Black-hole Thermodynamics of Higher Derivative Theories of Gravity

By

Anirban Dinda

PHYS11201704007

National Institute of Science Education and Research, Bhubaneswar

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

May, 2023

Homi Bhaba National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Anirban Dinda entitled "Black-hole Thermodynamics of Higher Derivative Theories of Gravity" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman - Prof. Sudipta Mukherji	S. Mukherji	Date 9/5/23
Guide / Convener - Dr. Sayantani Bhattacharyya	Sayantani B	Date 9/5/23
Examiner - Dr. Nabamita Banerjee	N Banerjee	Date 9/5/23
Member 1 - Dr. Yogesh Kumar Srivastava	Yogesh	Date 9/5/23
Member 2 - Dr. Chethan N. Gowdigere	Chethan	Date 9/5/23
Member 3 - Dr. Shamik Banerjee	Shamik Banerjee	Date 9/5/23

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I/We hereby certify that I/we have read this thesis prepared under my/our direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date : 9/5/23

Place : NISER, Bhubaneswar

Sayantani B.

Dr. Sayantani Bhattacharyya(Guide)

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Anirban Dinda

DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

Anirban Dinda

List of Publications arising from the thesis

Journal

1. *"An entropy current for dynamical black holes in four-derivative theories of gravity"*, Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Anirban Dinda and Nilay Kundu, JHEP 06 (2020) 017.

2. *"The zeroth law of black hole thermodynamics in arbitrary higher derivative theories of gravity"*, Sayantani Bhattacharyya, Parthajit Biswas, Anirban Dinda and Nilay Kundu,
arXiv:2205.01648 [hep-th]

Other publications (not included in thesis)

1. *"An entropy current and the second law in higher derivative theories of gravity"*, Sayantani Bhattacharyya , Prateksh Dhivakar , Anirban Dinda , Nilay Kundu, Milan Patra and Shuvayu Roy, JHEP09(2021)169

2. *"Fluid-gravity and membrane-gravity dualities Comparison at subleading orders"*, Sayantani Bhattacharyya, Parthajit Biswas, Anirban Dinda and Milan Patra, JHEP05(2019)054

3. *"The large D black hole dynamics in AdS/dS backgrounds"*, Sayantani Bhattacharyya, Parthajit Biswas, Bidisha Chakrabarty, Yogesh Dandekar and Anirban Dinda, JHEP10(2018)033.

Anirban Dinda

Anirban Dinda

Presentations at conferences and workshops

1. An Entropy Current for dynamical black holes in four-derivative theories of gravity
   Poster presentationat National String Meeting 2019, December 23-29, 2019, IISER Bhopal, India

2. Solving Einstein Equation Using Large D Approximation
   Poster presentation and gong show talk at Kavli Asian Winter School on Strings, Particles and Cosmology 2019, January 7-17, 2019, Sogang University, Seoul, South Korea

DEDICATIONS

To My Mother

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my thesis advisor Dr. Sayantani Bhattacharyya for her incessant support, patience, and encouragement throughout my Ph.D. years. I am very grateful to have Nilay Kundu and Jyotirmoy Bhattacharya as my collaborators from whom I have learned a lot.
I would like to thank my friends and collaborators Parthajit Biswas, Yogesh Dandekar, Bidisha Chakrabarty, Shuvayu Roy, Prateksh Dhivakar, Milan Patra.
I would like to thank my friends at NISER and outside: Suchetan, Nabu, Ritabrata, Semanti, Korak, Sayak, Somnath, Amlan, Ritam, Biggi, Sonu, Soheli, Pooja, Jabu, Tapas, Samapan, Samir, Dipika, Unmesh, Utsabda, Abhishek, Himu, Bisu. I would like to thank all the juniors of room-409 for making the room vibrant. I must thank two of my friends who helped me to go through one of the toughest times of my life: Anirban Samaddar and Prantik Saha.

Without the sacrifices made by my mother, I could never have reached here. I would like to thank my mother for her unconditional support, affection, and encouragement throughout these years.

Contents

Summary

We consider diffeomorphism invariant theories of gravity with arbitrary higher derivative terms in the Lagrangian as corrections to Einstein's general relativity. We construct a proof of the zeroth law of black hole thermodynamics in such theories. We assume that a stationary black hole solution in an arbitrary higher derivative theory can be obtained by starting with the corresponding stationary solution in general relativity and correcting it order by order in a perturbative expansion in the coupling constants of the higher derivative Lagrangian. We prove that surface gravity(which is the definition of temperature) remains constant on its horizon when computed for such stationary black holes, which is the zeroth law. Our proof for the zeroth law is valid up to arbitrary order in the expansion in the higher derivative couplings. Now we will move on to the second law for a specific higher derivative theory of gravity. We propose an entropy current for dynamical black holes in a theory with arbitrary four derivative corrections to Einstein's gravity linearized around a stationary black hole. The Einstein-Gauss-Bonnet theory is a special case of the class of theories that we consider. Within our linearized approximation, our construction allows us to write down a completely local version of the second law of black hole thermodynamics in the presence of the higher derivative corrections considered here. This ultra-local, stronger form of the second law is a generalization of a weaker form, applicable to the total entropy, integrated over a compact 'time-slice' of the horizon, a proof of which has been recently presented in [1]. We also provide a general algorithm to construct the entropy current for the four derivative theories, which may be straightforwardly generalized to arbitrary higher derivative corrections to Einstein's gravity. This algorithm highlights the possible ambiguities in defining the entropy current.

List of Figures

List of Tables

Chapter 1

Introduction

General Relativity(Einstein-Hilbert Theory) is one of the most successful theories of physics. Einstein formulated this theory in 1915 using the general covariance property of physics under any coordinate transformations. This theory is governed by the Einstein Fields equations. The Einstein field equations can be derived from the Einstein-Hilbert action

$$S_{E.H.} = \int_M d^d x \sqrt{-g} R \tag{1.1}$$

General Relativity describes gravity as a curvature of space-time. This theory predicts many experimental results from gravitational waves to black holes.

But if one tries to quantize it, one faces many problems, including ultraviolet(UV) problem. Even at the classical level it has singularities like black hole singularities, cosmological singularities where the physics of that theory breaks down. One of many candidates to resolve these divergences is String Theory which culminates in AdS/CFT conjecture[2]. String theory predicts at low energy limits, the Einstein-Hilbert action gets modified and gets corrections of higher derivatives of the metric tensors instead of the usual two derivatives. Sometimes it is called α corrections. In other words, Einstein-Hilbert action is the leading term in an infinite series of correction terms built out the metric tensor and curvature tensors.

Black holes are not only fascinating objects in an astronomical sense but also they are very intriguing in a purely theoretical sense. What lies behind the horizon baffles theoreticians for decades. On the other hand, all the three forces of nature can be described by quantum theory but gravity. Black hole thermodynamics interests theorists with the hope that it might light us the very nature of Quantum gravity. Though black holes are purely geometrical objects, thermodynamic properties must come from some microscopic degrees of freedom. So black holes are valuable objects if one wants to study quantum gravity. We must add higher derivative corrections to the Einstein-Hilbert action to have a consistent theory of quantum gravity. If we add higher derivative corrections in the Lagrangian, then the thermodynamic properties are not well understood, unlike General Relativity, where these properties are very well understood. We may consider the higher derivative correction as an effective field theory coming from the low energy limit of some UV complete quantum gravity. A priori, we can't say what kind of higher derivative terms can appear in that effective theory. One effective way is to study the black hole solutions of the theory and their thermodynamic properties. If one black hole solution of any higher derivative theory of gravity fails to maintain the thermodynamic properties, we can say that higher derivative term can not appear in the low energy limit. But there is no guarantee that a small correction to the Lagrangian will lead to a small correction to the solution of the unperturbed theory. In fact, higher derivative theories of gravity admit a whole new class of solutions that are not present in the General Relativity[3][4][5]. But, these solutions are not analytic in the coefficients of the higher derivative corrections appearing in the Lagrangian. So if we consider these higher derivative theories of gravity as an effective field theory, we can discard these solutions as unphysical[6]. There are systematic procedures how to get physical solutions from a higher derivative theory of gravity solution[6][7]. These higher derivative theories of gravity exhibit similar types of physics like General Relativity. They do have black hole like solutions at least if we consider the corrections perturbatively. In some cases, like Lanczos-Lovelock gravity(the most general second-order higher derivative gravity theories), there are known exact black hole solutions[4].

Black holes are fascinating gravitational objects with many properties that are very surprising and peculiar. However, black holes share one common property with ordinary matter: they also behave as thermodynamic objects, such as the area of the event horizon as its entropy, the surface gravity as the temperature, etc. All these thermodynamic properties of black holes are very well understood in General Relativity. As we have said earlier, General Relativity can't be the complete story. We need to add higher derivative corrections to it, treating it as an Effective Field Theory. In that case, when we go beyond General Relativity, the thermodynamic properties of black holes are not very well understood. In a full quantum theory of gravity, black holes would be an ensemble of microscopic states. General Relativity is the leading term in the finite series of higher derivative corrections in the low energy limit. We know that black holes, in General Relativity, exhibit thermodynamic properties coming from an ensemble of states. Adding small higher derivative correction terms in the Lagrangian would lead us to a black hole solution of that theory of gravity having slightly changed in the number of microstates in the ensemble. So we can hope to have thermodynamic properties maintained by the black holes of that theory. In this thesis, we are going to address these problems of satisfying these thermodynamic properties by black holes in higher derivative theories of gravity, try to define and prove some of the thermodynamic properties.

Chapter 2

Background

2.1 Black hole thermodynamics of Einstein-Hilbert Gravity

The main dynamical object of any gravity theory is the 2-tensor metric tensor($g_{\mu\nu}$).The action of a metric theory is constructed out of solely metric and its derivatives. One such example is the Einstein-Hilbert action, where the action depends only on the Ricci scalar(if not considering the cosmological constant).

$$S_{E.H.} = \int_M d^d x\sqrt{-g}(R + \mathcal{L}_m) \tag{2.1}$$

the integration is over some d-dimensional manifold. $\mathcal{L}_m$ is the matter sector of the action. If we vary the above action with respect to metric $g_{\mu\nu}$ and neglect the boundary terms, we get the Einstein Field equations

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = T_{\mu\nu} \tag{2.2}$$

Black holes are solutions of these Einstein field equations. They behave like thermodynamic objects having temperature, energy, and entropy. In 1973, Bardeen, Carter, and Hawking gave given a set of four laws that describe the behavior of black holes as thermodynamic objects[8, 9, 10]. These laws are a mere analogy to the four laws of thermodynamics. These four laws are

1. Zeroth Law: The surface gravity of a stationary black hole is constant over the entire event horizon

2. First Law: If a stationary black hole of mass M, charge Q and angular momentum J, with the future event horizon of surface gravity κ, electric surface potential Φ_H and angular velocity Ω_H, is perturbed such that it settles down to another black hole with mass $M + \delta M$.charge $Q + \delta Q$ and angular momentum $J + \delta J$, then

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ \tag{2.3}$$

3. Second Law: If $T_{\mu\nu}$ satisfies the null energy condition, and assuming that the cosmic censorship hypothesis is true, then the area of the future event horizon of an asymptotically flat space-time is non-decreasing.This is called Hawking Area Theorem.

4. Third Law: The surface gravity of a black hole can not be reached to zero within a finite amount of time.

2.2 Higher Derivative Theories of Gravity

As a modified gravity, the higher derivative gravity plays a crucial role. As higher derivative theories are also metric theory, whatever corrections we add to Einstein-Hilbert action, it must be constructed out of metric, its derivative, Riemann Tensors and its derivatives. At the end everything must be contracted appropriately to make a scalar as the action is a scalar quantity. Keeping all in mind, the action of most general diffeomorphism invariant theory of gravity in d space-time dimensions is

$$I = \int d^d x \sqrt{-g} (\mathcal{L}_{grav} + \mathcal{L}_{mat})$$

The gravity part of the Lagrangian is constructed out of metric tensor, Riemann tensors, and their derivatives

$$\mathcal{L}_{grav} = \mathcal{L}_{grav}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\gamma R_{\mu\nu\rho\sigma}, ...)$$

The covariant derivatives(∇) are compatible with the metric tensor $g_{\mu\nu}$

The equation of motions following from the above action has schematic form

$$\mathcal{E}_{\mu\nu} = E_{\mu\nu} - T_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + E^{HD}_{\mu\nu} - T_{\mu\nu}$$

For the matter sector, we will assume that the stress tensor always satisfies the null energy condition while we are considering the 2nd law and the dominant energy condition while we are considering the Zeroth law. These conditions are

Null Energy Condition (NEC): $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ for all null (lightlike) vectors ξ^μ.

Dominant Energy Condition (DEC):$T_{\mu\nu}\xi_1^\mu\xi_2^\nu \geq 0$ for all future directed timelike vectors ξ_1^μ, ξ_2^μ.

2.3 Wald Entropy

We will be considering the classical theory of gravity, arising from a diffeomorphism invariant Lagrangian in d space-time dimension[11, 12]. For a theory like this, one can associate a local symmetry for a vector field ξ^μ. Using Noether procedure, one can construct out of the vector field ξ^μ and the fields appearing in the Lagrangian, Noether Current, and Noether Charge for that symmetry. The variation of Lagrangian associated with a vector field ξ^μ is

$$\delta\mathcal{L} = E^{\mu\nu}\delta g_{\mu\nu} + \nabla_\mu\theta^\mu \tag{2.4}$$

Since ξ^μ generates the diffeomorphism, the variation of the Lagrangian must be a total derivative

$$\delta(\sqrt{-g}L) = \sqrt{-g}\nabla_\mu(\xi^\mu L) \tag{2.5}$$

where $\mathcal{L} = \sqrt{-g}L$. The change in metric is $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. Putting all these together, we get

$$\begin{aligned} \nabla_\mu(\xi^\mu L) &= 2E^{\mu\nu}\nabla_\mu \xi_\nu + \nabla_\mu \theta^\mu \\ \nabla_\mu(\theta^\mu - \xi^\mu L) + 2\nabla_\mu(E^{\mu\nu}\xi_\mu) &= \nabla_\mu J^\mu \\ J^\mu = -\xi^\mu L + \theta^\mu + 2E^{\mu\nu}\xi_\nu &= \nabla_\nu Q^{\mu\nu} \end{aligned} \tag{2.6}$$

where $Q^{\mu\nu} = -Q^{\nu\mu}$ and, we have used the Bianchi Identity($\nabla_\mu E^{\mu\nu} = 0$). $Q^{\mu\nu}$ and J^μ are called Noether Charge and Noether current respectively.

The Noether charge can always be written in the most general form as [12]

$$Q^{\mu\nu} = W^{\mu\nu\rho}\xi_\rho - 2E^{\mu\nu\rho\sigma}\nabla_{[\rho}\xi_{\sigma]} + Y^{\mu\nu} + \nabla_\rho Z^{\mu\nu\rho} \tag{2.7}$$

where $E_R^{\mu\nu\rho\sigma}$ is the equations of motion if we vary the Action w.r.t $R_{\mu\nu\rho\sigma}$ treating as an independent function.

$$E^{\mu\nu\rho\sigma} = \frac{\partial I}{\partial R_{\mu\nu\rho\sigma}} \tag{2.8}$$

Then the definition of Wald entropy is as follows

$$S_{Wald} = -2\pi \int_{\sum_v} d^{d-2}x\sqrt{h}E_R^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}$$

where $\epsilon_{\mu\nu} = \nabla_\mu \xi_\nu$ is the bi-normal to the constant time slice $\sum_v$ and normalized $\epsilon^{\mu\nu}\epsilon_{\mu\nu} = -2$. ξ^μ is the null generator of the Killing horizon. The integration is over a constant time slice of the event horizon.

Let's look at one example of Wald entropy computations. The action for Einstein-Gauss-Bonnet theory is the following

$$I = \int d^dx\sqrt{-g}\left(R + a_{gb}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right), \tag{2.9}$$

where a_{gb} is a constant Gauss-Bonnet parameter. The corresponding equations of motion are

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + E^{HD}_{\mu\nu} = 0, \tag{2.10}$$

where

$$\begin{aligned} E^{HD}_{\mu\nu} = a_{gb} \Big(& 2RR_{\mu\nu} - 4R^{\alpha\beta} R_{\mu\alpha\nu\beta} - 4R_{\mu}{}^{\alpha} R_{\nu\alpha} + 2R_{\mu}^{\alpha\beta\sigma} R_{\nu\alpha\beta\sigma} \\ & - \frac{1}{2} g_{\mu\nu} (R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\rho}R^{\alpha\beta\gamma\rho}) \Big) . \end{aligned} \tag{2.11}$$

The explicit vv-component of the equations of motion is

$$\begin{aligned} & E_{vv} = R_{vv} + E^{\text{HD}}_{vv} = 0, \\ & E^{\text{HD}}_{vv} = a_{gb} \left(2RR_{vv} - 4R^{\alpha\beta} R_{v\alpha v\beta} - 4R^{\alpha}_{v} R_{v\alpha} + 2R_{v}{}^{\alpha\beta\sigma} R_{v\alpha\beta\sigma} \right) \end{aligned} \tag{2.12}$$

Lets concentrate particularly in $d = (3 + 1)$ space-time dimensions, where the Gauss-Bonnet term becomes topological. In this case, the Gauss-Bonnet term becomes a total derivative term, and therefore, it does not contribute to the equations of motion, i.e. $E^{\text{HD}}_{vv} = 0$ identically. However, if one can use the Wald entropy as the equilibrium definition of black hole entropy (4.5), there is a finite non-vanishing contribution to it even from the topological Gauss-Bonnet part of the Lagrangian. The Wald entropy density s^{HD}_{w} (see (4.42)) for this case is given by the Ricci scalar of the co-dimension-2 spatial slice of the horizon $\mathcal{H}_v$,

$$s^{\text{HD}}_{w} = 2\, a_{gb}\, \mathcal{R}\,, \tag{2.13}$$

where a_{gb} is the Gauss-Bonnet parameter appearing in (4.89). Since $\mathcal{H}_v$, in this case, is a 2-dimensional manifold, the integrated total entropy S_W becomes the topological Euler number of $\mathcal{H}_v$.

This Wald entropy is a generalization to stationary black hole solutions in higher derivative theories of gravity in [11, 12], in such a way that it satisfied the first law of thermodynamics [1]. Now, the first law of thermodynamics relates the infinitesimal shifts in the

[1] See [13, 14] for the latest review of black hole thermodynamics in higher derivative theories of gravity.

parameters of two different but nearby equilibrium configurations. Therefore, the Wald entropy, whose construction was solely based on consideration of first law alone, does not unambiguously extend to dynamical situations. Indeed, as it was pointed out in [15, 16, 17], there were ambiguities associated with Wald entropy for non-stationary black hole solutions with dynamical event horizons. All these ambiguities vanished for stationary solutions. We shall refer to these ambiguities as the JKM ambiguities.

Unlike two derivative Einstein's theory, it is not a priori clear whether Wald entropy satisfies the second law of thermodynamics.

The second law says *the area of the future event horizon of an asymp- totically flat space-time is non-decreasing*. In the weakest version, the second law could be stated as follows. Consider two equilibrium configurations (in our case, two black hole solutions, not necessarily close by in any sense) B_1 and B_2 such that if one perturbs B_1 in certain ways it is possible to reach B_2 eventually. Then the entropy evaluated on the solution B_2 must be strictly greater than the entropy of B_1.

Though the above formulation of the second law does not really need a definition of entropy away from stationarity, it clearly refers to dynamics. One way towards a proof would be to show that there exists some extension of Wald entropy to dynamical situations so that the second law is satisfied. It is natural to expect that this extension (if at all possible) might fix those ambiguities related to the definition of entropy, which only arises in non-stationary situations (i.e. the JKM ambiguities). We will talk about the second law in detail in chapter-4.

2.4 Zeroth Law

Our aim is to focus on the Zeroth law of black hole thermodynamics in higher derivative theories of gravity. As in an ordinary thermodynamic system, the Zeroth law for black

hole mechanics is a characteristic signature of stationary or equilibrium black hole configurations. Stationary black holes have a space-time metric that admits a null hypersurface known as the Killing horizon, where a Killing vector becomes null. Using the fact that event horizons for stationary black holes are Killing horizons (due to the rigidity theorems[18]), and also that the temperature of the black hole is given by the surface gravity for such stationary black hole metrics, one can make a precise statement of zeroth law as follows: *the surface gravity of a stationary black hole is constant over the entire event horizon*.
This statement has been proven for two derivative theories of gravity, with an additional assumption of the dominant energy condition for the matter stress tensor [9], by analyzing the equations of motion in General Relativity. Alternative proofs have been constructed [19, 20], without any use of the equations of motion of the theory but assuming extra symmetries of space-time.

If we do not use any additional symmetry of the black hole space-time as mentioned above, it is an interesting question to ask if one can extend the proof of the Zeroth law to theories of gravity beyond General Relativity. Recently in [21] such proof was given for stationary black hole solutions in Gauss-Bonnet and Lovelock theories of gravity by modifying and improving upon a previously reported negative result in such theories [22]; also see [23] for a similar result. As these results were worked out for particular models of higher derivative theories of gravity, to the best of our knowledge, a similar result is not yet known for arbitrary diffeomorphism invariant theories of gravity.
In this thesis, we address this particular question and find that the answer to this is in the affirmative: *we have been able to construct a proof for the zeroth law in an arbitrary diffeomorphism invariant theories of gravity where the higher derivative terms in the Lagrangian are added as a correction to the leading two derivative theory of General Relativity*.

2.5 First Law

In the literature, there are many versions of first law of black hole thermodynamics e.g. equilibrium state version and physical process law. For the definitions and more elaborate discussions, we refer to the review article[24]. Here we only review the ”physical process version”. The formulation of the ‘physical process version ’ of the first law uses exactly the same setup as the one used in [1]. Here also one perturbs the stationary black hole out of equilibrium and lets it settle to another nearby stationary solution with slightly shifted parameters. The first law is a relation between these shifts of parameters, which characterize the two equilibrium solutions[2]. In the arguments leading to this physical process version of the first law, the external agent which drives the system out of equilibrium is a very specific one - some small matter (associated with a small shift in matter stress tensor) entering the system through asymptotic infinity. The similarity between the two set up of the second law and the physical process version of the first law is very suggestive of the fact that the structural nature of the terms in entropy, which play a major role in the proof of the physical process version of first law, would also be extremely important in the proof of the second law. We shall refer to such terms as ‘zero boost terms’, the justification of such terminology would be explained later in the main text.

After this extensive review of [1], we shall closely study how the physical version of the first law constrains these ‘zero boost terms’. We shall find that locally the required ‘time-time’ component of the equation of motion (let us denote this ‘time as v and the relevant component of the equation of motion as E_{vv}) need not have the form specified form which is naively implied by the physical process version of the first law. This naive expectation would be that the zero boost terms in E_{vv} , has two ‘time’ derivatives acting on

[2]We would like to emphasize that though the proofs of both first law and second law use the same set-up, they are very different in terms of details. In particular, Wall’s construction could fix many more terms in entropy (usually denoted as JKM ambiguities in literature) that do not contribute to the first law at all.

some local quantity (let us denote it as J^v) defined on the horizon. This naive expectation is not accurate, since any term that could be expressed as a single 'time' derivative acting on the spatial divergence of some space current (denoted here as J^i) may also be present in E_{vv}, without affecting the first law. This is because the physical process version of the first law deals with a total change in entropy (along with the charge and mass) as the black hole evolves from one equilibrium to another. The total entropy always comes with integration over all spatial section of the horizon $\mathcal{H}_v$. In that case, any such total divergence term would just integrate to zero.

It should be noted that such a term will not affect the argument of [1], which proves a weaker version of the second law, in which the total entropy (integrated over $\mathcal{H}_v$) has been considered. This weaker form is a local statement in time (i.e, total entropy increases at every instant of time) but not in space. At every stage of the arguments in [1], the integration over the spatial sections of the horizon played an important role.

Once we have realized that, it is possible to introduce the notion of a spatial entropy current, without affecting the proof of both the first law and the second law (even a strong ultra-local form of it), the next immediate question is whether such a spatial entropy current is necessary. In other words, we should investigate that, if we were to write down an ultra-local version of the second law, largely following the procedure of [1], can we do it without introducing the entropy current, in any higher derivative theory of gravity. In more practical terms, we need to check whether the relevant 'zero boost terms' in the equation of motion E_{vv} for a given higher derivative theory of gravity, does indeed have terms which give rise to the spatial entropy current J^i. To answer this question, we specialize to four-derivative theories of gravity. In §4.2.1, we explicitly compute the relevant component of the equations of motion and we see that there has to be a spatial entropy current in some of these four derivative theories, if we want a completely local version of the second law to be true. Besides achieving manifest locality, our procedure of constructing the entropy current might play a crucial role in providing an alternative proof of the second law, without invoking the ‘physical process’ version of the first law, the use of which has been a necessary input for the proof presented in [1].

Chapter 3

Zeroth Law of arbitrary diffeomorphism invariant Theories of Gravity

This chapter is based on [25]

3.1 Introduction

It has long been understood that the laws of black hole mechanics can be viewed as laws of thermodynamics [8, 9, 10]. We also know from [26] that this similarity is not an analogy. However, indeed one can derive the temperature of a black hole related to its surface gravity in a rigorous way.

We must think of General Relativity as an effective theory valid at low energies or large length scales. In a complete theory of quantum gravity, one can take its low energy limit and would get general relativity as the leading theory. Following this procedure, one would also generate corrections to general relativity. Without detailed knowledge of the UV complete theory and the process of taking a low energy limit, we cannot be sure what corrections to be added to the leading two derivative theory. Nevertheless, on general grounds, we expect that one would get various higher derivative terms in the Lagrangian in addition to the Einstein-Hilbert piece. Different higher derivative corrections will come with different dimensionful parameters as coefficients in the Lagrangian, and this will signify the length scale, say $l_{\rm HD}$ at which the higher derivative terms would be as important as the leading Einstein gravity piece. We denote the higher derivative couplings collectively as the dimensionless parameter α [1].

Once we extend the scope of gravity theories by including arbitrary higher derivative corrections in addition to the leading two derivative theory, the black holes still remain to be solutions of these new theories, and they should also retain their thermodynamic properties. Therefore although the laws of black hole mechanics were first understood in general relativity, one can not ignore the importance of understanding the validity of a similar set of laws for black hole thermodynamics in such higher derivative theories of gravity. In [11, 12], it was shown that a version of the first law of black hole thermodynamics could indeed be argued for an arbitrary diffeomorphism invariant theory of gravity. This construction also suggested a geometric object defined on the horizon of the black hole as the generalized definition of black hole entropy. This definition of black hole entropy is known as the Wald entropy in the literature. It says that the Noether charge associated with the Killing symmetry generator of the null horizon should be identified as the entropy of black holes in such arbitrary diffeomorphism invariant theory of gravity. Of course, this definition of entropy reduces to the area of the horizon as one considers black holes in general relativity. However, once out-of-equilibrium dynamic processes involving black holes are considered, the Wald entropy suffers from possible ambiguities known as the JKM ambiguities [15, 16, 17]. Additionally, there is no general proof that the Wald entropy satisfies the second law of thermodynamics. There have been various attempts at designing a proof for the second law that will be valid for arbitrary higher derivative theories of gravity [12, 15, 16, 17, 27, 28, 1, 29, 30][2]. Also, recently the construction of an entropy current in such theories was studied [31, 32, 33], following the work of [1].

An important assumption in our construction is the fact that it applies to theories where

[1]The higher derivative coupling will have dimensions in general, however, with the use of appropriate powers of l_{HD} we can define a dimensionless coupling α and also choose units by putting $l_{\text{HD}} = 1$.

[2]See the recent reviews [13, 14] and the references therein for a detailed discussion on black hole thermodynamics for higher curvature theory of gravity.

all higher derivative terms, appearing in the Lagrangian associated with a coupling parameter α, are treated as corrections to a leading two derivative theory of gravity, namely Einstein's general relativity. In operational terms, this means that for theories that we consider, a smooth limit of taking the higher derivative coupling $\alpha \to 0$ exists, and in that limit, we recover the general relativity as the leading candidate theory. This, in particular, enables us to obtain stationary black hole metrics as solutions in arbitrary higher derivative theories of gravity as they can be constructed perturbatively around some known stationary black hole solutions in two derivative general relativity when $\alpha = 0$. It is, however, important to highlight that our proof only requires the existence of this perturbative higher derivative coupling α. However, it is valid for all orders in this α-expansion and is also valid for any number of higher derivative coupling.

Let us now mention some of the salient features of the technical tools that we have used in constructing our proof. We will be very brief here, and all of these issues will be discussed in great detail in the subsequent sections. Firstly, we will work with a particular choice for the metric of stationary black holes. This does not lose any generality as one can always make these gauge choices for any stationary black holes. For our analysis, we will focus on Killing horizons where a Killing vector becomes null on the co-dimension one null hypersurface [3]. Further, we will associate the constancy of surface gravity on the horizon with specific components of the equations of motion. In other words, the off-shell structure of some specific components of the equations of motion, when evaluated at the horizon, will be related to (actually, be proportional to) the derivative of the surface gravity with respect to the coordinates on the horizon. Once we can establish this, the zeroth law would follow automatically by equating these components of equations of motion to

[3] Sometimes, we also call it the event horizon of the black hole. However, one needs to account for various global issues in the form of rigidity theorems to ensure that the local definition of a Killing horizon can be associated with the global concept of an event horizon. It is known to be true for general relativity, but it is still an open question to prove rigidity theorems beyond general relativity. Therefore, to be precise, we will actually be working with the Killing horizon in this paper

zero. However, we must point out that we do not explicitly use equations of motion in our analysis apart from this last step. Also, the main point here is to establish the following fact - *it is always possible to express the off-shell structure of a particular component of the equations of motion in arbitrary higher derivative theories of gravity (with the assumption of them augmenting the leading two derivative theory) in a form such that they get related to the derivative of surface gravity with respect to coordinates tangent to the horizon* - this is the main result of our analysis in this paper.

As mentioned before, we organize our calculations in a perturbative expansion in the higher derivative coupling. Within such a perturbative framework, we will use the method of induction to prove that a particular component of the equation of motion has the desired off-shell structure at arbitrary order. We first argue that at the leading order, i.e., when $\alpha = 0$, the equations of motion are indeed of the form expected, as this is just reviewing the proof of zeroth law known in the literature. Next, we assume that the proof works at an arbitrary order in the α-expansion, say at $\mathcal{O}(\alpha^m)$. Then we show that the proof will also work at the next order $\mathcal{O}(\alpha^{m+1})$. Therefore, following the method of induction, we can conclude that the proof will work up to any arbitrary higher-order in the α-expansion.

In establishing our result, a crucial input used a residual gauge invariance for our choice of the metric, named the boost symmetry. This boost symmetry is the consequence of a Killing isometry for stationary black holes, and the Killing horizon is mapped to itself under the flow generated by this boost transformation. Any covariant tensor, e.g., the equations of motion, will transform in a particular way under this boost transformation. This symmetry was an essential input for several recent works in the context of black hole thermodynamics. For example, in [1], a proof of linearized second law for arbitrary higher derivative theories of gravity was developed using this symmetry. Also, in [31] and [32], it was crucial to determine the structure of the equations of motion to construct an entropy current with non-negative divergence. In our present paper, assuming that the zeroth law is being satisfied at the order, $\mathcal{O}(\alpha^m)$ of the α-expansion, this boost-symmetry enables us to constrain the off-shell structure of the equations of motion at the next order $\mathcal{O}(\alpha^{m+1})$ as the desired one.

Finally, we end this section with an overview of how the paper is structured. We begin with a description of the basic setup and an operational statement of the problem at hand in §3.2. Here we discuss the particular choice of horizon adapted coordinates that we will use throughout this paper and present a schematic sketch of how various quantities can be organized in the perturbative expansion in the higher derivative coupling α. In the following section §3.3, we present a detailed description of the boost-symmetry and the basic rules following as a consequence of this, in connection to a stationary black hole and the zeroth law. In the next section §3.4 we briefly discuss and summarise the basic strategy of our proof without getting involved in the technical details of it. This is followed by a technically rigorous presentation of the main proof in §3.5. We divide this into several sub-sections, each corresponding to various steps in the analysis following a method of induction. We conclude this paper with some discussions in §4.3. Important supplementary material with various technical results is presented in the Appendices §A.1-§A.4.

3.2 Basic set-up and statement of the problem

In this section, we start by describing the basic setup of our analysis, and we will make a precise statement of the problem using that.

We are considering any arbitrary higher curvature theory of gravity without any matter couplings in d space-time dimensions with coordinates denoted by x^μ. Following [12], the requirement of diffemorphism invariance restricts the Lagrangian for such theories to be of the following form

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu},\, R_{\mu\nu\alpha\beta},\, D_\sigma R_{\mu\nu\alpha\beta}, \cdots) \tag{3.1}$$

However, for our analysis in this paper, we will work with theories such that the gravity action has the following form

$$I = \frac{1}{4\pi} \int d^d x \sqrt{-g} \left(R + \sum_{m=1}^{\infty} \alpha^m \mathcal{L}_{2m+2} \right) \tag{3.2}$$

where the higher derivative couplings in the theory are denoted by the parameter α. The other parameter present in the Lagrangian (i.e. m) counts the order of derivatives on the metric tensor (i.e. $g_{\mu\nu}$), the field variable in our theory. Therefore, it should be clear that $\mathcal{L}_{2m+2}$ is the $(2m+2)$-th order higher derivative term in the Lagrangian involving $(2m+2)$-derivatives acting on $g_{\mu\nu}$. The leading term, i.e. $m = 0$, gives us the standard Einstein-Hilbert Lagrangian for general relativity. It is important to mention that, apart from having $(2m + 2)$ number of derivatives on $g_{\mu\nu}$, $\mathcal{L}_{2m+2}$ has no other restrictions and is, therefore, completely arbitrary.

Ideally, all such higher derivative terms can, in principle, appear in the Lagrangian with different numerical coefficients. Hence, one should allow for different coupling constants for each of them in different order of the parameter m. Even within one same order of m-th derivative coupling, different possible terms can appear with different coupling coefficients but with the same dimensionality [4]. However, as we will see in the later parts of our analysis, the only important thing for us is to have the Einstein-Hilbert term as the leading contribution in a limiting sense when the higher derivative couplings are taken to be small. In other words, all we need is to have theories with arbitrary higher curvature terms in the Lagrangian, but any higher derivative couplings can be taken to zero in a smooth limit, leaving us with two derivative classical general relativity as the most significant one. Therefore, without any loss of generality, we collectively denote every possible higher derivative coupling by α^m for $m = 1, 2, \cdots$, with a specific number of derivatives on $g_{\mu\nu}$ determined

[4] For example, let us consider the two terms at $\mathcal{O}(\alpha^2)$ in the Lagrangian: $R^{\nu}_{\mu} R^{\rho}_{\nu} R^{\mu}_{\rho}$ and $D_\mu R_{\nu\rho} D^\mu R^{\nu\rho}$. Both of them have six derivatives and hence can appear in the Lagrangian in the following way

$$\alpha^2 \mathcal{L}_6 \sim \alpha^2 (c_1 \, R^{\nu}_{\mu} R^{\rho}_{\nu} R^{\mu}_{\rho} + c_2 \, D_\mu R_{\nu\rho} D^\mu R^{\nu\rho}) \,,$$

where c_1 and c_2 are two different but $\mathcal{O}(1)$ coefficients.

by the corresponding value of m. We will treat α as a small parameter allowing ourselves to perform a perturbative expansion in it. However, our analysis will be valid for arbitrary higher-order in that α expansion, as we have already mentioned before.

As we have described, we will be working in a perturbative expansion in the parameter α; it is obvious that the equations of motion (EoM) will have the following structure,

$$E_{\mu\nu} = E^{(0)}_{\mu\nu} + \alpha\, E^{(1)}_{\mu\nu} + \alpha^2\, E^{(2)}_{\mu\nu} + \cdots\,, \tag{3.3}$$

where, $E^{(0)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, is the EoM coming from Einstein's general relativity.

Next, we would like to comment on another essential ingredient in setting up our analysis related to obtaining stationary black hole solutions in arbitrary higher derivative theories of gravity. For purposes of the arguments presented in this paper, we do not need to know the exact form of the stationary black hole metric as a solution to the equations of motion. However, we assume that such solutions must exist in the higher derivative theory of gravity that we are considering. One should, quite naturally, be able to construct such solutions [34, 35] within our setup of perturbative expansion in α, the coupling of the higher derivative terms in our theory.

Let us suppose we start with a given stationary black hole solution, denoted by $g^{(bh)}_{\mu\nu}$, in the leading order theory in α expansion, which is Einstein's general relativity. It is obvious that the stationary $g_{\mu\nu} = g^{(bh)}_{\mu\nu}$ solves the equation of motion $E^{(0)}_{\mu\nu} = 0$. As a consequence of this, $g^{(bh)}_{\mu\nu}$ will have a Killing horizon - a null hypersurface generated by a global Killing vector field which we will denote by ∂_τ. By definition, ∂_τ will be a null geodesic on the horizon, and all the metric components in $g^{(bh)}_{\mu\nu}$ will be independent of the τ coordinate. In the following paragraphs, we will make this more precise.

We will be working with a particular horizon adapted set of the space-time coordinates along with a particular gauge choice for the metric $g^{(bh)}_{\mu\nu}$. In a d-dimensional space-time we can always choose a coordinate system $x^\mu = \{\tau, \rho, x^i\}$, where $i = 1, ..., d-2$, so that the stationary metric $g^{(bh)}_{\mu\nu}$ takes the following form

$$ds^2 = g^{(bh)}_{\mu\nu}\, dx^\mu dx^\nu = 2\, d\tau\, d\rho - \rho\, X(\rho, x^i)\, d\tau^2 + 2\, \rho\, \omega_i(\rho, x^i)\, d\tau\, dx^i + h_{ij}(\rho, x^i)\, dx^i\, dx^j\, . \tag{3.4}$$

Let us briefly justify the gauge choice for the metric in eq.(3.4) (see section-(2.1) and Appendix-A of [36] for the details). The coordinates $\{\tau,\, x^i\}$ span the co-dimension one horizon which lies on $\rho = 0$. We should also note that $\rho = 0$ is a null hypersurface, for which the null generators are taken to be the vector $\xi = \partial_\tau$. By construction, this is normal to itself and the other spatial generators (∂_i) of the horizon. At constant values of the coordinates x^i, the parameter τ runs along one null generator, whereas, for a constant value of τ, the coordinates x^i parametrizes different null generators on the horizon. The coordinate τ is not necessarily affinely parametrized. To describe the geometry in the vicinity of a null hypersurface, we need two null normals to it. Hence, apart from ξ, we have considered the auxiliary vector $\chi = \partial_\rho$, which is also null. This gives us the coordinate ρ, which parametrizes the distance away from the null horizon. The coordinate ρ has been chosen to be affinely parametrized, and the inner products: $(\partial_\tau, \partial_\rho)|_{\rho=0} = 1$ and $(\partial_i, \partial_\rho)|_{\rho=0} = 0$, define the angles with which the null-vector ∂_ρ pierces through the horizon at $\rho = 0$.

Next, the additional requirement of stationarity should explain why the metric coefficients (the functions X, ω_i, and h_{ij}) are independent of the coordinate τ. To this we note that ξ is a Killing vector for the metric eq.(3.4), satisfying the Killing equation $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$, where D_μ is the covariant derivative with respect to the full black hole metric, $g^{(bh)}_{\mu\nu}$. The norm of this Killing vector vanishes on the surface $\rho = 0$. Thus, in our choice of coordinates, $\rho = 0$ hypersurface is a Killing horizon.

The vector field ξ^μ also satisfies the geodesic equation

$$\xi^\nu\, D_\nu \xi^\mu = \kappa\, \xi^\mu\, . \tag{3.5}$$

Note that, the RHS of the above equation is not zero since τ is not necessarily an affine parameter. This equation could be considered as the definition of the quantity κ, which is in general a function of the coordinates $(\tau,\, x^i)$ and is called the surface gravity. It can be straightforwardly shown that the surface gravity for the black hole space-time described by the metric given in eq.(3.4) can be written as

$$\kappa = \sqrt{-\frac{1}{2}\left(D_\mu \xi_\nu\right)\left(D^\mu \xi^\nu\right)}\Bigg|_{\text{horizon}} . \tag{3.6}$$

The surface gravity is related to the temperature of a stationary black hole, and thus to prove the zeroth law, we must show that κ is constant over the horizon. It means that the surface gravity is constant not only for evolutions along one null generator but also does not change across different null generators of the null horizon. In other words, we would aim to prove that, when evaluated on the horizon,

$$\partial_\tau \kappa = 0, \quad \text{and} \quad \partial_i \kappa = 0\,. \tag{3.7}$$

Following the definition in eq.(3.6), we can evaluate the surface gravity for our choice of metric eq.(3.4) for $\xi = \partial_\tau$, to get the following expression (see Appendix-A.1 for details of the calculation)

$$\kappa = \frac{1}{2} X(\rho, x^i)\Bigg|_{\rho=0} . \tag{3.8}$$

It is obvious from eq.(3.8) that κ is independent of the coordinate τ. Basically, since ξ is a Killing vector, we trivially obtain the τ independence of $X(\rho, x^i)$, and hence $\partial_\tau \kappa = 0$. Therefore, to prove the zeroth law we have to show the following on the horizon

$$\partial_i X(\rho, x^i)\,|_{\rho=0} = 0\,. \tag{3.9}$$

3.3 Boost symmetry in the context of the zeroth law and stationarity

As we have laid down the statement of the problem in operational terms, in this section, we would like to highlight one crucial significance of the zeroth law or, equivalently, the constancy of surface gravity over the horizon. Let us remind ourselves that the zeroth law is, in a sense, one particular manifestation of stationarity for black hole solutions in our theory. It is noteworthy that for our choice of the stationary black hole metric in eq.(3.4) the coordinate τ runs along the null generators of the horizon but is not affinely parametrized. However, a slightly different but very useful choice of coordinate system as written below

$$ds^2 = \tilde{g}^{(bh)}_{\mu\nu}\, dx^\mu dx^\nu = 2\, dv\, dr - r^2\, X(rv,\, x^i)\, dv^2 + 2\, r\, \omega_i(rv,\, x^i)\, dv\, dx^i + h_{ij}(rv,\, x^i)\, dx^i\, dx^j \,, \tag{3.10}$$

also describes metric of stationary black holes with the horizon being set at $r = 0$, see [1], [31], [32]. The crucial difference between this choice of metric in eq.(3.10), written in terms of the new coordinates $(r,\, v,\, x^i)$, compared to the one in eq.(3.4), is the fact that the v coordinate here is affinely parametrized along the null generators ∂_v of the horizon. It should also be noted that, although, for the choice of metric in eq.(3.4) the metric coefficients are independent of the parameter τ, in eq.(3.10) the metric coefficients are functions of the coordinate v. However, the dependence on v is not arbitrary but restricted to the product $r\, v$. The reason for this is the following, for stationary metrics, the Killing generator ∂_τ and the affinely parametrized null generators are not the same but proportional to each other, see Appendix-A of [31] for a detailed discussion on this.

Let us now highlight the usefulness of writing the stationary black hole metric in the form of eq.(3.10) with v being an affine parameter. This particular choice does not fix the gauge completely and one still has some residual freedom of performing further coordinate transformation. Particularly, one can do the following scaling of the coordinates $(r,\, v)$

$$r \to \lambda\, r \,, \quad \text{accompanied with} \quad v \to \frac{v}{\lambda} \,, \tag{3.11}$$

where λ is a constant parameter [5]. It should be convincing that this transformation should leave the metric invariant, since the metric functions depend on the coordinates $(r,\, v)$ only

[5] Actually, one can do a more general residual coordinate transformation $v \to f_1(x^i)v + f_2(x^i)$, along with appropriate redefinition of r, but here we have restricted ourselves to a subclass of it.

through their product. This is called the boost transformation and due to this the stationary black hole configurations are said to enjoy a boost symmetry, see [31], [32] for details.

Alternatively, we can also explain the boost symmetry, that we described above, in the following way. In the coordinate system $\{r,\, v,\, x\}$, a stationary black hole solution, as written in eq.(3.10), has a Killing vector

$$\xi = \xi^\mu \partial_\mu = (v\partial_v - r\partial_r)\,. \tag{3.12}$$

In other words, the metric eq.(3.10) satisfies the following

$$\mathcal{L}_\xi g^{bh}_{\mu\nu} = 0\,, \tag{3.13}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector ξ. It can be easily checked that ξ is also the generator of the infinitesimal version of the boost transformation eq.(3.11).

As a consequence of this Killing symmetry, we can also confirm that the Lie derivative of any arbitrary covariant tensor constructed out of the metric should also vanish. The boost-symmetry is extremely useful in determining how any general tensor quantity built out of the metric coefficients or various derivatives of them, would transform under the aforementioned boost-transformation. In particular, any covariant tensor, say $\mathcal{B}$, with all components lowered, would transform in the following way

$$\mathcal{B} \to \widetilde{\mathcal{B}} = \lambda^w\, \mathcal{B}, \quad \text{under} \quad \left(r \to \tilde{r} = \lambda\, r\,,\, v \to \tilde{v} = \frac{v}{\lambda}\right) \tag{3.14}$$

so that we define the boost-weight of $\mathcal{B}$ to be given by w. Alternatively, we can also show that the boost-weight of any covariant tensor would be given by the number of excess lower v-indices over the lower r-indices, see [32] and Appendix-A.2 for a justification in favor of this.

Let us mention one important result that follows from the set up of boost-symmetry discussed above, any quantity with positive boost-weight will always vanish when computed using metric corresponding to a stationary configurations and evaluated on the Killing horizon. In order to explain this statement, let us first note that, from the definition of boost weight given in eq.(3.14) it can be argued that the metric functions, $X(rv,\, x^i)$, $\omega_i(rv,\, x^i)$, $h_{ij}(rv,\, x^i)$ appearing in eq.(3.10), are all boost invariant objects. Additionally the derivatives ∂_v and ∂_r have boost weights given by $+1$ and -1 respectively,

$$\partial_v \to \lambda\, \partial_v\,, \quad \text{and} \quad \partial_r \to \lambda^{-1}\, \partial_r\,. \tag{3.15}$$

Therefore, any covariant tensor, say $\mathcal{B}(rv,\, x^i)$, with positive boost weight can generically be written as

$$\mathcal{B}(rv,\, x^i) \sim (\partial_r)^{m_r} (\partial_v)^{m_v}\, \widetilde{\mathcal{B}}(rv,\, x^i)\,, \quad \text{with} \quad m_v > m_r\,, \tag{3.16}$$

where $\widetilde{\mathcal{B}}(rv,\, x^i)$ can include derivatives with respect to the spatial coordinates, but not any ∂_v or ∂_r. The functional dependence of $\mathcal{B}(rv,\, x^i)$ or $\widetilde{\mathcal{B}}(rv,\, x^i)$ on the product of rv signifies that they are evaluated on stationary configurations. Because of $m_v > m_r$, $\mathcal{B}$ has positive boost weight equal to $(m_v - m_r)$. Now, it is easy to convince ourselves that whenever one operates $(\partial_r)^{m_r}(\partial_v)^{m_v}$ on $\widetilde{\mathcal{B}}(rv,\, x^i)$, or in that case any function of the product rv, $(m_v - m_r)$ factors of r will be obtained, and hence it will vanish when we further evaluate this on the horizon $r = 0$. This will also be very crucially used in our present paper.

In [31] and [32], this particular boost symmetry was used to construct a local entropy current with non-negative divergence on the horizon of a dynamically perturbed stationary black hole in an arbitrary diffeomorphism invariant theory of gravity. In order to study non-stationary dynamical processes, this boost-symmetry is broken slightly by some matter source hitting the stationary black hole space-time. One can organize the dynamics in a perturbative expansion around the initial stationary configuration in the small amplitude of the external matter disturbance. Up to linearized order in the expansion in this amplitude expansion, the vv-component of the equations of motion (EoM) in any diffeomorphism invariant theory of gravity attains a universal structure as given below

$$E_{vv} \sim \partial_v \left(\partial_v J^v + \nabla_i J^i \right) + \text{quadratic fluctuations} \,, \tag{3.17}$$

where the quantity J^v represents local entropy density, reproducing the Wald entropy expression upon taking the stationary limit. On the other hand, the spatial components J^i signify the spatial flow of entropy on constant v-slices of the horizon. Using this result obtained in general gravity theories, one further needs to use the null energy condition for the stress-energy tensor coming from the matter sector to construct a proof for the local version of a second law.

Through the discussions in the previous paragraphs, we are actually trying to emphasize the following point. It was, therefore, indeed essential for the analysis in [32] to have the stationary metric written in the form given in eq.(3.10). In this section, we will argue that if the zeroth law is satisfied one can perform a coordinate transformation that changes the space-time metric from eq.(3.4) to eq.(3.10). Although this was implicit in the calculations in Appendix-A of [32], here we would like to make it very explicit.

Once zeroth law is satisfied, we get the surface gravity constant over the horizon. Therefore, we should be able to solve eq.(3.9) and obtain athe general solution for the metric coefficient function $X(\rho,\, x^i)$

$$X(\rho,\, x^i) = c_1 + \rho\, f(\rho,\, x^i) \,, \tag{3.18}$$

where c_1 is an integration constant and $f(\rho,\, x^i)$ is some arbitrary function of $(\rho,\, x^i)$. Also, note that in order to satisfy eq.(3.8), the constant c_1 gets fixed as $c_1 = 2\,\kappa$. We can substitute this in eq.(3.4) to obtain

$$ds^2 = 2\, d\tau\, d\rho - \rho\, \left(c_1 + \rho\, f(\rho,\, x^i) \right)\, d\tau^2 + 2\, \rho\, \omega_i(\rho, x^i)\, d\tau\, dx^i + h_{ij}(\rho, x^i)\, dx^i\, dx^j \,. \tag{3.19}$$

Next we perform the following coordinate transformation from the coordinates $\{\rho,\, \tau,\, x^i\}$

to $\{r,\, v,\, x^i\}$ given by

$$\tau \to v = \frac{2}{c_1} \exp\left(\frac{c_1}{2}\tau\right), \quad \text{and}, \quad \rho \to r = \rho \exp\left(-\frac{c_1}{2}\tau\right) \tag{3.20}$$

to arrive at

$$ds^2 = 2\, dv\, dr - r^2\, f(c_1\, r\, v/2,\, x^i)\, dv^2 + r\, \omega_i(c_1\, r\, v/2,\, x^i)\, dv\, dx^i + h_{ij}(c_1\, r\, v/2,\, x^i)\, dx^i dx^j\,, \tag{3.21}$$

which is of the form eq.(3.10). Note that the horizon stays at $r = 0$ in the new coordinates, and, also, the fact that $c_1 = 2\kappa$ is a constant was crucially used while performing this coordinate transformation. Once written in this coordinate system, we can straightforwardly use the consequences of boost-symmetry that the metric in this form enjoys.

Finally, before we end this section, let us make one comment on how these results that one derives using boost invariance of a stationary black hole expressed in the coordinates as in eq.(3.21), would be helpful in the later sections of this paper as we aim to prove zeroth law. This may seem puzzling since, to derive these results, we have already used the zeroth law itself. However, as we will explain later, we will follow a methodology for the proof of zeroth law by organizing our calculations as a perturbative correction in the higher derivative coupling α correcting the leading order two derivative theory of general relativity. In that perturbative set-up, we will construct the proof by using a method of induction. More precisely, with the assumption that at n-th order in the α-expansion, our construction validates the zeroth law, we will aim to extend the proof to $n+1$-th order. Therefore, while working at $n+1$-th order, the truncated and corrected metric till the previous n-th order could be brought to the form as in eq.(3.21) and thus would satisfy boost-invariance under the transformation given in eq.(3.11). Consequently, when evaluated on the metric corrected and truncated up to n-th order in α-expansion, any covariant tensor would transform with a particular boost-weight entirely determined by its index structure alone. For our case, using these concepts, we will see that the (vi)-component of EoM, i.e., E_{vi}, would have a boost weight equal to $+1$ and would thus vanish for stationary black hole configurations.

3.4 Brief outline of the strategy

In this section, our goal is to present a broad outline of the strategy of the proof without getting into operational details. Following this, in the next section, we will construct a technically rigorous proof.

The crucial ingredient in our strategy to prove the zeroth law will be to argue that eq.(3.9) follows from the vanishing of a particular component of the equations of motion (EoM). More precisely, we will explicitly show that the LHS of eq.(3.9) must be expressed in terms of the $\{\tau\, i\}$-component of the EoM

$$E_{\tau i}\,|_{\rho=0} \sim \partial_i X(\rho, x^i)\,|_{\rho=0}\,, \tag{3.22}$$

upto numerical factors, where, following eq.(3.3), $E_{\tau i}$ must include contributions from all higher derivative terms present in the Lagrangian of the theory in addition to the leading Einstein-Hilbert term. Next, we should use $E_{\tau i} = 0$, as the stationary black hole space-times must solve the full EoM's. It is clear that eq.(3.9) follows immediately.

With the explanations so far, let us summarise the main goal that we will pursue in the rest of this paper: *In order to prove the zeroth law (or equivalently, for proof of eq.*(3.9)*), our primary goal would be to justify that the off-shell structure of the entire $E_{\tau i}$ (including corrections due to higher derivative coupling α as in eq.*(3.3)*) reproduces eq.*(3.22)*. To achieve this, we must evaluate $E_{\tau i}$ for a stationary black hole solution obtained by treating the higher derivative coupling α perturbatively in an expansion around a stationary black hole solution of the leading two-derivative theory.*

Before we proceed further with describing our strategy, let us take a small detour to highlight the significance of this particular component of EoM's, $E_{\tau i}$, in justifying the zeroth law. On general grounds, looking at eq.(3.9), we should expect that $\partial_i X(\rho, x^i)\,|_{\rho=0}$

should get related to some components of $E_{\mu\nu}$ in order to be vanishing when evaluated on on-shell configurations. The index structure then suggests that one of the two indices in $E_{\mu\nu}$ must be the spatial indices (say, $\mu = i$), leaving the other one to be either $\nu = \tau$, or, $\nu = \rho$. If we now focus on Einstein's gravity and compute the (τ, i) component of the Einstein tensor, which is the EoM, we can immediately check that eq.(3.22) is reproduced. Hence, the zeroth law is proved for the two derivative theory. In our set-up, we treat arbitrary theories of gravity as perturbative corrections in the higher derivative coupling α to a leading two derivative theory. Hence, it is expected that even in such general theories, we must look into the off-shell structure of $E_{\tau i}$ to justify eq.(3.9).

The arguments presented in the previous paragraph may appear to be heuristic. However, a more rigorous justification can be devised to support the following statement: for a proof of the zeroth law, one should investigate the off-shell structure of $(\tau\, i)$-component of EoM. This has already been noted in the literature. It is possible to show that (see [9] for proof)

$$e_i^\mu D_\mu \, \kappa = -R_{\mu\nu} \, \xi^\mu \, e_i^\nu \, , \tag{3.23}$$

where e_i^μ are the space-like tangent vectors to the horizon at $\rho = 0$. Most significantly, we should note that to derive eq.(3.23) one does not need to use any EoM, and hence this is valid universally in any theory of gravity. For two derivative Einstein gravity, once we use EoM, the RHS in eq.(3.23) gets related to components of the stress-energy tensor, $T_{\mu\nu}$, coming from the matter sector coupled to gravity, if any. One can further use the dominant energy condition for the stress-energy tensor, and consequently, the RHS in eq.(3.23) vanishes, proving the zeroth law. However, once we focus on higher derivative theories of gravity, in this process of substituting $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ we get extra contributions in the EoM due to the higher derivative terms in the Lagrangian

$$E^{(0)}_{\mu\nu} + \alpha \, E^{\text{HD}}_{\mu\nu} = T_{\mu\nu} \, , \tag{3.24}$$

where $E^{(0)}_{\mu\nu} = R_{\mu\nu} - (1/2)\, g_{\mu\nu}\, R$ is the Einstein tensor, and $E^{\text{HD}}_{\mu\nu}$ is the EoM coming from the higher derivative terms in the theory along with the coupling α. With this, we are convinced that in order to prove that the RHS of eq.(3.23) vanishes, we must need to investigate the off-shell structure of $E^{\text{HD}}_{\mu\nu}\, \xi^\mu\, e^\nu_i$, which, in our chosen coordinate system, is precisely the $E^{\text{HD}}_{\tau i}$.

With the set-up that we have discussed so far, we are now in a position to give a schematic overview of the operational strategy that will be followed to argue that $E_{\tau i}$ is indeed of the form given in eq.(3.22). The main idea will be to organize the analysis in a perturbative expansion in higher derivative coupling α around the leading two derivative theory. The EoM has already been written in eq.(3.3) in order by order expansion in α. In order to investigate the off-shell structure of $E_{\mu\nu}$ constructed out of the space-time metric $g_{\mu\nu}$ and derivatives acting on it, we would also need to take a similar ansatz for $g_{\mu\nu}$ expanded in powers of α

$$g^{(bh)}_{\mu\nu} = g^{(0)}_{\mu\nu} + \alpha\, g^{(1)}_{\mu\nu} + \alpha^2\, g^{(2)}_{\mu\nu} + \cdots \,, \tag{3.25}$$

where the superscript in $g^{(n)}_{\mu\nu}$ signifies that it corresponds to the n-th order in the expansion of α. We would demand that this ansatz for $g^{(bh)}_{\mu\nu}$ solves the EoM given in eq.(3.3) order by order in the expansion of α. This, in turn, would allow us to justify that eq.(3.22) is indeed true up to all orders in the perturbative expansion in α.

We will follow the method of induction to establish the desired off-shell structure of $E_{\tau i}$ up to arbitrary order in the α-expansion. First, we would show that it is indeed the case at the leading order with $\alpha = 0$ for two-derivative Einstein gravity. Then we will extend this to arbitrary order of $\mathcal{O}(\alpha^{m+1})$ in the α-perturbative expansion, assuming that things do work out till the previous order of $\mathcal{O}(\alpha^m)$. In this process, we will need to know specifics about the generic structure of $E_{\mu\nu}$ in an arbitrary order of the perturbation. To evaluate $E_{\mu\nu}$, we need to substitute for $g^{(bh)}_{\mu\nu}$, given in eq.(3.25), in eq.(3.3) and isolate the terms contributing at $\mathcal{O}(\alpha^{m+1})$. We will see that at this order (i.e. at $\mathcal{O}(\alpha^{m+1})$), $E_{\mu\nu}$ can be partitioned into two types of terms. The first type being the zeroth order EoM $E^{(0)}_{\mu\nu}$ evaluated on $g^{(0)}_{\mu\nu} + \alpha^{m+1} g^{(m+1)}_{\mu\nu}$, where we will treat $\alpha^{m+1} g^{(m+1)}_{\mu\nu}$ as linearized perturbation around $g^{(0)}_{\mu\nu}$. The second type of terms involve the coefficient of α^{m+1} in the full EoM evaluated on the metric corrected till the previous order, i.e. till $g^{(m)}_{\mu\nu}$. Schematically, this looks as the following

$$\text{At}\, \mathcal{O}(\alpha^{m+1}): \quad E^{(0)}_{\mu\nu}[g^{(0)}_{\mu\nu} + \alpha^{m+1}\, g^{(m+1)}_{\mu\nu}] + E_{\mu\nu}[g^{(0)}_{\mu\nu} + \alpha\, g^{(1)}_{\mu\nu} + \alpha^2\, g^{(2)}_{\mu\nu} + \cdots + \alpha^m\, g^{(m)}_{\mu\nu}] = \mathcal{O}(\alpha^{m+2})\,, \tag{3.26}$$

where for the second term on the LHS we should truncate it to $\mathcal{O}(\alpha^{m+1})$.

The first term on the LHS of eq.(3.26) has an universal structure as it is basically the Einstein's tensor linearized around $g^{(0)}_{\mu\nu}$ for a small perturbation given by $g^{(m+1)}_{\mu\nu}$.

To treat the second term on the LHS in eq.(3.26), however, we have to be more careful. Since this term has no universal structure like the first one, a further non-trivial argument must be invoked. We should take note of the fact that this second term is an arbitrary co-variant tensor of rank two, but most importantly, built out of metric coefficients truncated at $\mathcal{O}(\alpha^m)$. When we are looking at the order $\mathcal{O}(\alpha^{m+1})$, we will assume that eq.(3.22) and eq.(3.9) have been satisfied till the order of $\mathcal{O}(\alpha^m)$. This, in turn, enables us to ascertain that the surface gravity, κ, computed with the corrected metric till $\mathcal{O}(\alpha^m)$, will be constant over the horizon. As a consequence of this, we know that $\partial_i X(\rho,\, x^i)|_{\rho=0} = 0$ up to $\mathcal{O}(\alpha^m)$, and hence we can use the coordinate transformation eq.(3.50) to write the metric in terms of coordinates $(r,\, v,\, x^i)$, as in eq.(3.51). As we have discussed before, once we have succeeded in writing the space-time metric in this new coordinate system, we can use the boost-symmetry. Consequently, we would now be able to assign boost weights to various covariant tensors just by counting the difference in lower v and r components. By standard rule of how tensors should transform under coordinate transformation one can see that $E_{\tau i}$

in the original $(\rho,\, \tau,\, i)$-coordinates will be proportional to $\tilde{E}_{vi}$ - the (vi)-component of the new EoM in the new coordinates. Finally, we note that $\tilde{E}_{vi}$ has boost-weight$=\,+1$, and hence should vanish when computed for stationary configurations. Therefore we conclude that the second term on the LHS of eq.(3.26) will not contribute at $\mathcal{O}(\alpha^{m+1})$. With this, we will be able to show that $E_{\tau i}$ has indeed the form mentioned in eq.(3.22) and, thereby, we will be able to complete the proof of the zeroth law.

3.5 Constructing the proof for the zeroth law

This section will construct the proof for the zeroth law with all details. The discussions in this section will be divided into several sub-sections. Following the outline of our strategy presented in §3.4, we would start with understanding the general structure of the equations of motion (EoM) order by order in an expansion in the higher derivative coupling α. This would be followed by explicitly working out the leading order term in this expansion, which is actually the Einstein tensor coming from the Einstein-Hilbert Lagrangian. We would review how zeroth law is satisfied at this leading order. Next, we would extend this procedure to an arbitrary higher-order in the α-expansion adopting a method of induction. Finally, we will also explicitly see how the application of boost symmetry for stationary metric corrected up to a particular order of α-expansion helps us determine the (τi)-components of the EoM to the following order.

3.5.1 General structure for the equations of motion in α-expansion

As we have already mentioned before in §3.2, the key assumption in our working principle is that we could solve the EoM perturbatively in an expansion in the higher derivative coupling α. Thereby, the EoM has a structure given in eq.(3.3), which we rewrite here for convenience

$$E_{\mu\nu} = E^{(0)}_{\mu\nu} + \alpha\, E^{(1)}_{\mu\nu} + \alpha^2\, E^{(2)}_{\mu\nu} + \cdots\,, \tag{3.27}$$

where $E^{(0)}_{\mu\nu}$ is the Einstein tensor - the EoM in the two derivative theory of gravity. Also, $\alpha^k E^{(k)}_{\mu\nu}$, for $k \geq 1$, are all higher derivative corrections to the EoM. They depend on the details of the theory. We have also noticed in §3.4, that as a consequence, the metric will also admit a similar expansion, given in eq.(3.25). Here we present that as well

$$g^{(bh)}_{\mu\nu} = g^{(0)}_{\mu\nu} + \alpha\, g^{(1)}_{\mu\nu} + \alpha^2\, g^{(2)}_{\mu\nu} + \cdots . \tag{3.28}$$

In our choice of coordinates this will lead to an expansion of the metric components X, ω_i and h_{ij} as given below

$$\begin{aligned} X(\rho, x^i) &= X^{(0)}(\rho, x^i) + \alpha\, X^{(1)}(\rho, x^i) + \alpha^2\, X^{(2)}(\rho, x^i) + \cdots , \\ \omega_i(\rho, x^i) &= \omega^{(0)}_i(\rho, x^i) + \alpha\, \omega^{(1)}_i(\rho, x^i) + \alpha^2\, \omega^{(2)}_i(\rho, x^i) + \cdots , \\ h_{ij}(\rho, x^i) &= h^{(0)}_{ij}(\rho, x^i) + \alpha\, h^{(1)}_{ij}(\rho, x^i) + \alpha^2\, h^{(2)}_{ij}(\rho, x^i) + \cdots . \end{aligned} \tag{3.29}$$

We are viewing the EoM in eq.(3.27) to be evaluated on the metric $g^{(bh)}_{\mu\nu}$ in eq.(3.28), and the corresponding structures should be analysed order by order in the α-expansion. Let us now see what we can learn at the very leading order, i.e. at $\mathcal{O}(\alpha^0)$. At this order very leading order the metric functions $X^{(0)}(\rho, x^i)$, $\omega^{(0)}_i(\rho, x^i)$ and $h^{(0)}_{ij}(\rho, x^i)$ should be exact solutions of the zeroth order equation (i.e. the Einstein's equation for two derivative theory of gravity)

$$E^{(0)}_{\mu\nu}[g^{(0)}_{\mu\nu}] = 0 \tag{3.30}$$

Therefore, we should be viewing this as a differential equation for $g^{(0)}_{\mu\nu}$, which is the unknown variable at leading order, and by solving this, we would be able to fix $g^{(0)}_{\mu\nu}$.

Now, let us suppose that we want to solve the EoM $E_{\mu\nu} = 0$ upto the first sub-leading order (i.e., upto order $\mathcal{O}(\alpha^1)$). We have already determined $g^{(0)}_{\mu\nu}$ while working at the previous order $\mathcal{O}(\alpha^0)$. At this order of $\mathcal{O}(\alpha^1)$, we realise that $g^{(1)}_{\mu\nu}$ (or the metric functions $X^{(1)}(\rho, x^i)$, $\omega^{(1)}_i(\rho, x^i)$ and $h^{(1)}_{ij}(\rho, x^i)$) are the unknowns. To find out the relevant part of EoM from eq.(3.27) at this order, we will basically have to evaluate the tensor $E_{\mu\nu}$ on the metric $g^{(bh)}_{\mu\nu}$, neglecting all terms proportional to quadratic or higher powers of α. In other words, it is obvious that $E^{(n)}_{\mu\nu}$ and $g^{(n)}_{\mu\nu}$ for every $n \geq 2$ are negligible at this order of $\mathcal{O}(\alpha^1)$. As a result, the differential equation for the unknowns will have the following structure

$$E^{(0)}_{\mu\nu}\left[g^{(0)}_{\alpha\beta} + \alpha\, g^{(1)}_{\alpha\beta}\right] + \alpha\, E^{(1)}_{\mu\nu}\left[g^{(0)}_{\alpha\beta}\right] = \mathcal{O}(\alpha^2)\,, \tag{3.31}$$

Here, in eq.(3.31), the first term on the LHS is basically the Einstein tensor, linearised around $g^{(0)}_{\mu\nu}$ where $g^{(1)}_{\mu\nu}$ plays the role of the small fluctuation metric. The second term is actually not of any universal structure like Einstein tensor since the explicit form would depend on the type of higher derivative theory that we are focussing on at linear order in α. However, for our purpose, that is not at all a problem since we just need to know that $E^{(1)}_{\mu\nu}$ is a covariant tensor of rank two (constructed out of appropriate contractions of Riemann tensors and its covariant derivatives) evaluated on the exact stationary black hole solution, i.e., $g^{(0)}_{\mu\nu}$, of the two-derivative theory of gravity. From eq.(3.31), it is clear that this term will act as a source term in the inhomogeneous PDE for $g^{(1)}_{\mu\nu}$. The first term, on the other hand, is homogeneous in $g^{(1)}_{\mu\nu}$ and has a known and universal structure.

Consequently, at the next sub-leading order at $\mathcal{O}(\alpha^2)$, the unknowns are $g^{(2)}_{\mu\nu}$ (or, as usual, $X^{(2)}(\rho, x^i)$, $\omega^{(2)}_i(\rho, x^i)$ and $h^{(2)}_{ij}(\rho, x^i)$), and the PDE for them should look like

$$E^{(0)}_{\mu\nu}\left[g^{(0)}_{\alpha\beta} + \alpha^2\, g^{(2)}_{\alpha\beta}\right] + E_{\mu\nu}\left[g^{(0)}_{\alpha\beta} + \alpha\, g^{(1)}_{\alpha\beta}\right] = \mathcal{O}(\alpha^3)\,, \tag{3.32}$$

where the first term on LHS is the homogeneous piece and the other term is the source term for $g^{(2)}_{\mu\nu}$. The source term, again, is evaluated on metric functions $g^{(1)}_{\mu\nu}$ and $g^{(0)}_{\mu\nu}$, which are already solved in the previous iteration at $\mathcal{O}(\alpha^2)$.

Once we have studied the EoM to the second sub-leading order in the α-expansion, we should be able to extend this analysis to any arbitrary order in α. Let us assume that we are currently focussing on the $(m+1)$-th order term in the expansion. We have learned that if we would like to determine the solution correctly up to order $\mathcal{O}(\alpha^{m+1})$, we have to evaluate $E_{\mu\nu}$ on $g^{(bh)}_{\mu\nu}$ neglecting all terms of order $\mathcal{O}(\alpha^{m+2})$ and higher. At this order, i.e. at $\mathcal{O}(\alpha^{m+1})$, the unknowns would be the components of $g^{(m+1)}_{\mu\nu}$ or in our choice of gauge, the metric functions $X^{(m+1)}(\rho, x^i)$, $\omega^{(m+1)}_i(\rho, x^i)$ and $h^{(m+1)}_{ij}(\rho, x^i)$. Now, we would like to know what would be the structure of the EoM at this order. As it is true for any perturbative solution technique, the homogeneous part of the equation at every order has an universal structure. In this case it is the Einstein equation $E^{(0)}_{\mu\nu}$ linearized around the zeroth order black hole metric $g^{(0)}_{\mu\nu}$, but now the role of the fluctuation metric will be played by $g^{(m+1)}_{\mu\nu}$ [6].

But the source terms (analogous to the second term in eq.(3.31)) will not have this universal form. At order $\mathcal{O}(\alpha^{m+1})$, where the metric upto order $\mathcal{O}(\alpha^m)$ is already fixed by solving the equations at previous orders, the source terms will be the coefficient of α^{m+1} in $E_{\mu\nu}$ once evaluated on $g^{(bh)}_{\mu\nu}$ as in eq.(3.28) but corrected upto $\mathcal{O}(\alpha^m)$, that is

$$g^{(bh)}_{\mu\nu}\,|_{\text{corrected upto } \mathcal{O}(\alpha^m)} = g^{(0)}_{\mu\nu} + \alpha\, g^{(1)}_{\mu\nu} + \cdots + \alpha^m\, g^{(m)}_{\mu\nu}\,.$$

Therefore, at $\mathcal{O}(\alpha^{m+1})$, the equation looks like the following

$$E^{(0)}_{\mu\nu}\left[g^{(0)}_{\mu\nu} + \alpha^{m+1}\, g^{(m+1)}_{\mu\nu}\right] + E_{\mu\nu}\left[g^{(0)}_{\mu\nu} + \alpha\, g^{(1)}_{\mu\nu} + \alpha^2\, g^{(2)}_{\mu\nu} + \cdots + \alpha^m\, g^{(m)}_{\mu\nu}\right] = \mathcal{O}(\alpha^{m+2})\,, \tag{3.33}$$

However, all the terms contributing to the source term in eq.(3.33) are obtained from metric functions, which are all solved until the previous order in this iterative construction. Interestingly, for our proof, we do not need the details of the source term, except for the fact that at any given order, it is a covariant tensor evaluated on a metric that solves the EoM up to the previous order [7].

[6] The reason for this universality is as follows. The correction to the solution i.e., $g^{(m+1)}_{\mu\nu}$ already carries a factor of α^{m+1}. Since we are interested in evaluating the equation at order $\mathcal{O}(\alpha^{m+1})$ and also if we want to collect only those terms that involves $g^{(m+1)}_{\mu\nu}$, everything else in the equation must be of zeroth order in α. It follows that at order $\mathcal{O}(\alpha^{m+1})$ none of the $E^{(m)}_{\mu\nu}$, for $m > 0$ can contribute to terms that has $g^{(m+1)}_{\mu\nu}$ and the same is true for product terms of the form $g^{(m)}\, g^{(n)}$, for $m > 0$. Therefore at order $\mathcal{O}(\alpha^{m+1})$, terms that contain $g^{(m+1)}_{\mu\nu}$ can only come from the EoM at zeroth order linearized around the zeroth order solution.

[7] We must emphasize that for this perturbative technique to work at a given order (say $\mathcal{O}(\alpha^{m+1})$), EoM

3.5.2 Zeroth law for two derivative theories of gravity, at leading order in α-expansion

In the previous subsection, we have described the general structure of the EoM at any given order in α-expansion. We have seen that the starting point must be Einstein's two derivative gravity, and $g^{(0)}_{\mu\nu}$ must be an exact stationary black hole solution of the Einstein equations $E^{(0)}_{\mu\nu}$.

It is well known that in the two derivative theory of gravity, the temperature of a stationary black hole is constant over the horizon. In this sub-section, we will review, following the strategy outlined in §3.4, how this can be proved in our choice of coordinate system eq.(3.4). As we have already mentioned, to achieve this, we must look into the off-shell structure of the (τi) component of the zeroth-order EoM $E^{(0)}_{\mu\nu}$. It will turn out that $E^{(0)}_{\tau i}$ is indeed of the form eq.(3.22). When EoMs are satisfied by stationary black hole configurations, we will readily obtain eq.(3.9). This is, therefore, enough to prove the desired result in two-derivative theories of gravity. In the following, we will argue that eq.(3.22) is indeed true.

Let us consider two derivative theories of gravity without any matter field. To be more explicit, let us reiterate that the equation of motion eq.(3.3) is

$$E_{\mu\nu} = E^{(0)}_{\mu\nu} \,. \tag{3.34}$$

The metric eq.(3.4) upto order $\mathcal{O}(\alpha^0)$ in the horizon adapted coordinate system is

$$ds^2 = 2d\tau\, d\rho - \rho X^{(0)}(\rho, x^i)d\tau^2 + 2\rho\, \omega^{(0)}_i(\rho, x^i)d\tau dx^i + h^{(0)}_{ij}(\rho, x^i)dx^i dx^j \,. \tag{3.35}$$

We would calculate τi component of the EoM on the horizon.

$$E_{\tau i} = R_{\tau i} - \frac{1}{2}R g_{\tau i} \quad \Rightarrow \quad E_{\tau i}|_{\rho=0} = R_{\tau i}|_{\rho=0} \,. \tag{3.36}$$

must be solved till the previous order. This will ensure that the source term i.e., $E_{\mu\nu}$, evaluated on $(g^{(0)}_{\mu\nu} + \alpha\, g^{(1)}_{\mu\nu} + \cdots + \alpha^m\, g^{(m)}_{\mu\nu})$ will be non-zero only at order $\mathcal{O}(\alpha^{m+1})$.

We must now compute $R_{\tau i}$ for our choice of metric eq.(3.35). Using the expression of $R_{\tau i}$ (see appendix A.3), we get

$$E_{\tau i}|_{\rho=0} = -\frac{1}{2}(\partial_i X^{(0)})|_{\rho=0} \tag{3.37}$$

Using EoM, we can straightforwardly conclude

$$\partial_i X^{(0)}(\rho,\, x^i)|_{\rho=0} = 0\,, \tag{3.38}$$

which is basically eq.(3.9) upto $\mathcal{O}(\alpha^0)$, and therefore, implies zeroth law at the same order.

3.5.3 Zeroth law for higher curvature theories of gravity at arbitrary order in α-expansion

After establishing the zeroth law at the leading order in α-expansion for two derivative theories, in this section, we aim to extend this to arbitrary higher-order in the perturbative α expansion. We will construct our proof using a method of induction. It will be shown that if the temperature is constant over the horizon till order $\mathcal{O}(\alpha^n)$, then it will remain constant at order $\mathcal{O}(\alpha^{n+1})$. Following our strategy described in §3.4, and just like what we did at the zeroth order, we will again use the off-shell structure of the (τi) component of the EoM to show this.

For convenience, let us first re-write the metric and its α-expansion, eq.(3.28) and eq.(3.40),

$$ds^2 = g^{(bh)}_{\mu\nu}dx^\mu dx^\nu = 2\,d\tau\,d\rho - \rho\,X(\rho,x^i)d\tau^2 + 2\rho\,\omega_i(\rho,x^i)d\tau dx^i + h_{ij}(\rho,x^i)dx^i dx^j\,, \tag{3.39}$$

where,

$$\begin{aligned} X(\rho,x^i) &= X^{(0)}(\rho,x^i) + \alpha\,X^{(1)}(\rho,x^i) + \alpha^2\,X^{(2)}(\rho,x^i) + \cdots \\ \omega_i(\rho,x^i) &= \omega_i^{(0)}(\rho,x^i) + \alpha\,\omega_i^{(1)}(\rho,x^i) + \alpha^2\,\omega_i^{(2)}(\rho,x^i) + \cdots \\ h_{ij}(\rho,x^i) &= h_{ij}^{(0)}(\rho,x^i) + \alpha\,h_{ij}^{(1)}(\rho,x^i) + \alpha^2\,h_{ij}^{(2)}(\rho,x^i) + \cdots\,. \end{aligned} \tag{3.40}$$

We start with the statement that we have solved the EoM accurately upto order $\mathcal{O}(\alpha^m)$. Also, following the same logic, we are assuming that the temperature is constant on the horizon upto order $\mathcal{O}(\alpha^m)$. In terms of equation it implies

$$\partial_i \left(X^{(0)}(\rho,\, x^i) + \alpha\, X^{(1)}(\rho,\, x^i) + \cdots + \alpha^m\, X^{(m)}(\rho,\, x^i) \right) \Big|_{\rho=0} = 0, \tag{3.41}$$

Given this, now, we would like to solve the EoM at order $\mathcal{O}(\alpha^{m+1})$. As we have discussed in the previous sub-section §3.5.1, in the context of the general structure of the EoM at an arbitrary order of the α-expansion, working at $\mathcal{O}(\alpha^{m+1})$, we will get a linear partial differential equation for the unknown $g^{(m+1)}_{\mu\nu}$. This will be a linear PDE with two types of terms; one is the homogeneous term along with another source term. In the following, we will analyze these two terms one by one.

From eq.(3.33) we have learned that the homogeneous part of the equation could be universally evaluated as linearisation of the Einstein tensor $\left(\text{i.e., } E^{(0)}_{\mu\nu}\right)$ around $g^{(0)}_{\mu\nu}$,

$$\text{Homogeneous part of the PDE at } \mathcal{O}\left(\alpha^{m+1}\right) \;=\; E^{(0)}_{\mu\nu}\left[g^{(0)}_{\mu\nu} + \alpha^{m+1} g^{(m+1)}_{\mu\nu}\right] + \mathcal{O}(\alpha^{m+2})\,. \tag{3.42}$$

Note in the above equation the RHS will have the leading contribution at order $\mathcal{O}(\alpha^{m+1})$ since by construction $E^{(0)}_{\mu\nu}\left[g^{(0)}_{\mu\nu}\right] = 0$. For our purpose, we just need to look at the (τi) component of the EoM. By explicit evaluation in our choice of coordinate system we could show (see appendix A.4)

$$E^{(0)}_{\tau i}\left[g^{(0)}_{\mu\nu} + \alpha^{m+1} g^{(m+1)}_{\mu\nu}\right]_{\rho=0} = -\,\frac{1}{2}\,\alpha^{m+1}\,\left(\partial_i X^{(m+1)}\right)\Big|_{\rho=0} + \mathcal{O}(\alpha^{m+2}) \tag{3.43}$$

It should be noted that in deriving eq.(3.43), we have used the result obtained in eq.(3.38) for the leading order two derivative theory.

Now we come to the source terms. These are the known terms at order $\mathcal{O}(\alpha^{m+1})$. These could be computed by evaluating the EoM, keeping terms up to order $\mathcal{O}(\alpha^{m+1})$, and ignoring all higher-order terms on the metric corrected up to order $\mathcal{O}(\alpha^m)$.

Before we proceed, let us introduce a new notation here for convenience. For any function Y that admits an α expansion, $Y^{(m)}$ denotes the coefficient of α^m and $Y^{[m]}$ denotes the expansion of Y correct upto order $\mathcal{O}(\alpha^m)$. In other words, if Y could be written as $Y = \sum_{m=0}^{\infty} \alpha^m \, Y^{(m)}$, then $Y^{[m]}$ denotes

$$Y^{[m]} \equiv \sum_{k=0}^{m} \alpha^k \, Y^{(k)} \, . \tag{3.44}$$

According to this notation, then, for the corrected and truncated metric and EoM till order $\mathcal{O}(\alpha^m)$, we get

$$g_{\mu\nu}^{[m]} \equiv \sum_{i=0}^{m} \alpha^m \, g_{\mu\nu}^{(m)}, \quad \text{and} \quad E_{\mu\nu}^{[m]} \equiv \sum_{i=0}^{m} \alpha^m \, E_{\mu\nu}^{(m)} \, . \tag{3.45}$$

Using this new notation, let us now write down the source term in the PDE for $g_{\mu\nu}^{(m)}$, working at order $\mathcal{O}(\alpha^{m+1})$, as the following

$$\text{Source terms of the PDE at } \mathcal{O}(\alpha^{m+1}) \; = \; E_{\mu\nu}^{[m+1]} \Big|_{\text{evaluated on } g_{\mu\nu}^{[m]}} + \mathcal{O}(\alpha^{m+2}) \tag{3.46}$$

Note that according to our assumptions, $g_{\mu\nu}^{[m]}$ solves the EoM up to order $\mathcal{O}(\alpha^m)$. It follows that the source terms as written above will have the first non-zero contribution at order $\mathcal{O}(\alpha^{m+1})$. As we have mentioned before, the source terms do not have any universal structure, unlike the homogeneous piece. However, for the constancy of the temperature, we need to analyze only the (τi)-component of the EoM and that too only at the horizon, i.e., $\rho = 0$ hypersurface in our choice of coordinates. This will simplify our analysis.

Boost symmetry for the truncated metric and vanishing of the source term

In the previous sub-section, we have shown that the homogeneous part of the (τi) component of the EoM at this order is simply proportional to $\partial_i X^{(m+1)}$, see eq.(3.43). Therefore, what is left to be checked is that the source term in the PDE for $g_{\mu\nu}^{(m+1)}$ vanishes at this order $\mathcal{O}(\alpha^{m+1})$. In the following, we will argue that this is exactly what will turn out to be true.

As we have described before, the source term at order $\mathcal{O}(\alpha^{(m+1)})$, given in eq.(3.46), is simply the leading piece (in terms of α-expansion) in $E^{[m]}_{\mu\nu}$ evaluated on $g^{[m]}_{\mu\nu}$. We should remember that, the corrected space-time metric $g^{[m]}_{\mu\nu}$ is truncated at $\mathcal{O}(\alpha^m)$. Also, it is corrected, because, it solves the EoM till the same order. The truncated black hole metric $g^{[m]}_{\mu\nu}$, leads to the following line element

$$ds^2_{[m]} = 2\,d\tau\,d\rho - \rho\,X^{[m]}(\rho,x^i)\,d\tau^2 + 2\,\rho\,\omega^{[m]}_i(\rho,x^i)\,d\tau\,dx^i + h^{[m]}_{ij}(\rho,x^i)\,dx^i dx^j \quad (3.47)$$

As a consequence of our assumption in eq.(3.41), the zeroth law can be assumed to be satisfied for the metric $g^{[m]}_{\mu\nu}$ till $\mathcal{O}(\alpha^m)$. So, we are allowed to use the fact that $X^{[m]}(\rho = 0,\, x^i)$ is constant over the horizon ($\rho = 0$ hypersurface),

$$\partial_i \left(\sum_{m\leq n} \left(X^{(0)}(\rho,x^i) + \alpha^m X^{(m)}(\rho,x^i) \right) \right) \Bigg|_{\rho=0} = \mathcal{O}(\alpha^{m+1})\,. \quad (3.48)$$

This in turn enables us to ascertain that the surface gravity, κ, computed with the corrected metric till $\mathcal{O}(\alpha^m)$, will be constant over the horizon. In other words $X^{[m]}$ could be written, by solving eq.(3.48), as

$$X^{[m]}(\rho,x^i) = C^{[m]} + \rho\,F^{[m]}(\rho,x^i)\,, \quad (3.49)$$

where $C^{[m]}$ is a constant, and $F^{[m]}(\rho,x^i)$ is an arbitrary function of (ρ,x^i), and both of them are corrected upto $\mathcal{O}(\alpha^m)$.

Following our discussion in §3.3 (see eq.(3.20)) we would like to perform the following coordinate transformation from $x^\mu = \{\tau,\rho,x^i\}$ to $\widetilde{x}^\mu = \{v,r,x^i\}$ where, v is the affine parameter along the null generator of the horizon,

$$v = \frac{2}{C^{[m]}} \exp\left(\frac{C^{[m]}}{2}\tau\right), \quad r = \rho\,\exp\left(-\frac{C^{[m]}}{2}\tau\right)\,. \quad (3.50)$$

The truncated metric in the new coordinates takes the form

$$ds^2_{[m]} = 2dv\,dr - r^2\,F^{[m]}(C^{[m]}rv/2,x^i)dv^2 + 2r\,\omega^{[m]}_i(C^{[m]}rv/2,x^i)dv\,dx^i + h^{[m]}_{ij}(C^{[m]}rv/2,x^i)dx^i dx^j \quad (3.51)$$

Now, $E^{[m]}_{\mu\nu}$ is just a covariant tensor of rank two, constructed out of appropriate contractions of the product of Riemann tensors and/or their covariant derivatives. So, without knowing any details about it, we could tell how its component would transform under the above-mentioned coordinate transformation. By which we mean that $\widetilde{E}^{[m]}_{\mu\nu}$ in the new coordinates will be related to $E^{[m]}_{\mu\nu}$ in the old coordinates, as follows

$$E^{[m]}_{\mu\nu}(\rho,\tau,x^i) = \frac{\partial \widetilde{x}^\alpha}{\partial x^\mu}\frac{\partial \widetilde{x}^\beta}{\partial x^\nu}\,\widetilde{E}^{[m]}_{\alpha\beta}(r,,v,x^i)\,. \tag{3.52}$$

For our purpose we just need to study the τi-component of $E^{[m]}_{\mu\nu}(\rho,\tau,x^i)$. Also, we can readily obtain the relevant components of $\frac{\partial \widetilde{x}^\alpha}{\partial x^\mu}$ from eq.(3.50),

$$\frac{\partial v}{\partial \tau} = \exp\left(\frac{C^{[m]}}{2}\tau\right), \quad \text{and} \quad \frac{\partial r}{\partial \tau} = -\rho\frac{C^{[m]}}{2}\exp\left(-\frac{C^{[m]}}{2}\tau\right). \tag{3.53}$$

Using them we obtain

$$E^{[m]}_{\tau i}(\rho,\tau,x^i)\Big|_{\rho=0} = \exp\left(\frac{C^{[m]}}{2}\tau\right)\widetilde{E}^{[m]}_{vi}(r,,v,x^i)\Big|_{r=0} \tag{3.54}$$

It is important to note that we have obtained $E^{[m]}_{\tau i}$ is proportional to $\widetilde{E}^{[m]}_{vi}$ when evaluated on the horizon. To decide about $\widetilde{E}^{[m]}_{vi}$ in the new coordinate system $(r,,v,x^i)$ we can directly use the boost-invariance of the metric eq.(3.47). As we can see that $\widetilde{E}^{[m]}_{vi}$ contains one extra lower v-index compared to r-index. According to the arguments due to this boost-symmetry the boost-weight assigned to $\widetilde{E}^{[m]}_{vi}$ comes out to be $+1$. Therefore, if we compute $\widetilde{E}^{[m]}_{vi}$ on the stationary metric eq.(3.21) at $r = 0$ it will simply vanish. This in turn shows that, by using eq.(3.54), in our old (ρ,τ,x^i) coordinates $E^{[m]}_{\tau i}$ also vanishes.

Therefore, we have now established the fact that at $\mathcal{O}(\alpha^{m+1})$ the source term contribution to the PDE for $g^{(m+1)}_{\mu\nu}$ vanishes. The homogeneous piece is the only contribution and that too is of the form argued in eq.(3.43). With this we have also successfully demonstrated that

$$\partial_i X^{(k)}(\rho,\,x^i)|_{\rho=0} = 0, \quad \text{for} \quad k = (m+1)\,, \tag{3.55}$$

once we start with the assumption of $\partial_i X^{(k)}(\rho,\, x^i)|_{\rho=0} = 0$ for $k \leq m$. Finally, by method of induction, we, therefore also prove that, starting with a positive result in the leading two derivative gravity, the zeroth law is true upto all orders in the perturbative expansion in the higher derivative coupling α.

3.6 Discussions

In this paper, we have worked out a proof for the zeroth law of black hole thermodynamics in diffeomorphism invariant theories of gravity. Our analysis crucially depends on the fact that we consider only such theories of gravity where arbitrary higher derivative theories of gravity augment the leading two derivative theory of general relativity. We assumed that the higher derivative coupling (denoted by α in this paper) could be taken to zero in a smooth limit leaving us with the leading two derivative theory. This, in turn, allows us to organize our analysis in a perturbative expansion in the higher derivative coupling α. For example, suppose we have an exact solution in the form of a black hole metric of the equations of motion coming from the two-derivative Einstein's equation. We can correct this solution in that perturbation scheme and expect to obtain the corresponding black hole solution in the higher derivative theory of gravity. The zeroth law is a statement about stationary configurations. We used a particular coordinate system, like choosing a particular gauge, to write down the space-time metric of a stationary black hole. Since the temperature of a black hole is identified with the geometric quantity called surface gravity, working within our choice of metric gauge, our main aim was to prove that the surface gravity is constant over the horizon for stationary black holes. We want to stress here that for our construction of the proof we did not need to use any extra symmetry, we have only used the boost-symmetry which follows from stationarity.

The crucial ingredient in our construction for the proof was to use specific components of the equations of motion (EoM). We expanded the EoM order by order in a perturbation series in the higher derivative coupling α, with the leading term (with $\alpha = 0$) being Einstein's equation. The metric was also expanded in a similar perturbative expansion in α, with the leading order term being the stationary black hole solution in Einstein gravity. We followed a method of induction for the proof. First, we showed that the components of EoM have the desired off-shell structure needed for the proof to go through at the leading order in Einstein's gravity. Then we assumed that this is true at the n-th order, and we argued that it should be satisfied at the following order in α-expansion.

Working at $\mathcal{O}(\alpha^{m+1})$, once we assumed that the zeroth law is satisfied at the previous order, i.e., till $\mathcal{O}(\alpha^{m})$, we made use of a specific residual gauge symmetry to perform a coordinate transformation. In this new coordinate system, the coordinate along the null generators of the horizon happens to be affinely parametrized, and the new metric enjoys a symmetry called the boost-symmetry of the stationary black holes. Using this symmetry, we could predict the structure of the components of the EoM without knowing its explicit form. In other words, we viewed the EoM at $\mathcal{O}(\alpha^{m+1})$ as a covariant tensor built out of the metric components corrected till $\mathcal{O}(\alpha^{m})$ to satisfy the zeroth law. Then, by knowing how the EoM for any arbitrary higher derivative theory should transform under the coordinate transformation (boost transformation), we could prove that it indeed has the required structure to satisfy the zeroth law at $\mathcal{O}(\alpha^{m+1})$.

It is essential to highlight that this particular boost symmetry can be used only when the metric can be cast in the new coordinate system we mentioned above. This was crucially used in constructing the entropy current, using which the local version of the second law was argued for arbitrary diffeomorphism invariant theory of gravity. We also understood that one could write down the stationary black hole metric in these new coordinates if the zeroth law is satisfied. This was an important assumption in constructing the entropy current in [31] and [32]. Therefore, our proof of zeroth law in this paper justifies this important assumption that was made in those works aimed at proving the second law.

Another critical point in constructing our proof in this paper is the assumption that it only applies to such theories where, in an appropriate α expansion, the leading order piece has to be Einstein's two derivative theory. It was one significant input in our proof. However, we have not argued that there cannot be any other proof that will not require this assumption of starting the perturbation series from Einstein's gravity. From the perspective of a UV complete theory of quantum gravity, it is pretty natural to expect that the low energy effective theories following from any quantum theory of gravity would organize themselves in such a perturbative framework starting with two derivative general relativity. However, it is exciting to note that without having access to the details of how UV completion is achieved and staying entirely within a low energy perspective, principles like the laws of black hole thermodynamics also hint toward general relativity as the leading theory in a perturbative framework.

Although, our proof is perturbative in higher derivative coupling constant α, we want to stress that it works up to arbitrary order in the perturbative expansion. With this statement, we might hope that finding a proof of zeroth law for theories non-perturbatively connected to general relativity will be worth exploring.

In this work, we have used surface gravity as the definition of the temperature of the black holes. We consider here only the gravitational part of the Lagrangian. We believe that our proof will be valid if there is some matter sector obeying Dominant energy condition. But there are theories like Horndeski Theory, where this definition of temperature as surface gravity does not hold[37]. It will be very interesting to find out how our proof will go through for these types of theories.

Chapter 4

Entropy Current for Four Derivative Theories of Gravity

This chapter is based on[31]

It is widely believed that Einstein's theory of gravity must admit an adequate UV completion when we approach length scales comparable to Planck length. Such a putative UV complete theory of quantum gravity, at large length scales, must reduce to the weakly coupled, two derivative Einstein's theory, which has been exhaustively verified in the IR, by several experiments. However, at intermediate length scales, we may encounter a regime, where gravity is still weakly coupled and the quantum corrections are suppressed, but higher derivative corrections to Einstein equations cannot be ignored [1]. In this regime, gravity would be described by an arbitrary diffeomorphism invariant classical theory

$$\mathcal{L} = \mathcal{L}_g\left(g_{\mu\nu}, R_{\mu\nu\lambda\sigma}, \nabla_\alpha R_{\mu\nu\lambda\sigma}, \nabla_\beta\nabla_\alpha R_{\mu\nu\lambda\sigma} + \dots\right) + \mathcal{L}_m, \tag{4.1}$$

where, the gravity part of the Lagrangian would admit a derivative expansion, as follows

$$\mathcal{L}_g = R + \left(\alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) + \text{higher derivatives} \tag{4.2}$$

[1] Such an intermediate regime exists, for instance, in string theory, which is a prominent candidate for the UV complete theory of quantum gravity. In string theory, within this regime, the string coupling $g_s \to 0$, implying the quantum corrections are suppressed. While, the ratio $\ell_s/\mathfrak{R}$ is non-negligible, $\mathfrak{R}$ being the length scale associated with space-time curvatures, while ℓ_s is the string length. Whether such an intermediate regime exists in the real world, or whether Einstein's description is a good description right up to the Planck scale, can only be answered with precision experiments of the future.

Here, in (4.1), $\mathcal{L}_m$ represents the matter part the theory [2]. Any non-minimal coupling to gravity, i.e., the terms involving curvature and the matter fields are also included in $\mathcal{L}_m$. Note that, the specific values of the coefficients α_1, α_2 and α_3 depend on the details of the UV complete theory. These are dimension full constants and therefore must be proportional to the (square of) some fundamental length scale of the UV complete theory. For example, in the case of string theory, all of these coefficients will have the form $\alpha_i \sim l_s^2 \tilde{\alpha}_i$ where l_s is the string length and $\tilde{\alpha}_i$s are some numbers. The limit that we shall consider here, is the one, where the length scale associated with the curvatures of the space-time and those associated with the variations of the matter fields, are much larger compared to l_s. In other words, all the higher derivative corrections will be more and more suppressed, with the increase in the number of derivatives.

We know that the two derivative Einstein's theory admits black holes solutions with Killing event horizons. These solutions can be understood as macroscopic manifestation of an ensemble of many microscopic degrees of freedom of the more fundamental theory of gravity, in thermodynamic equilibrium, at finite temperature. If this statistical picture of the black hole is correct, then stationary black hole solutions should exist even after we add higher derivative corrections to the gravity action. Also, within these higher derivative theories of gravity, we should be able to construct macroscopic quantities, such as entropy, for the black hole solutions, which will satisfy the laws of thermodynamics.

In two derivative theory of Einstein's gravity, we have a candidate for entropy that satisfies both the first and second law of thermodynamics. The entropy, in this case, is given by the area of a 'time-slice' of the event horizon [39, 40, 41] (also see [42, 43, 44]). We shall denote the even horizon with $\mathcal{H}$ and the time-slice of it with $\mathcal{H}_v$. The second law,

[2]Note that, there would certainly be some phenomenological restrictions on $\mathcal{L}_m$. For example, it should reduce to the standard model Lagrangian at large length scales. Beside this, $\mathcal{L}_m$ here also incorporates all the matter fields, that may be required to make the higher derivative theory of gravity (4.2) well defined. For instance, It was pointed out in [38], that causality constrains on tree level graviton three point functions imply that that $\mathcal{L}_m$ should incorporate higher spin fields.

for this entropy, followed from the famous area increase theorem for black holes [39, 42], which assumes that the matter energy-momentum tensor obeys the null-energy condition. Throughout our discussion in this note, we shall assume this condition to be valid for the matter part of the Lagrangian $\mathcal{L}_m$.

Now it is very difficult to analyze dynamical black hole solutions even in two derivative Einstein's theory of gravity. People usually take recourse to several perturbation schemes, around the stationary solutions that are known exactly. In this context, the simplest situation that comes to mind is the case where some stationary black hole is slightly perturbed by some external agent so that the resulting dynamical black hole metric could be decomposed as a sum of stationary part and the time dependent part with small amplitude. Then one can analyze the equations in an expansion in terms of the amplitude [3].

In [1] the author has used this expansion to construct one 'out of equilibrium' extension of Wald entropy, which, up to linear order in amplitude expansion, is monotonically increasing at every instant of 'time' and therefore satisfies the second law in a stronger sense (also see [27, 28, 46, 30]). This locality in time is not entirely unexpected in this type of set-up where the space-time is 'near' some equilibrium or stationary solution at every instant of time. Following the same intuition, we could also say that for such slow enough, 'near equilibrium' time evolution, where we could assume that different subregions of a large macroscopic system are in approximate equilibrium with its immediate neighborhood, at every instant of time, we should also expect a spatial locality, in the formulation of second law. This expectation is completely consistent with the scenario in Einstein's theory of gravity, where the area increase theorem is valid locally, for every infinitesimal area element of a 'time-slice' of the horizon. Our expectation for a local, stronger form of second

[3] Under this approximation, it is not possible to study violent processes such as the formation of black holes or merger of two black holes. Throughout this chapter, we shall work only up to linear order in the amplitude of fluctuations and therefore, our results would not directly apply to these violent scenarios. See [45], where similar questions relating to entropy and the second law, for processes involving the merger of black holes have been addressed.

law, is very much motivated by this example of Einstein's theory. Now, in a more general setting involving higher derivative corrections to Einstein's equation, during slow time evolutions, besides entropy production in every infinitesimal subregion, we have to be also open to the possibility that entropy could be redistributed between the neighboring regions, by flowing in or out via some spatial current. The necessity of having such non-zero spatial current for entropy, and the existence of a strong ultra-local form of the second law of thermodynamics, in higher derivative theories of gravity as well, are the key points of our investigation here.

In this chapter, we shall demonstrate that it is possible to formulate the second law in its strongest form, so that at least for 'slow enough' dynamical situations, entropy is produced at every point of the evolving space-time, up to a possible inflow and outflow via some spatial current[4]. We will explicitly construct this spatial current for entropy flow, in the most general four derivative theory of gravity.

Let us now outline the organization of this chapter, along with a brief summary of the key arguments and results in the various sections.

At first, in §4.1, we shall review the paper [1] in detail. The author in [1] has shown that at the leading order in amplitude expansion, certain 'time-time' component of the equation of motion of any higher derivative theories of gravity could always be written as two 'time derivatives' acting on some quantity. Then he could further argue that if one identifies the integral of this quantity (the expression on which the two time derivatives are acting) over $\mathcal{H}_v$, with the entropy of the gravitational theory, then it will satisfy the second law at least at the leading order within this approximation. Following [1] we have first set

[4]An entropy current with non-negative divergence certainly exists in near equilibrium states for theories that do not include dynamical gravity [47, 48, 49, 50]. In the case of gravity, where the space-time itself becomes the fundamental dynamical object, the concept of locality might become a bit confusing. The locality in space-time in some sense becomes analogous to some form of locality in the space of fundamental fields of non-gravitational theories. However, the kind of perturbation that we are considering here, there is always a stationary base metric which could play the role of the background and the above mentioned issues could be avoided.

up an appropriate coordinate system, thus defining the 'time' mentioned above. Next, by using a symmetry of the horizon geometry (referred to as 'boost symmetry' in [1]), we have classified the terms that can appear in that particular 'time-time' component of the equations of motion, according to their weight under this boost transformation. We shall see that the argument and construction in [1] smoothly goes through for all the higher weight terms except the one that appears at zero boost weight.

This point was noted in [1] and it has been argued that if these 'zero boost terms' are not of the correct form (i.e., two time derivatives acting on some quantity) it would amount to the violation of the first law itself, once viewed in the 'physical version' formulation of it [17, 51] (also see [52, 53, 54, 55, 56]). Therefore, though the central argument in [1] naively break down for these special 'zero boost terms', it must work out in actual theories, where the physical process version of the first law is valid.

In this context, the four-dimensional Gauss-Bonnet theory requires a special mention. We have discussed this case in details in §4.2.6. It is well known that in four dimensions Gauss-Bonnet action is a total derivative and therefore does not contribute to the equation of motion. However, from Wald's analysis, we know that the entropy of the black holes in Gauss-Bonnet theory does receive correction which is proportional to the intrinsic Ricci-scalar evaluated on the two-dimensional spatial section of the horizon. Integration of this quantity over a compact two dimensional manifold results in a topological quantity, the Euler characteristics, that does not change under small continuous deformation of the horizon caused due to the perturbation. This is perfectly consistent with the fact that 4-d Gauss-Bonnet term does not introduce any correction to the equation of motion and so (following the argument of [1]) no correction to the change in total entropy during time evolution. However, if we are thinking of in terms of the entropy density (i.e., the same intrinsic Ricci scalar without the integration over all spatial sections of the horizons), it does evolve with time. But, this entropy density, clearly, would not satisfy the ultra-local version of the second law. However, the validity of the local version of the second law is restored, if we also consider a spatial entropy current. It is satisfying to check that, for the $3+1$ dimension the v-derivative of the Ricci scalar is identical to the time derivative of the divergence of a spatial current (given in terms of the extrinsic curvatures of $\mathcal{H}_v$). These two contributions from the entropy density and the spatial entropy current, to the equation of motion, cancel out each other. This cancellation is off-shell and specific to $3+1$ dimensions only. Thus, in this simple example, it is easy to recognize the necessity of the entropy current, even before performing the detailed calculation. As we will see in more detail in §4.2.6, this example also helps us to identify an ambiguity present in the definition of the entropy current.

In §4.2.2, §4.2.3 and §4.2.4, we go on to develop a general algorithm for constructing the spatial components of the entropy current for arbitrary four derivative theory. From this exercise, we learn that the most general form of the relevant equation of motion $E_v v$, which is consistent with the boost symmetry, has a structure which is more general compared to what would be essential for defining the entropy current. In other words, the fact we have an entropy current and consequently a local second law, puts very non-trivial constrains on the most general possible structure of E_{vv} [5]. Although at the moment we do not have a precise explanation regarding the physical origin of these constraints, we believe that these may arise due to some residual gauge freedom. We think understanding the exact mathematical reason behind these constrains, would lead us to a proof of the local second law through the construction of the entropy current, without invoking the first law at all.

This general algorithm has also helped us to understand the ambiguities related to the construction of the current more clearly. In §4.2.7, we report on one of the primary source of such ambiguities. Finally in §4.3, we conclude and discuss possible future directions.

Before concluding this section, we would like to mention that, the notion of an entropy

[5] As we have mentioned above, an explicit calculation for the four derivative theories demonstrates that, these constraints are automatically met for these theories.

current for black hole dynamics in higher derivative theories of gravity, is not completely new in this note. This idea has been previously introduced in [57, 58, 59]. In [57, 58] it was primarily motivated by the entropy current, constructed in the context of the fluid gravity correspondence [47]. While in [59], the entropy current was constructed exploiting the membrane-gravity duality, using an expansion in inverse powers of space-time dimension. Although, the exact context of these constructions are different from our considerations here, but there are some similarities in the basic idea (see §4.3 for further discussions on this). The exact relation between our construction and that reported in these papers is a topic of our current investigation and we hope to report on it in the near future.

4.1 A comprehensive review of [1]

As we have discussed in the last section in an arbitrary diffeomorphism invariant theory of gravity, a proof of the second law for dynamical black holes was provided in [1]. Let us review the details of the proof here, which would serve as a useful prelude to the subsequent discussion of our entropy current.

Let us first choose a coordinate system. Let ∂_v be the null generators of the event horizon, where v is the affine parameter. Let ∂_is denote the rest of the spatial tangents of the horizon. Integral curves of ∂_is are the spatial coordinates along the constant v slices of the horizon. Then from every point on the horizon, we shoot off a set of null geodesics, making a precise angle with the coordinates on the horizon. We label each of these new set of null geodesics (null everywhere) by the coordinates of the point at which it intersects the horizon. We denote the affine parameter along the null geodesics to be r which is the coordinate, away from the horizon. The most general metric with this choice of coordinates would have the following structure (see appendix of [36])

$$ds^2 = 2dv\, dr - r^2 X(r, v, x^i)\, dv^2 + 2r\, \omega_i(r, v, x^i)\, dv\, dx^i + h_{ij}(r, v, x^i)\, dx^i\, dx^j \quad (4.3)$$

Here we have chosen the horizon to be at $r = 0$ (a choice for the origin of the affine parameter along each null geodesic ∂_r). Note that this choice of gauge is slightly different from that of [1]. We have set $g_{rv} = 1$ throughout space-time, but in [1] this condition was set only on the future horizon $\mathcal{H}$. In Appendix A of [36], it was demonstrated that this choice of metric (4.3), is possible without any loss of any generality, even for dynamical black holes. We would also like to emphasize that this difference in gauge choice, do not affect the arguments in [1], in any way. We shall work with this slight difference in this note, simply because we prefer to work with a metric where the gauge fixing is more complete.

Given the form of the metric (4.3), let us now outline the broad strategy of the proof of second law provided in [1].

4.1.1 Strategy of the proof of second law of black hole thermodynamics

The general strategy of the proof follows that of area increase theorem for dynamical black holes in Einstein gravity [39, 42]. Following [1] we shall consider small time dependent fluctuations about stationary black holes. Let us denote the amplitude of the fluctuation to be ϵ. All the analysis would be linear in the amplitude (denoted by ϵ) of this fluctuation.

Let us denote a v-slice of the horizon of the dynamical black hole to be $\mathcal{H}_v$. In [1] $\mathcal{H}_v$ has been considered to be compact, an assumption which played an extremely important role in the proof [6]. Then, under the approximations considered here, let us schematically write down the entropy of the black hole, in an out of equilibrium scenario, to be

$$S = \int_{\mathcal{H}_v} \sqrt{h}(1 + s_n) \tag{4.4}$$

where

$$s_n = s_w^{\text{HD}} + s_c.$$

[6]This assumption ensured certain boundary terms to vanish. Therefore, even if the horizon was non-compact, but those boundary terms continued to vanish, the proof of [1] is completely valid. Our local statement of the second law should not be sensitive to the compact nature of the horizon, or depend on the vanishing of such boundary terms.

Here s_w^{HD} are the corrections to the area law, coming from the Wald entropy formula due to the presence of the higher derivative corrections to the Einstein-Hilbert action. Note that, in this notation the Wald entropy is given by [7]

$$S_W = \int_{\mathcal{H}_v} \sqrt{h}(1 + s_w^{\mathrm{HD}}) = \int_{\mathcal{H}_v} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}, \tag{4.5}$$

where $\epsilon_{\mu\nu}$ are the bi-normal to $\mathcal{H}_v$, the co-dimension-2 spatial slicing of the horizon. Here, s_c are further corrections to the Wald entropy, which are a part of the JKM ambiguity. One of the central idea of the proof, is to choose an appropriate s_c so as to ensure $\partial_v S \geq 0$. This in turn, therefore, fixes the ambiguity.

Now let us act (4.4) with a v-derivative, which can be moved inside the integral in the RHS, since the integral is over a v-slice of the horizon. We have

$$\partial_v S = \int_{\mathcal{H}_v} \partial_v \left(\sqrt{h}(1 + s_n)\right) \equiv \int_{\mathcal{H}_v} \sqrt{h}\, \vartheta, \tag{4.6}$$

where h_{ij} is the induced metric on $\mathcal{H}_v$, and

$$\vartheta = \vartheta_E + \frac{1}{\sqrt{h}}\partial_v \left(\sqrt{h} s_n\right),$$

with ϑ_E being the contribution coming from the area form, which is present even in pure Einstein gravity without any higher derivative corrections. We can show that $\vartheta_E = \frac{1}{2}h^{ij}\partial_v h_{ij}$ is the expansion of the congruence of the null generators of the horizon [44].

Following the proof of the black hole area increase theorem[39, 42] for Einstein gravity, the general strategy for proving $\partial_v S \geq 0$, is to demonstrate that $\partial_v \vartheta \leq 0$. This, together with the additional physical expectation [8] $\vartheta|_{v\to+\infty} \to 0$, implies that $\vartheta \geq 0$, for all $v \geq 0$.

Now at linear order in amplitude both inequalities (i.e., $\partial_v S \geq 0$ and $\partial_v \vartheta \leq 0$), must be some equality relation since terms linear in amplitude (ϵ) could have any sign depending on

[7] Here, we have treated the area term corresponding to Einstein theory separately, to facilitate comprehension for our readers who are familiar with the area increase theorem.

[8] Here, the physical expectation is that the dynamical black hole will settle to a stationary metric with a Killing horizon at $v \to \infty$, leading to the vanishing of $\partial_v S$ at $v \to \infty$.

the sign of ϵ. In other words, the only way the inequalities could be satisfied is to set them to zero at linear order in ϵ.

$$\partial_v \vartheta = \mathcal{O}\left(\epsilon^2\right) \tag{4.7}$$

We shall try to choose s_n such that $\partial_v \vartheta = \mathcal{O}\left(\epsilon^2\right)$ is ensured. More precisely, by equation (4.7), what we mean is the following.

- We shall consider only those dynamics where every metric component G_{AB}, that are not already fixed by our gauge choice, could be decomposed as

  $$G_{AB} = G^{(0)}_{AB} + \epsilon\, \delta G_{AB}$$

  where $G^{(0)}_{AB}$ is the non dynamical part of the metric and has a time-like Killing vector with a Killing horizon. δG_{AB} is time-dependent. ϵ is the small parameter encoding the amplitude of dynamics, which could be of either sign but always small. All terms quadratic or higher order in ϵ would be neglected.

- We further demand that $G^{(0)}_{AB}$ is an exact solution of Einstein equation with appropriate higher derivative corrections and also relevant matter stress tensor. Additionally, G_{AB} also solves the same equations but upto corrections of order $\mathcal{O}\left(\epsilon^2\right)$.

- Now our goal is to construct an s_n out of $G^{(0)}_{AB}$ and δG_{AB} such that if we just blindly evaluate $\partial_v \vartheta$ and impose equations of motion it turns out to be order $\mathcal{O}\left(\epsilon^2\right)$ (or just vanishes within our approximation).

One of the key points of [1] is to provide an algorithm to construct such an s_n in all possible higher derivative theories of gravity.

At this stage we would like to emphasize that equation (4.7) is a necessary condition for second law, but certainly not sufficient, even within this perturbative treatment. Sufficiency would demand a particular sign for the ceofficient of the $\mathcal{O}\left(\epsilon^2\right)$ term and in those special space-time points where this coefficient also vanishes, one has to keep track of even the higher order terms. However, as it is the case in [1], in this note we shall confine ourselves to computations only up to order $\mathcal{O}(\epsilon)$. They themselves turn out to be constraining enough to fix a large part of ambiguities that are there in the form of gravitational entropy for higher derivative theories.

Now let us process equation (4.7) little further, which will finally tell us how, manipulating a particular component of equations of motion, we could construct some s_n that satisfies equation (4.7).

$$
\begin{aligned}
\partial_v\vartheta &= \partial_v\vartheta_E + \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}s_n\right)\right) \\
&= -R_{vv} + \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}s_n\right)\right) + \mathcal{O}\left(\epsilon^2\right) \\
&= -T_{vv} + E^{\text{HD}}_{vv} + \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}s_n\right)\right) + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
\tag{4.8}
$$

Here T_{vv} denotes the vv component of the matter stress tensor and E^{HD}_{vv} is the vv component of the higher derivative corrections to the gravity part of the equations of motion. In the second line we have used the fact

$$
\partial_v\vartheta_E = -R_{vv} + \mathcal{O}\left(\epsilon^2\right)
$$

This is essentially the Raychaudhury equation for the congruence of null geodesics and this is an off-shell equation - it does not require the metric to satisfy any particular equation of motion. We have used the equation of motion while going from second to the last line of equation (4.8)

$$
R_{vv} + E^{\text{HD}}_{vv} = T_{vv} \tag{4.9}
$$

In all of our analysis this the only place where we shall use the on-shell condition on the metric components.

Now let us analyse the ϵ dependence of T_{vv}. We would like to argue that T_{vv} is also of order $\mathcal{O}(\epsilon^2)$ and therefore does not contribute within our approximation.

In case of higher derivative theory the definition of matter stress tensor might become a bit confusing. Our convention is the following. If we vary the action with respect to the metric fluctuation, the resultant two-indexed tensors could be categorized in two different classes; terms that depend only on the metric components and terms that along with the metric components, also depend on the matter fields. All the higher derivative terms that are of the first category, are together called as E^{HD}_{AB} and the matter stress tensor T_{AB} consists of all the terms in the second category.

Clearly if we want to know the ϵ dependence of T_{vv}, we need to fix the ϵ dependence of the matter fields. Let Φ denotes all the matter fields (collectively) and let's assume that it also admits the following expansion.

$$\Phi = \Phi^{(0)} + \epsilon\, \delta\Phi$$

Here $\delta\Phi$ encodes the dynamics and $\Phi^{(0)}$ is the value of Φ on the stationary situation i.e., when all field configurations, including both metric and the matter fields, admit a Killing vector. As in the case of metric, we want $\Phi^{(0)}$ to satisfy the equations of motion (on the background of stationary metric $G^{(0)}_{AB}$) for the matter field exactly and $\delta\Phi$ upto linear order in ϵ.

We shall consider only those matter stress tensors that satisfy the null energy condition. In our context it implies that as long as the matter fields satisfy their equations of motion (in any smooth background geometry that need not be dynamical), the vv component of the stress tensor is always non-negative.

$$T_{vv} \geq 0$$

We would like to stress again that the validity of the above condition requires only the matter fields to be on-shell, but the metric need not be. Now one can argue that in a stationary situation (i.e., in the limit of $\epsilon \to 0$) T_{vv} simply vanishes (see appendix §B.2 for the details). Any quantity that satisfies some positivity condition and also vanishes in stationary situation, must be quadratic in the amplitude of dynamics, since the linear term could have either sign. It follows (exactly for the same reason as in equation-(4.7)) that T_{vv} is also of order $\mathcal{O}\left(\epsilon^2\right)$ at every order in derivatives.

Now equation (4.8) could be simply satisfied for some choice of s_n, provided E_{vv}^{HD} has the following off-shell form[9]

$$E_{vv}^{\text{HD}}\|_{\text{offshell}} = \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\varsigma\right)\right) + \mathcal{O}(\epsilon^2). \tag{4.10}$$

If equation (4.10) is true, then one could just choose $\left[\int_{\mathcal{H}_v}\sqrt{h}\,s_n\right]$ to be minus of $\left[-\int_{\mathcal{H}_v}\sqrt{h}\,\varsigma\right]$ upto correction of order $\mathcal{O}(\epsilon^2)$.

Let us pause here for a moment, to make an important observation about a very special situation. Imagine a situation in which, a fluctuation in the matter field with a very small amplitude sources the metric, through its energy momentum tensor. The first correction to the zeroth order stationary metric is entirely due to back-reaction from this source. In such a situation, the first change in the matter fields is of order ϵ, and the energy momentum tensor is of order ϵ^2; but, the first correction to the metric would be of order ϵ^2. The $\mathcal{O}(\epsilon)$ piece for the metric would be trivially zero, in this special case, when the boundary conditions are chosen suitably. So everything we have said so far, for the $\mathcal{O}(\epsilon)$ coefficient in the metric remains true, but trivially true.

Now, in this special case, since the first correction to the metric occurs at $\mathcal{O}(\epsilon^2)$, all our conclusions in this note, regarding the linearized corrections to metric, would then be applicable to the $\mathcal{O}(\epsilon^2)$ terms. The only crucial difference would be that, instead of the equality (4.7), we would now get an inequality for the coefficient of ϵ^2 in $\partial_v\vartheta$, i.e. $\partial_v\vartheta|_{\epsilon^2} \leq 0$. This

[9] Note it is very important that the form predicted in (4.10) is an off-shell requirement on E_{vv}^{HD}. Since we have already argued that T_{vv} is of order $\mathcal{O}\left(\epsilon^2\right)$, equations of motion for the metric (4.9) ensures that on-shell E_{vv}^{HD} must be of order $\mathcal{O}\left(\epsilon^2\right)$. In other words, just like the solutions of any other differential equations, on-shell we do not have the freedom of determining the ϵ dependence of terms involving large number of derivatives, once the lower derivatives are fixed. However, our final goal is to construct an expression for s_n and we can actually achieve this goal by treating E_{vv}^{HD} off-shell, where the naive ϵ counting works.

happens because of (4.8), where, due to the null-energy condition, T_{vv} now contributes positively, at the same order at which the metric receives its first corrections. The cancellation between E_{vv}^{HD} and $\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}s_n\right)\right)$ is realized in exactly the same way as discussed earlier. This, in turn, implies that $\partial_v S \geq 0$, the inequality being important even for the linear (first non-trivial) corrections to the metric. Note that, this situation is perhaps physically important, since this is one of the simplest situations where we can realize a dynamical event horizon, by throwing a tiny amount of matter towards the black hole.

In [1], author has explicitly shown that equation (4.10) is true. As we have mentioned in the introduction, he has used the transformation property of E_{vv}^{HD} under certain boost symmetry for his proof. His key argument works barring few 'leading terms' in E_{vv}^{HD}, for which the author has used the 'physical process formulation' of the first law as an extra input. As we shall see below, for particularly these terms the integration over the constant v slices of the horizon turns out to be very crucial.

4.1.2 An entropy for non-stationary horizons obeying the second law

In this subsection, we shall review the arguments in [1], which establishes that most of the terms in E_{vv}^{HD} could be recast in the form (4.10).

A residual coordinate redefinition freedom

Let us recall that in (4.3), the coordinate v constitutes an affine parameter along the null generators of the horizon, while the coordinate x^i labels the individual generators. But this definition does not completely fix the coordinates on the horizon $\mathcal{H}$. We still have the freedom of the following two classes of coordinate redefinition.

1. We can perform an affine re-parametrization of the generators of horizon $\mathcal{H}$, through the transformation

$$v \to \tilde{v} = v\, p_1(x^j) + p_2(x^j). \tag{4.11}$$

Note that, an arbitrary re-parametrization of this form (4.11), may not be compatible with the gauge choice as used in (4.3). Therefore, along with the transformation (4.11), it may be required to transform the r coordinate as well, so that the gauge choice of (4.3) is retained, even after the coordinate transformation (4.11). This point is explicated further below, in a special case of (4.11).

2. We can also relabel the generators as follows

$$x^i \to \tilde{x}^i = f^i(x^j). \tag{4.12}$$

The transformation (4.12) does not change the constant v slices of the horizon; consequently $\mathcal{H}_v$ is invariant under it. Also, for the choice of the metric (4.3), covariance under (4.12) may be implemented by ensuring that all the spatial i-indices are covariant; especially, the covariant derivatives along x^i, should be compatible with the metric h_{ij}. For this reason, ensuring invariance (covariance) of entropy (or the second law) under (4.12) is relatively easy. However, covariance under (4.11) is extremely non-trivial and leads to constraints, that were exploited in [1] to fix the form of E_{vv}^{HD} and hence the correction to the entropy.

In [1] only a special case of (4.11) was considered under which $p_1 = a,\ p_2 = 0$, so that v is rescaled as $v \to \tilde{v} = av$, a being a constant. Now, in order to ensure that our coordinate redefinition is compatible with the gauge choice of (4.3), we must rescale the r coordinate suitably. For instance, in order to ensure that $g_{rv} = 1$ everywhere, even after rescaling v, we must simultaneously rescale [10]

$$v \to \tilde{v} = av, \quad r \to \tilde{r} = \frac{1}{a} r. \tag{4.13}$$

In the new coordinates (4.13) the metric takes the following form

$$\begin{aligned} ds^2 =& 2\, d\tilde{v}\, d\tilde{r} - \tilde{r}^2\, X(\lambda\tilde{r}, \frac{\tilde{v}}{\lambda}, x^i) d\tilde{v}^2 \\ &+ 2\tilde{r}\, \omega_i(\lambda\tilde{r}, \frac{\tilde{v}}{\lambda}, x^i) d\tilde{v}\, dx^i + h_{ij}(\lambda\tilde{r}, \frac{\tilde{v}}{\lambda}, x^i) dx^i\, dx^j \end{aligned} \tag{4.14}$$

[10] In [1], this rescaling has been referred to as 'boosts', while the quantities invariant under this rescaling has been referred to as 'boost invariant'.

Note that, for the parametrization (4.3), the metric looks almost invariant under this coordinate transformation (4.13), however, the arguments of the metric functions are appropriately scaled. In particular, on the horizon $\mathcal{H}$, the induced metric in the new coordinates takes the following form

$$ds^2_{\mathcal{H}} = 2\, d\tilde{v}\, d\tilde{r} + h_{ij} dx^i\, dx^j\ , \tag{4.15}$$

which has an identical structure as compared to that in the old coordinates.

Structural form of E^{HD}_{vv}

At first, let us enlist the various derivatives and functions that may occur in E^{HD}_{vv}, for any general diffeomorphism invariant theory of gravity (4.1). These building blocks for constructing E^{HD}_{vv} include

1. The metric functions X, ω_i and h_{ij}.
2. The covariant derivative ∇_i with respect to x^i, compatible with the metric h_{ij}, which can act on the above metric functions.
3. The partial derivatives ∂_r and ∂_v, which may also act on the metric functions.

Let us immediately note that among these building blocks, it is only ∂_v and ∂_r that transform non-trivially under the coordinate rescaling (4.13). These transform as

$$\partial_r \to \partial_{\tilde{r}} = \lambda \partial_r,\ \ \partial_v \to \partial_{\tilde{v}} = \frac{1}{\lambda}\partial_v \tag{4.16}$$

As is apparent from (4.14), the rest of the building blocks of E^{HD}_{vv}, which include the metric functions and the covariant derivative ∇_i, remain invariant under (4.13).

Let us now note that under the coordinate rescaling (4.13), E^{HD}_{vv}, being the vv-component of a covariant tensor, must transform as

$$E^{\text{HD}}_{vv} \to E^{\text{HD}}_{\tilde{v}\tilde{v}} = \frac{1}{\lambda^2} E^{\text{HD}}_{vv}\,. \tag{4.17}$$

Let us define the weight of a quantity to be the power of λ by which the quantity rescales under the transformation (4.13). In this sense, the weight of $E_{vv}^{\rm HD}$ is -2.

Now, from the transformation property of the building blocks, it is clear that only ∂_v has a negative weight under (4.13). Hence, it follows that, every term in $E_{vv}^{\rm HD}$ must have at least two ∂_v. At linear order in ϵ, the most general schematic structure of any term in $E_{vv}^{\rm HD}$ would be

$$E^{(m,n,k)} = \partial_r^k \left[(\partial_v\partial_r)^m P\right] \; \partial_v^{k+2} \; \left[(\partial_v\partial_r)^n Q\right] + \mathcal{O}(\epsilon^2), \tag{4.18}$$

where m, n and k are positive integers including zero. Here we have kept all the ∂_r and ∂_v derivatives explicit [11]. P and Q are appropriate structures built out of rest of the building blocks, which consist of the metric functions and ∇_i acting on them. They do not contain any further ∂_v or ∂_r derivatives. Thus the most general structure of $E_{vv}^{\rm HD}$ would be

$$E_{vv}^{\rm HD} = \sum_{m,n,k} E^{(m,n,k)} \tag{4.19}$$

The upper limits of these sum would be fixed by the number of derivative on the metric in the gravity Lagrangian (4.1).

Now we shall manipulate $E^{(m,n,k)}$ to demonstrate that $E_{vv}^{\rm HD}$ as given by (4.19) can be cast into the form (4.10). At first, we shall consider $E^{(m,n,k)}$ for $k \neq 0$, and derive a recursion relation for this quantity. This recursion relation would be used to derive a general structure for $E^{(m,n,k)}$ and hence for $E_{vv}^{\rm HD}$. Certain terms corresponding to the case $k = 0$ would require special treatment, and after invoking the 'physical process' version of the first law, we shall demonstrate that entire sum (4.19), and hence the most general form of $E_{vv}^{\rm HD}$, can be cast into the form (4.10). This would complete our objective as laid out in §4.1.1, thus proving the second law of thermodynamics in the linearized case. This would also provide us with

[11] It may seem at first glance that the term in (4.18) is already second order in ϵ for any non-zero m, since in that case, it becomes a product of two terms, on which some v-derivative has acted. However, it should be noted that when both the derivatives $(\partial_v\partial_r)$ act together on P, it can be non-zero on $\mathcal{H}$ even in equilibrium, i.e. it can be an $\mathcal{O}(\epsilon^0)$ term (see appendix B.1).

an explicit construction to compute the corrections to the Wald entropy, for an arbitrary theory (4.1).

At first, let us note that, any term of the form $\partial_v^m \partial_r^n P$, where $n \geq m$ and P does not contain any further ∂_r or ∂_v derivative, would generically be non-zero once evaluated on the Killing horizon of the stationary solution. On the other hand, such terms would vanish on the Killing horizon if $m > n$ (see appendix B.1 for a demonstration of this fact). Therefore, it follows that, in a dynamical situation, when the amplitude of the time dependent perturbation ϵ is small, $(\partial_v^m \partial_r^n P)$ must be of order $\mathcal{O}(\epsilon)$, whenever $m > n$. Consequently, all terms of the product form $(\partial_v^m \partial_r^n P)(\partial_v^{m'} \partial_r^{n'} Q)$ with $m > n$ and $m' > n'$, are of order $\mathcal{O}(\epsilon^2)$, and therefore, can be neglected in our linearized analysis.

Now let us turn our attention back to the expression (4.18), for $k \neq 0$. In (4.18), we can take the v-derivatives acting on the term involving Q and transfer them onto the term involving P at the expense of a minus sign and a total derivative term. By repeating this operation, even on the terms that are generated due to previous such operations, it is possible to reduce (4.18) to the following recurrence relation

$$\begin{aligned} E^{(m,n,k)} = \partial_v^2 \Bigg(\sum_{p=0}^{k-1} (-1)^p \, \partial_r^{k-p} \left[(\partial_v \partial_r)^{m+p} P \right] \, \partial_v^{k-p} \, \left[(\partial_v \partial_r)^n Q \right] \Bigg) \\ + (-1)^k \partial_v \Bigg(\left[(\partial_v \partial_r)^{m+k} P \right] \, \partial_v \, \left[(\partial_v \partial_r)^n Q \right] \Bigg) - E^{(m+1,n,k-1)} \end{aligned} \tag{4.20}$$

Now, we can use this recursion relation (4.20) itself, to evaluate $E^{(m+1,n,k-1)}$

$$\begin{aligned} E^{(m+1,n,k-1)} = \sum_{p=0}^{k-2} (-1)^p \, \partial_v^2 \Bigg(\partial_r^{k-p-1} \left[(\partial_v \partial_r)^{m+p+1} P \right] \, \partial_v^{k-p-1} \, \left[(\partial_v \partial_r)^n Q \right] \Bigg) \\ + (-1)^{k-1} \partial_v \Bigg(\left[(\partial_v \partial_r)^{m+k} P \right] \, \partial_v \, \left[(\partial_v \partial_r)^n Q \right] \Bigg) - E^{m+2,n,k-2} \end{aligned} \tag{4.21}$$

Using the recursion relation repeatedly, we can recast $E^{(m,n,k)}$ into the form

$$E^{(m,n,k)} = \sum_{q=0}^{k-1}\sum_{p=0}^{k-q-1}(-1)^{q+p}\,\partial_v^2\Big(\partial_r^{k-q-p}\left[(\partial_v\partial_r)^{m+q+p}P\right]\,\partial_v^{k-q-p}\,\left[(\partial_v\partial_r)^n Q\right]\Big)$$
$$+\Big(\underbrace{(-1)^k-(-1)^{k-1}+(-1)^{k-2}-\dots}_{k\text{-terms}}\Big)\partial_v\Big(\left[(\partial_v\partial_r)^{m+k}P\right]\,\partial_v\,\left[(\partial_v\partial_r)^n Q\right]\Big) \tag{4.22}$$

Hence, after performing the sum in the second term, we have

$$E^{(m,n,k)} = \sum_{q=0}^{k-1}\sum_{p=0}^{k-q-1}(-1)^{q+p}\,\,\partial_v^2\Big(\partial_r^{k-q-p}\left[(\partial_v\partial_r)^{m+q+p}P\right]\,\partial_v^{k-q-p}\,\left[(\partial_v\partial_r)^n Q\right]\Big)$$
$$+\,k(-1)^k\partial_v\Big(\left[(\partial_v\partial_r)^{m+k}P\right]\,\partial_v\,\left[(\partial_v\partial_r)^n Q\right]\Big) \tag{4.23}$$

Note that, inside the sum, the first term has the structure $\partial_v^2\left(\partial_r^\ell X^{(\ell)}\right)\,\left(\partial_v^\ell\,Y^{(\ell)}\right)$, where the lowest value of $\ell = 1$. If we now, plug in (4.23) back into the sum (4.19), a similar structure of this first term would be maintained, again with ℓ starting from 1. Note that, here, we are only considering values of k starting from 1, in the sum (4.19). As we mentioned previously, the $k = 0$ terms needs to be treated separately.

But now, let us also note from (4.18), that the $k = 0$ term can always be recast into the same form as the second term in (4.23). This is true only in the linearized approximation.

Thus, from (4.19) and (4.23) it is clear that, at linear order in amplitude, $E_{vv}^{\rm HD}$ could always be written in the following form

$$\begin{aligned} E_{vv}^{\rm HD} &= \partial_v\left[A\,\partial_v\,B\right]+\partial_v^2\left[\sum_{k=1}\left(\partial_r^k A^{(k)}\right)\,\left(\partial_v^k\,B^{(k)}\right)\right]+\mathcal{O}(\epsilon^2)\\ &= \partial_v\left[A\,\partial_v\,B\right]+\partial_v\left[\left(\frac{1}{\sqrt{h}}\right)\partial_v\sum_{k=1}\sqrt{h}\left(\partial_r^k A^{(k)}\right)\,\left(\partial_v^k\,B^{(k)}\right)\right]+\mathcal{O}(\epsilon^2)\end{aligned} \tag{4.24}$$

where A, B, $A^{(k)}$ and $B^{(k)}$ are appropriate structures as implied by (4.19) and (4.23), which do not transform under rescaling (4.13). Let us re-emphasize that, although (4.23) has been derived under the assumption $k \neq 0$, the form of $E_{vv}^{\rm HD}$ in (4.24), also incorporates the $k = 0$ term, in the sum (4.19).

As argued above (also see appendix B.1), we know that the action of a v-derivative, which does not appear with a compensating r-derivative, on a quantity that is invariant under the rescaling (4.13), must be of $\mathcal{O}(\epsilon)$ on $\mathcal{H}$, in the amplitude expansion. This is because, such a quantity should vanish on the Killing horizon, and so must be at least $\mathcal{O}(\epsilon)$ for dynamical horizons. Consequently, whenever one or more ∂_v act on any one of these A, B, $A^{(k)}$ or $B^{(k)}$, it must be $\mathcal{O}(\epsilon)$. This also justifies appropriate incorporation of the factors of $\sqrt{h}$ in (4.24). We do not get any additional terms due to these factors of $\sqrt{h}$, since we are working in the linearized approximation in ϵ.

Thus to conclude, we have obtained a precise structural form of $E_{vv}^{\rm HD}$ in (4.24). At this stage, we observe that the second term of $E_{vv}^{\rm HD}$ in (4.24) is already in the desired form (4.10). So our objective would be accomplished, if we are able to argue that the first term in (4.24), can also be written in the form (4.10), i.e. as two v-derivatives acting on a quantity which is invariant under the rescaling (4.13).

The physical process version of First law and its implications

In [1] it was argued that the structure of the quantities A and B in (4.24), must be such, that $E_{vv}^{\rm HD}$ has the form (4.10). This conclusion followed from the *physical process* version of the first law of black hole thermodynamics [51, 17] (also see [54, 60]), which was assumed to be applicable for the theory of gravity under consideration. It was demonstrated in [1], that if A and B in (4.24) did not have the requisite structures, then the physical process version of the first law would be invalidated. Let us now review this argument, as presented in [1].

Let us consider a stationary black hole, which is perturbed by small fluctuations in the matter sector. The amplitude of such perturbations is assumed to be small. For instance, this could be some small amount of matter falling into the black hole. The matter stress tensor would back-react onto the metric and produces fluctuations in it, which would also be small. These fluctuations would result in a non-stationary fluctuating black hole. However, in a physical situations, it may be expected that, at late times, these fluctuations would die down and the black hole would again become stationary. Such a dynamical process, where the black hole is stationary, both at early and late times, is referred to as a 'physical process'.

The new stationary black hole at late times would have slightly different parameters compared to the one at early times (such as, mass or angular momentum). The overall shift in the mass (energy) of the black hole would be given by integrating specific component of the energy-momentum tensor over the horizon. This shift in mass, must be related to the shift in entropy of the black hole through the first law of black hole thermodynamics $T\Delta S = \Delta\mathcal{E}$. Therefore, we can express the change in entropy during this physical process ΔS, to the integrated energy-momentum tensor, in the following way

$$\Delta S = -\frac{2\pi}{\kappa}\int_{\mathcal{H}} \Delta T_{ab}\, \xi^a\, d\Sigma^b \tag{4.25}$$

Here, ΔT_{ab} is the part of the energy momentum tensor that has initiated the dynamics of the black hole horizon. ξ^a is the generator of the future horizon $\mathcal{H}$; it is a Killing generator at early and late times, when the black hole is stationary. Also, $d\Sigma^b$ is the area element along the horizon. The parameter κ is the surface gravity of the black hole and is proportional to the temperature of the black hole[12].

The equation (4.25) is referred to as the physical process version of the first law of black hole thermodynamics. For a more complete and detailed discussion of this, see section (2) of [17]. We should note that because the initial and final states are stationary, the ΔS in (4.25) is expected to be given by change in Wald entropy, which, by construction, satisfies the *usual* form of the first law for stationary black holes. However, whether Wald entropy

[12]It turns out that the combination $\Delta T_{ab}\xi^a d\Sigma^b$ itself is of the order of amplitude of the perturbation. Therefore, as long as we are working at linear order in the amplitude of perturbation, the difference in the value of κ for the initial and final stationary black holes is negligible. Thus, within this approximation, κ can be taken to be constant throughout the duration of the physical process.

does satisfy the physical process version of the first law (4.25) does not immediately follow from its construction, and we require additional arguments to establish this [].

This version of the first law (4.25) now enables us to make further deductions regarding the structural form of A and B appearing in (4.24). With our choice of coordinates (4.3), ξ^a is related to the affinely parametrized null generators of the horizon ∂_v, in the following way

$$\xi^a \partial_a = \kappa \, v \, \partial_v \tag{4.26}$$

While, in our coordinates (4.3), the area element on the horizon $\mathcal{H}$ is given by

$$d\Sigma^b \partial_b = -\sqrt{h} \, d^{d-2}x \, dv \, \partial_v \tag{4.27}$$

Using (4.26) and (4.27) back in (4.25), we have

$$\Delta S = 2\pi \int_{\mathcal{H}} \sqrt{h} \, d^{d-2}x \, dv \, v \, \Delta T_{vv} = 2\pi \int_{\mathcal{H}} \sqrt{h} \, d^{d-2}x \, dv \, v \, (R_{vv} + E^{\text{HD}}_{vv}) \tag{4.28}$$

Here, we have used the equation of motion to rewrite the stress tensor in terms of geometric quantities. Now, if entropy S has the form (4.4), ΔS in (4.28) can be split into two parts $\Delta S = \Delta S_E + \Delta S_{\text{HD}}$. ΔS_E being the change in the integrated area of $\mathcal{H}_v$, responsible for the change in entropy in two derivative Einstein theory, while ΔS_{HD} is the change in entropy due to higher derivative terms. Clearly, the terms proportional to R_{vv} on the RHS of (4.28), must be equal to ΔS_E on the LHS of (4.28). This is manifest in the limit when the higher derivative corrections to Einstein's gravity vanishes. Using (4.24), we can therefore write

$$\begin{aligned}
\Delta S_{\text{HD}} &= 2\pi \int_{\mathcal{H}} \sqrt{h} \, d^{d-2}x \, dv \, v \, (E^{\text{HD}}_{vv}) \\
&= 2\pi \int_{H} \sqrt{h} \, d^{d-2}x \, dv \, v \, \partial_v \left[A \, \partial_v \, B + \left(\frac{1}{\sqrt{h}} \right) \sum_{k=1} \partial_v \left(\sqrt{h} \, \partial_r^k A^{(k)} \, \partial_v^k B^{(k)} \right) \right] \\
&\quad + \mathcal{O}\left(\epsilon^2\right)
\end{aligned} \tag{4.29}$$

It is extremely important for the subsequent arguments to realize that ΔS_{HD} must be non-zero in general, and should be expressible in terms of some geometrical quantity integrated over $\mathcal{H}_v$. This is clear from the fact that for an arbitrary higher derivative corrections to Einstein's gravity s_n in (4.4) is non-zero even for stationary black holes. In the stationary case, it is expected to be given by Wald entropy (4.5), which in general is different than area of the horizon. Thus, if s_n is non-trivial, it is expected that, in general, in a dynamical scenario, ΔS_{HD} must be non-trivial. In fact, as pointed out earlier, ΔS_{HD} should be given by change in the corresponding Wald entropy, since the state both at early and late times are stationary states.

Now the contribution from the second term in (4.29) vanishes. This can be seen by manipulating the term as follows.

$$\begin{aligned}
&\int_{\mathcal{H}} \sqrt{h}\, d^{d-2}x\, dv\, v\, \partial_v \left[\frac{1}{\sqrt{h}}\, \partial_v(\sqrt{h}\, X)\right] \\
&= \int_{\mathcal{H}} d^{d-2}x\, dv\, \partial_v \left[v\, \partial_v(\sqrt{h}\, X) - (\sqrt{h}\, X)\right] + \mathcal{O}\left(\epsilon^2\right) \\
&= \int_{v=-\infty}^{v=\infty} dv\, \partial_v \left(\int_{\mathcal{H}_v} d^{d-2}x \left[v\, \partial_v(\sqrt{h}\, X) - (\sqrt{h}\, X)\right]\right) + \mathcal{O}\left(\epsilon^2\right) \\
&= \left[\int_{\mathcal{H}_v} d^{d-2}x\, v\, \partial_v(\sqrt{h}\, X) - \int_{\mathcal{H}_v} d^{d-2}x(\sqrt{h}\, X)\right]_{v=-\infty}^{v=\infty}
\end{aligned} \tag{4.30}$$

where

$$X = \sum_{k=1}^{N} \left[\left(\partial_r^k A^{(k)}\right)\ \left(\partial_v^k\, B^{(k)}\right)\right] + \mathcal{O}\left(\epsilon^2\right) \tag{4.31}$$

Note that both the terms in the last line of equation (4.30) contains more than one ∂_v derivatives on expressions that are invariant under the rescaling (4.13). Therefore, they must vanish in the two limits of far past and far future, where we have a stationary black hole with a Killing horizon. It follows that, these terms do not contribute to ΔS_{HD} in (4.29). Hence, (4.29) can be always reduced to an integral of the form

$$\Delta S_{\text{HD}} = 2\pi \int_{\mathcal{H}} \sqrt{h}\, d^{d-2}x\, dv\, v\, \partial_v\left[A\, \partial_v\, B\right] \tag{4.32}$$

Again by performing an integration by parts, we obtain

$$\Delta S_{\text{HD}} = 2\pi \left[\int_{\mathcal{H}_v} \sqrt{h}\, d^{d-2}x\, v\ \left(A\, \partial_v\, B\right)\right]_{v=-\infty}^{v=\infty} - 2\pi \int_{\mathcal{H}_v} \sqrt{h}\, d^{d-2}x\, dv\ \left[A\, \partial_v\, B\right] \tag{4.33}$$

However, to have a consistent first law, the second term also should be writeable as a total ∂_v derivative so that the v integration of this term from infinite past to infinite future finally would give the net change of some geometric quantity - entropy, defined on the constant v slices of the horizon[13]. Naively this is possible if A is of the form $\left[\text{some constant}/\sqrt{h}\right]$ and B has the form $\left[\sqrt{h}\ \tilde{B}\right]$ where $\tilde{B}$ is a scalar under $\{x^i\} \to \{y^i\} = y^i\left(\{\vec{x}\}\right)$.

If it is indeed the case that A is always a constant times $\left(1/\sqrt{h}\right)$ (let's choose the constant to be one without any loss of generality), then the schematic form of E^{HD}_{vv}

$$E^{\text{HD}}_{vv} = \partial_v \left(\frac{\partial_v \left(\sqrt{h}\ \tilde{B}\right)}{\sqrt{h}} \right) + \partial_v \left[\left(\frac{1}{\sqrt{h}}\right) \partial_v \left(\sqrt{h} \sum_{k=1}^{N} \partial_r^k A^{(k)}\ \partial_v^k B^{(k)} \right) \right] + \mathcal{O}(\epsilon^2) \quad (4.34)$$

If E^{HD}_{vv} does have the form of equation (4.34), not only the 'physical process version' of the first law but also the second law as argued in [1] will be true with the following identification for correction to the total entropy (see equation (4.10)).

$$\delta S_{\text{HD}} = \int_{\mathcal{H}_v} \left[\tilde{B} + \sum_{k=1}^{N} \partial_r^k A^{(k)}\ \partial_v^k B^{(k)} \right] \quad (4.35)$$

Note that $\tilde{B}$ is the only term that is non-zero even in equilibrium. This term must match with Wald entropy[14]. Rest of the terms (for $k \geq 1$) vanish on stationary metric and therefore are part of JKM ambiguities.

So in summary, [1] has given a constructive proof for the second law for dynamical black hole solutions in higher derivative theories of gravity provided the physical process formulation of the first law is true for these solutions. The validity of the ' physical process formulation of the first law' requires a very specific structure for a certain term in the equation of motion (the first term in (4.24) must take the form of the first term in (4.34)),

[13] At this stage, by 'geometric quantity' we simply mean some expression in terms of the metric components and their derivatives that is invariant under any diffeomorphism, mixing only the spatial coordinates of the constant v slices of the horizon.

[14] Though we have explained the argument here, specializing to higher derivatives theories, it is trivially true for two derivative theories of gravity where $\tilde{B}$ is simply 0, and entropy is simply given by $S = \int_{\mathcal{H}_v} \sqrt{h}$.

which does not follow from the boost-symmetry (4.13) alone (the only symmetry that is considered in [1]) [15]. For the convenience of reporting, we shall refer to the first term in (4.34) (or the first term in (4.24)) as the 'zero boost term'. This nomenclature is inspired by the fact that $\tilde{B}$ in (4.34) (or A and B in (4.24)) does not have any v or r derivative. Apart from this, there is absolutely no other physical motivation behind this nomenclature. The reader must not confuse the phrase 'zero boost term' to be a synonym for boost invariant term. By boost invariant terms we shall continue to mean such terms which are invariant under the rescaling symmetry (4.13).

4.2 An entropy current for four derivative theories of gravity

In the previous section §4.1, we have reviewed the proof of the second law for linearised fluctuations, following [1]. As emphasized earlier, this proof is designed to prove a second law for the 'total entropy' of the system. The proof crucially involves an integration over the full spatial slice of the horizon, which defines the 'total entropy'. Therefore, it is insensitive to any total (spatial) derivative term, that may be present in the integrand, which is derived from the equation of motion. This drawback exits even in the proof for the physical version of the first law.

In this section, we shall carefully re-examine this particular subtlety. In explicit examples of four-derivative theories of gravity, we shall demonstrate that such total derivative terms do exist if we follow the algorithm of [1], and their inclusion would naturally lead to a construction of an entropy current. With the help of this entropy current, we can imme-

[15] This special structure of the first term in (4.34) has only been verified in specific theories of gravity where the physical process version of the first law has been proven (for instance, see [17]). To our knowledge, a complete proof demonstrating this special structure of the zero boost term in a general higher derivative theory of gravity does not exist. It would be interesting to explore, if it is possible to arrive at such a proof using the residual gauge transformations (4.11), which is more general than the boost symmetry (4.13) (see §4.3 for further discussion).

diately prove an ultra-local version of second law, associated with any dynamical horizon $\mathcal{H}$.

Let us now elaborate this point further. In (4.34), we have shown that a special structure for E^{HD}_{vv} is necessary for the validity of both the second law, as well as the physical process version of the first law. As we have explained in §4.1, the structure of the second term in (4.34) is fixed by the 'boost symmetry' (4.13), upto higher order corrections in the amplitude of fluctuations ϵ. But the same is not true for the first term in (4.34), which we have named as 'zero-boost terms' of E^{HD}_{vv}.

In §4.1, we have argued that the physical version of the first law, and consequently, the second law, would be true if the 'zero boost term' in E^{HD}_{vv} has the following schematic structure (see (4.34))

$$E^{\text{HD}}_{vv}\Big|_{\text{zero boost}} \sim \partial_v \left(\frac{1}{\sqrt{h}}\partial_v \left(\sqrt{h}\ \widetilde{B}\right)\right) , \tag{4.36}$$

where $\widetilde{B}$ is some scalar, which is invariant under spatial diffeomorphism.

However, the above form of the zero-boost term, though sufficient for the validity of the physical process version of the first law note1 and the second law, it is neither necessary, nor does it follow in any way, from the boost-symmetry (4.13). In this section, to begin with, our goal is just to verify (4.36). We shall explicitly compute the equation of motion, and in particular the zero-boost terms, in all possible four-derivative theories of gravity. From this explicit computation in four derivative theories of gravity we will show that (4.36) is not true in general. There exist cases where the zero boost terms in E^{HD}_{vv} could not be recast in the above form. In fact, the zero boost terms in E^{HD}_{vv} consists of additional terms, which can never be cast into the form (4.36).

Being motivated by this observation, we investigate the structural nature of the zero boost terms of E^{HD}_{vv}, to understand why such additional terms do not affect validity of the first and the second law. The possibility of non-zero spatial components of the entropy current arises here very naturally. Finally, through a general algorithm, we shall establish that, the zero boost terms of E_{vv}^{HD} for every four derivative theories of gravity, could be rendered into a form, which guarantees an ultra local version of the second law, in terms of an entropy current with non-zero spatial components.

4.2.1 Explicit calculation of E_{vv}^{HD} and the entropy current for theories with four derivative corrections to Einstein gravity

In this subsection, we shall compute the 'vv'-component of the equation of motion, E_{vv}, for all possible four derivative theories of gravity. We shall immediately find that it is possible to rearrange the terms so that upto corrections of order $\mathcal{O}(\epsilon^2)$ it takes the form

$$E_{vv} = -\,\partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,J^v\right) + \nabla_i J^i\right] + \mathcal{O}\left(\epsilon^2\right)\ . \tag{4.37}$$

Once we could rewrite E_{vv} in this form [16] , it is very natural to identify J^v with the entropy density and J^i as the spatial entropy current, capturing the in-flow and out-flow of entropy. Vanishing of E_{vv} at order $\mathcal{O}\left(\epsilon\right)$ would then correspond to a locally conserved entropy current and therefore an ultra local version of the second law (see §4.1 for details of this argument)[17]. More explicitly, once E_{vv} has the form (4.37), the standard arguments

[16]Note that for Einstein gravity E_{vv} takes the simple form

$$E_{vv}^{\text{Einstein}} = -\,\partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\right)\right].$$

Hence, when we consider higher derivative corrections to Einstein's equations, the terms in this equation arising out of these corrections also has a similar form

$$E_{vv}^{\text{HD}} = -\,\partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\tilde{J}^v\right) + \nabla_i J^i\right] + \mathcal{O}\left(\epsilon^2\right),$$

where $J^v - \tilde{J}^v = 1$. For most of our analysis, especially in the abstract manipulations, we have used E_{vv}^{HD}, instead of E_{vv}.

[17]The calculations here clearly suggest about the existence of an entropy current for some of these higher derivative theories. However, it requires a bit of clever manipulation. See the following subsections for a more algorithmic method which clearly exhibits that we need the spatial entropy current, which in turn provides us with an ultra local version of the second law.

outlined in §4.1 would imply that [18]

$$\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,J^v\right)+\nabla_i J^i=\mathcal{O}(\epsilon^2). \tag{4.39}$$

There are only three possible covariant terms which can appear in the gravity Lagrangian with 4-derivatives on the metric. These are given by: $R^2,\ R_{\mu\nu}R^{\mu\nu},\ R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$. In the following subsections, we shall separately consider three different four derivative theories of gravity

1. Ricci scalar squared theory: $\mathcal{I}^{(1)}=\int d^dx\sqrt{-g}\,(R+a_1\,R^2)$,
2. Ricci tensor squared theory: $\mathcal{I}^{(2)}=\int d^dx\sqrt{-g}(R+a_2\,R_{\mu\nu}R^{\mu\nu})$,
3. Riemann tensor squared theory: $\mathcal{I}^{(3)}=\int d^dx\sqrt{-g}(R+a_3\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$,

and explicitly compute the 'vv'-component of the respective equations of motion, E_{vv}, for each of them. After some algebraic manipulations on E_{vv}, in each of these cases, we shall write down the entropy current, It is then trivial to combine these results, to give us the entropy current for any arbitrary four derivative theory of gravity. The final result is tabulated in Table-(4.1).

For each of the three four-derivative theories mentioned above, if we just evaluate the equation of motion on our gauge fixed metric (4.3), it turns out to be an extremely complicated expression, even after we restrict it to the horizon. In general, just by inspection, it is quite difficult to rearrange the terms to arrive at the form (4.37). However, we know that in stationary situation, at least J^v should reduce to the well-known form of Wald entropy and

[18] If we consider special processes where the metric is entirely sourced by a small matter energy momentum tensor, so that both the first correction to the metric as well as the matter energy momentum tensor are of $\mathcal{O}(\epsilon^2)$ (the $\mathcal{O}(\epsilon)$ correction to the metric being zero), then for the ϵ^2 coefficient, (4.39) would be modified to the inequality

$$\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,J^v\right)+\nabla_i J^i\right)\Big|_{\epsilon^2}\geq 0. \tag{4.38}$$

Note that, while deducing this inequality, we have assumed that there exist other matter fields satisfying the null energy condition. See §4.1.1 for a more detailed discussion of this point.

the rest of the terms must be such that they vanish in a stationary situation. We shall use this fact to guide our intuition about the form of the entropy density and then finally deduce the form of the entropy current. More precisely, we shall obtain the following constituents for J^v

$$J^v = \sqrt{h}\,\left(s_w + s_c\right) \tag{4.40}$$

where s_w is the Wald entropy density for the stationary black holes defined as

$$s_w = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\,\epsilon_{\mu\nu}\epsilon_{\rho\sigma}, \tag{4.41}$$

where $\epsilon_{\mu\nu}$ are the bi-normal to $\mathcal{H}_v$, the co-dimension-2 spatial slicing of the horizon. Also, s_c is the non-stationary corrections to s_w. As we have argued before, s_c will vanish once we take stationary limit. Let us also define the contribution to s_w from the higher derivative part of the action as s_w^{HD}

$$s_w^{\text{HD}} = \frac{\partial \mathcal{L}^{\text{HD}}}{\partial R_{\mu\nu\rho\sigma}}\,\epsilon_{\mu\nu}\epsilon_{\rho\sigma}. \tag{4.42}$$

At this point, let us clarify one subtlety regarding the split mentioned in (4.40). It turns out that if we evaluate s_w on any dynamical metric, along with the terms that contribute in stationary situation, it will also have terms that vanish in the stationary limit. For convenience, let us name such terms as 'off-equilibrium' structures. Such off-equilibrium terms in the entropy suffer from the well-known class of JKM ambiguities, which arises as soon as we try to extrapolate Wald's formalism to non-stationary solutions. In our identification of the entropy density, we have used the fact that, in the stationary limit, it should reduce to Wald entropy. As we will see in the later sections, this requirements also fixes one class of ambiguities, in defining the entropy current.

For convenience, let us separate out the contribution of Wald entropy density s_w^{HD} to E_{vv}^{HD} and define the quantity $\mathbb{E}_{vv}^{\text{HD}*}$ as follows

$$\mathbb{E}_{vv}^{\text{HD}*} \equiv E_{vv}^{\text{HD}} + \partial_v\left[\frac{1}{\sqrt{h}}\,\partial_v\left(\sqrt{h}\,s_w^{\text{HD}}\right)\right] \tag{4.43}$$

Then, from the definition (4.40) it follows that

$$\mathbb{E}_{vv}^{\text{HD}*} = -\,\partial_v \left[\frac{1}{\sqrt{h}} \left(\partial_v \sqrt{h}\, s_c\right) + \nabla_i J^i\right] + \mathcal{O}\left(\epsilon^2\right). \tag{4.44}$$

It turns out that, in the examples that we consider, algebraically it is comparatively easier to recast $\mathbb{E}_{vv}^{\text{HD}*}$ in the form (4.44), instead of dealing with the full E_{vv}^{HD}.

We would like to emphasize here that, the procedure adopted in this subsection, is a set of intuitive manipulations and educated guess-work. It gives us an explicit demonstration that for the theories that we consider here, it is possible to lift both the first and the second law to an ultra-local form, by entertaining the possibility of non-zero spatial components of the entropy current, which captures the effect of the inflow and outflow of the entropy from any arbitrary local subregion. However, at this stage, we would not be able to say, whether the spatial components of the current is an absolute necessity, or there exist other possible rearrangement of terms, such that we can avoid the spatial components of the current altogether. In the later subsections, we shall repeat the same analysis more systematically, and for the four derivative theories, we shall be able to quantify these ambiguities involved in defining the entropy current more precisely. We shall conclude that, although there are some ambiguities in defining the entropy current, its non-zero spatial components are an unavoidable feature of the ultra local form of the second law.

Ricci scalar square theory

The action for Ricci scalar square theory is

$$\mathcal{I}^{(1)} = \int d^d x \sqrt{-g}\,\left(R + a_1\, R^2\right) \tag{4.45}$$

where a_1 is an arbitrary constant. The equations of motion which follows from the action (4.45), is given by

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + E_{\mu\nu}^{\text{HD}} = 0, \tag{4.46}$$

where

$$E^{\rm HD}_{\mu\nu} = a_1 \left(2RR_{\mu\nu} - 2D_\mu D_\nu R + 2g_{\mu\nu} D^\rho D_\rho R - \frac{1}{2} g_{\mu\nu} R^2\right) \tag{4.47}$$

are the higher derivative corrections to the Einstein equation. The explicit form of the vv-component of the equations of motion, on the horizon, is

$$E_{vv} = R_{vv} + E^{\rm HD}_{vv}\,, \tag{4.48}$$

where

$$E^{\rm HD}_{vv} = a_1 \left(2RR_{vv} - 2D_v D_v R\right). \tag{4.49}$$

The Wald entropy for this theory happens to be

$$S_w = \int_{\mathcal{H}_v} d^{d-2}x\, \sqrt{h}\, (1 + 2\, a_1\, R). \tag{4.50}$$

Once we have the Wald entropy, we could compute $\mathbb{E}^{\rm HD*}_{vv}$. In this case it simply vanishes implying that we do not need to add any current, neither do we get any correction to entropy density, beyond what is given by the Wald entropy.

Ricci tensor square theory

In this theory, the Ricci tensor square is added to the Einstein-Hilbert action, as a higher derivative correction. We have

$$I = \int d^dx \sqrt{-g}\, (R + a_2\, R_{\mu\nu} R^{\mu\nu}) \tag{4.51}$$

The equations of motion, for this theory are given by

$$\begin{aligned} &E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + E^{\rm HD}_{\mu\nu} = 0, \;\; \text{where} \\ &E^{\rm HD}_{\mu\nu} = a_2 \bigg(2R^{\alpha\beta} R_{\mu\alpha\nu\beta} - D_\mu D_\nu R + D^\alpha D_\alpha R_{\mu\nu} + \frac{1}{2}\, g_{\mu\nu}\, D^\alpha D_\alpha R - \frac{1}{2}\, g_{\mu\nu}\, R_{\alpha\beta} R^{\alpha\beta}\bigg) \end{aligned} \tag{4.52}$$

The explicit form of the vv-component of the equations of motion on the horizon, is as given below

$$\begin{aligned} E_{vv} =& R_{vv} + E^{\text{HD}}_{vv} = 0, \\ E^{\text{HD}}_{vv} =& a_2\big(2R^{\alpha\beta} R_{v\alpha v\beta} - D_v D_v R + D^{\alpha} D_{\alpha} R_{vv}\big). \end{aligned} \tag{4.53}$$

The Wald entropy for this theory is given by

$$S_w = \int_{\mathcal{H}_v} d^{d-2}x \, \sqrt{h} \, (1 + 2\, a_2 \, R_{rv}), \tag{4.54}$$

so that $s^{\text{HD}}_w = 2\, a_2 \, R_{rv}$. Once we have obtained the Wald entropy, we can compute $\mathbb{E}^{\text{HD}*}_{vv}$. Using the form of the metric (4.3) and the formulae provided in appendix B.3, we evaluate $\mathbb{E}^{\text{HD}*}_{vv}$ explicitly in terms of metric functions and their derivatives.

$$\mathbb{E}^{\text{HD}*}_{vv} = a_2\, \partial_v \left[\frac{1}{\sqrt{h}} \partial_v \big(\sqrt{h}\, K \bar{K}\,\big)\right] + a_2\, \partial_v \left[\nabla_i \left(\nabla^i K + \, h^{ij} \partial_v \omega_j - 2\, \nabla_j K^{ij}\right)\right]. \tag{4.55}$$

Now, we could easily re-express $\mathbb{E}^{\text{HD}*}_{vv}$ in the form of (4.44). Subsequently, it is straightforward to identify the current as

$$\begin{aligned} J^v =& - \, s^{\text{HD}}_w - a_2\, K \bar{K}, \\ J^i =& a_2 \, \left(2\, \nabla_j K^{ij} - \nabla^i K - \, h^{ij} \partial_v \omega_j\right). \end{aligned} \tag{4.56}$$

Riemann tensor square theory

The action for Riemann tensor square theory is

$$I = \int d^d x \sqrt{-g} (R + a_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})$$

The corresponding equations of motion are

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + E^{\text{HD}}_{\mu\nu} = 0, \tag{4.57}$$

where

$$\begin{aligned} E^{\text{HD}}_{\mu\nu} = a_3 \Big(& 4R^{\alpha\beta} R_{\mu\alpha\nu\beta} - 2D_{\mu} D_{\nu} R + 4D^{\alpha} D_{\alpha} R_{\mu\nu} - 4R^{\alpha}_{\mu} R_{\nu\alpha} \\ & - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta\gamma\sigma} R^{\alpha\beta\gamma\sigma} + 2R^{\alpha\beta\sigma}_{\mu} R_{\nu\alpha\beta\sigma} \Big). \end{aligned} \tag{4.58}$$

The vv-component of the equations of motion is

$$\begin{aligned} E_{vv} =& R_{vv} + E^{\text{HD}}_{vv} = 0, \\ E^{\text{HD}}_{vv} =& a_3 \bigg(4 R^{\alpha\beta} R_{v\alpha v\beta} - 2 D_v D_v R + 4 D^\alpha D_\alpha R_{vv} - 4 R^\alpha_v R_{v\alpha} \\ & + 2 R_v^{\alpha\beta\sigma} R_{v\alpha\beta\sigma} \bigg). \end{aligned} \tag{4.59}$$

The Wald entropy for this theory will be

$$s_w = \int_{\mathcal{H}_v} d^{d-2}x \sqrt{h}\, (1 - 4\, a_3\, R_{rvrv}),$$

such that $s_w^{\text{HD}} = -4\, a_3\, R_{rvrv}$.

Once we have Wald entropy, it is easy to compute $\mathbb{E}^{\text{HD}*}_{vv}$. Using the form of the metric (4.3), and the formulae provided in appendix B.3, we can evaluate $\mathbb{E}^{\text{HD}*}_{vv}$, explicitly in terms of metric functions and their derivatives. We find that

$$\mathbb{E}^{\text{HD}*}_{vv} = 4\, a_3\, \partial_v \left[\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h}\, K_{ij} \bar{K}^{ij} \right) \right] + 4\, a_3\, \partial_v \left[\nabla_i \left(h^{ij} \partial_v \omega_j - \nabla_j K^{ij} \right) \right], \tag{4.60}$$

which has been expressed in the structural form (4.44). From this, it is again straightforward to read off the entropy current to be

$$\begin{aligned} J^v =& - \ s_w^{\text{HD}} - 4\, a_3\, K_{ij} \bar{K}^{ij}, \\ J^i =& - 4\, a_3\, \left(h^{ij} \partial_v \omega_j - \nabla_j K^{ij} \right). \end{aligned} \tag{4.61}$$

4.2.2 The most general structure of the 'zero boost term' in E^{HD}_{vv}

Determining the equation of motion and in particular, its 'vv'-component, given the coordinate choice in (4.3), are, in principle, a straightforward task. But, it becomes increasingly tedious with the number of derivatives present in the action. Also, as we have seen in §4.2.4, the unambiguous definition of the spatial components of the entropy current arises out of the zero boost term in E^{HD}_{vv}. This implies that, for the construction of the entropy current, we do not need the equation of motion in its every detail. What we need is a very specific

1	Ricci scalar square theory: $I = \int d^dx\sqrt{-g}\left(R + a_1 R^2\right)$	$E_{vv}^{\text{HD}} = -\partial_v\Theta + \mathcal{O}[\epsilon]^2$ $\Theta = \frac{2\,a_1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,R\right)$
2	Ricci tensor square theory: $I = \int d^dx\sqrt{-g}\left(R + a_2 R_{\mu\nu}R^{\mu\nu}\right)$	$E_{vv}^{\text{HD}} = -\partial_v\Theta - \partial_v\left(\nabla_i J^i\right) + \mathcal{O}[\epsilon]^2$ $\Theta = \frac{a_2}{\sqrt{h}}\partial_v\left[\sqrt{h}\left(2R_{rv} - \bar{K}K\right)\right]$ $J_i = a_2\left[2\nabla^j K_{ij} - \nabla_i K - \partial_v\omega_i\right]$
3	Riemann tensor square theory: $I = \int d^dx\sqrt{-g}\left(R + a_3 R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}\right)$	$E_{vv}^{\text{HD}} = -\partial_v\Theta - \partial_v\left(\nabla_i J^i\right) + \mathcal{O}[\epsilon]^2$ $\Theta = \frac{4\,a_3}{\sqrt{h}}\partial_v\left(\sqrt{h}\left(-R_{rvrv} + \bar{K}_{ij}K^{ij}\right)\right)$ $J_i = 4a_3\left[\nabla^j K_{ij} - \partial_v\omega_i\right]$

Table 4.1: Table showing the higher derivative corrections to Einstein's equations, for all possible 4-derivative theories of gravity.

set of terms in $E^{\rm HD}_{vv}$, namely the terms that could be written in the form of the first term in (4.24). For convenience, we are re-writing (4.24) here again

$$E^{\rm HD}_{vv} = \underbrace{\partial_v \left[A\, \partial_v\, B\right]}_{\left(=E^{\rm HD}_{vv}\big|_{\text{zero boost}}\right)} + \underbrace{\frac{1}{\sqrt{h}}\,\partial_v^2 \left[\sum_{k\geq 1} \sqrt{h}\, \left(\partial_r^k A^{(k)}\right)\, \left(\partial_v^k\, B^{(k)}\right)\right]}_{\left(=E^{\rm HD}_{vv}\big|_{\text{higher boost}}\right)} + \mathcal{O}(\epsilon^2)\,. \tag{4.62}$$

In this subsection, we would like to develop an algorithm that would isolate out these zero-boost terms in $E^{\rm HD}_{vv}$. The most important feature of these terms is that at linear order in amplitude expansion of the perturbations, it is always possible to rewrite them as

$$E^{\rm HD}_{vv}\Big|_{\text{zero boost}} = \partial_v \left[A\, \partial_v B\right] \sim A\, \partial_v^2 B + \mathcal{O}(\epsilon^2)\,, \tag{4.63}$$

where both A and B are boost invariant quantities and they are non-vanishing on the stationary solutions. Hence here our main focus would be to search for terms of the form $A\, \partial_v^2 B$ in the 'vv'-component of the linearised equation of motion, $E^{\rm HD}_{vv}$. However, before proceeding to extract the zero boost terms from $E^{\rm HD}_{vv}$, let first point out an important ambiguity in defining the zero boost terms, through the structure (4.63).

Generating terms like $A\, \partial_v^2 B$ from the $k = 1$ terms in (4.62)

Before proceeding further with the zero boost terms, we would like to discuss one subtle point that will be important in our attempt to separate out the $k = 0$ terms (i.e. the zero boost sector) from the $k \neq 0$ ones in (4.62). Recall that our final goal is to determine the form of the boost invariant terms A and B in (4.62) and we plan to do that by keeping track of the terms of form $A\, \partial_v^2 B$ in the linearised $E^{\rm HD}_{vv}$. However, the strategy mentioned above to uniquely extract out the zero boost terms from linearised $E^{\rm HD}_{vv}$ would be unsuccessful if there is a possibility of generating terms of the form $A\, \partial_v^2 B$ (with A and B being boost invariant) from the second term in equation (4.62). As we will see now, there is indeed such a possibility of contamination arising from the term $k = 1$ in the summation on the RHS of

(4.62). Let us analyse this term more carefully

$$\begin{aligned} E_{vv}^{\text{HD}}\Big|_{k=1} &= \frac{1}{\sqrt{h}}\,\partial_v^2\left[\sqrt{h}\,\left(\partial_r A^{(1)}\right)\left(\partial_v\, B^{(1)}\right)\right] + \mathcal{O}(\epsilon^2) \\ &= 2\left(\partial_v\partial_r A^{(1)}\right)\left(\partial_v^2\, B^{(1)}\right) + \left(\partial_r A^{(1)}\right)\left(\partial_v^3\, B^{(1)}\right) + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{4.64}$$

In (4.64) above the first term is precisely of the form $\sim X\partial_v^2 Y$, where $\left(\partial_v\partial_r A^{(1)}\right)$ and $B^{(1)}$ respectively can be added to A and B of the zero boost terms. Thus, the terms of our interest $A\partial_v^2 B$, which we are looking for in E_{vv}^{HD} can be contaminated by terms generated from $\partial_v\left[B^{(1)}\,\partial_v\left(\partial_v\partial_r A^{(1)}\right)\right]$. This clearly demonstrates that it is impossible to uniquely determine A and B, appearing in the first term of (4.62), just by looking at the terms of the form $A\,\partial_v^2 B$ alone in E_{vv}^{HD}; it would be difficult to know if they arise from $k=0$ or $k=1$ terms in our classification (4.62) for the terms in E_{vv}^{HD}.

With this subtlety in mind, let us also comment on the way to tackle this issue. We can subtract off the contributions coming from the $k=1$ terms, that are of the same form as the $k=0$ terms in (4.62). This could be done easily by noting that whenever such a term is generated from $k=1$ piece, it will also generate the second term in equation (4.64). Hence to determine A and B unambiguously and construct $E_{vv}^{\text{HD}}\big|_{\text{zero boost}}$, we will have to isolate out few special terms of the form $(\partial_r X)(\partial_v^3 Y)$ in E_{vv}^{HD}, with X and Y being boost invariant.

Note that, due to the structural nature of the terms, a similar issue may also arise from the $k=2$ term in (4.62). However, we are not discussing the $k=2$ case in greater detail here, because such terms would not arise in four derivative theories of gravity. This is because, there are a total of six derivatives in the $k=2$ terms.

Algorithm to uniquely extract the terms like $A\,\partial_v^2 B$ from linearised E_{vv}^{HD}

Our job will now be to develop an algorithm, to determine the most general structure of this $k=0$ 'zero boost term' appearing in (4.64), keeping in mind the above mentioned subtlety.

It is clear from our previous discussions that for constructing the entropy current, which satisfies the strongest form of the second law, we need the knowledge of the zero boost term in E_{vv}^{HD}, only on the horizon $\mathcal{H}$. This in turn means that, the $g_{vi} = r\,\omega_i$ component of the metric (4.3) can appear in the zero boost term, only after differentiation with one ∂_r, and the $g_{vv} = r^2 X$ component can appear only after the action of two ∂_r. The spatial components of the metric h_{ij} can appear without any derivative acting on it. So the basic building blocks, for constructing the zero boost term on the horizon, are given in Table-(4.2).

	Candidate terms	Derivative counting	Boost weight
1.	h_{ij}	zero	zero
2.	ω_i	one	zero
3.	X	two	zero

Table 4.2: the basic building blocks

Let us first concentrate on terms of the form $(X\partial_v^2 Y)$ and isolate such terms in E_{vv}^{HD} when we have a four derivative theory of gravity. These terms in E_{vv}^{HD} can be constructed by applying ∂_r, ∂_v and ∇_i on these building blocks [19], so that the total number of derivatives are always equal to four[20], when we restrict to the four derivative theories of gravity.

For convenience, we shall now classify the data in two categories:

1. *equilibrium* data and 2. *off-equilibrium* data.

As it is clear from the names, 'equilibrium data' are those structures that are non-vanishing even in a stationary situation, whereas 'off-equilibrium data' vanishes when stationary limits are taken. Now, from the discussion in appendix-(B.1) it follows that 'equilibrium data' must be 'boost-invariant' and therefore could have definite structures and their appropriate products as listed in Table-(4.3).

[19]To begin with the structures that appear in E_{vv}^{HD} will have only ∂_i. However, we know that E_{vv}^{HD} is a scalar with respect to the coordinate transformation that only mixes the $\{x^i\}$ coordinates among themselves. If we want to construct scalars out of the horizon data with spatial derivatives on the three building blocks, it must be combined with appropriate spatial derivatives of h_{ij} so that it finally becomes a covariant derivative with respect to h_{ij}. This covariance with respect to the mixing of $\{x^i\}$ tells us that just spatial derivatives of h_{ij} need not be taken as any independent data.

Also note that in our set-up r and v are genuinely distinguished coordinates and we do not demand any covariance with respect to transformation that mixes these two coordinates among themselves and others.

	Candidate structures	Derivative counting	Boost weight
1.	$\left(\nabla_{j_1}\cdots\nabla_{j_p}\right)(\partial_r\partial_v)^{m_1}h_{ij}$	$p+2\,m_1$	zero
2.	$(\partial_r\partial_v)^{m_2}\left(\nabla_{j_1}\cdots\nabla_{j_q}\right)\omega_i$	$q+2\,m_2$	zero
3.	$(\partial_r\partial_v)^{m_3}\left(\nabla_{j_1}\cdots\nabla_{j_r}\right)X$	$r+2\,m_3$	zero

Table 4.3: 'equilibrium' and 'boost-invariant' structures built out of the basic building blocks.

On the other hand, the 'off-equilibrium data' are not boost-invariant, i.e., the total number of ∂_v should be more than the total number of ∂_r, when we consider these two derivatives as operators acting on the three basic building blocks listed above. In general there are many many possibilities for such 'off-equilibrium data'. However, here we are interested in a very specific term in $E^{\rm HD}_{vv}$, where the total number of ∂_v's is exactly two more than the number of ∂_r's (again considering them as operators on the basic building blocks and not directly on the metric components). Also both of these two extra ∂_v's must be acting on the same structure, otherwise it would generate a term which is second order in terms of the amplitude expansion we are considering here. 'Off-equilibrium data' with this property could have the following structures in general as given in Table-(4.4):

	Candidate structures	Derivative counting	Boost weight
1.	$\left(\nabla_{j_1}\cdots\nabla_{j_p}\right)\partial_v^2(\partial_r\partial_v)^{m_1}h_{ij}$	$p+2\,m_1+2$	two
2.	$\partial_v^2(\partial_r\partial_v)^{m_2}\left(\nabla_{j_1}\cdots\nabla_{j_q}\right)\omega_i$	$q+2\,m_2+2$	two
3.	$\partial_v^2(\partial_r\partial_v)^{m_3}\left(\nabla_{j_1}\cdots\nabla_{j_r}\right)X$	$r+2\,m_3+2$	two

Table 4.4: The list of 'off-equilibrium' and 'boost-weight= 2' data built out of the basic building blocks.

Finally, we have to contract the 'equilibrium data' and 'off-equilibrium data' appropriately to get the scalar term in $E^{\rm HD}_{vv}$. Since in this note we are focusing only on the four-derivative theories of gravity, every term in $E^{\rm HD}_{vv}$ contains four-derivatives on the metric components. So the relevant equilibrium data can have a maximum of two derivatives acting on the metric components, the possible structures are listed in Table-(4.5). Following

Therefore the derivatives with respect to r and v would remain as simple ∂_r and ∂_v.

[20]In this derivative counting ω_i and X must be taken as one derivative and two derivative data, respectively

the same argument to maintain the derivative counting, the relevant 'off-equilibrium data' are listed below in Table-(4.6).

Equilibrium and boost-invariant data			Number of derivatives
1.	Tensor structures:	$T^{(1)}_{ij} \equiv \partial_r \partial_v \, h_{ij}$	2
		$T^{(2)}_{ij} \equiv \nabla_i \omega_j$	2
		$T^{(3)}_{ij} \equiv \mathcal{R}_{ij}$	2
2.	Vector structure:	$V^{(1)}_i \equiv \omega_i$	1
3.	Scalar Structure:	$S^{(1)} \equiv X$	2

Table 4.5: Relevant equilibrium and boost invariant data with maximum number of derivatives= 2, in four-derivative theories of gravity

Off-equilibrium and boost-weight= 2 data			Number of derivatives
1.	Tensor structures:	$T^{(4)}_{ij} \equiv \nabla_i \nabla_j \, (\partial_v^2 \, h_{kl})$	4
		$T^{(5)}_{ij} \equiv \nabla_i \, (\partial_v^2 \, h_{jk})$	3
		$T^{(6)}_{ij} \equiv \partial_v^2 \, (\partial_r \partial_v h_{ij})$	4
		$T^{(7)}_{ij} \equiv \partial_v^2 \, (\nabla_i \omega_j)$	4
		$T^{(8)}_{ij} \equiv \partial_v^2 h_{ij}$	2
2.	Vector structure:	$V^{(2)}_i \equiv \partial_v^2 \omega_i$	3
3.	Scalar Structure:	$S^{(2)} \equiv \partial_v^2 X$	4

Table 4.6: Relevant off-equilibrium data with maximum number of derivatives= 4 and boost-weight= 2, in four-derivative theories of gravity. Let us emphasise that within the tensor structures there are three types of terms: (i) 4-index structure: $T^{(4)}_{ij}$, (ii) 3-index structure: $T^{(5)}_{ij}$, (iii) 2-index structure: $T^{(6)}_{ij}$, $T^{(7)}_{ij}$ and $T^{(8)}_{ij}$.

Now our job is to contract these two sets of data as given in Table-(4.5) and Table-(4.6), to get the candidate scalar terms in E^{HD}_{vv}, maintaining the count of total number of derivatives equal to four. This could be done systematically as outlined below:

- The four-indexed tensor structure $T^{(4)}_{ij}$ itself has four derivatives. Therefore the free indices have to be contracted with zero derivative 'equilibrium-data' or just among themselves. Now, there is no 'equilibrium-data' that has zero derivatives, see Table-(4.5). Therefore, self contraction of the indices in $T^{(4)}_{ij}$ is the only possibility here and it could be done in two ways leading to two different scalar structures:

$$T_1 = h^{ij}h^{kl}\nabla_i\nabla_j\left(\partial_v^2\, h_{kl}\right), \;\; T_2 = h^{ik}h^{jl}\nabla_i\nabla_j\left(\partial_v^2\, h_{kl}\right)$$

- The three-indexed 'off-equilibrium data' $T_{ij}^{(5)}$ has three derivatives and therefore it has to be contracted with one derivative 'equilibrium data' ($V_i^{(1)} = \omega_i$). Here also, two different types of contractions are possible leading to two different scalars:

$$T_3 = h^{ij}h^{kl}\omega_i\nabla_j\left(\partial_v^2\, h_{kl}\right), \;\; T_4 = h^{ik}h^{jl}\omega_i\nabla_j\left(\partial_v^2\, h_{kl}\right)$$

- The two 'off-equilibrium' tensor structures with 2 indices, $T_{ij}^{(6)}$ and $T_{ij}^{(7)}$, themselves have four derivatives and therefore the free indices have to be contracted among themselves. In each case there is only one way the contraction could be done. The resultant scalars are

$$T_5 = h^{ij}\partial_v^2\left(\partial_r\partial_v h_{ij}\right), \quad T_6 = h^{ij}\partial_v^2\left(\nabla_i\omega_j\right)$$

- The last 'off equilibrium' tensor structure $T_{ij}^{(8)}$ has two derivatives. It has to be contracted with two derivative 'equilibrium-data' and also the equilibrium data must have even number (in this case it could be either zero or two) of free indices so that contraction is possible. Here we get the following structures:

$$\begin{aligned}
&T_7 = X\; h^{ij}\left(\partial_v^2 h_{ij}\right)\,, \\
&T_8 = h^{ij}h^{kl}\left(\nabla_i\omega_j\right)\left(\partial_v^2 h_{kl}\right), \quad T_9 = h^{ik}h^{jl}\left(\nabla_i\omega_j\right)\left(\partial_v^2 h_{kl}\right)\,, \\
&T_{10} = h^{ij}h^{kl}\left(\omega_i\omega_j\right)\left(\partial_v^2 h_{kl}\right), \quad T_{11} = h^{ik}h^{jl}\left(\omega_i\omega_j\right)\left(\partial_v^2 h_{kl}\right)\,, \\
&T_{12} = h^{ij}h^{kl}\left(\partial_r\partial_v h_{ij}\right)\left(\partial_v^2 h_{kl}\right), \quad T_{13} = h^{ik}h^{jl}\left(\partial_r\partial_v h_{ij}\right)\left(\partial_v^2 h_{kl}\right)
\end{aligned}$$

- The 'off-equilibrium' vector data, $V_i^{(2)}$, is a three-derivative structure therefore it has to be contracted with one derivative 'equilibrium data' $V_i^{(1)} = \omega_i$, leading to the following scalar structure

$$T_{14} = h^{ij}\omega_i\partial_v^2\omega_j$$

- The 'off-equilibrium' scalar data, $S^{(2)}$, itself is a four-derivative and no contraction is needed.

$$T_{15} = \partial_v^2 X$$

- Considering possible contractions between the 'equilibrium data' $T_{ij}^{(3)}$ (given in terms of the intrinsic curvature of $\mathcal{H}_v$), and the 'off equilibrium' tensor structure $T_{ij}^{(8)}$, we can also get two more terms as given below

$$T_{16} = h^{ik}\, h^{jl}\, \mathcal{R}_{kl}\, \partial_v^2 h_{ij}; \quad T_{17} = h^{ij}\, h^{kl}\, \mathcal{R}_{kl}\, \partial_v^2 h_{ij}\,.$$

- Finally, as we have already mentioned in the beginning of this subsection, to determine the boost-invariant A and B in (4.62) unambiguously, we also need to keep track of terms of the form $(\partial_r X)(\partial_v^3 Y)$ where X and Y are boost invariant. These are the terms which will contribute to the $k = 1$ sector of linearised E_{vv}^{HD}. Although, we are interested in finding out the $k = 0$ zero-boost sector of the same, we need to track these specific $k = 1$ terms (see (4.64)) as they will be needed to separate out the boost-invariant A and B in (4.62). In case of four-derivative theories we have only two possibilities for these terms as listed below

$$\widetilde{T}_1 = h^{ij}h^{kl}(\partial_r h_{ij})(\partial_v^3 h_{kl}), \quad \widetilde{T}_2 = h^{ik}h^{jl}(\partial_r h_{ij})(\partial_v^3 h_{kl})\,. \tag{4.65}$$

It is important to note that in this list of structures we have not counted h_{ij} and the determinant of h_{ij} as independent structures. All possible occurrences of these two pieces of data are automatically taken care of in the way we have listed our data. For example h_{ij} could only occur in contraction of other indices and all possible contractions of indices are already counted in our listing. Finally, all the nineteen possible candidate terms (seventeen of the T_i's and two of the $\widetilde{T}_i$'s) to appear in $E_{vv}^{\text{HD}}\big|_{\text{zero boost}}$, are listed in Table-(4.7).

At this stage our claim is that the first term in (4.62), i.e., the term of the form $\partial_v\,(A\,\partial_v B) \sim A\,\partial_v^2 B + \mathcal{O}(\epsilon^2)$, for any four-derivative theory could always be expressed as a sum of these

$T_1 = h^{ij}h^{kl}\nabla_i\nabla_j\,(\partial_v^2\, h_{kl})$	$T_2 = h^{ik}h^{jl}\nabla_i\nabla_j\,(\partial_v^2\, h_{kl})$
$T_3 = h^{ij}h^{kl}\omega_i\nabla_j\,(\partial_v^2\, h_{kl})$	$T_4 = h^{ik}h^{jl}\omega_i\nabla_j\,(\partial_v^2\, h_{kl})$
$T_5 = h^{ij}\partial_v^2\partial_r\partial_v h_{ij}$	$T_6 = h^{ij}\partial_v^2\,(\nabla_i\omega_j)$
$T_7 = X\; h^{ij}\,(\partial_v^2 h_{ij})$	$T_8 = h^{ij}h^{kl}\,(\nabla_i\omega_j)\,(\partial_v^2 h_{kl})$
$T_9 = h^{ik}h^{jl}\,(\nabla_i\omega_j)\,(\partial_v^2 h_{kl})$	$T_{10} = h^{ij}h^{kl}\,(\omega_i\omega_j)\,(\partial_v^2 h_{kl})$
$T_{11} = h^{ik}h^{jl}\,(\omega_i\omega_j)\,(\partial_v^2 h_{kl})$	$T_{12} = h^{ij}h^{kl}\,(\partial_r\partial_v h_{ij})\,(\partial_v^2 h_{kl})$
$T_{13} = h^{ik}h^{jl}\,(\partial_r\partial_v h_{ij})\,(\partial_v^2 h_{kl})$	$T_{14} = h^{ij}\omega_i\partial_v^2\omega_j$
$T_{15} = \partial_v^2 X$	$T_{16} = h^{ik}\, h^{jl}\,\mathcal{R}_{kl}\,\partial_v^2 h_{ij}$
$T_{17} = h^{ij}\, h^{kl}\,\mathcal{R}_{kl}\,\partial_v^2 h_{ij}$	
$\widetilde{T}_1 = h^{ij}h^{kl}(\partial_r h_{ij})(\partial_v^3 h_{kl})$	$\widetilde{T}_2 = h^{ik}h^{jl}(\partial_r h_{ij})(\partial_v^3 h_{kl})$

Table 4.7: Listing the seventeen T_i's and two $\widetilde{T}_i$'s, the possible 4-derivative scalar data with boost weight $= 2$. They are candidate terms that appear in E_{vv}^{HD} for 4-derivative theories of gravity. The seventeen T_i terms will contribute to $k = 0$ sector, and the two $\widetilde{T}_i$ terms will contribute to $k = 1$ sector of E_{vv}^{HD}.

seventeen terms listed in Table-(4.7) with constant coefficients. Further, we claim that the contribution of the $k = 1$ piece from the second term of (4.62) (written in the form of a sum over several k values) could also be expressed in terms of these seventeen structures plus two more, listed in equation (4.65)

$$E_{vv}^{\text{HD}} = -\sum_{i=1}^{17} a_i\, T_i - \sum_{i=1}^{2} \tilde{a}_i\, \widetilde{T}_i + \cdots\,, \tag{4.66}$$

where $\cdots$ denote the terms that do not matter for the proof of the physical process version of the first law. The negative sign on the RHS of (4.66) is chosen for convenience. The specific values of these seventeen a_i and two $\tilde{a}_i$ coefficients appearing in (4.66), will of course vary from theory to theory. As we have mentioned before, the above classification of terms have been done keeping in mind the four derivative theories of gravity. The most general four derivative theory of pure gravity could have three more terms apart from the standard two derivative term in Einstein gravity. In Table-(4.8) we are listing the values of a_i's and $\tilde{a}_i$'s for each of these three cases. These set of values of the a_i coefficients are obtained by comparing (4.66) with the explicit calculation of E_{vv}^{HD} for each of the three four derivative theories of gravity, which was performed in §4.2.1, §4.2.1 and §4.2.1.

	Different theories $\left(\mathcal{I}^{(i)} = \int d^dx\sqrt{-g}\,\mathcal{L}^{(i)}\right)$	The calculated values of the coefficients a_i and $\tilde{a}_i$'s
1.	$\mathcal{L}^{(1)} = R^2$ (Ricci scalar squared)	$a_1 = 2,\ a_2 = -2,\ a_3 = 0,\ a_4 = 0,$ $a_5 = 4,\ a_6 = -4,\ a_7 = 1,$ $a_8 = -2,\ a_9 = 4,\ a_{10} = 3/2,$ $a_{11} = -3,\ a_{12} = 4,$ $a_{13} = -10,\ a_{14} = 6,\ a_{15} = 2,$ $a_{16} = 2,\ a_{17} = -1,$ $\tilde{a}_1 = 1,\ \tilde{a}_2 = -3.$
2.	$\mathcal{L}^{(2)} = R_{\mu\nu}R^{\mu\nu}$ (Ricci tensor squared)	$a_1 = 1/2,\ a_2 = -1,\ a_3 = -(1/2),$ $a_4 = 1,\ a_5 = 1,\ a_6 = 0,$ $a_7 = 1/2,\ a_8 = -(1/2),\ a_9 = 1,$ $a_{10} = 1/2,\ a_{11} = -1,\ a_{12} = 1,$ $a_{13} = -2,\ a_{14} = 2,\ a_{15} = 1,$ $a_{16} = 0,\ a_{17} = 0,$ $\tilde{a}_1 = 1/4\,,\ \tilde{a}_2 = -(1/2).$
3.	$\mathcal{L}^{(3)} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (Riemann tensor squared)	$a_1 = 0,\ a_2 = -2,\ a_3 = -2,$ $a_4 = 4,\ a_5 = 0,\ a_6 = 4,\ a_7 = 1,$ $a_8 = 0,\ a_9 = 0,\ a_{10} = 1/2,$ $a_{11} = -1,\ a_{12} = 0, a_{13} = 2,$ $a_{14} = 2,\ a_{15} = 2,\ a_{16} = 0,\ a_{17} = 0,$ $\tilde{a}_1 = 0,\ \tilde{a}_2 = 1..$

Table 4.8: Explicit calculation for each of the three theories produces these values of the coefficients a_i, appearing in $E_{vv}^{\text{HD}} = -\sum_{i=1}^{17} a_i\, T_i - \sum_{i=1}^{2} \tilde{a}_i\, \widetilde{T_i}$, for 4-derivative theories of gravity.

4.2.3 Constraints on the 'zero boost terms' in $E_{vv}^{\textbf{HD}}$

A very specific structure for the zero boost terms in E_{vv}^{HD} is predicted in (4.36). This structure does not follow automatically just from the boost transformation property, which we have used to classify terms in the previous subsection. Clearly, imposing (4.36) would impose further constraints on the seventeen coefficients mentioned above. In this subsection, we shall first find those constraints. We shall list the most general possible structure for $\widetilde{B}$, defined in (4.36), which is a two-derivative scalar with vanishing boost weight. According to the terminology of the previous subsection it must be an 'equilibrium data'. It turns out that $\widetilde{B}$ could have only five independent structures which are non-vanishing at equilibrium (see table -(4.9)).

Candidate terms for $\widetilde{B}$	
Equilibrium data:	1. $h^{ij}\,(\partial_r\partial_v h_{ij})$, 2. $h^{ij}\,\nabla_i\,\omega_j$, 3. $h^{ij}\,\omega_i\,\omega_j$, 4. $\mathcal{R}$, 5. X,
Off-equilibrium data:	6. $h^{ij}\,h^{kl}\,(\partial_v h_{ij})\,(\partial_r h_{kl})$, 7. $h^{ik}\,h^{jl}\,(\partial_v h_{ij})\,(\partial_r h_{kl})$

Table 4.9: Possible structures that can appear in $\widetilde{B}$: each of them has two derivatives and boost weight$=0$.

Using linear combinations of the independent structures presented in Table-(4.9) we can now write down the most general structure of $\widetilde{B}$, if it exists, as follows

$$\widetilde{B} = A_1\, h^{ij}\partial_r\partial_v h_{ij} + A_2\, h^{ij}\nabla_i\omega_j + A_3\, h^{ij}\omega_i\omega_j + A_4\, X + A_5\,\mathcal{R}\,. \tag{4.67}$$

The first term in $\widetilde{B}$ needs a special attention. This is the term whose contribution to E_{vv}^{HD} could get mixed with some the $k = 1$ term (see equation (4.62) and the discussion after that). To see this more explicitly, let us write down the contribution to E_{vv}^{HD} coming from the term $\widetilde{B} = A_1\, h^{ij}\partial_r\partial_v h_{ij}$

$$\begin{aligned}\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\widetilde{B}\right)\right) &\sim A_1\,\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,h^{ij}\partial_r\partial_v h_{ij}\right)\right)\\ &= A_1\left(T_5 + \frac{T_{12}}{2} - T_{13}\right).\end{aligned} \tag{4.68}$$

It can be easily checked that the terms T_{12} and T_{13} could also be generated as $k = 1$ terms in E_{vv}^{HD} from the following two off-equilibrium candidates for $\widetilde{B}$ (see the list of off-equilibrium data in table-4.9)

$$(i)\; h^{ij}h^{kl}\,(\partial_v h_{ij})\,(\partial_r h_{kl})\,, \quad (ii)\; h^{ik}h^{jl}\,(\partial_v h_{ij})\,(\partial_r h_{kl})\;.$$

We assume that in E_{vv}^{HD}, these two terms mentioned above contribute with coefficients A_6 and A_7 respectively as written below [21]

$$E_{vv}^{\text{HD}}\big|_{k=1} = -A_6\left(2\,T_{12} + \tilde{T}_1\right) - A_7\left(2\,T_{13} + \tilde{T}_2\right). \tag{4.69}$$

From the above equation it is clear that A_6 and A_7 could be simply fixed by comparing the coefficients of $\tilde{T}_1$ and $\tilde{T}_2$ respectively in the $k = 1$ sector of (4.66) and (4.69),

$$A_6 = \tilde{a}_1\,, \quad A_7 = \tilde{a}_2\,. \tag{4.70}$$

We have now extracted out the $k = 1$ part of E_{vv}^{HD} in (4.69) which has the form of $A\,\partial_v^2 B$, as desired from (4.62). Next, we subtract off (4.69) from (4.66) and obtain the part of E_{vv}^{HD} that is entirely generated from zero boost sector. This could be written as

$$\begin{aligned} E_{vv}^{\text{HD}}\big|_{\text{zero boost}} &= -\sum_{i=1}^{11} a_i\,T_i - (a_{12} - 2\,\tilde{a}_1)\,T_{12} \\ &\quad - (a_{13} - 2\,\tilde{a}_2)\,T_{13} - \sum_{i=14}^{17} a_i\,T_i\,. \end{aligned} \tag{4.71}$$

At this point, it is important to note that although in (4.66) there were nineteen terms to begin with, the zero boost sector $E_{vv}^{\text{HD}}\big|_{k=0}$ is constructed out of seventeen terms T_i's appearing on the RHS of (4.71). We, therefore, have to deal with seventeen coefficients as well. The easiest way to understand this is by realising that the coefficients $\tilde{a}_1$ and $\tilde{a}_2$ do not count as

[21] To obtain the expressions in (4.71) we have used the following relations

$$\begin{aligned} \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\;h^{ij}h^{kl}\,(\partial_v h_{ij})\,(\partial_r h_{kl})\right)\right) &= 2\,T_{12} + \widetilde{T}_1\,, \\ \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\;h^{ik}h^{jl}\,(\partial_v h_{ij})\,(\partial_r h_{kl})\right)\right) &= 2\,T_{13} + \widetilde{T}_2\,. \end{aligned}$$

additional ones since they will always appear in the combination $(a_{12}-2\,\tilde{a}_1)$ and $(a_{13}-2\,\tilde{a}_2)$ respectively [22].

On the other hand, from (4.67) we know that the number of free coefficients in $\widetilde{B}$ thus turns out to be five, namely the A_i's (for $i = 1, \cdots, 5$). As of now the A_i's are free coefficients and we want to solve them in terms of the a_i's appearing in (4.71). To do that, we first substitute $\widetilde{B}$ from (4.67) in (4.36) and write it in terms of the basis of T_i structures, as listed in Table-(4.7) and obtain the following

$$\begin{aligned} E^{\text{HD}}_{vv}\big|_{\text{zero boost}} &\sim -\,\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\widetilde{B}\right)\right) \\ &= -\Bigg[A_1\left(T_5+\frac{T_{12}}{2}-T_{13}\right)+A_2\left(T_6+\frac{T_8}{2}-T_9\right) \\ &\quad + A_3\left(2T_{14}+\frac{T_{10}}{2}-T_{11}\right)+A_4\left(T_{15}+\frac{T_7}{2}\right) \\ &\quad - A_5\left(T_{16}-\frac{T_{17}}{2}+T_1-T_2\right)\Bigg]. \end{aligned} \tag{4.72}$$

In deriving (4.72), we have used the following relations

$$\begin{aligned} \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,h^{ij}\,\partial_r\partial_v h_{ij}\right)\right) &= T_5+\frac{T_{12}}{2}-T_{13}\,, \\ \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,h^{ij}\,\nabla_i\omega_j\right)\right) &= T_6+\frac{T_8}{2}-T_9\,, \\ \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,h^{ij}\,\omega_i\omega_j\right)\right) &= 2\,T_{14}+\frac{T_{10}}{2}-T_{11}\,, \\ \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,X\right)\right) &= T_{15}+\frac{T_7}{2}\,, \\ \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\mathcal{R}\right)\right) &= -\left(T_{16}-\frac{T_{17}}{2}+T_1-T_2\right)\,. \end{aligned} \tag{4.73}$$

We now compare (4.72) with (4.71) and equate the coefficients of T_i's on both sides, which gives us seventeen relations between the A_i $(i = 1, \cdots, 5)$ and a_j $(j = 1, \cdots, 17)$, $\tilde{a}_i$ $(i =$

[22] In what follows, whenever we refer to the seventeen coefficients a_i's, it will be implied that there are actually nineteen coefficients (the a_i's and the $\tilde{a}_i$'s) but the two coefficients $\tilde{a}_1$ and $\tilde{a}_2$ will always appear being paired with a_{12} and a_{13} respectively, see (4.71), and hence the independent coefficients will be counted as seventeen.

1, 2) [23]. We can solve the five A_i's in terms of the a_i's and then we will be left with twelve constraints on the coefficients a_i's which ensure the consistency of (4.36). These twelve constraints on a_i's are listed below,

$$\begin{aligned} &a_1 = a_{16},\ 2\,a_{10} = -a_{11},\ a_{14} = 4\,a_{10},\ a_{15} = 2\,a_7,\ 2\,a_{17} = -a_1, \\ &a_2 = -a_1,\ a_3 = 0,\ a_4 = 0,\ a_6 = 2\,a_8,\ 2\,a_8 = -a_9\,, \\ &2\,(a_{12} - 2\,\tilde{a}_1) = -(a_{13} - 2\,\tilde{a}_2),\ a_5 = 2\,(a_{12} - 2\,\tilde{a}_1)\,. \end{aligned} \tag{4.74}$$

Finally, we would like to check whether the a_i's as given in Table-(4.8) satisfy the constraints given in (4.74). Remember that in the previous subsection, we have already calculated the allowed values of the a_i's for each of the three different 4-derivative theory of gravity, see Table-(4.8). Upon inspection, we can convince ourselves that for Ricci scalar squared theory the constraints in (4.74) are satisfied, where as for both of the other two four derivative theories of gravity, namely the Ricci tensor squared and the Riemann tensor squared theories, the constraints in (4.74) are simply not satisfied. Therefore, we convince ourselves that the constraints obtained in (4.74) are not correct, as they are not satisfied by the results obtained by explicit calculation of E^{HD}_{vv} which is the content of Table-(4.8) for the most general four derivative theory of gravity. We should keep this in mind that these constraints were derived from demanding the consistency of (4.36). As a result, we are led to the conclusion that the general structure of E^{HD}_{vv} in the zero boost sector, as predicted in (4.36), is not generically true for the most general four derivative theory of gravity.

4.2.4 The general strategy for constructing the entropy current maintaining the boost symmetry

From the analysis of the previous subsection, we have established the fact that the zero boost terms in E^{HD}_{vv} does not always follow the structure predicted in (4.36). Motivated by this observation, in this subsection our goal will therefore be to explore what are the further

[23] As we have mentioned before, we have seventeen coefficients on the RHS of (4.71), and not nineteen, because the coefficients $(a_{12} - 2\,\tilde{a}_1)$ and $(a_{13} - 2\,\tilde{a}_2)$ always comes in this particular combination.

structures, if any, that we could allow for the zero boost terms in E_{vv}^{HD} without affecting the proofs for the physical version of the first law and second law.

As we have explained before, the first law is a statement about the total change in the thermodynamic parameters like entropy, energy etc., characterising two nearby equilibrium solutions connected by dynamics. Hence its formulation always involves an integration over all space and therefore is usually insensitive to any boundary terms. The same is true for black hole mechanics. The total change in entropy as described in (4.33) has an integration over the spatial slices of the horizon. If the horizon is compact, this integration would be insensitive to any boundary term that appears in E_{vv}^{HD}. It follows that the zero boost term in E_{vv}^{HD}, in addition to the term already mentioned in (4.36), could also have a structure of the form

$$E_{vv}^{\text{HD}}\big|_{\text{zero boost}} \sim -\,\partial_v\left(\nabla_i J^i\right) = -\,\partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\,J^i\right)\right) \tag{4.75}$$

where J^i is some spatial current with boost weight 1 (i.e., it must contain an explicit ∂_v that could not be paired up with any ∂_r). On compact horizons such a term would clearly integrate to zero and therefore will not contribute to the total change in entropy (see the derivation of (4.33)).

It is worth noting that the compatibility with the first law also allows a term, generically of the form $\nabla_i Y^i$ in E_{vv}^{HD} where Y^i is some arbitrary vector quantity, i.e. a spatial current with boost weight equal to 2. However, the manipulation that follows from (4.18) shows that working upto linear order of amplitude perturbations we could always re-arrange the terms in E_{vv}^{HD} (including the possible $\nabla_i Y^i$ term) in a form where there is an overall ∂_v outside[24]. It is important to stress that although the first law itself does not require this rearrangement

[24] This can be schematically presented as

$$E_{vv}^{\text{HD}}\big|_{\text{zero boost}} \sim \nabla_i Y^i \sim \partial_v\left(\nabla_i \widetilde{Y}^i\right) + \mathcal{O}(\epsilon^2), \tag{4.76}$$

where $\widetilde{Y}^i$ is some spatial vector with boost weight equal to one, since one ∂_v is extracted from Y^i which has boost weight equal to one.

as in (4.76), it is a must to proceed towards an argument for the second law. Therefore, in our classification, we shall not consider such terms for which this rearrangement is not true. This, in particular, allows us not to consider the term that we have just mentioned above in (4.75) as a possible term in E_{vv}^{HD}.

Combining equations (4.36) and (4.75), it follows that, both the first and the second law would be satisfied, at least at the linear order in amplitude of time dependent perturbations, provided the zero boost terms in E_{vv}^{HD} has the following form

$$E_{vv}^{\text{HD}}\big|_{\text{zero boost}} \sim -\partial_v \left(\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h}\, \widetilde{B} \right) + \nabla_i J^i \right) . \tag{4.77}$$

Interestingly, we should note that on the RHS of (4.77), the term inside the parenthesis (i.e. ignoring the overall ∂_v), looks exactly like the divergence of a 'four-current', let us call it S^A, such that it's v and i components are respectively given by

$$\begin{aligned} E_{vv}^{\text{HD}}\big|_{\text{zero boost}} &\sim -\partial_v \left(\nabla_A S^A \right) , \\ \text{such that } \; S^v_{(k=0)} = \widetilde{B} \, , &\quad S^i_{(k=0)} = J^i \, , \end{aligned} \tag{4.78}$$

where, the the index $A = v, x^i$ and we have also used $k = 0$ as a subscript in $S^A_{(k=0)}$ to denote the fact that we are only looking at the zero boost terms in E_{vv}^{HD}.

Next, we would like to see how the seventeen structures, listed in Table-(4.7) in §4.2.2, should combine so that the zero boost terms in E_{vv}^{HD} could be recast in the form of (4.77). In other words, if the form of E_{vv}^{HD} as proposed in (4.77) is correct, we will be using that to derive the constraints that the seventeen coefficients a_i should satisfy. As it appears in (4.77), $\widetilde{B}$ is a boost invariant scalar data and J^i is a vector data with boost weight one. In the previous subsection, we have already argued the most general structure of $\widetilde{B}$ in (4.67). Now J^i is an off-equilibrium data and from the counting of boost-weight we could see it must have exactly one ∂_v derivative, which is not paired with an ∂_r provided we are considering them as operators acting on the three basic building blocks, namely h_{ij}, ω_i and X. Taking all these facts into account, we could construct the five possible structures for a candidate term in J_i, as listed in Table-(4.10) below.

Candidate terms for J_i	
Off-equilibrium data:	1. $\partial_v \omega_i$ 2. $h^{jk}\nabla_j\left(\partial_v h_{ki}\right)$ 3. $h^{jk}\nabla_i\left(\partial_v h_{jk}\right)$ 4. $h^{jk}\,\omega_j\left(\partial_v h_{ki}\right)$ 5. $h^{jk}\,\omega_i\left(\partial_v h_{jk}\right)$

Table 4.10: Possible structures that can appear in J_i: each one of them has two derivatives and boost weight$=1$.

It is now straightforward to write down the most general form of J^i using linear combinations of the structures written in Table-(4.10)

$$\begin{aligned} J^i =& B_1\, h^{ij}\partial_v\omega_j + B_2\, h^{il}h^{jk}\,\nabla_j\left(\partial_v h_{kl}\right) + B_3\, h^{il}h^{jk}\,\nabla_l\left(\partial_v h_{jk}\right) \\ &+ B_4\, h^{il}h^{jk}\,\omega_j\left(\partial_v h_{kl}\right) + B_5\, h^{il}h^{jk}\,\omega_l\left(\partial_v h_{jk}\right), \end{aligned} \tag{4.79}$$

where the coefficients B_i for $i = 1, \cdots, 5$, are, as of now, arbitrary constant coefficients. Our aim will now be to fix them in terms of the coefficients a_i's ($i = 1, \cdots, 17$), just like the coefficients A_i's, appearing in (4.67), were fixed in the previous subsection. To achieve this we will calculate the second term on the RHS of (4.77), with J^i being substituted from (4.79). We express the resulting expression in terms of the T_i's, listed in Table-4.7, and obtain the following relation

$$\begin{aligned} E^{\text{HD}}_{vv}\Big|^{(J^i\text{ part})}_{\text{zero boost}} \sim -\,\partial_v\left(\nabla_i J^i\right) =& -\Bigg[B_1\left(T_6+T_4-\frac{T_3}{2}\right)+B_2\,T_2 \\ &+B_3\,T_1+B_4\left(T_4+T_9\right)+B_5\left(T_3+T_8\right)\Bigg]. \end{aligned} \tag{4.80}$$

In deriving (4.80) we have used the following relations

$$\begin{aligned} &\partial_v\left[\nabla_i(h^{ij}\partial_v\omega_j)\right] = T_6+T_4-(T_3/2)\,,\ \ \partial_v\left[\nabla_i(h^{il}h^{jk}\nabla_j\left(\partial_v h_{kl}\right))\right] = T_2, \\ &\partial_v\left[\nabla_i(h^{il}h^{jk}\nabla_l\left(\partial_v h_{jk}\right))\right] = T_1\,,\ \ \partial_v\left[\nabla_i(h^{il}h^{jk}\omega_j\left(\partial_v h_{kl}\right))\right] = T_4+T_9, \\ &\partial_v\left[\nabla_i(h^{il}h^{jk}\omega_l\left(\partial_v h_{jk}\right))\right] = T_3+T_8\,. \end{aligned} \tag{4.81}$$

Once we have obtained (4.72) and (4.80), we shall combine them to obtain a complete expression for the zero boost part (i.e. $k=0$) of E^{HD}_{vv} in terms of the T_i's as follows

$$
\begin{aligned}
E^{\rm HD}_{vv}\big|_{\text{zero boost}} = & -\Bigg[A_1\left(T_5 + \frac{T_{12}}{2} - T_{13}\right) + A_2\left(T_6 + \frac{T_8}{2} - T_9\right) \\
& + A_3\left(2T_{14} + \frac{T_{10}}{2} - T_{11}\right) + A_4\left(T_{15} + \frac{T_7}{2}\right) \\
& - A_5\left(T_{16} - \frac{T_{17}}{2} + T_1 - T_2\right) + B_1\left(T_6 + T_4 - \frac{T_3}{2}\right) + B_2\, T_2 + B_3\, T_1 \\
& + B_4\left(T_4 + T_9\right) + B_5\left(T_3 + T_8\right)\Bigg] .
\end{aligned}
\tag{4.82}
$$

It is obvious from the RHS of (4.82) above that we still have ten undetermined coefficients, five of the A_i's and five of the B_i's. We therefore conclude that, if we want the first term in (4.62), to have a form such that it is compatible with the physical process version of the first law, then it can have twelve independent coefficients $(A_i,\ B_i)$ for any four derivative theories of gravity. On the other hand, just from the consideration of boost symmetry, a total of seventeen terms are allowed in $E^{\rm HD}_{vv}$, see (4.66). Clearly, even after the inclusion of the spatial current in (4.77), the compatibility with the physical version of the first law would imply some constraints between a_i's (though it would certainly be less in number than what we have derived in the previous subsection). A naive counting suggests that there must be $(17-10) \ = \ 7$ relations among the seventeen possible coefficients a_i. However, as it turns out, there is a redundancy in our counting of independent structures that could appear in the expression of entropy density ($\widetilde{B}$) and spatial entropy current (J^i). In other words, not all of the ten $A_i,\ B_i$'s are independently and one of them, the term with A_2 as coefficient, can be absorbed into others by redefining some of the B_i coefficients. It is easy to check that if we redefine the coefficients $B_1,\ B_4$ and B_5 in the following way,

$$
\widehat{B}_1 = B_1 + A_2\,, \quad \widehat{B}_4 = B_4 - A_2\,, \quad \widehat{B}_5 = B_5 + \frac{A_2}{2}\,, \tag{4.83}
$$

the term with coefficient A_2 in (4.82) disappears and we are left with

$$
\begin{aligned}
E_{vv}^{\text{HD}}\big|_{\text{zero boost}} = & -\Big[A_1\left(T_5 + \frac{T_{12}}{2} - T_{13}\right) + A_3\left(2T_{14} + \frac{T_{10}}{2} - T_{11}\right) \\
& + A_4\left(T_{15} + \frac{T_7}{2}\right) - A_5\left(T_{16} - \frac{T_{17}}{2} + T_1 - T_2\right) \\
& + \widehat{B}_1\left(T_6 + T_4 - \frac{T_3}{2}\right) + B_2\,T_2 + B_3\,T_1 + \widehat{B}_4\,(T_4 + T_9) \\
& + \widehat{B}_5\,(T_3 + T_8)\Big].
\end{aligned}
\tag{4.84}
$$

As the independent terms on the RHS of (4.84) has now been reduced to nine, we should obtain eight relations among the coefficients a_i, which are given by

$$
\begin{aligned}
& a_4 = a_6 + a_9, \quad a_3 = -\frac{a_6}{2} + a_8, \quad a_{16} = -2\,a_{17}, \quad a_{15} = 2\,a_7, \\
& a_{11} = -\frac{a_{14}}{2}, \quad a_{10} = \frac{a_{14}}{4}, \quad 2\,(a_{12} - 2\,\tilde{a}_1) = a_5\,, \quad a_{13} - 2\,\tilde{a}_2 = a_5\,.
\end{aligned}
\tag{4.85}
$$

Furthermore, once the a_i's satisfy the identities given in (4.85), we can solve the A_i's and B_i's in terms of the a_i's, as given below [25]

$$
\begin{aligned}
& A_1 = a_5\,, \quad A_3 = \frac{a_{14}}{2}, \quad A_4 = 2\,a_7\,, \quad A_5 = 2\,a_{17}\,, \\
& A_2 = \text{free/undetermined}, \\
& B_1 = a_6 - A_2, \quad B_2 = a_2 - 2\,a_{17}\,, \quad B_3 = a_1 + 2\,a_{17}\,, \quad B_4 = a_9 + A_2\,, \\
& B_5 = a_8 - \frac{A_2}{2}\,.
\end{aligned}
\tag{4.86}
$$

It is worth mentioning that in deriving the identities in (4.85) and the solutions in (4.86) we have not assumed any particular form of the four derivative gravity Lagrangian. In other words these relations are true for any four derivative theory of gravity.

Once we have obtained the coefficients A_i and B_i, one can readily derive the entropy density $\widetilde{B}$ and the entropy currents J^i in terms of the coefficients a_i. Since specific values for the set of coefficients a_i corresponds to specific four derivative theories of gravity, (see

[25] In (4.86) we are still writing in terms of the coefficients $B_1,\ B_2,\ B_3$, instead of writing them in terms of the redefined $\widehat{B}_1,\ \widehat{B}_2,\ \widehat{B}_3$. This makes the appearance of the undetermined coefficient A_2 explicit, and is just a matter of convenient choice for us.

Table-(4.8)), we can substitute them for a_i's in (4.86) to obtain the specific values of A_i and B_i for each of the three individual four derivative theories of gravity. We present them in Table-(4.11).

	Different theories $\left(\mathcal{I}^{(i)} = \int d^dx\sqrt{-g}\,\mathcal{L}^{(i)}\right)$	Values of the coefficients $A_i,\ B_i$
1	$\mathcal{L}^{(1)} = R^2$ (Ricci scalar squared)	$A_1 = 4,\ A_2 =$ undetermined, $A_3 = 3,\ A_4 = 4,\ A_5 = -2,$ $B_1 = -4 - A_2,\ B_2 = 0,\ B_3 = 0,$ $B_4 = 4 + A_2,\ B_5 = -2 - (A_2/2).$
2	$\mathcal{L}^{(2)} = R_{\mu\nu}R^{\mu\nu}$ (Ricci tensor squared)	$A_1 = 1,\ A_2 =$ undetermined, $A_3 = 1,\ A_4 = 2,\ A_5 = 0,$ $B_1 = -A_2,\ B_2 = -1,\ B_3 = 1/2,$ $B_4 = 1 + A_2,\ B_5 = -(1/2) - (A_2/2).$
3	$\mathcal{L}^{(3)} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (Riemann tensor squared)	$A_1 = 0,\ A_2 =$ undetermined, $A_3 = 1,\ A_4 = 4,\ A_5 = 0,$ $B_1 = 4 - A_2,\ B_2 = -2,\ B_3 = 0,$ $B_4 = A_2,\ B_5 = -(A_2/2).$

Table 4.11: $A_i,\ B_i$'s for different 4-derivative theories of gravity.

Finally, we conclude this sub-section with the following remarks:

- **More details on the redundancy in the parameter A_2:** We have already mentioned before that our analysis in this subsection to classify possible candidate terms in the zero boost sector of E_{vv}^{HD} solely based on boost symmetry, cannot fix the coefficient A_2 in (4.82). As a result it remained undetermined in (4.86). We have also seen that this redundancy in fixing A_2 is actually related to a proper count of the independent data in $\widetilde{B}$ and J^i.

  In order to make it explicitly manifest, let us now consider the specific terms written below and their combinations as candidates for $\widetilde{B}$ and J^i [26]

$$\widetilde{B}_{(*)} = h^{ij}\nabla_i\omega_j\,, \quad J^i_{(*)} = -h^{ij}\partial_v\omega_j + h^{il}h^{jk}\omega_j\partial_v h_{kl} - \frac{1}{2}h^{il}h^{jk}\omega_l\partial_v h_{jk}\,,$$

and with this choices it can be shown that

$$\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\widetilde{B}_{(*)}\right) + \nabla_i J^i_{(*)} = 0\,. \tag{4.87}$$

The interesting thing to note about the combination written in (4.87) is that it identically vanishes without any use of the gravity equations of motion and therefore we could add the v-derivative of this combination (so that it has the appropriate boost weight$=2$) to any expression for E^{HD}_{vv}, without affecting the equation of motion and dynamics. Because of this, among the twelve terms that appeared on the RHS of (4.82) above we could hope to fix only eleven of them by comparing with the E^{HD}_{vv} of a given four derivative theory, (4.66) [27]. Also, for the same reason, we have seen that in each of the three cases tabulated in Table-(4.11), the coefficient A_2 could not be fixed as it could combine with few spatial currents to give vanishing contribution to E^{HD}_{vv}.

- **The redundancy in A_2 is fixed by matching the equilibrium limit of $\widetilde{B}$ with the equilibrium Wald entropy density:** Having realised the fact that only boost symmetry alone can not fix the coefficient A_2, let us now focus on the implications of this redundancy in the coefficient A_2 beyond boost symmetry and try to explore if there is any other principle that can fix it. Looking at the Table-(4.9) and (4.67), we remind ourselves that, by construction, the scalar structures appearing in $\widetilde{B}$ does not

[26] Note that this $\widetilde{B}_{(*)}$ appears in the expression of $\widetilde{B}$ in (4.67) with the coefficient A_2.

[27] Actually, we can make use of this redundancy to reorganize (4.67) and (4.79) with the following redefinition of $\widetilde{B}$ and J^i

$$\widetilde{B} \to \widetilde{B}\,; \quad J^i \to J^i + \alpha_*\,A_2\,J^i_{(*)}, \tag{4.88}$$

where α_* being a tunable free parameter and thus enabling us to fix the value of the coefficient A_2 to any specific number. In particular by making the choice of $\alpha_* = 1$, we can even make the coefficient A_2 not contributing to (4.82), as in that case, as A_2 disappears from E^{HD}_{vv}.

vanish when evaluated on a stationary solution. Most importantly, the term that appears in $\tilde{B}$, (4.67), with coefficient A_2 is generically non-zero in the equilibrium limit. Therefore the redundancy in the coefficient A_2 discussed in detail above, implies that possible different choices of A_2 would amount to having different expressions for the equilibrium entropy density, s_w, of the same configuration. Though the difference does not persist in the expression of total entropy S_W, since this density turns out to be a total derivative term: $(\nabla \cdot \omega)$ in this case.

Motivated by the arguments given above and based on general grounds, we should, therefore, also require that once the equilibrium limit is considered, the entropy density $\widetilde{B}$ in (4.67), should reproduce the appropriate Wald entropy density. This should be satisfied by the $\widetilde{B}$ apart from being constructed following the boost symmetry. As we will see now, at least for the cases that we are studying in this note, this additional requirement uniquely fixes the ambiguity related to the coefficient A_2. Thus, the important point to note here is that the Wald's formula (4.41) picks up a very specific value for A_2 for every cases that we have discussed here, and in some sense fixes this ambiguity which clearly could not be fixed just by imposing first or second law of thermodynamics even in its ultra local version. For example, in R^2 theory, once we demand matching with (4.41), A_2 gets fixed to a specific numerical value $A_2 = -4$, implying that there is no spatial current, which is actually consistent with what we have found in subsection §4.2.1 [28].

The consistency with Wald's formula in the equilibrium limit, forces the entropy density $\widetilde{B}$, that we have obtained in this subsection, to reduce to the stationary limit of s^{HD}_{Wald} (see (4.42)), which we derived in subsection §4.2.1, upto the ambiguity of A_2.

[28] A first glance at the non-zero values of the coefficients B_1, B_4 and B_5 for the R^2 theory in Table-(4.11) might naively suggest that there is a non-zero current for the R^2 theory. However once we make the choice of $A_2 = -4$ in order to match with the equilibrium Wald entropy density s^{HD}_w, it can be verified that there is no spatial current in this case, but a finite non-equilibrium correction s_{cor} to s^{HD}_w, see (4.40).

More precisely, if we take the expressions of s^{HD}_{Wald} as computed in subsections §4.2.1, §4.2.1 and §4.2.1 and simply remove the terms that would vanish in stationary situations (for example, a term like $\mathcal{K}\bar{\mathcal{K}}$ would be ignored), the resultant expressions should exactly match with the corresponding $\tilde{B}$'s derived in this subsection with a specific choice of the coefficient A_2 for every case [29]. It turns out that they indeed match provided we choose the coefficient A_2 to be as follows:

1. for R^2 theory: $A_2 = -4$,
2. for $R_{\mu\nu}R^{\mu\nu}$ theory: $A_2 = -1$,
3. for $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ theory: $A_2 = 0$.

This matching serves as a consistency check for our results. Therefore, once we use the values of the coefficient A_2 for different cases, as written above, in Table-(4.11) and further using (4.67) and (4.79) the specific expressions for $\widetilde{B}$ and J^i can be derived as listed in Table-(4.12).

- **Constraints on a_i's satisfied:** In §4.2.2 we computed the specific values of the coefficients a_i's and tabulated them in the Table-(4.8) for three different four derivative theories of gravity. It is now straightforward to check that the constraints derived in (4.85) are indeed satisfied by all of the four derivative theories of gravity. In other words, the physical process version of the first law holds for all of these theories once we allow for the spatial current term in E^{HD}_{vv}, (4.77).

[29] Though a mismatch at this stage would have been a serious contradiction with the existing literature and Wald's formalism, we still do not have any abstract proof for it, applicable to any higher derivative theories of gravity. According to our understanding, this would essentially amount to showing a step by step equivalence between the proof of physical version of the first law and the Wald formalism. We could not find it in literature and leave it for future work.

	Different theories $\left(\mathcal{I}^{(i)} = \int d^dx\sqrt{-g}\,\mathcal{L}^{(i)}\right)$	Expressions for $\widetilde{B}\,,\ J^i$
1.	$\mathcal{L}^{(1)} = R^2$	$\widetilde{B} = 4\,h^{ij}\,\partial_r\partial_v h_{ij} - 4\,h^{ij}\nabla_i\omega_j + 3\,h^{ij}\,\omega_i\omega_j + 4\,X - 2\,\mathcal{R}$
		$J^i = 0$
2.	$\mathcal{L}^{(2)} = R_{\mu\nu}R^{\mu\nu}$	$\widetilde{B} = h^{ij}\,\partial_r\partial_v h_{ij} + h^{ij}\,\omega_i\omega_j + 2\,X$
		$J^i = -\,h^{il}h^{jk}\,\nabla_j\,(\partial_v h_{kl}) + \frac{1}{2}\,h^{il}h^{jk}\nabla_l\,(\partial_v h_{jk}) + h^{il}h^{jk}\,\omega_j\,\left(\partial_v h_{kl}\right) - \frac{1}{2}\,h^{il}h^{jk}\,\omega_l\,\left(\partial_v h_{jk}\right)$
3.	$\mathcal{L}^{(3)} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$	$\widetilde{B} = h^{ij}\,\omega_i\omega_j + 4\,X$
		$J^i = 4\,h^{ij}\,\partial_v\omega_j - 2\,h^{il}h^{jk}\,\nabla_j\,(\partial_v h_{kl})$

Table 4.12: $\widetilde{B}$ and J^i's for different four derivative theory of gravity. While writing the expressions we have used the values for the coefficient A_2 in each of the three cases as following : (i) for R^2 theory: $A_2 = -4$, (ii) for $R_{\mu\nu}R^{\mu\nu}$ theory: $A_2 = 0$, and (iii) for $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ theory: $A_2 = 0$.

4.2.5 Einstein-Gauss-Bonnet gravity in $d \geq (4+1)$

The Einstein-Gauss-Bonnet theory has been extensively studied as a prototype of higher derivative corrections to Einstein's gravity and has been accorded significant importance in the relevant literature. It is also a theory with 4-derivative correction to Einstein's gravity, where the 4-derivative term is a specific combination of the three terms, that has been discussed in §4.2.1, §4.2.1 and §4.2.1. This linear combination is such that, although the Einstein-Hilbert action has 4-derivatives corrections, the equations of motion that follow from it, only have two derivatives on the metric, just like Einstein equations. Since the Gauss-Bonnet term is simply a specific linear combination of the four derivative terms discussed in the previous sections, the analysis for the Einstein-Gauss-Bonnet theory can be done quickly by considering the same linear combination of the results we obtained before. In this section, we state our results explicitly for this theory.

The Einstein-Gauss-Bonnet theory is non-trivial in any dimensions greater than $3+1$. In $3+1$ dimension the 4-derivative term is a total derivative (and is therefore a topological surface term). In lower dimensions, it vanishes as an identity. Let us first consider this theory in space-time dimensions $d \geq 4+1$; we shall discuss the the special case of $d = 3+1$ in the next subsection.

The action for Einstein-Gauss-Bonnet theory is given by

$$I = \int d^d x \sqrt{-g} \left(R + a_{gb} \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right), \tag{4.89}$$

where a_{gb} is a constant Gauss-Bonnet parameter. The corresponding equations of motion are

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + E^{HD}_{\mu\nu} = 0, \tag{4.90}$$

where

$$\begin{aligned} E^{HD}_{\mu\nu} = a_{gb} \Big(& 2RR_{\mu\nu} - 4R^{\alpha\beta}R_{\mu\alpha\nu\beta} - 4R_{\mu}{}^{\alpha}R_{\nu\alpha} + 2R_{\mu}^{\alpha\beta\sigma}R_{\nu\alpha\beta\sigma} \\ & - \frac{1}{2} g_{\mu\nu} (R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\rho}R^{\alpha\beta\gamma\rho}) \Big). \end{aligned} \tag{4.91}$$

The explicit vv-component of the equations of motion is

$$\begin{aligned} E_{vv} &= R_{vv} + E^{\text{HD}}_{vv} = 0, \\ E^{\text{HD}}_{vv} &= a_{gb} \left(2RR_{vv} - 4R^{\alpha\beta}R_{v\alpha v\beta} - 4R^{\alpha}_{v}R_{v\alpha} + 2R_{v}{}^{\alpha\beta\sigma}R_{v\alpha\beta\sigma} \right) \end{aligned} \tag{4.92}$$

By explicitly computing E^{HD}_{vv} in terms of the metric components (4.3) and their derivatives, it is possible to rewrite E_{vv} for the Einstein-Gauss-Bonnet theory into the form (4.37). Subsequently, we can read off the entropy current from it and we have

$$\begin{aligned} J^v &= \left(1 + 2a_{gb}(\mathcal{R} - 2\bar{K}_{AB}K^{AB} + 2K\bar{K}) \right) \\ J^i &= -4a_{gb} \nabla^j \left(K h_{ij} - K_{ij} \right) \end{aligned} \tag{4.93}$$

Note that, this entropy density and spatial entropy current for the Einstein-Gauss-Bonnet theory has been constructed following the philosophy of §4.2.1. In the next subsection we shall do a systematic study of this entropy current, concentrating particularly in $d = (3+1)$ space-time dimensions, where the Gauss-Bonnet term becomes topological.

4.2.6 The Einstein-Gauss-Bonnet theory in $d = 3 + 1$

The Gauss-Bonnet theory in $(3+1)$ space-time dimensions needs a separate discussion. In this case, the Gauss-Bonnet term becomes a total derivative term and therefore it does not contribute to the equations of motion, i.e. $E_{vv}^{\text{HD}} = 0$ identically. However, if one uses the Wald entropy as the equilibrium definition of black hole entropy (4.5), there is a finite non-vanishing contribution to it even from the topological Gauss-Bonnet part of the Lagrangian. The Wald entropy density s_w^{HD} (see (4.42)) for this case, is given by the Ricci scalar of the co-dimension-2 spatial slice of the horizon $\mathcal{H}_v$,

$$s_w^{\text{HD}} = 2\, a_{gb}\, \mathcal{R}\,, \tag{4.94}$$

where a_{gb} is the Gauss-Bonnet parameter appearing in (4.89). Since $\mathcal{H}_v$ in this case is a 2-dimensional manifold, the integrated total entropy S_W becomes the topological Euler number of $\mathcal{H}_v$.

Once we consider dynamical black hole solutions in this theory and restrict ourselves to consider perturbations characterised by small amplitudes around a stationary configurations, the total integrated Wald entropy S_W doesn't change with time as long as the perturbation is small and therefore, does not affect topology of $\mathcal{H}_v$. However, if we consider the local Wald entropy density s_w (without being integrated on the spatial slice $\mathcal{H}_v$), it does indeed change with time and therefore has a non-zero contribution to $\partial_v(\sqrt{h}\,\widetilde{B})$. With these in mind let us look at (4.77), which is the main result of this note and rewrite it here again for convenience

$$E_{vv}^{\text{HD}}\big|_{\text{zero boost}} \sim -\,\partial_v \left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\widetilde{B}\right)\right) - \partial_v\left(\nabla_i J^i\right). \tag{4.95}$$

From the above discussion it is clear that for $(3+1)$ space-time dimensions, the LHS of (4.95) vanishes identically. However, the first term on the RHS is non-zero, making us wonder how to make sense of this equation if we had not included the second term on RHS involving the spatial entropy current. As we will see, both the term on the RHS of the above equation are non-zero but they will precisely cancel each other, and that is how this equation will be satisfied. In other words, we are left with verifying that the RHS. of (4.95) vanishes identically without using any on-shell gravity equations of motion, up to $\mathcal{O}(\epsilon^2)$ corrections.

We start by noting that the values of the coefficients a_i presented in Table-(4.8) are achieved by explicitly computing the $E_{vv}^{\rm HD}$ for different four derivative gravity theories and for our metric choice (4.3), but most importantly, the results are not limited to the space-time dimensions we are working in. Therefore the same results (presented in Table-(4.8)) holds for $(3+1)$-dimensional space-time as well. The specific values of these coefficients for Gauss-Bonnet theory turns out to be the following:

$$\begin{aligned} & a_{16} = 2,\ a_{17} = -1,\ a_i = 0\ (\text{for all } i = 1, \cdots, 15)\,, \\ \text{such that,} \quad & E_{vv}^{\rm HD} = a_{16}\, T_{16} + a_{17}\, T_{17} = 2\left(\mathcal{R}^{ij} - \frac{1}{2} h^{ij}\mathcal{R}\right)\partial_v^2 h_{ij}\,. \end{aligned} \tag{4.96}$$

However, for 2-dimensional space-time one can show that the following relation is identically true,

$$\mathcal{R}^{ij} - \frac{1}{2} h^{ij}\mathcal{R} = 0\,. \tag{4.97}$$

This is true because, in 2-dimensional space we could always choose a coordinate system where the metric is conformally flat and Einstein tensor vanishes on any 2-dimensional conformally flat space-time.

Let us now consider the expression $(1/\sqrt{h})\,\partial_v(\sqrt{h}\,\mathcal{R})$. It is well-known that the linear variation of Ricci scalar around any metric generates a term proportional to the Einstein tensor plus a total derivative term. Because of the fact mentioned above, without doing any further calculation, we could say

$$\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,\mathcal{R}\right) = \left(\mathcal{R}^{ij} - \frac{1}{2} h^{ij}\mathcal{R}\right)(\partial_v h_{ij}) + \nabla_i Z^i\,, \tag{4.98}$$

where Z^i is some spatial current characterising the total derivative term, which could be easily fixed as follows. Using table Table-(4.11) we could find the list of values for A_i, B_i

for Gauss-Bonnet theory

$$A_5 = -2,\; B_2 = 2,\; B_3 = -2, \tag{4.99}$$

leading to the following expression for Z^i

$$Z^i = 2\nabla_j \left(K^{ij} - h^{ij}\, K\right) \tag{4.100}$$

Again in $(3+1)$-dimensions, where $\{i, j\}$ indices run over $\{1, 2\}$, the first term in the RHS of (4.98) identically vanishes. Therefore, we can immediately rewrite (4.98) as

$$\frac{1}{\sqrt{h}}\partial_v \left(\sqrt{h}\,\mathcal{R}\right) - \nabla_i Z^i = 0, \quad \text{(only in } (3+1)\text{-dimensions)}$$

This looks exactly like a divergence of a four-current and identically vanishes in $(3 + 1)$-dimensions. In this particular case of $(3+1)$-dimensional space-time, the above expression has exactly the same status as that of (4.87), or the structure multiplying the coefficient A_2 in the expression of $E^{\rm HD}_{vv}$ (4.82), see also (4.72) and (4.73). In other words, in $(3+1)$-dimension we are free to add $\{\mathcal{R}, Z^i\}$ to the expression of entropy density and spatial entropy current respectively, with any arbitrary overall coefficient. Such an addition will not affect the ultra-local version of the second law or the physical process version of the first law and this is true for all theories as long as we are restricting ourselves to $(3 + 1)$ dimensions. However, just like in case of A_2, the Wald entropy formalism fixes that arbitrary coefficient to a very specific value.

To summarise, the main physical interpretation that one can draw from the arguments presented above is the following. For Gauss-Bonnet theory in $(3 + 1)$ dimensions $E^{\rm HD}_{vv}$ vanishes identically and that is related to the fact that the total integrated Wald entropy S_W is not changing due to time dependent perturbations. This is because the Wald entropy S_W in this case is given by topological Euler number of the 2-dimensional $\mathcal{H}_v$ and we are considering small amplitude approximation for the perturbations, which are too weak to change the topology of $\mathcal{H}_v$. However, even in that approximation, the local change of entropy density is not vanishing. This necessitates the introduction of the idea of a spatial entropy current, that quantifies the inflow or outflow of local entropy density and cancels the change in local entropy density, within any infinitesimal region in $\mathcal{H}_v$. This analysis, at least for the situation considered in this subsection, therefore plays an important role in motivating the need for an spatial entropy current.

4.2.7 Comments on entropy current for higher boost terms in E_{vv}^{HD}

Once we have analysed the zero-boost terms, the next immediate question is to analyse the contribution of higher boost terms of E_{vv}^{HD}, to the entropy current (4.78). As we have observed in §4.1, the arguments in [1] for second law, works smoothly, for all the higher-boost terms in E_{vv}^{HD}. The contribution from these higher boost terms, to the total entropy falls within the class of JKM ambiguities, and they do not contribute to the physical process version of the first law. Unlike the zero-boost terms, nothing necessitates the existence of a spatial component of the current for these higher boost terms. Both the first and second law would remain valid, if we simply declare that these terms would just modify the entropy density as in (4.35), and they do not affect the spatial components of the current. However, the spatial components of the current could still exist, even for the higher boost terms as we now demonstrate.

Before we proceed it is worth clarifying that we will not be doing an exhaustive classification of all such possible higher boost terms in E_{vv}^{HD}. Our aim here is just to present an argument based on analysing a candidate term as an example that justifies the above mentioned statement. We postpone a more detailed study of this aspect to future work.

Schematically, the higher boost terms have the following structure (see (4.24) or (4.62))

$$E_{vv}^{\text{HD}}\Big|_{\text{higher boost}} \sim \partial_v^2 \left[\partial_r^k A^{(k)}\, \partial_v^k B^{(k)}\right] + \mathcal{O}\left(\epsilon^2\right) , \tag{4.101}$$

where $A^{(k)}$ and $B^{(k)}$ are boost-invariant. Now it turns out that the same higher boost term could be recast in different ways, upto corrections that are quadratic or higher order, in the amplitude of the dynamics. This allows us to absorb certain higher boost terms (the ones that have at least one ∇_i) either entirely within the correction to entropy density, or partially in entropy density and partially in the spatial components of the current. Let us explain this ambiguity more specifically.

Consider a typical higher boost term in $E_{vv}^{\rm HD}$, as in (4.101), where the term $\partial_v^k B^{(k)}$ could be expressed as divergence of a spatial current with boost-weight $k > 1$,

$$\partial_v^k B^{(k)} \sim \vec{\nabla} \cdot \vec{J}^{(k)},$$

and substituting it in the expression of $E_{vv}^{\rm HD}$ we find

$$E_{vv}^{\rm HD}\big|_{\text{higher boost}} \sim \partial_v^2 \left[\partial_r^k A^{(k)}\, \vec{\nabla}\cdot\vec{J}^{(k)}\right] \tag{4.102}$$

$$= \partial_v\left(\vec{\nabla}\cdot\left[\partial_v\left(\vec{J}^{(k)}\,\partial_r^k A^{(k)}\right)\right]\right) + \partial_v^2\left[-\vec{J}^{(k)}\cdot\vec{\nabla}\left(\partial_r^k A^{(k)}\right)\right] + O\left(\epsilon^2\right) . \tag{4.103}$$

On one hand, from the first line of (4.102), we can conclude that the contribution of this term to entropy current is simply [30]

$$\text{From (4.102):}\quad S^v_{k\geq 1} = -\,\partial_r^k A^{(k)}\;\; \vec{\nabla}\cdot\vec{J}^{(k)}, \quad S^i_{k\geq 1} = 0\,. \tag{4.104}$$

with no spatial current. On the other hand, from the second line (4.103), we may infer that, this term contributes to the spatial components of the entropy current, apart from the contribution to the entropy density, which is different from the previous case (4.104). That is, we can write the contribution to entropy current also in the following way

$$\text{From (4.103):}\quad S^v_{k\geq 1} = \vec{J}^{(k)}\cdot\vec{\nabla}\left(\partial_r^k A^{(k)}\right), \quad \vec{S}_{k\geq 1} = -\,\partial_v\left(\vec{J}^{(k)}\,\partial_r^k A^{(k)}\right) . \tag{4.105}$$

We would like to emphasize that the above manipulation, which is essentially an interchange of ∂_v and $\vec{\nabla}$, crucially uses the fact that any term, generated due to the non zero commutator

[30] The quantity S^A has been introduced in (4.78). The subscript $k \geq 1$ in S^A is to denote that this is the contribution from the higher boost terms in $E_{vv}^{\rm HD}$.

of these two types of derivatives, would be of higher order in amplitude. This is because the commutator itself is of boost weight one, for higher boost terms,

$$\partial_v(\vec{\nabla}.\vec{J}^k) \sim \vec{\nabla}.(\partial_v \vec{J}^k) + (\partial_v \Gamma^i_{ij})\, J^{(k)^j} \sim \vec{\nabla}.(\partial_v \vec{J}^k) + \mathcal{O}(\epsilon^2)\,. \tag{4.106}$$

Also note that, this is true only for the higher-boost terms, and in particular, not true for the boost invariant terms, for which the presence non-zero spatial entropy current was unambiguous.

The two different choices of entropy density in (4.104) and (4.105), are related by a total spatial derivative, as expected. This ensures that, in the integrated (weak) version of the second law, this difference would have no impact. However, in the ultra local version, where we demand the entropy to be produced at every point in space and time, this difference is significant. This leads to an ambiguity in the definition of our entropy current, which cannot be fixed, merely from the transformation property of $E^{\rm HD}_{vv}$ under boost (4.13).

It is possible that, if we keep track of the higher order terms in amplitude expansion, this ambiguity may be removed. Alternatively, it is also possible that some suitable extension of the boost symmetry, like (4.11), which preserves our global choice of coordinates, might constraint the structure of our entropy current further, and consequently fix this ambiguity. We would like to explore this point further in our future work.

4.3 Discussions and Future directions

In this note, we have demonstrated that the intricacies in the arguments involved in the proof of the physical process version of the first law, and the second law, naturally lead us to the notion of a spatial entropy current on the horizon. This spatial entropy current captures the inflow or outflow of entropy from any subregion of $\mathcal{H}_v$ - the horizon v-slice. For most of our analysis in this note, we consider dynamical black holes which can be treated within the linearized approximation, where the amplitude of the 'time' dependent metric fluctuations, about a given stationary black hole solution, is small. Under this approximation, we are able to establish that the entropy density and the spatial components of the entropy current, constructed through our algorithm, satisfies an ultra-local stronger version of the second law of black hole thermodynamics. The validity of this local form of the second law is ensured by the equations of motion for the higher derivative theories of gravity, and therefore, true for any metric that solves these classical equations, at the linearized level.

The construction of our entropy current is not unique. All the ambiguities in defining the current can be traced back to the fact that there exist certain terms $\mathcal{T}_{\text{amb}}$ which can be simultaneous written in two ways. We can write $\mathcal{T}_{\text{amb}} = \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{J}^v\right)$, but also we can write the same term as $\mathcal{T}_{\text{amb}} = -\nabla_i\mathcal{J}^i$, for some choice of $\mathcal{J}^v$ and some choice of $\mathcal{J}^i$. Obviously, it follows that $\mathcal{J}^v$ must have at least one spatial derivative, while J^i must have at least one v-derivative. If we have such terms, appearing in the equation of motion as $\partial_v\mathcal{T}_{\text{amb}}$, then it becomes unclear whether to write it as $\mathcal{T}_{\text{amb}} = \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{J}^v\right)$ and consider it to be a part of the entropy density. Or to write it as $\mathcal{T}_{\text{amb}} = -\nabla_i\mathcal{J}^i$ and interpret it to be being a part of the spatial components of the entropy current. A third possibility is to split this term up, into the entropy density and the entropy current. In §4.2.7, we have discussed these kind of ambiguities in detail.

Again, if we indeed have terms like $\mathcal{T}_{\text{amb}}$ which can be written in both these ways, we can always add a $0 = \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{J}^v\right) + \nabla_i\mathcal{J}^i\right)$, to the equation of motion, and subsequently include $\mathcal{J}^v$ and $\mathcal{J}^i$ into the definition of the entropy density and the spatial entropy current. Neither the equation of motion nor any of the laws of thermodynamics would be affected by this operation. This kind of ambiguity arises, for example, in the Einstein-Gauss-Bonnet theory in $3+1$ dimensions, discussed in detail in §4.2.6 [31]. If a term like $\mathcal{T}_{\text{amb}}$ is such that, $\mathcal{J}^v$ is non-zero on stationary solutions, then it would contribute to Wald entropy as well. In such a case, the ambiguity corresponding to this term may be

[31] Also see the ambiguity related to the parameter A_2, discussed in detail in §4.2.4

removed by demanding that our entropy reduces to Wald entropy on stationary solutions. But if $\mathcal{J}^v$ vanishes in equilibrium then this additional criterion would remain ineffective in fixing it.

It should also be noted that it may be possible to write down a particular term simultaneously in both the forms, only at the linearized order in perturbations. Such an equivalence may cease to be true once we proceed to consider corrections which are higher-order in amplitudes. In that case, these ambiguities would only be a linear order artefact, and would disappear once we are able to construct the full non-linear current. However, some of these ambiguities of the entropy current may remain, even in the full non-linear construction.

Having highlighted the ambiguities of the entropy density and the corresponding current, we must point out that, every member of this ambiguous class, have the property that the total entropy reduces to Wald entropy for stationary black holes. For the non-stationary dynamical black holes, all these entropy density and currents also satisfy a local second law. Hence, every such entropy density and entropy current are perfectly well defined macroscopic entities that can provide excellent effective thermodynamics description of the system. Some additional microscopic information is likely to make one of them special, and it can stand out as the correct definition of entropy density and entropy current away from equilibrium. Therefore, despite these ambiguities, it appears to us, that the notion of the spatial components of the entropy current and a local second law on a dynamical horizon is a concept of significant importance in the thermodynamic description of black holes.

This note is essentially a series of observations, on the evolution of black hole entropy in dynamical scenarios, in a specific set of examples of higher derivative theories of gravity. Through explicit calculations, we have been able to test our hypothesis about the spatial components of the entropy current, only in four derivative theories of gravity. This is a small step towards formulating an ultra-local version of the second law in gravitational theories (if it exists in full non-perturbative sense) and deciphering all its physical ramifications.

Clearly, there are several directions in which this work needs to be extended, so that a more complete picture of this whole mechanism may emerge. Here is a brief list of related questions, which we would like to investigate in the near future.

1. The reader may have noticed that, throughout this note, we have used the word 'time' under a quotation mark. This is because our 'time' here is not really a parameter along a time-like vector field; rather, it is the affine parameter along a 'distinguished' null direction that generates the event horizon. Therefore, the expression appearing in the local version of the second law is not the $d+1$ dimensional [32] covariant divergence of a covariant current. This is quite unlike the standard way in which the local version of the second law is expressed, for near-equilibrium dynamics of non-gravitational theories, where the $d+1$ dimensional Lorentz covariance is manifestly maintained.

   On the event horizon, we do not have a time-like direction, so the question of Lorenz invariance does not arise here. However, our construction has used a specific choice of coordinates and physically we expect some form of invariance should exist once we choose a different coordinate system - for example, a different spatial slicing of the null generators.

   It would be extremely important to explore whether any such invariance exists and if it exists, then how does it control our construction.

2. Another question related to the above is as follows. We have seen that our construction of the spatial entropy current mainly involves the 'zero boost terms' in the equation of motion. For this construction to work, these 'zero boost terms' were required to have a specific form. This requirement may be viewed as a set of constraints on the most general structure of the relevant component of the equation of motion (see

[32] Remember if we are working in $D+1$ dimensional space-time, then the current is expected to have $d+1$ components, with $d = D-1$. This is because this current would be defined on the event horizon, which is a co-dimension one surface.

§4.2.3 and §4.2.4). The physical origin of these constraints is, at the moment, unclear. We suspect that the reason behind these constraints could be the set of residual gauge invariance, expressed in (4.11), which is a generalization of the boost symmetry (4.13). Whether this suspicion is true, or there is a completely different reason for these constraints must be investigated through explicit computations.

3. The 'zero boost terms' in the equation of motion, which are central to our analysis in this note, are also relevant for the physical process version of the first law, and hence, control the definition of entropy in stationary situations.

   Now, it is well known that Wald's formalism [11, 12] also determines this same equilibrium entropy in a covariant fashion using the conserved Noether current corresponding to the diffeomorphism symmetry. It would be extremely interesting to clearly establish a connection between these two methods. In particular, if it is possible to identify our spatial current within Wald's construction, it would probably lead to a more satisfying covariant construction of the entropy current. This may help us arrive at an abstract proof for the existence of this entropy current and the local second law, for any higher derivative theory of gravity.

   In absence of any such concrete proof, it would be quite useful to gather more data, simply by repeating the exercise presented in this note, for theories of gravity with 6 or more derivatives.

4. Another obvious generalization would be to extend our construction to non-linear order in amplitude ϵ. This can potentially fix the ambiguity related to the construction of the spatial components of the entropy current, which arises for higher boost ($k \geq 0$) terms (see the discussion in §4.2.7).

   But more importantly, it might provide us with further insights, which can help us formulate a non-perturbative proof of second law for higher derivative theories. For Einstein's theory, entropy production is ensured by the famous 'horizon area increase theorem', which is proved for any dynamical situation, in full non-perturbative way. It would be nice to have a similar proof (or a clean counter-example) for higher derivative theories of gravity.

5. Naively, it might seem that, at non-linear order, we do not have to worry about the second law, since for Einstein's theory itself, the entropy production takes place at quadratic order in amplitude. Now because higher derivative corrections are always suppressed compared to the leading order piece corresponding to Einstein's theory, they cannot reverse the sign which guarantees entropy production.

   However, if we are interested in an ultra-local form, then during a non-trivial 'time' evolution, the contribution to entropy due to Einstein's theory could vanish locally at a given point. Then, for the question of entropy production and the second law, we must take the higher-derivative corrections seriously. See [36] for the construction of entropy in dynamical black holes for the Einstein-Gauss-Bonnet theory, where these subtle issues have been addressed. The construction of entropy in [36] did not yield a second law, for the most generic dynamical situation. But, in [36] this idea of a spatial entropy current was not used. It would be very interesting to revisit [36], and check if the obstruction is resolved when the spatial entropy current is incorporated into the statement of the second law.

6. Within the framework of gauge gravity duality, a precise correspondence exists between slowly varying fluctuations of a black hole and the hydrodynamic fluctuations of the boundary fluid. Since the boundary fluid dynamics comes equipped with a local entropy current, there exists a dual of this current, for the black hole in the bulk [47, 57, 58]. This dual also constitutes a gravitational entropy current, in this par-

ticular context. In [47], the construction has been done for two derivative Einstein's theories. While in [57, 58], it has been extended to higher derivative theories of gravity, following Wald's formalism of Noether current. All these constructions use the derivative expansion extensively and their validity is restricted to this particular case of fluid-gravity correspondence. Therefore, although these constructions of the entropy current relate to horizon dynamics, it subtly uses the asymptotic AdS conditions, which ensures black-brane solutions exist and the fluid-gravity correspondence could be formulated in a clean fashion. For example, the entropy current constructed in these papers is clearly a (3+1) dimensional current (for 5 bulk dimension) with one component (the entropy density) clearly along with a time-like direction. This is achieved by lifting the null coordinate along the horizon, to the time-like direction of the boundary through the fluid-gravity map. This time-like direction also serves to formulate an unambiguous statement of the second law, in terms of the divergence of this entropy current.

On the other hand, our construction is completely confined to the horizon, it does not have any time-like direction, to begin with. As we have explained before, in absence of any Lorentz symmetry it is not straight-forward to interpret our result as a covariant 'four'-current. Also, we do not need any assumption about the asymptotic structure of spatial infinity.

Our construction looks quite different from what has been done in [47, 57, 58]. But it is also clear that there must be some relation between these two constructions. This question is a topic of our ongoing investigation.

7. Very recently, one candidate entropy current has been constructed in [59], for Gauss-Bonnet theory within the framework of membrane-gravity duality, in an expansion in inverse powers of space-time dimension D. This is a duality that gives a pre-

cise correspondence between the dynamics of a membrane (a time-like hypersurface, embedded in flat space-time) and that of the horizon, in the large D expansion. Unlike our construction, which does not rely on any duality, in [59] the entropy current has been constructed in the dual picture of the membrane. Their entropy current is entirely confined within the membrane and has the usual property of Lorentz invariance. In their case, the non-negative divergence of the entropy current follows from the membrane dynamics governed by those membrane equations, which have been derived from the dual gravity picture.

Moreover, within their approximation, the authors of [59] have also shown that the existence of a Killing vector is a consequence of no entropy production. They have also demonstrated that the charge corresponding to this conserved entropy current reduces to the well-known expression of Wald entropy, in a stationary situation.

It would be extremely interesting to see how the entropy current of [59] compares with ours. In particular, we would like to explore, if this membrane-gravity duality can be used to formulate a principle which can fix the ambiguities of the entropy density and spatial entropy current.

8. We have constructed a proof of the 2nd law by introducing an Entropy Current for four derivative theories of gravity. A natural question comes to mind the generalization of this proof to any arbitrary diffeomorphism invariant theory of gravity. In this paper [32], the authors have addressed this question and constructed a proof of the 2nd law for any arbitrary diffeomorphism invariant theories of gravity. In our work, we have calculated a particular component of the equations of motion for four derivative theories of gravity, then we found the entropy current arranging the terms that appeared in the particular component of the equation of motion. It is a brute-force calculation. But in this work[32], the authors classified the terms that can appear in

that particular component of the equations of motion by analyzing the boost weight. Then they showed the existence of an entropy current on the horizon to have a consistent 2nd law. By constructions, this entropy current is divergenceless and gives an ultra-local version of the second law. This proof is still in the regime where the perturbation is linearized around a stationary black hole.

Chapter 5

Conclusion

In this thesis, we have made some progress in the understanding of black hole mechanics; particularly, we focused on the Zeroth Law and the second Law in the presence of higher derivative corrections to the usual Einstein-Hilbert action. As we said earlier, the thermodynamic properties of black holes become difficult to comprehend in the presence of higher derivative corrections; one probable and profound reason is that the Raychowdhary equations are not well understood beyond General Relativity. In this work, we attempted to find the status of the black hole thermodynamics in the presence of higher derivative corrections.

First, we looked for a zeroth Law of black hole mechanics. We believe we have given a proof of the zeroth law for generic higher derivative theories of gravity with some reasonable physical assumptions. The proof includes an inductive method where we assume that the higher derivative corrections have a smooth limit to General Relativity and also we choose some specific but most general gauge conditions. In other words, the black hole solutions have a smooth limit to the solutions of General Relativity. The Zeroth law is valid in General Relativity. If so, our induction method works order by order in the coupling of the higher derivatives leading to a proof of Zeroth law for generic higher derivative theories of gravity.

The second part of the thesis deals with the second Law. The first law has already been proven for generic higher derivative theories so we are analyzing the second law. In the case of the second Law, we just concentrated on a specific higher derivative theory, the four-derivative theory. But as we know, the second law is a dynamical statement. We have to go beyond stationary metric. We make our stationary system non-stationary by throwing some matter into the system, and the metric becomes dynamical. We have introduced Entropy current to achieve a proof of the Second Law. We have used the same setups and gauge conditions as we used before to prove Zeroth law. We have considered linear perturbation around a stationary black hole in our study of the second law. The proofs of the second law that are known are an integrated version, but our proof is an ultra-local version of the second law. Entropy Current, which is a (d-1)-dimensional vector field defined on the horizon, played a crucial role in achieving this ultra-local version of the second law. The divergence of the entropy current is zero, giving us the ultra-local version of the second law for four derivative theories of gravity. The work of the thesis on the second law is still in the realm of linearization around a stationary black hole, but the authors of this paper[61], took a step toward a non-linear region but treated the higher derivative terms as an Effective field theory. We always assumed that the matter sector obeys the null energy condition. But if we consider the matter sector coupled non-minimally to gravity, the null energy condition does not hold anymore. In [62], the authors investigated if the second law is still valid or not in the case where the matter sector is coupled with gravity non-minimally. They found that, indeed, the theory satisfies the second law, and they found the entropy current for that theory.

References

[1] A. C. Wall, *A Second Law for Higher Curvature Gravity*, *Int. J. Mod. Phys.* **D24** (2015), no. 12 1544014, [arXiv:1504.08040].

[2] J. Maldacena, *The large-n limit of superconformal field theories and supergravity*, *International journal of theoretical physics* **38** (1999), no. 4 1113–1133.

[3] A. A. Starobinsky, *A new type of isotropic cosmological models without singularity*, *Physics Letters B* **91** (1980), no. 1 99–102.

[4] D. G. Boulware and S. Deser, *String-generated gravity models*, *Physical Review Letters* **55** (1985), no. 24 2656.

[5] R. C. Myers, *Black holes in higher curvature gravity*, in *Black Holes, Gravitational Radiation and the Universe*, pp. 121–136. Springer, 1999.

[6] J. Z. Simon, *Higher-derivative lagrangians, nonlocality, problems, and solutions*, *Physical Review D* **41** (1990), no. 12 3720.

[7] J. Z. Simon, *Stability of flat space, semiclassical gravity, and higher derivatives*, *Physical Review D* **43** (1991), no. 10 3308.

[8] S. W. Hawking, *Gravitational radiation from colliding black holes*, *Phys. Rev. Lett.* **26** (1971) 1344–1346.

[9] J. M. Bardeen, B. Carter, and S. W. Hawking, *The Four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161–170.

[10] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333–2346.

[11] R. M. Wald, *Black hole entropy is the noether charge*, *Phys. Rev. D* **48** (Oct, 1993) R3427–R3431.

[12] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D50** (1994) 846–864, [gr-qc/9403028].

[13] A. C. Wall, *A Survey of Black Hole Thermodynamics*, arXiv:1804.10610.

[14] S. Sarkar, *Black Hole Thermodynamics: General Relativity and Beyond*, *Gen. Rel. Grav.* **51** (2019), no. 5 63, [arXiv:1905.04466].

[15] T. Jacobson and R. C. Myers, *Black hole entropy and higher curvature interactions*, *Phys. Rev. Lett.* **70** (1993) 3684–3687, [hep-th/9305016].

[16] T. Jacobson, G. Kang, and R. C. Myers, *On black hole entropy*, *Phys. Rev.* **D49** (1994) 6587–6598, [gr-qc/9312023].

[17] T. Jacobson, G. Kang, and R. C. Myers, *Increase of black hole entropy in higher curvature gravity*, *Phys. Rev.* **D52** (1995) 3518–3528, [gr-qc/9503020].

[18] S. W. Hawking, *Black holes in general relativity*, *Communications in Mathematical Physics* **25** (1972), no. 2 152–166.

[19] I. Rácz and R. M. Wald, *Global extensions of spacetimes describing asymptotic final states of black holes*, *Classical and Quantum Gravity* **13** (mar, 1996) 539–552.

[20] Y. Xie, J. Zhang, H. O. Silva, C. de Rham, H. Witek, and N. Yunes, *Square Peg in a Circular Hole: Choosing the Right Ansatz for Isolated Black Holes in Generic Gravitational Theories*, *Phys. Rev. Lett.* **126** (2021), no. 24 241104, [arXiv:2103.03925].

[21] R. Ghosh and S. Sarkar, *Black Hole Zeroth Law in Higher Curvature Gravity*, *Phys. Rev. D* **102** (2020), no. 10 101503, [arXiv:2009.01543].

[22] S. Sarkar and S. Bhattacharya, *Issue of zeroth law for Killing horizons in Lanczos-Lovelock gravity*, *Phys. Rev. D* **87** (2013), no. 4 044023, [arXiv:1205.2042].

[23] A. Sang and J. Jiang, *Black hole zeroth law in the Horndeski gravity*, *Phys. Rev. D* **104** (2021), no. 8 084092, [arXiv:2110.00903].

[24] S. Sarkar, *Black hole thermodynamics: general relativity and beyond*, *General Relativity and Gravitation* **51** (2019), no. 5 1–28.

[25] S. Bhattacharyya, P. Biswas, A. Dinda, and N. Kundu, *The zeroth law of black hole thermodynamics in arbitrary higher derivative theories of gravity*, *arXiv preprint arXiv:2205.01648* (2022).

[26] S. W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199–220. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[27] S. Sarkar and A. C. Wall, *Generalized second law at linear order for actions that are functions of Lovelock densities*, *Phys. Rev.* **D88** (2013) 044017, [arXiv:1306.1623].

[28] S. Bhattacharjee, S. Sarkar, and A. C. Wall, *Holographic entropy increases in quadratic curvature gravity*, *Phys. Rev.* **D92** (2015), no. 6 064006, [arXiv:1504.04706].

[29] S. Bhattacharjee, A. Bhattacharyya, S. Sarkar, and A. Sinha, *Entropy functionals and c-theorems from the second law*, *Phys. Rev.* **D93** (2016), no. 10 104045, [arXiv:1508.01658].

[30] A. C. Wall, *A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices*, *Phys. Rev.* **D85** (2012) 104049, [arXiv:1105.3445]. [erratum: Phys. Rev.D87,no.6,069904(2013)].

[31] J. Bhattacharya, S. Bhattacharyya, A. Dinda, and N. Kundu, *An entropy current for dynamical black holes in four-derivative theories of gravity*, *JHEP* **06** (2020) 017, [arXiv:1912.11030].

[32] S. Bhattacharyya, P. Dhivakar, A. Dinda, N. Kundu, M. Patra, and S. Roy, *An entropy current and the second law in higher derivative theories of gravity*, *JHEP* **09** (2021) 169, [arXiv:2105.06455].

[33] S. Bhattacharyya, P. Jethwani, M. Patra, and S. Roy, *Reparametrization Symmetry of Local Entropy Production on a Dynamical Horizon*, arXiv:2204.08447.

[34] R.-G. Cai, *Gauss-bonnet black holes in AdS spaces*, *Physical Review D* **65** (mar, 2002).

[35] L. Ma, Y.-Z. Li, and H. Lü, *$D = 5$ rotating black holes in einstein-gauss-bonnet gravity: mass and angular momentum in extremality*, *Journal of High Energy Physics* **2021** (jan, 2021).

[36] S. Bhattacharyya, F. M. Haehl, N. Kundu, R. Loganayagam, and M. Rangamani, *Towards a second law for Lovelock theories*, *JHEP* **03** (2017) 065, [arXiv:1612.04024].

[37] K. Hajian, S. Liberati, M. Sheikh-Jabbari, and M. Vahidinia, *On black hole temperature in horndeski gravity*, *Physics Letters B* **812** (2021) 136002.

[38] X. O. Camanho, J. D. Edelstein, J. Maldacena, and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, *JHEP* **02** (2016) 020, [arXiv:1407.5597].

[39] S. W. Hawking, *Gravitational radiation from colliding black holes*, *Phys. Rev. Lett.* **26** (May, 1971) 1344–1346.

[40] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (Apr, 1973) 2333–2346.

[41] S. W. Hawking and J. B. Hartle, *Energy and angular momentum flow into a black hole*, *Communications in Mathematical Physics* **27** (Dec, 1972) 283–290.

[42] R. M. Wald, *General relativity*. Chicago Univ. Press, Chicago, IL, 1984.

[43] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*. Chicago Lectures in Physics. University of Chicago Press, Chicago, IL, 1995.

[44] M. Blau, *Lecture notes on general relativity*. http://www.blau.itp.unibe.ch/Lecturenotes.html.

[45] S. Sarkar and A. C. Wall, *Second Law Violations in Lovelock Gravity for Black Hole Mergers*, *Phys. Rev.* **D83** (2011) 124048, [arXiv:1011.4988].

[46] S. Chatterjee and M. Parikh, *The second law in four-dimensional Einstein-Gauss-Bonnet gravity*, *Class. Quant. Grav.* **31** (2014) 155007, [arXiv:1312.1323].

[47] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita, M. Rangamani, and H. S. Reall, *Local Fluid Dynamical Entropy from Gravity*, *JHEP* **06** (2008) 055, [arXiv:0803.2526].

[48] S. Bhattacharyya, *Constraints on the second order transport coefficients of an uncharged fluid*, *JHEP* **07** (2012) 104, [arXiv:1201.4654].

[49] S. Bhattacharyya, *Entropy Current from Partition Function: One Example*, *JHEP* **07** (2014) 139, [arXiv:1403.7639].

[50] S. Bhattacharyya, *Entropy current and equilibrium partition function in fluid dynamics*, *JHEP* **08** (2014) 165, [arXiv:1312.0220].

[51] S. Gao and R. M. Wald, *The 'Physical process' version of the first law and the generalized second law for charged and rotating black holes*, *Phys. Rev.* **D64** (2001) 084020, [gr-qc/0106071].

[52] A. J. Amsel, D. Marolf, and A. Virmani, *The Physical Process First Law for Bifurcate Killing Horizons*, *Phys. Rev.* **D77** (2008) 024011, [arXiv:0708.2738].

[53] S. Bhattacharjee and S. Sarkar, *Physical process first law and caustic avoidance for Rindler horizons*, *Phys. Rev.* **D91** (2015), no. 2 024024, [arXiv:1412.1287].

[54] A. Mishra, S. Chakraborty, A. Ghosh, and S. Sarkar, *On the physical process first law for dynamical black holes*, *JHEP* **09** (2018) 034, [arXiv:1709.08925].

[55] A. Chatterjee and S. Sarkar, *Physical process first law and increase of horizon entropy for black holes in Einstein-Gauss-Bonnet gravity*, *Phys. Rev. Lett.* **108** (2012) 091301, [arXiv:1111.3021].

[56] S. Kolekar, T. Padmanabhan, and S. Sarkar, *Entropy Increase during Physical Processes for Black Holes in Lanczos-Lovelock Gravity*, *Phys. Rev.* **D86** (2012) 021501, [arXiv:1201.2947].

[57] S. Chapman, Y. Neiman, and Y. Oz, *Fluid/Gravity Correspondence, Local Wald Entropy Current and Gravitational Anomaly*, *JHEP* **07** (2012) 128, [arXiv:1202.2469].

[58] C. Eling, A. Meyer, and Y. Oz, *Local Entropy Current in Higher Curvature Gravity and Rindler Hydrodynamics*, *JHEP* **08** (2012) 088, [arXiv:1205.4249].

[59] Y. Dandekar and A. Saha, *Large D membrane for Higher Derivative Gravity and Black Hole Second Law*, arXiv:1910.10964.

[60] S. Kolekar, T. Padmanabhan, and S. Sarkar, *Entropy increase during physical processes for black holes in lanczos-lovelock gravity*, *Phys. Rev. D* **86** (Jul, 2012) 021501.

[61] S. Hollands, Á. D. Kovács, and H. S. Reall, *The second law of black hole mechanics in effective field theory*, *Journal of High Energy Physics* **2022** (2022), no. 8 1–63.

[62] P. Biswas, P. Dhivakar, and N. Kundu, *Non-minimal coupling of scalar and gauge fields with gravity: an entropy current and linearized second law*, *Journal of High Energy Physics* **2022** (2022), no. 12 1–56.

Appendix A

(For Chapter-3)

A.1 Computing the Christoffel symbols and the surface gravity for the metric eq.(3.4)

A.1.1 Computing the Christoffel symbols

In this appendix, we will calculate the Christoffel symbols for the metric eq.(3.4) upto order $\mathcal{O}(\alpha)$.

$$ds^2 = 2d\tau\, d\rho - \rho\left[X^{(0)}(\rho,x^i) + \alpha X^{(1)}(\rho,x^i)\right]d\tau^2 + 2\rho\left[\omega_i^{(0)}(\rho,x^i) + \alpha\omega_i^{(1)}(\rho,x^i)\right]d\tau dx^i$$
$$+\left[h_{ij}^{(0)}(\rho,x^i) + \alpha h_{ij}^{(1)}(\rho,x^i)\right]dx^i dx^j \tag{A.1}$$

Different components of the metric are

$$g_{\tau\tau} = -\rho\left[X^{(0)}(\rho,x^i) + \alpha X^{(1)}(\rho,x^i)\right],\;\; g_{\tau\rho} = 1,\;\; g_{\tau i} = \rho\left[\omega_i^{(0)}(\rho,x^i) + \alpha\omega_i^{(1)}(\rho,x^i)\right],$$
$$g_{\rho\rho} = 0,\;\; g_{\rho i} = 0,\;\; g_{ij} = \left[h_{ij}^{(0)}(\rho,x^i) + \alpha h_{ij}^{(1)}(\rho,x^i)\right] \tag{A.2}$$

Different components of the inverse metric up to order $\mathcal{O}(\alpha)$ are

$$\begin{aligned} & g^{\tau\tau} = 0,\;\; g^{\tau\rho} = 1,\;\; g^{\tau i} = 0, \\ & g^{\rho\rho} = \rho\left[X^{(0)} + \alpha X^{(1)}\right] + \rho^2 h_{(0)}^{ij}\omega_i^{(0)}\omega_j^{(0)} + \alpha\rho^2\left[2h_{(0)}^{ij}\omega_i^{(0)}\omega_j^{(1)} - h_{(1)}^{ij}\omega_i^{(0)}\omega_j^{(0)}\right] \\ & g^{\rho i} = -\rho\left[h_{(0)}^{ij}\omega_j^{(0)} + \alpha\left(h_{(0)}^{ij}\omega_j^{(1)} - h_{(1)}^{ij}\omega_j^{(0)}\right)\right],\;\; g^{ij} = h_{(0)}^{ij} - \alpha h_{(1)}^{ij} \end{aligned} \tag{A.3}$$

where, $h_{(0)}^{ij}$ is defined as $h_{(0)}^{ik}h_{kj}^{(0)} = \delta_j^i$ and $h_{(1)}^{ij}$ is defined as $h_{(1)}^{ij} = h_{(0)}^{im}h_{(0)}^{jn}h_{mn}^{(1)}$.

Now we will compute different components of Christoffel symbols. We would require the expression of one component of the Christoffel symbol $\Gamma^{\rho}_{i\tau}$ off the horizon. The expression of $\Gamma^{\rho}_{i\tau}$ up to order $\mathcal{O}(\rho)$ is

$$\Gamma^{\rho}_{i\tau} = -\frac{\rho}{2}\partial_i\left(X^{(0)} + \alpha X^{(1)}\right) - \frac{1}{2}\rho\left(X^{(0)}\omega_i^{(0)} + \alpha X^{(1)}\omega_i^{(0)} + \alpha X^{(0)}\omega_i^{(1)}\right) \tag{A.4}$$

The rest of the components are on the horizon

$$\Gamma^{\rho}_{\rho\tau} = -\frac{1}{2}\left[X^{(0)} + \alpha X^{(1)}\right], \;\; \Gamma^{\tau}_{i\tau} = -\frac{1}{2}\left[\omega^{(0)}_i + \alpha\omega^{(1)}_i\right], \;\; \Gamma^{\rho}_{\rho j} = \frac{1}{2}\left[\omega^{(0)}_i + \alpha\,\omega^{(1)}_i\right],$$
$$\Gamma^{j}_{i\tau} = 0, \;\; \Gamma^{j}_{\rho\tau} = \frac{1}{2}\left[\omega^{j}_{(0)} + \alpha\,\omega^{j}_{(1)} - \alpha h^{jk}_{(1)}\omega^{(0)}_k\right], \;\; \Gamma^{\rho}_{ij} = 0, \;\; \Gamma^{\tau}_{\tau\tau} = \frac{1}{2}\left[X^{(0)} + \alpha X^{(1)}\right],$$
$$\Gamma^{\tau}_{\tau\rho} = 0, \;\; \Gamma^{\tau}_{i\rho} = 0, \;\; \Gamma^{j}_{\tau\tau} = 0, \;\; \Gamma^{\tau}_{ij} = -\frac{1}{2}\left(\partial_\rho h^{(0)}_{ij} + \alpha\partial_\rho h^{(1)}_{ij}\right), \;\; \Gamma^{\rho}_{\tau\tau} = 0, \;\; \Gamma^{i}_{\tau\tau} = 0, \tag{A.5}$$

Where, $\omega^i_{(0)}$ and $\omega^i_{(1)}$ are defined as $\omega^i_{(0)} = h^{ij}_{(0)}\omega^{(0)}_j$ and $\omega^i_{(1)} = h^{ij}_{(0)}\omega^{(1)}_j$

A.1.2 Computing the surface gravity

The metric of the space-time is given in eq.(3.4) and we write it here again for convenience

$$ds^2 = 2d\tau\, d\rho - \rho X(\rho, x^i)d\tau^2 + 2\rho\,\omega_i(\rho, x^i)d\tau dx^i + h_{ij}(\rho, x^i)dx^i dx^j \tag{A.6}$$

This metric admits a Killing vector $\xi = \partial_\tau$ with the horizon being chosen to be at $\rho = 0$. The definition of surface gravity is given by

$$\kappa = \sqrt{-\frac{1}{2}(\nabla_\mu\xi_\nu)(\nabla^\mu\xi^\nu)}\,\Bigg|_{\rho=0}\,. \tag{A.7}$$

We use the inverse metric expressions written in eq.(A.3) to obtain the following components of ξ_μ,

$$\xi_\rho = 1, \; \xi_\tau = -\rho\, X(\rho, x^i), \; \xi_i = \rho\,\omega_i(\rho, x^i)\,. \tag{A.8}$$

Next we compute the components of $\nabla_\mu\xi_\nu$ evaluated on $\rho = 0$ and the non-vanishing components are as follows

$$\begin{aligned}\nabla_\tau\xi_\rho|_{\rho=0} &= -\nabla_\rho\xi_\tau|_{\rho=0} = -\frac{1}{2}\,X(\rho = 0, x^i)\,,\\ \nabla_\rho\xi_i|_{\rho=0} &= -\nabla_i\xi_\rho|_{\rho=0} = \frac{1}{2}\,\omega_i(\rho = 0, x^i)\,,\end{aligned} \tag{A.9}$$

Using these, we obtain

$$(\nabla_\mu\xi_\nu)(\nabla^\mu\xi^\nu)\,|_{\rho=0} = 2g^{\tau\rho}g^{\tau\rho}\,(\nabla_\tau\xi_\rho)\,(\nabla_\rho\xi_\tau)\,|_{\rho=0} = -\frac{1}{2}\,X^2(\rho = 0, x^i) \tag{A.10}$$

Finally, we obtain the surface gravity as the following

$$\kappa = \frac{1}{2} X(\rho, x^i)\Big|_{\rho=0} \, . \tag{A.11}$$

A.2 Few details regarding the boost weight of covariant tensors

In this appendix we aim to provide some more detail regarding the boost invariance of the stationary metric written in eq.(3.10). We write the metric here again for convenience,

$$ds^2 = \tilde{g}^{(bh)}_{\mu\nu}\, dx^\mu dx^\nu = 2\, dv\, dr - r^2\, X(rv,\, x^i)\, dv^2 + 2\, r\, \omega_i(rv,\, x^i)\, dv\, dx^i + h_{ij}(rv,\, x^i)\, dx^i\, dx^j \, . \tag{A.12}$$

The vector ξ defined in eq.(3.12) ,generates Killing symmetry of the stationary background with the metric in eq.(A.12). Due to this, as we have already mentioned before, when we operate Lie derivative with respect to ξ on any covariant tensor constructed out of the stationary metric eq.(A.12), will vanish. To be more precise, acting with the Lie derivative with respect to ξ, on a covariant tensor, say $\mathcal{B}_{\mu_1\mu_2\cdots\mu_k}$ with all lowered indices, will produce the following,

$$\begin{aligned} \mathcal{L}_\xi \mathcal{B}_{\mu_1\mu_2\cdots\mu_k} =& \xi^\beta \partial_\beta \mathcal{B}_{\mu_1\mu_2\cdots\mu_k} + \left(\partial_{\mu_1}\xi^\beta\right) \mathcal{B}_{\beta\mu_2\cdots\mu_k} + \left(\partial_{\mu_2}\xi^\beta\right) \mathcal{B}_{\mu_1\beta\cdots\mu_k} + \cdots \\ & + \left(\partial_{\mu_k}\xi^\beta\right) \mathcal{B}_{\mu_1\mu_2\cdots\beta} \, . \end{aligned} \tag{A.13}$$

Furthermore, when we evaluate this for the metric eq.(A.12), and with xi given in eq.(3.12), we will get

$$\mathcal{L}_\xi \mathcal{B}_{\mu_1\mu_2\cdots\mu_k} = \left[w + (v\partial_v - r\partial_r)\right] \mathcal{B}_{\mu_1\mu_2\cdots\mu_k} \, , \tag{A.14}$$

where w is the boost weight of $\mathcal{B}_{\mu_1\mu_2\cdots\mu_k}$ and from eq.(A.14) we can also confirm that w counts the excess number of lower v indices compared to lower r indices in $\mathcal{B}_{\mu_1\mu_2\cdots\mu_k}$. Following this argument, it is also obvious that the vi-component of EoM, E_{vi} will have boost weight equal to $+1$, and hence, will vanish for stationary configurations when evaluated on the horizon. This is the main ingredient that we have used in §3.5.3.

Before we conclude this appendix let us summarise the useful points that we should remember while using the boost weight analysis,

1. We should think about any component of a covariant tensor to have a structure with some number of ∂_r, ∂_v and ∇_i operators acting on the metric coefficients in eq.(A.12): (X, ω_i, and h_{ij}) or product of such structures.

2. The boost weight of any covariant tensor can be obtained by looking at the factor w in eq.(A.14), when a Lie derivative $\mathcal{L}_\xi$, with respect to ξ $(= v\partial_v - r\partial_r)$, acts on it.

3. Any expression with positive boost weight will vanish when evaluated on the horizon for a stationary metric.

For more details we refer the reader to section-(2.3) and Appendix-B of [32].

A.3 More detailed calculation for Einstein's gravity

In this appendix, we will calculate τi component of equation of motion $E^{(0)}_{\tau i}$ for Einstein's gravity.

$$R_{\tau i} = R^{\tau}{}_{\tau\tau i} + R^{\rho}{}_{\tau\rho i} + R^{j}{}_{\tau j i} \tag{A.15}$$

Using the expressions of Christoffel symbols computed in Appendix-(A.1.1), we can calculate different components of Riemann tensor upto order $\mathcal{O}(\alpha^0)$

$$\begin{aligned} R^{\tau}{}_{\tau\tau i}|_{\rho=0} &= \partial_\tau \Gamma^{\tau}_{i\tau} - \partial_i \Gamma^{\tau}_{\tau\tau} + \Gamma^{\tau}_{\tau E}\Gamma^{E}_{i\tau} - \Gamma^{\tau}_{iE}\Gamma^{E}_{\tau\tau} \\ &= -\frac{1}{2}\partial_i X^{(0)} \end{aligned} \tag{A.16}$$

$$\begin{aligned} R^{\rho}{}_{\tau\rho i}|_{\rho=0} &= \partial_\rho \Gamma^{\rho}_{i\tau} - \partial_i \Gamma^{\rho}_{\rho\tau} + \Gamma^{\rho}_{\rho E}\Gamma^{E}_{i\tau} - \Gamma^{\rho}_{iE}\Gamma^{E}_{\rho\tau} \\ &= 0 \end{aligned} \tag{A.17}$$

$$R^j{}_{\tau ji}|_{\rho=0} = \partial_j \Gamma^j_{i\tau} - \partial_i \Gamma^j_{j\tau} + \Gamma^j_{jE}\Gamma^E_{i\tau} - \Gamma^j_{iE}\Gamma^E_{j\tau} \\ = 0 \tag{A.18}$$

Finally we get

$$E_{\tau i}|_{\rho=0} = R_{\tau i}|_{\rho=0} = -\frac{1}{2}(\partial_i X^{(0)})|_{\rho=0} \tag{A.19}$$

A.4 Calculation of the homogeneous part

In this appendix, we will derive the expression of the homogeneous part eq.(3.43). As has been discussed in sub-section §3.5.1, we have to linearize $E^{(0)}_{\mu\nu}$ around $g^{(0)}_{\alpha\beta}$. We have to calculate $E^{(0)}_{\mu\nu}\left[g^{(0)}_{\alpha\beta} + \delta g_{\alpha\beta}\right]$, where, we will treat $\delta g_{\alpha\beta} \equiv \alpha^{m+1} g^{(m+1)}_{\alpha\beta}$ as linearized perturbations around $g^{(0)}_{\alpha\beta}$. $E^{(0)}_{\mu\nu}$ is the Einstein's tensor

$$E^{(0)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{A.20}$$

As, $g^{(0)}_{\mu\nu}$ is an exact solution of $E^{(0)}_{\mu\nu}$

$$E^{(0)}_{\mu\nu}\left[g^{(0)}_{\alpha\beta} + \delta g_{\alpha\beta}\right] \equiv \delta E^{(0)}_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} \delta R - \frac{1}{2} R^{(0)} \delta g_{\mu\nu} \tag{A.21}$$

$R^{(0)}$ is the Ricci scalar evaluated on the metric $g^{(0)}_{\mu\nu}$. We have to calculate τi component of the above equation at $\rho = 0$. We can compute $\delta E^{(0)}_{\tau i}$ off the horizon, but for our purpose that is not required.

$$\delta E^{(0)}_{\tau i}|_{\rho=0} = \delta R_{\tau i} - \frac{1}{2} g^{(0)}_{\tau i} \delta R - \frac{1}{2} R^{(0)} \alpha^{m+1} g^{(m+1)}_{\tau i} = \delta R_{\tau i} \tag{A.22}$$

Since we have denoted the coordinates by $\{\tau, \rho, x^i\}$, for notational convenience, instead of using μ, ν we will be denoting the spacetime coordinates by $\{A, B, C...\}$. We will be using this notation only for this appendix. If we calculate the Christoffel symbols on $g^{(0)}_{AB} + \alpha^{m+1} g^{(m+1)}_{AB}$ we can decompose it as follows

$$\Gamma^A_{BC} = \bar{\Gamma}^A_{BC} + \delta\Gamma^A_{BC} \tag{A.23}$$

where $\bar{\Gamma}^A_{BC}$ is the Christoffel symbols for $g^{(0)}_{\mu\nu}$. Linearized Ricci tensor is

$$\delta R_{AB} = \nabla_D \delta\Gamma^D_{AB} - \nabla_B \delta\Gamma^D_{AD} \tag{A.24}$$

We can very easily read-off the expressions of $\bar{\Gamma}^A_{BC}$ and $\delta\Gamma^A_{BC}$ from eq.(A.5).

$$\delta R_{AB} = \partial_D \left(\delta\Gamma^D_{AB}\right) + \bar{\Gamma}^D_{DE}\delta\Gamma^E_{AB} - \bar{\Gamma}^E_{DA}\delta\Gamma^D_{EB} - \bar{\Gamma}^E_{DB}\delta\Gamma^D_{AE} - \partial_B\delta\Gamma^D_{AD} + \bar{\Gamma}^E_{BA}\delta\Gamma^D_{ED} \tag{A.25}$$

$$\delta R_{\tau i} = \partial_D \left(\delta\Gamma^D_{\tau i}\right) + \bar{\Gamma}^D_{DE}\delta\Gamma^E_{\tau i} - \bar{\Gamma}^E_{D\tau}\delta\Gamma^D_{Ei} - \bar{\Gamma}^E_{Di}\delta\Gamma^D_{\tau E} - \partial_i\delta\Gamma^D_{\tau D} + \bar{\Gamma}^E_{i\tau}\delta\Gamma^D_{ED} \tag{A.26}$$

Now, we will compute different terms of the above equation on $\rho = 0$ separately

$$\begin{aligned}\partial_D\delta\Gamma^D_{\tau i} &= \partial_\rho\delta\Gamma^\rho_{\tau i} + \partial_\tau\delta\Gamma^\tau_{\tau i} + \partial_j\delta\Gamma^j_{\tau i}\\ &= -\frac{1}{2}\alpha^{m+1}\partial_i\left(X^{(m+1)}\right) - \frac{1}{2}\alpha^{m+1}\left(X^{(m+1)}\omega^{(0)}_i + X^{(0)}\omega^{(m+1)}_i\right)\end{aligned} \tag{A.27}$$

$$\begin{aligned}\bar{\Gamma}^D_{DE}\delta\Gamma^E_{\tau i} &= \bar{\Gamma}^D_{D\tau}\delta\Gamma^\tau_{\tau i}\\ &= \left(\bar{\Gamma}^\tau_{\tau\tau} + \bar{\Gamma}^\rho_{\rho\tau}\right)\delta\Gamma^\tau_{\tau i}\\ &= \left(\frac{1}{2}X^{(0)} - \frac{1}{2}X^{(0)}\right)\left(-\frac{1}{2}\alpha^{m+1}\omega^{(m+1)}_i\right)\\ &= 0\end{aligned} \tag{A.28}$$

$$\begin{aligned}\bar{\Gamma}^E_{D\tau}\delta\Gamma^D_{Ei} &= \bar{\Gamma}^\tau_{D\tau}\delta\Gamma^D_{\tau i} + \bar{\Gamma}^\rho_{D\tau}\delta\Gamma^D_{\rho i} + \bar{\Gamma}^j_{D\tau}\delta\Gamma^D_{ji}\\ &= \bar{\Gamma}^\tau_{\tau\tau}\delta\Gamma^\tau_{\tau i} + \bar{\Gamma}^\rho_{\rho\tau}\delta\Gamma^\rho_{\rho i} + \bar{\Gamma}^j_{\rho\tau}\delta\Gamma^\rho_{ji}\\ &= \frac{1}{2}X^{(0)}\left(-\frac{1}{2}\alpha^{m+1}\omega^{(m+1)}_i\right) - \frac{1}{2}X^{(0)}\frac{1}{2}\alpha^{m+1}\omega^{(m+1)}_i\\ &= -\frac{1}{2}X^{(0)}\alpha^{m+1}\omega^{(m+1)}_i\end{aligned} \tag{A.29}$$

$$\begin{aligned}\bar{\Gamma}^E_{Di}\delta\Gamma^D_{\tau E} &= \bar{\Gamma}^\tau_{Di}\delta\Gamma^D_{\tau\tau} + \bar{\Gamma}^\rho_{Di}\delta\Gamma^D_{\tau\rho} + \bar{\Gamma}^j_{Di}\delta\Gamma^D_{\tau j}\\ &= \bar{\Gamma}^\tau_{\tau i}\delta\Gamma^\tau_{\tau\tau} + \left(\bar{\Gamma}^\rho_{\rho i}\delta\Gamma^\rho_{\tau\rho} + \bar{\Gamma}^\rho_{ji}\delta\Gamma^j_{\tau\rho}\right) + \bar{\Gamma}^j_{\tau i}\delta\Gamma^\tau_{\tau j}\\ &= -\frac{1}{2}\omega^{(0)}_i\frac{1}{2}\alpha^{m+1}X^{(m+1)} + \frac{1}{2}\omega^{(0)}_i\left(-\frac{1}{2}\alpha^{m+1}X^{(m+1)}\right)\\ &= -\frac{1}{2}\omega^{(0)}_i\alpha^{m+1}X^{(m+1)}\end{aligned} \tag{A.30}$$

$$\begin{aligned}\partial_i\delta\Gamma^D_{\tau D} &= \partial_i\left(\delta\Gamma^\rho_{\tau\rho} + \delta\Gamma^\tau_{\tau\tau}\right)\\ &= \partial_i\left(-\frac{1}{2}\alpha^{m+1}X^{(m+1)} + \frac{1}{2}\alpha^{m+1}X^{(m+1)}\right)\\ &= 0\end{aligned} \tag{A.31}$$

$$\begin{aligned}\bar{\Gamma}^{E}_{i\tau}\delta\Gamma^{D}_{ED} &= \bar{\Gamma}^{\tau}_{i\tau}\delta\Gamma^{D}_{\tau D} \\ &= \bar{\Gamma}^{\tau}_{i\tau}\left(\delta\Gamma^{\rho}_{\tau\rho} + \delta\Gamma^{\tau}_{\tau\tau}\right) \\ &= -\frac{1}{2}\omega^{(0)}_{i}\left(-\frac{1}{2}\alpha^{m+1}X^{(m+1)} + \frac{1}{2}\alpha^{m+1}X^{(m+1)}\right) \\ &= 0\end{aligned} \tag{A.32}$$

Substituting eq.(A.27) - eq.(A.32) in eq.(A.26) we get

$$\delta R_{\tau i} = -\frac{1}{2}\alpha^{m+1}\partial_{i}\left(X^{(m+1)}\right) \tag{A.33}$$

Appendix B

(For Chapter-4)

B.1 A general stationary metric can have v dependent components

In a black hole usually the Killing generators of the horizon cannot be affinely parametrized maintaining the Killing conditions. In other words, the components of the stationary metric are independent of the Killing coordinate - τ, but they are not independent of the affine parameter v along the generators of the Killing horizons. Though in a stationary metric with a Killing horizon, the Killing vector field - ∂_τ and the affinely parametrized null generators ∂_v are proportional to each other and there exists a precise relation between them. In this appendix, we shall use this relation to fix the v dependence of the stationary metric.

More precisely, we would like to determine how the components of a generic stationary metric, written in the gauge of (4.3), could depend on the v-coordinate.

Consider a generic stationary black hole with a Killing horizon, i.e, there exists a coordinate τ such that

1. All metric components are independent of τ
2. ∂_τ is time like everywhere outside the horizon.
3. ∂_τ becomes null on the event horizon.

Now we could do exactly same construction as in case of the metric (4.3), the only difference being that now the coordinates on the horizon would be ∂_τ and ∂_i, instead of the affinely parametrized ∂_v. Let ρ be the coordinate that denotes distances away from the horizon.

Now also we could choose ρ to be the affine parameter along the set of null geodesics, intersecting the horizon at fixed angles with ∂_τ and ∂_i and labelled by the coordinates of the intersection point. Following the same logic as before, the metric in τ, x^i and ρ coordinate will have almost the same structure as that of (4.3). The $\tau\tau$ and τi components of the metric will again vanish on the horizon ($\rho = 0$) owing to the fact that it is a null hypersurface. But since ∂_τ is not affinely parametrized, unlike (4.3), the first ρ derivative of the ($\tau\tau$) component of the metric (let us denote it by $g_{\tau\tau}(\rho, x^i)$) will not vanish on the horizon. However for stationary black holes, $\partial_\rho g_{\tau\tau}$ is related to the temperature of the black hole and the zeroth law of Black hole mechanics ensures $\left[\partial_\rho g_{\tau\tau}|_{\rho=0} \equiv C\right]$ is a constant, i.e., independent of the spatial coordinates x^is. Putting all these facts together we finally write the most general stationary metric in our gauge.

$$ds^2 = 2\, d\tau\, d\rho - \left(\rho\, C + \rho^2 X(\rho)\right)\, d\tau^2 + 2\rho\, \omega_i(\rho)\, d\tau\, dx^i + h_{ij}(\rho)\, dx^i dx^j \tag{B.1}$$

Now we have to transform this metric to the gauge of (4.3) where the null coordinate along the horizon is affine parameter of the null generators v. Our final goal is to find out how the metric components of an arbitrary stationary metric will depend on v.

The coordinate transformation which fulfils this objective is given by

$$\rho = \frac{C}{2} r\, v, \quad \tau = \frac{2}{C} \log\left(\frac{C\, v}{2}\right) \tag{B.2}$$

The metric in the new coordinate takes the following form

$$ds^2 = 2\, dv\, dr - r^2 X\left(Crv/2\right) dv^2 + r\omega_i\left(Crv/2\right)\, dv\, dx^i + h_{ij}\left(Crv/2\right)\, dx^i\, dx^j \tag{B.3}$$

To get the above metric, we have crucially used the fact that C is independent of v and x^i (C is independent of ρ by construction).

The most important noteworthy feature of this metric (B.3) is that, the metric components are explicitly dependent on the v-coordinate, although it describes a stationary black hole because it is a mere coordinate transformation of the most general stationary metric (B.1). However, though imposing the condition of stationarity on the general form of the metric (4.3), *does not* imply that the metric functions X, ω_i and h_{ij} should be independent of the v coordinate, there are some constraints on the v dependence of the stationary metrics. Here the metric components never depend on r and v independently, but always on the product rv. In other words, on any metric of the form (B.3), with components depending only on the product rv, we could always apply the inverse of the coordinate transformation (B.2) to take it to a form where redefined coordinate - τ is manifestly the Killing coordinate.

Now for the proof of second law it is crucial that the ∂_v of entropy vanishes on stationary black holes attained at $v \to \infty$. Naively the form of the stationary metric (B.3) contradicts this step of the argument. But note that any ∂_v derivative on the metric components in (B.3), will also bring down a factor of r and therefore will vanish on $\mathcal{H}$ (the hyper-surface at $r = 0$), unless there is also one ∂_r derivative present along with every ∂_v derivative. Thus we may conclude that, the terms of the form $((\partial_r\partial_v)^m P)$, where P is a function of the metric components in (4.3) and their ∇_i derivatives (without any ∂_r or ∂_v derivatives), can be non-zero on a generic Killing horizon. Note that, all such terms are invariant under the λ scaling (4.13). It also implies that the terms of the form $(\partial_r^n\partial_v^m P)$, with $m > n$ must vanish on a Killing horizon. This is because, as is apparent from (B.3), the higher number of v-derivatives would give rise to factors of r, which will force the entire term to zero on the $r = 0$ hyper surface.

B.2 Arguments leading to vanishing of T_{vv} on any Killing horizon

In this section we would like to argue that the vv component of the matter stress tensor vanishes on Killing horizons.

We shall use the boost transformation property of T_{vv} to reach this conclusion. Note that just like the gravity part of the equation of motion (i.e., E_{vv} or $E^{\rm HD}_{vv}$) , matter stress tensor itself is a covariant with nice transformation properties under any coordinate transformation, in particular the λ scaling described in equation (4.13). T_{vv} should transform exactly the way E_{vv} or $E^{\rm HD}_{vv}$ transforms, namely

$$T_{vv} \to T_{\tilde{v}\tilde{v}} = \frac{1}{\lambda^2} T_{vv} \tag{B.4}$$

Now we shall consider only those stress tensors that are regular on the event horizons at least in those coordinate systems where the full dynamical metric is regular, everywhere apart from the black hole singularity. This is certainly the case in the coordinate system we have chosen in our metric (4.3). It follows that T_{vv} must admit a Taylor series expansion around the horizon at $r = 0$. Equation (4.13) and equation (B.4) together suggest the following expansion for T_{vv}

$$T_{vv} = \frac{1}{v^2} \sum_{k=0}^{\infty} (rv)^k \, w^{(k)}(\vec{x}) \tag{B.5}$$

where $w^{(k)}$ s are scalar functions of only the spatial coordinates $\{x^i\}$, (i,e., they are both boost invariant and also invariant under any coordinate transformation that mixes only the $\{x^i\}$ coordinates among themselves). Exactly on the horizon only the leading terms of the above expansion will contribute.

$$T_{vv}\big|_{\rm horizon} = \frac{w^{(0)}(\vec{x})}{v^2} \tag{B.6}$$

Note that both (B.5) and (B.6) do not need any stationarity for their validity.

Now let us specialize to stationary cases. Here we have a Killing vector (∂_τ). All relevant fields including the matter fields are independent of this τ coordinate and the same is true for T_{vv}, as well. In terms of equation it implies

$$\partial_\tau T_{vv}\big|_{\rm stationary} = 0 \tag{B.7}$$

From equation (B.2) it follows

$$\partial_\tau = \frac{C}{2}\,(v\,\partial_v - r\,\partial_r) \tag{B.8}$$

Equation (B.8) clearly contradicts equation (B.7) unless $w^{(k)} = 0$ for every k. It follows that T_{vv} vanishes identically on any configuration with a Killing vector.

We would like to emphasize that in the above arguments the key elements are

1. The existence of the event horizon (or more precisely a null hyper-surface at $r = 0$) so that the horizon-adapted coordinate choice in the metric (4.3) and consequently the boost symmetry is meaningful.

2. Stationarity or the existence of a Killing vector, which is proportional to the null generators of the horizon.

We have not used the fact that T_{vv} is stress tensor, neither that fact that the field configuration (including the metric) satisfy any particular equation. What we have argued is that whenever there is one Killing vector field, the vv component of any covariant tensor identically vanishes in the vicinity of the horizon (where the Taylor expansion in equation (B.5) makes sense) and it is a completely off-shell statement.

B.3 Conventions, notations and useful formulae

In this appendix we summarise our conventions, write down various notations and collectively represent several important formulae that we have used in the note.

- The coordinate choice:

$$
\begin{aligned}
x^\mu &= \text{The full space-time coordinates in } (d+1)\text{-dimensions : } \{v,\, r,\, x^i\}, \\
v &= \text{The EddingtonâFinkelstein type time coordinate}, \\
r &= \text{The radial coordinate}, \\
x^i &= \text{The } (d-1) \text{ spatial coordinates},
\end{aligned}
$$

- The choice for the space-time metric:

$$
\begin{aligned}
ds^2 =& \, 2\, dv\, dr - r^2\, X(r,v,x^i)\, dv^2 + 2\, r\, \omega_i(r,v,x^i)\, dv\, dx^i \\
&+ \, h_{ij}(r,v,x^i)\, dx^i dx^j
\end{aligned}
\tag{B.9}
$$

- Useful notations and conventions:

1. $\mathcal{H} =$ The co-dimension one horizon, which we choose to be at the radial coordinate $r = 0$,
2. $\mathcal{H}_v =$ The co-dimension two, constant v-slice of the horizon,
3. $h =$ Determinant of the induced metric, h_{ij}, on $\mathcal{H}$,
4. The total integrated Wald entropy at equilibrium is defined as

$$
S_W = -\, 2\, \pi \int_{\mathcal{H}_v} d^{d-2}x\, \sqrt{h}\, \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\, \epsilon_{\mu\nu}\epsilon_{\rho\sigma} = -\, 2\, \pi \int_{\mathcal{H}_v} d^{d-2}x\, \sqrt{h}\, s_w\, ,
$$

   where $\epsilon_{\mu\nu} =$ Bi-normal to $\mathcal{H}_v$,

5. $s_w = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\, \epsilon_{\mu\nu}\epsilon_{\rho\sigma} =$ The Wald entropy density,
6. $S_W^{\rm HD}$, $s_w^{\rm HD}$ $=$ Contributions to integrated Wald entropy (S_W) and Wald entropy density (s_w) from the higher derivative part of the gravity Lagrangian $\mathcal{L}^{\rm HD}$.
   It can be shown that: $s_w = 1 + s_w^{\rm HD}$, such that for Einstein gravity one obtains $s_w = 1$.

7. The density of time variation of Wald entropy is denoted by ϑ, defined as

$$\partial_v S_W = \int_{\mathcal{H}_v} d^{d-2}x\, \sqrt{h}\, \vartheta\, ; \quad \vartheta = \vartheta_E + \vartheta^{\rm HD},$$

such that $\vartheta_E = \frac{1}{\sqrt{h}}\partial_v(\sqrt{h})$ = contribution from Einstein gravity, and $\vartheta^{\rm HD} = \frac{1}{\sqrt{h}}\partial_v(\sqrt{h}\, s_w^{\rm HD})$ = contribution from higher derivative part of the Lagrangian $\mathcal{L}^{\rm HD}$,

8. s_c = correction to the entropy density which vanish on stationary solutions,

9. $E_{vv}^{\rm HD}$ = 'vv' component of the equation of motion, getting contribution only from the higher derivative part of the Lagrangian $\mathcal{L}^{\rm HD}$,

- Useful definitions:

  1. The extrinsic curvatures of the horizon $\mathcal{H}$:

     (a). $\mathcal{K}_{ij} = \frac{1}{2}\partial_v h_{ij}$; $\qquad \mathcal{K}^{ij} = -\frac{1}{2}\partial_v h^{ij}$,

     (b). $\overline{\mathcal{K}}_{ij} = \frac{1}{2}\partial_r h_{ij}$; $\qquad \overline{\mathcal{K}}^{ij} = -\frac{1}{2}\partial_r h^{ij}$.

  2. The trace of the extrinsic curvatures:

     (a). $\mathcal{K} = \frac{1}{2}h^{ij}\partial_v h_{ij} = \frac{1}{\sqrt{h}}\partial_v\sqrt{h}$,

     (b). $\overline{\mathcal{K}} = \frac{1}{2}h^{ij}\partial_r h^{ij} = \frac{1}{\sqrt{h}}\partial_r\sqrt{h}$.

- Expressions for the components of Riemann tensors, Ricci tensors and Ricci scalar on the horizon:

$$
\begin{aligned}
R_{rvrv} &= X + \frac{1}{4}\omega^2 \\
R_{rvri} &= -\partial_r \omega_i + \frac{1}{2}\omega^j_{ij} \\
R_{rvvi} &= -\frac{1}{2}\left(\partial_v \omega_i + \omega^j_{ij}\right) \\
R_{rirj} &= -\partial_{r ij} +^{k}_{ik\ j} \\
R_{rivj} &= -\partial_{r ij} + \frac{1}{2}\nabla_j \omega_i - \frac{1}{4}\omega_i \omega_j +^{k}_{jk\ i} \\
R_{vivj} &= -\partial_{v ij} +^{k}_{ik\ j} \\
R_{ijvk} &= \nabla_{j ik} - \nabla_{i jk} - \frac{1}{2}\omega_i \mathcal{K}_{jk} + \frac{1}{2}\omega_j \mathcal{K}_{ik} \\
R_{ijrk} &= \left(\nabla_j - \frac{1}{2}\omega_j\right)_{ik} - \left(\nabla_i - \frac{1}{2}\omega_i\right)_{jk} \\
R_{ijkl} &= \mathcal{R}_{ijkl} -_{ik\ jl} -_{ik\ jl} +_{il\ jk} +_{il\ jk}
\end{aligned}
\tag{B.10}
$$

where ∇_i is the covariant derivative with respect to the induced metric h_{ij}.

- Expressions for the components of Ricci tensors on the horizon:

$$
\begin{aligned}
R_{rr} &= -\partial_r -_{ij}\ ^{ij} \\
R_{rv} &= -X - \frac{1}{2}\,\omega^2 - \partial_r -_{ij}\ ^{ij} + \frac{1}{2}\,\nabla^i\,\omega_i \\
R_{ri} &= \partial_r \omega_i - \frac{1}{2}\,^{j}_{i}\,\omega_j + \left(\nabla_j - \frac{1}{2}\,\omega_j\right)^j_i - \left(\nabla_i - \frac{1}{2}\,\omega_i\right) \\
R_{vv} &= -\partial_v -_{ij}\ ^{ij} \\
R_{vi} &= -\frac{1}{2}\,\partial_v \omega_i - \frac{1}{2}\,^{j}_{i}\,\omega_j + \left(\nabla_j + \frac{1}{2}\,\omega_j\right)^j_i - \left(\nabla_i + \frac{1}{2}\,\omega_i\right) \\
R_{ij} &= \mathcal{R}_{ij} - 2\,\partial_{r ij} + \frac{1}{2}\left(\nabla_j \omega_i + \nabla_i \omega_j - \omega_i\,\omega_j\right) -\ _{ij} -\ _{ij} \\
&\quad + 2\left(_{ik}\,^{k}_{j} +_{jk}\ ^{k}_{i}\right)
\end{aligned}
\tag{B.11}
$$

- Expressions for the Ricci scalar on the horizon:

$$
R = \mathcal{R} - 2X - \frac{3}{2}\,\omega^2 - 4\,\partial_r \mathcal{K} + 2(\nabla \cdot \omega) - 2\,\bar{\mathcal{K}}_{ij}\mathcal{K}^{ij} - 2\,\mathcal{K}\,\overline{\mathcal{K}}
\tag{B.12}
$$

B.4 Detailed expressions

B.4.1 Expressions of Riemann tensors and Ricci tensors off the horizon

As we will compute the 'vv'-component of the equations of motion E_{vv}^{HD}, we will need the following expressions for the components of Riemann tensors and Ricci tensor calculated off the horizon, i.e. without imposing $r = 0$,

$$
\begin{aligned}
R_{rvrv} &= X + \frac{1}{4}\omega^2 \\
R_{rviv} &= \frac{1}{2}\nabla_i(2rX + r^2\partial_r X) + \frac{1}{2}\partial_v(\omega_i + r\partial_r\omega_i) + \frac{r^2}{2}(2X + r\partial_r X)\omega^j \bar{\mathcal{K}}_{ij} \\
&- r(\partial_v\omega_j)\bar{\mathcal{K}}_i^j - \frac{r^2}{2}(\nabla_j X)\bar{\mathcal{K}}_i^j + \frac{r}{4}(\omega^j(\omega_j + r\partial_r\omega_j))(\omega_i + r\partial_r\omega_i) \\
&- \frac{r}{4}(\omega^j + r\partial_r\omega^j)(\nabla_j\omega_i - \nabla_i\omega_j - \mathcal{K}_{\rangle|}) \\
4\,R_{vivj} &= r^2\,(\nabla^m\omega_i)\,(\nabla_m\omega_j) + r^2\,(\nabla_i\omega^m)\,(\nabla_j\omega_m) - r^2(\nabla_m\omega_i)(\nabla_j\omega^m) \\
&- r^2(\nabla_m\omega_j)(\nabla_i\omega^m) - 2r(\nabla_m\omega_i)\mathcal{K}_j^m - 2r(\nabla_m\omega_j)\mathcal{K}_i^m + 4\,\mathcal{K}_{im}\mathcal{K}_j^m \\
&+ r^2(\nabla_i\omega^m)\mathcal{K}_{jm} + 2r(\nabla_j\omega^m)\mathcal{K}_{im} + 2\,\bar{\mathcal{K}}_{ij}\big[r^2\omega^2(2rX + r^2\partial_r X) \\
&+ r^2X(2rX + r^2\partial_r X) - r^3(\omega\cdot\nabla)X - 2\,r^2\omega^m(\partial_v\omega_m) - (\partial_v r^2 X)\big] \\
&+ \ (\omega_j + r\partial_r\omega_i)\,\big[2\,r\omega^m\mathcal{K}_{im} - r^2(\omega\cdot\nabla)\omega_i + r^2\omega^m(\nabla_i\omega_m) + (\nabla_i r^2 X)\big] \\
&+ 4r^2\,(r^2X + r^r\omega^2)(\omega_i + r\partial_r\omega_i)(\omega_j + r\partial_r\omega_j) \\
&+ \ (\omega_i + r\partial_r\omega_i)\,\big[2\,r\omega^m\mathcal{K}_{jm} - r^2(\omega\cdot\nabla)\omega_j + r^2\omega^m(\nabla_j\omega_m) + (\nabla_j r^2 X)\big] \\
&- r^2(2X + r\partial_r X)\,[\nabla_i\omega_j + \nabla_j\omega_i] \\
&+ 2r^2\nabla_i\nabla_j X + 2(2rX + r^2\partial_r X)\mathcal{K}_{ij} - 4\,\partial_v\mathcal{K}_{ij} \\
&+ 2r\,\nabla_i(\partial_v\omega_j) + 2\,r\nabla_j(\partial_v\omega_i)
\end{aligned} \tag{B.13}
$$

$$
\begin{aligned}
R_{vv} = {} & r^2(X+\omega^2)\left[\frac{1}{2}(2X+4rX+r^2\partial_r^2X)+\frac{1}{2}(\omega^i+r\partial_r\omega^i)(\omega_i+r\partial_r\omega_i)\right. \\
& \left.+\frac{1}{2}\bar{\mathcal{K}}(2rX+r^2\partial_rX)\right]-r^2(\omega\cdot\nabla)(2X+r\partial_rX)-r\omega^i(\partial_v(\omega_i+r\partial_r\omega_i)) \\
& -r^3(2X+r\partial_rX)(\omega^i\bar{\mathcal{K}}_{ij}\omega^j)+2\,r^2(\partial_v\omega_i)\omega_j\bar{\mathcal{K}}^{ij}+r^3(\nabla_jX)\omega_i\bar{\mathcal{K}}^{ij} \\
& -\frac{r^2}{2}(\omega^i(\omega_i+r\partial_r\omega_i))^2+r^2\omega^i((\omega+r\partial_r\omega)\cdot\nabla)\omega_i \\
& -r^2(\omega^i+r\partial_r\omega^i)(\omega\cdot\nabla)\omega_i+\frac{r^2}{2}(\nabla^i\omega^j)(\nabla_i\omega_j)-\frac{r^2}{2}(\nabla_j\omega^i)(\nabla_i\omega^j) \\
& -\partial_v\mathcal{K}-\mathcal{K}_{ij}\mathcal{K}^{ij}+\frac{r^2}{2}(\nabla^2X)+\frac{1}{2}(2rX+r^2\partial_rX)\mathcal{K} \\
& +r\nabla^i(\partial_v\omega_i)-\frac{r^2}{2}\bar{\mathcal{K}}(\omega\cdot\nabla)X-\frac{r^2}{2}\bar{\mathcal{K}}(\partial_vX)-r^2\bar{\mathcal{K}}\omega^i(\partial_v\omega_i) \\
& +\frac{r^2}{2}((\omega+r\partial_r\omega)\cdot\nabla)X-\frac{r^2}{2}(2X+r\partial_rX)(\nabla\cdot\omega)
\end{aligned}
\tag{B.14}
$$

B.4.2 Relevant terms on the horizon $\mathcal{H}$, to compute $E_{vv}^{\mathbf{HD}}$ for different theories

$$
\begin{aligned}
R_v^{\alpha}\,R_{\alpha v} &= (\partial_v\mathcal{K})\left(2\,X+\omega^2-\nabla\cdot\omega+2\,\partial_v\bar{\mathcal{K}}\right) \\
R_v{}^{\alpha\beta\gamma}\,R_{v\alpha\beta\gamma} &= \omega^i\omega^j\partial_v\mathcal{K}_{ij}-2\,\nabla^i\omega^j\partial_v\mathcal{K}_{ij}+\,4\,\partial_v\overline{\mathcal{K}}_{ij}\,\partial_v\mathcal{K}^{ij} \\
R^{\alpha\beta}\,R_{v\alpha v\beta} &= -\left(X+\frac{\omega^2}{4}\right)(\partial_v\mathcal{K})-\mathcal{R}^{ij}\,\partial_v\mathcal{K}_{ij}-(\nabla^i\omega^j)\partial_v\mathcal{K}_{ij} \\
&\quad+\frac{1}{2}\omega^i\omega^j\partial_v\mathcal{K}_{ij}+2\,\partial_r\mathcal{K}_{ij}\,\partial_v\mathcal{K}^{ij} \\
D_vD_vR &= \partial_v^2\mathcal{R}-2\,\partial_v^2X-3\,\omega^i\partial_v^2\omega_i+3\,\omega^i\omega^j\partial_v\mathcal{K}_{ij} \\
&\quad+2\,\partial_v^2(\nabla\cdot\omega)-4\,\partial_r\partial_v^2\mathcal{K}-2\,\overline{\mathcal{K}}_{ij}\partial_v^2\mathcal{K}^{ij} \\
&\quad-2\,\overline{\mathcal{K}}\,\partial_v^2\mathcal{K}-4\,\partial_v\overline{\mathcal{K}}_{ij}\,\partial_v\mathcal{K}^{ij}-4\,\partial_v\mathcal{K}\,\partial_v\bar{\mathcal{K}} \\
D_\alpha D^\alpha R_{vv} &= \left(\frac{3}{2}\,\omega^2-\nabla\cdot\omega+4\,X\right)(\partial_v\mathcal{K})+(\omega\cdot\nabla)\partial_v\mathcal{K} \\
&\quad-2\omega^i\,\nabla^j(\partial_v\mathcal{K}_{ij})-2\,\partial_r\partial_v^2\mathcal{K}-\omega^i\,\partial_v^2\omega_i-\nabla^2(\partial_v\mathcal{K}) \\
&\quad+2\,\nabla^i(\partial_v^2\omega_i)-\overline{\mathcal{K}}\partial_v^2\mathcal{K}-4\,\partial_r\mathcal{K}_{ij}\,\partial_v\mathcal{K}^{ij}\,.
\end{aligned}
\tag{B.15}
$$

Note that, to obtain the last two expressions above we need to first evaluate $D_\mu D_\nu R_{\alpha\beta}$ where the indices μ,ν runs over the full space-time coordinates: v, r, x^i and D_μ is covariant derivative with respect to the full space-time metric $g_{\mu\nu}$.

B.4.3 Ricci scalar square Theory

Following the discussions in sections §4.2.1 and §4.2.1 here we write down the detailed expressions for various quantities for the Ricci Scalar squared theory.

The 'vv'-component of $E^{\rm HD}_{\mu\nu}$ from (4.49)

$$E^{\rm HD}_{vv} = a_1\Bigg[\partial_v\left(\frac{1}{\sqrt{h}}\partial_v(\sqrt{h}\,(\,3\,\omega^2 - 4\,(\nabla\cdot\omega) + 4\,X\,)\right) - 2\,\mathcal{R}\,\partial_v\mathcal{K} - 2\,\partial_v^2\mathcal{R} \\ + 8\,\partial_r\partial_v^2\mathcal{K} + 4\,\bar{\mathcal{K}}_{ij}\,\partial_v^2\mathcal{K}^{ij} + 4\,\bar{\mathcal{K}}\,\partial_v^2\mathcal{K} + 8\,\partial_v\bar{\mathcal{K}}_{ij}\,\partial_v\mathcal{K}^{ij} + 16\,\partial_v\mathcal{K}\,\partial_v\bar{\mathcal{K}}\Bigg] + \mathcal{O}[\epsilon^2]\,. \tag{B.16}$$

From (4.50) we know the Wald entropy density as

$$s^{\rm HD}_w = 2\,a_1\,R\,, \tag{B.17}$$

and therefore, we immediately obtain

$$\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,s^{\rm HD}_w\right)\right) = 2\,a_1\,\partial_v\Bigg(\frac{1}{\sqrt{h}}\,\partial_v\Bigg(\sqrt{h}\,(\mathcal{R} - 2\,X - \frac{3}{2}\,\omega^2 - 4\,\partial_r\mathcal{K} \\ + 2(\nabla\cdot\omega) - 2\,\bar{\mathcal{K}}_{ij}\mathcal{K}^{ij} - 2\,\mathcal{K}\bar{\mathcal{K}})\Bigg)\Bigg). \tag{B.18}$$

We can now use (4.43) and after some algebraic manipulation we obtain

$$\mathbb{E}^{\rm HD*}_{vv} = \mathcal{O}[\epsilon^2]. \tag{B.19}$$

Finally, comparing with (4.44) we see that for Ricci scalar squared theory there is no spatial entropy current

$$J^i = 0. \tag{B.20}$$

B.4.4 Ricci tensor squared Theory

The 'vv'-component of $E^{\rm HD}_{\mu\nu}$ for Ricci tensor squared theory, following (4.49) as discussed in §4.2.1, comes out to be

$$\begin{aligned} E^{\rm HD}_{vv} = a_2 \Bigg[& \partial_v \left(\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h} \left(\omega^2 + 2\, X - 2\, \nabla \cdot \omega + 2\, \partial_v \bar{\mathcal{K}} + \bar{\mathcal{K}}\, \mathcal{K} + 2\, \bar{\mathcal{K}}_{ij}\, \mathcal{K}^{ij} \right) \right) \right) \\ & + \partial_v \big(\nabla_i \left(2\, h^{ij}\, \partial_v \omega_j + \omega^i\, \mathcal{K} - h^{ij}\, \nabla_j \mathcal{K} - 2\, \omega_j\, \mathcal{K}^{ij} \right) \big) \\ & - 2\, \partial_v \big(\nabla_i \nabla_j \big(\mathcal{K}^{ij} - \mathcal{K}\, h^{ij} \big) \big) \Bigg] \end{aligned} \tag{B.21}$$

From (4.54) we recognise that the Wald entropy density coming from the higher derivative part of the Lagrangian is

$$s^{\rm HD}_w = 2\, a_2\, R_{rv} \tag{B.22}$$

and therefore we compute

$$\begin{aligned} \partial_v \left(\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h}\, s^{\rm HD}_w \right) \right) = 2\, a_2\, \partial_v \Bigg(\frac{1}{\sqrt{h}}\, \partial_v \Bigg(\sqrt{h} \Bigg(& - X - \frac{1}{2}\, \omega^2 - \partial_r \mathcal{K} \\ & - \bar{\mathcal{K}}_{ij}\, \mathcal{K}^{ij} + \frac{1}{2}\, \nabla^i \omega_i \Bigg) \Bigg) \Bigg) . \end{aligned} \tag{B.23}$$

Using the definition as given in (4.43) we calculate the following

$$\mathbb{E}^{\rm HD*}_{vv} = a_2\, \partial_v \left[\frac{1}{\sqrt{h}}\, \partial_v \big(\sqrt{h}\, \bar{\mathcal{K}}\, \mathcal{K} \big) + \nabla_i \big(h^{ij}\, \nabla_j \mathcal{K} + h^{ij}\, \partial_v \omega_j - 2\, \nabla_j \mathcal{K}^{ij} \big) \right], \tag{B.24}$$

where to derive this we have used the identity

$$\partial_v \left[\frac{1}{\sqrt{h}}\, \partial_v \big(\sqrt{h}\, \nabla \cdot \omega \big) \right] = \nabla_i \big(h^{ij}\, \partial_v \omega_j - 2\, \mathcal{K}^{ij} \omega_j + \omega^i\, \mathcal{K} \big) \tag{B.25}$$

Therefore the spatial entropy current turns out to be

$$\begin{aligned} J^v &= -\, s^{\rm HD}_w - a_2\, \bar{\mathcal{K}}\, \mathcal{K}\,, \\ J^i &= a_2 \left(2\, \nabla_j \mathcal{K}^{ij} - h^{ij}\, \nabla_j \mathcal{K} - h^{ij}\, \partial_v \omega_j \right). \end{aligned} \tag{B.26}$$

B.4.5 Riemann tensor squared Theory

Following the same steps as followed in the previous subsections for the cases of Ricci scalar squared and Ricci tensor squared theory, we compute the 'vv'-component of equations of motion for Riemann tensor squared theory, previously discussed in §4.2.1, as follows

$$\begin{aligned} E^{\rm HD}_{vv} = a_3 \bigg[&\partial_v \left(\frac{1}{\sqrt{h}} \, \partial_v \left(\sqrt{h} \left(\omega^2 + 4\, X - 4\, \nabla \cdot \omega + 4\, \bar{\mathcal{K}}_{ij} \, \mathcal{K}^{ij} \right) \right) \right) \\ &+ 4\, \partial_v \big(\nabla_i \left(2\, h^{ij} \, \partial_v \omega_j + \omega^i \, \mathcal{K} - h^{ij} \, \nabla_j \mathcal{K} - 2\, \omega_j \, \mathcal{K}^{ij} \right) \big) \\ &- 4\, \partial_v \big(\nabla_i \nabla_j \big(\mathcal{K}^{ij} - \mathcal{K} \, h^{ij} \big) \big) \bigg] \end{aligned} \tag{B.27}$$

We take note of the fact that in this case the Wald entropy density for the higher derivative part of the Lagrangian is

$$s^{\rm HD}_w = -4\, a_3 \, R_{rvrv} \tag{B.28}$$

and using this we compute

$$\partial_v \left(\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h} \, s^{\rm HD}_w \right) \right) = -4\, a_3 \, \partial_v \left(\frac{1}{\sqrt{h}} \, \partial_v \left(\sqrt{h} \left(X + \frac{1}{4} \, \omega^2 \right) \right) \right). \tag{B.29}$$

Next we compute $\mathbb{E}^{\rm HD*}_{vv}$ defined in (4.43) as given below,

$$\mathbb{E}^{\rm HD*}_{vv} = 4\, a_3 \, \partial_v \left[\frac{1}{\sqrt{h}} \, \partial_v \big(\sqrt{h} \, \bar{\mathcal{K}}^{ij} \, \mathcal{K}_{ij} \big) + \nabla_i \big(h^{ij} \, \partial_v \omega_j - \nabla_j \mathcal{K}^{ij} \big) \right]. \tag{B.30}$$

Finally we are now at a stage to write down the expressions for the components of the entropy current

$$\begin{aligned} J^v &= -\, s^{\rm HD}_w - 4\, a_3 \, \bar{\mathcal{K}}^{ij} \, \mathcal{K}_{ij} \, , \\ J^i &= 4\, a_3 \left(\nabla_j \mathcal{K}^{ij} - h^{ij} \, \partial_v \omega_j \right) \end{aligned} \tag{B.31}$$