}\frac{dt}{t}-\frac{a_j(0)}{s}.\end{aligned}\tag{4.8}$$

So we have

$$\begin{aligned}\Lambda_N(f_j,s)&=\int_1^\infty\left(f_j\left(\frac{it}{\sqrt{N}}\right)-a_j(0)\right)t^s\frac{dt}{t}\\&\quad+\sum_{j\le l\le p}\binom{l}{j}(-1)^{l-j}N^{\frac{l-j}{2}}i^{l+j+k}\int_1^\infty\left(g_l\left(\frac{it}{\sqrt{N}}\right)-b_l(0)\right)t^{k-l-j-s}\frac{dt}{t}\\&\quad-\frac{a_j(0)}{s}-\sum_{j\le l\le p}\binom{l}{j}(-1)^{l-j}N^{\frac{l-j}{2}}i^{l+j+k}\frac{b_l(0)}{k-l-j-s}.\end{aligned}$$

Simplifying the last sum of the right hand side of the above identity, we deduce that

$$\begin{aligned}\Lambda_N(f_j,s)&=\int_1^\infty\left(f_j\left(\frac{it}{\sqrt{N}}\right)-a_j(0)\right)t^s\frac{dt}{t}\\&\quad+\sum_{j\le l\le p}\binom{l}{j}(-1)^{l-j}N^{\frac{l-j}{2}}i^{l+j+k}\int_1^\infty\left(g_l\left(\frac{it}{\sqrt{N}}\right)-b_l(0)\right)t^{k-l-j-s}\frac{dt}{t}\\&\quad-\frac{a_j(0)}{s}-\sum_{0\le l\le p-j}\binom{l+j}{j}N^{\frac{l}{2}}i^{k+2j-l}\frac{b_l(0)}{k-l-2j-s}.\end{aligned}\tag{4.9}$$

Similarly for $\mathrm{Re}(s)>\nu+1$, one deduces that

$$\begin{aligned}\Lambda_N(g_j,s)&=\int_1^\infty\left(g_j\left(\frac{it}{\sqrt{N}}\right)-b_j(0)\right)t^s\frac{dt}{t}\\&\quad+\sum_{j\le l\le p}\binom{l}{j}(-1)^{l-j}N^{\frac{l-j}{2}}i^{l+j-k}\int_1^\infty\left(f_l\left(\frac{it}{\sqrt{N}}\right)-a_l(0)\right)t^{k-l-j-s}\frac{dt}{t}\\&\quad-\frac{b_j(0)}{s}-\sum_{0\le l\le p-j}\binom{l+j}{j}N^{\frac{l}{2}}i^{-(k-2j)-l}\frac{a_l(0)}{k-l-2j-s}.\end{aligned}\tag{4.10}$$

The expressions deduced in (4.9) and (4.10) provide the claimed analytic properties of $\Lambda_N(f_j,s)$ and $\Lambda_N(g_j,s)$ of meromorphic continuation and boundedness in vertical strips.

Now we establish the claimed functional equation. From (4.8) we obtain

$$\Lambda_N(f_j,s)=\int_1^\infty\left(f_j\left(\frac{it}{\sqrt{N}}\right)-a_j(0)\right)t^s\frac{dt}{t}+i^{2j+k}\int_1^\infty\left(g_j\left(\frac{it}{\sqrt{N}}\right)-b_j(0)\right)t^{k-2j-s}\frac{dt}{t}$$
$$+\sum_{j+1\le l\le p}\binom{l}{j}(-1)^{l-j}N^{\frac{l-j}{2}}i^{l+j+k}\int_1^\infty g_l\left(\frac{it}{\sqrt{N}}\right)t^{k-l-j-s}\frac{dt}{t}-\frac{a_j(0)}{s}-\frac{b_j(0)i^{2j+k}}{k-2j-s}. \tag{4.11}$$

Using (4.3), we observe that for each $1\le l\le p$, we have

$$g_l\left(\frac{it}{\sqrt{N}}\right)=\sum_{l\le m\le p}\binom{m}{l}(-1)^{m-l}N^{\frac{m-l}{2}}i^{m+l-k}t^{m+l-k}f_m\left(\frac{i}{\sqrt{N}t}\right).$$

Using the above identity in (4.11), we obtain

$$\Lambda_N(f_j,s)$$
$$=\int_1^\infty\left(f_j\left(\frac{it}{\sqrt{N}}\right)-a_j(0)\right)t^s\frac{dt}{t}+i^{2j+k}\int_1^\infty\left(g_j\left(\frac{it}{\sqrt{N}}\right)-b_j(0)\right)t^{k-2j-s}\frac{dt}{t}$$
$$+\sum_{j+1\le l\le p}\binom{l}{j}(-1)^{l-j}N^{\frac{l-j}{2}}i^{l+j+k}\int_1^\infty\sum_{l\le m\le p}\binom{m}{l}(-1)^{m-l}N^{\frac{m-l}{2}}i^{m+l-k}f_m\left(\frac{i}{\sqrt{N}t}\right)t^{m-j-s}\frac{dt}{t}$$
$$-\frac{a_j(0)}{s}-\frac{b_j(0)i^{2j+k}}{k-2j-s}.$$

Interchanging the summations in the last integral of the right hand side of the above identity, using the combinatorial identity $\binom{n}{j}\binom{m}{l}=\binom{m}{j}\binom{m-j}{m-l}$ and changing the variable $t\to 1/t$, we obtain

$$\begin{aligned}\Lambda_N(f_j,s)&=\int_1^\infty\left(f_j\left(\frac{it}{\sqrt{N}}\right)-a_j(0)\right)t^s\frac{dt}{t}+i^{2j+k}\int_1^\infty\left(g_j\left(\frac{it}{\sqrt{N}}\right)-b_j(0)\right)t^{k-2j-s}\frac{dt}{t}\\&+i^k\sum_{j+1\le m\le p}(-1)^{m-j}N^{\frac{m-j}{2}}\binom{m}{j}i^{m+j-k}\sum_{j+1\le l\le m}\binom{m-j}{m-l}i^{2l}\int_0^1 f_m\left(\frac{it}{\sqrt{N}}\right)t^{s+j-m}\frac{dt}{t}\\&-\frac{a_j(0)}{s}-\frac{b_j(0)i^{2j+k}}{k-2j-s}.\end{aligned} \tag{4.12}$$

Now observe that

$$\begin{aligned}\sum_{j+1\le l\le m}\binom{m-j}{m-l}i^{2l}&=i^{2j}\sum_{1\le l\le m-j}\binom{m-j}{m-j-l}(-1)^l\\&=i^{2j}\sum_{0\le l\le m-j}\binom{m-j}{l}(-1)^l-i^{2j}\\&=0-i^{2j}=-i^{2j}.\end{aligned} \tag{4.13}$$

Using (4.13) in (4.12), we obtain

$$\begin{aligned}
\Lambda_N(f_j,s) &= \int_1^\infty \left(f_j\left(\frac{it}{\sqrt{N}}\right) - a_j(0)\right) t^s \frac{dt}{t} + i^{2j+k}\int_1^\infty \left(g_j\left(\frac{it}{\sqrt{N}}\right) - b_j(0)\right) t^{k-2j-s}\frac{dt}{t} \\
&\quad - i^{2j+k}\sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}N^{\frac{m-j}{2}} i^{m+j-k}\int_0^1 f_m\left(\frac{it}{\sqrt{N}}\right) t^{s+j-m}\frac{dt}{t} \\
&\quad - \frac{a_j(0)}{s} - \frac{b_j(0)i^{2j+k}}{k-2j-s} \\
&= \int_1^\infty \left(f_j\left(\frac{it}{\sqrt{N}}\right) - a_j(0)\right) t^s \frac{dt}{t} + i^{2j+k}\int_1^\infty \left(g_j\left(\frac{it}{\sqrt{N}}\right) - b_j(0)\right) t^{k-2j-s}\frac{dt}{t} \\
&\quad - i^{2j+k}\sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}N^{\frac{m-j}{2}} i^{m+j-k}\int_0^1 \left(f_m\left(\frac{it}{\sqrt{N}}\right) - a_m(0)\right) t^{s+j-m}\frac{dt}{t} \\
&\quad - \frac{a_j(0)}{s} - \frac{b_j(0)i^{2j+k}}{k-2j-s} - i^{2j+k}\sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}N^{\frac{m-j}{2}} i^{m+j-k}\frac{a_m(0)}{s+j-m}.
\end{aligned} \tag{4.14}$$

Similar to the expression for $\Lambda_N(f_j,s)$ in (4.11), we obtain the following expression for $\Lambda_N(g_j,s)$ in a similar way.

$$\begin{aligned}
\Lambda_N(g_j,s) &= \int_1^\infty \left(g_j\left(\frac{it}{\sqrt{N}}\right) - b_j(0)\right) t^s \frac{dt}{t} + i^{2j-k}\int_1^\infty \left(f_j\left(\frac{i}{\sqrt{N}t}\right) - a_j(0)\right) t^{k-2j-s}\frac{dt}{t} \\
&\quad + \sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}N^{\frac{m-j}{2}} i^{m+j-k}\int_1^\infty \left(f_m\left(\frac{it}{\sqrt{N}}\right) - a_m(0)\right) t^{k-m-j-s}\frac{dt}{t} \\
&\quad - \frac{b_j(0)}{s} - \frac{a_j(0)i^{2j-k}}{k-2j-s} - \sum_{j+1\le m\le p}(-1)^{m-j}N^{\frac{m-j}{2}} i^{m+j-k}\frac{a_m(0)}{k-j-m-s}.
\end{aligned} \tag{4.15}$$

Now from (4.14) and (4.15), we obtain

$$\begin{aligned}
&\Lambda_N(g_j,s) - i^{2j-k}\Lambda_N(f_j,k-2j-s) \\
&\quad = \sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}N^{\frac{m-j}{2}} i^{m+j-k}\int_0^\infty \left(f_m\left(\frac{it}{\sqrt{N}}\right) - a_m(0)\right) t^{k-m-j-s}\frac{dt}{t} \\
&\quad = \sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}N^{\frac{m-j}{2}} i^{m+j-k}\Lambda_N(f_m,k-m-j-s).
\end{aligned}$$

Rearranging the terms we get

$$\Lambda_N(g_j,s) = \sum_{0\le m\le p-j} i^{-(k-2j)-m}N^{\frac{m}{2}}\binom{j+m}{m}\Lambda_N(f_{m+j},k-2j-m-s).$$

Similarly we obtain

$$\Lambda_N(f_j, s) = \sum_{m=0}^{p-j} i^{k-2j-m} N^{\frac{m}{2}} \binom{j+m}{m} \Lambda_N(g_{m+j}, k-2j-m-s).$$

We now prove (2) $\implies$ (1). If $p = k/2$ then the condition (4.7) implies that $a_p(0) = b_p(0)$ and hence $f_p = g_p = a_p(0) = b_p(0)$. If $p = k/2$ and $j \in \{0, 1, \dots, p-1\}$ or $0 \leq p < k/2$ and $j \in \{0, 1, \dots, p-1\}$, then we need to show that

$$g_j(z) = \sum_{j \leq l \leq p} \binom{l}{j} (-1)^{l-j} N^{k/2-j} f_l(-1/Nz)(Nz)^{l+j-k}. \tag{4.16}$$

Since both sides of (4.16) are holomorphic functions, it suffices to show the equality (4.16) on the vertical line $z = it/\sqrt{N}, t > 0$. For $\sigma > \nu + 1$, we have

$$g_j\left(\frac{it}{\sqrt{N}}\right) = b_j(0) + \sum_{n \geq 1} b_j(n) e^{-\frac{2\pi n t}{\sqrt{N}}} = b_j(0) + \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\sigma} \Lambda_N(g_j, s) t^{-s} ds.$$

Since $L(g_j, s)$ is bounded on $\mathrm{Re}(s) = \sigma$, by (4.5), we deduce that on $\mathrm{Re}(s) = \sigma$, for any $\mu > 0$, we have

$$|\Lambda_N(g_j, s)| = O(|\mathrm{Im}(s)|^{-\mu}) \text{ as } |\mathrm{Im}(s)| \to \infty.$$

Similarly one proves that on $\mathrm{Re}(s) = \sigma$, for any $\mu > 0$, we have

$$|\Lambda_N(f_j, s)| = O(|\mathrm{Im}(s)|^{-\mu}) \text{ as } |\mathrm{Im}(s)| \to \infty.$$

Now choose any real number δ such that $k - 2j - p - \delta > \nu + 1$. Then by the functional equation, on $\mathrm{Re}(s) = \delta$, for any $\mu > 0$, we have

$$|\Lambda_N(g_j, s)| \leq \sum_{m=0}^{p-j} N^{\frac{m}{2}} \binom{j+m}{m} |\Lambda_N(f_{j+m}, k-2j-m-s)| = O(|\mathrm{Im}(s)|^{-\mu}) \text{ as } |\mathrm{Im}(s)| \to \infty.$$

By assumption, the function

$$h(s) := \Lambda_N(g_j, s) + \frac{b_j(0)}{s} + \sum_{m=0}^{p-j} \binom{j+m}{m} \frac{i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0)}{k-2j-m-s}$$

is bounded and holomorphic on the vertical strip $\delta \leq \mathrm{Re}(s) \leq \sigma$. Therefore by applying Phragmén-Lindelöf theorem (Theorem 4.2.11) for $h(s)$, we deduce that in the domain $\delta \leq \mathrm{Re}(s) \leq \sigma$, we have

$$\left|\Lambda_N(g_j, s) + \frac{b_j(0)}{s} + \sum_{m=0}^{p-j} \binom{j+m}{m} \frac{i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0)}{k-2j-m-s}\right| = O(|\mathrm{Im}(s)|^{-1}) \text{ as } |\mathrm{Im}(s)| \to \infty.$$

From this we deduce that in the domain $\delta \leq \mathrm{Re}(s) \leq \sigma$, we have $\Lambda_N(g_j, s) = O(|\mathrm{Im}(s)|^{-1})$ as $|\mathrm{Im}(s)| \to \infty$. Without loss of generality we assume that $\sigma > k$ and $\delta < \min\{0, k-3p\}$. The function $\Lambda_N(g_j, s)t^{-s}$ has simple poles at $s = 0$ and $s = k-2j-m$ for $m = 0, 1, \cdots, p$ with the respective residues $-b_j(0)$ and $\binom{j+m}{m} i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0) t^{2j+m-k}$. Now shifting the path of integration from $\mathrm{Re}(s) = \sigma$ to $\mathrm{Re}(s) = \delta$, we obtain

$$g_j\left(\frac{it}{\sqrt{N}}\right) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}(s)=\delta} \Lambda_N(g_j, s)t^{-s} ds + \sum_{m=0}^{p-j} \binom{j+m}{m} i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0) t^{2j+m-k}.$$

Using the functional equation (4.7), we obtain

$$\begin{aligned} g_j\left(\frac{it}{\sqrt{N}}\right) = & \sum_{m=0}^{p-j} i^{-(k-2j)-m} N^{\frac{m}{2}} \binom{j+m}{m} \frac{1}{2\pi i} \int\limits_{\mathrm{Re}(s)=\delta} \Lambda_N(f_{j+m}, k-2j-m-s) t^{-s} ds \\ & + \sum_{m=0}^{p-j} \binom{j+m}{m} i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0) t^{2j+m-k}. \end{aligned}$$

With the change of variable $k-2j-m-s \mapsto s$ in the integral above, we get

$$\begin{aligned} g_j\left(\frac{it}{\sqrt{N}}\right) = & \sum_{m=0}^{p-j} i^{-(k-2j)-m} N^{\frac{m}{2}} \binom{j+m}{m} \frac{1}{2\pi i} \int\limits_{\mathrm{Re}(s)=k-2j-m-\delta} \Lambda_N(f_{j+m}, s) t^{s-k+2j+m} ds \\ & + \sum_{m=0}^{p-j} \binom{j+m}{m} i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0) t^{2j+m-k} \\ = & \sum_{m=0}^{p-j} i^{-(k-2j)-m} N^{\frac{m}{2}} \binom{j+m}{m} t^{2j+m-k} \left(a_{j+m}(0) + \frac{1}{2\pi i} \int\limits_{\mathrm{Re}(s)=k-2j-m-\delta} \Lambda_N(f_{j+m}, s) t^s ds \right). \end{aligned}$$

Also one has

$$\begin{aligned} f_{j+m}\left(\frac{i}{\sqrt{N}t}\right) &= a_{j+m}(0) + \sum_{n=1}^{\infty} a_{j+m}(n) e^{-\frac{2\pi n}{\sqrt{N}}t} \\ &= a_{j+m}(0) + \frac{1}{2\pi i} \int\limits_{\mathrm{Re}(s)=k-2j-m-\delta} \Lambda_N(f_{j+m}, s) t^s ds. \end{aligned}$$

Therefore we have

$$g_j\left(\frac{it}{\sqrt{N}}\right) = \sum_{m=0}^{p-j} i^{2j-m-k} N^{\frac{m}{2}} \binom{j+m}{m} f_{j+m}\left(\frac{i}{\sqrt{N}t}\right) t^{2j+m-k}.$$

□

4.4 Weil's converse theorem for quasimodular forms

Let f_j $(0 \leq j \leq p)$ be as defined in (4.6). For any Dirichlet character ψ of conductor m_ψ, we twist the Fourier series of each f_j to get the twisted Fourier series

$$f_{j,\psi}(z) := \sum_{n=0}^{\infty} \psi(n) a_j(n) q^n. \tag{4.17}$$

The twisted Dirichlet series associated to f_j by the character ψ is the same as the Dirichlet series attached to $f_{j,\psi}$, that is,

$$L(f_j, s, \psi) = \sum_{n=1}^{\infty} \frac{\psi(n) a_j(n)}{n^s}.$$

For $N \geq 1$, we put

$$\Lambda_N(f_j, s, \psi) = \left(\frac{2\pi}{m_\psi \sqrt{N}}\right)^{-s} \Gamma(s) L(f_j, s, \psi).$$

The twist of $\vec{f} = (f_0, f_1, \ldots, f_p)$ by the character ψ is defined by $\vec{f_\psi} = (f_{0,\psi}, f_{1,\psi} \ldots, f_{p,\psi})$. Let F be the function which is a polynomial in $1/y$ associated to $(f_0, f_2, \ldots f_p)$ as in Proposition 4.2.7. Then the twist of F by the character ψ is defined by

$$F_\psi(z) = \sum_{0 \leq \ell \leq p} f_{\ell,\psi}(z)(2iy)^{-\ell}. \tag{4.18}$$

Proposition 4.4.1. *Let k, N be positive integers and let p be a non-negative integer with $p \leq \frac{k}{2}$. For each $j \in \{0, 1, \ldots, p\}$, let $(a_j(n))_{n\geq 0}, (b_j(n))_{n\geq 0}, f_j, g_j, \vec{f}, \vec{g}$ be as in Theorem 4.3.1. Let ψ be a primitive Dirichlet character of conductor $m_\psi(> 1)$. Then the following two statements are equivalent.*

(1) $\vec{f}_\psi|_k \widetilde{W}_{Nm_\psi^2} = C_\psi \vec{g}_{\overline{\psi}}$.

(2) *The completed Dirichlet series $\Lambda_N(f_j, s, \psi)$ and $\Lambda_N(g_j, s, \psi), 0 \leq j \leq p$, can be analytically continued to the whole s-plane, are bounded on any vertical strip, and satisfy the functional equation*

$$\Lambda_N(f_j, s, \psi) = C_\psi \sum_{l=0}^{p-j} i^{k-2j-l} (m_\psi^2 N)^{\frac{l}{2}} \binom{j+l}{l} \Lambda_N(g_{j+l}, k-2j-l-s, \overline{\psi}), \tag{4.19}$$

with

$$C_\psi = \chi(m_\psi)\psi(-N)\tau(\psi)/\tau(\overline{\psi}) = \chi(m_\psi)\psi(N)\tau(\psi)^2/m_\psi, \tag{4.20}$$

where $\tau(\psi) = \sum_{a=1}^{m_\psi} \overline{\psi}(a) e^{2\pi i a/m_\psi}$ is the Gauss sum associated to ψ.

Proof. This follows from Theorem 4.3.1 by taking $f_j = f_{j,\psi}, g_j = C_\psi g_{j,\overline{\psi}}$ and $N = Nm_\psi^2$. □

We next state the following result [23, Lemma 4.3.10]. In the statement of [23, Lemma 4.3.10], it is assumed that the function f is holomorphic. But the proof works even if f is not holomorphic.

Lemma 4.4.2. *Let $k \geq 1$ be any integer and let f be a (not necessarily holomorphic) function on $\mathbb{H}$ with Fourier series expansion $f(z) = \sum_{n=0}^{\infty} a(n, y) q^n$. Let ψ be a primitive Dirichlet character of conductor m_ψ, and let $f_\psi = \sum_{n=0}^{\infty} \psi(n) a(n, y) q^n$. For any real number r, let $T^r = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$. Then we have*

$$f_\psi = \tau(\overline{\psi})^{-1} \sum_{u=1}^{m_\psi} \overline{\psi}(u) \left(f|_k T^{u/m_\psi}\right),$$

where $\tau(\overline{\psi}) = \sum_{a=1}^{m_\psi} \overline{\psi}(a) e^{2\pi i a/m_\psi}$ *is the Gauss sum associated to the character* $\overline{\psi}$*.*

Proposition 4.4.3. *Let* $f \in M_{k,p}^{\mathrm{qm}}(N, \chi)$ *with components* $f_0, f_1, \ldots, f_p$*. Let* m_χ *be the conductor of the character* χ *and let* ψ *be a primitive Dirichlet character of conductor* m_ψ*. Let* $M = \mathrm{lcm}(N, m_\psi^2, m_\psi m_\chi)$*. If* $p = k/2$ *and* $m_\psi > 1$*, then* $f_\psi \in M_{k,p}^{\mathrm{qm}}(M, \chi\psi^2)$ *with components* $f_{0,\psi}, f_{1,\psi}, \ldots, f_{p,\psi}$*. If* $p < k/2$ *or* $m_\psi = 1$*, then* $f_\psi \in M_{k,p-1}^{\mathrm{qm}}(M, \chi\psi^2)$ *with components* $f_{0,\psi}, f_{1,\psi}, \ldots, f_{p-1,\psi}$*.*

Proof. If $p = k/2$ then f_p is a constant and therefore $f_{p,\psi}$ is zero for nontrivial ψ. If $p < k/2$ or ψ is trivial then $f_{p,\psi}$ is non-zero. It is clear that all the functions $f_{0,\psi}, f_{1,\psi}, \ldots, f_{p,\psi}$ are polynomially bounded functions. Therefore we only need to establish the transformation property of f_ψ. Let $\gamma = \left(\begin{smallmatrix} a & b \\ cM & d \end{smallmatrix}\right) \in \Gamma_0(M)$. Then it is easy to verify that

$$\gamma' = T^{u/m_\psi} \gamma T^{-d^2u/m_\psi} \in \Gamma_0(M) \subseteq \Gamma_0(N).$$

If $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, then

$$d' = d - cd^2uM/m_\psi \equiv d \pmod{m_\chi}.$$

Therefore we have

$$f|_k\gamma' = \chi(d) \sum_{j=0}^{p} f_j \left(\frac{cM}{cMz + d - \frac{cd^2uM}{m_\psi}} \right)^j. \tag{4.21}$$

Since

$$T^{u/m_\psi}\gamma = T^{u/m_\psi}\gamma T^{-d^2u/m_\psi} T^{d^2u/m_\psi},$$

we also have

$$f|_k T^{u/m_\psi}\gamma = \left(f|_k T^{u/m_\psi}\gamma T^{-d^2u/m_\psi} \right) |_k T^{d^2u/m} = f|_k \gamma' T^{d^2u/m_\psi}.$$

Now by (4.21), we have

$$\begin{aligned} f|_k\left(T^{u/m_\psi}\gamma\right) &= \left(\chi(d)\sum_{j=0}^{p} f_j\left(\frac{cM}{cMz+d-\frac{cd^2uM}{m_\psi}}\right)^j\right)|_k T^{d^2u/m_\psi} \\ &= \chi(d)\sum_{j=0}^{p}\left(f_j|_{k-2j}T^{d^2u/m_\psi}\right)\left(\frac{cM}{cMz+d}\right)^j. \end{aligned} \tag{4.22}$$

By (4.22) and Lemma 4.4.2, we have

$$\begin{aligned} f_\psi|_k\gamma &= \tau(\overline{\psi})^{-1}\sum_{u=1}^{m_\psi}\overline{\psi}(u)\left(\chi(d)\sum_{j=0}^{p}\left(f_j|_{k-2j}|T^{d^2u/m_\psi}\right)\left(\frac{cM}{cMz+d}\right)^j\right) \\ &= \chi(d)\psi(d^2)\sum_{j=0}^{p}\left(\tau(\overline{\psi})^{-1}\sum_{u=1}^{m_\psi}\overline{\psi}(d^2u)\left(f_j|_{k-2j}|T^{d^2u/m_\psi}\right)\right)\left(\frac{cM}{cMz+d}\right)^j. \end{aligned}$$

For each j, we have $f_j(z+1)=f_j(z)$. Also, if u runs over all residue classes modulo m_ψ then d^2u runs over the same. Therefore we have

$$f_\psi|_k\gamma = \chi(d)\psi(d^2)\sum_{j=0}^{p}\left(\tau(\overline{\psi})^{-1}\sum_{u=1}^{m_\psi}\overline{\psi}(u)\left(f_j|_{k-2j}|T^{u/m_\psi}\right)\right)\left(\frac{cM}{cMz+d}\right)^j.$$

Again by applying Lemma 4.4.2, we have

$$f_\psi|_k\gamma = (\chi\psi^2)(\gamma)\sum_{j=0}^{p} f_{j,\psi}(z)\left(\frac{cM}{cMz+d}\right)^j.$$

□

Proposition 4.4.4. *Let f and g be two quasimodular forms of weight k, depth p, level N and characters χ and $\overline{\chi}$ and components $f_0,\dots,f_p$ and $g_0,\dots,g_p$ respectively. Let ψ be a primitive Dirichlet character of conductor m_ψ. If $(m_\psi,N)=1$ and $\vec{f}|_k\widetilde{W}_N=\vec{g}$, then we have*

$$\vec{f}_\psi|_k\widetilde{W}_{Nm_\psi^2} = C_\psi\vec{g}_{\overline{\psi}}, \tag{4.23}$$

where C_ψ is as in (4.20).

Proof. Let

$$F(z)=\sum_{0\le\ell\le p} f_\ell(z)(2iy)^{-\ell} \quad\text{and}\quad G(z)=\sum_{0\le\ell\le p} g_\ell(z)(2iy)^{-\ell}. \tag{4.24}$$

Since $\vec{f}|_k\widetilde{W}_N = \vec{g}$, we have $F|_kW_N = G$. Now by Lemma 4.4.2 and (4.18), we have

$$\begin{aligned}\tau(\overline{\psi})^{-1}\sum_{u=1}^{m_\psi}\overline{\psi}(u)\left(F|_kT^{u/m_\psi}\right) &= \sum_{0\le\ell\le p}\left(\tau(\overline{\psi})^{-1}\sum_{u=1}^{m_\psi}\overline{\psi}(u)\left(f_\ell|_{k-2\ell}T^{u/m_\psi}\right)\right)(2iy)^{-\ell}\\ &= \sum_{0\le\ell\le p} f_{\ell,\psi}(2iy)^{-\ell}\\ &= F_\psi.\end{aligned} \tag{4.25}$$

Similarly, we have

$$\tau(\overline{\psi})^{-1}\sum_{u=1}^{m_\psi}\overline{\psi}(u)\left(G|_kT^{u/m_\psi}\right) = \sum_{0\le\ell\le p} g_{\ell,\psi}(2iy)^{-\ell} = G_\psi.$$

For any integer u with $(u, m_\psi) = 1$, let n and v be integers such that $nm_\psi - Nuv = 1$. Observe that

$$T^{u/m_\psi}W_{Nm_\psi^2} = \begin{pmatrix} m_\psi & 0\\ 0 & m_\psi\end{pmatrix} W_N \begin{pmatrix} m_\psi & -v\\ -uN & n\end{pmatrix} T^{v/m_\psi}.$$

Therefore we have

$$F|_kT^{u/m_\psi}W_{Nm_\psi^2} = G|_k\begin{pmatrix} m_\psi & -v\\ -uN & n\end{pmatrix} T^{v/m_\psi}.$$

Since G is a nearly holomorphic modular form of weight k, depth p, level N and character $\overline{\chi}$, we have

$$F|_kT^{u/m_\psi}W_{Nm_\psi^2} = \overline{\chi}(n)G|_kT^{v/m_\psi} = \chi(m_\psi)G|_kT^{v/m_\psi}. \tag{4.26}$$

By (4.25) and (4.26), we have

$$\begin{aligned}\tau(\overline{\psi})F_\psi|_kW_{Nm_\psi^2} &= \sum_{u=1}^{m_\psi}\overline{\psi}(u)F|_k\left(T^{u/m_\psi}W_{Nm_\psi^2}\right)\\ &= \chi(m_\psi)\psi(-N)\sum_{v=1}^{m_\psi}\psi(v)G|_kT^{v/m_\psi}\\ &= \chi(m_\psi)\psi(-N)\tau(\psi)G_{\overline{\psi}}.\end{aligned}$$

Therefore

$$F_\psi|_kW_{Nm_\psi^2} = C_\psi G_{\overline{\psi}},$$

and hence we conclude that

$$\vec{f}_\psi|_k\widetilde{W}_{Nm_\psi^2} = C_\psi \vec{g}_{\overline{\psi}},$$

as desired. $\square$

For the next two lemmas, let us fix some terminology. Let k, N be positive integers, p be a non-negative integer with $p \leq \frac{k}{2}$ and χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. For any two integers m, v with $(m, Nv) = 1$, let $n, u \in \mathbb{Z}$ such that $mn - uNv = 1$. Let $\gamma(m, v) := \left(\begin{smallmatrix} m & -v \\ -Nu & n \end{smallmatrix}\right) \in \Gamma_0(N)$. Clearly $\gamma(m, v)$ is not uniquely determined but u mod m is so. We have the following identity.

$$T^{u/m}W_{Nm^2} = mW_N\gamma(m, v)T^{v/m}. \tag{4.27}$$

Lemma 4.4.5. *Let m be an odd prime number or 4 prime to N. Let $\vec{f} = (f_0, \dots f_p)$ and $\vec{g} = (g_0, \dots g_p)$ be any two tuples of holomorphic functions satisfying $\vec{f}|_k\widetilde{W}_N = \vec{g}$ and (4.23) for all primitive Dirichlet characters ψ with conductor $m_\psi = m$. Let F and G be the associated polynomials in $1/y$ to $\vec{f}$ and $\vec{g}$ respectively given by (4.24). Then we have*

$$G|_k(\chi(m) - \gamma(m, u'))T^{u'/m} = G|_k(\chi(m) - \gamma(m, v'))T^{v'/m}$$

for any two integers u' and v' coprime to m_ψ.

Proof. In view of the definition (4.18) and by (4.23), we have

$$F_\psi|_kW_{Nm_\psi^2} = C_\psi G_{\overline{\psi}}.$$

Now by using Lemma 4.4.2, we obtain

$$\sum_{u=1}^{m_\psi} \overline{\psi}(u)F|_kT^{u/m_\psi}W_{Nm_\psi^2} = \chi(m_\psi)\psi(-N)\sum_{u=1}^{m_\psi}\psi(u)G|_kT^{u/m_\psi}. \tag{4.28}$$

For each u with $(u, m_\psi) = 1$, let v be such that $-Nuv \equiv 1 \pmod{m_\psi}$. Then by (4.27) we have

$$F|_k T^{u/m_\psi} W_{Nm_\psi^2} = G|_k \gamma(m_\psi, v) T^{v/m_\psi}. \tag{4.29}$$

Since the left-hand side of (4.29) is independent of the choice of a representative of $u \pmod{m_\psi}$, so is the right-hand side of the choice of $\gamma(m_\psi, v)$. Using (4.29), from (4.28) we obtain

$$\sum_v \psi(v) G|_k (\chi(m_\psi) - \gamma(m_\psi, v)) T^{v/m_\psi} = 0. \tag{4.30}$$

Here v runs over a complete set of representatives of $\mathbb{Z}/m_\psi\mathbb{Z}$. We note that (4.30) is independent of the choice of representative of $\mathbb{Z}/m_\psi\mathbb{Z}$. Let v_1, v_2 be two integers coprime to m_ψ. Multiplying both sides of (4.30) by $\overline{\psi}(v_1) - \overline{\psi}(v_2)$, taking the summation with respect to all nontrivial Dirichlet characters $\psi \pmod{m_\psi}$, we get

$$\sum_{\substack{\psi \\ \text{non-trivial}}} \overline{\psi}(v_1) - \overline{\psi}(v_2) \sum_v \psi(v) G|_k (\chi(m_\psi) - \gamma(m_\psi, v)) T^{v/m_\psi} = 0. \tag{4.31}$$

Using the facts that $\overline{\psi}(v_1) - \overline{\psi}(v_2) = 0$ if ψ is trivial and all non-trivial Dirichlet characters ψ mod m_ψ are primitive characters as m_ψ is an odd prime or 4, we obtain from (4.31) that

$$G|_k (\chi(m_\psi) - \gamma(m_\psi, u)) T^{u/m_\psi} = G|_k (\chi(m_\psi) - \gamma(m_\psi, v)) T^{v/m_\psi}.$$

□

Lemma 4.4.6. *Let m and n (not necessarily distinct) be odd prime numbers or 4 coprime to N. Let $\vec{f}$ and $\vec{g}$ be as in Lemma 4.4.5 satisfying $\vec{f}|_k \widetilde{W}_N = \vec{g}$ and (4.23) for all primitive Dirichlet characters ψ with conductor m_ψ equal to m or n. Assume that f_p and g_p are constants and χ is trivial if $p = k/2$. Then for any $j \in \{0, 1, \ldots p\}$, we have*

$$g_j|_{k-2j} \gamma = \overline{\chi}(\gamma) \sum_{j \le \ell \le p} \binom{\ell}{j} g_\ell(z) \left(X(\gamma)(z)\right)^{\ell - j} \tag{4.32}$$

for every $\gamma \in \Gamma_0(N)$ of the form $\gamma = \left(\begin{smallmatrix} m & -v \\ -Nu & n \end{smallmatrix}\right)$.

Proof. Let G be the polynomial associated to $\vec{g}$ given by (4.24). Put

$$h = G|_k(\chi(m) - \gamma) = \chi(m)G - G|_k\gamma.$$

Using a similar technique as in the proof of [23, Lemma 4.3.14], we get $h|_k\beta = h$, where

$$\beta = \gamma^{-1}T^{-2v/n}\gamma' T^{-2v/m} = \begin{pmatrix} 1 & -2v/m \\ 2uN/n & 4/mn - 3 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} m & v \\ Nu & n \end{pmatrix}.$$

Now, we have

$$\begin{aligned} G|_k\gamma &= \left(\sum_{0\le\ell\le p} g_\ell(z)(2iy)^{-\ell}\right)|_k\gamma \\ &= \sum_{0\le\ell\le p} g_\ell(\gamma z)j(\gamma,z)^{\ell-k}(j(\gamma,z)-2iy)^\ell(2icy)^{-\ell} \\ &= \sum_{0\le\ell\le p} g_\ell(\gamma z)\sum_{0\le j\le\ell}\binom{\ell}{j} j(\gamma,z)^{\ell+j-k}(-c)^{\ell-j}(2iy)^{-j}. \end{aligned} \tag{4.33}$$

Simplifying the right hand side of (4.33), we obtain

$$\begin{aligned} G|_k\gamma &= \sum_{0\le j\le p}\left(\sum_{j\le\ell\le p}\binom{\ell}{j}(-1)^{\ell-j}\left(j(\gamma,z)^{2\ell-k}g_\ell(\gamma z)\right)j(\gamma,z)^{-(\ell-j)}c^{\ell-j}\right)(2iy)^{-j} \\ &= \sum_{0\le j\le p}\left(\sum_{j\le\ell\le p}\binom{\ell}{j}(-1)^{\ell-j}\left(g_\ell|_{k-2\ell}\gamma\right)(z)(X(\gamma))^{\ell-j}\right)(2iy)^{-j}. \end{aligned}$$

Therefore, we have

$$h(z) = \sum_{0\le j\le p} h_j(z)(2iy)^{-j},$$

where

$$h_j(z) := \chi(m)g_j(z) - \sum_{j\le\ell\le p}\binom{\ell}{j}(-1)^{\ell-j}\left(g_\ell|_{k-2\ell}\gamma\right)(z)(X(\gamma)(z))^{\ell-j}. \tag{4.34}$$

By [10, Theorem 5.1.22], we have

$$h_j|_{k-2j}\beta = \sum_{j\le\ell\le p}\binom{\ell}{j}h_\ell\left(X\left(\gamma\right)(z)\right)^{\ell-j}.$$

In particular, we get that h_p is invariant under β. By the assumption on m and n, we obtain that $|\mathrm{tr}(\beta)| = |4/mn - 2| < 2$ and $|\mathrm{tr}(\beta)| \neq 0, 1$. Therefore β is an elliptic matrix and any eigenvalue of β is not a root of unity. Let z_0 be the unique point of $\mathbb{H}$ fixed by β. Put

$$\rho = (z_0 - \bar{z_0})^{-1}\begin{pmatrix} 1 & -z_0 \\ 1 & -\bar{z_0} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

and

$$\mathfrak{p}(w) = (h_p|_{k-2p}\rho^{-1})(w) = j(\rho^{-1}, w)^{-k+2p} h_p(\rho^{-1}w), \quad w \in \mathbb{D},$$

where $\mathrm{GL}_2(\mathbb{C})$ is the group of 2×2 complex invertible matrices and $\mathbb{D}$ is the unit disc. The function $\mathfrak{p}(w)$ is holomorphic on $\mathbb{D}$. Writing

$$\rho\beta\rho^{-1} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \quad \textit{with } \zeta \in \mathbb{C},$$

we have

$$\begin{aligned} \mathfrak{p}(\rho\beta\rho^{-1}w) &= (h_p|_{k-2p}\rho^{-1})(\rho\beta\rho^{-1}w) \\ &= j(\rho\beta\rho^{-1}, w)^{k-2p}\left(j(\rho\beta\rho^{-1}, w)^{-k+2p}(h_p|_{k-2p}\rho^{-1})(\rho\beta\rho^{-1}w)\right) \\ &= j(\rho\beta\rho^{-1}, w)^{k-2p}(h_p|_{k-2p}\beta\rho^{-1})(w). \end{aligned}$$

Since h_p is invariant under β, we get

$$\mathfrak{p}(\rho\beta\rho^{-1}w) = j(\rho\beta\rho^{-1}, w)^{k-2p}(h_p|_{k-2p}\rho^{-1})(w).$$

Thus, we get that

$$\mathfrak{p}(\rho\beta\rho^{-1}w) = j(\rho\beta\rho^{-1}, w)^{k-2p}\mathfrak{p}(w).$$

Hence we have

$$\mathfrak{p}(\zeta^2 w) = \zeta^{-k+2p}\mathfrak{p}(w).$$

Since $\mathfrak{p}(w)$ is holomorphic on $\mathbb{D}$, it has a Taylor expansion around $w = 0$. Let $\mathfrak{p}(w) = \sum_{n=0}^{\infty} a_n w^n$ be the Taylor expansion of $\mathfrak{p}(w)$ around $w = 0$. Then we have $\zeta^{2n}a_n = \zeta^{-k+2p}a_n$ for all $n \geq 0$. Since ζ is an eigenvalue of β which is not a root of unity, we get $a_n = 0$ for all $n \geq 0$ when $p < k/2$. This implies that $h_p \equiv 0$ if $p < k/2$. If $p = k/2$, then from (4.34) it is trivial to see that $h_p \equiv 0$ as g_p is constant and χ is trivial. Thus we have

$$h_{p-1}|_{k-2(p-1)}\beta = h_{p-1}.$$

Following the above arguments again, we get $h_{p-1} \equiv 0$. Proceeding recursively, we see that $h_j \equiv 0$ for every $0 \leq j \leq p$. This means that

$$\chi(m)g_j = \sum_{j \leq \ell \leq p} \binom{\ell}{j} (-1)^{\ell-j} \left(g_\ell|_{k-2\ell}\gamma\right) \left(X(\gamma)(z)\right)^{\ell-j}, \quad 0 \leq j \leq p.$$

Let $P(Y) = \sum\limits_{j=0}^{p} g_j Y^j$ be a polynomial in Y. By using the above identity, we have

$$P(Y) = \sum_{0 \leq j \leq p} g_j Y^j = \overline{\chi}(m) \sum_{0 \leq \ell \leq p} \left(g_\ell|_{k-2\ell}\gamma\right) \left(Y - X(\gamma)(z)\right)^{\ell}.$$

Putting $Y = Y + X(\gamma)(z)$ in the above equation, we get

$$\sum_{0 \leq j \leq p} g_j \left(Y + X(\gamma)(z)\right)^j = \overline{\chi}(m) \sum_{0 \leq \ell \leq p} \left(g_\ell|_{k-2\ell}\gamma\right) Y^{\ell}.$$

By comparing the coefficients of Y^j both sides and using the fact that $\chi(m) = \overline{\chi}(n) = \overline{\chi}(\gamma)$, we have (4.32). $\square$

Now we are ready to state the main theorem of this chapter [8, Theorem 1.1] which can be considered as an analogue of Weil's converse theorem for quasimodular forms. We denote $\mathcal{P}$ for a subset of positive integers such that any element of $\mathcal{P}$ is either an odd prime or 4 which is relatively prime to N and for any two relatively prime positive integers a and b, the intersection of $\mathcal{P}$ with the set $\{a + nb : n \in \mathbb{Z}\}$ is non-empty.

Theorem 4.4.7. *Let k, N be positive integers and let p be a non-negative integer with $p \leq k/2$. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$, and χ is trivial when $p = k/2$. Let $\vec{f} = (f_0, f_1, \ldots, f_p)$ and $\vec{g} = (g_0, g_1, \ldots, g_p)$ be two* *vectors, each one consisting of* $p+1$ *functions given by the Fourier expansion* (4.6) *corresponding to the sequences* $(a_j(n))$ *and* $(b_j(n))$ *respectively,* $0 \leq j \leq p$. *Moreover, let* f_p *and* g_p *be non-zero constant functions if* $p = k/2$. *Assume that* $a_j(n)$ *and* $b_j(n)$ *are bounded by* $O(n^{\nu})$ *for some* $\nu > 0$. *Then the following two statements are equivalent.*

(1) *The functions* f_0 *and* g_0 *are quasimodular forms of weight* k, *depth* p, *level* N *and characters* χ *and* $\overline{\chi}$, *component functions* $f_0, f_1, \ldots, f_p$ *and* $g_0, g_1, \ldots, g_p$ *respectively. Moreover, we have* $\vec{f}\underset{k}{|}\widetilde{W}_N = \vec{g}$.

(2) (a) *For each* $j \in \{0, 1, \cdots, p\}$, *the completed Dirichlet series* $\Lambda_N(f_j, s)$ *and* $\Lambda_N(g_j, s)$ *admit meromorphic continuations to the whole* s*-plane and they satisfy the following functional equations.*

$$\Lambda_N(f_j, s) = \sum_{m=0}^{p-j} i^{k-2j-m} N^{\frac{m}{2}} \binom{j+m}{m} \Lambda_N(g_{j+m}, k-2j-m-s).$$

Moreover, for each $j \in \{0, 1, \ldots, p\}$, *the functions*

$$\Lambda_N(f_j, s) + \frac{a_j(0)}{s} + \sum_{m=0}^{p-j} \binom{j+m}{m} \frac{i^{k-2j-m} N^{\frac{m}{2}} b_{j+m}(0)}{k-2j-m-s}$$

$$\Lambda_N(g_j, s) + \frac{b_j(0)}{s} + \sum_{m=0}^{p-j} \binom{j+m}{m} \frac{i^{-(k-2j)-m} N^{\frac{m}{2}} a_{j+m}(0)}{k-2j-m-s}$$

are holomorphic on the whole s*-plane and bounded on any vertical strip.*

(b) *For any primitive Dirichlet character* ψ *whose conductor* $m_\psi \in \mathcal{P}$, *each of the completed Dirichlet series* $\Lambda_N(f_j, s, \psi)$ *and* $\Lambda_N(g_j, s, \psi)$ *can be analytically continued to the whole* s*-plane, bounded on any vertical strip, and satisfies the following functional equation.*

$$\Lambda_N(f_j, s, \psi) = C_\psi \sum_{l=0}^{p-j} i^{k-2j-l} (m_\psi^2 N)^{\frac{l}{2}} \binom{j+l}{l} \Lambda_N(g_{j+l}, k-2j-l-s, \overline{\psi}),$$

where C_ψ *is same as in* (4.20).

Proof. We observe that the implication (1) $\implies$ (2) follows immediately from our previous results. By Theorem 4.3.1, we have (2)(a). By Proposition 4.4.4 and Proposition 4.4.1, we have (2)(b).

Next we prove (2) $\implies$ (1). By Lemma 4.2.6 and Lemma 4.2.3, we see that $f_0, f_1, \cdots, f_p$ and $g_0, g_1, \cdots, g_p$ define holomorphic functions on $\mathbb{H}$ and all are polynomially bounded functions. Now, by Theorem 4.3.1 and Proposition 4.4.1, we have

$$\vec{f}|_k \widetilde{W}_N = \vec{g}, \quad \vec{f_\psi}|_k \widetilde{W}_{Nm_\psi^2} = C_\psi \vec{g_{\overline{\psi}}}$$

for any primitive Dirichlet character ψ whose conductor $m_\psi \in \mathcal{P}$. The constant C_ψ is same as in (4.20). We need to establish the quasimodular transformation properties for f_0 and g_0. We first prove that for each j with $0 \leq j \leq p$, we have

$$g_j|_{k-2j}\gamma = \overline{\chi}(\gamma) \sum_{j \leq \ell \leq p} \binom{\ell}{j} g_\ell \left(X(\gamma)(z)\right)^{\ell-j} \quad \text{for all} \quad \gamma \in \Gamma_0(N). \tag{4.35}$$

Since each g_j has Fourier series expansion given by (4.6), we have $g_j(z+1) = g_j$. Therefore, if

$$\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)$$

and $c = 0$, then we have (4.35) as $\chi(-1) = (-1)^k$. Now assume that $c \neq 0$. Since $(a, cN) = (d, cN) = 1$, there exist integers s, t such that $a + tcN \in \mathcal{P}$ and $d + scN \in \mathcal{P}$. Put

$$\begin{aligned} m &= a + tcN, \\ n &= d + scN, \\ u &= -c, \\ v &= -(b + sm + stuN + nt). \end{aligned}$$

Then we have

$$\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m & -v \\ -uN & n \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}.$$

By Lemma 4.4.6, we have

$$\begin{aligned}
g_j|_{k-2j}\gamma &= g_j|_{k-2j}\begin{pmatrix} m & -v \\ -uN & n \end{pmatrix}\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \\
&= \left(\overline{\chi}(n)\sum_{j\le \ell\le p}\binom{\ell}{j} g_\ell\left(\frac{-uN}{-uNz+n}\right)^{\ell-j}\right)|_{k-2j}\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \\
&= \overline{\chi}(d)\sum_{j\le \ell\le p}\binom{\ell}{j} g_\ell\left(\frac{-uN}{-uN(z-s)+n}\right)^{\ell-j} \\
&= \overline{\chi}(\gamma)\sum_{j\le \ell\le p}\binom{\ell}{j} g_\ell\left(\frac{cN}{cNz+d}\right)^{\ell-j}.
\end{aligned}$$

This implies that $g_0 \in M_{k,p}^{\mathrm{qm}}(N,\overline{\chi})$ with components $g_0, g_1, \ldots, g_p$. Since

$$\vec{f} = (-1)^k \vec{g}|_k \widetilde{W}_N,$$

by Theorem 4.2.10, we deduce that $f_0 \in M_{k,p}^{\mathrm{qm}}(N,\chi)$ with components $f_0, f_1, \ldots, f_p$.

$\square$

4.5 Applications

In this section, we provide some applications of Theorem 4.4.7. In his celebrated 1916 paper [26], Ramanujan introduced the following function. For any two non-negative integers k and ℓ with $k \le \ell$, define

$$\Phi_{k,\ell}(z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^k m^\ell q^{mn} = \sum_{n=1}^{\infty} n^k\left(\sum_{0<d|n} d^{\ell-k}\right) q^n.$$

If $0 \le \ell < m$ and $\ell + m$ is odd then $\Phi_{\ell,m}$ is (up to a constant) the ℓ-th derivative of Eisenstein series $E_{m-\ell+1}$. The Eisenstein series $E_{m-\ell+1}$ is a modular form on $SL_2(\mathbb{Z})$ if $m - \ell + 1 \ge 4$ is even. If $m - \ell + 1 = 2$, $E_{m-\ell+1}$ is a quasimodular on $SL_2(\mathbb{Z})$. For $\ell = 0$ and m odd, $\Phi_{\ell,m}$ is a scalar multiple of E_{m+1} up to an additive constant. From this, it follows that for $0 \le \ell < m$ and $\ell + m$ odd, $\Phi_{\ell,m}$ is a quasimodular form of weight $\ell + m + 1$ and depth less than or equal to $\ell + 1$ on $SL_2(\mathbb{Z})$. When $\ell + m$

is even, it was proved in [1] that $c + \Phi_{\ell,m}$ is not a quasimodular form on $SL_2(\mathbb{Z})$ for any complex number c. But it is not proved if $c + \Phi_{\ell,m}$ is not a quasimodular form of level N for any positive integer $N > 1$. We have the following result [8, Corollary 1.4] which proves exactly the same.

Corollary 4.5.1. *Let $0 \leq \ell \leq m$ be such that $\ell + m$ is even. For any integer $N \geq 1$, any Dirichlet character χ modulo N and any constant $c \in \mathbb{C}$, the function $c + \Phi_{\ell,m}(z)$ is not a quasimodular form of level N and character χ.*

Proof. Suppose that $\ell + m$ is even and there exists a constant $c \in \mathbb{C}$ such that $c + \Phi_{\ell,m}$ is a quasimodular form of some weight and some depth, level N and character χ for some $N \geq 1$. Note that the completed Dirichlet series attached to $c + \Phi_{\ell,m}$ is

$$\Lambda_{\Phi_{\ell,m}}(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s)\zeta(s-\ell)\zeta(s-m).$$

It is clear that $\Lambda_{\Phi_{\ell,m}}$ has poles at $-r$ for each positive integer r such that $-r - \ell$ and $-r - m$ are both odd. By Theorem 4.4.7, $\Lambda_{\Phi_{\ell,m}}(s)$ can not have a pole at any negative integer due to the quasimodularity assumption on $c + \Phi_{\ell,m}$. This gives a contradiction. $\square$

Next, we deduce the oscillatory behaviour of the Fourier coefficients of certain quasimodular forms. Following [25], we say a sequence of complex numbers $(a_n)_{n\geq 1}$ is oscillatory if for each real number $\phi \in [0, \pi)$, either the sequence $\left(\mathrm{Re}\left(e^{-i\phi}a_n\right)\right)_{n\geq 1}$ has infinitely many sign changes or is trivial. Observe that $\mathrm{Re}\left(e^{-i\phi}a_n\right)$ is equal to $\mathrm{Re}(a_n)$ and $\mathrm{Im}(a_n)$ if ϕ is equal to 0 and $\pi/2$ respectively. We have the following result which is special case of [25, Theorem 1].

Theorem 4.5.2. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. Let $D(s) = \sum\limits_{n=1}^{\infty} \frac{a_n}{n^s}$ be a non-trivial Dirichlet series which converges somewhere. If the function D is* *holomorphic on the whole real line and has infinitely many real zeros, then* $(a_n)_{n\geq 1}$ *is oscillatory.*

We have the following result [8, Corollary 1.5] which proves that the sequence of Fourier coefficients of a non-zero quasimodular forms of level N is oscillatory.

Corollary 4.5.3. *Let* f *be a non-zero quasimodular form of weight* k*, depth* p*, level* N *and character* χ *such that the constant Fourier coefficient of* f *and that of all the component functions of* $f|_k\widetilde{W}_N$ *are zero. Then the sequence of Fourier coefficients* $(a(n))_{n\geq 1}$ *of* f *is oscillatory.*

Proof. Since the constant Fourier coefficients of f and all the component functions of $f|_k\widetilde{W}_N$ are zero, by the direct part of Theorem 4.4.7 we have that $\Lambda_N(f,s)$ is holomorphic on the whole complex plane. Since $\Gamma(s)$ has poles at all the non-positive integers, the corollary follows from Theorem 4.5.2. □

Chapter 5

Converse theorem for weakly holomorphic quasimodular forms

5.1 Introduction

In this chapter, we define L-functions associated to weakly holomorphic quasimodular forms and derive functional equations of these L-functions. We also obtain a converse theorem for weakly holomorphic quasimodular forms.

Unlike modular forms, the usual L-series associated to a weakly holomorphic modular form is a nowhere convergent series since the Fourier coefficients grow exponentially. In [2], certain L-functions associated to weakly holomorphic modular forms have been defined. But the analytic properties of these L-functions have not been studied. In [12], Diamantis et al. defined L-functions associated to weakly holomorphic modular forms and obtained their functional equations. The L-functions studied in [12] are generalizations of the L-functions studied in [2]. In [12], Diamantis et al. also obtained a converse theorem for weakly holomorphic modular forms.

A weakly holomorphic quasimodular form is a certain generalization of a quasimodular form (see Definition 1.3.2). In [32], Wang and Zhang defined and studied weakly holomorphic quasimodular forms on $SL_2(\mathbb{Z})$. Following the methods of [2], they defined L-functions associated to weakly holomorphic quasimodular forms on

$SL_2(\mathbb{Z})$ and obtained functional equation for these L-functions. Using the methods of [12], we define L-functions associated to weakly holomorphic quasimodular forms on $\Gamma_0(N)$. Our L-functions are generalization of the L-functions defined in [32]. We obtain functional equations for the L-functions associated to weakly holomorphic quasimodular forms on $\Gamma_0(N)$. We also obtain a converse theorem for weakly holomorphic quasimodular forms of level N. The results of this chapter are contained in [5].

5.2 Notations and preliminaries

5.2.1 Weakly holomorphic quasimodular forms

Let k, N be positive integers and let p be a non-negative integer. Let χ be a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. Let $f \in M_{k,p}^{\mathrm{qm},!}(N, \chi)$ with components $f_0, f_1, \ldots, f_p$. We also denote f by $\vec{f} = (f_0, f_1, \ldots, f_p)$. As in the case of quasimodular forms, we have $f_0 = f$ in this case also. Moreover, the following proposition [5, Proposition 2.2] shows that each component f_j of f is again a weakly holomorphic quasimodular form of weight $k-2j$ and depth $p-j$. The proof of this result is similar to the proof of [29, Proposition 3.3]. Therefore we omit the proof here.

Proposition 5.2.1. *Let $f \in M_{k,p}^{\mathrm{qm},!}(N, \chi)$ with components $f_0, f_1, \ldots, f_p$. Then for every $0 \leq j \leq p$, we have*

$$f_j|_{k-2j}\gamma(z) = \chi(\gamma) \sum_{v=0}^{p-j} \binom{j+v}{v} f_{j+v}(z) \left(X(\gamma)(z)\right)^v, \quad \textit{for all } \gamma \in \Gamma_0(N).$$

Now, we prove the following theorem [5, Theorem 2.3].

Theorem 5.2.2. *Let $f \in M_{k,p}^{\mathrm{qm},!}(N, \chi)$ with components $f_0, f_1, \ldots, f_p$. Then each f_j has a Fourier expansion of the form*

$$f_j(z) = \sum_{n=-n_0}^{\infty} a_j(n) q^n,$$

where $a_j(n) \in \mathbb{C}$ *and* $n_0 \geq 0$ *with*

$$a_j(n) = O(e^{C_j\sqrt{|n|}}) \text{ for some } C_j > 0. \tag{5.1}$$

Proof. By Proposition 5.2.1, each f_j is a weakly holomorphic quasimodular form. Therefore each f_j is holomorphic, periodic and it satisfies the condition (2) of Definition 1.3.2. Now using the same idea used in [3, pp. 55], we obtain the required Fourier expansion of f_j. The bound for the Fourier coefficients of f_j follows from [3, Lemma 3.4]. □

We finish this subsection by stating a lemma [5, Lemma 2.4] which will be useful in establishing Theorem 5.4.1.

Lemma 5.2.3. *For a sequence* $(a_j(n))_{n\geq -n_0}$ *of complex numbers, let*

$$f(z) = \sum_{n=-n_0}^{\infty} a(n)q^n.$$

If $a(n) = O(e^{C\sqrt{|n|}})$ *for some* $C > 0$*, then the above series defining* $f(z)$ *converges absolutely and uniformly on any compact subset of* $\mathbb{H}$ *and hence* $f(z)$ *is holomorphic on* $\mathbb{H}$*. Moreover,* $f(z) - P(z) = O\left(e^{-2\pi y}\right)$ *as* $y \to \infty$ *and* $f(z) = O(e^{\epsilon/y})$ *as* $y \to 0$ *uniformly on* $\mathrm{Re}(z)$*, where* $P(z) = \sum_{n=0}^{n_0} a(-n)e^{-2\pi inz}$ *and some* $\epsilon > 0$*.*

Proof. Let $z = x + iy$. We have

$$\sum_{n=-n_0}^{\infty} |a(n)||e^{2\pi iz}| \leq \sum_{n=-n_0}^{m-1} e^{C\sqrt{|n|}} e^{2\pi|n|y} + \sum_{n=m}^{\infty} e^{C\sqrt{n}} e^{-2\pi ny}, \tag{5.2}$$

where m is a sufficiently large positive integer. Now on any compact subset of $\mathbb{H}$, we get

$$\sum_{n=m}^{\infty} e^{C\sqrt{n}} e^{-2\pi ny} \leq \sum_{n=m}^{\infty} e^{L(p\sqrt{n}-n)}, \tag{5.3}$$

where L is a positive constant and p is a positive integer. Therefore from (5.2) and (5.3), we get that $f(z)$ is convergent absolutely and uniformly on any compact subset of $\mathbb{H}$. Similarly we obtain that $f(z) - P(z)$ is bounded when $y \to \infty$. Now put

$$g(z) = \sum_{n=0}^{\infty} a(n+1)e^{2\pi i n z}.$$

Then $g(z)$ is also bounded as $y \to \infty$. Therefore we obtain

$$f(z) - P(z) = e^{2\pi i z} g(z) = O(e^{-2\pi y}) \text{ as } y \to \infty.$$

Now from (5.2), we get

$$\begin{aligned} |f(z)| &\leq \sum_{n=-n_0}^{m-1} e^{C\sqrt{|n|}} e^{2\pi|n|y} + \sum_{n=m}^{\infty} e^{C\sqrt{n}} e^{-2\pi n y} \\ &\leq \sum_{n=-n_0}^{m-1} e^{C\sqrt{|n|}} e^{2\pi|n|y} + \sum_{n=m}^{\infty} e^{-2\pi y\sqrt{n}} \\ &\leq \sum_{n=-n_0}^{m-1} e^{C\sqrt{|n|}} e^{2\pi|n|y} + \frac{M}{y^4}. \end{aligned}$$

Since $1/y \leq e^{1/y}$ as $y \to 0$, we get $f(z) = O(e^{\epsilon/y})$ as $y \to 0$ for some $\epsilon > 0$. □

5.2.2 Nearly weakly holomorphic modular forms

In this subsection, we briefly review some results on nearly weakly holomorphic modular forms and their relation with weakly holomorphic quasimodular forms. If $F \in M_{k,p}^{\mathrm{nh},!}(N, \chi)$, then we write

$$F(z) = \sum_{0 \leq j \leq p} f_j(z)(2iy)^{-j}$$

for some holomorphic functions f_j on $\mathbb{H}$ which satisfy condition (2) of Definition 1.3.2. We have the following result which provides a relation between nearly weakly holomorphic modular forms and weakly holomorphic quasimodular forms. The proof of this result is similar to the proof of Proposition 4.2.7. Therefore we omit the proof here.

Proposition 5.2.4. *Let $f_0, f_1, \ldots, f_p$ be holomorphic functions on $\mathbb{H}$ satisfying condition (2) of Definition 1.3.2. Define the function $F : \mathbb{H} \longrightarrow \mathbb{C}$ by*

$$F(z) = \sum_{0 \le j \le p} f_j(z)(2iy)^{-j}.$$

Then the following two statements are equivalent.

(1) *The function $F \in M_{k,p}^{\mathrm{nh},!}(N, \chi)$.*

(2) *The function $f_0 \in M_{k,p}^{\mathrm{qm},!}(N, \chi)$ with components $f_0, f_1, \ldots, f_p$.*

As in the case of quasimodular forms, the image of a weakly holomorphic quasimodular form of level N under the usual Fricke involution operator W_N is not a weakly holomorphic quasimodular form. We use the same method of Chapter 4 to define the operator W_N appropriately with the help of Proposition 5.2.4 to overcome this difficulty. Let

$$F(z) = \sum_{0 \le m \le p} f_m(z)(2iy)^{-m} \in M_{k,p}^{\mathrm{nh},!}(N, \chi).$$

For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$, we have

$$F|_k\gamma = \sum_{0 \le \ell \le p} \left(\sum_{\ell \le m \le p} \binom{m}{\ell} (\det\gamma)^{k/2-m} f_m(\gamma z) j(\gamma, z)^{m+\ell-k} (-c)^{m-\ell} \right) (2iy)^{-\ell}.$$

In particular, for $\gamma = W_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$, we have

$$F|_k W_N = \sum_{0 \le \ell \le p} \widetilde{f}_\ell(z)(2iy)^{-\ell}, \tag{5.4}$$

where

$$\widetilde{f}_\ell(z) = \sum_{\ell \le m \le p} \binom{m}{\ell} (-1)^{m-\ell} N^{k/2-\ell} f_m\left(-\frac{1}{Nz}\right) (Nz)^{m+\ell-k}. \tag{5.5}$$

By the transformation property of F with respect to the group $\Gamma_0(N)$ and (5.5), we get the following [5, Lemma 2.8].

Lemma 5.2.5. *If $F \in M_{k,p}^{\mathrm{nh},!}(N,\chi)$ then $F|_k W_N \in M_{k,p}^{\mathrm{nh},!}(N,\overline{\chi})$.*

In the view of Proposition 5.2.4 and Lemma 5.2.5, we define the operator $\widetilde{W}_N$ on weakly holomorphic quasimodular forms of level N, which serves our purpose.

Definition 5.2.6. *Let $f \in M_{k,p}^{\mathrm{qm},!}(N,\chi)$ with components $f_0, f_1, \ldots, f_p$ and let $\vec{f} = (f_0, f_1, \ldots, f_p)$. Then the action of $\widetilde{W}_N$ on the weakly holomorphic quasimodular form $\vec{f}$ is defined by*

$$\vec{f}|_k\widetilde{W}_N = (\widetilde{f_0}, \widetilde{f_1}, \ldots, \widetilde{f_p}),$$

where

$$\widetilde{f_\ell}(z) = \sum_{\ell \le j \le p} \binom{j}{\ell} (-1)^{j-\ell} N^{k/2-\ell} (Nz)^{j+\ell-k} f_j\left(-\frac{1}{Nz}\right), \quad 0 \le \ell \le p. \tag{5.6}$$

Proposition 5.2.7. *If $\vec{f} = (f_0, f_1, \ldots, f_p) \in M_{k,p}^{\mathrm{qm},!}(N,\chi)$ then $\vec{f}|_k\widetilde{W}_N = (\widetilde{f_0}, \widetilde{f_1}, \ldots, \widetilde{f_p}) \in M_{k,p}^{\mathrm{qm},!}(N,\overline{\chi})$, where $\widetilde{f_\ell}$ is defined by (5.6). Moreover, for $0 \le \ell \le p$, we have*

$$f_\ell(z) = i^{2k} \sum_{\ell \le j \le p} \binom{j}{\ell} (-1)^{j-\ell} N^{k/2-\ell} (Nz)^{j+\ell-k} \widetilde{f_j}\left(-\frac{1}{Nz}\right), \tag{5.7}$$

and $\vec{f}|_k\widetilde{W}_N|_k\widetilde{W}_N = (-1)^k \vec{f}$.

Proof. Since each component of $\vec{f}$ satisfies condition (2) of Definition 1.3.2, from (5.6) it is clear that each component of $\vec{f}|_k\widetilde{W}_N$ also satisfies condition (2) of Definition 1.3.2. By Proposition 5.2.4, we get that the function $F(z) := \sum_{0 \le j \le p} f_j(z)(2iy)^{-j} \in M_{k,p}^{\mathrm{nh},!}(N,\chi)$. Now by Lemma 5.2.5 and again Proposition 5.2.4, we see that $\widetilde{f} := \widetilde{f_0} \in M_{k,p}^{\mathrm{qm},!}(N,\overline{\chi})$ with components $\widetilde{f_0}, \widetilde{f_1}, \ldots, \widetilde{f_p}$. Remaining part of the proof is similar to the proof of Proposition 4.2.10. $\square$

5.3 L-series associated to weakly holomorphic quasimodular forms

Let $C(\mathbb{R},\mathbb{C})$ be the space of piecewise smooth complex-valued functions on $\mathbb{R}$. The Laplace transform of a piecewise smooth complex-valued function φ on $\mathbb{R}$ is given by

$$(\mathcal{L}\varphi)(s) := \int_0^\infty e^{-st}\varphi(t)dt \tag{5.8}$$

for each $s \in \mathbb{C}$ for which the integral converges absolutely. Let f be a function on $\mathbb{H}$ which is given by an absolutely convergent series

$$f(z) = \sum_{n=-n_0}^{\infty} a(n)q^n. \tag{5.9}$$

Let $\mathcal{F}_f$ be the space of functions $\varphi \in C(\mathbb{R},\mathbb{C})$ such that the integral defining $(\mathcal{L}\varphi)(s)$ and the series

$$\sum_{n=-n_0}^{\infty} |a(n)|(\mathcal{L}|\varphi|)\,(2\pi n) \tag{5.10}$$

converge.

Definition 5.3.1. *Let f be a function on $\mathbb{H}$ given by the series expansion as in* (5.9)*. The L-series of f is defined to be the map $L_f\colon \mathcal{F}_f \to \mathbb{C}$ such that for each $\varphi \in \mathcal{F}_f$,*

$$L_f(\varphi) = \sum_{n=-n_0}^{\infty} a(n)(\mathcal{L}\varphi)(2\pi n). \tag{5.11}$$

Lemma 5.3.2. *Let f be a function on $\mathbb{H}$ given by the series expansion as in* (5.9)*. For any $\varphi \in \mathcal{F}_f$, the L-series $L_f(\varphi)$ is given by*

$$L_f(\varphi) = \int_0^\infty f(it)\varphi(t)dt. \tag{5.12}$$

Proof. By Definition 5.3.1, for $\varphi \in \mathcal{F}_f$,

$$L_f(\varphi) = \sum_{n=-n_0}^{\infty} a(n)(\mathcal{L}\varphi)(2\pi n) \tag{5.13}$$

and this series converges absolutely. Now by (5.8), we have

$$(\mathcal{L}\varphi)(2\pi n) = \int_0^\infty e^{-2\pi n t}\varphi(t)dt. \tag{5.14}$$

By using the above expression in (5.13), we get

$$L_f(\varphi) = \sum_{n=-n_0}^{\infty} \int_0^\infty a(n)e^{-2\pi n t}\varphi(t)dt. \tag{5.15}$$

Since $\varphi \in \mathcal{F}_f$, we can interchange the order of summation and integration and we get the result. □

Our goal in the remainder of this section is to obtain a functional equation for the L-series $L_f(\varphi)$, where $f \in M_{k,p}^{\mathrm{qm},!}(N,\chi)$. Let $f_0, f_1, \ldots, f_p$ be the component functions of f. Then by Theorem 5.2.2, we have that for each $0 \leq j \leq p$, f_j has a Fourier expansion of the form

$$f_j(z) = \sum_{n=-n_0}^{\infty} a_j(n)q^n. \tag{5.16}$$

The L-series of each f_j is defined to be the map $L_{f_j} : \mathcal{F}_{f_j} \to \mathbb{C}$ such that, for $\varphi \in \mathcal{F}_{f_j}$,

$$L_{f_j}(\varphi) = \sum_{n=-n_0}^{\infty} a_j(n)(\mathcal{L}\varphi)(2\pi n). \tag{5.17}$$

By Lemma 5.3.2, we have

$$L_{f_j}(\varphi) = \int_0^\infty f_j(it)\varphi(t)dt. \tag{5.18}$$

Let D be a positive integer. For a Dirichlet character ψ modulo D, the twisted function $f_{j,\psi}$ is defined by

$$f_{j,\psi}(z) := \sum_{u=1}^{D} \overline{\psi(u)} \left(f_j|_k T^{u/D}\right)(z) = \sum_{n=-n_0}^{\infty} \tau_{\bar{\psi}}(n)a_j(n)q^n, \tag{5.19}$$

where $T^r = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$ for any real number r and $\tau_\psi(n) = \sum_{u=1}^{D} \psi(u)e^{2\pi i n \frac{u}{D}}$, the generalized Gauss sum. Then for each $\varphi \in \mathcal{F}_{f_{j,\psi}}$, the L-series associated to $f_{j,\psi}$ is given by

$$L_{f_{j,\psi}}(\varphi) = \sum_{n=-n_0}^{\infty} \tau_{\bar{\psi}}(n)a_j(n)(\mathcal{L}\varphi)(2\pi n). \tag{5.20}$$

By Lemma 5.3.2, we have

$$L_{f_{j,\psi}}(\varphi) = \int_0^\infty f_{j,\psi}(it)\varphi(t)dt. \tag{5.21}$$

Let the twist of $\vec{f} = (f_0, f_1, \dots, f_p)$ by the character ψ be $\vec{f_\psi} = (f_{0,\psi}, f_{1,\psi} \dots, f_{p,\psi})$. Let F be the function which is a polynomial in $1/y$ associated to $(f_0, f_1 \dots, f_p)$ as in Proposition 5.2.4. Then the twist of F by the character ψ is defined by

$$F_\psi(z) = \sum_{0\le \ell\le p} f_{\ell,\psi}(z)(2iy)^{-\ell}. \tag{5.22}$$

Proposition 5.3.3. *Let f and g be two quasimodular forms of weight k, depth p, level N with characters χ and $\overline{\chi}$ and components $f_0, \dots, f_p$ and $g_0, \dots, g_p$ respectively. Let ψ be a Dirichlet character modulo D with $(D,N) = 1$. If $\vec{f}|_k\widetilde{W}_N = \vec{g}$, then we have*

$$\vec{f_\psi}|_k\widetilde{W}_{ND^2} = \chi(D)\psi(-N)\vec{g}_{\overline{\psi}}. \tag{5.23}$$

Proof. Let

$$F(z) = \sum_{0\le \ell\le p} f_\ell(z)(2iy)^{-\ell} \quad \text{and} \quad G(z) = \sum_{0\le \ell\le p} g_\ell(z)(2iy)^{-\ell}. \tag{5.24}$$

Since $\vec{f}|_k\widetilde{W}_N = \vec{g}$, we have $F|_kW_N = G$. Now

$$F_\psi = \sum_{0\le \ell\le p} f_{\ell,\psi}(2iy)^{-\ell} = \sum_{0\le \ell\le p} \left(\sum_{u=1}^{D} \overline{\psi}(u)\left(f_\ell|_{k-2\ell}T^{u/D}\right)\right)(2iy)^{-\ell} = \sum_{u=1}^{D} \overline{\psi}(u)\left(F|_kT^{u/D}\right). \tag{5.25}$$

Similarly, we have

$$G_\psi = \sum_{0\le \ell\le p} g_{\ell,\psi}(2iy)^{-\ell} = \sum_{u=1}^{D} \overline{\psi}(u)\left(G|_kT^{u/D}\right). \tag{5.26}$$

For any integer u with $(u,D) = 1$, let n and v be integers such that $nD - Nuv = 1$. Observe that

$$T^{u/D}W_{ND^2} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} W_N \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D}. \tag{5.27}$$

Therefore we have

$$F|_k T^{u/D} W_{ND^2} = G|_k \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D}.$$

Since G is a nearly weakly holomorphic modular form of weight k, depth p, level N and character $\overline{\chi}$, we have

$$F|_k T^{u/D} W_{ND^2} = \overline{\chi}(n) G|_k T^{v/D} = \chi(D) G|_k T^{v/D}. \tag{5.28}$$

By (5.25) and (5.28), we have

$$F_\psi|_k W_{ND^2} = \chi(D)\psi(-N) \sum_{v=1}^{D} \psi(v) G|_k T^{v/D} = \chi(D)\psi(-N) G_{\overline{\psi}}.$$

Therefore $F_\psi|_k W_{ND^2} = \chi(D)\psi(-N)G_{\overline{\psi}}$ and hence we conclude that $\vec{f}_\psi|_k W_{ND^2} = \chi(D)\psi(-N)\vec{g}_{\overline{\psi}}$. □

For each $a \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\varphi\colon \mathbb{R}_+ \to \mathbb{C}$, we define

$$(\varphi|_a W_N)(x) := (Nx)^{-a} \varphi\left(\frac{1}{Nx}\right) \qquad \text{for all } x > 0. \tag{5.29}$$

Since this action applies to functions on $\mathbb{R}_+$ and the action (1.2) to complex functions, the use of the same notation should not cause a confusion but some caution is advised.

We also define a set of "test functions" we will be using in most of the remaining results. Let $S_c(\mathbb{R}_+)$ be a set of complex-valued, compactly supported and piecewise smooth functions on $\mathbb{R}_+$ which satisfy the following condition: for any $t \in \mathbb{R}_+$, there exists $\varphi \in S_c(\mathbb{R}_+)$ such that $\varphi(t) \neq 0$. We are ready to prove the following result [5, Theorem 1.1] which gives functional equations for the L-function $L_f(\varphi)$ and its twists.

Theorem 5.3.4. *Let f be a weakly holomorphic quasimodular form of weight k, depth p, level N and character χ with component functions $f_0, f_1, \ldots, f_p$ and let ψ be* *a Dirichlet character modulo D with $(D,N)=1$. Also let the tuple $\vec{f}=(f_0,\dots,f_p)$ satisfy $\vec{f}|_k\widetilde{W_N}=\vec{g}$, where $\vec{g}=(g_0,\dots,g_p)$. For each $j\in\{0,1,\dots,p\}$, consider the map $L_{f_{j,\psi}}:\mathcal{F}_{f_{j,\psi}}\to\mathbb{C}$ given in* (5.20)*. Set*

$$\mathcal{F}_{f,g}:=\bigcap_{j=0}^{p}\bigcap_{m=0}^{p-j}\left\{\varphi\in\mathcal{F}_{f_j}\ :\ \varphi|_{2-(k-m-2j)}W_N\in\mathcal{F}_{g_{j+m}}\right\}.$$

Then $\mathcal{F}_{f,g}\neq\{0\}$ and for any $\varphi\in\mathcal{F}_{f,g}$ and $j\in\{0,1,\dots,p\}$, we have

$$L_{f_{j,\psi}}(\varphi)=\chi(D)\psi(-N)\sum_{m=0}^{p-j}i^{k-2j-m}(ND^2)^{1+m-\frac{k-2j}{2}}\binom{j+m}{m}L_{g_{j+m,\bar{\psi}}}(\varphi|_{2-(k-m-2j)}W_{ND^2}).$$

Proof. Let $\varphi\in S_c(\mathbb{R}_+)$, with $\mathrm{Supp}(\varphi)\subset(c_1,c_2)$, where c_1 and c_2 are positive real numbers satisfying $c_1<c_2$, then for all $x>0$, we get

$$\mathcal{L}(|\varphi|)(x)=\int_{c_1}^{c_2}|\varphi(t)|e^{-xt}dt\ll_{c_1,c_2,\varphi}e^{-xc_1}. \tag{5.30}$$

Now by (5.1), we deduce that the series

$$\sum_{n=-n_0}^{\infty}|a_j(n)|(\mathcal{L}|\varphi|)(2\pi n) \tag{5.31}$$

is convergent for all $0\le j\le p$. Therefore $S_c(\mathbb{R}_+)\subset\mathcal{F}_{f_j}$ for all $0\le j\le p$. Since $S_c(\mathbb{R}_+)$ is closed under the action of W_N, we have $S_c(\mathbb{R}_+)\subset\mathcal{F}_{f,g}$. We further note that if $\varphi\in\mathcal{F}_{f_j}$, then $\varphi\in\mathcal{F}_{f_{j,\psi}}$ for all ψ. This follows from (5.20) and the boundedness of $\tau_{\bar{\chi}}(n)$.

Now we obtain the functional equations for $L_{f_{j,\psi}}(\varphi)$. From (5.21), we get

$$L_{f_{j,\psi}}(\varphi)=\int_0^{\infty}f_{j,\psi}(it)\varphi(t)dt. \tag{5.32}$$

By changing the variable from t to $\frac{1}{ND^2t}$ in the above equation, we obtain

$$L_{f_{j,\psi}}(\varphi)=\int_0^{\infty}f_{j,\psi}\left(\frac{i}{ND^2t}\right)\varphi\left(\frac{1}{ND^2t}\right)(ND^2)^{-1}t^{-2}dt.$$

Using Proposition 5.3.3, we obtain

$$L_{f_{j,\psi}}(\varphi)=\chi(D)\psi(-N)\sum_{j\le m\le p}\binom{m}{j}(-1)^{m-j}(ND^2)^{\frac{k-2j}{2}}i^{m+j+k}$$
$$\times\int_0^{\infty}g_{m,\overline{\psi}}(it)\varphi\left(\frac{1}{ND^2t}\right)\frac{dt}{ND^2t^{2+m+j-k}}.$$

Simplifying the right hand side of the above identity, we obtain

$$
\begin{aligned}
L_{f_{j,\psi}}(\varphi) = \chi(D)\psi(-N) \sum_{j\le m\le p} \binom{m}{j} (-1)^{m-j} (ND^2)^{m+1-\frac{k}{2}} i^{m+j+k} \\
\times \int_0^\infty g_{m,\overline{\psi}}(it)\varphi\left(\frac{1}{ND^2t}\right) \frac{dt}{(ND^2t)^{2+m+j-k}}.
\end{aligned} \tag{5.33}
$$

Using (5.29) in (5.33), we obtain

$$
\begin{aligned}
L_{f_{j,\psi}}(\varphi) &= \chi(D)\psi(-N) \sum_{j\le m\le p} \binom{m}{j} (-1)^{m-j} (ND^2)^{m+1-\frac{k}{2}} i^{m+j+k} \\
&\qquad \times \int_0^\infty g_{m,\overline{\psi}}(it)(\varphi|_{2+m+j-k}W_{ND^2})(t)dt \\
&= \chi(D)\psi(-N) \sum_{j\le m\le p} \binom{m}{j} (-1)^{m-j} (ND^2)^{m+1-\frac{k}{2}} i^{m+j+k} \\
&\qquad \times L_{g_{m,\overline{\psi}}}(\varphi|_{2+m+j-k}W_{ND^2}).
\end{aligned}
$$

Rearranging the terms, we get

$$
L_{f_{j,\psi}}(\varphi) = \chi(D)\psi(-N) \sum_{0\le m\le p-j} \binom{m+j}{j} i^{k-m-2j} (ND^2)^{m+1-\frac{k-2j}{2}} L_{g_{m+j,\overline{\psi}}}(\varphi|_{2+m+2j-k}W_{ND^2}).
$$

$\square$

For any $s \in \mathbb{C}$, we define

$$
\varphi_s(x) := \varphi(x)x^{s-1}. \tag{5.34}
$$

Note that $\varphi_1 = \varphi$. We have the following result [5, Theorem 1.2].

Theorem 5.3.5. *Let f be a weakly holomorphic quasimodular form of weight k, depth p, level N and character χ with component functions $f_0, f_1, \ldots, f_p$. Set $\vec{g} := (g_0, g_1, \ldots, g_p)$ and $\vec{g} = \vec{f}|_k\widetilde{W}_N$. Let n_0 be a natural number such that $f(z)$ and $g(z)$ are $O(e^{2\pi n_0 y})$ as $y = \mathrm{Im}(z) \to \infty$. Suppose that $\varphi \in C(\mathbb{R}, \mathbb{C})$ is a non-zero function such that, for some $\epsilon > 0$, $\varphi(x)$ and $\varphi(x^{-1})$ are $o(e^{-2\pi(n_0+\epsilon)x})$ as $x \to \infty$. We further assume that for each $j \in \{0, 1, \cdots, p\}$, the series*

$$
\sum_{n=-n_0}^{\infty} |a_j(n)| \left((\mathcal{L}|\varphi|^2)(2\pi n)\right)^{\frac{1}{2}} \tag{5.35}
$$

converges. Then for each $j \in \{0, 1, \cdots, p\}$*, the series*

$$L(s, f_j, \varphi) := L_{f_j}(\varphi_s) \tag{5.36}$$

converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}$*, has an analytic continuation to all* $s \in \mathbb{C}$ *and satisfies the functional equation*

$$L(s, f_j, \varphi) = \sum_{0 \le m \le p-j} i^{m+2j+k} N^{\frac{m}{2}} \binom{j+m}{m} L(1-m-2j-s, g_{m+j}, \varphi|_{1-k} W_N). \tag{5.37}$$

Proof. By following in a similar way as in the proof of [12, Theorem 4.6, pp. 18], we get that $\varphi_s \in \mathcal{F}_{f_j}$ for $\mathrm{Re}(s) > \frac{1}{2}$ and $0 \le j \le p$. Therefore recalling the integral representation of $L_{f_j}(\varphi_s) = L(s, f_j, \varphi)$ in (5.18), we have

$$L(s, f_j, \varphi) = \int_0^\infty f_j\,(it)\,\varphi(t) t^s \frac{dt}{t}.$$

With the change of variable $t \mapsto 1/Nt$, we obtain

$$L(s, f_j, \varphi) = \int_{\sqrt{N}^{-1}}^\infty f_j(it)\varphi(t) t^s \frac{dt}{t} + \int_{\sqrt{N}^{-1}}^\infty f_j\left(\frac{i}{Nt}\right) \varphi\left(\frac{1}{Nt}\right) (Nt)^{-s} \frac{dt}{t}.$$

Since $\vec{f}|_k \widetilde{W} = \vec{g}$, using (5.7) we obtain

$$\begin{aligned} L(s, f_j, \varphi) = \int_{\sqrt{N}^{-1}}^\infty f_j(it)\varphi(t) t^s \frac{dt}{t} &+ \sum_{j \le l \le p} \binom{l}{j} (-1)^{l-j} N^{\frac{k-2j-2s}{2}} i^{l+j+k} \\ &\times \int_{\sqrt{N}^{-1}}^\infty g_l(it) \varphi\left(\frac{1}{Nt}\right) t^{k-l-j-s} \frac{dt}{t}. \end{aligned} \tag{5.38}$$

Recall that

$$\varphi\left(\frac{1}{Nt}\right) = \left(\varphi|_a W_N\right)(t)(Nt)^a$$

for any $a \in \mathbb{Z}$. With $a = 1-k$, we get, for $\mathrm{Re}(s) > \frac{1}{2}$,

$$\begin{aligned} L(s, f_j, \varphi) = \int_{\sqrt{N}^{-1}}^\infty f_j(it)\varphi(t) t^s \frac{dt}{t} &+ \sum_{j \le l \le p} \binom{l}{j} (-1)^{l-j} N^{\frac{2-2j-2s-k}{2}} i^{l+j+k} \\ &\times \int_{\sqrt{N}^{-1}}^\infty g_l(it) \left(\varphi|_{1-k} W_N\right) t^{1-l-j-s} \frac{dt}{t}. \end{aligned} \tag{5.39}$$

Similarly for $\mathrm{Re}(s) > \nu + 1$, we deduce that

$$L(s, g_j, \varphi) = \int_{\sqrt{N}^{-1}}^{\infty} g_j(it)\varphi(t)t^s \frac{dt}{t} + \sum_{j \leq l \leq p} \binom{l}{j} (-1)^{l-j} N^{\frac{2-2j-2s-k}{2}} i^{l+j-k} \times \int_{\sqrt{N}^{-1}}^{\infty} f_l(it) \left(\varphi|_{1-k} W_N\right) t^{1-l-j-s} \frac{dt}{t}. \tag{5.40}$$

From (5.39), we see that $L(s, f_j, \varphi)$ has analytic continuation to all $s \in \mathbb{C}$.

Now we establish the claimed functional equation. From (5.40) we obtain

$$\begin{aligned} L(s, g_j, \varphi) = &\int_{\sqrt{N}^{-1}}^{\infty} g_j(it)\varphi(t)t^s \frac{dt}{t} + i^{2j-k} N^{\frac{2-2j-2s-k}{2}} \int_{\sqrt{N}^{-1}}^{\infty} f_l(it) \left(\varphi|_{1-k} W_N\right) t^{1-l-j-s} \frac{dt}{t} \\ &+ \sum_{j+1 \leq l \leq p} \binom{l}{j} (-1)^{l-j} N^{\frac{2-2j-2s-k}{2}} i^{l+j-k} \int_{\sqrt{N}^{-1}}^{\infty} f_l(it) \left(\varphi|_{1-k} W_N\right) t^{1-l-j-s} \frac{dt}{t}. \end{aligned} \tag{5.41}$$

Using (5.7), we observe that for each $1 \leq l \leq p$, we have

$$f_l(it) = \sum_{l \leq m \leq p} \binom{m}{l} (-1)^{m-l} N^{\frac{2m-k}{2}} i^{m+l+k} t^{m+l-k} g_m\left(\frac{i}{Nt}\right).$$

Using the above identity in (5.41), we obtain

$$\begin{aligned} L(s, g_j, \varphi) = &\int_{\sqrt{N}^{-1}}^{\infty} g_j(it)\varphi(t)t^s \frac{dt}{t} + i^{2j-k} N^{\frac{2-2j-2s-k}{2}} \int_{\sqrt{N}^{-1}}^{\infty} f_l(it) \left(\varphi|_{1-k} W_N\right) t^{1-l-j-s} \frac{dt}{t} \\ &+ \sum_{j+1 \leq l \leq p} \binom{l}{j} (-1)^{l-j} N^{\frac{2-2j-2s-k}{2}} i^{l+j-k} \\ &\times \int_{\sqrt{N}^{-1}}^{\infty} \sum_{l \leq m \leq p} \binom{m}{l} (-1)^{m-l} N^{\frac{2m-k}{2}} i^{m+l+k} g_m\left(\frac{i}{Nt}\right) \left(\varphi|_{1-k} W_N\right) t^{1+m-j-k-s} \frac{dt}{t}. \end{aligned}$$

Interchanging the summations in the last integral of the right-hand side of the above identity, using the combinatorial identity $\binom{l}{j}\binom{m}{l} = \binom{m}{j}\binom{m-j}{m-l}$ and changing the variable $t \to 1/Nt$, we obtain

$$\begin{aligned} L(s, g_j, \varphi) = &\int_{\sqrt{N}^{-1}}^{\infty} g_j(it)\varphi(t)t^s \frac{dt}{t} + i^{2j-k} N^{\frac{2-2j-2s-k}{2}} \int_{\sqrt{N}^{-1}}^{\infty} f_l(it) \left(\varphi|_{1-k} W_N\right) t^{1-l-j-s} \frac{dt}{t} \\ &+ i^{-k} \sum_{j+1 \leq m \leq p} (-1)^{m-j} \binom{m}{j} i^{m+j+k} \sum_{j+1 \leq l \leq m} \binom{m-j}{m-l} i^{2l} \int_{0}^{\sqrt{N}^{-1}} g_m(it)\, \varphi(t) t^{s+j-m} \frac{dt}{t}. \end{aligned} \tag{5.42}$$

Using (4.13) in (5.42), we obtain

$$L(s,g_j,\varphi)=\int_{\sqrt{N}^{-1}}^{\infty} g_j(it)\varphi(t)t^s\frac{dt}{t}+i^{2j-k}N^{\frac{2-2j-2s-k}{2}}\int_{\sqrt{N}^{-1}}^{\infty} f_l(it)\left(\varphi|_{1-k}W_N\right)t^{1-l-j-s}\frac{dt}{t}$$
$$-i^{2j-k}\sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}i^{m+j+k}\int_0^{\sqrt{N}^{-1}} g_m(it)\,\varphi(t)t^{s+j-m}\frac{dt}{t}. \tag{5.43}$$

Similar to the expression we have got for $L(s,g_j,\varphi)$ in (5.41), we obtain the following expression for $L(s,f_j,\varphi)$:

$$L(s,f_j,\varphi)=\int_{\sqrt{N}^{-1}}^{\infty} f_j(it)\varphi(t)t^s\frac{dt}{t}+i^{2j+k}N^{\frac{2-2j-2s-k}{2}}\int_{\sqrt{N}^{-1}}^{\infty} g_j(it)\left(\varphi|_{1-k}W_N\right)t^{1-2j-s}\frac{dt}{t}$$
$$+\sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}N^{\frac{2-2j-2s-k}{2}}i^{m+j+k}\int_{\sqrt{N}^{-1}}^{\infty} g_m(it)\left(\varphi|_{1-k}W_N\right)t^{1-m-j-s}\frac{dt}{t}. \tag{5.44}$$

Now from (5.43) and (5.44), we obtain

$$\begin{aligned}
L(s,f_j,\varphi)&-i^{2j+k}N^{\frac{2-2j-2s-k}{2}}L(1-2j-s,g_j,\varphi|_{1-k}W_N)\\
&=\sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}i^{m+j+k}N^{\frac{2-2j-2s-k}{2}}\int_0^{\infty} g_m(it)\left(\varphi|_{1-k}W_N\right)(t)t^{1-m-j-s}\frac{dt}{t}\\
&=\sum_{j+1\le m\le p}\binom{m}{j}(-1)^{m-j}i^{m+j+k}N^{\frac{2-2j-2s-k}{2}}L(1-m-j-s,g_m,\varphi|_{1-k}W_N)
\end{aligned}$$

Rearranging the terms we get

$$L(s,f_j,\varphi)=\sum_{0\le m\le p-j} i^{m+2j+k}N^{\frac{m}{2}}\binom{j+m}{m}L(1-m-2j-s,g_{m+j},\varphi|_{1-k}W_N).$$

□

5.4 Converse theorem

In this section, we obtain the converse of Theorem 5.3.4. The theorem is as follows [5, Theorem 1.3].

Theorem 5.4.1. *Let k,p and N be integers with $p\ge 0, N\ge 1$ and χ let be a Dirichlet character modulo N. For each integer $0\le j\le p$, let $(a_j(n))_{n\ge -n_0}$ and*

$(b_j(n))_{n\geq -n_0}$ *for some integer* n_0*, be a pair of sequence of complex numbers such that* $a_j(n) = O(e^{C\sqrt{|n|}})$ *and* $b_j(n) = O(e^{C\sqrt{|n|}})$ *for some* $C > 0$*. Put*

$$f_j(z) = \sum_{n=-n_0}^{\infty} a_j(n)q^n, \quad g_j(z) = \sum_{n=-n_0}^{\infty} b_j(n)q^n, \quad 0 \leq j \leq p.$$

For any $\varphi \in S_c(\mathbb{R}_+)$ *and any Dirichlet character* ψ *modulo* D *with* $D \in \{1, 2, \ldots, N^2 - 1\}$ *and* $(D, N) = 1$*, we assume that*

$$L_{f_{j,\psi}}(\varphi) = \chi(D)\psi(-N) \sum_{0\leq m\leq p-j} i^{k-2j-m}(ND^2)^{1+m-\frac{k-2j}{2}} \binom{j+m}{m} \times L_{g_{j+m,\bar{\psi}}}(\varphi|_{2-(k-m-2j)}W_{ND^2}). \tag{5.45}$$

Then the function $f_0(z)$ *is a weakly holomorphic quasimodular form of weight* k*, depth* p*, level* N *and characters* χ *with component functions* $f_0, f_1, \ldots, f_p$ *and* $\vec{f}|_k\widetilde{W}_N = \vec{g} := (g_0, g_1, \ldots, g_p)$*, where* $\vec{f} := (f_0, f_1, \ldots, f_p)$*.*

Proof. By using Lemma 5.2.3, we see that $f_0, f_1, \cdots, f_p$ and $g_0, g_1, \cdots, g_p$ define holomorphic functions on $\mathbb{H}$ and all satisfy the condition (2) of Definition 1.3.2. Likewise, for any Dirichlet character ψ modulo D, recall that, for $0 \leq j \leq p$, by definition

$$f_{j,\psi}(z) = \sum_{n=-n_0}^{\infty} \tau_{\bar{\psi}}(n)a_j(n)q^n, \tag{5.46}$$

$$g_{j,\psi}(z) = \sum_{n=-n_0}^{\infty} \tau_{\bar{\psi}}(n)b_j(n)q^n \tag{5.47}$$

are absolutely convergent. Our aim is to show that for each $j = 0, 1, \cdots, p$, we have

$$f_{j,\psi}(z) = \chi(D)\psi(-N)i^{2k} \sum_{j\leq m\leq p} \binom{m}{j}(-1)^{m-j}(ND^2)^{k/2-m}(ND^2z)^{m+j-k} g_{m+j,\overline{\psi}}\left(\frac{-1}{ND^2z}\right). \tag{5.48}$$

Since both sides of (5.48) are holomorphic functions, it suffices to show the equality (5.48) on the vertical line $z = it, t > 0$. Note that for any $s \in \mathbb{C}$ and $\varphi \in S_c(\mathbb{R}_+)$, $\varphi_s(t) = t^{s-1}\varphi(t) \in S_c(\mathbb{R}_+)$. We first show that φ_s satisfies (5.10) for $f_{j,\psi}$ and $g_{j,\psi}$, $j = 0, 1, \cdots, p$ and hence belongs to $\bigcap_{j=0}^{p}(\mathcal{F}_{f_{j,\psi}} \cap \mathcal{F}_{g_{j,\psi}})$. Indeed, since $\varphi \in S_c(\mathbb{R}_+)$, there exist real numbers c_1 and c_2 with $0 < c_1 < c_2$ and $C > 0$ such that $\mathrm{Supp}(\varphi) \subset [c_1, c_2]$ and $|\varphi(t)| \leq C$ for any $t > 0$. Then for $n > 0$,

$$\begin{aligned} |a_j(n)| (\mathcal{L}|\varphi_s|)\,(2\pi n) &\leq C|a_j(n)| \int_{c_1}^{c_2} t^{\mathrm{Re}(s)} e^{-2\pi n t} \frac{dt}{t} \\ &\leq C|a_j(n)| e^{-2\pi n c_1} (c_2 - c_1) \max\{c_1^{\mathrm{Re}(s)-1}, c_2^{\mathrm{Re}(s)-1}\}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=-n_0}^{\infty} |\tau_{\bar{\psi}}(n)||a_j(n)|(\mathcal{L}|\varphi_s|)(2\pi n) &\leq \sum_{n=-n_0}^{0} |\tau_{\bar{\psi}}(n)||a_j(n)|(\mathcal{L}|\varphi_s|)(2\pi n) \\ &+ C(c_2 - c_1) \max\{c_1^{\mathrm{Re}(s)-1}, c_2^{\mathrm{Re}(s)-1}\} \sum_{n=1}^{\infty} |\tau_{\bar{\psi}}(n)||a_j(n)| e^{-2\pi n c_1} < \infty \quad (5.49) \end{aligned}$$

for any $s \in \mathbb{C}$ and any Dirichlet character ψ modulo D. Similarly, for any $s \in \mathbb{C}$ and any Dirichlet character ψ modulo D, we obtain

$$|b_j(n)| (\mathcal{L}|\varphi_s|)\,(2\pi n) \leq C|a_j(n)| e^{-2\pi n c_1} (c_2 - c_1) \max\{c_1^{\mathrm{Re}(s)-1}, c_2^{\mathrm{Re}(s)-1}\} \quad (5.50)$$

and

$$\begin{aligned} \sum_{n=-n_0}^{\infty} |\tau_{\bar{\psi}}(n)||a_j(n)|(\mathcal{L}|\varphi_s|)(2\pi n) &\leq \sum_{n=-n_0}^{0} |\tau_{\bar{\psi}}(n)||a_j(n)|(\mathcal{L}|\varphi_s|)(2\pi n) \\ &+ C(c_2 - c_1) \max\{c_1^{\mathrm{Re}(s)-1}, c_2^{\mathrm{Re}(s)-1}\} \sum_{n=1}^{\infty} |\tau_{\bar{\psi}}(n)||a_j(n)| e^{-2\pi n c_1} < \infty. \quad (5.51) \end{aligned}$$

Thus $\varphi_s \in \bigcap_{j=0}^{p} (\mathcal{F}_{f_{j,\psi}} \cap \mathcal{F}_{g_{j,\psi}})$ and by Weierstrass theorem, we see that for each $j = 0, 1, \cdots, p$, as functions of s, $L_{f_{j,\psi}}(\varphi_s)$ and $L_{g_{j,\psi}}(\varphi_s)$ are analytic functions. Then, by inverse Mellin transform, we get

$$f_{j,\psi}(it)\varphi(t) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}(s)=\sigma} L_{f_{j.\psi}}(\varphi_s) t^{-s} ds \quad (5.52)$$

for all $\sigma \in \mathbb{R}$.

Now we will show that $L_{f_{j\chi}}(\varphi_s) \to 0$ as $|\operatorname{Im}(s)| \to \infty$, uniformly for $\mathrm{Re}(s)$, in any compact set in $\mathbb{C}$. Indeed, from (5.18), we obtain

$$L_{f_{j,\psi}}(\varphi_s) = \int_0^{\infty} f_{j,\psi}(it)\varphi(t) t^s \frac{dt}{t}.$$

Using integration by parts and the fact that $\varphi(t)$ vanishes in $(0,\epsilon)\cup(1/\epsilon,\infty)$ for some $\epsilon>0$, we obtain

$$L_{f_{j,\psi}}(\varphi_s) = -\frac{1}{s}\int_0^\infty \frac{d}{dt}\Big(f_{j.\psi}(it)\varphi(t)\Big)t^s dt.$$

Then

$$\Big|L_{f_{j,\psi}}(\varphi_s)\Big| \leq \frac{1}{|s|}\int_0^\infty \left|\frac{d}{dt}\Big(f_{j,\psi}(it)\varphi(t)\Big)\right| t^{\mathrm{Re}(s)}dt \to 0 \tag{5.53}$$

as $|\operatorname{Im}(s)|\to\infty$.

We can therefore move the line of integration in (5.52) from $\mathrm{Re}(s)=\sigma$ to $\mathrm{Re}(s)=\delta$.

Thus we have

$$f_{j,\psi}(it)\varphi(t) = \frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=\delta} L_{f_{j.\psi}}(\varphi_s)t^{-s}ds. \tag{5.54}$$

Applying the functional equation (5.45) in (5.54), we obtain

$$\begin{aligned} f_{j,\psi}(it)\varphi(t) = \chi(D)\psi(-N) &\sum_{0\leq m\leq p-j}\binom{m+j}{j} i^{k-m-2j}(ND^2)^{m+1-\frac{k-2j}{2}} \\ &\times \frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=\delta} L_{g_{m+j,\overline{\psi}}}(\varphi_s|_{2+m+2j-k}W_{ND^2})t^{-s}ds. \end{aligned} \tag{5.55}$$

Changing the variable from s to $k-m-2j-s$ in the above integral, we get

$$\begin{aligned} f_{j,\psi}(it)\varphi(t) =&\chi(D)\psi(-N)\sum_{0\leq m\leq p-j}\binom{m+j}{j} i^{k-m-2j}(ND^2)^{m+1-\frac{k-2j}{2}} \\ &\times \frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=k-m-2j-\delta} L_{g_{m+j,\overline{\psi}}}(\varphi_{k-m-2j-s}|_{2+m+2j-k}W_{ND^2})t^{s-k+m+2j}ds. \end{aligned} \tag{5.56}$$

Now for each $t>0$, we have

$$\begin{aligned} (\varphi_{k-m-2j-s}|_{2+m+2j-k}W_{ND^2})(t) &= (ND^2t)^{k-m-2j-2}\varphi_{k-m-2j-s}\left(\frac{1}{ND^2t}\right) \\ &= (ND^2t)^{s-1}\varphi\left(\frac{1}{ND^2t}\right). \end{aligned} \tag{5.57}$$

By (5.57) , we obtain

$$\begin{aligned} L_{g_{m+j,\overline{\psi}}}(\varphi_{k-m-2j-s}|_{2+m+2j-k}W_{ND^2}) &= \int_0^\infty g_{m+j,\overline{\psi}}(it)(\varphi_{k-m-2j-s}|_{2+m+2j-k}W_{ND^2})(t)dt \\ &= \int_0^\infty g_{m+j,\overline{\psi}}(it)(ND^2t)^{s-1}\varphi\left(\frac{1}{ND^2t}\right)dt. \end{aligned} \tag{5.58}$$

Changing the variable from t to $1/ND^2t$ in the above integral, we obtain

$$L_{g_{m+j,\overline{\psi}}}(\varphi_{k-m-2j-s}|_{2+m+2j-k}W_{ND^2}) = \frac{1}{ND^2}\int_0^\infty g_{m+j,\overline{\psi}}\left(-\frac{1}{iND^2t}\right)\varphi(t)t^{-s-1}dt. \tag{5.59}$$

Now by the inverse Mellin transform, we have

$$\frac{1}{ND^2}g_{m+j,\overline{\psi}}\left(\frac{-1}{iND^2t}\right)\varphi(t) = \frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=k-m-2j-\delta} L_{g_{m+j,\overline{\psi}}}(\varphi_{k-m-2j-s}|_{2+m+2j-k}W_{ND^2})t^s ds. \tag{5.60}$$

Using (5.60) in (5.56), we obtain

$$\begin{aligned} f_{j,\psi}(it)\varphi(t) = \chi(D)\psi(-N)\sum_{0\le m\le p-j}\binom{m+j}{j} i^{k-m-2j}(ND^2)^{m-\frac{k-2j}{2}}t^{m+2j-k} \\ \times g_{m+j,\overline{\psi}}\left(\frac{-1}{iND^2t}\right)\varphi(t). \end{aligned} \tag{5.61}$$

Therefore if $t \in \mathbb{R}_+$ such that $\varphi(t) \neq 0$, then from the above identity, we have

$$f_{j,\psi}(it) = \chi(D)\psi(-N)\sum_{0\le m\le p-j}\binom{m+j}{j} i^{k-m-2j}(ND^2)^{m-\frac{k-2j}{2}}t^{m+2j-k}g_{m+j,\overline{\psi}}\left(\frac{-1}{iND^2t}\right). \tag{5.62}$$

Now from the definition of $S_c(\mathbb{R}_+)$, for each $t \in \mathbb{R}_+$ there exists $\varphi \in S_c(\mathbb{R}_+)$ such that $\varphi(t) \neq 0$. Thus (5.62) is true for all $t \in \mathbb{R}_+$. Therefore we get

$$f_{j,\psi}(z) = \chi(D)\psi(-N)i^{2k}\sum_{0\le m\le p-j}\binom{m+j}{j}(-1)^m(ND^2)^{m-\frac{k-2j}{2}}z^{m+2j-k}g_{m+j,\overline{\psi}}\left(-\frac{1}{ND^2z}\right). \tag{5.63}$$

Rearranging the terms, we obtain

$$f_{j,\psi}(z) = \chi(D)\psi(-N)i^{2k}\sum_{j\le m\le p}\binom{m}{j}(-1)^{m-j}(ND^2)^{k/2-m}(ND^2z)^{m+j-k}g_{m+j,\overline{\psi}}\left(-\frac{1}{ND^2z}\right). \tag{5.64}$$

By Proposition 5.2.7, we have

$$\vec{f}_{j,\psi} = \chi(D)\psi(-N)(\vec{g}_{j,\bar{\psi}}|_k\widetilde{W}_{ND^2}^{-1}). \tag{5.65}$$

Let

$$F_\psi(z) = \sum_{0\le\ell\le p} f_{\ell,\psi}(z)(2iy)^{-\ell} \quad \text{and} \quad G_\psi(z) = \sum_{0\le\ell\le p} g_{\ell,\psi}(z)(2iy)^{-\ell}. \tag{5.66}$$

Then by (5.65), we get

$$F_\psi = \chi(D)\psi(-N)G_{\bar\psi}|_k W_{ND^2}^{-1}. \tag{5.67}$$

By (5.26), we have

$$G_{\bar\psi} = \sum_{u=1}^{D} \psi(u) G|_k T^{u/D}. \tag{5.68}$$

For any integer u with $(u, D) = 1$, let n and v be integers such that $nD - Nuv = 1$. Observe that

$$T^{u/D} W_{ND^2}^{-1} = \begin{pmatrix} 1/D & 0 \\ 0 & 1/D \end{pmatrix} W_N^{-1} \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D}. \tag{5.69}$$

Therefore using the identity $F = G|_k W_N^{-1}$ (deduced by applying (5.67) with $D = 1$), we have

$$G|_k T^{u/D} W_{ND^2}^{-1} = F|_k \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D}. \tag{5.70}$$

Hence

$$G_{\bar\psi}|_k W_{ND^2}^{-1} = \sum_{u=1}^{D} \psi(u) F|_k \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D}. \tag{5.71}$$

Now from (5.67) and (5.71), we obtain

$$\begin{aligned} F_\psi &= \chi(D)\psi(-N) \sum_{u=1}^{D} \psi(u) F|_k \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D} \\ &= \chi(D) \sum_{v=1}^{D} \overline{\psi(v)} F|_k \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D}. \end{aligned} \tag{5.72}$$

By (5.25), we have

$$F_\psi = \sum_{v=1}^{D} \overline{\psi(v)} F|_k T^{v/D}. \tag{5.73}$$

Now from (5.72) and (5.73), we obtain

$$\sum_{v=1}^{D} \overline{\psi(v)} F|_k T^{v/D} = \chi(D) \sum_{v=1}^{D} \overline{\psi(v)} F|_k \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix} T^{v/D}. \tag{5.74}$$

By the orthogonality of the multiplicative characters, after taking the sum over all characters modulo D, we deduce that, for each integer u and v such that $-Nuv \equiv 1 (\operatorname{mod} D)$, we have

$$F = \chi(D) F|_k \begin{pmatrix} D & -v \\ -uN & n \end{pmatrix}, \tag{5.75}$$

which implies

$$F|_k \begin{pmatrix} n & v \\ uN & D \end{pmatrix} = \chi(D) F. \tag{5.76}$$

For each positive m, let t, s be any integers satisfying the condition that $\begin{pmatrix} t & s \\ mN & D \end{pmatrix} \in \Gamma_0(N)$. For each congruence class modulo mN, let S_m be the set consisting of exactly one of these $\begin{pmatrix} t & s \\ mN & D \end{pmatrix}$. By [28, Proposition 3], we know that the set

$$\bigcup_{m=1}^{N} S_m \cup \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \tag{5.77}$$

generates $\Gamma_0(N)$. Therefore from (5.76), we obtain

$$F|_k \gamma = \chi(D) F \tag{5.78}$$

for all $\gamma \in \Gamma_0(N)$. Therefore F is a nearly weakly holomorphic modular form of weight k, depth p, level N and character χ. Now by Proposition 5.2.4, we get that the function $f_0(z)$ is a weakly holomorphic quasimodular form of weight k, depth p, level N and characters χ with component functions $f_0, f_1, \ldots, f_p$. By using (5.65) with $D = 1$, we get $\vec{f}|_k \widetilde{W}_N = \vec{g}$. $\square$

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