# SOLUTIONS IN THE CLASS OF MEASURES FOR SOME HYPERBOLIC SYSTEMS OF CONSERVATION LAWS AND SCALAR CONSERVATION LAWS WITH DISCONTINUOUS FLUX

By

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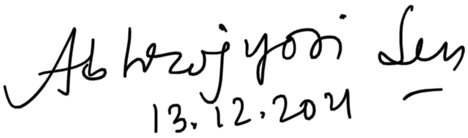

Abhrojyoti Sen

## DEDICATIONS

Foremost, I would like to dedicate this thesis to my ***Father and Mother*** whose immense love, affection, encouragement and prays of day and night made me what I am today. They have always been cynosure to my life and will remain the same. Also this work is dedicated to Pradip Kumar Ray (jethu), Tripti Ray(jethima), Partha Sarathi Ray (partha dada) and Shikha Ray (mom didi), who introduced me with good literature, films and music without which I would remain incomplete and finally to my all childhood friends for gifting me fantastic memories.

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# ABSTRACT

The objective of this thesis is manifold. Based on the different problems and methods, we divide the thesis into four parts. In the first part, we considered some strictly hyperbolic systems adjoined with Riemann-type initial data and studied the limiting behavior of their solutions. We showed that the solution of a non-strictly hyperbolic system can be obtained by studying the limits of the solutions for a strictly hyperbolic system. In the second part, we consider the 1D Saint-Venant model which is a non-strictly hyperbolic system of balance laws. The third part consists of the one-dimensional gas dynamics equation in the quarter plane. Finally in the fourth part, measured valued solutions for scalar conservation laws with discontinuous flux are obtained. In chapter 1, we collect the standard results for the system of conservation laws, numerical approximation for conservation laws, and some introductory material for non strictly hyperbolic systems. Chapter 2 and chapter 3 are devoted to studying the limiting behavior of the solutions for the Euler equation of compressible fluid flow and its generalization. Moreover, in chapter 2, we provide an example of a strictly hyperbolic system whose solution contains $\delta$-measure. In chapter 4, We study the kinematic 1D Saint-Venant model and use the vanishing viscosity method to obtain an explicit formula for the model. In chapter 5, the question of solvability of the initial-boundary value problem for the 1D gas dynamics equation is addressed. Using the method of generalized potentials and characteristic triangles, extended to the boundary value case, an explicit way of constructing measure-valued solutions is presented. The prescription of boundary data is shown to depend on the behavior of the generalized potentials at the boundary. In chapter 6, we study the scalar conservation laws with discontinuous flux with *overcompressive flux pair*. A generalized solution containing $\delta$-measure is proposed. The vanishing viscosity method is used for the linear fluxes and a Lax-Oleinik type formula is obtained for general fluxes. An explicit numerical scheme is developed to capture the $\delta$-shock solution efficiently.

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# Summary

The well-established theory for the system of conservation laws generally assumes that the system is strictly hyperbolic and the characteristic fields are either genuinely non-linear or linearly degenerate. Moreover, the classical existence theories are applicable when the total variation of the initial data is small. It is well known that the solution of conservation laws may develop discontinuities after a finite time even though the initial data is smooth. In that case, the solution space is generally $L^p, 1 \leq p \leq \infty$ or $BV$, the space of the functions of bounded variation. The solution is understood in a weak sense and in general the weak solutions are not unique. In the literature, several admissible criteria have been developed to establish the uniqueness of the solution depending upon its physical relevance. In practice, there are systems that may violate both of the conditions (strict hyperbolicity and small total variation of initial data) assumed in these theories. The solution class for these systems may be wider than $L^\infty$ or $BV$. Korchinski first observed that the solution of a $2 \times 2$ non strictly hyperbolic system does not lie in $BV(\mathbb{R})$ but a Borel measure.

This thesis is concerned about the existence and uniqueness of the non-classical (measure-valued) solutions for the following conservation laws arising from various physical models: (i) Euler equations of compressible fluid flow and its generalization, (ii) system of non-strictly balance laws arising from 1D Saint-Venant model, (iii) initial-boundary value problems for 1D pressureless gas dynamics model and (iv) scalar conservation laws with discontinuous flux function. The main focus of this thesis is to obtain the explicit formulae for the above systems. For the system mentioned in (i), we used the shadow-wave method and a vanishing pressure limit approach which can be viewed as a variant of the vanishing viscosity method. In the presence of small pressure, the system is strictly hyperbolic thus by using Lax theory we constructed a solution and found its distributional limit as the pressure term approaches zero. The obtained distributional limit satisfies as the solution in a weak sense. For the system mentioned in (ii), we used the vanishing viscosity method. To be precise, we mainly used a transformation similar to Hopf-Cole transformation to linearize the parabolic approximation (viscous approximation) of the system, and then by passing to the limit we got an explicit formula. Using Volpert's product we showed that the ob-

tained explicit formula is a solution. The initial value problem for the system mentioned in (iii) in one and higher space dimension has been extensively pursued in the literature in the past decades but to the best of our knowledge, no attempts have been made so far to solve the initial-boundary value problem. We extended the work Huang by introducing a second type of potential-boundary potential. Furthermore, the boundary condition is designed in a physically meaningful way. For the scalar conservation laws with discontinuous flux (for *overcompressive flux pair*) mentioned in (iv), first, we used the vanishing viscosity method to obtain an explicit formula for a special type of flux function. For general *overcompressive flux pair*, a weak formulation is proposed which allows concentration along t-axis and an explicit formula is constructed satisfying the weak formulation. A numerical scheme is proposed which effectively captures the solution and its convergence analysis is carried out.

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# Chapter 1

# System of conservation laws

In this chapter, we recollect the well-known results about the general theory of the hyperbolic system of conservation laws and review the general solution of the Riemann problem. Also, we introduce some preliminary facts about the non-strictly hyperbolic systems and numerical approximation of conservation laws. The proofs are mostly taken from [1, 2]. For further reference, see Lax [3], Glimm [4].

## 1.1 Mathematical preliminaries

In this section, we recall some of the basic facts and tools which will be useful in the general theory of conservation laws.

**Theorem 1.1.1** (Implicit function theorem)**.** *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open sets and $f : U \times V \to \mathbb{R}^m$ be a $k$-times continuously differentiable function with $k \geq 1$, such that $f(a, b) = 0$ for some point $(a, b) \in U \times V$. Assume also $D_x f(a, b)$ is invertible. Then there exists a neighbourhood $W \subset V$ of $b$ and a $k$-times differentiable function $g : W \to U$ such that*

$$g(b) = a \quad \text{and} \quad f(g(y), y) = 0$$

*for every $y \in W$.*
*The derivative of $g$ at the point $b$ is a $n \times m$ matrix and is given by*

$$Dg(b) = -[D_x f(a, b)]^{-1} D_y f(a, b).$$

The next result is the parameterized version of the implicit function theorem, meaning $f$ depends also on a parameter $z$.

**Theorem 1.1.2** (Parametarized version of implicit function theorem). *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^m$ are open sets and $f : U \times V \times W \to \mathbb{R}^m$ is a $k$-times continuously differentiable map, with $k \geq 1$. Let $\xi : W \to U \times V$ be a $k$-times continuously differentiable map such that $f(\xi(z), z) = 0$ for every $z \in W$. If the Jacobian $D_x f(\xi(z), z)$ is invertible for every $z$ in a compact set $K \subset W$. Then there exists a small $\epsilon > 0$ and a $k$-time continuously differentiable map $\varphi : V \times W \to U$ such that*

$$\varphi(\xi_2(z), z) = \xi_1(z), \quad f(\varphi(y, z), y, z) = 0$$

*where $z \in K$ and $|y - \xi_2(z)| < \epsilon$.*

Let $A$ be a $m \times m$ matrix where the entries $a_{ij}$'s are smooth functions on $\mathbb{R}^m$, that is $A : \mathbb{R}^m \to \mathbb{M}^{m \times m}$. We assume the following property of $A$.
*For each $z \in \mathbb{R}^m$, the eigenvalues of $A(z)$ are real and distinct. Thus the eigenvalues $\lambda_i(z)$ can be arranged in an increasing order*

$$\lambda_1(z) < \lambda_2(z) < \cdots < \lambda_m(z), \quad (z \in \mathbb{R}^m) \tag{1.1.1}$$

Let $r_i(z), i = 1, 2, .., m$ are the corresponding right eigenvectors such that

$$A(z)r_i(z) = \lambda_i(z)r_i(z), \quad i = 1, 2, ..., m \text{ and } (z \in \mathbb{R}^m)$$

Since we always assume (1.1.1), the eigenvectors $\{r_i(z)\}_{i=1}^m$ span the whole space of $\mathbb{R}^m$. We also introduce left eigenvectors $\{l_i(z)\}_{i=1}^m, i = 1, 2, ...., m$ such that

$$l_i(z)A(z) = \lambda_i(z)l_i(z), \quad i = 1, 2, ..., m \text{ and } (z \in \mathbb{R}^m)$$

One can easily observe that

$$\begin{aligned} \lambda_i(z)[l_j(z) \cdot r_i(z)] &= l_j(z)[A(z) \cdot r_i(z)] \\ &= \lambda_j(z)[l_j(z) \cdot r_i(z)] \end{aligned}$$

Since $\lambda_j(z) \neq \lambda_i(z)$ for $i \neq j$, we get $l_j(z) \cdot r_i(z) = 0$. We impose the conditions on the eigenvectors

$$|r_i(z)| = |l_i(z)| = 1, \;\; l_j(z) \cdot r_i(z) = \begin{cases} 0, & if\ i \neq j \\ 1 & if\ i = j \end{cases} \tag{1.1.2}$$

Now we study the behavior of $\lambda_i(z), r_i(z), l_i(z)$ as the parameter $z$ changes. We will see that if $A$ is a matrix of smooth functions depending on a parameter $z$, the eigenvalues and the corresponding eigenvectors are also smooth functions depending on $z$.

**Theorem 1.1.3.** *Let $A = [a_{i,j}]_{i,j=1}^m$ is a matrix where $a_{ij}(z)$ are $k$-times continuously differentiable function for $z \in \mathbb{R}^m$ and $A$ satisfies the condition* (1.1.1). *Then*
(a) *The eigenvalues $\{\lambda_i(z)\}_{i=1}^m$ are $k$-times continuously differentiable functions.*
(b) *The corresponding right and left eigenvectors $\{r_i(z)\}_{i=1}^m$ and $\{l_i(z)\}_{i=1}^m$ are also $k$-times continuously differentiable functions.*

*Proof.* Since $A$ satisfies the condition (1.1.1), for a fixed $\bar{z} \in \mathbb{R}^m$ we find

$$\lambda_1(\bar{z}) < \lambda_2(\bar{z}) < \cdots\cdot < \lambda_m(\bar{z}).$$

For a fixed $i \in \{1, 2...., m\}$ and $\bar{z} \in \mathbb{R}^m$, let $r_i(\bar{z})$ satisfy

$$A(\bar{z}) r_i(\bar{z}) = \lambda_i(\bar{z}) r_i(\bar{z}).$$

Without loss of any generality, one may assume

$$r_i(\bar{z}) = (0, ...., 1).$$

We show that for $|z - \bar{z}| < \epsilon$, there exists $k$-times differentiable maps $\lambda_i(z)$ and $r_i(z)$ such that

$$A(z) r_i(z) = \lambda_i(z) r_i(z). \tag{1.1.3}$$

First let us define $\chi : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^{m+1}$ by

$$\chi(\xi, \lambda, z) = \Big( A(z)\xi - \lambda I, |\xi|^2 \Big)$$

Simplifying the map and rewriting, we have

$$\chi(\xi,\lambda,z)=\Big(\xi_1 a_{11}+\cdot\cdot+\xi_m a_{1m}-\lambda\xi_1,\cdot\cdot\cdot,\xi_1 a_{m1}+\cdot\cdot+\xi_m a_{mm}-\lambda\xi_m,|\xi|^2\Big)$$

Now we calculate

$$\frac{\partial\chi(\xi,\lambda,z)}{\partial(\xi,\lambda)}=\begin{bmatrix}\frac{\partial\chi_1}{\partial\xi_1} & \frac{\partial\chi_1}{\partial\xi_2} & \cdots & \frac{\partial\chi_1}{\partial\lambda}\\ \cdots & & & \\ \cdots & & & \\ \frac{\partial\chi_{m+1}}{\partial\xi_1} & \frac{\partial\chi_{m+1}}{\partial\xi_2} & \cdots & \frac{\partial\chi_{m+1}}{\partial\lambda}\end{bmatrix}$$
$$=\begin{bmatrix} & & & -\xi_1\\ & A(z)-\lambda I & & \\ & & & -\xi_m\\ 2\xi_1 \;.. & 2\xi_2.. & 2\xi_m & 0\end{bmatrix}$$

Since the eigenvenctor corresponding to $\lambda_i(\bar{z})$ is taken as $(0,...,1)$, we will able to apply implicit function theorem if we have,

$$\det\begin{bmatrix} & & & 0\\ & A(\bar{z})-\lambda_i(\bar{z})I & & \\ & & & -1\\ 0 \;.. & 0.. & 2 & 0\end{bmatrix}\neq 0 \tag{1.1.4}$$

To prove the above assertion we construct a perturbed matrix

$$A^\delta(z)=A(z)-(\lambda_i(z)+\delta)I,\;\;\text{for } \delta>0 \text{ small.}$$

We note that $A^\delta(z)$ is invertible for sufficiently small $\delta$. Indeed, suppose $A^\delta$ is not invertible, so we have det$(A(z)-(\lambda_i(z)+\delta)I)=0$ which implies $\lambda_i(z)+\delta$ is an eigenvalue of $A(z)$. Thus $\lambda_i(z)+\delta=\lambda_j(z)$ for some $j\neq i$. But we can choose $\delta$ sufficiently small so that the above relation does not hold for any $j$. Since $r_i(\bar{z})=(0,..,1)$, we also have

$$A^\delta(\bar{z})r_i(\bar{z})=-\delta r_i(\bar{z}). \tag{1.1.5}$$

Now using (1.1.5) one can easily observe that

$$\begin{bmatrix} & & & 0\\ & A^\delta(\bar{z}) & & \\ & & & -1\\ 0 \;.. & 0.. & 2 & 0\end{bmatrix}\begin{bmatrix} & & & 0\\ & I & & \\ & & & -\frac{1}{\delta}\\ 0 \;.. & 0.. & 2 & 1\end{bmatrix}=\begin{bmatrix} & & & 0\\ & A^\delta(\bar{z}) & & \\ & & & 0\\ 0 \;.. & 0.. & 2 & -\frac{2}{\delta}\end{bmatrix}$$

Thus

$$\det \begin{bmatrix} & & & 0 \\ & A^{\delta}(\bar{z}) & & \\ & & & -1 \\ 0 & .. & 0.. \quad 2 & 0 \end{bmatrix} = \frac{2}{\delta}\det(A^{\delta}(\bar{z})) = 2\prod_{i\neq j}(\lambda_j(\bar{z}) - (\lambda_i(\bar{z}) + \delta))$$

Now as $\delta \to 0$, the last expression tends to $2\prod_{i\neq j}(\lambda_j(\bar{z}) - \lambda_i(\bar{z})$ and since $A(\bar{z})$ satisfies (1.1.1), we get that the expression is nonzero which proves our claim (1.1.4). Thus applying implicit function theorem we find $\lambda_i(z), r_i(z)$ for $|z - \bar{z}| < \epsilon$ satisfying (1.1.3).

To show the global existence of $\lambda_i(z), r_i(z)$, we define

$$R = \sup_{r>0}\Big\{r : \lambda_i(z), r_i(z) \text{ exist and } k\text{-times continuously differentiable on } B(0,r)\Big\}$$

If $R$ is finite, we can cover $\partial B(0, R)$ by a finite number of open sets where we can extend $\lambda_i(z), r_i(z)$ continuously. But this gives a contradiction to the definition of $R$. Thus $R = \infty$ and we are done. □

## 1.2 The notion of weak solution

In this section, we will discuss the notion of the weak solution and Rankine-Hugoniot condition for a solution. Let us start with reviewing the definition of conservation laws. A scalar conservation law is a first-order PDE of the form

$$u_t + f(u)_x = 0 \tag{1.2.1}$$

where $u(x, t)$ is a locally integrable function defined on $U \subset \mathbb{R} \times \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. Here $u$ and $f$ are the conserved quantity and flux function respectively. Integrating (1.2.1) over an interval $[a, b]$ we find

$$\frac{d}{dt}\int_a^b u(x,t)dx = -\int_a^b f(u(x,t))dx = f(u(a,t)) - f(u(b,t)).$$

This shows that the change of $u$ only depends on the flow through the endpoints of an interval. This calculation justifies the name *conservation laws*. Next, we will be studying the $n \times n$ *system of conservation laws* defined below:

$$\begin{cases} \frac{\partial u_1}{\partial t} + \frac{\partial f^1(u_1,...u_m)}{\partial x} = 0 \\ \cdots \\ \cdots \\ \frac{\partial u_m}{\partial t} + \frac{\partial f^m(u_1,...u_m)}{\partial x} = 0 \end{cases} \tag{1.2.2}$$

where $u = (u_1, ...., u_m)$ is a vector in $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth map. For smooth solutions the above system (1.2.2) is equivalent to

$$\frac{\partial}{\partial t}u + A(u)\frac{\partial}{\partial x}u = 0$$

where

$$A(u) := Df(u) = \begin{pmatrix} \frac{\partial f^1}{\partial u_1} & \frac{\partial f^1}{\partial u_2} & \cdots\cdot & \frac{\partial f^1}{\partial u_m} \\ \cdots & & & \\ \cdots & & & \\ \frac{\partial f^m}{\partial u_1} & \frac{\partial f^m}{\partial u_2} & \cdots\cdot & \frac{\partial f^m}{\partial u_m} \end{pmatrix},$$

is $m \times m$ Jacobian matrix of $f$ at the point $u$.

**Definition 1.2.1** (Strictly hyperbolic systems)**.** *A system of conservation laws is said to be strictly hyperbolic if for every $u \in \mathbb{R}^m$, the Jacobian matrix $A(u) := Df(u)$ has $m$ real distinct roots with $\lambda_1(u) < ..... < \lambda_m(u)$.*

The next theorem shows that the strictly hyperbolic property is independent of coordinate transformation.

**Theorem 1.2.2.** *Let $u(x, t)$ be a smooth solution for the strictly hyperbolic system* (1.2.2) *and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ be a smooth diffeomorphism such that $\phi(v(x, t)) = u(x, t)$. Then $v(x, t)$ solves a strictly hyperbolic system*

$$v_t + B(v)v_x = 0.$$

*Proof.* A straightforward calculation shows that $u_t = D\phi(v(x, t))v_t$ and $u_x = D\phi(v(x, t))v_x$.

Then

$$
\begin{aligned}
0 = u_t + A(u)u_x &= D\phi(v)v_t + A(\phi(v))D\phi(v)v_x \\
&= v_t + D\phi(v)^{-1}A(\phi(v))D\phi(v)v_x \\
&= v_t + B(v)v_x
\end{aligned}
$$

It remains to show that $v_t + B(v)v_x$ is strictly hyperbolic. Since $A(u)$ is strictly hyperbolic we have

$$A(u)r_i(u) = \lambda_i(u)r_i(u)$$

for some eigenvalue $\lambda_i(u)$ and corresponding eigenvector $r_i(u)$. Now we set

$$r_i(v) = D\phi(v)^{-1}r_i(\phi(v)), \quad \lambda_i(v) = \lambda_i(\phi(v))$$

A simple calculation gives,

$$B(v)r_i(v) = D\phi(v)^{-1}A(\phi(v))r_i(\phi(v)) = \lambda_i(\phi(v))r_i(v).$$

A similar proof holds for the left eigenvector $l_j(u)$. This completes the proof. □

Next, we describe a fundamental phenomenon of the Cauchy problem

$$
\begin{aligned}
&u_t + f(u)_x = 0 \\
&u(x,0) = u_0(x).
\end{aligned}
$$

where $f : \mathbb{R} \to \mathbb{R}$ a nonlinear smooth function and initial data $u_0(x)$ is also smooth. For example, we show that even the initial data is smooth, the solution can develop discontinuity after a finite time.

**Example 1.2.3.** *Let us consider the Burgers' equation with a continuous initial data,*

$$
\begin{aligned}
&u_t + \left(\frac{u^2}{2}\right)_x = 0; \\
&u(x,0) = \begin{cases} 1, & x \le 0; \\ 1 - x, & 0 \le x \le 1; \\ 0, & x \ge 1. \end{cases}
\end{aligned}
$$

*We use the method of characteristic to solve the Cauchy problem up to the time when characteristics do not meet. Thus, we have*

$$\frac{dx(t)}{dt} = u(x(t), t), \quad x(0) = \eta.$$

*Now*

$$\frac{d}{dt}u(x(t), t) = u_t(x(t), t) + \frac{dx(t)}{dt}u_x((t), t) = 0.$$

*So the solution is constant along the characteristic curve and one finds*

$$X(\eta, t) = \eta + tu_0(\eta), \quad \text{that is;}$$
$$X(\eta, t) = \begin{cases} \eta + t, & \eta \leq 0; \\ \eta + t(1 - \eta), & 0 \leq \eta \leq 1; \\ \eta, & \eta \geq 1. \end{cases}$$

*We observe that the characteristics meet at time $t = 1$ and for $t < 1$, solution i given by*

$$u(x, t) = \begin{cases} 1, & \eta \leq 0 \\ \frac{1-x}{1-t}, & 0 \leq \eta \leq 1 \\ 0 & \eta \geq 0. \end{cases}$$

*But after time $t > 1$ and in the region $\{1 \leq x \leq t\}$ the solution $u(x, t)$ becomes double-valued function. So discontinuity must develop in this region. This phenomenon leads us to seek global (in time) solutions into the class of discontinuous functions and to define the solution in a weak sense.*

**Definition 1.2.4.** *Given an initial data*

$$u(x, 0) = u_0(x) \tag{1.2.3}$$

*with $u_0 \in \mathbb{L}^1_{loc}(\mathbb{R}; \mathbb{R}^m)$, a function $u : [0, T] \times \mathbb{R} \to \mathbb{R}^m$ is said to be a distributional solution for the Cauchy problem* (1.2.2)-(1.2.3) *if the following integral identity holds:*

$$\int_0^T \int_{-\infty}^{\infty} [u(x, t)\varphi_t(x, t) + f(u)\varphi_x(x, t)]dxdt + \int_{-\infty}^{\infty} u_0(x)\varphi(x, 0)dx = 0$$

*for every test function $\varphi \in C_c^\infty(\mathbb{R} \times [0, T))$.*

Next, we define the notion of weak solution which is somewhat a stronger concept in the sense that, we demand the continuity of $u$ as a function of $t$, with values into $\mathbb{L}^1_{loc}(\mathbb{R})$.

**Definition 1.2.5.** *A function $u : [0,T]\times\mathbb{R} \to \mathbb{R}^m$ is said to be a weak solution of the Cauchy problem* (1.2.2)-(1.2.3) *if $u(\cdot,t) : [0,T] \to \mathbb{L}^1_{loc}(\mathbb{R})$ is continuous, the initial condition* (1.2.3) *holds and $u|_{(0,T)\times\mathbb{R}}$ is a distributional solution.*

**Theorem 1.2.6.** *Every weak solution is a distributional solution but not vice-versa.*

*Proof.* See [2] □

**Theorem 1.2.7** (Rankine-Hugoniot condition)**.** *Let $U \subset \mathbb{R}^m \times [0,\infty)$ is an open set and $u : U \to \mathbb{R}^m$ be a continuously differentiable function except a finite number of continuously differentiable curves $C_i : (0,\infty) \to \mathbb{R}$. We define*

$$\lim_{x\to C_i(t)+} u(x,t) = u_i^+(t), \qquad \lim_{x\to C_i(t)-} u(x,t) = u_i^-(t).$$

*Then the following statements are equivalent:*

*(a) $u(x,t)$ is a distributional solution of* (1.2.2).

*(b) $u(x,t)$ satisfies the equation*

$$u_t + A(u)u_x = 0$$

*for almost every $(x,t)$ and for each $i$ and almost every $t \in (a_i,b_i)$ we have*

$$\dot{C}_i(t)[u_i^+(t) - u_i^-(t)] = f(u_i^+(t)) - f(u_i^-(t)$$

*Proof.* See [2] □

## 1.3 Admissibility conditions

The solution of conservation laws may not be unique in practice. Thus to choose a 'physically relevant solution' we need certain conditions, the so-called admissibility conditions.

### 1.3.1 Lax admissibility condition

Let us first begin by defining the averaged matrix for $Df(u) = A(u)$.

**Definition 1.3.1.** *Given two fixed state $u, v \in \mathbb{R}^m$, we define the averaged matrix for $Df$ by*

$$A(u, v) = \int_0^1 Df(\theta u + (1-\theta)v)d\theta.$$

*and $\lambda_i(u, v)$ for $i = 1, .., m$ are called the eigenvalues of $A(u, v)$.*

Now take a piecewise constant function

$$u(x,t) = \begin{cases} u_l, & x < \lambda t \\ u_r, & x > \lambda t. \end{cases}$$

By Rankine-Hugoniot condition and the fundamental theorem of calculus we have

$$\begin{aligned} \lambda(u_l - u_r) = f(u_l) - f(u_r) &= \int_0^1 f'(\theta u_l + (1-\theta)u_r)d\theta \\ &= \int_0^1 Df(\theta u_l + (1-\theta)u_r) \cdot (u_l - u_r)d\theta \\ &= A(u_l, u_r)(u_l - u_r) \end{aligned}$$

The last equality shows that the speed of the discontinuity curve $\lambda$ corresponds to the eigenvalue $\lambda(u_l, u_r)$ of the averaged matrix $A(u_l, u_r)$ with the eigenvector $(u_l - u_r)$. Now we state Lax admissibility condition.

Let $u(x,t)$ is a weak solution of (1.4.1)-(1.4.2) of the form

$$u(x,t) = \begin{cases} u_l, & x < \xi(t) \\ u_r, & x > \xi(t) \end{cases}$$

where $x = \xi(t)$ is a discontinuity curve of $u(x,t)$. Then from the above calculation we conclude that $\dot{\xi}(t) = \lambda_i(u_l, u_r)$ for some $i \in \{1, .., m\}$. The weak solution $u(x,t)$ is said to be admissible in Lax's sense if the following condition holds:

$$\lambda_i(u_l) > \dot{\xi}(t)(= \lambda_i(u_l, u_r)) > \lambda_i(u_r). \tag{1.3.1}$$

### 1.3.2 Vanishing viscosity and traveling waves

Now we consider the following regularised system

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon \tag{1.3.2}$$

where the term $\epsilon u_{xx}^\epsilon$ is interpreted as a viscosity. Our plan is to study the limit of $u^\epsilon$ as $\epsilon \to 0$. If $||u^\epsilon - u||_{\mathbb{L}^1_{loc}} \to 0$ as $\epsilon \to 0$, the $u$ is the admissible solution of (1.4.1). In the next theorem we demonstrate a connection between vanishing viscosity limit and Lax admissibility condition. Let us start with a piecewise constant function

$$u(x,t) = \begin{cases} u_l & x < \lambda t \\ u_r & x > \lambda t \end{cases}$$

The function $u(x,t)$ satisfies travelling wave entropy condition if $u^\epsilon(x,t) \to u(x,t)$ a.e as $\epsilon \to 0$ where

$$u^\epsilon(x,t) = v\Big(\frac{x-\lambda t}{\epsilon}\Big)$$

and $u^\epsilon$ is a classical solution of (1.3.2). Inserting $v\Big(\frac{x-\lambda t}{\epsilon}\Big)$ into the equation (1.3.2) we find

$$-\lambda \dot{v}(s) + \frac{\partial}{\partial s} f(v(s)) = \ddot{v}(s)$$

where $s = \frac{x-\lambda t}{\epsilon}$. Integrating the above equation once, we get

$$\dot{v}(s) = -\lambda v(s) + f(v(s)) + const. \tag{1.3.3}$$

Now since

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = \lim_{\epsilon\to 0} v\Big(\frac{x-\lambda t}{\epsilon}\Big) = \begin{cases} u_l & x < \lambda t \\ u_r & x > \lambda t, \end{cases}$$

so we have

$$\lim_{s\to-\infty} v(s) = u_l, \quad \lim_{s\to+\infty} v(s) = u_r.$$

Now employing the above limits in the equation (1.3.3) we get

$$0 = \lim_{s\to\pm\infty} \dot{v}(s), \; const. = \lambda u_l - f(u_l) = \lambda u_r - f(u_r)$$

Hence from the above equations we have

$$\lambda(u_l - u_r) = f(u_l) - f(u_r) \text{ and } \begin{cases} \dot{v}(s) = -\lambda(v(s) - u_l) + f(v(s)) - f(u_l) \\ \lim_{s\to-\infty} v(s) = u_l, \quad \lim_{s\to+\infty} v(s) = u_r. \end{cases} \tag{1.3.4}$$

**Theorem 1.3.2.** *Let the $i$-th characteristic field be genuinely nonlinear and $|u_l - u_r| < \varepsilon$. If there exists a travelling wave solution connecting $u_l$ to $u_r$ then $\lambda_i(u_l) > \lambda(= \lambda_i(u_l, u_r)) > \lambda_i(u_r)$.*

*Proof.* From the above calculation we note that $u_r = S_i(\xi)(u_l)$ for some $i \in \{1, ..., n\}$. Now it is enough to show that $u_r = S_i^-(\xi)(u_l)$. Setting $g(z) := f(z) - f(u_l) - \lambda(z - u_l)$, the above ODE in (1.3.4) takes the form

$$\dot{v}(s) = g(v(s)), \text{ and } g(u_l) = g(u_r) = 0$$

Now since $Dg(u_l) = A(u_l) - \lambda I$, the eigenvalues of $Dg(u_l)$ are $\lambda_i(u_l) - \lambda$ and the corresponding left and right eigenvectors are $\{l_i(u_l)\}_{i=1}^m$ and $\{r_i(u_l)\}_{i=1}^m$ respectively. Next we state a fact

$$\lambda = \frac{\lambda_i(u_l) + \lambda_i(u_r)}{2} + \mathcal{O}(\xi^2), \ \xi \to 0.$$

The proof of the above fact shall be presented in the next section, see (1.4.28). Assuming the above and $|u_l - u_r| < \varepsilon$ fact we find

$$\lambda_i(u_l) - \lambda = \frac{\lambda_i(u_l) - \lambda_i(u_r)}{2} + \mathcal{O}(|u_l - u_r|^2)$$

since $v(-\infty) = u_l$ and $v(\infty) = u_r$ then we have $\lambda_i(u_l) - \lambda > 0$ and $\lambda_i(u_r) - \lambda < 0$ in respectively. This proves the claim $u_r = S_i^-(u_l)$. □

### 1.3.3 Entropy and entropy flux pairs

**Definition 1.3.3.** *A smooth function $\eta : \mathbb{R}^m \to \mathbb{R}$ is called an entropy for the system $u_t + f(u)_x = 0$, with an entropy flux $q : \mathbb{R}^m \to \mathbb{R}$ if $\eta$ is convex and*

$$D\eta(z)Df(z) = Dq(z), \ \ z \in \mathbb{R}^m.$$

Once again let us consider the viscous equation

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon.$$

Assume that $u^\epsilon$ is uniformly bounded for sufficiently small $\epsilon$ and $u^\epsilon \to u$ in $\mathbb{L}^1_{loc}$ as $\epsilon \to 0$. The function $u$ is the vanishing viscosity limit which is a solution for the system $u_t + f(u)_x = 0$. Let us choose any smooth entropy-entropy flux pairs $(\eta, q)$. Left multiplying $D\eta(u^\epsilon)$ in the above viscous equation, we get

$$\begin{aligned}
&D\eta(u^\epsilon)u_t^\epsilon + D\eta(u^\epsilon)f(u^\epsilon)_x = \epsilon D\eta(u^\epsilon)u_{xx}^\epsilon \\
&\implies \eta(u^\epsilon)_t + D\eta(u^\epsilon)Df(u^\epsilon)u_x^\epsilon = \epsilon D\eta(u^\epsilon)u_{xx}^\epsilon. \\
&\implies \eta(u^\epsilon)_t + Dq(u^\epsilon)u_x^\epsilon = \epsilon D\eta(u^\epsilon)u_{xx}^\epsilon. \\
&\implies \eta(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon D\eta(u^\epsilon)u_{xx}^\epsilon.
\end{aligned}$$

Since

$$\eta(u^\epsilon)_{xx} = \Big(D^2\eta(u^\epsilon)u_x^\epsilon\Big) \cdot u_x^\epsilon + D\eta(u^\epsilon)u_{xx}^\epsilon,$$

the last equality can be written as

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon\eta(u^\epsilon)_{xx} - \epsilon\Big(D^2\eta(u^\epsilon)u_x^\epsilon\Big) \cdot u_x^\epsilon \tag{1.3.5}$$

The convexity of $\eta$ implies the Hessian matrix of $\eta$ to be positive semi definite, that is $\Big(D^2\eta(u^\epsilon)u_x^\epsilon\Big) \cdot u_x^\epsilon \geq 0$. Multiplying a test function $\varphi(x,t) \in C_c^\infty(\mathbb{R} \times (0,\infty))$ with $\varphi \geq 0$ and using integration by parts we get

$$\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty \eta(u^\epsilon)\varphi_t + q(u^\epsilon)\varphi_x dxdt &= \int_0^\infty \int_{-\infty}^\infty \epsilon\Big(D^2\eta(u^\epsilon)u_x^\epsilon\Big) \cdot u_x^\epsilon v - \epsilon\eta(u^\epsilon)v_{xx} dxdt \\
&\geq -\epsilon\eta(u^\epsilon)v_{xx} dxdt.
\end{aligned}$$

In the last inequality, we used the convexity of $\eta$ and positivity of $\varphi$. Now using the inequality on compact subsets of $\mathbb{R} \times (0,\infty)$

$$\int_0^\infty \int_{-\infty}^\infty |\eta(u^\epsilon) - \eta(u)| \leq ||\eta||_\infty \int_0^\infty \int_{-\infty}^\infty |u^\epsilon - u| \to 0$$

as $\epsilon \to 0$, we find

$$\int_0^\infty \int_{-\infty}^\infty \eta(u)\varphi_t + q(u)\varphi_x dxdt \geq 0.$$

Summing up the above calculations we write

**Definition 1.3.4** (Entropy inequality)**.** *A weak solution $u(x,t)$ of the system $u_t + f(u)_x = 0$ is said to be entropy admissible if*

$$\eta(u)_t + q(u)_x \leq 0$$

*holds in the sense of distribution, for every entropy-entropy flux pairs $(\eta, q)$ with convex entropy $\eta$ and the corresponding flux pair $q$.*

## 1.4 The Riemann problem

In this section, we investigate the structure of the solution for a system of conservation law when the initial data is piece-wise constant. More precisely, we will find the weak solutions to the Riemann problem for the system of conservation laws

$$u_t + f(u)_x = 0 \tag{1.4.1}$$

with the initial data

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0, \end{cases} \tag{1.4.2}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ smooth map and $u_l, u_r \in \mathbb{R}^m$. Let us start with seeking the solutions of (1.4.1) of the form $u(x,t) = V\left(\frac{x}{t}\right), (x,t) \in \mathbb{R} \times (0,\infty)$ where $v = (v^1, ..., v^m) : \mathbb{R} \to \mathbb{R}^m$ Now substituting $u(x,t) = v\left(\frac{x}{t}\right)$ into the equation (1.4.1) we find

$$\frac{1}{t}\Big[A\big(v\big(\frac{x}{t}\big)\big)\dot{v}\big(\frac{x}{t}\big) - \frac{x}{t}\dot{v}\big(\frac{x}{t}\big)\Big] = 0$$

This implies

$$\dot{v}\big(\frac{x}{t}\big) = r_i\big(v\big(\frac{x}{t}\big)\big), \quad \lambda_i\big(v\big(\frac{x}{t}\big)\big) = \frac{x}{t}, \quad \text{for some } i \in \{1, ..., m\}. \tag{1.4.3}$$

On the other hand, if we employ a change of variable $\frac{x}{t} = \xi$, from (1.4.1) we find

$$
\begin{aligned}
0 &= \dot{v}(\xi)\xi_t + A(v(\xi))\dot{v}(\xi)\xi_x \\
&= \dot{v}(\xi)\big[\xi_t + \lambda_i(v(\xi))\xi_x\big] \\
&= r_i(v(\xi))\big[\xi_t + \lambda_i(v(\xi))\xi_x\big],
\end{aligned}
$$

since eigenvectors are non-zero, we get

$$\xi_t + \lambda_i(v(\xi))\xi_x = 0.$$

Note that the above equation can be written in a form of a scalar conservation law

$$\xi_t + f_i(\xi)_x = 0$$

where

$$f_i(\xi) = \int_0^{\xi} \lambda_i(v(s))ds.$$

Since the general theory of scalar conservation law is developed [1] under the condition of convex or concave fluxes, we compute,

$$f_i''(\xi) = D\lambda_i(v(\xi)) \cdot \dot{v}(\xi) = D\lambda_i(v(\xi)) \cdot r_i(v(\xi)).$$

Thus we obtain the following possibilities,

$$D\lambda_i(v(\xi)) \cdot r_i(v(\xi)) > 0 \ \text{ or } \ D\lambda_i(v(\xi)) \cdot r_i(v(\xi)) < 0.$$

Also for linear fluxes we get $D\lambda_i(v(\xi)) \cdot r_i(v(\xi)) = 0$. All the computations above lead us to define the following:

**Definition 1.4.1.** *(a) For $i \in \{1, .., m\}$, the $i$-th characteristic field is said to be genuinely non-linear if the pair $(\lambda_i(z), r_i(z))$ satisfies*

$$D\lambda_i(z) \cdot r_i(z) \neq 0, \ \forall \ z \in \mathbb{R}^m.$$

*(b) For $i \in \{1, .., m\}$, the $i$-th characteristic field is said to be linearly degenarate if the pair $(\lambda_i(z), r_i(z))$ satisfies*

$$D\lambda_i(z) \cdot r_i(z) = 0, \ \forall \ z \in \mathbb{R}^m.$$

### 1.4.1 Rarefaction waves

Motivated by the previous computations and from the equation (1.4.3) we define

**Definition 1.4.2.** *For a given fixed state $\bar{z} \in \mathbb{R}^m$, we define the $i$-rarefaction curve by $\xi \mapsto R_i(\xi)(\bar{z})$ which is a solution of the ODE in* (1.4.3) *and passes through $\bar{z}$. If $i$-th characteristic field is genuinely nonlinear, we write*

$$R_i^+(\xi)(\bar{z}) := \{z \in R_i(\xi)(\bar{z}) | \lambda_i(z) > \lambda_i(\bar{z})\}$$

*and*

$$R_i^-(\xi)(\bar{z}) := \{z \in R_i(\xi)(\bar{z}) | \lambda_i(z) < \lambda_i(\bar{z})\}.$$

*Then the rarefaction curve is expressed as*

$$R_i(\xi)(\bar{z}) = R_i^+(\xi)(\bar{z}) \cup \{\bar{z}\} \cup R_i^-(\xi)(\bar{z})$$

**Remark 1.4.3.** *Note that when the $i$-th characteristic field is genuinely non-linear, $\lambda_i(z)$ is increasing along the integral curve of $r_i(z)$. Indeed, since $\lambda_i(z)$ is smooth, we calculate $\frac{d}{d\xi}\lambda_i(v(\xi)) = D\lambda_i(v(\xi)) \cdot r_i(v(\xi)) > 0$ for all $v \in \mathbb{R}^m$, by changing the sign of the eigenvector $r_i$, if necessary. On the other hand, when the $i$-th characteristic field is linearly degenerate, $\lambda_i(z)$ is constant along the integral curve of $r_i$.*

**Theorem 1.4.4** (Existence of Rarefaction waves)**.** *Let $u_l, u_r$ are given as* (1.4.2) *and the $i$-th characteristic field is genuinely non-linear. Assume that there exists $\bar{\xi} \geq 0$ such that $u_r = R_i^+(\xi)(u_l)$. Then for $t > 0$,*

$$u(x,t) = \begin{cases} u_l, & x < \lambda_i(u_l)t \\ R_i^+(\xi)(u_l), & \lambda_i(u_l)t < x < \lambda_i(u_r)t \\ u_r, & x > \lambda_i(u_r)t. \end{cases} \tag{1.4.4}$$

*defines a weak solution of* (1.4.1)-(1.4.2).

*Proof.* Since $u_r = R_i^+(\bar{\xi})(u_l)$, we have $\lambda_i(u_r) > \lambda_i(u_l)$ and we define a map $\Phi : [0, \bar{\xi}] \to [\lambda_i(u_l), \lambda_i(u_r)]$ such that $\Phi(\xi) = \lambda_i(R_i^+(\xi)(u_l))$. Again since $\Phi(0) = \lambda_i(u_l), \Phi(\bar{\xi}) = \lambda_i(u_r)$ and $\Phi$ is increasing due to the genuinely non-linearity, therefore $\Phi$ is onto. In the regions $\{x < \lambda_i(u_l)t\}$ and $\{x > \lambda_i(u_r)t\}$, $u(x,t)$ satisfies the equation (1.4.1) trivially. Now let us consider the region $\{\lambda_i(u_l) < x < \lambda_i(u_r)\}$. Since we are interested in self similar solutions, that is $u(x,t) = v(\frac{x}{t})$, clearly $u(x,t)$ is constant $x = ct$ line for any constant $c$. Hence along this line we have

$$0 = \frac{d}{dt}u(ct,t) = u_t(ct,t) + \frac{x}{t}u_x(ct,t). \tag{1.4.5}$$

Observe that in the region $\{\lambda_i(u_l) < x < \lambda_i(u_r)\}$, we have $\frac{x}{t} = \lambda_i(R_i^+(\xi)(u_l))$ for some $\xi \in [0, \bar{\xi}]$ becasue the map $\Phi$ is onto. Now we calculate,

$$\begin{aligned} \frac{\partial}{\partial x}u(x,t) = \frac{\partial}{\partial x}[R_i^+(\xi)(u_l)] &= \frac{dR_i^+(\xi)(u_l)}{d\xi} \cdot \frac{d\xi}{d\lambda_i} \cdot \frac{d\lambda_i}{dx} \\ &= r_i(R_i^+(\xi)(u_l))\big[D\lambda_i(R_i^+(\xi)(u_l)) \cdot r_i(R_i^+(\xi)(u_l))\big]^{-1}\frac{1}{t} \end{aligned}$$

Since the $i$-th characteristic field is genuinely non linear and $t > 0$ the last expression is parallal to $r_i(u), u \in \mathbb{R}^m$. Therefore it is also an eigenvector of $A(u)$ corresponding to the eigen value $\lambda_i(u), u \in \mathbb{R}^m$, that is $A(u)u_x = \lambda_i(u)u_x$. Thus from the equation (1.4.5), we find

$$u_t + \frac{x}{t}u_x = u_t + \lambda_i(R_i^+(\xi)(u_l))u_x = u_t + A(u)u_x = 0.$$

Now it remains to show $||u(x,t) - u(x,0)||_{\mathbb{L}^1_{loc}} \to 0$ as $t \to 0^+$. In the regions $\{x < \lambda_i(u_l)t\}$ and $\{x > \lambda_i(u_r)t\}$, the solution is defined by the initial data, thus the assertion holds trivially. Now consider the region $\{\lambda_i(u_l) < x < \lambda_i(u_r)\}$. Note that

$$||u(x,t) - u(x,0)||_{\mathbb{L}^1_{loc}} = \int_{\lambda_i(u_l)t}^{\lambda_i(u_r)t} |R_i^+(\xi)(u_l) - u(x,0)|dx \to 0$$

as $t \to 0^+$. This completes the proof. □

**Remark 1.4.5.** *It is worth full to note that if $u_r = R_i^-(\bar{\xi})(u_l)$ (equivalent to saying $\bar{\xi} < 0$), then by the definition of $R_i^z(\xi)(z), z \in \mathbb{R}^m$, we have $\lambda_i(u_l)t > \lambda_i(u_r)t$. Thus from the above* *construction of solution we find* $u(x,t) = \begin{cases} u_l \\ u_r \end{cases}$ *in the region* $\{\lambda_i(u_l)t > x > \lambda_i(u_r)t\}$ *which is absurd.*

### 1.4.2 Shock waves

In this section, we want to describe the solution for (1.4.1)-(1.4.2) which is of the form

$$u(x,t) = \begin{cases} u_l, & x < \lambda t \\ u_r, & x > \lambda t \end{cases} \tag{1.4.6}$$

for some $\lambda \in \mathbb{R}$. Then by Rankine-Hugoniot condition one must have

$$f(u_l) - f(u_r) = \lambda(u_l - u_r). \tag{1.4.7}$$

Now given a state $u_l \in \mathbb{R}^m$, we want to find all $u_r$ and $\lambda$ such that the solution can be expressed in the above form (1.4.6). But in the above equation (1.4.7), $n$ equations and $n+1$ unknowns. So to obtain a solution it is natural to invoke implicit function theorem.

**Theorem 1.4.6.** *Let the system* (1.4.1) *be strictly hyperbolic. Then for a fixed* $u_l \in \mathbb{R}^m$ *there exists* $\bar{\xi} > 0$, *$n$ number of smooth curves* $S_i : [-\bar{\xi}, \bar{\xi}] \to \mathbb{R}^m$ *and $n$ number of scalar functions* $\lambda_i : [-\bar{\xi}, \bar{\xi}] \to \mathbb{R}$ *such that*

$$f(u_l) - f(S_i(\xi)(u_l)) = \lambda_i(\xi)(u_l - S_i(\xi)(u_l)), \quad \xi \in [-\bar{\xi}, \bar{\xi}]. \tag{1.4.8}$$

*Furthermore, we have*

*(a)* $|\frac{dS_i(\xi)(u_l)}{d\xi}| = 1$ *and* $S_i(0)(u_l) = u_l$, $\lambda_i(0) = \lambda_i(u_l)$.

*(b)* $\frac{dS_i(\xi)(u_l)}{d\xi}|_{\xi=0} = r_i(u_l)$.

*(c)* $\frac{d\lambda_i(\xi)(u_l)}{d\xi}|_{\xi=0} = \frac{1}{2} D\lambda_i(u_l) \cdot r_i(u_l)$.

*(d)* $\frac{d^2 S_i(\xi)(u_l)}{d\xi^2}|_{\xi=0} = Dr_i(u_l) \cdot r_i(u_l)$.

*Proof.* For any two pints $u, v \in \mathbb{R}^m$, define

$$A(u,v) = \int_0^1 A(\theta u + (1-\theta)v)d\theta.$$

Since $A(u)$ consists of smooth entries, as $v \to u$ $A(u,v) \to A(u)$. $A(u)$ has $n$ real distinct smooth eigenvalues with $\lambda_1(u) < ... < \lambda_m(u)$. We want to show that there exists a neighbourhood $B(u,\delta)$ of $u$ such that for any $v \in B(u,\delta)$, matrix $A(u,v)$ has $n$ real distinct smooth eigenvalues with $\lambda_1(u,v) < ... < \lambda_m(u,v)$. Consider the polynomial

$$P(v,\lambda) \doteq \det(A(u,v) - \lambda I).$$

For a fixed $i \in \{1,...,m\}$, we have $P(u,\lambda_i(u)) = \det(A(u) - \lambda_i(u)I) = 0$ and $\frac{\partial P}{\partial \lambda}(u,\lambda_i(u)) \neq 0$. Thus by implicit function theorem we get a smooth map $v \to \lambda_i(u,v)$ such that $P(v,\lambda_i(u,v)) = 0$. This implies $\lambda_i(u,v)$ is an eigenvalue of $A(u,v)$. Also following the similar arguments of theorem 1, we get that the corresponding left eigenvectors $\{l_i(u,v)\}_{i=i}^m$ and right eigenvectors $\{r_i(u,v)\}_{i=1}^m$ are smooth satisfying the conditions (1.1.2). Now from fundamental theorem of caluculas, we have

$$f(u) - f(u_l) = \int_0^1 Df(\theta u + (1-\theta)u_l)(u-u_l)d\theta = A(u,u_l)(u-u_l).$$

So R-H condition to be satisfied, we must have

$$f(u) - f(u_l) = \lambda(u-u_l) = A(u,u_l)(u-u_l)$$

which means that $(u-u_l)$ is a right eigenvector of $A(u,u_l)$ with the eigenvalue $\lambda_i(u,u_l)$ for some $i \in \{1,..,m\}$. Multiplying a left eigenvector $l_j(u,u_l)$ in the above equation we have

$$\begin{aligned} & l_j(u,u_l)A(u,u_l)(u-u_l) = \lambda_i(u,u_l)l_j(u,u_l)(u-u_l) \\ \implies & \lambda_j(u,u_l)l_j(u,u_l)(u-u_l) = \lambda_i(u,u_l)l_j(u,u_l)(u-u_l) \\ \implies & l_j(u,u_l)(u-u_l)(\lambda_j(u,u_l) - \lambda_i(u,u_l)) = 0. \end{aligned}$$

Since $\lambda_j(u,u_l) \neq \lambda_i(u,u_l)$, to find the $i$-th eigenvector of $A(u,u_l)$ it is enough to solve the equation $l_j(u,u_l)(u-u_l) = 0$ for $j \neq i$. Note that there are $m$ unknowns and $m-1$ equations, thus to apply implicit function theorem let us define $\Phi : \mathbb{R}^m \to \mathbb{R}^{m-1}$

$$\Phi(u) = \Big(l_1(u,u_l)(u-u_l), .., l_{i-1}(u,u_l)(u-u_l), l_{i+1}(u,u_l)(u-u_l), .., l_m(u,u_l)(u-u_l)\Big))$$

We observe that $\Phi(u_l) = 0$ and $D\Phi(u_l) = \Big(l_1(u_l), .., l_{i-1}(u_l), l_{i+1}(u_l).., l_m(u_l)\Big)_{(m-1)\times m}$. Since $l_j(u_l)_{j=1}^m$ forms a basis of $\mathbb{R}^m$, $rank\{D\Phi(u_l)\} = m-1$. Hence by implicit function theorem we can find a smooth function (with a choice of parametarisation $\xi$ and for a fixed $i$) $S_i(.)(u_l) : [-\bar{\xi}, \bar{\xi}] \to \mathbb{R}^m$ such that

$$\Phi(S_i(\xi)(u_l)) = 0, \quad S_i(0)(u_l) = u_l. \tag{1.4.9}$$

and

$$|\frac{dS_i(\xi)(u_l)}{d\xi}| = 1.$$

Therefore $f(S_i(\xi)(u_l)) - f(u_l) = \lambda_i(S_i(\xi)(u_l), u_l)(S_i(\xi)(u_l) - u_l)$ holds and denoting $\lambda_i(S_i(\xi)(u_l), u_l) = \lambda_i(\xi)$ we write

$$f(S_i(\xi)(u_l)) - f(u_l) = \lambda_i(\xi)(S_i(\xi)(u_l) - u_l) \tag{1.4.10}$$

holds. Now the equation (1.4.9) implies $\Phi_j(S_i(\xi)(u_l)) = l_j(S_i(\xi)(u_l), u_l)(S_i(\xi)(u_l) - u_l), j \neq i$. Since $r_i(.)$ is perpendicular to $l_j(.)$ we find $(S_i(\xi)(u_l) - u_l)$ is parallel to $r_i(S_i(\xi)(u_l), u_l)$. So there exists a function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$S_i(\xi)(u_l) = u_l + \alpha(\xi) \cdot r_i(S_i(\xi)(u_l), u_l).$$

which satisfies $\alpha(0) =$ and $\dot{\alpha}(0) = 1$. Thus

$$\frac{dS_i(\xi)(u_l)}{d\xi}|_{\xi=0} = \alpha(\xi) \cdot \frac{d}{d\xi} r_i(S_i(\xi)(u_l), u_l)|_{\xi=0} + r_i(S_i(\xi)(u_l), u_l)\dot{\alpha(\xi)}|_{\xi=0} = r_i(u_l)$$

Next, differentiating the equation (1.4.10) w.r.t $\xi$ we get

$$Df(S_i(\xi)(u_l))\dot{S}_i(\xi)(u_l) = \lambda_i(\xi)\dot{S}_i(\xi)(u_l) + \dot{\lambda}_i(\xi)(S_i(\xi)(u_l) - u_l)$$

To simplify the notations, setting $A(S_i(\xi)(u_l)) = A_i(\xi)$ and dropping the dependence on $\xi$, we write

$$A_i\dot{S}_i = \dot{\lambda}_i(S_i - u_l) + \lambda_i\dot{S}_i \tag{1.4.11}$$

Differentiating once again, we get

$$A_i\ddot{S}_i + \dot{S}_i\dot{A}_i = 2\dot{\lambda}_i\dot{S}_i + \ddot{\lambda}_i(S_i - u_l) + \lambda_i\ddot{S}_i \tag{1.4.12}$$

Since $\dot{S}_i(0) = r_i(u_l)$, calculating (1.4.12) at the point $\xi = 0$ we find

$$A_i(0)\ddot{S}_i(0)(u_l) + \dot{A}_i(0)r_i(u_l) = 2\dot{\lambda}_i(0)r_i(u_l) + \lambda_i(u_l)\ddot{S}_i(0)(u_l) \tag{1.4.13}$$

Let us consider

$$A_i(\xi)r_i(S_i(\xi)) = \lambda_i(S_i(\xi))r_i(S_i(\xi))$$

Differentiating the above equation at $\xi = 0$, we find

$$\begin{aligned} &A_i(0)Dr_i(u_l)\cdot r_i(u_l) + \dot{A}_i(0)r_i(u_l) \\ &= \Big(D\lambda_i(u_l)\cdot r_i(u_l)\Big)r_i(u_l) + \lambda_i(u_l)\Big(Dr_i(u_l)\cdot r_i(u_l)\Big) \end{aligned} \tag{1.4.14}$$

Now substituting $A_i(0)r_i(u_l)$ from (1.4.14) to (1.4.13) we get

$$\begin{aligned} &A_i(0)\ddot{S}_i(0)(u_l) + \Big(D\lambda_i(u_l)\cdot r_i(u_l)\Big)r_i(u_l) + [\lambda_i(u_l) - A_i(0)]\Big(Dr_i(u_l)\cdot r_i(u_l)\Big) \\ &= 2\dot{\lambda}_i(0)r_i(u_l) + \lambda_i(u_l)\ddot{S}_i(0)(u_l). \end{aligned} \tag{1.4.15}$$

Since $A_i(0) = A(u_l)$, multiplying the above equation by $l_i(u_l)$ we get

$$\dot{\lambda}_i(0) = \frac{1}{2}D\lambda_i(u_l)\cdot r_i(u_l). \tag{1.4.16}$$

Next using (1.4.16) in the equation (1.4.15) we find

$$\Big(A(u_l) - \lambda_i(u_l)\Big)\Big(\ddot{S}_i(0)(u_l) - Dr_i(u_l)\cdot r_i(u_l)\Big) = 0. \tag{1.4.17}$$

This implies $\ddot{S}_i(0)(u_l) - Dr_i(u_l)\cdot r_i(u_l)$ is an eigenvector of $A(u_l)$ with the eigenvalue $\lambda_i(u_l)$. Thus $\ddot{S}_i(0)(u_l) - Dr_i(u_l)\cdot r_i(u_l) = \beta r_i(u_l)$ for some $\beta \in \mathbb{R}$. Taking inner product with $r_i(u_l)$

$$\begin{aligned} \langle r_i(u_l), \beta r_i(u_l)\rangle &= \langle r_i(u_l), \ddot{S}_i(0)(u_l) - Dr_i(u_l)\cdot r_i(u_l)\rangle \\ &= \langle r_i(u_l), \ddot{S}_i(0)(u_l)\rangle - \langle r_i(u_l), Dr_i(u_l)\cdot r_i(u_l)\rangle \end{aligned} \tag{1.4.18}$$

Now we use the following identities:

(i) Let $X : \mathbb{R}^m \to \mathbb{R}^m$ , then

$$\langle X, DX \cdot X\rangle = \frac{1}{2} D\langle X, X\rangle \cdot X. \tag{1.4.19}$$

(ii) Let $\xi \mapsto X(\xi) : \mathbb{R} \to \mathbb{R}^m$, then

$$\langle \dot{X}, \ddot{X}\rangle = \frac{1}{2}\frac{d}{d\xi}\langle \dot{X}, \dot{X}\rangle \tag{1.4.20}$$

The last expression of (1.4.18) can be written as

$$\begin{aligned} &\langle \dot{S}_i(0)(u_l), \ddot{S}_i(0)(u_l)\rangle - \langle r_i(u_l), Dr_i(u_l) \cdot r_i(u_l)\rangle \\ &= \frac{1}{2}\frac{d}{d\xi}\langle S_i(\dot{0})(u_l), S_i(\dot{0})(u_l)\rangle - \frac{1}{2} D\langle r_i(u_l), r_i(u_l)\rangle \cdot r_l(u_l) = 0, \end{aligned} \tag{1.4.21}$$

as $|\dot{S}_i(0)(u_l)| = |r_i(u_l)| = 1$. □

Thus from equation (1.4.18) we have $\langle r_i(u_l), \beta r_i(u_l)\rangle = 0$, hence $\beta$ must be zero. So we get the identity (d).

**Theorem 1.4.7.** *If the $i$-th characteristic field is linearly degenerate then $R_i(\xi)(u_l) = S_i(\xi)(u_l)$.*

*Proof.* Suppose $v(\xi)$ solves the ode

$$\dot{v}(\xi) = r_i(v(\xi))$$

$$v(0) = u_l$$

Since $\lambda_i(\cdot)$ is constant in this case, we calculate

$$\begin{aligned} f(v(\xi)) - f(u_l) = \int_0^\xi \frac{d}{d\xi} f(v(s))ds = \int_0^\xi Df(v(s))\dot{v}(s)ds = \int_0^\xi Df(v(s))r_i(s)ds \\ = \int_0^\xi \lambda_i(v(s))r_i(s)ds \\ = \int_0^\xi \lambda_i(u_l)r_i(s)ds \\ = \lambda_i(u_l)(v(\xi) - u_l) \end{aligned}$$

Hence $v(\xi)$ lies in the $i$-th shock curve through $u_l$. □

Now let us assume the $i$-th characteristic field is genuinely nonlinear. By the previous theorem, we observe that

$$\frac{d\lambda_i(\xi)}{d\xi}|_{\xi=0} = \frac{1}{2} D\lambda_i(u_l) \cdot r_i(u_l) > 0 \tag{1.4.22}$$

and

$$\frac{d\lambda_i(S_i(\xi))(u_l)}{d\xi}|_{\xi=0} = D\lambda_i(u_l) \cdot r_i(u_l) > 0 \tag{1.4.23}$$

Since $\lambda_i$ and $S_i$ are smooth function, thus in a neighbourhood of $0$, $\frac{d\lambda_i(\xi)}{d\xi}$ and $\frac{d\lambda_i(S_i(\xi))(u_l)}{d\xi}$ will be positive, in other words $\lambda_i(\xi)$ and $\lambda_i(S_i(\xi)(u_l))$ are increasing in a neighbourhood of $0$. Now let us define a smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ given by

$$\Psi(\xi) = 2\lambda_i(\xi) - \lambda_i(S_i(\xi)(u_l))$$

Using a Taylor expansion of $\Psi$ near $0$, we get

$$\begin{aligned}
\Psi(\xi) &= \Psi(0) + \xi\dot{\Psi}(0) + \mathcal{O}(\xi^2) \\
&= 2\lambda_i(0) - \lambda_i(u_l) + \xi\Big[2\frac{d\lambda_i(\xi)}{d\xi}|_{\xi=0} - \frac{d\lambda_i(S_i(\xi))(u_l))}{d\xi}|_{\xi=0}\Big] + \mathcal{O}(\xi^2) \\
&= \lambda_i(u_l) + \mathcal{O}(\xi^2)
\end{aligned}$$

The last equality holds because of $\lambda_i(0) = \lambda_i(u_l)$ and the equations (1.4.22)-(1.4.23). Hence

$$2\lambda_i(\xi) - \lambda_i(u_l) - \lambda_i(S_i(\xi)(u_l)) = \mathcal{O}(\xi^2) \tag{1.4.24}$$

for $\xi$ near to $0$. Next we observe that for $\xi \in (0, \bar{\xi}]$, we have

$$\lambda_i(u_l) < \lambda_i(\xi) < \lambda_i(S_i(\xi)(u_l)). \tag{1.4.25}$$

Indeed, Since $\lambda_i(\xi)$ and $\lambda_i(S_i(\xi))$ is increasing near $0$, we find

$$\lambda_i(u_l) = \lambda_i(0) < \lambda_i(\xi) \tag{1.4.26}$$

and

$$\lambda_i(u_l) = \lambda_i(S_i(0)(u_l)) < \lambda_i(S_i(\xi)(u_l)) \tag{1.4.27}$$

Now from the equation (1.4.24) and (1.4.27) we have

$$\begin{aligned}\lambda_i(\xi) &= \frac{\lambda_i(u_l) + \lambda_i(S_i(\xi)(u_l))}{2} + \mathcal{O}(\xi^2) \\ &< \lambda_i(S_i(\xi)(u_l)) + \mathcal{O}(\xi^2)\end{aligned} \tag{1.4.28}$$

Thus combining (1.4.26) and (1.4.28) we get (1.4.25). Using similar arguments we get

$$\lambda_i(u_l) > \lambda_i(\xi) > \lambda_i(S_i(\xi)(u_l)) \tag{1.4.29}$$

for $\xi \in [-\bar{\xi}, 0)$ for some suitably small $\bar{\xi}$. Note that the inequality in (1.4.29) corresponds to *Lax admissibility condition.*

**Definition 1.4.8.** *For a given fixed state $\bar{z} \in \mathbb{R}^m$, we define the i-shock curve by $\xi \mapsto S_i(\xi)(\bar{z})$ which satisfies the equation* (1.4.8)*. If the $i$-th characteristic field is genuinely nonlinear, we write*

$$S_i^+(\xi)(\bar{z}) := \{z \in S_i(\xi)(\bar{z}) | \lambda_i(\bar{z}) < \lambda_i(\xi) < \lambda_i(S_i(\xi)(\bar{z}))\}$$

*and*

$$S_i^-(\xi)(\bar{z}) := \{z \in S_i(\xi)(\bar{z}) | \lambda_i(\bar{z}) > \lambda_i(\xi) > \lambda_i(S_i(\xi)(\bar{z}))\}$$

*Then the shock curve is expressed as*

$$S_i(\xi)(\bar{z}) := S_i^+(\xi)(\bar{z}) \cup \{\bar{z}\} \cup S_i^-(\xi)(\bar{z})$$

Now by the construction of the theorem (1.4.6) the shock solution $u(x,t)$ of the system (1.4.1) is defined as

**Definition 1.4.9.** *Assume that there exists a $\bar{\xi} < 0$ such that $S_i^-(\bar{\xi})(u_l) = u_r$, then the function*

$$u(x,t) = \begin{cases} u_l, & x < t\lambda_i(\xi) \\ S_i^-(\xi)(u_l), & x > t\lambda_i(\xi) \end{cases}$$

*is said to be a shock solution if it is a weak solution of* (1.4.1)-(1.4.2).

### 1.4.3 The general solution of the local Riemann problem

In this section we discuss the existence of a general solution to the local Riemann problem, more precisely we provide a unique weak solution of (1.4.1)-(1.4.2). In the previous sections, we discussed the solutions when $u_r$ in either on a rarefaction curve or on a shock curve passing through $u_l$. Here we use the previous construction to provide a solution for any Riemann type initial data, $u_l, u_r$ with $|u_l - u_r|$ small. Let us start with the following definition.

**Definition 1.4.10.** *(a) If the $i$-th characteristic field is genuinely nonlinear, for a fixed $\bar{z} \in \mathbb{R}^m$ we write*

$$T_i(\xi)(\bar{z}) := R_i^+(\xi)(\bar{z}) \cup \{\bar{z}\} \cup S_i^-(\xi)(\bar{z})$$

*(b) If $i$-th characteristic filed is linearly degenerate, for a fixed $\bar{z} \in \mathbb{R}^m$ we write*

$$T_i(\xi)(\bar{z}) := R_i^+(\xi)(\bar{z}) = S_i^-(\xi)(\bar{z}).$$

**Theorem 1.4.11.** *Assume that the $i$-th characteristic field is either genuinely nonlinear or linearly degenerate. Then for every compact set $K \subset \mathbb{R}^m$, there exists $\epsilon > 0$ such that the Riemann problem* (1.4.1)-(1.4.2) *has a unique weak solution whenever $u_l, u_r \in K$ and $|u_l - u_r| < \epsilon$.*

*Proof.* We plan to use implicit function theorem to a function $\Lambda : \mathbb{R}^m \to \mathbb{R}^m$ defined in a neighborhood of $0$, in the following manner.
Given a $\xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m$ with $|\xi|$ small. Now fixing the state $z_0 = u_l$ we will choose the states $z_0, ...., z_m$ such that :

$$z_1 = T_1(\xi_1)(z_0), \;\; \lambda_1(z_1) - \lambda_1(z_0) = \xi_1,$$

$$z_2 = T_2(\xi_2)(z_1), \;\; \lambda_2(z_2) - \lambda_1(z_1) = \xi_2,$$

.

.

$$z_m = T_{m-1}(\xi_m)(z_{m-1}), \;\; \lambda_m(z_m) - \lambda_m(z_{m-1}) = \xi_m.$$

and we define that map $\Lambda(\xi_1, ..., \xi_m)(z_0) = z_m$. Next we observe that by the definition of $R_i^+(\xi)(\bar{z})$ and $S_i^-(\xi)(\bar{z})$, the chosen state $z_1, ...., z_m$ can be represented as:

$$z_1 = \begin{cases} R_1^+(\xi_1)(z_0), & \xi_1 \geq 0, \\ S_1^-(\xi_1)(z_0), & \xi_1 < 0. \end{cases}$$

$$z_2 = \begin{cases} R_2^+(\xi_2)(z_1), & \xi_2 \geq 0, \\ S_2^-(\xi_2)(z_1), & \xi_2 < 0. \end{cases}$$

and finally,

$$z_m = \begin{cases} R_m^+(\xi_m)(z_{m-1}), & \xi_m \geq 0, \\ S_m^-(\xi_m)(z_{m-1}), & \xi_m < 0. \end{cases}$$

Thus $\Lambda$ can be represented as:

$$\Lambda(\xi_1, ..., \xi_m)(z_0) = T_m(\xi_m) \circ T_{m-1}(\xi_{m-1}) \circ \cdots \circ T_1(\xi_1)(z_0)$$

Now we observe that

$$\Lambda(0, .., 0)(z_0) = z_0$$

and

$$\frac{\partial \Lambda}{\partial \xi_i}|_{\xi=0} = r_i(z_0). \tag{1.4.30}$$

Indeed, we calculate,

$$\Lambda(0, ..., \xi_i, ..0)(z_0) - \Lambda(0, ..., 0)(z_0)$$

$$= T_i(\xi_i)(z_0) - z_0.$$

$T_i(\xi_i)(z_0)$ is either $R_i^+(\xi_i)(z_0)$ or $S_i^-(\xi_i)(z_0)$ depending on $\xi_i \geq 0$ or $\xi < 0$. Now by definition of rarefaction curve, we get

$$\begin{aligned} R_i^+(\xi_i)(z_0) &= R_i^+(0)(z_0) + \xi_i \frac{\partial}{\partial \xi_i} R_i^+(\xi_i)(z_0)|_{\xi_i=0} + \mathcal{O}(\xi_i^2). \quad 1 \\ &= z_0 + \xi_i r_i(z_0) + \mathcal{O}(\xi_i^2). \end{aligned}$$

Similarly, using the theorem (1.4.6), we have

$$\begin{aligned} S_i^-(\xi_i)(z_0) &= S_i^-(0)(z_0) + \xi_i \frac{\partial}{\partial \xi_i} S_i^-(\xi_i)(z_0)|_{\xi_i=0} + \mathcal{O}(\xi_i^2). \\ &= z_0 + \xi_i r_i(z_0) + \mathcal{O}(\xi_i^2). \end{aligned}$$

So in any case ($\xi \geq 0$ or $\xi < 0$) we have,

$$\Lambda(0, ..., \xi_i, ..0) - \Lambda(0, ..., 0) = \xi_i r_i(z_0) + \mathcal{O}(\xi_i^2)$$

Thus (1.4.30) is proved. Therefore the matrix $D\Lambda(0) = [\{r_i(z_0)\}_{i=1}^m]_{m\times m}$ is invertible since $r_i(z_0)$ forms a basis of $\mathbb{R}^m$ for each $i$. Now employing implicit function theorem we have a unique $\xi$ near $0$ such that $\Lambda(\xi)(u_l) = z_m = u_r$ whenever $|u_l - u_r| < \epsilon$. Now we will define explicit form of a weak solution $u(x,t)$ by combining rarefaction and shock waves. By previous theorems, we know that each Riemann problem

$$\begin{aligned} &u_t + f(u)_x = 0 \\ &u(x,0) = \begin{cases} z_{i-1}, & x < 0 \\ z_i, & x > 0. \end{cases} \end{aligned}$$

has a unique entropy solution consists of rarefaction and shock waves.

*Case I:* If the $i$-th characteristic field is genuinely nonlinear and $\xi_i \geq 0$, then the solution consists of a rarefaction wave and it is given by

$$u(x,t) = \begin{cases} z_{i-1}, & x < \lambda_i(z_{i-1})t \\ R_i^+(\xi)(z_{i-1}) & \lambda_i(z_{i-1})t < x < \lambda_i(z_i)t \\ z_i & x > \lambda_i(z_i)t \end{cases}$$

*Case II:* If the $i$-th characteristic curve is genuinely nonlinear and $\xi_i < 0$ or the $i$-th characteristic curve is linearly degenerate the solution is given by

$$u(x,t) = \begin{cases} z_{i-1}, & x < \lambda_i(z_{i-1}, z_i) \\ z_i, & x > \lambda_i(z_{i-1}, z_i) \end{cases}$$

where $\lambda_i(z_{i-1}) > \lambda_i(z_{i-1}, z_i) > \lambda_i(z_i)$ and $\lambda_i(z_{i-1}) = \lambda_i(z_{i-1}, z_i) = \lambda_i(z_i)$ hold respectively. Now for $(\xi_1, .., \xi_m)$ near $0$, using the strict hyperbolicity and continuity of $\lambda_i$'s we conclude that the intervals $[\lambda_i(z_{i-1}), \lambda_i(z_i)]$ would not intersect each other. Hence denoting $\lambda_i^+ = \lambda_i(z_i)$ and $\lambda_i^- = \lambda_i(z_{i-1})$ we have

$$\lambda_1^- \leq \lambda_1^+ \leq \lambda_2^- \leq \lambda_2^+ \leq \cdot\cdot \leq \lambda_m^- \leq \lambda_m^+.$$

Now the weak solution $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}^n$ is defined as

$$u(x,t) = \begin{cases} u_l, & -\infty < x < \lambda_1^- t \\ R_i^+(s)(z_{i-1}) & \lambda_i^- t < x < \lambda_i^+ t \\ z_i, & \lambda_i^+ t < x < \lambda_{i+1}^- t \\ u_r, & \lambda_m^+ t < x < \infty. \end{cases}$$

□

## 1.5 Finite Difference Schemes for Approximating Scalar Conservation Laws

Here we restrict ourselves to explicit one step finite difference schemes. Discretize the $x-$axis by a sequence $\{x_{i+\frac{1}{2}}\}$ with $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h, i \in Z, h > 0$ and $t-$axis by $\{t_n\}$ with $t_n = n\Delta t, n = 0, 1, 2, ..., \Delta t > 0$. $\Delta t$ and $h$ are called time step and spacial mesh size respectively. Let $\lambda = \frac{\Delta t}{h}$ and $x_i = \frac{x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}}{2}$.

To approximate the conservation law

$$\begin{cases} u_t + f(u)_x = 0 \text{ in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R} \end{cases} \tag{1.5.1}$$

we introduce $(2k + 1)$ point scheme of the form

$$v_i^{n+1} = H(v_{i-k}^n, v_{i-k+1}^n, ....., v_i^n, ....., v_{i+k-1}^n, v_{i+k}^n) \tag{1.5.2}$$

where $H : \mathbb{R}^{2k+1} \to \mathbb{R}$ is a continuous function and $v_i^n$ denotes the approximation of the exact solution $u$ at the grid point $(x_{i+\frac{1}{2}}, t_n)$. Initial data $\{v_i^0\}$ is defined by

$$v_i^0 = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(x)\,dx. \tag{1.5.3}$$

If $u_0(x)$ is continuous,then one can take

$$v_i^0 = u_i^0 = u_0(x_i)\ \forall i$$

**Definition 1.5.1.** *A difference scheme (1.5.2) is said to be in the conservative form,if there exists a continuous function $F : \mathbb{R}^{2k} \to \mathbb{R}$ such that*

$$H(v_{i-k}^n, ..., v_{i+k}^n) = v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{1.5.4}$$

*where $F_{i+\frac{1}{2}}^n = F(v_{i-k+1}^n, ..., v_{i+k}^n)$. The function $F$ is called* **numerical flux**.

**Definition 1.5.2.** *The difference scheme (1.5.4) is said to be consistent with the equation (1.5.1) if*

$$F(v, ..., v) = f(v)\ \ \forall v \in \mathbb{R} \tag{1.5.5}$$

$$i.e., \quad H(v, ..., v) = v\ \ \forall v \in \mathbb{R} \tag{1.5.6}$$

In order to analyse the convergence of the solution $\{v_i^n\}$ of the difference scheme (1.5.4) we introduce the piecewise constant function $v_h$ defined a.e. in $\mathbb{R} \times (0, \infty)$ by

$$v_h(x,t) = v_i^n \ \text{ for }\ (x,t) \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [t_n, t_{n+1}) \tag{1.5.7}$$

**Theorem 1.5.3** (Lax-Wendroff)**.** *Let $v_h$ be the numerical solution obtained from the scheme (1.5.2) which is in the conservative form and consistent with equation (1.5.1). Assume that there exists a sequence $\{h_k\}$ which tends to $0$ as $k \to \infty$ such that, if we set $\Delta_k t = \lambda h_k$ ($\lambda = \frac{\Delta_k t}{h_k}$, kept constant)*
*(i)$||v_{h_k}||_{L^\infty(\mathbb{R}\times(0,\infty))} \leq C$ for some constant $C > 0$,*
*(ii) $v_{h_k}$ converges in $L^1_{loc}(\mathbb{R} \times (0, \infty))$ and a.e to a function $v$. Then $v$ is a weak solution of (1.5.1).*

For detail see [5, Chapter 3, Theorem 1.1]

### 1.5.1 Examples of 3-point scheme:

The entropy solution of the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u(x,0) = \begin{cases} u_l & \text{if} \quad x < 0 \\ u_r & \text{if} \quad x > 0 \end{cases} \end{cases} \tag{1.5.8}$$

is self-similar,i.e., of the form

$$u(x,t) = w_R(\frac{x}{t}, u_l, u_r) \tag{1.5.9}$$

where $w_R$ depends only on the function $f$ and consists of two constant states $u_l$ and $u_r$. Now observe that $\xi \to f(w_R(\xi, u_l, u_r))$ is a continuous function at $\xi = 0$. Indeed if $w_R(\xi, u_l, u_r)$ is discontinuous at $\xi = 0$,this means that this discontinuity is **stationary**(the corresponding discontinuous wave moves with zero speed). Hence by Rankine-Hugoniot condition and continuity of $f(w_R(\xi, u_l, u_r))$ at $\xi = 0$ we have

$$f(w_R(0-, u_l, u_r)) = f(w_R(0+, u_l, u_r)) = f(w_R(0, u_l, u_r)) \tag{1.5.10}$$

**Example 1.5.4** (The Godunov scheme). *Now* **Godunov scheme** *is given by*

$$v_i^{n+1} = v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{1.5.11}$$

*where its numerical flux*

$$F_{i+\frac{1}{2}}^n = F^G(v_i^n, v_{i+1}^n) = f(w_R(0, v_i^n, v_{i+1}^n)).$$

*Godunov has given the following simple formula for a general $f$ to evaluate the numerical flux*

$$F^G(u,v) = f(w_R(0,u,v)) \tag{1.5.12}$$

$$= \begin{cases} \min_{w \in [u,v]} f(w) & \text{if } u \leq v \\ \max_{w \in [v,u]} f(w) & \text{if } u \geq v \end{cases} \tag{1.5.13}$$

*Godunov scheme (1.5.11) is in* **conservative form, consistent** *with the conservation law (1.5.1) and also an* **upwind scheme** *i.e.,*

$$F^G(u,v) = \begin{cases} f(u)\ if\ f\ is\ a\ nondecreasing\ function\ between\ u\ and\ v \\ f(v)\ if\ fis\ a\ nonincreasing\ function\ between\ u\ and\ v \end{cases} \tag{1.5.14}$$

**Remark 1.5.5.** *If $f$ is a convex function and $f(\theta) = \min_{w\in\mathbb{R}} f(w)$,then the Godunov flux (1.5.13) can be expressed in more simpler form*

$$F^G(u,v) = \max(f(\max(u,\theta)), f(\min(v,\theta))).$$

*Similarly if $f$ is concave and $f(\theta) = \max_{w\in\mathbb{R}} f(w)$,then*

$$F^G(u,v) = \min(f(\min(u,\theta)), f(\max(v,\theta))).$$

**Example 1.5.6** (The Lax-Friedrichs scheme)**.** *This is the simplest centered finite scheme which is given by*

$$\begin{aligned} v_i^{n+1} &= v_i^n - \frac{\lambda}{2}(f(v_{i+1}^n) - f(v_{i-1}^n)) \\ &= v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \end{aligned} \tag{1.5.15}$$

*where*

$$F_{i+\frac{1}{2}}^n = F^{LF}(v_i^n, v_{i+1}^n) = \frac{1}{2}(f(v_i^n) + f(v_{i+1}^n) - \frac{1}{\lambda}(v_{i+1}^n - v_i^n))$$

*This is in a conservative form and is consistent with conservation law (1.5.1) but is not an upwind scheme.*

## 1.5.2 Monotone and TVD Schemes

**Definition 1.5.7.** *The finite difference scheme (1.5.2) is said to be* **monotone** *if $H$ is non decreasing function of each of its arguments.*

Notation: Let $v = (v_i)_{i\in Z}$ and $w = (w_i)_{i\in Z}$ then $v \geq w$ means $v_i \geq w_i\ \forall i \in Z$.

Let $(H_\Delta(v))_i = H(v_{i-k}, v_{i-k+1}, ....., v_i, ....., v_{i+k-1}, v_{i+k})$ and $H_\Delta(v) = ((H_\Delta(v))_i)_{i\in Z}$ then scheme (1.5.2) is monotone means,

$$v \geq w \;\Rightarrow\; H_\Delta(v) \geq H_\Delta(w)$$

**Examples of monotone schemes:** Godunov,Lax-Friedrichs and Enquist-Osher schemes are monotone under the CFL like condition

$$\lambda \sup_{v_j} |f'(v_j)| \leq 1. \tag{1.5.16}$$

**Theorem 1.5.8.** *If a scheme is in conservative form, consistent, and monotone. Then we have the following*

1. *If a 3-point scheme, then the numerical flux $F(u, v)$ is an increasing function in its first argument and a decreasing function in its second argument.*

2. *($l^\infty$ stability):Let $v^n = (v_i^n)_{i\in Z} \in l^\infty$, then $v^{n+1} = (v_i^{n+1})_{i\in Z} \in l^\infty$ and*

$$||v^{n+1}||_\infty \leq ||v^n||_\infty \tag{1.5.17}$$

3. *($l^1$ contraction) Suppose $v^n \in l^\infty$ . Then $H_\Delta : l^1 \to l^1$ is a mapping which preserves the integral and for any sequence $u$ and $v$ in $l^1$ we have*

$$||H_\Delta(u) - H_\Delta(v)||_1 \leq ||u - v||_1 \tag{1.5.18}$$

For detail see [5, Chapter 3]

**Definition 1.5.9.** *A scheme is said to be* **total variation diminishing(TVD)** *if*

$$TV(v^{n+1}) \leq TV(v^n)\; \forall n = 0, 1, 2, ....$$

**Lemma 1.5.10.**

1. *(TVD). Any conservative, consistent monotone scheme satisfies total variation diminishing property i.e.,*

$$\sum_{i\in Z} |v_i^{n+1} - v_{1-1}^{n+1}| = \sum_{i\in Z} |v_i^n - v_{i-1}^n|.$$
$$i.e.,\ Tv(v^{n+1}) \leq TV(v^n) \tag{1.5.19}$$

2. *(Time estimate). Let scheme (1.5.2) be in conservative form, consistent with (1.5.1) and TVD with a Lipschitz continuous numerical flux $F_{i+\frac{1}{2}}$. Assume moreover that scheme is $l^\infty$ stable. Then there exists a constant $C > 0$ such that $\forall\ 0\ \ \leq m \leq n$,*

$$h\sum_{i\in Z} |v_i^m - v_i^n| \ \leq\ C(m-n)\Delta t TV(v^0) \tag{1.5.20}$$

For detail see [5, Chapter 3]

**Theorem 1.5.11** (Existence of a weak solution)**.** *Let for any $T > 0$,*
*(a) $u_0 \in L^\infty(\mathbb{R})$ and $BV(\mathbb{R})$,a functions of bounded variations and $v^0$ given by (1.5.3).*
*(b) $v_h(x,t)$ be an approximate solution obtained from a difference scheme (1.5.2) which is in conservative form and consistent with (1.5.1).*
*(c) $||v_h(.,t)||_\infty < \infty$ and a $TV(v_h(.,t)) < \infty$ for $0 \leq t \leq T$.*
*(d)*

$$||v_h(.,t) - v_h(.,s)||_{L^1} \leq C(|t-s| + \Delta t)TV(v_h(.,0)$$

*for some constant $C > 0, 0 \leq s \leq t \leq T$.*

*Then there exists a sequence $h_k \to 0$ such that if we set,$\Delta_k t = \lambda h$,with $\lambda$ being kept constant,the sequence $v_{h_k}$ converges in $L^\infty(0,T; L^1_{loc}(\mathbb{R}))$ , say to $u$. This limit $u$ is a weak solution of (1.5.1).*

For detail see [5, Chapter 3, Theorem 3.3, Theorem 3.4]

## 1.6 Non-strictly hyperbolic systems

In this section, we move on to describe *non-strictly or weakly hyperbolic systems*. In the previous sections, we presented an overview of the existing theory for the system of conservation laws that generally assumes that the system is strictly hyperbolic and the characteristic fields are either genuinely non-linear or linearly degenerate. Furthermore, the classical existence theories are applicable when the total variation of the initial data is small. It is well known that the solution of conservation laws may develop discontinuities after a finite time even though the initial data lies in the space $C^\infty(\mathbb{R})$. In that case, the solution space is generally $L^p, 1 \leq p \leq \infty$ or $BV$, the space of the functions of bounded variation. The solution is understood in a weak sense and in general the weak solutions are not unique. In the literature, several admissible criteria [6, 7, 8] have been developed to establish the uniqueness of the solution depending upon its physical relevance. In practice, some systems may violate both of the conditions (strict hyperbolicity and small total variation of initial data) assumed in these theories. The solution class for these systems may be wider than $L^\infty$ or $BV$. It is important to note that, unlike the strictly hyperbolic systems, there is no single theory or framework towards the well-posedness of solutions for non-strictly hyperbolic systems.

**Definition 1.6.1.** *A system of conservation laws* (1.2.2) *is called non strictly hyperbolic if the eigenvectors of the Jacobian matrix $Df(u)$ does not form a complete basis, consequently there exists $u \in \mathbb{R}^m$ such that at least two eigenvalues of the Jacobian matrix $Df(u)$ are equal, i.e, if $\lambda_1(u), \cdots, \lambda_m(u)$ are the eigenvalues of $Df(u)$, then $\lambda_i(u) = \lambda_j(u)$ for some $i \neq j$.*

**Example 1.6.2.** *(i) The first example of non strictly hyperbolic system appeared in the thesis* *of Korchinski[9]. The following system was considered*

$$u_t + (\frac{u^2}{2})_x = 0$$
$$\rho_t + (\rho u)_x = 0$$

*Observe that the both eigenvalues for this system are $u$ and in the solution, the second component $\rho$ contains $\delta-$ measure. Other equations of this type which arise from various physical models, were studied by many authors, e.g LeFloch [10], Joseph [11], Tan et.al. [12],. Danilov and Shelkovich [13].*

*(ii) Another well-studied non strictly hyperbolic system is the Euler equation of gas dynamics, see [14],*

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2)_x = 0. \end{aligned} \tag{1.6.1}$$

For works on initial value problems of more singular type, see Joseph [15], Shelkovich [16] and the references therein.

### 1.6.1 Approximation processes for non strictly hyperbolic systems

This subsection contains the approximation processes which are commonly used in literature and relevant to this thesis to solve non strictly hyperbolic systems.

1. **Vanishing Viscosity method :** The vanishing viscosity method is already described in section 1.3.2. Among all the methods, vanishing viscosity is considered as the most "physically relevant" one.

2. **Strictly hyperbolic approximation :** Solution for some strictly hyperbolic systems can be thought of as an approximation for the solution of non-strictly hyperbolic systems. For example, Chen and Liu [17] considered the scaled (i.e considering the pressure term as a function of some small parameter $\epsilon > 0$) system of gas dynamics equations

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2 + \epsilon p(\rho))_x = 0. \end{aligned} \tag{1.6.2}$$

and it was shown that the solution of (1.6.2) convergers to the solution of (1.6.1) as $\epsilon \to 0$.

3. **Shadow waves :** Shadow wave solutions (SDW) are constructed by a family of piecewise constant functions with respect to the time variable depending on a small parameter $\epsilon > 0$. Primarily SDW is aimed to approximate $\delta$-shocks in a $\epsilon$-neighbourhood of the shock location. Outside of that $\epsilon$-neighbourhood, they are just defined as a classical solution of the system. Shadow waves are introduced by M.Nedeljkov[18] in 2010.

### 1.6.2 BV function and Volpert's product

Let $\Omega \subset \mathbb{R}^m$ be open.

**Definition 1.6.3** (BV function). *A locally integrable function $u : \Omega \to \mathbb{R}$ has locally bounded variation if for every compact set $K \subset \mathbb{R}^m$, there exists a constant $C_k$ such that*

$$|\int u \cdot \frac{\partial \varphi}{\partial y_i} dy| \leq C_k ||\varphi||_{C^0}$$

*for each $i$ and for every $\varphi \in C_c^1$ with compact support in $K$.*

**Remark 1.6.4.** *When $u$ is a function of bounded variation its distributional derivatives $D_{x_i} u$ for $i = 1, 2, .., m$ are measures.*

**Definition 1.6.5** (Approximate jump). *A function $u$ has an approximate jump discontinuity at the point $\bar{y}$ if there exists vectors $u_r \neq u_l$ and a unit vector $\eta \in \mathbb{R}^m$ such that, setting*

$$U(y) = \begin{cases} u_l & if \quad y \cdot \eta < 0 \\ u_r & if \quad y \cdot \eta > 0 \end{cases}$$

*the following holds*

$$\lim_{R \to 0} \frac{1}{R^m} \int_{|y|<R} |u(\bar{y} + y) - U(y)| dy = 0.$$

*Furthermore, when $u_l = u_r$, we say $u$ is approximately continuous at $\bar{y}$.*

The next theorem describes the structure of BV functions.

**Theorem 1.6.6.** *Let $\Omega \in \mathbb{R}^m$ be open and $u : \Omega \to \mathbb{R}^n$ be a BV function. The there exists a set $\mathcal{N} \subset \Omega$ with $\mathcal{H}^{m-1}(\mathcal{N}) = 0$ such that at each point $y \notin \mathcal{N}$, $u$ is either approximately continuous or has approximate jump discontinuity.*

**Definition 1.6.7.** *Let $\bar{y}$ be a point of approximate jump discontinuity for $u = (u_1, ..., u_n)$ and $f : \mathbb{R}^n \to \mathbb{R}$ be $C^1$ function. We define the functional superimposition of $f(u(\bar{y}))$ as $\bar{f}(u(\bar{y}))$ in the following manner.*

$$\bar{f}(u(\bar{y})) = \int_0^1 f(tu_l + (1-t)u_r)dt$$

*where $u_l, u_r$ are defined as before.*

The next theorem states that the functional superposition is measurable.

**Theorem 1.6.8.** *If $u = (u_1, ..., u_n) \in BV(\Omega; \mathbb{R}^n)$ and $v \in BV(\Omega)$, then $\bar{f}(u(x))$ ia a measurable with respect to the measure $v_x$ in any bounded region whose closer lies in $\Omega$.*

Now we define Volpert's product as follows.

**Theorem 1.6.9.** *If $u = (u_1, ..., u_n) \in BV(\Omega; \mathbb{R}^n)$ and $v \in BV(\Omega)$, then the product of $f(u)$ and the measure $v_x$ is a Borel measure $\mu$ and given by*

$$\mu(U) = \int_U \bar{f}(u)v_x,$$

*for any Borel measurable subset $U$ of $\Omega$.*

### 1.6.3 Conservation laws with discontinuous flux

The scalar conservation laws with discontinuous flux reads

$$\begin{aligned} &u_t + (F(x,u))_x = 0, \\ &u(x,0) = u_0(x) \end{aligned} \tag{1.6.3}$$

where $F(x, u) = H(x)f(u) + (1 - H(x))g(u)$, $H$ is a Heaviside function, $u_0$ is bounded measurable function and $f$,$g$ are locally Lipschitz function in general. Non-strictly hyperbolic systems and scalar conservation laws with discontinuous flux are closely related. For instance, consider the following non strictly hyperbolic system

$$u_t + (f(u))_x = 0$$

$$\rho_t + (\rho f'(u))_x = 0$$

where $f : \mathbb{R} \to \mathbb{R}$ a Lipschitz function and $f'(u)$ is possibly discontinuous. In that case, the second equation of the above system becomes a transport equation with a discontinuous coefficient. A detailed discussion about the related problems can be found in Chapter 6.

# Chapter 2

# Limiting behavior of some strictly hyperbolic systems of conservation laws

## 2.1 Introduction

This chapter deals with the limiting behavior of two strictly hyperbolic systems of different nature. The first one is the one-dimensional model for the Euler equation of compressible fluid flow and the second one is a perturbed version of a non-strictly hyperbolic system of conservation laws, called one dimensional model for the large scale structure formation of the universe, which was first studied by Korchinski[9]. Both the characteristic fields of the first system are genuinely nonlinear whereas the second one does not possess the same property. For the second system, the first characteristic field is genuinely nonlinear and the second characteristic field is linearly degenerate.

Euler equation of one-dimensional compressible fluid flow reads

$$\begin{cases} u_t + (\frac{u^2}{2} + P(\rho))_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{2.1.1}$$

We take the initial conditions

$$u(x,0) = u_0(x), \ \rho(x,0) = \rho_0(x). \tag{2.1.2}$$

The equation (2.1.1) was first derived by S. Earnshaw [19, 20] for isentropic flow. It is a scaling limit system of Newtonian dynamics with long-range interaction for a continuous distribution of mass [21, 22]. This equation is also hydrodynamic limit of Vlasov equation [23]. For smooth nonzero solutions the system(2.1.1) is equivalent to isentropic gas dynamics equation, namely

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + \tilde{P}(\rho))_x = 0, \ \tilde{P}(\rho) = \displaystyle\int_0^\rho sP'(s)ds. \end{cases} \tag{2.1.3}$$

This is evident from the following.

$$(\rho u)_t + (\rho u^2 + (\tilde{P}(\rho))_x = u\big[\rho_t + (\rho u)_x\big] + \rho\big[u_t + (\frac{u^2}{2} + P(\rho))_x\big].$$

But for shock solutions, the above identity is no more valid and the two systems (2.1.1) and (2.1.3) are completely different.

The existence of viscosity solution of (2.1.1) with initial data $\rho_0(x) > 0$, was shown in [24] and existence of global weak solutions for locally finite bounded variation initial data for the equation (2.1.1) was by DiPerna [25], where he took $p(\rho) = k^2\rho^\gamma, \gamma \in (1, 3)$.

Here we are interested in the limiting behavior of the solutions of (2.1.1) as the pressure term $P$ approaches zero. For that purpose we take scalar function $P$ is not only a function of density $\rho$ but also a small parameter $\epsilon > 0$, satisfying $\lim_{\epsilon \to 0} P(\rho, \epsilon) = 0$ and we take $P(\rho, \epsilon) = \epsilon p(\rho)$, where $p(\rho)$ is a twice differentiable function and satisfies

$$p'(\rho) > 0, \ \ 3p'(\rho) + \rho p''(\rho) > 0. \tag{2.1.4}$$

For the calculation of the "entropy-entropy flux pair" we use the following particular form of $p$ which also satisfies (2.1.4).

$$p(\rho) = \int_0^\rho \frac{q^{'}(\xi)}{\xi} d\xi, \ \text{ where } \ q(\rho) = \int_0^\rho \xi^2 \exp(\xi) d\xi. \tag{2.1.5}$$

At this point, the system(2.1.1) can be expressed as

$$\begin{cases} u_t + (\frac{u^2}{2} + \epsilon p(\rho))_x = 0 \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{2.1.6}$$

One can readily see that as $\epsilon \to 0$, formally the system (2.1.6) becomes

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, \ \ x \in \mathbb{R}, t > 0 \\ \rho_t + (\rho u)_x = 0, \ \ x \in \mathbb{R}. \end{cases} \tag{2.1.7}$$

The above equation (2.1.7) is a one-dimensional model for the large-scale structure formation of the universe[26]. This is an example of a non-strictly hyperbolic system, which got widespread attention, started with the work of Korchinski[9]. For some interesting articles regarding this system we cite[11, 27, 28, 29]. The importance of this system lies in the fact that the solution is not a function of bounded variation, rather the second component $\rho$ is a measure.

Physical significance for introducing the term $P(\rho, \epsilon) = \epsilon p(\rho)$ are as follows: For $\epsilon > 0$, i.e, in the presence of pressure there is no concentration of mass in the solution as we can observe in section 3. As pressure vanishes($\epsilon \to 0$), the solution contains a concentration of mass whenever there is a jump in the velocity component $u$(see Section 3, Theorem(2.3.4)). Also, (2.1.6) can be thought of as a strictly hyperbolic system of conservation laws approximating the non-strictly hyperbolic system of conservation laws(2.1.7).

In this chapter, we study the existence of solution for the equation(2.1.6) for Riemann type initial data, namely,

$$\begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_l \\ \rho_l \end{pmatrix}, & \text{if} \quad x < 0 \\ \begin{pmatrix} u_r \\ \rho_r \end{pmatrix}, & \text{if} \quad x > 0. \end{cases} \tag{2.1.8}$$

Note that for $\epsilon > 0$, the system (2.1.6) is strictly hyperbolic and both the characteristics fields are genuinely nonlinear. For a strictly hyperbolic system whose characteristics field are either genuinely nonlinear or linearly degenerate, the theory[2, 30] demonstrates the existence of a solution for close-by Riemann type initial data. But for our system(2.1.6), large Riemann data is not an obstruction.

In this chapter first, we find the solution for the system(2.1.6) for any Riemann type initial data and the solution is a combination of shock and rarefaction waves. Then we study the limiting behavior of these solutions as the parameter $\epsilon$ approaches to zero. It turns out that this limit is a solution for (2.1.7) and agrees with the vanishing viscosity limit

[11]. This kind of method is not very common in the literature and can be used to construct solutions for non-strictly hyperbolic systems. In this regard, we refer to two interesting articles ([17],[31]) on isentropic gas dynamics.

The second strictly hyperbolic system we study in this chapter is

$$\begin{cases} u_t + (\frac{(u+\epsilon)^2}{2})_x = 0 \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{2.1.9}$$

Though the system (2.1.9) is strictly hyperbolic for $\epsilon > 0$, it can be solved only for close by Riemann data. We observe that if $u_l - u_r \geq 2\epsilon$, one can not get Lax type solution consisting of shock and rarefaction waves. This is an example of a system where a smallness condition is required on the initial data to get Lax type solution. For large Riemann data, the solution is not a function of bounded variation. There are many methods such as Colombeau generalized functions[32], weak asymptotic method[33], Volpert product [34, 35]and shadow wave approach[18] to overcome such difficulties. We cite [36] which deals with a highly non-strictly hyperbolic system of conservation laws using some of these methods. Here in this case *shadow wave approach* [18] will be our method of choice.

Shadow wave is a family of piecewise continuous functions $(u^\eta, \rho^\eta),\ \eta > 0$ such that the equations (2.1.9) holds in the sense of distribution as $\eta$ approaches zero. It turns out that the distributional limit of $(u^\eta, \rho^\eta)$ as $\eta$ tends to zero satisfies the equation.

This chapter is structured as follows. In section 2, shock and rarefaction curves are described for the system(2.1.6). In section 3, shock-wave solution is constructed for (2.1.6)-(2.1.8), when $u_l > u_r$ and the distributional limit is obtained when the parameter $\epsilon$ approaches to zero and it is shown that limit satisfies (2.1.7) in the sense of distribution. In section 4, entropy-entropy flux pair is found for (2.1.6) and it satisfies entropy condition for small $\epsilon$. In section 5, the solution for the case $u_l \leq u_r$ is obtained by using other elementary waves. Finally, in section 6, we explicitly determine the solution for the system (2.1.9) for any Riemann type initial data, and also the distributional limit of the solutions as $\epsilon$ vanishes, are obtained.

## 2.2 The Riemann solution

The co-efficient matrix $A(u,\rho)$ of the equation (2.1.6) is given by

$$A(u,\rho) = \begin{pmatrix} u & \epsilon p'(\rho) \\ \rho & u \end{pmatrix}.$$

Eigenvalues for this co-efficient matrix are the following: $\lambda_1(u,\rho) = u - \sqrt{\epsilon p'(\rho)\rho}$ and $\lambda_2(u,\rho) = u + \sqrt{\epsilon p'(\rho)\rho}$ and the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ are $X_1 = (-\sqrt{\frac{\epsilon p'(\rho)}{\rho}}, 1)$ and $X_2 = (\sqrt{\frac{\epsilon p'(\rho)}{\rho}}, 1)$ respectively and $\nabla\lambda_i.X_i \neq 0$ for $i = 1, 2$.
Each characteristic field is genuinely nonlinear for problem (2.1.6).

**Shock curves**: The shock curves $s_1$,$s_2$ through $(u_l, \rho_l)$ are derived from the Rankine-Hugoniot conditions

$$\begin{aligned} \lambda(u-u_l) =& (\frac{u^2}{2} + \epsilon p(\rho)) - (\frac{u_l^2}{2} + \epsilon p(\rho_l)), \\ \lambda(\rho - \rho_l) =& \rho u - \rho_l u_l. \end{aligned} \tag{2.2.1}$$

Eliminating $\lambda$ from (2.2.1), the admissible part of the shock curves passing through $(u_l, \rho_l)$ are computed as

$$s_1 = \{(u,\rho) : (u-u_l)^2 \frac{(\rho+\rho_l)}{2} = \epsilon(\rho-\rho_l)(p(\rho)-p(\rho_l)), \ \ \rho > \rho_l; \ \ u < u_l\},$$

$$s_2 = \{(u,\rho) : (u-u_l)^2 \frac{(\rho+\rho_l)}{2} = \epsilon(\rho-\rho_l)(p(\rho)-p(\rho_l)), \ \ \rho < \rho_l; \ \ u < u_l\}.$$

**Rarefaction curves**: The Rarefaction curves $R_1$, $R_2$ passing through $(u_l, \rho_l)$ are the following :

*1- Rarefaction curve*: The first Rarefaction curve passing through $(u_l, \rho_l)$ is derived by solving

$$\frac{du}{d\rho} = -\sqrt{\frac{\epsilon p'(\rho)}{\rho}}, \qquad u(\rho_l) = u_l;$$

$$R_1 = \big\{(u,\rho) : u - u_l = -\int_{\rho_l}^{\rho} \sqrt{\frac{\epsilon p'(\xi)}{\xi}} d\xi, \;\; \rho < \rho_l\big\}.$$

*2- Rarefaction curve*: The second Rarefaction curve $R_2$ passing through $(u_l, \rho_l)$ is derived by solving

$$\frac{du}{d\rho} = \sqrt{\frac{\epsilon p'(\rho)}{\rho}}, \qquad u(\rho_l) = u_l;$$

$$R_2 = \big\{(u,\rho) : u - u_l = \int_{\rho_l}^{\rho} \sqrt{\frac{\epsilon p'(\xi)}{\xi}} d\xi, \;\; \rho > \rho_l\big\}.$$

To solve the equation (2.1.6) with (2.1.8), three cases are required to be considered, that is (I) $u_l > u_r$, (II) $u_l = u_r$ and (III) $u_l < u_r$. In case (I) for sufficiently small $\epsilon$, we have solutions as a combination of two shock waves, if the Riemann type initial data are fixed. For case (II) solutions are given as the combination of 1-rarefaction and 2-shock curves or 1-shock and 2-rarefaction curves depending upon $\rho_l > \rho_r$ or $\rho_l < \rho_r$ respectively. And finally, in case (III) for sufficiently small $\epsilon$ and with fixed Riemann type initial data, the solution consists of two rarefaction waves and a vacuum state. We obtain the limit for the solutions in each case and it is exactly equal to the vanishing viscosity limit found in [11] which satisfies the equation in the sense of definition(2.3.5).

## 2.3 Formation of shock waves for $u_l > u_r$

In this section the limiting behavior for the solution of the equations (2.1.6)-(2.1.8) for $u_l > u_r$ as $\epsilon$ tends to zero has been studied. We assume $p(\rho)$ is a twice differentiable function and satisfies (2.1.4). First, we find solution for the system (2.1.6) satisfying Lax-entropy condition for the case $u_l > u_r$. $\rho_l$ and $\rho_r$ are taken positive through out this section. The key result of this section is the following:

**Theorem 2.3.1.** *If $u_l > u_r$, there exists an $\eta > 0$ such that for any $\epsilon < \eta$, we have a unique intermediate state $(u^*_\epsilon, \rho^*_\epsilon)$ which connects $(u_l, \rho_l)$ to $(u^*_\epsilon, \rho^*_\epsilon)$ by 1-shock and $(u^*_\epsilon, \rho^*_\epsilon)$ to $(u_r, \rho_r)$ by 2-shock and satisfies Lax-entropy condition.*

*Proof.* The admissible 1-shock curve passing through $(\bar{u}, \bar{\rho})$ satisfies the following:

$$\begin{aligned}(u-\bar{u})s_1 =&(\frac{u^2}{2}+\epsilon p(\rho)) - (\frac{\bar{u}^2}{2}+\epsilon p(\bar{\rho})),\\ (\rho-\bar{\rho})s_1 =&\rho u - \bar{\rho}\bar{u},\end{aligned} \tag{2.3.1}$$

and satisfies the inequality

$$s_1 < \lambda_1(\bar{u}, \bar{\rho}),\ \ \lambda_1(u,\rho) < s_1 < \lambda_2(u,\rho). \tag{2.3.2}$$

Eliminating $s_1$ from (2.3.1) and simplifying yields

$$(u-\bar{u})^2 = 2\epsilon\frac{\rho-\rho_l}{\rho+\rho_l}(p(\rho)-p(\rho_l)). \tag{2.3.3}$$

We show that for a given $u < \bar{u}$, there exists a unique $\rho > \bar{\rho}$ such that equation (2.3.3) holds. For that let us define a function

$$F(\rho) := 2\epsilon\frac{\rho-\bar{\rho}}{\rho+\bar{\rho}}(p(\rho)-p(\bar{\rho})). \tag{2.3.4}$$

We see that $F(\bar{\rho}) = 0$ and $F(\rho) \to \infty$ as $\rho \to \infty$. So by intermediate value theorem we have $F([\bar{\rho}, \infty)) = [0, \infty)$. Hence for a given $u$ there exist a $\rho > \bar{\rho}$ such that

$$F(\rho) = (u-\bar{u})^2.$$

This proves the existence of $\rho$. To prove the uniqueness, now we differentiate the equation (2.3.4) with respect to $\rho$ to get

$$F'(\rho) = 2\epsilon\frac{2\bar{\rho}}{(\rho+\bar{\rho})^2}(p(\rho)-p(\bar{\rho})) + 2\epsilon\frac{\rho-\bar{\rho}}{\rho+\bar{\rho}}p^{'}(\rho).$$

As $\rho>\bar{\rho}$ and $p^{'}(\rho) > 0$, $F'(\rho)$ is positive. So $(u-\bar{u})^2$ will be achieved only once in the interval $[\bar{\rho}, \infty)$, which proves the uniqueness. The conditions (2.3.1) and (2.3.2) hold if and only if $u < \bar{u}$ and $\rho > \bar{\rho}$. In fact, $s_1$ satisfies (2.3.2) if

$$\begin{aligned} &\frac{\rho u - \bar{\rho}\bar{u}}{\rho - \bar{\rho}} < \bar{u} - \sqrt{\epsilon p^{'}(\bar{\rho})\bar{\rho}} \quad , \\ &u - \sqrt{\epsilon p^{'}(\rho)\rho} < \frac{\rho u - \bar{\rho}\bar{u}}{\rho - \bar{\rho}} < u + \sqrt{\epsilon p'(\rho)\rho}. \end{aligned} \tag{2.3.5}$$

Now from the first inequality of (2.3.5) one can get,

$$\frac{\rho(u-\bar{u})}{(\rho-\bar{\rho})} < -\sqrt{\epsilon p'(\bar{\rho})\bar{\rho}}.$$

Since $u < \bar{u}$, the above inequality implies

$$\frac{\rho^2(u-\bar{u})^2}{(\rho-\bar{\rho})^2} > \epsilon p'(\bar{\rho})\bar{\rho}. \tag{2.3.6}$$

Then the equation (2.3.3) yields

$$\frac{2\epsilon\rho^2(p(\rho)-p(\bar{\rho}))}{(\rho+\bar{\rho})(\rho-\bar{\rho})} > \epsilon p'(\bar{\rho})\bar{\rho}. \tag{2.3.7}$$

To show that the inequality (2.3.7) holds let us define

$$G(\rho) := 2\rho^2(p(\rho)-p(\bar{\rho})) - p'(\bar{\rho})\bar{\rho}(\rho^2-\bar{\rho}^2).$$

Differentiating the above equation one can get

$$G'(\rho) := 2p'(\rho)\rho(\rho-\bar{\rho}) + 4\rho(p(\rho)-p(\bar{\rho})) > 0.$$

The above inequality is evident since $p$ is an increasing function and $\rho > \bar{\rho}$. So, $G$ is an increasing function and $G(\rho) > G(\bar{\rho}) = 0$. Thus we are done.

Again from the second inequality of (2.3.5), one can get

$$\frac{\bar{\rho}^2(u-\bar{u})^2}{(\rho-\bar{\rho})^2} < \epsilon p'(\rho)\rho.$$

To prove this inequality, let us define the following:

$$H(\rho) := 2\bar{\rho}^2(p(\rho) - p(\bar{\rho})) - p'(\rho)\rho(\rho^2 - \bar{\rho}^2).$$

Differentiating the above equation,

$$H'(\rho) = -(\rho^2 - \bar{\rho}^2)(3p'(\rho) + p''(\rho)\rho) < 0,$$

since $\rho > \bar{\rho}$ and $3p'(\rho)+p''(\rho)\rho > 0$. So, $H$ is a decreasing function and $H(\rho) < H(\bar{\rho}) = 0$. Therefore, the branch of the curve satisfying (2.3.1) and (2.3.2) can be parameterized by a $C^1$ function $\rho_1 : (-\infty, \bar{u}] \to [\bar{\rho}, \infty)$ with the parameter $u$.

From the equation (2.3.3), $\rho_1(u)$ satisfies

$$\frac{(u-\bar{u})^2}{\epsilon}\frac{\bar{\rho}+\rho_1(u)}{2(\rho_1(u)-\bar{\rho}))} + p(\bar{\rho}) = p(\rho_1(u)). \tag{2.3.8}$$

Differentiating the above equation with respect to $u$, we get

$$\frac{(u-\bar{u})(\bar{\rho}+\rho_1(u))}{\epsilon(\rho_1(u)-\bar{\rho})} + \frac{-\bar{\rho}\rho_1'(u)(u-\bar{u})^2}{\epsilon(\rho_1(u)-\bar{\rho})^2} = p'(\rho_1(u))\rho_1'(u).$$

Since $\rho_1(u) > \bar{\rho}$ , $\bar{\rho}+\rho_1(u)$ and $-\bar{\rho}+\rho_1(u)$ are positive, left-hand side of the above equation is negative. This implies $\rho_1'(u)$ is negative, because $p'$ is positive.

Similarly, the branch of the curve satisfying

$$s_1 > \lambda_2(u,\rho),\ \ \lambda_1(\bar{u},\bar{\rho}) < s_1 < \lambda_2(\bar{u},\bar{\rho}),$$

is the admissible 2-shock curve which can be parameterized by a $C^1$ function $\rho_2 : (-\infty, \bar{u}] \to (-\infty, \bar{\rho}]$ with the parameter $u$.

Also $\rho_2$ satisfies the following equation:

$$\frac{(u-\bar{u})^2}{\epsilon}\frac{\bar{\rho}+\rho_2(u)}{2(\rho_2(u)-\bar{\rho}))} + p(\bar{\rho}) = p(\rho_2(u)). \tag{2.3.9}$$

Differentiating the above equation (2.3.9) with respect to $u$, we get

$$\frac{(u-\bar{u})(\bar{\rho}+\rho_2(u))}{\epsilon(\rho_2(u)-\bar{\rho})} + \frac{-\bar{\rho}\rho_2'(u)(u-\bar{u})^2}{\epsilon(\rho_2(u)-\bar{\rho})^2} = p'(\rho_2(u))\rho_2\prime(u).$$

That is,

$$\frac{(u-\bar{u})(\bar{\rho}+\rho_2(u))}{\epsilon(\rho_2(u)-\bar{\rho})} = \rho_2'(u)\left[p'(\rho_2(u)) + \frac{\bar{\rho}(u-\bar{u})^2}{\epsilon(\rho_2(u)-\bar{\rho})^2}\right] \tag{2.3.10}$$

From (2.3.9), we get

$$\frac{(u-\bar{u})}{\epsilon}\frac{\bar{\rho}+\rho_2(u)}{2(\rho_2(u)-\bar{\rho}))} = \frac{p(\rho_2(u))-p(\bar{\rho})}{u-\bar{u}}. \tag{2.3.11}$$

Since $p$ is increasing and $u < \bar{u}$, from (2.3.11) and (2.3.10)

$$\rho_2'(u)\left[p'(\rho_2(u)) + \frac{\bar{\rho}(u-\bar{u})^2}{\epsilon(\rho_2(u)-\bar{\rho})^2}\right] > 0.$$

This implies $\rho_2'(u) > 0$ on $(-\infty, \bar{u})$.

Consider the branch of the curve passing through $(u_r, \rho_r)$ satisfying the condition $u > u_r,\ \rho > \rho_r$. In a similar way as above it can be parameterized by a $C^1$- curve $\rho_2^* : [u_r, \infty) \to [\rho_r, \infty)$. The part of the curve $\rho_2^*$ from $(w, z)$ to $(u_r, \rho_r)$ will be the admissible 2-shock curve connecting $(w, z)$ to $(u_r, \rho_r)$. So it is clear that $\rho_2^{*\prime}(u)$ is positive.

Let us denote the admissible 1-shock curve passing through $(u_l, \rho_l)$ as $\rho_1^*$. From the previous analysis, this is parameterized by a $C^1$ curve $\rho_1^* : (-\infty, u_l] \to [\rho_l, \infty)$ and satisfies $\rho_1^{*\prime}(u) < 0$ .

$\rho_1^*(u_r)$ satisfies (2.3.8) with $\rho_1(u)$ and $u$ replaced by $\rho_1^*(u_r)$ and $u_r$ respectively, and $\bar{u}$, $\bar{\rho}$ replaced by $u_l$ and $\rho_l$ respectively, i.e.,

$$\frac{(u_r-u_l)^2}{\epsilon}\frac{\rho_l+\rho_1^*(u_r)}{2(\rho_1^*(u_r)-\rho_l)} + p(\rho_l) = p(\rho_1^*(u_r)).+ \tag{2.3.12}$$

Again $\rho_2^*(u_l)$ satisfies (2.3.9) with $\rho_2(u)$ and $u$ replaced by $\rho_2^*(u_l)$ and $u_l$ respectively, and $\bar{u}, \bar{\rho}$ replaced by $u_r$ and $\rho_r$ respectively, i.e.,

$$\frac{(u_l-u_r)^2}{\epsilon}\frac{\rho_l+\rho_2^*(u_l)}{2(\rho_2^*(u_l)-\rho_r)} + p(\rho_r) = p(\rho_2^*(u_l)). \tag{2.3.13}$$

It is evident from (2.3.12) and (2.3.13) that $\rho_1^*(u_r)$ and $\rho_2^*(u_l)$ tend to $\infty$ as $\epsilon$ tends to zero. Therefore there exists an $\eta > 0$ such that $\epsilon < \eta$, we have $\rho_2^*(u_l) > \rho_l$ and $\rho_1^*(u_r) > \rho_r$.

Now let us consider the function $\rho_1^* - \rho_2^*$. Since $\rho_1^*(u_l) - \rho_2^*(u_l) = \rho_l - \rho_2^*(u_l) < 0$ and $\rho_1^*(u_r) - \rho_2^*(u_r) = \rho_1^*(u_r) - \rho_r > 0$, by intermediate value theorem there exists a point $u_\epsilon^*$ such that $\rho_1^*(u_\epsilon^*) = \rho_2^*(u_\epsilon^*) = \rho_\epsilon^*$(say). $\rho_\epsilon^*$ is unique because $\rho_1^*$ is strictly decreasing and $\rho_2^*$ is strictly increasing. Since we are considering only admissible curves, the Lax entropy condition holds. This completes the proof.

□

Now we determine the limit of the problem (2.1.6) for the shock case. For this, first, we will define $\delta$-distribution followed by a simple technical lemma which will be useful later.

**Definition 2.3.2.** *A weighted $\delta$-distribution "$d(t)\delta_{x=c(t)}$" concentrated on a smooth curve $x = c(t)$ can be defined by*

$$\langle\, d(t)\delta_{x=c(t)}, \varphi(x,t)\rangle = \int_0^\infty d(t)\varphi(c(t),t)dt$$

*for all $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$.*

**Lemma 2.3.3.** *Suppose $a_\epsilon(t)(> 0)$ and $b_\epsilon(t)(> 0)$ converge uniformly to $0$ on compact subsets of $(0,\infty)$ as $\epsilon$ tends to zero. Also assume that $d_\epsilon(t)$ converges to $d(t)$ uniformly on compact subsets of $(0,\infty)$ as $\epsilon$ tends to zero. Then*

$$\frac{1}{b_\epsilon(t) + a_\epsilon(t)} d_\epsilon(t)\chi_{(c(t)-a_\epsilon(t),c(t)+b_\epsilon(t))}(x)$$

*converges to $d(t)\delta_{x=c(t)}$ in the sense of distribution.*

*Proof.* Let us denote

$$\Psi(x,t) = \frac{1}{b_\epsilon(t) + a_\epsilon(t)} d_\epsilon(t)\chi_{(c-a_\epsilon(t),c+b_\epsilon(t))}(x).$$

Let us now consider the integral

$$\begin{aligned} &\Big|\int_0^\infty \int_{-\infty}^\infty \big(\Psi(x,t)\varphi(x,t)dxdt - \int_0^\infty d(t)\varphi(c(t),t)dt\big)\Big| \\ &\leq \int_0^\infty \Big|\frac{1}{b_\epsilon(t) + a_\epsilon(t)} \int_{c(t)-a_\epsilon(t)}^{c(t)+b_\epsilon(t)} d_\epsilon(t)\varphi(x,t) - d(t)\varphi(c(t),t)\Big|\, dx\, dt. \end{aligned}$$

Now, since $\varphi(x,t)$ has compact support and $d_\epsilon(t)$ converges to $d(t)$ uniformly on compact sets as $\epsilon \to 0$, the integral in the the right hand side of the inequality above converges to 0. Since this is true for all test function $\varphi$, the proof of this lemma is completed. □

**Theorem 2.3.4.** *The pointwise limit $u^\epsilon$ is $u$ which is also a distributional limit and is given by*

$$u(x,t) = \begin{cases} u_l, & \text{if } x < \frac{u_l+u_r}{2}t \\ \frac{u_l+u_r}{2}, & \text{if } x = \frac{u_l+u_r}{2}t \\ u_r, & \text{if } x > \frac{u_l+u_r}{2}t. \end{cases}$$

*The distributional limit of $\rho^\epsilon$ is $\rho$ and is given by*

$$\rho(x,t) = \begin{cases} \rho_l, & \text{if } x < \frac{u_l+u_r}{2}t \\ (u_l-u_r)\frac{\rho_l+\rho_r}{2}t\delta_{x=\frac{u_l+u_r}{2}t}, & \text{if } x = \frac{u_l+u_r}{2}t \\ \rho_r, & \text{if } x > \frac{u_l+u_r}{2}t. \end{cases}$$

*Proof.* From the above theorem, $(u_\epsilon^*, \rho_\epsilon^*)$ satisfies the following conditions:

$$\begin{aligned} (u_\epsilon^* - u_l)\frac{\rho_\epsilon^* u_\epsilon^* - \rho_l u_l}{\rho_\epsilon^* - \rho_l} &= (\frac{u_\epsilon^{*2}}{2} + \epsilon p(\rho_\epsilon^*)) - (\frac{u_l^2}{2} + \epsilon p(\rho_l)), \\ (u_\epsilon^* - u_r)\frac{\rho_\epsilon^* u_\epsilon^* - \rho_r u_r}{\rho_\epsilon^* - \rho_r} &= (\frac{u_\epsilon^{*2}}{2} + \epsilon p(\rho_\epsilon^*)) - (\frac{u_r^2}{2} + \epsilon p(\rho_r)). \end{aligned} \tag{2.3.14}$$

We know $u_\epsilon^* \in (u_r, u_l)$. So the sequence $u_\epsilon^*$ is bounded. We claim that $\rho_\epsilon^*$ *is unbounded as $\epsilon$ tends to zero*. In fact, if $\rho_\epsilon^*$ is bounded, then it has a convergent subsequence still denoted by $\rho_\epsilon^*$ and it converges to $\rho^*$ as $\epsilon$ tends to zero. Then from the equation (2.3.3), we get that $\rho_\epsilon^*$ satisfies:

$$(u_\epsilon^* - u_l)^2\frac{(\rho_\epsilon^* + \rho_l)}{2} = \epsilon(\rho_\epsilon^* - \rho_l)(p(\rho_\epsilon^*) - p(\rho_l)).$$

Now as $\epsilon \to 0$, the above equation becomes

$$(u^* - u_l)^2\frac{(\rho^* + \rho_l)}{2} = 0,$$

as right hand side of the equation is bounded. Now since $\rho^* + \rho_l > 0$, we get $u^* = u_l$. Again since $\rho_\epsilon^*$ satisfies:

$$(u_\epsilon^* - u_r)^2\frac{(\rho_\epsilon^* + \rho_r)}{2} = \epsilon(\rho_\epsilon^* - \rho_r)(p(\rho_\epsilon^*) - p(\rho_r)).$$

By the similar argument as above, we get, $u^* = u_r$. This implies $u_l = u_r$, leads to a contradiction.

So for subsequence of $u^*_\epsilon$ and $\rho^*_\epsilon$ still labeled as $u^*_\epsilon$ and $\rho^*_\epsilon$ respectively and that $u^*_\epsilon$ converges to $u^*$ and $\rho^*_\epsilon$ tend to $+\infty$. Passing to the limit for this subsequence in (2.3.14), we get

$$u^*(u^* - u_l) = \frac{u^{*2}}{2} - \frac{{u_l}^2}{2} + l$$
$$u^*(u^* - u_r) = \frac{u^{*2}}{2} - \frac{{u_r}^2}{2} + l,$$

where $\lim\limits_{\epsilon\to 0}\epsilon p(\rho^*_\epsilon) = l$. Solving the above two equations we get

$$u^* = \frac{u_l + u_r}{2} \text{ and } l = \frac{1}{8}(u_l - u_r)^2. \tag{2.3.15}$$

The solution for $(u^\epsilon, \rho^\epsilon)$ is given by

$$u^\epsilon(x,t) = \begin{cases} u_l & \text{if } x < (\frac{{u^*}_\epsilon + u_l}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_l))}{u^*_\epsilon - u_l})t \\ u^*_\epsilon & \text{if } (\frac{{u^*}_\epsilon + u_l}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_l))}{u^*_\epsilon - u_l})t < x < (\frac{{u^*}_\epsilon + u_r}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_r))}{u^*_\epsilon - u_r})t \\ u_r & \text{if } x > (\frac{{u^*}_\epsilon + u_r}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_r))}{u^*_\epsilon - u_r})t, \end{cases} \tag{2.3.16}$$

$$\rho^\epsilon(x,t) = \begin{cases} \rho_l & \text{if } x < (\frac{{u^*}_\epsilon + u_l}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_l))}{u^*_\epsilon - u_l})t \\ \rho^*_\epsilon & \text{if } (\frac{{u^*}_\epsilon + u_l}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_l))}{u^*_\epsilon - u_l})t < x < (\frac{{u^*}_\epsilon + u_r}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_r))}{u^*_\epsilon - u_r})t \\ \rho_r & \text{if } x > (\frac{{u^*}_\epsilon + u_r}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_r))}{u^*_\epsilon - u_r})t. \end{cases} \tag{2.3.17}$$

As $u^*_\epsilon$ converges to $u^* = \frac{u_l + u_r}{2}$ as $\epsilon \to 0$, we have the limit for $u(x,t)$ as stated in the theorem.

From (2.3.14) and (2.3.15), one can show that

$$\lim_{\epsilon\to 0}\Big[\frac{{u^*}_\epsilon + u_l}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_l))}{u^*_\epsilon - u_l}\Big] = \frac{u_l + u_r}{2},$$

and

$$\lim_{\epsilon\to 0}\Big[\frac{{u^*}_\epsilon + u_r}{2} + \frac{\epsilon(p(\rho^*_\epsilon) - p(\rho_r))}{u^*_\epsilon - u_r}\Big] = \frac{u_l + u_r}{2}.$$

Let us denote

$$\begin{aligned}
c(t) &= \frac{u_l+u_r}{2}t,\\
a_\epsilon(t) &= c(t) - \Big(\frac{{u^*}_\epsilon + u_l}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_l))}{u_\epsilon^* - u_l}\Big)t,\\
b_\epsilon(t) &= \Big(\frac{{u^*}_\epsilon + u_r}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_r))}{u_\epsilon^* - u_r}\Big)t - c(t),\\
d_\epsilon(t) &= \Big[\frac{u_r-u_l}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_r))}{u_\epsilon^* - u_r} - \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_l))}{u_\epsilon^* - u_l}\Big]\rho_\epsilon^* t.
\end{aligned}$$

With the above notations, the formula for $\rho^\epsilon$ in equation (2.3.17) can be written in the following form as in the Lemma(3.2):

$$\begin{aligned}
\rho^\epsilon =& \rho_l \chi_{(-\infty, c(t)-a_\epsilon(t))}(x) + \frac{d_\epsilon(t)}{b_\epsilon(t)+a_\epsilon(t)}\chi_{(c(t)-a_\epsilon(t), c(t)+b_\epsilon(t))}(x)\\
&+ \rho_r \chi_{(c(t)+b_\epsilon(t),\infty)}(x).
\end{aligned} \tag{2.3.18}$$

Observe that $a_\epsilon(t)$ and $b_\epsilon(t)$ satisfies the condition of the lemma, i.e, $a_\epsilon(t) > 0$ and $b_\epsilon(t) > 0$ for small $\epsilon$.

Now we will determine the limit of $d_\epsilon(t)$ as $\epsilon \to 0$.

The equation (2.3.14) can also be written in the following form:

$$\begin{aligned}
(\rho_\epsilon^* - \rho_l)\Big\{\frac{u_\epsilon^* + u_l}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_l))}{u_\epsilon^* - u_l}\Big\} &= \rho_\epsilon^* u_\epsilon^* - \rho_l u_l\\
(\rho_\epsilon^* - \rho_r)\Big\{\frac{u_\epsilon^* + u_r}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_r))}{u_\epsilon^* - u_r}\Big\} &= \rho_\epsilon^* u_\epsilon^* - \rho_r u_r.
\end{aligned} \tag{2.3.19}$$

Subtracting second equation from the first in (2.3.19), we get

$$\begin{aligned}
&\Big[\frac{u_r-u_l}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_r))}{u_\epsilon^* - u_r} - \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_l))}{u_\epsilon^* - u_l}\Big]\rho_\epsilon^*\\
&= \rho_l u_l - \rho_r u_r + \rho_r\Big(\frac{u_\epsilon^* + u_r}{2}\Big) - \rho_l\Big(\frac{u_\epsilon^* + u_l}{2}\Big)\\
&+ \rho_r \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_r))}{u_\epsilon^* - u_r} - \rho_l \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_l))}{u_\epsilon^* - u_l}.
\end{aligned}$$

Passing to the limit as $\epsilon \to 0$, we get

$$\lim_{\epsilon\to 0}\Big[\frac{u_r-u_l}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_r))}{u_\epsilon^* - u_r} - \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_l))}{u_\epsilon^* - u_l}\Big]\rho_\epsilon^* = \frac{1}{2}(u_l - u_r)(\rho_l + \rho_r). \tag{2.3.20}$$

This implies

$$\lim_{\epsilon\to 0} d_\epsilon(t) = \frac{1}{2}(u_l - u_r)(\rho_l + \rho_r)t. \tag{2.3.21}$$

Here in the calculation of (2.3.21), we have used the fact that $\lim_{\epsilon\to 0}\epsilon p(\rho_\epsilon^*) = \frac{1}{8}(u_l - u_r)^2$ and $\lim_{\epsilon\to 0} u_\epsilon^* = \frac{u_l + u_r}{2}$ from the equation (2.3.15).

The first and the third terms of (2.3.18) converge to $\rho_l \chi_{(-\infty, \frac{u_l+u_r}{2}t)}(x)$ and $\rho_r \chi_{(\frac{u_l+u_r}{2}t, \infty)}(x)$ respectively. Hence, employing the above lemma to the second term of (2.3.18), we get the distribution limit $\rho(x,t)$ as given in the theorem. Note that all the analysis has been done for a subsequence. Since the limit is the same for any subsequence, this implies the sequence itself converges to the same limit. Thus the proof of theorem 3.3 is completed. □

The limit $(u, \rho)$ satisfies the equation in the sense of Volpert is available in [37, 38]. There it was shown that $R_t + \overline{u}R_x = 0$ , where $\rho = R_x$ and $\overline{u}R_x$ is known as Volpert product [34]. Then $\rho = R_x$ satisfies the equation (2.1.7) in the sense of distribution. For completeness, we intend here to show that the limit $(u, \rho)$ satisfies the equation by formulating it as follows.

**Definition 2.3.5.** *Let $u$ is a Borel measurable function and $\rho = d\nu$ is a Radon measure on $\mathbb{R} \times [0, \infty)$. Then $(u, \rho = d\nu)$ is said to be a solution for the system* (2.1.7) *with initial data* (2.1.2) *if the following conditions hold.*

$$\begin{aligned} &\int_{\mathbb{R}\times[0,\infty)} (u\phi_t + u\phi_x)dxdt + \int_{\mathbb{R}} u_0(x)\phi(x,0)dx = 0 \\ &\int_{\mathbb{R}\times[0,\infty)} (\phi_t + u\phi_x)d\nu + \int_{\mathbb{R}} \rho_0(x)\phi(x,0)dx = 0, \end{aligned} \tag{2.3.22}$$

*for any test function $\phi$ supported in $\mathbb{R} \times [0, \infty)$.*

**Theorem 2.3.6.** *For $u_l > u_r$, the pointwise limit $u$ of $u^\epsilon$ and distributional limit of $\rho$ of $\rho^\epsilon$ satisfies the equation*(2.3.22).

*Proof.* From Theorem(3.3), the pointwise limit $u$ of $u^\epsilon$ and the distributional limit $\rho$ of $\rho^\epsilon$ are given by

$$u(x,t)=\begin{cases} u_l, & \text{if} x<st\\ s, & \text{if } x=st\\ u_r, & \text{if} x>st \end{cases} \qquad \rho(x,t)=\begin{cases} \rho_l, & \text{if } x<st\\ (u_l-u_r)\frac{\rho_l+\rho_r}{2}t\delta_{x=st}, & \text{if } x=st\\ \rho_r, & \text{if } x>st, \end{cases}$$

where $s=\frac{u_l+u_r}{2}$.

Let $\phi$ be any test function supported in$\mathbb{R}\times[0,\infty)$. It is well known that $u$ satisfies the first equation of (2.3.22). Now we show that $(u,\rho)$ satisfies the second equation of (2.3.22). Observe that the limit $\rho$ is a Radon measure and can be written in the following way.

$$\rho=d\nu=\{\rho_l+(\rho_r-\rho_l)H(x-st)\}dxdt+(u_l-u_r)\frac{\rho_l+\rho_r}{2}t\delta_{x=st}dt,$$

where $H$ is the Heaviside function and $\delta$ is the Dirac delta distribution. Note that $u$ is a Borel measurable function and defined everywhere on the domain. We calculate

$$\begin{aligned}&\int_{\mathbb{R}\times[0,\infty)}\phi_t d\nu\\ &=\int_{x<st}\rho_l\phi_t dxdt+\int_{x>st}\rho_r\phi_t dxdt+(u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty\int_{\mathbb{R}}t\phi_t(x,t)\delta_{x=st}dt \quad (2.3.23)\\ &=\int_{x<st}\rho_l\phi_t dxdt+\int_{x>st}\rho_r\phi_t dxdt+(u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty t\phi_t(st,t)dt.\end{aligned}$$

Using integration by parts for the first two integrals of the equation(2.3.23), we get

$$\begin{aligned}&\int_{\mathbb{R}\times[0,\infty)}\phi_t d\nu\\ &=-s\rho_l\int_0^\infty\phi(st,t)dt+s\rho_r\int_0^\infty\phi(st,t)dt-\int_{\mathbb{R}}\rho_0(x)\phi(x,0)dx\\ &+(u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty t\phi_t(st,t)dt\\ &=s(\rho_r-\rho_l)\int_0^\infty\phi(st,t)dt-\int_{\mathbb{R}}\rho_0(x)\phi(x,0)dx+(u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty t\phi_t(st,t)dt, \quad (2.3.24)\end{aligned}$$

where $\rho_0(x)=\rho_l+(\rho_r-\rho_l)H(x)$. Similarly we calculate

$$
\begin{aligned}
&\int_{\mathbb{R}\times(0,\infty)} u\phi_x d\nu \\
&= \int_{x<st} \rho_l u_l \phi_x dx dt + \int_{x>st} \rho_r u_r \phi_x dx dt + (u_l - u_r)s\frac{\rho_l+\rho_r}{2}\int_0^\infty t\phi_x(st,t)dt \\
&= \rho_l u_l \int_0^\infty \phi(st,t)dt - \rho_r u_r \int_0^\infty \phi(st,t)dt + s(u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty t\phi_x(st,t)dt \\
&= (\rho_l u_l - \rho_r u_r)\int_0^\infty \phi(st,t)dt + s(u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty t\phi_x(st,t)dt,
\end{aligned}
\tag{2.3.25}
$$

where in the third step we used integration by parts. From equations (2.3.24) and (2.3.25), we get

$$
\begin{aligned}
&\int_{\mathbb{R}\times[0,\infty)} (\phi_t + u\phi_x)d\nu + \int_{\mathbb{R}} \rho_0(x)\phi(x,0)dx \\
=&\big[s(\rho_r-\rho_l) + \rho_l u_l - \rho_r u_r\big]\int_0^\infty \phi(st,t)dt \\
&+ (u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty t(s\phi_x(st,t)+\phi_t(st,t))dt \\
=&\big[s(\rho_r-\rho_l) + \rho_l u_l - \rho_r u_r\big]\int_0^\infty \phi(st,t)dt \\
&+ (u_l-u_r)\frac{\rho_l+\rho_r}{2}\int_0^\infty t\frac{d}{dt}(\phi(st,t))dt \\
=&\big[s(\rho_r-\rho_l) + \rho_l u_l - \rho_r u_r - (u_l-u_r)\frac{\rho_l+\rho_r}{2}\big]\int_0^\infty \phi(st,t)dt = 0.
\end{aligned}
$$

This completes the proof. □

## 2.4 Entropy and entropy flux pairs

In this section, we show that the solution constructed for the system (2.1.6) for Riemann type data is entropy admissible. For the sake of completeness, we start with the following definitions[2] restricted to the $2\times 2$ system, namely

$$
\begin{aligned}
&u_t + (f_1(u,\rho))_x = 0 \\
&\rho_t + (f_2(u,\rho))_x = 0.
\end{aligned}
\tag{2.4.1}
$$

**Definition 2.4.1.** *A continuously differentiable function* $\eta : \mathbb{R}^2 \mapsto \mathbb{R}$ *is called an entropy for the system*(2.4.1) *with entropy flux* $q : \mathbb{R}^2 \mapsto \mathbb{R}$ *if*

$$D\eta(u,\rho).Df(u,\rho) = Dq(u,\rho),$$

*where* $f(u,\rho) = (f_1(u,\rho), f_2(u,\rho))$. *We say* $(\eta, q)$ *as entropy-entropy flux pair of the system*(2.4.1).

**Definition 2.4.2.** *A weak solution* $(u,\rho)$ *of the system* (2.4.1) *is called entropy admissible if*

$$\iint_{\mathbb{R}\times(0,\infty)} \eta(u,\rho)\varphi_t + q(u,\rho)\varphi_x \, dx\, dt \geq 0,$$

*for every non-negative test function* $\varphi : \mathbb{R}\times(0,\infty) \to \mathbb{R}$ *with compact support in* $\mathbb{R}\times(0,\infty)$*, where* $(\eta, q)$ *is the entropy-entropy flux pair as in the definition*(2.4.1).

For our system (2.1.6), $f(u,\rho) = (\frac{u^2}{2} + \epsilon p(\rho), u\rho)$, where $p$ is of the form (2.1.5). Therefore $(\eta, q)$ will be an entropy-entropy flux pair of (2.1.6) if

$$\left(\frac{\partial \eta}{\partial u}u + \frac{\partial \eta}{\partial \rho}\rho,\ \epsilon\rho e^{\rho}\frac{\partial \eta}{\partial u} + u\frac{\partial \eta}{\partial \rho}\right) = \left(\frac{\partial q}{\partial u}, \frac{\partial q}{\partial \rho}\right).$$

That is,

$$\begin{aligned} \frac{\partial q}{\partial u} &= \frac{\partial \eta}{\partial u}u + \frac{\partial \eta}{\partial \rho}\rho, \\ \frac{\partial q}{\partial \rho} &= \epsilon\rho e^{\rho}\frac{\partial \eta}{\partial u} + u\frac{\partial \eta}{\partial \rho}. \end{aligned} \tag{2.4.2}$$

Eliminating $q$ from (2.4.2), we have

$$\frac{\partial^2 \eta}{\partial \rho^2} = \epsilon e^{\rho}\frac{\partial^2 \eta}{\partial u^2}.$$

One can see that

$$\eta(u,\rho) = \frac{1}{2}u^2 + \epsilon e^{\rho}$$

is a solution of above the equation which is strictly convex (since $D^2\eta > 0$) and the corresponding entropy flux is

$$q(u,\rho) = \frac{1}{3}u^3 + \epsilon\rho u e^{\rho}.$$

We show here that our solution constructed in the previous section for Riemann type initial data ($u_l > u_r$) is entropy admissible as in the definition(2.4.1).

We calculate

$$\begin{aligned}\eta_t + q_x =& - s_1\left(\frac{1}{2}u_\epsilon^{*2} + \epsilon e^{\rho_\epsilon^*} - \frac{1}{2}u_l^2 - \epsilon e^{\rho_l}\right)\delta_{x=s_1t}\\ &- s_2\left(\frac{1}{2}u_r^2 + \epsilon e^{\rho_r} - \frac{1}{2}u_\epsilon^{*2} - \epsilon e^{\rho_\epsilon^*}\right)\delta_{x=s_2t}\\ &+ \left(\frac{1}{3}u_\epsilon^{*3} + \epsilon\rho_\epsilon^* u_\epsilon^* e^{\rho_\epsilon^*} - \frac{1}{3}u_l^3 - \epsilon\rho_l u_l e^{\rho_l}\right)\delta_{x=s_1t}\\ &+ \left(\frac{1}{3}u_r^3 - \epsilon\rho_r u_r e^{\rho_r} - \frac{1}{3}u_\epsilon^{*3} - \epsilon\rho_\epsilon^* u_\epsilon^* e^{\rho_\epsilon^*}\right)\delta_{x=s_2t},\end{aligned} \tag{2.4.3}$$

where

$$\begin{aligned} s_1 &= (\frac{u^*{}_\epsilon + u_l}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_l))}{u_\epsilon^* - u_l}),\\ s_2 &= (\frac{u^*{}_\epsilon + u_r}{2} + \frac{\epsilon(p(\rho_\epsilon^*) - p(\rho_r))}{u_\epsilon^* - u_r}).\end{aligned}$$

One should note that to show $\eta(u,\rho)$ and $q(u,\rho)$ satisfies the entropy inequality for small $\epsilon$, we separately show that the coefficients $\delta_{x=s_1t}$ and $\delta_{x=s_2t}$ are negative. Therefore it is enough if the limiting values of the coefficients are negative as $\epsilon$ tends to zero.

Coefficient of $\delta_{x=s_1t}$

$$= \underbrace{-s_1\left(\frac{1}{2}u_\epsilon^{*2} + \epsilon e^{\rho_\epsilon^*} - \frac{1}{2}u_l^2 - \epsilon e^{\rho_l}\right)}_{\text{I}} + \underbrace{\left(\frac{1}{3}u_\epsilon^{*3} + \epsilon\rho_\epsilon^* u_\epsilon^* e^{\rho_\epsilon^*} - \frac{1}{3}u_l^3 - \epsilon\rho_l u_l e^{\rho_l}\right)}_{\text{II}} \tag{2.4.4}$$

From (2.3.15),

$$\epsilon e^{\rho_\epsilon^*}(\rho_\epsilon^* - 1) \to \frac{(u_l - u_r)^2}{8} \;\text{ as }\; \epsilon \to 0. \tag{2.4.5}$$

Since $\rho_\epsilon^*$ is unbounded, we get

$$\epsilon e^{\rho_\epsilon^*} = \frac{\epsilon e^{\rho_\epsilon^*}(\rho_\epsilon^* - 1)}{(\rho_\epsilon^* - 1)} \to 0 \;\text{ as }\; \epsilon \to 0. \tag{2.4.6}$$

Now using (2.4.5) and observing that $s_1 \to \frac{(u_l+u_r)}{2}$, one can see

$$I \to \frac{-(u_l + u_r)(3u_l + u_r)(u_r - u_l)}{16} \quad \text{as } \epsilon \to 0.$$

Again using (2.4.5) and (2.4.6), a simple calculation yields

$$II \to \frac{(u_r - u_l)(7u_l^2 + 4u_l u_r + u_r^2)}{24} + \frac{(u_r - u_l)(u_r^2 - u_l^2)}{16} \quad \text{as } \epsilon \to 0.$$

Therefore from the equation(2.4.4),

$$\text{Coefficient of } \delta_{x=s_1 t} = I + II \to \frac{2(u_r - u_l)(u_l - u_r)^2}{48} \quad \text{as } \epsilon \to 0.$$

Since $u_l > u_r$, Coefficient of $\delta_{x=s_1 t} = I + II < 0$ for small $\epsilon$. In a similar way, the coefficients of $\delta_{x=s_2 t}$ can be handled.

**Remark 2.4.3.** *It is well known that if $\eta$ be a smooth entropy of the system* (2.1.6) *with the entropy flux $q$ and if one assumes that the Hessian $D^2\eta > 0$, then for genuinely non-linear characteristic fields the entropy inequality $\eta(u)_t + q(u)_x \leq 0$ is satisfied for sufficiently close initial data. For details, one can see [2]. But in our case, initial data need not to be sufficiently close. Our proof relies only on the smallness of $\epsilon$.*

## 2.5 Solution for the case $u_l \leq u_r$

This section is devoted to discuss other two cases, i.e, $u_l = u_r$ and $u_l < u_r$. We assume the same conditions on $p(\rho)$ as in section 3, i.e., $p(\rho)$ is a twice differentiable function and satisfies (2.1.4). In this section our proof goes in the spirit of [31].

**Case I** ($u_l = u_r$): For $u_l = u_r$, initial data is

$$\begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_l \\ \rho_l \end{pmatrix}, & \text{if} \quad x < 0 \\ \begin{pmatrix} u_l \\ \rho_r \end{pmatrix}, & \text{if} \quad x > 0. \end{cases}$$

Now if $\rho_l = \rho_r$, we have the trivial solution $u(x,t) = u_l$ and $\rho(x,t) = \rho_l$. Another two possibilities are $\rho_r < \rho_l$ or $\rho_r > \rho_l$.

*Subcase I($\rho_r < \rho_l$)*: For this case, we start traveling from the state $(u_l, \rho_l)$ in the curve $R_1$ to reach at $(u_\epsilon^*, \rho_\epsilon^*)$, then from $(u_\epsilon^*, \rho_\epsilon^*)$ we travel by $S_2$ to reach at $(u_l, \rho_r)$. 1-rarefaction curve $R_1$ through $(u_l, \rho_l)$ is obtained solving the differential equation

$$\frac{du}{d\rho} = -\sqrt{\frac{\epsilon p'(\rho)}{\rho}}, \qquad u(\rho_l) = u_l. \tag{2.5.1}$$

So, the branch of the curve satisfying (2.5.1) can be parameterized by a continuous function $u_1 : [\rho_r, \rho_l] \to [u_l, \infty)$ with parameter $\rho$. Since $p^{'}(\rho) > 0$, we see that $u_1$ is decreasing. Therefore, $u_1(\rho_r) > u_l$.

Any state $(u, \rho)$ connected to the end state $(u_l, \rho_r)$ by admissible 2-shock curve $S_2$ satisfies the following equation:

$$(u - u_l)\frac{\rho u - \rho_r u_l}{\rho - \rho_r} = (\frac{u^2}{2} + \epsilon p(\rho)) - (\frac{u_l^2}{2} + \epsilon p(\rho_r)), \ u > u_l, \text{ and } \rho > \rho_r, \tag{2.5.2}$$

and

$$s > \lambda_2(u, \rho), \ \lambda_1(u_l, \rho_r) < s < \lambda_2(u_l, \rho_r), \text{ where } s = \frac{\rho u - \rho_r u_l}{\rho - \rho_r}. \tag{2.5.3}$$

Equation (2.5.2) implies

$$(u - u_l)^2 = 2\epsilon \frac{(\rho - \rho_r)}{(\rho + \rho_r)}(p(\rho) - p(\rho_r)). \tag{2.5.4}$$

Our claim is that for every fixed $\rho > \rho_r$ there exists a unique $u > u_l$ such that the equation (2.5.2) holds. Let us define

$$F(u) := (u - u_l)^2.$$

Since $F(u_l) = 0$ and $F(u) \to \infty$ as $u \to \infty$, we have $F([u_l, \infty)) = [0, \infty)$. Since $p$ is increasing and $\rho > \rho_r$, right hand side of (2.5.4) is positive. Therefore for the given $\rho > \rho_r$, there exists a $u > u_l$ such that

$$F(u) = 2\epsilon \frac{(\rho - \rho_r)}{(\rho + \rho_r)}(p(\rho) - p(\rho_r))$$

Also since $F(u)$ is an increasing function for $u > u_l$, $u$ is unique for the given $\rho$.

Similarly in Theorem 3.1, the branch of the curve satisfying (2.5.2) and (2.5.3) can be parameterized by a $C^1$-function $u_2(\rho) = u_2 : [\rho_r, \rho_l] \to [u_l, \infty)$ satisfying

$$F(u_2(\rho)) = (u_2(\rho) - u_l)^2 = 2\epsilon \frac{(\rho - \rho_r)}{(\rho + \rho_r)}(p(\rho) - p(\rho_r)). \tag{2.5.5}$$

Note that $u_2(\rho_r) = u_l$ and it is clear that the function $u_2$ is well defined. The function $u_2$ is increasing in the interval $(\rho_r, \rho_l)$. Infact, differentiating the above equation (2.5.5) we get,

$$(u_2(\rho) - u_l){u_2}'(\rho) = \epsilon \left[ p'(\rho)\frac{\rho - \rho_r}{\rho + \rho_r} + (p(\rho) - p(\rho_r))\frac{2\rho_r}{(\rho + \rho_r)^2} \right].$$

Since $\rho > \rho_r$ and $p(\rho)$ is an increasing function, i.e, $p'(\rho) > 0$, RHS of above equation is positive for small $\epsilon > 0$. That is, $(u_2(\rho) - u_l){u_2}'(\rho) > 0$. Since $u_2(\rho) > u_l$, ${u_2}'(\rho) > 0$.

From the above analysis, there exists an intermediate state $\rho_\epsilon^* \in (\rho_r, \rho_l)$ such that $u_1(\rho_\epsilon^*) = u_2(\rho_\epsilon^*) = u_\epsilon^*$. Hence the solution for (2.1.6) is given by:

$$u^\epsilon = \begin{cases} u_l & \text{if } x < \lambda_1(u_l, \rho_l)t \\ R_1^u(x/t)(u_l, \rho_l) & \text{if } \lambda_1(u_l, \rho_l)t < x < \lambda_1(u_\epsilon^*, \rho_\epsilon^*)t \\ u_\epsilon^* & \text{if } \lambda_1(u_\epsilon^*, \rho_\epsilon^*)t < x < \frac{\rho_r u_l - \rho_\epsilon^* u_\epsilon^*}{\rho_r - \rho_\epsilon^*}t \\ u_r & \text{if } x > \frac{\rho_r u_l - \rho_\epsilon^* u_\epsilon^*}{\rho_r - \rho_\epsilon^*}t \end{cases}$$

and

$$\rho^\epsilon = \begin{cases} \rho_l & \text{if } x < \lambda_1(u_l, \rho_l)t \\ R_1^\rho(x/t)(u_l, \rho_l) & \text{if } \lambda_1(u_l, \rho_l)t < x < \lambda_1(u_\epsilon^*, \rho_\epsilon^*)t \\ \rho_\epsilon^* & \text{if } \lambda_1(u_\epsilon^*, \rho_\epsilon^*)t < x < \frac{\rho_r u_l - \rho_\epsilon^* u_\epsilon^*}{\rho_r - \rho_\epsilon^*}t \\ \rho_r & \text{if } x > \frac{\rho_r u_l - \rho_\epsilon^* u_\epsilon^*}{\rho_r - \rho_\epsilon^*}t. \end{cases}$$

Where $R_1(\xi)(\bar{u}, \bar{\rho}) = (R_1^u(\xi)(\bar{u}, \bar{\rho}), R_1^\rho(\xi)(\bar{u}, \bar{\rho}))$ and $R_1^u(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{du}{d\xi} = \frac{2p^{'}(\rho)}{3p^{'}(\rho) + \rho p''(\rho)}, \quad u(\lambda_1(\bar{u}, \bar{\xi})) = \bar{u}.$$

and $R_1^\rho(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{d\rho}{d\xi} = -\frac{2\sqrt{\epsilon p^{'}(\rho)\rho}}{3\epsilon p^{'}(\rho) + \epsilon\rho p''(\rho)}, \quad \rho(\lambda_1(\bar{u}, \bar{\xi})) = \bar{\rho}.$$

*Subcase II ($\rho_l < \rho_r$)*: This can be handled in a similar way. In fact, here we start from $(u_l, \rho_l)$ and reach at $(u_\epsilon^*, \rho_\epsilon^*)$ by $S_1$ and from $(u_\epsilon^*, \rho_\epsilon^*)$ to $(u_l, \rho_r)$ by $R_2$. So, the solution is given by :

$$u^\epsilon = \begin{cases} u_l & \text{if } x < \frac{\rho_\epsilon^* u_\epsilon^* - \rho_l u_l}{\rho_\epsilon^* - \rho_l} t \\ u_\epsilon^* & \text{if } \frac{\rho_\epsilon^* u_\epsilon^* - \rho_l u_l}{\rho_\epsilon^* - \rho_l} t < x < \lambda_2(u_\epsilon^*, \rho_\epsilon^*) t \\ R_2^u(x/t)(u_\epsilon^*, \rho_\epsilon^*) & \text{if } \lambda_2(u_\epsilon^*, \rho_\epsilon^*) t < x < \lambda_2(u_l, \rho_r) t \\ u_r & \text{if } x > \lambda_2(u_l, \rho_r) t \end{cases}$$

and

$$\rho^\epsilon = \begin{cases} \rho_l & \text{if } x < \frac{\rho_\epsilon^* u_\epsilon^* - \rho_l u_l}{\rho_\epsilon^* - \rho_l} t \\ \rho_\epsilon^* & \text{if } \frac{\rho_\epsilon^* u_\epsilon^* - \rho_l u_l}{\rho_\epsilon^* - \rho_l} t < x < \lambda_2(u_\epsilon^*, \rho_\epsilon^*) t \\ R_2^\rho(x/t)(u_\epsilon^*, \rho_\epsilon^*) & \text{if } \lambda_2(u_\epsilon^*, \rho_\epsilon^*) t < x < \lambda_2(u_l, \rho_r) t \\ \rho_r & \text{if } x > \lambda_2(u_l, \rho_r) t \end{cases}$$

where $R_2(\xi)(\bar{u}, \bar{\rho}) = (R_2^u(\xi)(\bar{u}, \bar{\rho}), R_2^\rho(\xi)(\bar{u}, \bar{\rho}))$ and $R_2^u(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{du}{d\xi} = \frac{2p'(\rho)}{3p'(\rho) + \rho p''(\rho)}, \quad u(\lambda_2(\bar{u}, \bar{\xi}) = \bar{u}.$$

and $R_2^\rho(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{d\rho}{d\xi} = \frac{2\sqrt{\epsilon p'(\rho)\rho}}{3\epsilon p'(\rho) + \epsilon \rho p''(\rho)}, \quad \rho(\lambda_2(\bar{u}, \bar{\xi}) = \bar{\rho}.$$

Now our aim is to find the limit of $(u^\epsilon, \rho^\epsilon)$ as $\epsilon \to 0$ in both of the above cases. Since $\rho_\epsilon^* \in (\rho_l, \rho_r)$ or $\rho_\epsilon^* \in (\rho_r, \rho_l)$ this implies $\rho_\epsilon^*$ is bounded. Also $\rho_\epsilon^*$ and $u_\epsilon^*$ satisfies

$$(u_\epsilon^* - u_l)^2 \frac{(\rho_\epsilon^* + \rho_r)}{2} = \epsilon(\rho_\epsilon^* - \rho_r)(p(\rho_\epsilon^*) - p(\rho_r)).$$

Since R.H.S is bounded, as $\epsilon \to 0$ we get,

$$\lim_{\epsilon \to 0} (u_\epsilon^* - u_l)^2 \frac{(\rho_\epsilon^* + \rho_r)}{2} = 0,$$

that is, $\lim_{\epsilon \to 0} u_\epsilon^* = u_l$. Therefore the solution $(u^\epsilon, \rho^\epsilon) \to (u, \rho)$ as $\epsilon \to 0$ where $(u, \rho)$ is given by:

$$\rho = \begin{cases} \rho_l & \text{if } x < u_l t \\ \rho_r & \text{if } x > u_l t. \end{cases}$$

and

$$u = \begin{cases} u_l & \text{if } x < u_l t \\ u_r & \text{if } x > u_l t. \end{cases}$$

Since here $u_l = u_r$ we have $u \equiv u_l$.

**Case II** $(u_l < u_r)$ : The 1st-rarefaction curve passing through $(u_l, \rho_l)$ is given by the solution of the following Cauchy problem:

$$\frac{du}{d\rho} = -\sqrt{\frac{\epsilon p'(\rho)}{\rho}}, \qquad \rho < \rho_l, \qquad u(\rho_l) = u_l.$$

Note that for this case it does not matter whether $\rho_l < \rho_r$ or $\rho_l > \rho_r$. So, W.L.O.G we can take $\rho_l > \rho_r > 0$. Now a branch of $R_1$ can be parameterized by a differentiable function $u_1 : [0, \rho_l] \to [u_l, \infty)$ with a parameter $\rho$. Explicitly $u_1$ can be written as

$$u_1(\rho) - u_l = -\int_{\rho_l}^{\rho} \sqrt{\frac{\epsilon p'(\xi)}{\xi}} d\xi.$$

Since $\rho \in [0, \rho_l]$ is bounded and $p$ is increasing, we have $u_1(\rho) \to u_l$ as $\epsilon \to 0$ decreasingly. Similarly, the 2nd-rarefaction curve is given by the solution of then Cauchy problem :

$$\frac{du}{d\rho} = \sqrt{\frac{\epsilon p'(\rho)}{\rho}}, \qquad \rho < \rho_r, \qquad u(\rho_r) = u_r. \tag{2.5.6}$$

Let $u_2 : [0, \rho_r] \to (-\infty, u_r]$ is differentiable and parameterized branch of $R_2$ satisfying (2.5.6) and can be written as

$$u_2(\rho) - u_r = \int_{\rho}^{\rho_r} \sqrt{\frac{\epsilon p'(\xi)}{\xi}} d\xi.$$

Since $\rho \in [0, \rho_r]$ and $p$ is increasing, we have $u_2(\rho) \to u_r$ as $\epsilon \to 0$ increasingly. Since $u_l < u_r$, by the above calculation one can see $u_1(0) < u_2(0)$ for small $\epsilon$. In this case the complete solution is the following:

$$u^\epsilon = \begin{cases} u_l & \text{if } x < \lambda_1(u_l, \rho_l)t \\ R_1^u(x/t)(u_l, \rho_l) & \text{if } \lambda_1(u_l, \rho_l)t < x < \lambda_1(u_\epsilon^{*(1)}, 0)t \\ x/t & \text{if } \lambda_1(u_\epsilon^{*(1)}, 0)t < x < \lambda_2(u_\epsilon^{*(2)}, 0)t \\ R_2^u(x/t)(u_\epsilon^{*(2)}, 0) & \text{if } \lambda_2(u_\epsilon^{*(2)}, 0)t < x < \lambda_2(u_r, \rho_r)t \\ u_r & \text{if } x > \lambda_2(u_r, \rho_r)t. \end{cases} \tag{2.5.7}$$

and

$$\rho^\epsilon = \begin{cases} \rho_l & \text{if } x < \lambda_1(u_l,\rho_l)t \\ R_1^\rho(x/t)(u_l,\rho_l) & \text{if } \lambda_1(u_l,\rho_l)t < x < \lambda_1(u_\epsilon^{*(1)},0)t \\ 0 & \text{if } \lambda_1(u_\epsilon^{*(1)},0)t < x < \lambda_2(u_\epsilon^{*(2)},0)t \\ R_2^\rho(x/t)(u_\epsilon^{*(2)},0) & \text{if } \lambda_2(u_\epsilon^{*(2)},0)t < x < \lambda_2(u_r,\rho_r)t \\ \rho_r & \text{if } x > \lambda_2(u_r,\rho_r)t, \end{cases}$$

where $R_1^u(.)$, $R_1^\rho(.)$, $R_2^u(.)$, $R_2^\rho(.)$ are as above.

Now it remains to find the limit of $(u^\epsilon,\rho^\epsilon)$ as $\epsilon \to 0$. Since $u_\epsilon^{*(1)} = u_1(0)$, we have $u_\epsilon^{*(1)} \to u_l$ and in the same way $u_\epsilon^{*(2)} \to u_r$ as $\epsilon \to 0$. Passing to the limit as $\epsilon$ tends to zero, we get

$$u(x,t) = \begin{cases} u_l & \text{if } x < u_l t \\ x/t & \text{if } u_l t < x < u_r t \\ u_r & \text{if } x > u_r t \end{cases}$$

and

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } x < u_l t \\ 0 & \text{if } u_l t < x < u_r t \\ \rho_r & \text{if } x > u_r t. \end{cases}$$

**Remark 2.5.1.** *In equation* (2.5.7)*, one has to take* $u^\epsilon(x,t) = \frac{x}{t}$ *in the region* $\lambda_1(u_\epsilon^{*(1)},0)t < x < \lambda_2(u_\epsilon^{*(2)},0)t$*. This kind of selection gives unique entropy solution. In fact, since* $\rho = 0$ *in this region, the first equation of* (2.1.6) *turns out to be the Burgers equation and* $u(x,t) = \frac{x}{t}$ *is the unique entropy solution for the rarefaction case.*

## 2.6 Limiting behavior of another strictly hyperbolic model: Shadow waves

This section aims to study the limiting behavior of the solution for the following strictly hyperbolic system.

$$\begin{aligned} u_t + \Big(\frac{(u+\epsilon)^2}{2}\Big)_x = 0 \\ \rho_t + (\rho u)_x = 0, \end{aligned} \tag{2.6.1}$$

with Riemann type initial data:

$$\begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_l \\ \rho_l \end{pmatrix}, & \text{if} \quad x < 0 \\ \begin{pmatrix} u_r \\ \rho_r \end{pmatrix}, & \text{if} \quad x > 0. \end{cases} \tag{2.6.2}$$

In comparison to the previous system dealt with in the first part of this article, this system is very much different in nature and only retains the strict hyperbolicity property. The eigenvalues and the eigenvectors for the system (2.6.1) are the following:
The first eigenvalue $\lambda_1(u) = u$ and the corresponding eigenvector is $r_1(u) = (0, 1)$ and the second eigenvalue $\lambda_2(u) = u + \epsilon$ and the corresponding eigenvector is $r_1(u) = (1, \rho/\epsilon)$. Again, $\nabla\lambda_1(u).r_1(u) = 0$ and $\nabla\lambda_2(u).r_2(u) = 1$. So, the first characteristic field is linearly degenerate and the second characteristic field is genuinely nonlinear, whereas the previous system is genuinely nonlinear in both of the characteristic fields. The main difficulty is that for certain cases, Lax-type solutions don't exist. In those cases we use a recent technique introduced in [18] called *Shadow Wave* solution. Now we describe explicitly shock and rarefaction curves for the system (2.6.1).

**1-rarefaction curve**: 1-rarefaction curve is the solution of the ODE;

$$\dot{w}(\xi) = r_1(w(\xi)), \quad w(\lambda_1(u_l, \rho_l)) = (u_l, \rho_l),$$

where $w(\xi) = (w_1(\xi), w_2(\xi))$. So, solving the following pair of ODE:

$$\dot{w}_1(\xi) = 0, \quad w_1(u_l) = u_l.$$

$$\dot{w}_2(\xi) = 1, \quad w_2(u_l) = \rho_l.$$

we get the 1-rarefaction curve $R_1$ passing through $(u_l, \rho_l)$,

$$R_1(\xi) = (u_l, \quad \xi + \rho_l - u_l).$$

**2-rarefaction curve**: 2-rarefaction curve is the solution of the ODE;

$$\dot{w}(\xi) = r_2(w(\xi)), \quad w(\lambda_2(u_l, \rho_l)) = (u_l, \rho_l),$$

where $w(\xi) = (w_1(\xi), w_2(\xi))$.

This gives the following system of ODEs with initial conditions.

$$\dot{w}_1(\xi) = 1, \quad w_1(u_l + \epsilon) = u_l.$$
$$\dot{w}_2(\xi) = \frac{w_2(\xi)}{\epsilon}, \quad w_2(u_l + \epsilon) = \rho_l.$$

Solving the above pair of ODEs, we get the 2-rarefaction curve $R_2$ passing through $(u_l, \rho_l)$.

$$R_2(\xi) = (\xi - \epsilon, \quad \rho_l . \exp(\frac{\xi - (u_l + \epsilon)}{\epsilon})). \tag{2.6.3}$$

The admissible part of the curve is $R_2^+$.

$$R_2^+(\xi) = (\xi - \epsilon, \quad \rho_l . \exp(\frac{\xi - (u_l + \epsilon)}{\epsilon})), \ \xi > u_l + \epsilon. \tag{2.6.4}$$

*Since the first characteristics field is linearly degenerate, the 1st-Shock curve and the 1st-Rarefaction curve will coincide, i.e., $R_1(\xi) = S_1(\xi)$.*

**Admissible 2-shock curve**: Admissible 2-shock curve passing through $(u_l, \rho_l)$ is given by:

$$\rho(u) = \frac{\rho_l(\frac{u-u_l}{2} + \epsilon)}{\epsilon - \frac{u-u_l}{2}}, \ \ u < u_l, \ \ u_l - u \leq 2\epsilon.$$

Now, a brief description of the concept of *Shadow waves* is given below. Shadow waves are constructed as families of functions that approximate delta shock waves (or, in fact any types of singular shocks) in a small neighborhood of the shock location. Outside that small neighborhood, they are classical solutions of the system.

**Definition 2.6.1.** *Let $f \in \mathcal{C}(\Omega : \mathbb{R}^n)$ and $U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n$ be a piecewise constant function given by*

$$U^\eta(x,t) = (u^\eta, \rho^\eta)(x,t) = \begin{cases} (u_l, \rho_l), & \text{if } x < (c - a_\eta)t \\ (u_1^\eta, \rho_1^\eta), & \text{if } (c - a_\eta)t < x < ct \\ (u_2^\eta, \rho_2^\eta), & \text{if } ct < x < (c + b_\eta)t \\ (u_r, \rho_r), & \text{if } x > (c + b_\eta)t. \end{cases}$$

*If* $(U^\eta)_t + (f(U^\eta)_x \to 0$ *in the sense of distribution, then* $U^\eta$ *is called a Shadow Wave solution to the conservation laws*

$$U_t + f(U)_x = 0$$

The main result of this section is the following.

**Theorem 2.6.2.** *The solutions* $(u^\epsilon, \rho^\epsilon)$ *of the system* (2.6.1) *with Riemann type initial data* (2.6.2) *are given below case by case:*

*For* $u_l < u_r$,

$$(u^\epsilon, \rho^\epsilon)(x,t) = \begin{cases} (u_l, \rho_l) & \text{if } x < u_l t \\ (u_l, \quad \rho_r.\exp(\frac{(u_l - u_r)}{\epsilon})) & \text{if } u_l t < x < (u_l + \epsilon)t \\ (x/t - \epsilon, \quad \rho_r.\exp(\frac{x/t - (u_r + \epsilon)}{\epsilon})) & \text{if } (u_l + \epsilon)t < x < (u_r + \epsilon)t \\ (u_r, \rho_r) & \text{if } x > (u_r + \epsilon)t. \end{cases} \tag{2.6.5}$$

*For* $u_l = u_r$, *the solution is given by*

$$(u^\epsilon, \rho^\epsilon)(x,t) = \begin{cases} (u_l, \rho_l) & \text{if } x < u_l t \\ (u_l, \rho_r) & \text{if } u_l t < x < (u_l + \epsilon)t \\ (u_l, \rho_r) & \text{if } x > (u_l + \epsilon)t. \end{cases} \tag{2.6.6}$$

*If* $u_l > u_r$ *and* $u_l - u_r \le 2\epsilon$, *then the solution is given by:*

$$(u^\epsilon, \rho^\epsilon)(x,t) = \begin{cases} (u_l, \rho_l) & \text{if } x < u_l t \\ (u_l, \rho^\star) & \text{if } u_l t < x < (\frac{u_l + u_r}{2} + \epsilon)t \\ (u_r, \rho_r) & \text{if } x > (\frac{u_l + u_r}{2} + \epsilon)t. \end{cases} \tag{2.6.7}$$

*where*

$$\rho^\star = \rho_r \frac{\frac{u_l - u_r}{2} + \epsilon}{\frac{u_r - u_l}{2} + \epsilon}.$$

*If* $u_l > u_r$ *and* $u_l - u_r > 2\epsilon$, *then the system admits shadow wave solution* $(u^{\eta,\epsilon}, \rho^{\eta,\epsilon})_{\eta>0}$. *The pointwise limit of* $u^{\eta,\epsilon}$ *is* $u^\epsilon$ *and the distributional limit of* $\rho^{\eta,\epsilon}$ *is* $\rho^\epsilon$ *and are given as follows:*

$$u^\epsilon(x,t) = \begin{cases} u_l & \text{if } \ x < (\frac{u_l + u_r}{2} + \epsilon)t \\ \frac{u_l + u_r}{2} + \epsilon, & \text{if } x = (\frac{u_l + u_r}{2} + \epsilon)t. \\ (u_r, \rho_r) & \text{if } \ x > (\frac{u_l + u_r}{2} + \epsilon)t, \end{cases} \tag{2.6.8}$$

$$\rho^\epsilon = \begin{cases} \rho_l & \text{if } \ x < (\frac{u_l+u_r}{2} + \epsilon)t \\ \big(\frac{(u_l-u_r)(\rho_l+\rho_r)}{2} + \epsilon(\rho_r - \rho_l)\big)t\delta_{x=(\frac{u_l+u_r}{2}+\epsilon)t}, & \text{if } x = (\frac{u_l+u_r}{2} + \epsilon)t. \\ \rho_r & \text{if } \ x > (\frac{u_l+u_r}{2} + \epsilon)t. \end{cases} \tag{2.6.9}$$

*Further* $(u^\epsilon, \rho^\epsilon)$ *satisfies the equation* (2.6.1) *with initial data* (2.6.2) *in the sense of the definition* (2.3.5).

*Proof.* **Case 1:** $u_l < u_r$: The state $(u_l, \rho_l)$ can be joined to $(u_l, \rho^*_\epsilon)$ by 1-shock curve and $(u_l, \rho^*_\epsilon)$ can be joined to $(u_r, u_r)$ by 2-rarefaction curve. Then by (2.6.3), $(u_l, \rho^*_\epsilon)$ will satisfy the following equations.

$$u_r = \xi - \epsilon, \ \ \rho_r = \rho^*_\epsilon . \exp(\frac{\xi - (u_l + \epsilon)}{\epsilon}).$$

Which yields

$$\xi = u_r + \epsilon, \ \ \rho^*_\epsilon = \rho_r . \exp(\frac{(u_l - u_r)}{\epsilon}).$$

So the solution for the perturbed problem is given by:

$$(u^\epsilon, \rho^\epsilon)(x,t) = \begin{cases} (u_l, \rho_l) & \text{if } \ x < \lambda_1(u_l,\rho_l)t \\ (u_l, \rho^*_\epsilon) & \text{if } \ \lambda_1(u_l,\rho_l)t < x < \lambda_2(u_l,\rho^*_\epsilon)t \\ R_2(\xi)(u_l,\rho^*_\epsilon) & \text{if } \ \lambda_2(u_l,\rho^*_\epsilon) < x < \lambda_2(u_r,\rho_r)t \\ (u_r, \rho_r) & \text{if } \ x > \lambda_2(u_r,\rho_r)t. \end{cases} \tag{2.6.10}$$

where $\lambda_2(R_2(\xi)(u_l, \rho_l) = x/t$, i.e, $\xi = x/t$. Using equation (2.6.3) and putting the values of $\lambda_1$ and $\lambda_2$ yields (2.6.5).

**Case 2:** $u_l = u_r$: The state $(u_l, \rho_l)$ is connected to $(u_l, \rho_r)$ by 1-shock and $(u_l, \rho_r)$ to $(u_l, \rho_r)$ by 2-shock,

$$(u^\epsilon, \rho^\epsilon)(x,t) = \begin{cases} (u_l, \rho_l) & \text{if } x < u_l t \\ (u_l, \rho_r) & \text{if } u_l t < x < (u_l + \epsilon)t \\ (u_l, \rho_r) & \text{if } \ x > (u_l + \epsilon)t. \end{cases} \tag{2.6.11}$$

**Case 3:** $u_l > u_r$**,** $u_l - u_r \leq 2\epsilon$ : In this case $(u_l, \rho_l)$ can be connected $(u_l, \rho^*)$ by 1-shock and $(u_l, \rho^*)$ can be connected to $(u_r, \rho_r)$ by 2-shock and a simple calculation yields

$$\rho^\star = \rho_r \frac{\frac{u_l - u_r}{2} + \epsilon}{\frac{u_r - u_l}{2} + \epsilon}.$$

**Case 4:** $u_l > u_r$**,** $u_l - u_r > 2\epsilon$ : In this case Lax method cannot be used. This situation is handled by using *shadow wave* approach. Use ansatz,

$$(u^\eta, \rho^\eta)(x,t) = \begin{cases} (u_l, \rho_l), & \text{if } x < (c-\eta)t \\ (u, \dfrac{\rho}{\eta}), & \text{if } (c-\eta)t < x < (c+\eta)t \\ (u_r, \rho_r), & \text{if } x > (c+\eta)t. \end{cases}$$

We want to determine $c$, $u$, and $\rho$ such that the following limits hold in the sense of distribution. As $\eta \to 0$,

$$\begin{cases} (u^\eta)_t + \Big(\dfrac{(u^\eta+\epsilon)^2}{2}\Big)_x \rightharpoonup 0, \\ (\rho^\eta)_t + (u^\eta \rho^\eta)_x \rightharpoonup 0. \end{cases}$$

That is, as $\eta \to 0$,

$$\begin{cases} \langle\, (u^\eta)_t + \Big(\dfrac{(u^\eta+\epsilon)^2}{2}\Big)_x, \varphi\rangle \to 0, \\ \langle\, (\rho^\eta)_t + (u^\eta \rho^\eta)_x, \varphi\rangle \to 0. \end{cases} \tag{2.6.12}$$

for all test function $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$. Now we calculate

$$\begin{aligned} \Big\langle\, (u^\eta)_t + \Big(\frac{(u^\eta+\epsilon)^2}{2}\Big)_x, \varphi \Big\rangle &= -(c-\eta)(u-u_l)\int_0^\infty \varphi((c-\eta)t,t)dt \\ &\quad -(c+\eta)(u_r-u)\int_0^\infty \varphi((c+\eta)t,t)dt \\ &\quad +\Big(\frac{(u+\epsilon)^2}{2} - \frac{(u_l+\epsilon)^2}{2}\Big)\int_0^\infty \varphi((c-\eta)t,t)dt \\ &\quad +\Big(\frac{(u_r+\epsilon)^2}{2} - \frac{(u+\epsilon)^2}{2}\Big)\int_0^\infty \varphi((c+\eta)t,t)dt. \end{aligned} \tag{2.6.13}$$

Since $\varphi$ has compact support, passing to the limit as $\eta$ tends to $0$, and using the first equation of (2.6.12) and equation (2.6.13), we get

$$\Big[ -c(u_r-u_l) + \frac{(u_r+\epsilon)^2}{2} - \frac{(u_l+\epsilon)^2}{2}\Big]\int_0^\infty \varphi(ct,t)dt = 0. \tag{2.6.14}$$

But equation (2.6.14) is true for all $\varphi$ having compact support in $\mathbb{R} \times (0,\infty)$. Therefore

$$c = \frac{u_l+u_r}{2} + \epsilon.$$

Next,

$$\begin{aligned}\Big\langle (\rho^\eta)_t + (u^\eta\rho^\eta)_x, \varphi\Big\rangle = &-(c-\eta)(\frac{\rho}{\eta}-\rho_l)\int_0^\infty \varphi((c-\eta)t,t)dt \\ &-(c-\eta)(\frac{\rho}{\eta}-\rho_l)\int_0^\infty \varphi((c+\eta)t,t)dt \\ &+(\frac{u\rho}{\eta}-u_l\rho_l)\int_0^\infty \varphi((c-\eta)t,t)dt \\ &+(u_r\rho_r-\frac{u\rho}{\eta})\int_0^\infty \varphi((c+\eta)t,t)dt.\end{aligned} \tag{2.6.15}$$

Taylor expansions of the functions $\varphi((c-\eta)t,t)$ and $\varphi((c+\eta)t,t)$ about $(ct,t)$ are

$$\varphi((c-\eta)t,t) = \varphi(ct,t) - at\varphi_x((ct,t) + \eta^2/2\int_0^t \varphi_{xx}(ct-\eta r,t)(t-r)^2 dr.$$

$$\varphi((c+\eta)t,t) = \varphi(ct,t) + at\varphi_x((ct,t) + \eta^2/2\int_0^t \varphi_{xx}(ct+\eta r,t)(t-r)^2 dr.$$

Using the expansions in (2.6.15), we get

$$\begin{aligned}&\Big\langle (\rho^\eta)_t + (u^\eta\rho^\delta)_x, \varphi\Big\rangle \\ &= \Big[-(c-\eta)(\rho/\eta-\rho_l) - (c+\eta)(\rho_r-\rho/\eta) - u_l\rho_l + u_r\rho_r\Big]\int_0^\infty \varphi(ct,t)dt \\ &+ t\Big[(c-\eta)(\rho-\rho_l\eta) - (u\rho-\eta u_l\rho_l) - (c+\eta)(\eta\rho_r-\rho) + (\eta u_r\rho_r - u\rho)\Big]\times \\ &\int_0^\infty \varphi_x(ct,t)dt + O(\eta).\end{aligned} \tag{2.6.16}$$

Passing to the limit as $\eta \to 0$ in the equation (2.6.16) and comparing the coefficients of the integrals as above, we get

$$u = c \text{ and } \rho = \frac{u_l\rho_l - u_r\rho_r - c(\rho_l-\rho_r)}{2}.$$

Now (2.6.8) and (2.6.9) follows easily. $(u,\rho)$ given in (2.6.8) and (2.6.9) satisfies the integral formulation(2.3.22) in the definition(2.3.5) is exactly similar to the proof of the Theorem(2.3.6) and is omitted. This completes the proof. □

**Remark 2.6.3.** *From the formula for $(u^\epsilon,\rho^\epsilon)$ given in the Theorem 2.6.2, one can easily verify that for the case $u_l \leq u_r$, the distributional limit of $(u^\epsilon,\rho^\epsilon)$ as $\epsilon$ tends to zero, converges* *to*

$$(u(x,t),\rho(x,t)) = \begin{cases} (u_l,\rho_l) & \textit{if } x < u_l t \\ (x/t,0) & \textit{if } u_l t < x < u_r t \\ (u_r,\rho_r) & \textit{if } x > u_r t \end{cases}$$

*and for the case* $u_l > u_r$*, the distributional limit of* $(u^\epsilon,\rho^\epsilon)$ *as* $\epsilon$ *tends to zero, converges to*

$$(u^\epsilon,\rho^\epsilon)(x,t) = \begin{cases} (u_l,\rho_l) \quad \text{if} \quad x < (\frac{u_l+u_r}{2})t \\ (\frac{u_l+u_r}{2}, \frac{(u_l-u_r)(\rho_l+\rho_r)}{2}\delta_{x=(\frac{u_l+u_r}{2})t}) \quad \text{if} \quad x = (\frac{u_l+u_r}{2})t \\ (u_r,\rho_r) \;\; \text{if} \;\; x > (\frac{u_l+u_r}{2})t \end{cases}$$

*in the sense of distribution. This is the vanishing viscosity limit for the large scale structure formation of the universe, see [11, 27]. This limit satisfies the equation* (2.1.7) *is already proved in Theorem*(2.3.6).

# Chapter 3

# Limiting behavior of scaled general Euler equations of compressible fluid flow

## 3.1 Introduction.

General Euler equations of compressible fluid flow read

$$\begin{cases} u_t + (\frac{u^2}{2} + f(\rho))_x = 0, \\ \rho_t + (u\rho + g(\rho))_x = 0. \end{cases} \tag{3.1.1}$$

We take the initial conditions

$$u(x,0) = u_0(x),\ \rho(x,0) = \rho_0(x) > 0. \tag{3.1.2}$$

For the above system, the assumptions on $f(\rho)$ and $g(\rho)$ are the following:

H1. $f, g \in C^3[0,\infty)$ and $f_1 = \frac{f'}{\rho} \in C^2[0,\infty)$, $g_1 = \frac{g'}{\rho} \in C^2[0,\infty)$.

H2. $f_1 \geq d$ and $2f_1' + g_1'(r_1 + g_1) \geq 0$, $2f_1' + g_1'(r_1 - g_1) \geq 0$, where $d$ is a fixed positive constant and $r_1 = \sqrt{g_1^2 + 4f_1}$.

Under these assumptions H1-H2, the system (3.1.1) is strictly hyperbolic and genuinely nonlinear in both of its characteristic fields[24]. Here we are interested in the system (3.1.1) with the following conditions on $f$ and $g$.

A1. $f \in C^2(0,\infty)$, $f' > 0$ and $f'' > 0$.

A2. $g \in C^1(0,\infty)$ and $g$ is any linear decreasing function.

It can be easily observed that our assumptions on $f$ and $g$ are compatible with H1 and H2. Since our $g$ is any linear decreasing function, it is enough to work with $g(\rho) = -\rho$. So the system (3.1.1) can be expressed as:

$$\begin{cases} u_t + (\frac{u^2}{2} + f(\rho))_x = 0, \\ \rho_t + (u\rho - \rho)_x = 0. \end{cases} \tag{3.1.3}$$

If $f(\rho) = \frac{\rho^2}{2}$, we get the following Brio system.

$$\begin{cases} u_t + (\frac{u^2 + \rho^2}{2})_x = 0, \\ \rho_t + (\rho(u-1))_x = 0. \end{cases} \tag{3.1.4}$$

Therefore the system (3.1.3) can be regarded as a generalization of the physically significant system known as Brio system (3.1.4). The Brio system (3.1.4) was first derived by M. Brio [39] and mainly arises as a simplified model in ideal magnetohydrodynamics(MHD). The study of MHD is based on the idea that the currents in the magnetic fields are inherent from moving electrically conducting fluids. In this system $(u, \rho)$ represents the velocity of the fluid whose dynamics is determined by magnetohydrodynamics forces. In [40], equation (3.1.4) was compared with a system whose first equation avoids the nonlinear term $\frac{1}{2}\rho^2$, such as

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, \\ \rho_t + (\rho(u-1))_x = 0. \end{cases} \tag{3.1.5}$$

It was shown in [40] that the solution for the Riemann problem of the system (3.1.5) contains $\delta$- shock waves. In [41], $\delta$-shock waves are observed in the solution of (3.1.4) by a complex-valued generalization of weak asymptotic method [42, 43] and in [44] similar result is obtained via a distributional product. Although uniqueness was an unresolved issue for both of them. Recently the question of uniqueness is also settled in [45] by introducing some nonlinear change of variable in the flux function of the first equation of (3.1.4).

In this chapter, we are interested in the limiting behavior of the solutions for the scaled version of (3.1.3) as the scaling parameter approaches zero. The scaled version of the system

(3.1.3) can be written as

$$\begin{cases} u_t + (\frac{u^2}{2} + \epsilon f(\rho))_x = 0, \\ \rho_t + (u\rho - \epsilon\rho)_x = 0, \end{cases} \tag{3.1.6}$$

where $\epsilon > 0$ is introduced as a scaling parameter. Recently [46] deals with the system

$$\begin{cases} u_t + (\frac{u^2}{2} + \epsilon f(\rho))_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{3.1.7}$$

One can see that the system (3.1.7) can be obtained by taking $g(\rho) = 0$ and introducing the scaling parameter $\epsilon$ in the system (3.1.1). It can be readily seen that as $\epsilon \to 0$, formally the above systems (3.1.6) and (3.1.7) becomes

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, \quad x \in \mathbb{R}, t > 0 \\ \rho_t + (\rho u)_x = 0, \quad x \in \mathbb{R}. \end{cases} \tag{3.1.8}$$

In [46], it is shown that the solution of the system (3.1.7) converges to the solution of (3.1.8) in the sense of distribution. As a continuation of [46], here our goal is to obtain the solution of (3.1.8) as a distributional limit of the solution of (3.1.6).

The above equation(3.1.8) is a one-dimensional model for the large-scale structure formation of the universe [26]. This is an example of a non-strictly hyperbolic system, which was studied by many authors [11, 36, 27, 28, 29, 12], started with the work of Korchinski [9]. We study the existence of solution for the equation (3.1.6) for Riemann type initial data, namely,

$$\begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_l \\ \rho_l \end{pmatrix}, & \text{if} \quad x < 0 \\ \begin{pmatrix} u_r \\ \rho_r \end{pmatrix}, & \text{if} \quad x > 0. \end{cases} \tag{3.1.9}$$

Note that for $\epsilon > 0$, the system (3.1.6) is strictly hyperbolic and both the characteristics fields are genuinely nonlinear (see section 2). For a strictly hyperbolic system whose characteristics fields are either genuinely nonlinear or linearly degenerate, the Lax theory[2, 30, 47] can be applied to show the existence of a solution for close-by Riemann type initial data, i.e., the Riemann type initial data has small total variation. But for our system(3.1.6) with a small epsilon which may depend upon the initial data, we show that the existence of a solution for any type of Riemann initial data. Summarizing the above paragraphs, the main result can be stated as follows.

**Theorem 3.1.1.** *The admissible solution of the system* (3.1.6) *with Riemann type initial data* (3.1.9) *converges to the solution (solution in the sense of definition* (2.3.5)*) of the non strictly hyperbolic model* (3.1.8) *in the sense of distribution when the parameter* $\epsilon$ *goes to zero.*

We propose a different regularization for the system (3.1.8) by introducing the parameter $\epsilon$ in the flux function of (3.1.1). Introduction of the scaling parameter $\epsilon > 0$ is motivated as follows: The flux $(\frac{u^2}{2} + \epsilon f(\rho), \rho u - \epsilon\rho)$ in (3.1.6) compared to the flux $(\frac{u^2}{2}, \rho u)$ in system (3.1.8) gives a more regularized effect. Besides this in presence of $\epsilon > 0$ there is no concentration of mass in the solution, however in the absence of $\epsilon$, the system (3.1.6) becomes (3.1.8) and concentration of mass can occur in the solution.

In this chapter first, we find the solution for the system(3.1.6) for *any Riemann type initial data* for small $\epsilon$ and the solution is a combination of shock and rarefaction waves. Then we study the limiting behavior of these solutions as the parameter $\epsilon$ approaches to zero. We show that the limit is a solution for (3.1.8) which is also the vanishing viscosity limit [11]. This type of singular flux function limit approach is very useful for certain systems and can be viewed as an alternative approach of vanishing viscosity to construct solution (which may be singular in nature) for non-strictly hyperbolic systems. In this regard, we refer [48] for the LeRoux system and [17, 49, 31, 50, 51] for isentropic and non isentropic systems of gas dynamics. On a slightly different note, one can see [52] where Riemann solution for (3.1.8) is obtained via a linear approximation of flux function.

This chapter is organized as follows. In section 2, shock and rarefaction curves are described for the system(3.1.6). In section 3, shock-wave solution is constructed for (3.1.6)-(3.1.9), when $u_l > u_r$ and the distributional limit is obtained when the parameter $\epsilon$ approaches to zero and it is shown that limit satisfies (3.1.8) in the sense of the definition(3.3.5). In section 4, an entropy-entropy flux pair is found for (3.1.4) which satisfies entropy condition for small $\epsilon$. In section 5, the solution for the case $u_l \leq u_r$ is obtained as a combination of other elementary waves. Lastly in section 6, we discuss the case when $f(\rho) = \frac{\rho^2}{2}$ and $g(\rho) = -\rho^2$. Further possibilities are also discussed for general $f$ and $g$.

## 3.2 The Riemann problem.

The co-efficient matrix $A(u,\rho)$ of the equation (3.1.6) is given by

$$A(u,\rho) = \begin{pmatrix} u & \epsilon f'(\rho) \\ \rho & u-\epsilon \end{pmatrix}.$$

Eigenvalues for this co-efficient matrix are the following: $\lambda_1(u,\rho) = u - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2}$ and $\lambda_2(u,\rho) = u - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2}$ and the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ are $X_1 = (\frac{\frac{\epsilon}{2}-\frac{1}{2}\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2}}{\rho}, 1)$ and $X_2 = (\frac{\frac{\epsilon}{2}+\frac{1}{2}\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2}}{\rho}, 1)$ respectively. Now consider,

$$\nabla\lambda_1.X_1 = \frac{\frac{\epsilon}{2}-\frac{1}{2}\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2}}{\rho} - \frac{\epsilon(\rho f''(\rho)+f'(\rho))}{\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2}}$$

As $f(\rho)$ and $f'(\rho)$ are increasing, we have $\nabla\lambda_1.X_1 < 0$. A similar calculation shows that $\nabla\lambda_2.X_2 > 0$. So each characteristic field is genuinely nonlinear for problem (3.1.6).

**Shock curves**: The shock curves $s_1$,$s_2$ through $(u_l,\rho_l)$ and $(u_r,\rho_r)$ are derived from the Rankine-Hugoniot conditions

$$\begin{aligned} \lambda(u_l-u_r) =& (\frac{1}{2}u_l^2+\epsilon f(\rho_l)) - (\frac{1}{2}u_r^2+\epsilon f(\rho_r)), \\ \lambda(\rho_l-\rho_r) =& (\rho_l u_l-\epsilon) - (\rho_r u_r - \epsilon). \end{aligned} \tag{3.2.1}$$

Eliminating $\lambda$ from (3.2.1) and simplifying further, one can get the following quadratic equation

$$(u_l-u_r)^2 + \Big(\frac{2\epsilon(\rho_r-\rho_l)}{\rho_l+\rho_r}\Big)(u_l-u_r) - 2\epsilon\frac{(\rho_l-\rho_r)(f(\rho_l)-f(\rho_r))}{\rho_l+\rho_r} = 0 \tag{3.2.2}$$

Solving the above equation (3.2.2), the admissible part of the shock curves passing through $(u_l, \rho_l)$ are computed as

$$s_1 = \Big\{(u,\rho) : (u - u_l) = \frac{\rho-\rho_l}{\rho+\rho_l}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\rho_l)(f(\rho)-f(\rho_l))}{(\rho-\rho_l)}}\Big],\ \ \rho > \rho_l;\quad u < u_l\Big\},$$

$$s_2 = \Big\{(u,\rho) : (u - u_l) = \frac{\rho-\rho_l}{\rho+\rho_l}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\rho_l)(f(\rho)-f(\rho_l))}{(\rho-\rho_l)}}\Big],\ \ \rho < \rho_l;\quad u < u_l\Big\}.$$

**Rarefaction curves**: The Rarefaction curves $R_1$, $R_2$ passing through $(u_l, \rho_l)$ are the following :

*1- Rarefaction curve*: The first Rarefaction curve passing through $(u_l, \rho_l)$ is derived by solving

$$\frac{du}{d\rho} = \frac{\epsilon - \sqrt{4\epsilon\rho f'(\rho) + \epsilon^2}}{2\rho}, \qquad u(\rho_l) = u_l;$$

$$R_1 = \Big\{(u,\rho) : u - u_l = \int_{\rho_l}^{\rho} \frac{\epsilon - \sqrt{4\epsilon\xi f'(\xi) + \epsilon^2}}{2\xi},\ \ \rho < \rho_l\Big\}.$$

*2- Rarefaction curve*: The second Rarefaction curve $R_2$ passing through $(u_l, \rho_l)$ is derived by solving

$$\frac{du}{d\rho} = \frac{\epsilon + \sqrt{4\epsilon\rho f'(\rho) + \epsilon^2}}{2\rho}, \qquad u(\rho_l) = u_l;$$

$$R_2 = \Big\{(u,\rho) : u - u_l = \int_{\rho_l}^{\rho} \frac{\epsilon + \sqrt{4\epsilon\xi f'(\xi) + \epsilon^2}}{2\xi},\ \ \rho > \rho_l\Big\}.$$

To solve the equation (3.1.6) with (3.1.9), three cases are required to be considered, that is (I) $u_l > u_r$, (II) $u_l = u_r$ and (III) $u_l < u_r$. In case (I) for sufficiently small $\epsilon(> 0)$, we have solutions as a combination of two shock waves. For case (II) solutions are given as the combination of 1-rarefaction and 2-shock curves or 1-shock and 2-rarefaction curves depending upon $\rho_l > \rho_r$ or $\rho_l < \rho_r$ respectively. Finally, in case (III) for sufficiently small $\epsilon(> 0)$, the solution consists of two rarefaction waves and a vacuum state. We obtain the limit for the solutions in each case and it exactly matches with the vanishing viscosity limit found in [11] which satisfies the equation in the sense of definition(3.3.5).

## 3.3 Formation of concentration for $u_l > u_r$.

In this section the limiting behavior for the solution of the equations (3.1.6)-(3.1.9) for $u_l > u_r$ as $\epsilon$ tends to zero has been studied. We find solution for the system (3.1.6) satisfying Lax- entropy condition for the case $u_l > u_r$. The first step towards this is to show the existence of the intermediate state. Note that $\rho_l$ and $\rho_r$ are taken positive throughout this section.

**Theorem 3.3.1.** *(Existence of an intermediate state).*
*If $u_l > u_r$, there exists an $\eta > 0$ such that for any $\epsilon < \eta$, we have a unique intermediate state $(u^*_\epsilon, \rho^*_\epsilon)$ which connects $(u_l, \rho_l)$ to $(u^*_\epsilon, \rho^*_\epsilon)$ by 1-shock and $(u^*_\epsilon, \rho^*_\epsilon)$ to $(u_r, \rho_r)$ by 2-shock which satisfies Lax-entropy condition.*

*Proof.* The admissible 1-shock curve passing through $(\bar{u}, \bar{\rho})$ satisfies the following:

$$\begin{aligned}(u-\bar{u})s_1 =& \frac{1}{2}(u^2-\bar{u}^2)+\epsilon(f(\rho)-f(\bar{\rho})),\\ (\rho-\bar{\rho})s_1 =& (\rho u-\bar{\rho}\bar{u})+\epsilon(\bar{\rho}-\rho),\end{aligned} \tag{3.3.1}$$

and satisfies the Lax entropy inequality

$$s_1 < \lambda_1(\bar{u},\bar{\rho}),\ \ \lambda_1(u,\rho) < s_1 < \lambda_2(u,\rho). \tag{3.3.2}$$

Eliminating $s_1$ from (3.3.1) and simplifying as in Section 2, we have

$$(u-\bar{u}) = \frac{\rho-\bar{\rho}}{\rho+\bar{\rho}}\Big[\epsilon-\sqrt{\epsilon^2+\frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}\Big]. \tag{3.3.3}$$

We show that for a given $u < \bar{u}$, there exists a unique $\rho > \bar{\rho}$ such that equation (3.3.3) holds. For that let us define a function

$$\begin{aligned}F(\rho) &= \frac{\rho-\bar{\rho}}{\rho+\bar{\rho}}\Big[\epsilon-\sqrt{\epsilon^2+\frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}\Big]\\ &= \frac{1-\frac{\bar{\rho}}{\rho}}{1+\frac{\bar{\rho}}{\rho}}\Big[\epsilon-\sqrt{\epsilon^2+\frac{2\epsilon(\rho+\bar{\rho})(\frac{f(\rho)}{\rho}-\frac{f(\bar{\rho})}{\rho})}{(1-\frac{\bar{\rho}}{\rho})}}\Big]\end{aligned} \tag{3.3.4}$$

As $\lim\limits_{\rho\to\infty}\dfrac{f(\rho)}{\rho} = \lim\limits_{\rho\to\infty} f'(\rho) \le \infty$, we have $\lim\limits_{\rho\to\infty} F(\rho) = -\infty$. Since $F(\bar{\rho}) = 0$, we have $F([\bar{\rho},\infty)) = (-\infty, 0]$. Hence the equation(3.3.3) is solvable for any given $u \in (-\infty, \bar{u}]$. To prove the uniqueness of $\rho$ in the interval $[\bar{\rho},\infty)$, observe that $F(\rho)$ satisfies the following equation:

$$F(\rho)^2 - \frac{2\epsilon(\rho-\bar{\rho})}{(\rho+\bar{\rho})}F(\rho) - \frac{2\epsilon(\rho-\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho+\bar{\rho})} = 0.$$

Differentiating the above equation, we have

$$F'(\rho)\Big[F(\rho) - \frac{\epsilon(\rho-\bar{\rho})}{(\rho+\bar{\rho})}\Big] = \frac{2\epsilon\bar{\rho}(f(\rho)+f(\bar{\rho})) + \epsilon(\rho^2-\bar{\rho}^2)f'(\rho) + 2\epsilon\bar{\rho}F(\rho)}{(\rho+\bar{\rho})^2} \tag{3.3.5}$$

Since $F(\rho) - \frac{\epsilon(\rho-\bar{\rho})}{(\rho+\bar{\rho})} < 0$ and $2\epsilon\bar{\rho}(f(\rho)+f(\bar{\rho})) + \epsilon(\rho^2-\bar{\rho}^2)f'(\rho) + 2\epsilon\bar{\rho}F(\rho) > 0$, from the above equation we conclude that $F(\rho)$ is decreasing in $[\bar{\rho},\infty)$. This shows the uniqueness of $\rho$. The conditions (3.3.1) and (3.3.2) hold if and only if $u < \bar{u}$ and $\rho > \bar{\rho}$. In fact, $s_1$ satisfies (3.3.2) if

$$\begin{aligned} \frac{\rho u - \bar{\rho}\bar{u}}{\rho-\bar{\rho}} - \epsilon < \bar{u} - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{4\epsilon\bar{\rho}f'(\bar{\rho}) + \epsilon^2}\quad, \\ u - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2} < \frac{\rho u - \bar{\rho}\bar{u}}{\rho-\bar{\rho}} - \epsilon < u - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{4\epsilon\rho f'(\rho)+\epsilon^2}. \end{aligned} \tag{3.3.6}$$

Now from the first inequality of (3.3.6) one can get,

$$\frac{\rho(u-\bar{u})}{(\rho-\bar{\rho})} - \frac{\epsilon}{2} < -\frac{1}{2}\sqrt{4\epsilon\bar{\rho}f'(\bar{\rho})+\epsilon^2}.$$

Using the equation (3.3.3) the above inequality can be rephrased as

$$\frac{\rho}{\rho+\bar{\rho}}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}\Big] - \frac{\epsilon}{2} < -\frac{1}{2}\sqrt{4\epsilon\bar{\rho}f'(\bar{\rho})+\epsilon^2}. \tag{3.3.7}$$

To prove the above inequality (3.3.7), we consider

$$G(\rho) = \frac{\rho}{\rho+\bar{\rho}}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}\Big] - \frac{\epsilon}{2}. \tag{3.3.8}$$

Now we claim that the above function $G(\rho)$ is decreasing. Assuming that the claim is true, let us complete the proof of the inequality (3.3.7). Since $G(\rho)$ is decreasing and $\rho > \bar{\rho}$, we have $G(\rho) < G(\bar{\rho})$. Note that, employing mean value theorem on $f(\rho)$, (3.3.8) can be written as

$$G(\rho) = \frac{\rho}{\rho+\bar{\rho}}\Big[\epsilon - \sqrt{\epsilon^2 + 2\epsilon(\rho+\bar{\rho})f'(\xi_\rho)}\Big] - \frac{\epsilon}{2}, \quad \xi_\rho \in [\bar{\rho}, \rho].$$

Therefore,

$$G(\bar{\rho}) = -\frac{1}{2}\sqrt{\epsilon^2 + 4\epsilon\bar{\rho}f'(\xi_{\bar{\rho}})}$$

As $G(\rho)$ is decreasing and $\rho > \bar{\rho}$, we have

$$G(\rho) < -\frac{1}{2}\sqrt{\epsilon^2 + 4\epsilon\bar{\rho}f'(\xi_{\bar{\rho}})}$$

So it is enough to show that

$$-\frac{1}{2}\sqrt{\epsilon^2 + 4\epsilon\bar{\rho}f'(\xi_{\bar{\rho}})} < -\frac{1}{2}\sqrt{\epsilon^2 + 4\epsilon\bar{\rho}f'(\bar{\rho})}$$

This is evident since $f'(\rho)$ is increasing. Now we show that $G(\rho)$ is a decreasing function. Differentiating (3.3.8) one can get

$$G'(\rho) = \frac{-\epsilon\rho\frac{d}{d\rho}\Big[\frac{(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}\Big]}{(\rho+\bar{\rho})\sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}} + \frac{\bar{\rho}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}\Big]}{(\rho+\bar{\rho})^2}. \tag{3.3.9}$$

Now let us analyze the numerator of the first term of (3.3.9). Consider,

$$\begin{aligned} &\frac{d}{d\rho}\Big[\frac{(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}\Big] \\ &= \frac{(\rho^2-\bar{\rho}^2)f'(\rho) - 2\bar{\rho}(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})^2}. \end{aligned} \tag{3.3.10}$$

Since $f'(\rho)$ is increasing, a use of mean value theorem on $f(\rho)$ in the interval $[\bar{\rho}, \rho]$ shows that $(\rho^2-\bar{\rho}^2)f'(\rho) - 2\bar{\rho}(f(\rho)-f(\bar{\rho})) > 0$. So from (3.3.9) we conclude that $G'(\rho) < 0$, i.e, $G(\rho)$ is decreasing. This proves our claim.

Now the second inequality of (3.3.6) can be rewritten as

$$\begin{aligned} \frac{\bar{\rho}(u-\bar{u})}{(\rho-\bar{\rho})} - \frac{\epsilon}{2} &< \frac{1}{2}\sqrt{\epsilon^2 + 4\epsilon\rho f'(\rho)}, \\ \frac{\bar{\rho}(u-\bar{u})}{(\rho-\bar{\rho})} - \frac{\epsilon}{2} &> -\frac{1}{2}\sqrt{\epsilon^2 + 4\epsilon\rho f'(\rho)}. \end{aligned} \tag{3.3.11}$$

As $\rho > \bar{\rho}$, $u < \bar{u}$, the first inequality of (3.3.11) is evident. Again using the equation (3.3.3), the second inequality of (3.3.11) can be written as

$$\frac{\bar{\rho}}{\rho+\bar{\rho}}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}\Big] - \frac{\epsilon}{2} > -\frac{1}{2}\sqrt{\epsilon^2 + 4\epsilon\rho f'(\rho)}.$$

To prove the above inequality, we consider,

$$H(\bar{\rho}) = \frac{\bar{\rho}}{\rho+\bar{\rho}}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\bar{\rho})(f(\rho)-f(\bar{\rho}))}{(\rho-\bar{\rho})}}\Big] - \frac{\epsilon}{2}. \tag{3.3.12}$$

In a similar way as above we can show that $H(\bar{\rho})$ is a decreasing function of $\bar{\rho}$ and since $\bar{\rho} < \rho$, we have $H(\bar{\rho}) > H(\rho)$. Now following the similar steps as above one gets the second inequality of (3.3.11). Note that the above inequality is independent of $\epsilon$ and holds for any $(u, \rho)$ and $(\bar{u}, \bar{\rho})$ satisfying the condition $u < \bar{u}$ and $\rho > \bar{\rho}$.

Therefore, the branch of the curve satisfying (3.3.1) and (3.3.2) can be parameterized by a $C^1$ function $\rho_1 : (-\infty, \bar{u}] \to [\bar{\rho}, \infty)$ with the parameter $u$.

From the equation (3.3.3), $\rho_1(u)$ satisfies

$$(u-\bar{u}) = \frac{\rho_1(u)-\bar{\rho}}{\rho_1(u)+\bar{\rho}}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_1(u)+\bar{\rho})(f(\rho_1(u))-f(\bar{\rho}))}{(\rho_1(u)-\bar{\rho})}}\Big]. \tag{3.3.13}$$

Differentiating the above equation (3.3.13) with respect to the parameter $u$, we have

$$\begin{aligned} 1 = \Big[&\frac{\rho_1(u)-\bar{\rho}}{\rho_1(u)+\bar{\rho}} \frac{-\epsilon\frac{d}{d\rho}\Big[\frac{(\rho_1(u)+\bar{\rho})(f(\rho_1(u))-f(\bar{\rho}))}{(\rho_1(u)-\bar{\rho})}\Big]}{\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_1(u)+\bar{\rho})(f(\rho_1(u))-f(\bar{\rho}))}{(\rho_1(u)-\bar{\rho})}}} \\ &+ \frac{2\bar{\rho}}{(\rho_1(u)+\bar{\rho})^2}\Big(\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_1(u)+\bar{\rho})(f(\rho_1(u))-f(\bar{\rho}))}{(\rho_1(u)-\bar{\rho})}}\Big)\Big]\rho_1'(u). \end{aligned} \tag{3.3.14}$$

Since $\rho_1(u) > \bar{\rho}$ and $f'(.)$ is increasing, from (3.3.10) we have

$$\frac{d}{d\rho}\Big[\frac{(\rho_1(u)+\bar{\rho})(f(\rho_1(u))-f(\bar{\rho}))}{(\rho_1(u)-\bar{\rho})}\Big] > 0.$$

Now since $\rho_1(u) > \bar{\rho}$, the first term in the right hand side of (3.3.14) is negative and the second term is also negative. Therefore we conclude that $\rho_1'(u) < 0$.

Similarly, the branch of the curve satisfying

$$s_1 > \lambda_2(u,\rho),\ \ \lambda_1(\bar{u},\bar{\rho}) < s_1 < \lambda_2(\bar{u},\bar{\rho}),$$

is the admissible 2-shock curve which can be parameterized by a $C^1$ function $\rho_2 : (-\infty,\bar{u}] \to (-\infty,\bar{\rho}]$ with the parameter $u$.

Also, $\rho_2$ satisfies the following equation:

$$(u-\bar{u}) = \frac{\rho_2(u)-\bar{\rho}}{\rho_2(u)+\bar{\rho}}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}\Big]. \tag{3.3.15}$$

Differentiating the above equation (3.3.15) we have,

$$\begin{aligned} 1 = \Big[&\frac{1}{(\rho_2(u)+\bar{\rho})}\Big\{\frac{\epsilon(\rho_2(u)-\bar{\rho})\frac{d}{d\rho}\Big[\frac{(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}\Big]}{\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}} \\ &+\Big(\epsilon+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}\Big)\Big\} \\ &-\frac{(\rho_2(u)-\bar{\rho})\Big[\epsilon+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}\Big]}{(\rho_2(u)+\bar{\rho})^2}\Big]\rho_2'(u). \end{aligned} \tag{3.3.16}$$

Note that, since $\rho_2(u) < \bar{\rho}$ the second term of the above equation (3.3.16) is positive. Now we determine the sign of the first term. To determine the sign, we calculate

$$\begin{aligned} &\frac{\epsilon(\rho_2(u)-\bar{\rho})\frac{d}{d\rho}\Big[\frac{(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}\Big]}{\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}} + \Big(\epsilon+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}\Big) \\ &= \frac{\epsilon(\rho_2(u)-\bar{\rho})\frac{d}{d\rho}\Big[\frac{(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}\Big] + \frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}{\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}} \\ &+ \frac{\epsilon\Big[\epsilon+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}\Big]}{\sqrt{\epsilon^2+\frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}}}. \end{aligned} \tag{3.3.17}$$

Now observe that, in the view of (3.3.10) and employing mean value theorem on $f(.)$ in the interval $[\rho_2(u),\bar{\rho}]$, the the numerator of the first term of the above equation, i.e

$$\epsilon(\rho_2(u)-\bar{\rho})\frac{d}{d\rho}\Big[\frac{(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})}\Big] + \frac{2\epsilon(\rho_2(u)+\bar{\rho})(f(\rho_2(u))-f(\bar{\rho}))}{(\rho_2(u)-\bar{\rho})} > 0.$$

To show the above inequality we also used the fact that $f(.)$ is increasing. Therefore from (3.3.16), we conclude that $\rho_2'(u) > 0$.

Now consider the branch of the curve passing through $(u_r, \rho_r)$ satisfying the condition $u > u_r, \ \rho > \rho_r$. In a similar way as above it can be parameterized by a $C^1$- curve $\rho_2^* : [u_r, \infty) \to [\rho_r, \infty)$. Then for any given point $(\alpha, \beta)$, the part of the curve $\rho_2^*$ connecting $(\alpha, \beta)$ to $(u_r, \rho_r)$ will be the admissible 2-shock curve. Let us denote the admissible 1-shock curve passing through $(u_l, \rho_l)$ as $\rho_1^*$. From the previous analysis, this is parameterized by a $C^1$ curve $\rho_1^* : (-\infty, u_l] \to [\rho_l, \infty)$. Then $\rho_1^*(u_r)$ satisfies (3.3.13) with $\rho_1(u)$ and $u$ replaced by $\rho_1^*(u_r)$ and $u_r$ respectively, and $\bar{u}$, $\bar{\rho}$ replaced by $u_l$ and $\rho_l$ respectively, i.e.,

$$(u_r - u_l) = \frac{\rho_1^*(u_r) - \rho_l}{\rho_1^*(u_r) + \rho_l}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_1^*(u_r) + \rho_l)(f(\rho_1^*(u_r)) - f(\rho_l))}{(\rho_1^*(u_r) - \rho_l)}}\Big]. \tag{3.3.18}$$

Again $\rho_2^*(u_l)$ satisfies (3.3.15) with $\rho_2(u)$ and $u$ replaced by $\rho_2^*(u_l)$ and $u_l$ respectively, and $\bar{u}, \bar{\rho}$ replaced by $u_r$ and $\rho_r$ respectively, i.e.,

$$(u_l - u_r) = \frac{\rho_2^*(u_l) - \rho_r}{\rho_2^*(u_l) + \rho_r}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_2^*(u_l) + \rho_r)(f(\rho_2^*(u_l)) - f(\rho_r))}{(\rho_2^*(u_l) - \rho_r)}}\Big]. \tag{3.3.19}$$

It is evident from (3.3.18) and (3.3.19) that $\rho_1^*(u_r)$ and $\rho_2^*(u_l)$ tend to $\infty$ as $\epsilon$ tends to zero. Suppose $\rho_1^*(u_r)$ and $\rho_2^*(u_l)$ are finite as $\epsilon$ tends to zero, then (3.3.18) and (3.3.19) implies $u_l = u_r$, which is not the case. Therefore there exists an $\eta > 0$ such that for any $\epsilon < \eta$, one has $\rho_2^*(u_l) > \rho_l$ and $\rho_1^*(u_r) > \rho_r$. Now let us consider the function $\rho_1^* - \rho_2^* : [u_r, u_l] \to \mathbb{R}$. Since $\rho_1^*(u_l) - \rho_2^*(u_l) = \rho_l - \rho_2^*(u_l) < 0$ and $\rho_1^*(u_r) - \rho_2^*(u_r) = \rho_1^*(u_r) - \rho_r > 0$, by intermediate value theorem there exists a point $u_\epsilon^*$ such that $\rho_1^*(u_\epsilon^*) = \rho_2^*(u_\epsilon^*) = \rho_\epsilon^*$(say). The uniqueness of $\rho_\epsilon^*$ follows from the fact that $\rho_1^*$ is strictly decreasing and $\rho_2^*$ is strictly increasing. Since we are considering only the admissible part of the curves, the Lax entropy condition holds. This completes the proof. □

The next task is to determine the limit of the problem (3.1.6) for the shock case. First, we recall the definition of $\delta$-distribution and state a Lemma from the Chapter 2 without proof .

**Definition 3.3.2.** *A weighted $\delta$-distribution "$d(t)\delta_{x=c(t)}$" concentrated on a smooth curve $x = c(t)$ can be defined by*

$$\langle\, d(t)\delta_{x=c(t)}, \varphi(x,t)\rangle = \int_0^\infty d(t)\varphi(c(t),t)dt$$

*for all $\varphi \in C_c^\infty(\mathbb{R}\times(0,\infty))$.*

**Lemma 3.3.3** **([46]).** *Suppose $a_\epsilon(t)(>0)$ and $b_\epsilon(t)(>0)$ converge uniformly to $0$ on compact subsets of $(0,\infty)$ as $\epsilon$ tends to zero. Also assume that $d_\epsilon(t)$ converges to $d(t)$ uniformly on compact subsets of $(0,\infty)$ as $\epsilon$ tends to zero. Then*

$$\frac{1}{b_\epsilon(t)+a_\epsilon(t)} d_\epsilon(t)\chi_{(c(t)-a_\epsilon(t),c(t)+b_\epsilon(t))}(x)$$

*converges to $d(t)\delta_{x=c(t)}$ in the sense of distribution, $\chi_{(a,b)}(\cdot)$ denotes the characteristic function on the interval $(a,b)$.*

**Theorem 3.3.4.** *(Limiting behavior as $\epsilon \to 0$ )*
*The distributional limit of $(u^\epsilon,\rho^\epsilon)$ is $(u,\rho)$ and is given by*

$$(u,\rho)(x,t) = \begin{cases} (u_l,\rho_l), \ \text{ if } x < \frac{u_l+u_r}{2}t \\ \left(\frac{u_l+u_r}{2}, (u_l-u_r)\frac{\rho_l+\rho_r}{2}t\delta_{x=\frac{u_l+u_r}{2}t}\right), \ \text{ if } \ x = \frac{u_l+u_r}{2}t \\ (u_r,\rho_r), \ \text{ if } \ x > \frac{u_l+u_r}{2}t. \end{cases}$$

*Proof.* From the previous Theorem 3.1, we have $(u_\epsilon^*,\rho_\epsilon^*)$ satisfies the following equations:

$$\begin{aligned} (u_\epsilon^* - u_l) &= \frac{\rho_\epsilon^* - \rho_l}{\rho_\epsilon^* + \rho_l}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_\epsilon^*+\rho_l)(f(\rho_\epsilon^*)-f(\rho_l))}{(\rho_\epsilon^*-\rho_l)}}\Big], \\ (u_\epsilon^* - u_r) &= \frac{\rho_\epsilon^* - \rho_r}{\rho_\epsilon^* + \rho_r}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_\epsilon^*+\rho_r)(f(\rho_\epsilon^*)-f(\rho_r))}{(\rho_\epsilon^*-\rho_r)}}\Big]. \end{aligned} \tag{3.3.20}$$

We know $u_\epsilon^* \in (u_r, u_l)$. So the sequence $u_\epsilon^*$ is bounded. We claim that $\rho_\epsilon^*$ *is unbounded as* $\epsilon$ *tends to zero*. In fact, if $\rho_\epsilon^*$ is bounded, then it has a convergent subsequence still denoted by $\rho_\epsilon^*$ and it converges to $\rho^*(\neq \rho_l, \rho_r)$ as $\epsilon$ tends to zero. Then passing to the limit as $\epsilon \to 0$ in the above equation (3.3.20), we have $u^* = u_l = u_r$. Now suppose $\rho^* = \rho_l$, since $\lim\limits_{\rho_\epsilon^* \to \rho_l} \dfrac{f(\rho_\epsilon^*) - f(\rho_l)}{\rho_\epsilon^* - \rho_l} > 0$, we have $u_\epsilon^* = u_l = u_r$. Similar argument works when $\rho^* = \rho_r$. In all of the cases we get a contradiction.

So for subsequence of $u_\epsilon^*$ and $\rho_\epsilon^*$ still denoted as $u_\epsilon^*$ and $\rho_\epsilon^*$ respectively we have that $u_\epsilon^*$ converges to $u^*$ and $\rho_\epsilon^*$ tend to $+\infty$ as $\epsilon \to 0$. Passing to the limit for this subsequence in (3.3.20), we get

$$(u^* - u_l) = -\sqrt{l}$$

$$(u^* - u_r) = \sqrt{l},$$

where $\lim\limits_{\epsilon\to 0} 2\epsilon\big(f(\rho_\epsilon^*) - f(\rho_l)\big) = \lim\limits_{\epsilon\to 0} 2\epsilon\big(f(\rho_\epsilon^*) - f(\rho_r)\big) = l$. Solving the above two equations one can easily find

$$u^* = \frac{u_l + u_r}{2} \text{ and } l = \frac{1}{4}(u_l - u_r)^2. \tag{3.3.21}$$

Now from the above Theorem 3.1, we see that the intermediate state $(u_\epsilon^*, \rho_\epsilon^*)$ satisfies the equation (3.3.1). That is,

$$\begin{aligned} (u - u_\epsilon^*)s_{1,\epsilon} =& \frac{1}{2}(u^2 - u_\epsilon^{*2}) + \epsilon(f(\rho) - f(\rho_\epsilon^*)), \\ (\rho - \rho_\epsilon^*)s_{1,\epsilon} =& (\rho u - \rho_\epsilon^* u_\epsilon^*) - \epsilon(\rho - \rho_\epsilon^*), \end{aligned} \tag{3.3.22}$$

where $s_{1,\epsilon}$ is the 1-shock speed. From the above equation we have,

$$s_{1,\epsilon} = \frac{\rho u - \rho_\epsilon^* u_\epsilon^*}{\rho - \rho_\epsilon^*} - \epsilon. \tag{3.3.23}$$

Now we observe that, using the first equation of (3.3.20)(with $u_l$ replaced by $u$) $s_{1,\epsilon}$ can be rewritten as

$$\begin{aligned} s_{1,\epsilon} =& \frac{\rho u - \rho_\epsilon^* u_\epsilon^*}{\rho - \rho_\epsilon^*} - \epsilon \\ =& \frac{(u + u_\epsilon^*)(\rho - \rho_\epsilon^*) + (u - u_\epsilon^*)(\rho + \rho_\epsilon^*)}{2(\rho - \rho_\epsilon^*)} - \epsilon \\ =& \frac{u + u_\epsilon^*}{2} - \frac{\epsilon}{2} - \frac{1}{2}\Big[\epsilon - \frac{(u - u_\epsilon^*)(\rho + \rho_\epsilon^*)}{\rho - \rho_\epsilon^*}\Big] \\ =& \frac{u + u_\epsilon^*}{2} - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho + \rho_\epsilon^*)(f(\rho) - f(\rho_\epsilon^*))}{(\rho - \rho_\epsilon^*)}}. \end{aligned} \tag{3.3.24}$$

Similarly using the second equation of (3.3.20) $s_{2,\epsilon}$ can be written as

$$s_{2,\epsilon} = \frac{u + u_\epsilon^*}{2} - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho + \rho_\epsilon^*)(f(\rho) - f(\rho_\epsilon^*))}{(\rho - \rho_\epsilon^*)}}. \tag{3.3.25}$$

where $s_{2,\epsilon}$ is the 2-shock speed.

The solution for $(u^\epsilon, \rho^\epsilon)$ is given by

$$(u^\epsilon, \rho^\epsilon)(x,t) = \begin{cases} (u_l, \rho_l) & \text{if } x < \Big(\frac{u_l + u_\epsilon^*}{2} - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_l + \rho_\epsilon^*)(f(\rho_l) - f(\rho_\epsilon^*))}{(\rho_l - \rho_\epsilon^*)}}\Big)t \\ (u_\epsilon^*, \rho_\epsilon^*) & \text{if } \Big(\frac{u_l + u_\epsilon^*}{2} - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_l + \rho_\epsilon^*)(f(\rho_l) - f(\rho_\epsilon^*))}{(\rho_l - \rho_\epsilon^*)}}\Big)t < x \\ & < \Big(\frac{u_r + u_\epsilon^*}{2} - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_r + \rho_\epsilon^*)(f(\rho_r) - f(\rho_\epsilon^*))}{(\rho_r - \rho_\epsilon^*)}}\Big)t \\ (u_r, \rho_r) & \text{if } x > \Big(\frac{u_r + u_\epsilon^*}{2} - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_r + \rho_\epsilon^*)(f(\rho_r) - f(\rho_\epsilon^*))}{(\rho_r - \rho_\epsilon^*)}}\Big)t, \end{cases} \tag{3.3.26}$$

As $u_\epsilon^*$ converges to $u^* = \frac{u_l + u_r}{2}$ as $\epsilon \to 0$, we have the limit for $u(x,t)$ as stated in the theorem.

From (3.3.21), one can show that

$$\lim_{\epsilon \to 0}\Big[\frac{u_l + u_\epsilon^*}{2} - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_l + \rho_\epsilon^*)(f(\rho_l) - f(\rho_\epsilon^*))}{(\rho_l - \rho_\epsilon^*)}}\Big] = \frac{u_l + u_r}{2},$$

and

$$\lim_{\epsilon \to 0}\Big[\frac{u_r + u_\epsilon^*}{2} - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_r + \rho_\epsilon^*)(f(\rho_r) - f(\rho_\epsilon^*))}{(\rho_r - \rho_\epsilon^*)}}\Big] = \frac{u_l + u_r}{2}.$$

Let us denote

$$
\begin{aligned}
c(t) &= \frac{u_l+u_r}{2}t,\\
a_\epsilon(t) &= \Big(\frac{u_l+u_\epsilon^*}{2}-\frac{\epsilon}{2}-\frac{1}{2}\sqrt{\epsilon^2+\frac{2\epsilon(\rho_l+\rho_\epsilon^*)(f(\rho_l)-f(\rho_\epsilon^*))}{(\rho_l-\rho_\epsilon^*)}}\Big)t-c(t),\\
b_\epsilon(t) &= c(t)-\Big(\frac{u_r+u_\epsilon^*}{2}-\frac{\epsilon}{2}+\frac{1}{2}\sqrt{\epsilon^2+\frac{2\epsilon(\rho_r+\rho_\epsilon^*)(f(\rho_r)-f(\rho_\epsilon^*))}{(\rho_r-\rho_\epsilon^*)}}\Big)t,\\
d_\epsilon(t) &= \Big[\frac{u_r-u_l}{2}+\frac{1}{2}\Big(\sqrt{\epsilon^2+\frac{2\epsilon(\rho_l+\rho_\epsilon^*)(f(\rho_l)-f(\rho_\epsilon^*))}{(\rho_l-\rho_\epsilon^*)}}\\
&\quad+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_r+\rho_\epsilon^*)(f(\rho_r)-f(\rho_\epsilon^*))}{(\rho_r-\rho_\epsilon^*)}}\Big)\Big]\rho_\epsilon^* t.
\end{aligned}
$$

With the above notations, the formula for $\rho^\epsilon$ in equation (3.3.26) can be written in the following form as in the Lemma(3.2):

$$
\begin{aligned}
\rho^\epsilon =&\rho_l\chi_{(-\infty,c(t)+a_\epsilon(t))}(x)+\frac{d_\epsilon(t)}{b_\epsilon(t)+a_\epsilon(t)}\chi_{(c(t)+a_\epsilon(t),c(t)-b_\epsilon(t))}(x)\\
&+\rho_r\chi_{(c(t)-b_\epsilon(t),\infty)}(x).
\end{aligned}
\tag{3.3.27}
$$

Note that $a_\epsilon(t)$ and $b_\epsilon(t)$ satisfies the condition of the lemma, i.e, $a_\epsilon(t)>0$ and $b_\epsilon(t)>0$ for small $\epsilon$.

Now we are in a position to determine the limit of $d_\epsilon(t)$ as $\epsilon\to 0$. The equation (3.3.20) can also be written in the following form:

$$
\begin{aligned}
(\rho_\epsilon^*+\rho_l)(u_\epsilon^*-u_l) &= (\rho_\epsilon^*-\rho_l)\Big[\epsilon-\sqrt{\epsilon^2+\frac{2\epsilon(\rho_\epsilon^*+\rho_l)(f(\rho_\epsilon^*)-f(\rho_l))}{(\rho_\epsilon^*-\rho_l)}}\Big]\\
(\rho_\epsilon^*+\rho_r)(u_\epsilon^*-u_r) &= (\rho_\epsilon^*-\rho_r)\Big[\epsilon+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_\epsilon^*+\rho_r)(f(\rho_\epsilon^*)-f(\rho_r))}{(\rho_\epsilon^*-\rho_r)}}\Big].
\end{aligned}
\tag{3.3.28}
$$

Subtracting second equation from the first in (3.3.28), we get

$$
\begin{aligned}
&\Big[(u_r-u_l)+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_\epsilon^*+\rho_l)(f(\rho_\epsilon^*)-f(\rho_l))}{(\rho_\epsilon^*-\rho_l)}}+\sqrt{\epsilon^2+\frac{2\epsilon(\rho_\epsilon^*+\rho_r)(f(\rho_\epsilon^*)-f(\rho_r))}{(\rho_\epsilon^*-\rho_r)}}\Big]\rho_\epsilon^*\\
&+\rho_l(u_\epsilon^*-u_l)-\rho_r(u_\epsilon^*-u_r)-\epsilon(\rho_r-\rho_l)\\
&=\rho_l\sqrt{\epsilon^2+\frac{2\epsilon(\rho_\epsilon^*+\rho_l)(f(\rho_\epsilon^*)-f(\rho_l))}{(\rho_\epsilon^*-\rho_l)}}+\rho_r\sqrt{\epsilon^2+\frac{2\epsilon(\rho_\epsilon^*+\rho_r)(f(\rho_\epsilon^*)-f(\rho_r))}{(\rho_\epsilon^*-\rho_r)}}.
\end{aligned}
$$

Passing to the limit as $\epsilon \to 0$, we get

$$\lim_{\epsilon\to 0}\Big[(u_r - u_l) + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_\epsilon^* + \rho_l)(f(\rho_\epsilon^*) - f(\rho_l))}{(\rho_\epsilon^* - \rho_l)}} + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_\epsilon^* + \rho_r)(f(\rho_\epsilon^*) - f(\rho_r))}{(\rho_\epsilon^* - \rho_r)}}\Big]\rho_\epsilon^*$$
$$= (u_l - u_r)(\rho_l + \rho_r). \tag{3.3.29}$$

This implies

$$\lim_{\epsilon\to 0} d_\epsilon(t) = \frac{1}{2}(u_l - u_r)(\rho_l + \rho_r)t. \tag{3.3.30}$$

Here in the calculation of (3.3.30), we have used the fact that $\lim\limits_{\epsilon\to 0} 2\epsilon\big(f(\rho_\epsilon^*) - f(\rho_l)\big) = \lim\limits_{\epsilon\to 0} 2\epsilon\big(f(\rho_\epsilon^*) - f(\rho_r)\big) = l = \frac{1}{4}(u_l - u_r)^2$ and $\lim\limits_{\epsilon\to 0} u_\epsilon^* = \frac{u_l + u_r}{2}$ from the equation (3.3.21). The first and the third terms of (3.3.27) converge to $\rho_l \chi_{(-\infty, \frac{u_l+u_r}{2}t)}(x)$ and $\rho_r \chi_{(\frac{u_l+u_r}{2}t, \infty)}(x)$ respectively. Hence, employing the above lemma to the second term of (3.3.27), we get the distribution limit $\rho(x,t)$ as given in the theorem. Note that all the analysis has been carried out for a subsequence. Since the limit is the same for any subsequence, this implies the sequence itself converges to the same limit. This completes the proof of the theorem. □

Now it remains to show that the limit $(u, \rho)$ found in the theorem above, satisfies the equation (3.1.8). The limit $(u, \rho)$ satisfies the equation in the sense of Volpert is available in [37]. There it was shown that $R_t + \overline{u}R_x = 0$ , where $\rho = R_x$ and $\overline{u}R_x$ is known as Volpert product [34]. Then $\rho = R_x$ satisfies the equation (3.1.8) in the sense of distribution. The limit $(u, \rho)$ satisfies the equation (3.1.8) is also shown in [46] the sense of the following definition.

**Definition 3.3.5** ([46]). *Let $u$ is a Borel measurable function and $\rho = d\nu$ is a Radon measure on $\mathbb{R} \times [0, \infty)$. Then $(u, \rho = d\nu)$ is said to be a solution for the system* (3.1.8) *with* *initial data* (3.1.2) *if the following conditions hold.*

$$\begin{aligned}&\int_{\mathbb{R}\times[0,\infty)}(u\phi_t+u\phi_x)dxdt+\int_{\mathbb{R}}u_0(x)\phi(x,0)dx=0\\&\int_{\mathbb{R}\times[0,\infty)}(\phi_t+u\phi_x)d\nu+\int_{\mathbb{R}}\rho_0(x)\phi(x,0)dx=0,\end{aligned} \tag{3.3.31}$$

*for any test function* $\phi$ *supported in* $\mathbb{R}\times[0,\infty)$.

Now we state the following theorem and the proof can be found in [46].

**Theorem 3.3.6** ([46]). *For* $u_l > u_r$*, the point wise limit* $u$ *of* $u^\epsilon$ *and distributional limit of* $\rho$ *of* $\rho^\epsilon$ *satisfies the equation*(3.3.31).

## 3.4 Entropy and entropy flux pairs.

This section is devoted to constructing an explicit entropy-entropy flux pairs for the system (3.1.6) when $f(\rho)=\frac{\rho^2}{2}$, i.e for Brio system. We start with the following definitions[2] restricted to the $2\times 2$ system, namely

$$\begin{aligned}&u_t+(f_1(u,\rho))_x=0\\&\rho_t+(f_2(u,\rho))_x=0.\end{aligned} \tag{3.4.1}$$

**Definition 3.4.1.** *A continuously differentiable function* $\eta:\mathbb{R}^2\mapsto\mathbb{R}$ *is called an entropy for the system*(3.4.1) *with entropy flux* $q:\mathbb{R}^2\mapsto\mathbb{R}$ *if*

$$D\eta(u,\rho).Df(u,\rho)=Dq(u,\rho),$$

*where* $f(u,\rho)=(f_1(u,\rho),f_2(u,\rho))$. *We say* $(\eta,q)$ *as entropy-entropy flux pair of the system*(3.4.1).

**Definition 3.4.2.** *A weak solution* $(u,\rho)$ *of the system* (3.4.1) *is called entropy admissible if*

$$\iint_{\mathbb{R}\times(0,\infty)}\eta(u,\rho)\varphi_t+q(u,\rho)\varphi_x\,dx\,dt\geq 0,$$

*for every non-negative test function* $\varphi:\mathbb{R}\times(0,\infty)\to\mathbb{R}$ *with compact support in* $\mathbb{R}\times(0,\infty)$*, where* $(\eta,q)$ *is the entropy-entropy flux pair as in the definition*(3.4.1).

Now for the system (3.1.6), $f(u,\rho)=\Big(\frac{u^2}{2}+\frac{\epsilon}{2}\rho^2,\quad u\rho-\epsilon\rho\Big)$. Therefore $(\eta,q)$ will be an entropy-entropy flux pair of (3.1.6) if

$$\Big(\frac{\partial\eta}{\partial u}u+\frac{\partial\eta}{\partial\rho}\rho\,,\ \epsilon\rho\frac{\partial\eta}{\partial u}+(u-\epsilon)\frac{\partial\eta}{\partial\rho}\Big)=\Big(\frac{\partial q}{\partial u},\frac{\partial q}{\partial\rho}\Big).$$

That is,

$$\begin{aligned}\frac{\partial q}{\partial u}&=\frac{\partial\eta}{\partial u}u+\frac{\partial\eta}{\partial\rho}\rho,\\ \frac{\partial q}{\partial\rho}&=\epsilon\rho\frac{\partial\eta}{\partial u}+(u-\epsilon)\frac{\partial\eta}{\partial\rho}.\end{aligned}\tag{3.4.2}$$

Eliminating $q$ from (3.4.2), we have

$$\epsilon\Big(\rho\frac{\partial^2\eta}{\partial u^2}-\frac{\partial^2\eta}{\partial\rho\partial\rho}\Big)-\rho\frac{\partial^2\eta}{\partial\rho^2}=0.$$

One can see that

$$\eta(u,\rho)=\frac{1}{2}u^2+\frac{\epsilon}{2}\rho^2$$

is a solution of above the equation which is strictly convex (since $D^2\eta>0$) and the corresponding entropy flux is

$$q(u,\rho)=\frac{1}{3}u^3+\Big(u-\frac{\epsilon}{2}\Big)\epsilon\rho^2.$$

By constructing an explicit entropy-entropy flux pair for the Brio system, we show here that our solution constructed in the previous section for Riemann type initial data ($u_l>u_r$) which can also be treated as a solution coming from scaled Brio system if we plug $f(\rho)=\frac{\rho^2}{2}$ into the equation(3.1.6), is entropy admissible in the sense of the above definition(3.4.2).

For that we calculate

$$\begin{aligned}\eta_t+q_x=&-s_{1,\epsilon}\Big(\frac{1}{2}u_\epsilon^{*2}+\frac{\epsilon}{2}\rho_\epsilon^{*2}-\frac{1}{2}u_l^2-\frac{\epsilon}{2}\rho_l^2\Big)\delta_{x=s_{1,\epsilon}t}\\ &-s_{2,\epsilon}\Big(\frac{1}{2}u_r^2+\epsilon e^{\rho_r}-\frac{1}{2}u_\epsilon^{*2}-\frac{\epsilon}{2}\rho_\epsilon^{*2}\Big)\delta_{x=s_{2,\epsilon}t}\\ &+\Big(\frac{1}{3}u_\epsilon^{*3}+(u_\epsilon^*-\frac{\epsilon}{2})\epsilon\rho_\epsilon^{*2}-\frac{1}{3}u_l^3-(u_l-\frac{\epsilon}{2})\epsilon\rho_l^2\Big)\delta_{x=s_{1,\epsilon}t}\\ &+\Big(\frac{1}{3}u_r^3-(u_r-\frac{\epsilon}{2})\rho_r^2-\frac{1}{3}u_\epsilon^{*3}-(u_\epsilon^*-\frac{\epsilon}{2})\epsilon\rho_\epsilon^{*2}\Big)\delta_{x=s_{2,\epsilon}t},\end{aligned}\tag{3.4.3}$$

where $s_{1,\epsilon}$ and $s_{2,\epsilon}$ denote 1-shock speed and 2-shock speed respectively. So from (3.3.24) $s_{1,\epsilon}$ and $s_{2,\epsilon}$ can be written as

$$s_{1,\epsilon} = \Big(\frac{u_l + u_\epsilon^*}{2} - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon(\rho_l + \rho_\epsilon^*)^2 + \epsilon^2}\Big),$$
$$s_{2,\epsilon} = \Big(\frac{u_r + u_\epsilon^*}{2} - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon(\rho_r + \rho_\epsilon^*)^2 + \epsilon^2}\Big).$$

One can observe that to show $\eta(u,\rho)$ and $q(u,\rho)$ satisfies the entropy inequality for small $\epsilon$, we must treat the coefficients $\delta_{x=s_1 t}$ and $\delta_{x=s_2 t}$ separately. We show that each of the coefficient will be negative as $\epsilon$ tends to zero. let us first consider the coefficient of $\delta_{x=s_1 t}$.

Coefficient of $\delta_{x=s_{1,\epsilon}t}$

$$= \underbrace{-s_1\left(\frac{1}{2}u_\epsilon^{*2} + \frac{\epsilon}{2}\rho_\epsilon^{*2} - \frac{1}{2}u_l^2 - \frac{\epsilon}{2}\rho_l^2\right)}_{\text{I}} + \underbrace{\left(\frac{1}{3}u_\epsilon^{*3} + (u_\epsilon^* - \frac{\epsilon}{2})\epsilon\rho_\epsilon^{*2} - \frac{1}{3}u_l^3 - (u_l - \frac{\epsilon}{2})\epsilon\rho_l^2\right)}_{\text{II}} \tag{3.4.4}$$

From (3.3.20) we have $(u_\epsilon^*, \rho_\epsilon^*)$ satisfies the following equations.

$$\begin{aligned} u_\epsilon^* - u_l &= \frac{\rho_\epsilon^* - \rho_l}{\rho_\epsilon^* + \rho_l}\Big[\epsilon - \sqrt{\epsilon^2 + \epsilon(\rho_\epsilon^* + \rho_l)^2}\Big], \\ u_\epsilon^* - u_r &= \frac{\rho_\epsilon^* - \rho_r}{\rho_\epsilon^* + \rho_r}\Big[\epsilon - \sqrt{\epsilon^2 + \epsilon(\rho_\epsilon^* + \rho_r)^2}\Big]. \end{aligned} \tag{3.4.5}$$

Now similarly as in Theorem 3.3 we have

$$u_\epsilon^* - u_l = -\sqrt{l}$$
$$u_\epsilon^* - u_r = \sqrt{l}$$

where

$$\lim_{\epsilon\to 0}\epsilon(\rho_\epsilon^* + \rho_l)^2 = \lim_{\epsilon\to 0}\epsilon(\rho_\epsilon^* + \rho_r)^2 = \lim_{\epsilon\to 0}\epsilon\rho_\epsilon^{*2} = \frac{1}{4}(u_r - u_l)^2. \tag{3.4.6}$$

Now using (3.4.6) and observing that $s_1 \to \frac{(u_l+u_r)}{2}$, one can see

$$I \to \frac{-(u_l + u_r)(u_r^2 - u_l^2)}{8} \quad \text{as } \epsilon \to 0.$$

Again using (3.4.6), a simple calculation yields

$$II \to \frac{(u_r^3 - u_l^3)}{6} \quad \text{as } \epsilon \to 0.$$

Therefore from the equation(3.4.4),

$$\text{Coefficient of } \delta_{x=s_1 t} = I + II \to \frac{(u_r - u_l)(u_l - u_r)^2}{24} \quad \text{as } \epsilon \to 0.$$

Since $u_l > u_r$, Coefficient of $\delta_{x=s_1 t} = I + II < 0$ for small $\epsilon$. Similarly, the coefficients of $\delta_{x=s_2 t}$ can be handled.

**Remark 3.4.3.** *It is well known that if $\eta$ be a smooth entropy of the system* (3.4.1) *with the entropy flux $q$ and if one assumes that the Hessian $D^2\eta > 0$, then for genuinely non-linear characteristic fields the entropy inequality $\eta(u)_t + q(u)_x \le 0$ is satisfied for Riemann type initial data having small total variation. Details can be found in [2]. Here we showed that the solution $(u^\epsilon, \rho^\epsilon)$ satisfies the entropy condition in the following sense: for any given initial data $(u_l, \rho_l)$ and $(u_r, \rho_r)$ there exists a $\mu > 0$ such that*

$$\iint_{\mathbb{R}\times(0,\infty)} \eta(u^\epsilon, \rho^\epsilon)\varphi_t + q(u^\epsilon, \rho^\epsilon)\varphi_x \, dx \, dt \ge 0$$

*holds for $\epsilon < \mu$ and for any test function $\varphi \ge 0$ compactly supported in $\mathbb{R} \times (0, \infty)$.*

## 3.5 Formation of contact discontinuity and cavitation for $u_l \le u_r$.

In this section we discuss other two cases, i.e, $u_l = u_r$ and $u_l < u_r$. The discussion in this section is a mere repetition of the steps [46] except for the fact that here we have two different shock speeds.

**Case I** $(u_l = u_r)$: For $u_l = u_r$, initial data is

$$\begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_l \\ \rho_l \end{pmatrix}, & \text{if} \quad x < 0 \\ \begin{pmatrix} u_l \\ \rho_r \end{pmatrix}, & \text{if} \quad x > 0. \end{cases}$$

Now if $\rho_l = \rho_r$, we have the trivial solution $u(x,t) = u_l$ and $\rho(x,t) = \rho_l$. Another two possibilities are $\rho_r < \rho_l$ or $\rho_r > \rho_l$.

*Subcase I($\rho_r < \rho_l$)*: In this case, we start traveling from the state $(u_l, \rho_l)$ in the curve $R_1$ to reach at $(u^*_\epsilon, \rho^*_\epsilon)$, then from $(u^*_\epsilon, \rho^*_\epsilon)$ we travel by $S_2$ to reach at $(u_l, \rho_r)$. 1-rarefaction curve $R_1$ through $(u_l, \rho_l)$ is obtained solving the differential equation

$$\frac{du}{d\rho} = \frac{\epsilon - \sqrt{4\epsilon\rho f'(\rho) + \epsilon^2}}{2\rho}, \qquad u(\rho_l) = u_l \tag{3.5.1}$$

Therefore the branch of the curve satisfying (3.5.1) can be parameterized by a $C^1$ function $u_1 : [\rho_r, \rho_l] \to [u_l, \infty)$ with parameter $\rho$. Since $\rho > 0$, we see that $u_1$ is decreasing. Therefore, $u_1(\rho_r) > u_l$.

Any state $(u, \rho)$ connected to the end state $(u_l, \rho_r)$ by admissible 2-shock curve $S_2$ satisfies the following equation:

$$(u - u_l) = \frac{\rho - \rho_r}{\rho + \rho_r}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{\epsilon(\rho+\rho_r)(f(\rho) - f(\rho_r))}{(\rho - \rho_r)}}\Big], \quad \rho_r < \rho < \rho_l; \quad u > u_l \tag{3.5.2}$$

and

$$s > \lambda_2(u, \rho), \;\; \lambda_1(u_l, \rho_r) < s < \lambda_2(u_l, \rho_r), \;\text{where}\;\; s = \frac{\rho u - \rho_r u_l}{\rho - \rho_r} - \epsilon. \tag{3.5.3}$$

Our claim is that for every fixed $\rho > \rho_r$ there exists a unique $u > u_l$ such that the equation (3.5.2) holds. For that let us define

$$F(u) := u - u_l.$$

Since $F(u_l) = 0$ and $F(u) \to \infty$ as $u \to \infty$, we have $F([u_l, \infty)) = [0, \infty)$. Since $\rho > \rho_r$, right hand side of (3.5.2) is positive. Therefore for the given $\rho > \rho_r$, there exists a $u > u_l$ such that

$$F(u) = \frac{\rho - \rho_r}{\rho + \rho_r}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\rho_r)(f(\rho) - f(\rho_r))}{(\rho - \rho_r)}}\Big].$$

Also observe that $F(u)$ is an increasing function for all $u$ since $F'(u) = 1$ , $u$ is unique for the given $\rho$.

Similarly in Theorem 3.1, the branch of the curve satisfying (3.5.2) and (3.5.3) can be parameterized by a $C^1$-function $u_2(\rho) = u_2 : [\rho_r, \rho_l] \to [u_l, \infty)$ satisfying

$$F(u_2(\rho)) = (u_2(\rho) - u_l) = \frac{\rho - \rho_r}{\rho + \rho_r}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho + \rho_r)(f(\rho) - f(\rho_r))}{(\rho - \rho_r)}}\Big] \qquad (3.5.4)$$

Note that $u_2(\rho_r) = u_l$ and it is clear from the above equation (3.5.4) that the function $u_2$ is well defined. One can easily check that the function $u_2$ is increasing in the interval $(\rho_r, \rho_l)$. In fact, differentiating the above equation (3.5.4) we get,

$$u_2{}'(\rho) = \frac{\epsilon(\rho - \rho_r)\frac{d}{d\rho}\Big[\frac{(\rho+\rho_r)(f(\rho)-f(\rho_r))}{(\rho-\rho_r)}\Big]}{\sqrt{\epsilon^2 + \frac{2\epsilon(\rho+\rho_r)(f(\rho)-f(\rho_r))}{(\rho-\rho_r)}}} + \Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho + \rho_r)(f(\rho) - f(\rho_r))}{(\rho - \rho_r)}}\Big].$$

Since $\rho > \rho_r$ and $\rho_r > 0$, in the view of (3.3.17) right hand side of above equation is positive for any $\epsilon > 0$. That is, $u_2{}'(\rho) > 0$.

From the above analysis, there exists an intermediate state $\rho^*_\epsilon \in (\rho_r, \rho_l)$ such that $u_1(\rho^*_\epsilon) = u_2(\rho^*_\epsilon) = u^*_\epsilon$. Hence the solution for (3.1.6) is given by:

$$(u^\epsilon, \rho^\epsilon) = \begin{cases} (u_l, \rho_l) & \text{if } x < \lambda_1(u_l, \rho_l)t \\ \big(R_1^u(x/t)(u_l, \rho_l), R_1^\rho(x/t)(u_l, \rho_l)\big) & \text{if } \lambda_1(u_l, \rho_l)t < x < \lambda_1(u^*_\epsilon, \rho^*_\epsilon)t \\ (u^*_\epsilon, \rho^*_\epsilon) & \text{if} \lambda_1(u^*_\epsilon, \rho^*_\epsilon)t < x < s_{2,\epsilon}(u_l, \rho_r, u^*_\epsilon, \rho^*_\epsilon)t \\ (u_r, \rho_r) & \text{if } x > s_{2,\epsilon}(u_l, \rho_r, u^*_\epsilon, \rho^*_\epsilon)t \end{cases}$$

Where $R_1(\xi)(\bar{u}, \bar{\rho}) = (R_1^u(\xi)(\bar{u}, \bar{\rho}), R_1^\rho(\xi)(\bar{u}, \bar{\rho}))$ and $R_1^u(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{du}{d\xi} = \frac{\epsilon - \sqrt{4\epsilon\rho f'(\rho) + \epsilon^2}}{2\rho}, \quad u(\lambda_1(\bar{u}, \bar{\xi})) = \bar{u}.$$

and $R_1^\rho(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{d\rho}{d\xi} = 1, \quad \rho(\lambda_1(\bar{u}, \bar{\xi})) = \bar{\rho}.$$

and

$$s_{2,\epsilon}(u_l, \rho_r, u^*_\epsilon, \rho^*_\epsilon) = \frac{u_l + u^*_\epsilon}{2} - \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_r + \rho^*_\epsilon)(f(\rho_r) - f(\rho^*_\epsilon))}{(\rho_r - \rho^*_\epsilon)}}.$$

*Sub-case II ($\rho_l < \rho_r$)*: In a similar way one can start from $(u_l, \rho_l)$ and reach at $(u_\epsilon^*, \rho_\epsilon^*)$ by $S_1$ and from $(u_\epsilon^*, \rho_\epsilon^*)$ to $(u_l, \rho_r)$ by $R_2$. Therefore the solution is given by :

$$(u^\epsilon, \rho^\epsilon) = \begin{cases} (u_l, \rho_l) & \text{if } x < s_{1,\epsilon}(u_l, \rho_l, u_\epsilon^*, \rho_\epsilon^*)t \\ (u_\epsilon^*, \rho_\epsilon^*) & \text{if } s_{1,\epsilon}(u_l, \rho_l, u_\epsilon^*, \rho_\epsilon^*)t < x < \lambda_2(u_\epsilon^*, \rho_\epsilon^*)t \\ \big(R_2^u(x/t)(u_\epsilon^*, \rho_\epsilon^*), R_2^\rho(x/t)(u_\epsilon^*, \rho_\epsilon^*)\big) & \text{if } \lambda_2(u_\epsilon^*, \rho_\epsilon^*)t < x < \lambda_2(u_l, \rho_r)t \\ (u_r, \rho_r) & \text{if } x > \lambda_2(u_l, \rho_r)t \end{cases}$$

where $R_2(\xi)(\bar{u}, \bar{\rho}) = (R_2^u(\xi)(\bar{u}, \bar{\rho}), R_2^\rho(\xi)(\bar{u}, \bar{\rho}))$ and $R_2^u(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{du}{d\xi} = \frac{\epsilon + \sqrt{4\epsilon\rho f'(\rho) + \epsilon^2}}{2\rho}, \quad u(\lambda_2(\bar{u}, \bar{\xi}) = \bar{u}.$$

and $R_2^\rho(\xi)(\bar{u}, \bar{\rho})$ is obtained by solving

$$\frac{d\rho}{d\xi} = 1, \quad \rho(\lambda_2(\bar{u}, \bar{\xi}) = \bar{\rho}.$$

and

$$s_{1,\epsilon}(u_l, \rho_l, u_\epsilon^*, \rho_\epsilon^*) = \frac{u_l + u_\epsilon^*}{2} - \frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon^2 + \frac{2\epsilon(\rho_l + \rho_\epsilon^*)(f(\rho_l) - f(\rho_\epsilon^*))}{(\rho_l - \rho_\epsilon^*)}}.$$

Now we aim to find the limit of $(u^\epsilon, \rho^\epsilon)$ as $\epsilon \to 0$ in both of the above cases. Since $\rho_\epsilon^* \in (\rho_l, \rho_r)$ or $\rho_\epsilon^* \in (\rho_r, \rho_l)$ this implies $\rho_\epsilon^*$ is bounded. Also from the above analysis it is evident that $\rho_\epsilon^*$ and $u_\epsilon^*$ satisfies 1-shock curve and 2-shock curve. This implies

$$\begin{aligned} (u_\epsilon^* - u_l) &= \frac{\rho_\epsilon^* - \rho_l}{\rho_\epsilon^* + \rho_l}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_\epsilon^* + \rho_l)(f(\rho_\epsilon^*) - f(\rho_l))}{(\rho_\epsilon^* - \rho_l)}}\Big], \;\; \rho_r > \rho_\epsilon^* > \rho_l; \;\; u_\epsilon^* < u_l \\ (u_\epsilon^* - u_l) &= \frac{\rho_\epsilon^* - \rho_r}{\rho_\epsilon^* + \rho_r}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho_\epsilon^* + \rho_r)(f(\rho_\epsilon^*) - f(\rho_r))}{(\rho_\epsilon^* - \rho_r)}}\Big], \;\; \rho_r < \rho_\epsilon^* < \rho_l; \;\; u_\epsilon^* > u_l \end{aligned} \tag{3.5.5}$$

Since right hand side of (3.5.5) is bounded, as $\epsilon \to 0$ we get, $\lim_{\epsilon\to 0} u_\epsilon^* = u_l$. Therefore the solution $(u^\epsilon, \rho^\epsilon) \to (u, \rho)$ as $\epsilon \to 0$ where $(u, \rho)$ is given by:

$$(u, \rho) = \begin{cases} (u_l, \rho_l) & \text{if } x < u_l t \\ (u_r, \rho_r) & \text{if } x > u_l t. \end{cases}$$

Since here $u_l = u_r$ we have $u \equiv u_l$.

**Case II** $(u_l < u_r)$ : It can be observed that solution for this case is exactly same as the solution for the case $u_l < u_r$ described in [46]. For the sake of completeness we include here that part of the result from [46]. The 1st-rarefaction curve passing through $(u_l, \rho_l)$ is given by the solution of the following Cauchy problem:

$$\frac{du}{d\rho} = \frac{\epsilon - \sqrt{4\epsilon\rho f'(\rho) + \epsilon^2}}{2\rho}, \qquad u(\rho_l) = u_l, \qquad \rho < \rho_l.$$

Note that for this case it does not matter whether $\rho_l < \rho_r$ or $\rho_l > \rho_r$. Therefore without loss of any generality one can take $\rho_l > \rho_r > 0$. Now a branch of $R_1$ can be parameterized by a differentiable function $u_1 : [0, \rho_l] \to [u_l, \infty)$ with a parameter $\rho$. Explicitly $u_1$ can be written as

$$u_1(\rho) - u_l = \int_{\rho}^{\rho_l} \frac{\epsilon - \sqrt{4\epsilon\xi f'(\xi) + \epsilon^2}}{2\xi} d\xi. \tag{3.5.6}$$

Since $\rho \in [0, \rho_l]$ is bounded and $\rho > 0$, the above integral goes to zero as $\epsilon$ approaches to zero. Therefore we have $u_1(\rho) \to u_l$ as $\epsilon \to 0$ decreasingly. Similarly, the 2nd-rarefaction curve is given by the solution of then Cauchy problem :

$$\frac{du}{d\rho} = \frac{\epsilon + \sqrt{4\epsilon\rho f'(\rho) + \epsilon^2}}{2\rho}, \qquad \rho < \rho_r, \qquad u(\rho_r) = u_r. \tag{3.5.7}$$

Let $u_2 : [0, \rho_r] \to (-\infty, u_r]$ is differentiable and parameterized branch of $R_2$ satisfying (3.5.7) and can be written as

$$u_2(\rho) - u_r = \int_{\rho}^{\rho_r} \frac{\epsilon + \sqrt{4\epsilon\xi f'(\xi) + \epsilon^2}}{2\xi} d\xi.$$

Since $\rho \in [0, \rho_r]$ and $\rho > 0$, using the same argument as above, we have $u_2(\rho) \to u_r$ as $\epsilon \to 0$ increasingly. Since $u_l < u_r$, by the above calculation one can see $u_1(0) < u_2(0)$ for small $\epsilon$. In this case the complete solution is given by:

$$(u^\epsilon, \rho^\epsilon) = \begin{cases} (u_l, \rho_l) & \text{if } x < \lambda_1(u_l, \rho_l)t \\ \big(R_1^u(x/t)(u_l, \rho_l), R_1^\rho(x/t)(u_l, \rho_l)\big) & \text{if } \lambda_1(u_l, \rho_l)t < x < \lambda_1(u_\epsilon^{*(1)}, 0)t \\ (x/t, 0) & \text{if } \lambda_1(u_\epsilon^{*(1)}, 0)t < x < \lambda_2(u_\epsilon^{*(2)}, 0)t \\ \big(R_2^u(x/t)(u_\epsilon^{*(2)}, 0), R_2^\rho(x/t)(u_\epsilon^{*(2)}, 0)\big) & \text{if } \lambda_2(u_\epsilon^{*(2)}, 0)t < x < \lambda_2(u_r, \rho_r)t \\ (u_r, rho_r) & \text{if } x > \lambda_2(u_r, \rho_r)t. \end{cases} \tag{3.5.8}$$

where $R_1^u(.)$, $R_1^\rho(.)$, $R_2^u(.)$, $R_2^\rho(.)$ are defined as above.

Now we find the limit of $(u^\epsilon, \rho^\epsilon)$ as $\epsilon \to 0$. Since $u_\epsilon^{*(1)} = u_1(0)$, we have $u_\epsilon^{*(1)} \to u_l$ and similarly $u_\epsilon^{*(2)} \to u_r$ as $\epsilon \to 0$. After passing to the limit in (3.5.8) as $\epsilon$ tends to zero, we get

$$(u, \rho)(x, t) = \begin{cases} (u_l, \rho_l) & \text{if } x < u_l t \\ (x/t, 0) & \text{if } u_l t < x < u_r t \\ (u_r, \rho_r) & \text{if } x > u_r t \end{cases}$$

**Remark 3.5.1.** *In equation* (3.5.8)*, one has to take* $u^\epsilon(x, t) = \frac{x}{t}$ *in the region* $\lambda_1(u_\epsilon^{*(1)}, 0)t < x < \lambda_2(u_\epsilon^{*(2)}, 0)t$. *This kind of choice gives an unique entropy solution. In fact, since* $\rho = 0$ *in this region, the first equation of* (3.1.6) *becomes the well known Burgers equation and* $u(x, t) = \frac{x}{t}$ *is the unique entropy solution for the rarefaction case of Burgers equation.*

# 3.6 Concluding remarks and further possibilities.

1. Theorem 1.1 can be achieved by combining the results of Theorem 3.1, Theorem 3.3 and the discussion in Section 5. In this article, we studied the generalized Euler system when $f(\rho)$ and $f'(\rho)$ both are increasing and $g(\rho)$ is any linear decreasing function. We observed that our analysis is still valid for some particular non linear decreasing $g(\rho)$ and particular $f(\rho)$ with the property stated above. For example, if we take $f(\rho) = \frac{\rho^2}{2}$ and $g(\rho) = -\rho^2$, the shock curves passing through $(u_l, \rho_l)$ are the following.

$$s_1 = \Big\{(u, \rho) : (u - u_l) = (\rho - \rho_l)\Big[\epsilon - \sqrt{\epsilon^2 + \epsilon}\Big], \;\; \rho > \rho_l; \;\; u < u_l\Big\},$$

$$s_2 = \Big\{(u, \rho) : (u - u_l) = (\rho - \rho_l)\Big[\epsilon + \sqrt{\epsilon^2 + \epsilon}\Big], \;\; \rho < \rho_l; \;\; u < u_l\Big\}$$

For the case $u_l > u_r$, one can have the existence of the intermediate state in the same way as in Theorem 3.1, however, in this case, calculations are simpler than the calculations presented here. One can show that $\lim_{\epsilon\to 0}\sqrt{\epsilon}\rho_\epsilon^*$ exists and following the steps of Theorem3.3, distributional limit of $(u_\epsilon, \rho_\epsilon)$ as $\epsilon \to 0$ can be determined. Finally, the case $u_l \le u_r$ can be handled in a similar way as in Section 5.

2. One can address a similar question with general $g(\rho)$. Note that the shock curves passing through $(u_l, \rho_l)$ for any general $f(\rho)$ and $g(\rho)$, can be found in the following manner.

$$s_1 = \Big\{(u,\rho) : (u-u_l) = \frac{g(\rho_l)-g(\rho)}{(\rho+\rho_l)}\Big[\epsilon - \sqrt{\epsilon^2 + \frac{2\epsilon(\rho^2-\rho_l^2)(f(\rho)-f(\rho_l))}{(g(\rho)-g(\rho_l))^2}}\Big]\Big\},$$

$$s_2 = \Big\{(u,\rho) : (u-u_l) = \frac{g(\rho_l)-g(\rho)}{(\rho+\rho_l)}\Big[\epsilon + \sqrt{\epsilon^2 + \frac{2\epsilon(\rho^2-\rho_l^2)(f(\rho)-f(\rho_l))}{(g(\rho)-g(\rho_l))^2}}\Big]\Big\}$$

The next difficulty is to choose the admissible shock curves satisfying Lax entropy inequality and show the existence of the intermediate state as in theorem (3.1). Then one needs to determine the proper growth condition on $g$ to find the distributional limit of solutions of the scaled system.

# Chapter 4

# Vanishing viscosity limit for a system of balance laws with general type initial data arising from 1D Saint-Venant model

## 4.1 Introduction

This chapter considers the following non-strictly hyperbolic system of balance laws

$$\begin{cases} u_t + uu_x = \Gamma(x,t), \\ \rho_t + (\rho u)_x = 0 \end{cases} \tag{4.1.1}$$

with initial data

$$u(x,0) = u_0(x), \;\; \rho(x,0) = \rho_0(x). \tag{4.1.2}$$

Here we assume the initial datum are measurable functions. We study (4.1.1)-(4.1.2) for the class of functions $\Gamma(x,t) = \Psi(t)x$, where $\Psi(t)$ belongs to the set

$$\mathcal{H} = \left\{ \frac{h''(t)}{h(t)} | h : [0,\infty) \to \mathbb{R} \quad \text{and} \quad h(t) \neq 0 \;\forall\; t \in [0,\infty), \;\; h(t) \in C^2[0,\infty) \right\}.$$

The above system (4.1.1) physically motivated by the following system known as 1-D Saint-Venant equation, namely

$$\begin{cases} A_t + (Au)_x = 0, \\ u_t + uu_x + g\xi_x = -\dfrac{P\tau}{Ad}. \end{cases} \tag{4.1.3}$$

The above is a model for incompressible fluid flow in an open channel of an arbitrary cross-section [53], where $A(x,t)$ denotes the cross sectional area, $u(x,t)$ is the velocity of the flow, $\xi(x,t)$ is the free surface elevation and $\tau(x,t)$ is stress along the perimeter $P(x,t)$ of the cross sectional area at $x$. Furthermore, $d$ denotes the constant density of the fluid and $g$ is the gravitational acceleration which may depend on the location $x$ (see Fig.1). Note that,

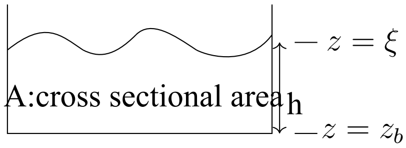


Figure 4.1: Cross section of an open channel

the first equation of the system (4.1.3) is a continuity equation and a substitution $A = \rho$ gives the second equation of the system (4.1.1). This system has extensive usage in various fields such as computer modelling [54], flood forecasting [55], dam breaking analysis and so on. If $S = -\frac{dz_b}{dx}$, $S_f = \frac{\tau}{\rho g R}$ and $R = \frac{A}{P}$, the second equation of the system(4.1.3) can be written as (see [56])

$$u_t + uu_x + gh_x + g(S_f - S) = 0,$$

where $h$ is the depth from the free surface. Now for the kinematic waves, it is assumed that $S_f$ (friction slope) is approximately equal to $S$ (slope of the channel) which simplifies the above equation as

$$u_t + uu_x + gh_x = 0.$$

Note that, if we consider $-gh_x$ as a function of $(x, t)$, say $\Gamma(x, t)$ we get the first equation of (4.1.1).

Moreover for the diffusive waves $gh_x + g(S_f - S) = 0$ and the system (4.1.3) turns out to be the well-known one dimensional model for the large scale structure formation of the universe[26]. In our case, this can be obtained by taking $h''(t) = 0$. For this case, there are literature which deals with the existence and uniqueness of solutions. For this, we cite [29, 57, 27, 11, 9, 28, 36, 12] and the references therein. One can also use the vanishing pressure limit approach[46, 58] to construct the solution.

In this chapter, we use the vanishing viscosity method to study the system(4.1.1)-(4.1.2) for a general type of initial data. The viscous form of (4.1.1) can be written as follows.

$$\begin{cases} u_t + (\frac{u^2}{2})_x = \Gamma(x,t) + \frac{\epsilon}{2} u_{xx} \\ \rho_t + (\rho u)_x = \frac{\epsilon}{2} u_{xx}. \end{cases} \tag{4.1.4}$$

The first equation of the above system(4.1.4) is a viscous Burgers equation with a source term $\Gamma(x,t)$. For $\Gamma(x,t) = 0$, the vanishing viscosity limit and large time behavior for the first equation are studied by E. Hopf[59].

Hopf [59], using a transformation (widely known as Hopf-Cole transformation) linearized the Burgers equation to heat equation and in this way obtained an explicit solution. Ding *et al.* [60] studied equation of the form

$$u_t + uu_x = \mu u_{xx} - kx, \;\; x \in \mathbb{R}, \;\; t > 0, \tag{4.1.5}$$

where $\mu$ and $k$ are positive constant. A Hopf-Cole transformation transformed the above equation into the following linear equation.

$$\varphi_t - \mu\varphi_{xx} = -x^2\varphi, \;\; x \in \mathbb{R}, \;\; t > 0.$$

Then using Hermite polynomials, they got an explicit formula for (4.1.5). In the same paper, they also analyzed large-time behavior for (4.1.5). In a subsequent paper, following the arguments of Hopf [59], Ding *et al.*[61] studied the limiting behavior of the solution as $\mu$ approaches to zero. They also showed that the distributional limit of the solution $u^\mu$ satisfies the inviscid forced Burgers equation $u_t + uu_x = kx$ when the initial data is locally bounded measurable and the growth is of order $o(x)$. In the same paper, they also observed the $\delta$-wave phenomenon for Riemann type initial data by coupling (4.1.5) with the equation

$$\rho_t + (u\rho)_x = 0, \;\; x \in \mathbb{R}, \;\; t > 0. \tag{4.1.6}$$

The equation(4.1.6) can be viewed as a conservation law with discontinuous flux when the initial data for $u$ is of Riemann type. In the case of overcompressiveness, it is interesting to observe $\delta$-waves for linear discontinuous fluxes, see[62]. For a general discontinuous flux, this is a difficult question. Extensive works on this discontinuous flux are being done, see [63] and references therein.

It is worth mentioning that the non homogeneous term $\Gamma(x,t)$ in the system (4.1.1) is not in $L^p(\mathbb{R}\times[0,\infty))$ for any $p\in[1,\infty]$. Also, it is not very common in the literature to deal with this kind of function depending upon both space and time-variable $(x,t)$ that is unbounded in any strip $0<t<T$. Oleinik[64] studied scalar conservation law in the Sobolev setting. However, due to the unboundedness of our non-homogeneous term, the first equation of (4.1.1) does not lie in Oleinki's framework.

An explicit formula for (4.1.4) is obtained by linearising the system. A use of a generalized Hopf-Cloe transformation to the first equation of the system (4.1.4) leads to a linear equation of the form

$$\phi_t - a\phi_{xx} = f(x,t)\phi, \quad x\in\mathbb{R}, \;\; t>0, \;\; a>0.$$

Apparently, one cannot expect to obtain an explicit solution for the above type of linear equation. Followed by another transformation, we are able to change it into a heat equation. In this way, we got an explicit formula for the component $u$ as well as the component $\rho$ and our first result is the following.

**Theorem 4.1.1.** *Let $u_0$ be a locally integrable function satisfying the growth condition $\int_0^x u_0(\xi)d\xi = o(x^2)$ and $\rho_0$ be a locally integrable function with $R_0(y) = \int_0^y \rho_0(\xi)d\xi = O(|y|^\beta)$, for any $\beta\in\mathbb{N}$. Then the explicit formulas*

$$u(x,t,\epsilon) = n(t)x + l(t)\frac{\int_{-\infty}^{\infty}\frac{l(t)x-y}{m(t)}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]}dy}{\int_{-\infty}^{\infty}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]}dy}$$

*and*

$$R(x,t,\epsilon) = \frac{\int_{-\infty}^{\infty}R_0(y)e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]}dy}{\int_{-\infty}^{\infty}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]}dy},$$

*are regular in the region* $t > 0$ *and satisfies the equation*(4.1.4) *and the initial condition* (4.1.2) *holds in the following sense.*

$$\begin{aligned} \int_0^x u(y,t,\epsilon)dy &\to \int_0^x u_0(y)dy \ \ as \ \ t \to 0 \\ \int_0^x \rho(y,t,\epsilon)dy &\to \int_0^x \rho_0(y)dy \ \ as \ \ t \to 0, \end{aligned} \tag{4.1.7}$$

*where* $l(t)$*,* $m(t)$ *and* $n(t)$ *are given by the equation* (4.2.11).

We also study the large time behavior for the solutions of (4.1.4) when the initial data $u_0$ and $\rho_0$ are lies in $L^1(\mathbb{R})$. In this regard, we have the following result.

**Theorem 4.1.2.** *Suppose the initial data* $(u_0,\ \rho_0) \in L^1(\mathbb{R})$ *and* $n(t)$ *defined by the equation*(4.2.11) *is integrable on* $[0,\infty]$*, then as* $t \to \infty$

$$\begin{aligned} &(i) \quad \sqrt{\frac{m(t)}{2}}\Big(u(x,t,\epsilon) - n(t)x\Big) \to l(\infty)\sqrt{\frac{\epsilon}{\pi}}\left[\frac{\varphi_0(-\infty) - \varphi_0(\infty)}{\varphi_0(-\infty) + \varphi_0(\infty)}\right] \\ &(ii) \quad R(x,t,\epsilon) \to \left[\frac{R_0(-\infty)\varphi_0(-\infty) + R_0(\infty)\varphi_0(\infty)}{\varphi_0(-\infty) + \varphi_0(\infty)}\right] \end{aligned} \tag{4.1.8}$$

*uniformly in compact sets, where* $\varphi_0(x) = e^{-\frac{1}{\epsilon}\int_0^x u_0(\xi)d\xi}$ *and* $R_0(y) = \int_0^y \rho_0(\xi)d\xi$*. Also for* $x \in \mathbb{R}$ *and* $t > 0$

$$\begin{aligned} \Big|u(x,t,\epsilon) - n(t)x\Big| &\leq \frac{C_1}{\sqrt{m(t)}}, \\ \Big|R(x,t,\epsilon)\Big| &\leq C_2 \end{aligned} \tag{4.1.9}$$

*for some constants* $C_1$ *and* $C_2$ *which depend only on the initial data and* $\epsilon$*.*

Our next aim is to find the the vanishing viscosity limit for $(u(x,t,\epsilon), \rho(x,t,\epsilon))$ to obtain the solution $(u(x,t), \rho(x,t))$ for (4.1.1). This is done through two steps. In the first step we find the vanishing viscosity limit for the solutions of the system(4.1.4) by imposing the same conditions on the initial data $u_0$ and $\rho_0$ that $\int_0^x u_0(z)dz = o(x^2)$, $\int_0^x \rho_0(z)dz = O(|x|^\beta)$, for any $\beta \in \mathbb{N}$. In this context we state the following theorem:

**Theorem 4.1.3.** *Assume $u_0$ and $\rho_0$ are locally integrable functions. Furthermore, assume $\int_0^x u_0(\xi)d\xi = o(x^2)$, $R_0(x) = \int_0^x \rho_0(\xi)d\xi = O(|x|^\beta)$, for any $\beta \in \mathbb{N}$ and Suppose $(u(x,t,\epsilon), \rho(x,t,\epsilon))$ be a solution of (4.1.4) subject to (4.1.2). Then we have,*

$$\begin{aligned} \lim_{\epsilon\to 0} u(x,t,\epsilon) &= n(t)x + \frac{l(t)}{m(t)}[l(t)x - y(x,t)] \\ R(x,t) &= \lim_{\epsilon\to 0} R(x,t,\epsilon) = R_0(y(x,t)). \end{aligned} \tag{4.1.10}$$

*for a.e. (x, t) in $\mathbb{R}\times(0,\infty)$. This in turn gives $\lim_{\epsilon\to 0}\rho(x,t,\epsilon) = \frac{\partial}{\partial x}(R(x,t))$. Here the partial derivative $\frac{\partial}{\partial x}$ is understood in the sense of distribution, where $n(t)$, $l(t)$, $m(t)$ are defined by*(4.2.11).

In the second step we show that the limits $u(x,t)$ and $R(x,t)$ satisfies the system (4.1.1). A localization technique is used to prove that the limit $u(x,t)$ for the first equation of (4.1.1) satisfies the weak formulation. To prove that $R(x,t)$ satisfies the equation, we employ the Volpert product[34]. First, this is done when the function $u(x,t)$ has a simple geometric structure and then extended for any $u(x,t)$ by writing the flux function in the second equation in a certain way. Now we mention some remarks in the following.

**Remark 4.1.4.** *The system* (4.1.1) *is more general than the system* (4.1.5) *that is considered in [60] and the non-homogeneous term depends both on time and space. For instance, the equation considered by Ding et al.[60, 61] can be obtained from the first equation of* (4.1.1) *if we take $h(t) = e^{-2t}$.*

**Remark 4.1.5.** *In [61] only the Riemann problem is solved for the de-coupled system, whereas we found an explicit formula for the de-coupled system for any general type of initial data.*

**Remark 4.1.6.** *We used a technique that is based on the idea of localization of the initial data. We prove Lemma 4.4.3 which plays an important role in the proof of the Theorem 4.4.4.*

*Our analysis differs from the usual way followed by [60, 61]. This approach may be used for a more general type of system as well.*

Moreover, we consider the following non-homogeneous system where the non-homogeneous term only depends on the variable $t$, namely,

$$\begin{cases} u_t + uu_x = h(t), \\ \rho_t + (\rho u)_x = 0 \end{cases} \tag{4.1.11}$$

with initial data

$$u(x,0) = u_0(x), \;\; \rho(x,0) = \rho_0(x). \tag{4.1.12}$$

The first equation of this system (4.1.11) is a generalized form of the equation studied in [65],namely,

$$u_t + uu_x = \frac{k}{\sqrt{(2\beta t + 1)}}.$$

We give an explicit solution for (4.1.11)-(4.1.12) by following a method introduced by Lax[47].

The plan of this chapter is as follows. In section $2$, we prove the Theorem 4.1.1using a generalized Hopf-Cole transformation. In section $3$, we give proof of the Theorem4.1.2. In section $4$, we prove Theorem 4.1.3and the vanishing viscosity limit satisfies the equation in a distributional sense is also shown. Finally, in section $5$ we consider the system (4.1.11)-(4.1.12) and obtain a solution characterized by a variational formula.

## 4.2 Explicit formula using Hopf-Cole transformation

In this section, we prove the Theorem 4.1.1. We use generalized Hopf-Cole transformation and obtain an explicit formula for the solution of the equation (4.1.4)-(4.1.2). Let us assume the following two conditions on $h(t)$:

- (H1) $h(t) \neq 0, 0 \leq t < \infty$ is a $C^2$- function.

- (H2) If $h'(0) \neq 0$, then for $t \geq 0$, $h(t)$ satisfies

$$\int_0^t \frac{1}{h^2(s)} ds - \frac{1}{h'(0)h(0)} \neq 0.$$

*Proof of Theorem 4.1.1.* Consider the equation (4.1.4) with initial data (4.1.2). Set $u = U_x$ and $\rho = R_x$, then $(U, R)$ satisfy

$$\begin{aligned} U_t + \frac{U_x^2}{2} &= \frac{\epsilon}{2} U_{xx} + \frac{h''(t)x^2}{2h(t)} \\ R_t + U_x R_x &= \frac{\epsilon}{2} R_{xx}, \end{aligned} \tag{4.2.1}$$

with initial conditions

$$U(x,0) = U_0(x) = \int_0^x u_0(y)dy, \ \ R_0(x) = \int_0^x \rho(y)dy. \tag{4.2.2}$$

Using the generalized Hopf-Cole transformation

$$\phi(x,t) = e^{\frac{-U}{\epsilon}}, \ \ \psi(x,t) = -\frac{R}{\epsilon} e^{\frac{-U}{\epsilon}}, \tag{4.2.3}$$

we obtain

$$U_t = -\epsilon \frac{\phi_t}{\phi}, \ \ U_x = -\epsilon \frac{\phi_x}{\phi}, \ \ U_{xx} = -\epsilon \frac{\phi_{xx}}{\phi} + \epsilon (\frac{\phi_x}{\phi})^2. \tag{4.2.4}$$

From equations (4.2.1) -(4.2.4), $\phi$ satisfies

$$\begin{aligned} \phi_t - \frac{\epsilon}{2}\phi_{xx} &= -\frac{h''(t)x^2}{2h(t)\epsilon}\phi \\ \phi(x,0) &= e^{-\frac{U_0(x)}{\epsilon}}. \end{aligned} \tag{4.2.5}$$

Using the transformation

$$\phi(x,t) = k(t) e^{-\frac{n(t)x^2}{2\epsilon}} \tilde{\phi}(l(t)x, m(t)), \tag{4.2.6}$$

we get

$$
\begin{aligned}
\phi_t(x,t) &= k(t)e^{-\frac{n(t)x^2}{2\epsilon}}\left[m'(t)\tilde{\phi}_t(l(t)x,m(t)) + l'(t)x\tilde{\phi}_x(l(t)x,m(t))\right] \\
&\quad + k'(t)e^{-\frac{n(t)x^2}{2\epsilon}}\tilde{\phi}(l(t)x,m(t)) - k(t)e^{-\frac{n(t)x^2}{2\epsilon}}\tilde{\phi}(l(t)x,m(t))\frac{n'(t)x^2}{2\epsilon} \\
\phi_{xx}(x,t) &= k(t)l^2(t))e^{-\frac{n(t)x^2}{2\epsilon}}\tilde{\phi}_{xx}(l(t)x,m(t)) - \frac{2k(t)l(t)n(t)x}{\epsilon}e^{-\frac{n(t)x^2}{2\epsilon}}\tilde{\phi}_x(l(t)x,m(t)) \\
&\quad - \frac{k(t)n(t)}{\epsilon}e^{-\frac{n(t)x^2}{2\epsilon}}\tilde{\phi}(l(t)x,m(t)) + \frac{k(t)n^2(t)x^2}{\epsilon^2}e^{-\frac{n(t)x^2}{2\epsilon}}\tilde{\phi}(l(t)x,m(t)).
\end{aligned}
\tag{4.2.7}
$$

Equations (4.2.5) and (4.2.7) yields,

$$
\begin{aligned}
&k(t)e^{-\frac{n(t)x^2}{2\epsilon}}\left[m'(t)\tilde{\phi}_t(l(t)x,m(t)) - l^2(t)\frac{\epsilon}{2}\tilde{\phi}_{xx}(l(t)x,m(t))\right] \\
&+ k(t)xe^{-\frac{n(t)x^2}{2\epsilon}}\left[l'(t)+n(t)l(t)\right]\tilde{\phi}_x + e^{-\frac{n(t)x^2}{2\epsilon}}\left[k'(t)+\frac{1}{2}k(t)n(t)\right]\tilde{\phi} \\
&- \frac{1}{2\epsilon}e^{-\frac{n(t)x^2}{2\epsilon}}k(t)x^2\left[n'(t)+n^2(t)-\frac{h''(t)}{h(t)}\right]\tilde{\phi} = 0.
\end{aligned}
\tag{4.2.8}
$$

In order to get a simplified equation of $\tilde{\phi}$, we take the following compatible conditions,

$$
\begin{cases}
k'(t) + \frac{1}{2}k(t)n(t) & = 0, \\
n'(t) + n^2(t) - \frac{h''(t)}{h(t)} & = 0, \\
k(t)\left\{l'(t) + n(t)l(t)\right\} & = 0, \\
m'(t) = l^2(t).
\end{cases}
\tag{4.2.9}
$$

We impose the following initial assumptions on $k(t), l(t), m(t)$ and $n(t)$,

$$
k(0) = 1, \;\; l(0) = 1, \;\; m(0) = 0, \;\; n(0) = 0. \tag{4.2.10}
$$

Solution for the equation(4.2.9) with initial conditions (4.2.10) are as follows.

$$
\begin{aligned}
n(t) &= \begin{cases} \frac{h'(t)}{h(t)}, & \text{if } h'(0) = 0 \\ \frac{h'(t)}{h(t)} + \dfrac{1}{h^2(t)\Big[\int_0^t \frac{1}{h^2(s)} ds - \frac{1}{h'(0)h(0)}\Big]}, & \text{if } h'(0) \neq 0 \end{cases} \\
l(t) &= \exp\left( - \int_0^t n(s) ds \right) \\
m(t) &= \int_0^t \left( \exp\Big( - \int_0^s n(r) dr \Big) \right)^2 ds.
\end{aligned}
\tag{4.2.11}
$$

After taking such combinations we see that (4.2.8) satisfies the usual heat equation:

$$
\tilde{\phi}_t - \frac{\epsilon}{2} \tilde{\phi}_{xx} = 0.
$$

Therefore,

$$
\tilde{\phi}(l(t)x, m(t)) = \int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy.
$$

From the given transformation (4.2.6) we get,

$$
\phi(x,t) = k(t) e^{-\frac{n(t)x^2}{2\epsilon}} \int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy.
$$

Now from (4.2.4), $U_x = u(x,t) = -\epsilon \frac{\phi_x}{\phi}$, then

$$
u(x,t,\epsilon) = n(t)x + l(t) \frac{\int_{-\infty}^{\infty} \frac{l(t)x-y}{m(t)} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy}. \tag{4.2.12}
$$

Now a direct calculation shows that $\psi$ satisfies the equation,

$$
\psi_t - \frac{\epsilon}{2}\psi_{xx} = -\frac{e^{-\frac{U}{\epsilon}}}{\epsilon}\Big[R_t + U_x R_x - \frac{\epsilon}{2} R_{xx}\Big] + \frac{R}{\epsilon} e^{-\frac{U}{\epsilon}} \Big[U_t + \frac{U_x^2}{2} - \frac{\epsilon}{2} U_{xx}\Big] = -\frac{h''(t)x^2}{2h(t)\epsilon}\psi,
$$

with the initial condition $\psi(x,0) = -\frac{R_0(x)}{\epsilon} e^{-\frac{U_0(x)}{\epsilon}}$. As before, the solution for $\psi$ is the following:

$$
\psi(x,t) = k(t) e^{-\frac{n(t)x^2}{2\epsilon}} \int_{-\infty}^{\infty} -\frac{R_0(y)}{\epsilon} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy.
$$

From the Hopf-Cole transformation(4.2.3), we get $R(x,t) = -\epsilon\frac{\psi}{\phi}$. Then

$$R(x,t,\epsilon) = \frac{\int_{-\infty}^{\infty} R_0(y) e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy}, \tag{4.2.13}$$

where $R_0(y) = \int_0^y \rho_0(z)dz$ and $\frac{\partial}{\partial x}(R(x,t,\epsilon)) = \rho(x,t,\epsilon)$.

The proof of the equation (4.1.7) is a mere repetition of the arguments of Hopf[59] and therefore omitted. Hence the equations (4.2.12) and (4.2.13) together with the arguments of Hopf[59] complete the proof of the theorem. □

**Remark 4.2.1.** *In general the linear equation* $\phi_t - \frac{\epsilon}{2}\phi_{xx} = \frac{f(x,t)}{\epsilon}\phi$ *cannot be solved explicitly. In our case* $f(x,t) = -\frac{h''(t)x^2}{2h(t)}$*, and is solved using a clever choice of the transformation* (4.2.6)*. Also we would like to mention that in this way we avoided the use of Hermite polynomials and got the explicit formula for more general class of non-homogeneous term in comparison to the article[60, 61]. Also note that if the condition* $H_2$ *is violated, i.e.,* $\int_0^{t_1} \frac{1}{h^2(s)}ds - \frac{1}{h'(0)h(0)} = 0$*, for some* $t = t_1 > 0$*, then the solution* $u$ *only valid in* $\{x \in \mathbb{R},\ t < t_1\}$.

## 4.3 Large time behavior

In this section, we discuss the large time behavior of the solution $u(x,t,\epsilon)$ and $R(x,t,\epsilon)$. To be more precise we prove the Theorem 4.1.2. Here we need an additional condition on $n(t)$ defined in (4.2.11) that $n(t)$ is integrable on $[0,\infty)$.

*Proof of Theorem 4.1.2.* Recall the explicit formula (4.2.12) for the first equation, derived in the previous section. Observe that it is enough to investigate the long time behavior for $u(x,t,\epsilon)$ and similar arguments hold for $R(x,t,\epsilon)$. From the explicit formula (4.2.12):

$$u(x,t,\epsilon) = n(t)x + l(t)\frac{\int_{-\infty}^{\infty} \frac{l(t)x-y}{m(t)} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\right]} dy}. \tag{4.3.1}$$

where $n(t)$, $l(t)$ and $m(t)$ are defined in (4.2.11).

First, we study the asymptotic behavior of the following term as $t \to \infty$,

$$\frac{\int_{-\infty}^{\infty} \frac{l(t)x-y}{m(t)} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]} dy}. \tag{4.3.2}$$

Using change of variable $\frac{l(t)x-y}{\sqrt{2m(t)}} = u$, we have

$$\begin{aligned}
&\frac{\int_{-\infty}^{\infty} \frac{l(t)x-y}{m(t)} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y u_0(z)dz\right]} dy} \\
&= \sqrt{\frac{2}{m(t)}} \frac{\int_{-\infty}^{\infty} u e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u} u_0(z)dz\right]} du}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u} u_0(z)dz\right]} du} \\
&= \sqrt{\frac{2}{m(t)}} \frac{\int_{-\infty}^{0} u e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u} u_0(z)dz\right]} du + \int_{0}^{\infty} u e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u} u_0(z)dz\right]} du}{\int_{-\infty}^{0} e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u} u_0(z)dz\right]} du + \int_{0}^{\infty} e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u} u_0(z)dz\right]} du} \\
&= \sqrt{\frac{2}{m(t)}} \frac{I_1+I_2}{I_3+I_4}.
\end{aligned} \tag{4.3.3}$$

Hence

$$\sqrt{\frac{m(t)}{2}}(u(x,t)-n(t)x) = l(t)\frac{I_1+I_2}{I_3+I_4}.$$

Since $u_0$ is integrable, using dominated convergence theorem in $I_i, i = 1,2,3,4$; yields

$$\lim_{t\to\infty} \sqrt{\frac{m(t)}{2}}(u(x,t)-n(t)x) = l(\infty)\sqrt{\frac{\epsilon}{\pi}}\left[\frac{\varphi_0(-\infty)-\varphi_0(\infty)}{\varphi_0(-\infty)+\varphi_0(\infty)}\right],$$

uniformly in compact sets. This proves first asymptotic behavior of (4.1.8). Second asymptotic behavior of (4.1.8) is a mere repetition of the above analysis.

To prove (4.1.9), note that from the previous calculation

$$\begin{aligned}\Big|u(x,t)-n(t)x\Big| &\leq \sqrt{\frac{2}{m(t)}}l(t)\frac{\int_{-\infty}^{\infty}|u|e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u}u_0(z)dz\right]}du}{\int_{-\infty}^{\infty}e^{-\frac{1}{\epsilon}\left[u^2+\int_0^{l(t)x-\sqrt{2m(t)}u}u_0(z)dz\right]}du}\\ &\leq \frac{C_1}{\sqrt{m(t)}},\end{aligned}$$

for some constant $C_1$, which may depend on the initial data and $\epsilon$. A very similar calculation for $R$ can lead to the estimate $|R(x,t)| \leq C_2$, for some constant $C_2$. □

**Remark 4.3.1.** *An immediate remark is that*

$$\begin{aligned}\sup_{x\in K}|u(x,t,\epsilon)-n(t)x| &\leq \frac{C(K,\epsilon)}{\sqrt{m(t)}}\left|\frac{\varphi_0(-\infty)-\varphi_0(\infty)}{\varphi_0(-\infty)+\varphi_0(\infty)}\right|\\ &\leq \frac{C(K,\epsilon)}{\sqrt{m(t)}},\end{aligned}$$

*as $t$ tends to infinity, where $C(K,\epsilon)$ is a constant which depends on the compact set $K$ and $\epsilon$, and is independent of the initial data.*

## 4.4 Vanishing viscosity behavior

This section aims to investigate the vanishing viscosity behavior of the system of equations. We start with two standard lemmas. Lemma 4.4.1 and Lemma 4.4.2 are used to proving the Theorem 4.1.3 in which vanishing viscosity limit is determined. In Theorem 4.4.4, we show that the vanishing viscosity limit $u(x,t)$ satisfies the equation. Theorem 4.4.5 contains the proof of the fact that the vanishing viscosity limit for the second component $R(x,t)$ satisfies the equation. This section finishes with a Riemann solution. Let us denote

$$G(x,y,t) = \frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz.$$

**Lemma 4.4.1.** *Suppose $G(x,y,t)$ attains minimum at $y_1$ for fixed $(x,t)$. Then $G(x',y_1,t) < G(x',y,t)$ holds for $y < y_1$ and $x < x'$.*

*Proof.* Since $G(x,y,t)$ attains minimum at $y_1$. Therefore

$$\frac{(l(t)x-y_1)^2}{2m(t)}+\int_0^{y_1} u_0(z)dz \;\leq \frac{(l(t)x-y)^2}{2m(t)}+\int_0^{y} u_0(z)dz. \tag{4.4.1}$$

Now observe that for $y<y_1$ and $x<x'$,

$$\frac{(l(t)x-y)^2}{2m(t)}+\frac{(l(t)x'-y_1)^2}{2m(t)}<\frac{(l(t)x-y_1)^2}{2m(t)}+\frac{(l(t)x'-y)^2}{2m(t)}. \tag{4.4.2}$$

Adding equation (4.4.1) and equation (4.4.2), we get

$$\begin{aligned}&\frac{(l(t)x-y_1)^2}{2m(t)}+\frac{(l(t)x-y)^2}{2m(t)}+\frac{(l(t)x'-y_1)^2}{2m(t)}+\int_0^{y_1} u_0(z)dz\\ &<\frac{(l(t)x-y)^2}{2m(t)}+\frac{(l(t)x-y_1)^2}{2m(t)}+\frac{(l(t)x'-y)^2}{2m(t)}+\int_0^{y} u_0(z)dz.\end{aligned} \tag{4.4.3}$$

Equation (4.4.3) leads after cancellations,

$$\frac{(l(t)x'-y_1)^2}{2m(t)}+\int_0^{y_1} u_0(z)dz)\leq\frac{(l(t)x'-y)^2}{2m(t)}+\int_0^{y} u_0(z)dz).$$

This implies, $G(x',y_1,t)<G(x',y,t)$.

□

For fixed $(x,t)$, define

$$y_-(x,t)=\inf\{y(x,t)\in\mathbb{R}|G(x,y,t) \text{ attains minimum at } y(x,t)\}$$

$$y_+(x,t)=\sup\{y(x,t)\in\mathbb{R}|G(x,y,t) \text{ attains minimum at } y(x,t)\}.$$

The following lemma is a straightforward application of Lemma (4.4.1).

**Lemma 4.4.2.** *For fixed t, $y_-(x,t)$ and $y_+(x,t)$ are monotonic increasing function in the variable $x$. Hence there is a unique minimizer of $G(x,y,t)$ for a.e. $(x,t)\in\mathbb{R}\times(0,\infty)$.*

### 4.4.1 Proof of Theorem 4.1.3

*Proof of Theorem 4.1.3.* For $t>0$, let $y(x,t)$ is a unique minimizer for $G(x,y,t)$, then it is enough to determine the limit of the expression

$$\frac{\int_{-\infty}^{\infty}\left(\frac{l(t)x-y}{m(t)}\right)\exp\left\{-\frac{1}{\epsilon}G(x,y,t)\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}G(x,y,t)\right\}dy}.$$

Now for any $\delta > 0$,

$$
\begin{aligned}
&\left|\frac{\int_{-\infty}^{\infty}\left(\frac{l(t)x-y}{m(t)}\right)\exp\left\{-\frac{1}{\epsilon}G(x,y,t)\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}G(x,y,t)\right\}dy} - \frac{l(t)x-y(x,t)}{m(t)}\right| \\
&= \left|\frac{\int_{-\infty}^{\infty}\left(\frac{y(x,t)-y}{m(t)}\right)\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}\right| \quad \leq I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{\int_{|y-y(x,t)|<\delta}\left|\frac{y(x,t)-y}{m(t)}\right|\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy} \\
I_2 &= \frac{\int_{|y-y(x,t)|>\delta}\left|\frac{y(x,t)-y}{m(t)}\right|\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}.
\end{aligned}
$$

It can be easily seen that $|I_1| \leq \frac{\delta}{m(t)}$. Now we have

$$
\lim_{|y|\to\infty}\frac{G(x,y,t)-G(x,y(x,t),t)}{(y-y(x,t))^2} = \frac{1}{2m(t)}.
$$

Since $y(x,t)$ is the unique minimizer, so $G(x,y,t)-G(x,y(x,t),t) > 0$. Therefore there exist an $a > 0$ such that

$$
\frac{G(x,y,t)-G(x,y(x,t),t)}{(y-y(x,t))^2} > a,
$$

for $|y-y(x,t)| > \delta$. Now consider,

$$
\begin{aligned}
I_2 &= \frac{\int_{|y-y(x,t)|>\delta}\left|\frac{y(x,t)-y}{m(t)}\right|\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy} \\
&= \frac{\int_{-\infty}^{y(x,t)-\delta}\left|\frac{y(x,t)-y}{m(t)}\right|\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy} \\
&+ \frac{\int_{y(x,t)+\delta}^{\infty}\left|\frac{y(x,t)-y}{m(t)}\right|\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\}dy}
\end{aligned}
$$

We estimate the first part of the above equality and the estimation of second part will follow similarly. For a small $\eta > 0$, since $y(x,t) < y < y(x,t)+\eta$, by continuity of $G(x,y,t)$ we have $0 < G(x,y,t) - G(x,y(x,t),t) < a\delta^2$. Now consider the first part of $I_2$,

$$\begin{aligned}
&\frac{\int_{-\infty}^{y(x,t)-\delta} \left|\frac{y(x,t)-y}{m(t)}\right| \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy} \\
&\leq \frac{\int_{-\infty}^{y(x,t)-\delta} \left|\frac{y(x,t)-y}{m(t)}\right| \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy}{\int_{y(x,t)}^{y(x,t)+\eta} \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy} \\
&\leq \frac{\int_{-\infty}^{y(x,t)-\delta} \frac{y(x,t)-y}{m(t)} \exp\left\{-\frac{1}{\epsilon}a(y-y(x,t))^2\right\} dy}{\eta e^{-\frac{a\delta^2}{\epsilon}}} \\
&= \frac{\int_{-\infty}^{-\delta} \frac{-z}{m(t)} \exp\left\{-\frac{1}{\epsilon}az^2\right\} dz}{\eta e^{-\frac{a\delta^2}{\epsilon}}} = \frac{\epsilon}{2am(t)\eta},
\end{aligned} \tag{4.4.4}$$

which tends to zero as $\epsilon$ tends to zero. Since $\delta$ is arbitrary, we have the first identity of (4.1.10).

Now we find the limit for the $R(x,t,\epsilon)$. Since $R_0$ is continuous, so for any given $\eta > 0$, there exist $\delta > 0$ such that

$|R_0(y) - R_0(y(x,t))| < \eta$ for $|y - y(x,t)| < \delta$.

$$\begin{aligned}
&\Big|R(x,t,\epsilon) - R_0(y(x,t))\Big| \\
&\leq \frac{\int_{-\infty}^{\infty} \Big|R_0(y) - R_0(y(x,t))\Big| \exp\left\{-\frac{1}{\epsilon}G(x,y,t)\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\epsilon}G(x,y,t)\right\} dy} \\
&= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{\int_{|y-y(x,t)|<\delta} \left|R_0(y) - R_0(y(x,t))\right| \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy} \\
I_2 &= \frac{\int_{|y-y(x,t)|>\delta} \left|R_0(y) - R_0(y(x,t))\right| \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\epsilon}[G(x,y,t)-G(x,y(x,t),t)]\right\} dy}.
\end{aligned}$$

It can be easily seen that $|I_1| \leq \eta$. Now from the previous analysis, there exist an $a > 0$, such that

$$\frac{G(x,y,t) - G(x,y(x,t),t)}{(y-y(x,t))^2} > a,$$

for $|y - y(x,t)| > \delta$. Since $R_0(y) = O(|y|^\beta)$, for any $\beta \in \mathbb{N}$, we can choose a large $C > 0$ such that the following holds,

$$
\begin{aligned}
I_2 &\leq C \frac{\int_{|y-y(x,t)|>\delta} |y(x,t) - y|^\beta \exp\left\{-\frac{1}{\epsilon}[G(x,y,t) - G(x,y(x,t),t)]\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\epsilon}[G(x,y,t) - G(x,y(x,t),t)]\right\} dy} \\
&= C \frac{\int_{-\infty}^{y(x,t)-\delta} |y(x,t) - y|^\beta \exp\left\{-\frac{1}{\epsilon}[G(x,y,t) - G(x,y(x,t),t)]\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\epsilon}[G(x,y,t) - G(x,y(x,t),t)]\right\} dy} \\
&+ C \frac{\int_{y(x,t)+\delta}^{\infty} |y(x,t) - y|^\beta \exp\left\{-\frac{1}{\epsilon}[G(x,y,t) - G(x,y(x,t),t)]\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\epsilon}[G(x,y,t) - G(x,y(x,t),t)]\right\} dy}.
\end{aligned}
\tag{4.4.5}
$$

A similar argument as (4.4.4), shows the expression (4.4.5) tends to zero as $\epsilon$ tends to zero. Therefore $\lim_{\epsilon \to 0} R(x,t,\epsilon) = R_0(y(t,x))$. □

Our next aim is to prove that the limit function $u$ satisfies a weak formulation (see equation(4.4.6) in Theorem (4.4.4)). For this purpose, we prove the following lemma regarding the bound on the minimizer $y(x,t)$.

**Lemma 4.4.3.** *If $x$ belongs to a bounded set $B$ and $0 < t \leq T$, then the minimizer $|y(x,t)| < M$, for some constant $M > 0$, which may depend on $B$ and $T$.*

*Proof.* For $x \in B$ and $0 < t \leq T$, choose large $M > 0$, such that for $|y| \geq M$, the following hold:

(i)

$$B \subset (-M, M).$$

(ii)

$$\left| \frac{\int_0^y u_0(z)dz}{(l(t)x - y)^2} 2m(t) \right| < \epsilon < 1,$$

for some $\epsilon > 0$.

(iii) For $0 < t \leq T$, $0 < m(t) \leq T$ and

$$\frac{(l(t)x - y)^2}{2T} \big[1 - \epsilon\big] \geq \sup_{x \in B} \Big( \int_0^{l(t)x} u_0(z)dz \Big).$$

Now for $|y| \geq M$ and $0 < t \leq T$,

$$\begin{aligned}
\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz &= \frac{(l(t)x-y)^2}{2m(t)}\Big[1 + \frac{\int_0^y u_0(z)dz}{(l(t)x-y)^2}2m(t)\Big] \\
&\geq \frac{(l(t)x-y)^2}{2m(t)}\big[1-\epsilon\big] \\
&\geq \frac{(l(t)x-y)^2}{2T}\big[1-\epsilon\big] \\
&\geq \sup_{x\in B}\Big(\int_0^{l(t)x} u_0(z)dz\Big).
\end{aligned}$$

Therefore, the minimum is achieved for $|y(x,t)| \leq M$, where $M$ depends only on $B$ and $T$. This proves the lemma. □

**Theorem 4.4.4.** *If $u_0(x)$ is locally bounded measurable function and $\int_0^x u_0(z)dz = o(x^2)$, then the limit function $u(x,t) = \lim_{\epsilon\to 0} u(x,t,\epsilon)$ satisfies the weak formulation in the following way.*

$$\int_0^\infty \int_{-\infty}^\infty [u(x,t)\varphi_t + \frac{u^2}{2}\varphi_x + \Gamma(x,t)\varphi]dxdt + \int_{-\infty}^\infty u(x,0)\varphi(x,0)dx = 0, \tag{4.4.6}$$

*for all test functions $\varphi$ compactly supported in $\mathbb{R}\times[0,\infty)$.*

*Proof.* We divide our proof into three steps.

**Step** 1. Let $B = (a,b)$, $M$ as in the previous theorem. Choose a cut off function $\zeta$ satisfying the conditions

$$\zeta \equiv 1 \text{ on } [-M,M], \; \zeta \equiv 0 \text{ on } \mathbb{R} - [-M,M].$$

Then for $x \in B$, $0 < t \leq T$, we have the following :

$$\min_{y\in\mathbb{R}}\Big[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y u_0(z)dz\Big] = \min_{y\in\mathbb{R}}\Big[\frac{(l(t)x-y)^2}{2m(t)} + \int_0^y \zeta(z)u_0(z)dz\Big]$$

and the minimum is achieved at the same point at $|y(x,t)| \leq M$. This follows easily from the previous lemma(4.4.3).

**Step** 2. Let us define

$$\tilde{u_0}(z) = \zeta(z)u_0(z).$$

Then clearly $\tilde{u_0}$ is a bounded function. Now consider the problem

$$\begin{cases} \tilde{u}_t + (\frac{\tilde{u}^2}{2})_x = \Gamma(x,t) + \frac{\epsilon}{2}\tilde{u}_{xx}, \\ \tilde{u}(x,0) = \tilde{u_0}(x). \end{cases} \tag{4.4.7}$$

One can derive an explicit formula for $\tilde{u}(x,t,\epsilon)$,

$$\tilde{u}(x,t,\epsilon) = n(t)x + l(t)\frac{\int_{-\infty}^{\infty}\frac{l(t)x-y}{m(t)}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y(\tilde{u_0}(z))dz\right]}dy}{\int_{-\infty}^{\infty}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y(\tilde{u_0}(z))dz\right]}dy}. \tag{4.4.8}$$

Now, observe that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{l(t)x-y}{m(t)}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y(\tilde{u_0}(z))dz\right]}dy \\
=&-\frac{\epsilon}{l(t)}\int_{-\infty}^{\infty}\frac{d}{dx}\left[e^{-\frac{1}{\epsilon}\frac{(l(t)x-y)^2}{2m(t)}}\right]e^{-\frac{1}{\epsilon}\int_0^y\tilde{u_0}(z)dz}dy \\
=&-\frac{\epsilon}{l(t)}\int_{-\infty}^{\infty}-l(t)\frac{d}{dy}\left[e^{-\frac{1}{\epsilon}\frac{(l(t)x-y)^2}{2m(t)}}\right]e^{-\frac{1}{\epsilon}\int_0^y\tilde{u_0}(z)dz}dy \\
=&\epsilon\int_{-\infty}^{\infty}\frac{d}{dy}\left[e^{-\frac{1}{\epsilon}\frac{(l(t)x-y)^2}{2m(t)}}\right]e^{-\frac{1}{\epsilon}\int_0^y\tilde{u_0}(z)dz}dy.
\end{aligned}$$

Using integration by parts, we have

$$\int_{-\infty}^{\infty}\frac{l(t)x-y}{m(t)}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y\tilde{u_0}(z)dz\right]}dy = \int_{-\infty}^{\infty}\tilde{u_0}(y)e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y \tilde{u_0}(z)dz\right]}dy.$$

From (4.4.8), we have

$$\begin{aligned}
\tilde{u}(x,t,\epsilon) &= n(t)x + l(t)\frac{\int_{-\infty}^{\infty}\frac{l(t)x-y}{m(t)}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y(\tilde{u_0}(z))dz\right]}dy}{\int_{-\infty}^{\infty}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y(\tilde{u_0}(z))dz\right]}dy} \\
&= n(t)x + l(t)\frac{\int_{-\infty}^{\infty}\tilde{u_0}(y)e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y(\tilde{u_0}(z))dz\right]}dy}{\int_{-\infty}^{\infty}e^{-\frac{1}{\epsilon}\left[\frac{(l(t)x-y)^2}{2m(t)}+\int_0^y(\tilde{u_0}(z))dz\right]}dy}.
\end{aligned}$$

Since $\tilde{u_0}(x)$ is bounded, $\tilde{u}(x,t,\epsilon)$ is locally bounded independent of $\epsilon$. Again by step $1$ one can observe that

$$\lim_{\epsilon\to 0}\tilde{u}(x,t,\epsilon) = \lim_{\epsilon\to 0}u(x,t,\epsilon) = u(x,t).$$

**Step** 3. Since $\tilde{u}(x,t,\epsilon)$ is smooth and satisfies (4.4.7), multiplying with test function $\phi$ supported in $B \times [0,T)$ and using integration by parts we get

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}[\tilde{u}(x,t,\epsilon)\phi_t + \frac{\tilde{u}(x,t,\epsilon)^2}{2}\phi_x + \Gamma(x,t)\phi]dxdt + \int_{-\infty}^{\infty}\tilde{u}(x,t_1,\epsilon)\phi(x,t_1)dx = -\frac{\epsilon}{2}\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\tilde{u}(x,t,\epsilon)\phi_{xx}dxdt.$$

This implies,

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}[\tilde{u}(x,t,\epsilon)\phi_t + \frac{\tilde{u}(x,t,\epsilon)^2}{2}\phi_x + \Gamma(x,t)\phi]dxdt + \int_{-\infty}^{\infty}\Big(\int_0^x \tilde{u}(y,t_1,\epsilon)dy\Big)_x\phi(x,t_1)dx = -\frac{\epsilon}{2}\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\tilde{u}(x,t,\epsilon)\phi_{xx}dxdt. \tag{4.4.9}$$

Applying integration by parts in (4.4.9), we have

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}[\tilde{u}(x,t,\epsilon)\phi_t + \frac{\tilde{u}(x,t,\epsilon)^2}{2}\phi_x + \Gamma(x,t)\phi]dxdt - \int_{-\infty}^{\infty}\int_0^x \tilde{u}(y,t_1,\epsilon)dy\phi_x(x,t_1)dx = -\frac{\epsilon}{2}\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\tilde{u}(x,t,\epsilon)\phi_{xx}dxdt. \tag{4.4.10}$$

As $\tilde{u}(x,t,\epsilon)$ is locally bounded and $\lim_{t\to 0}\int_0^x \tilde{u}(y,t,\epsilon)dy = \int_0^x \tilde{u_0}(y)dy$, applying limit $t_1 \to 0$ in (4.4.10), we get

$$\int_{0}^{\infty}\int_{-\infty}^{\infty}[\tilde{u}(x,t,\epsilon)\phi_t + \frac{\tilde{u}(x,t,\epsilon)^2}{2}\phi_x + \Gamma(x,t)\phi]dxdt - \int_{-\infty}^{\infty}\big(\int_0^x \tilde{u_0}(y)dy\big)\phi_x(x,0)dx = -\frac{\epsilon}{2}\int_{0}^{\infty}\int_{-\infty}^{\infty}\tilde{u}(x,t,\epsilon)\phi_{xx}dxdt. \tag{4.4.11}$$

Now, by step 2, $\lim_{\epsilon\to 0}\tilde{u}(x,t,\epsilon) = u(x,t)$ and $\tilde{u}(x,t,\epsilon)$ is locally bounded independent of $\epsilon$, passing to the limit $\epsilon \to 0$ in (4.4.11), we have the desired result. This completes the proof. □

Next we show that $R(x,t)$ satisfies the equation $R_t + uR_x = 0$ in the sense of Volpert. If $u$ is a function of bounded variation, we write

$$\mathbb{R} \times [0,\infty) = S_c \cup S_j \cup S_n,$$

where $S_c$ and $S_j$ are the set of points of approximate continuity and points of approximate jump discontinuity. $S_j$ can be expressed as a countable union of Lipschitz continuous curves. The set $S_n$ has one dimensional Hausdorff measure zero. At any point $(x,t) \in S_j$, $u(x-,t)$ and $u(x+,t)$ are respectively denote the left and right values of the jump discontinuity. For any continuous function $g : \mathbb{R} \to \mathbb{R}$, the averaged superposition $g(u)$(see Volpert [34]) is defined as

$$\overline{g(u)}(x,t) = \begin{cases} g(u(x,t)), & \text{if } (x,t) \in S_c \\ \int_0^1 g((1-\alpha)u(x-,t) + \alpha u(x+,t))d\alpha, & \text{if } (x,t) \in S_j. \end{cases} \tag{4.4.12}$$

For any $v \in BV(\mathbb{R} \times (0,\infty))$, $\overline{g(u)}$ is a Borel measurable function with respect to the measure $v_x$ and the product $g(u)v_x$ is defined as follows:

$$g(u)v_x(E) = \int_E \overline{g(u)} v_x \tag{4.4.13}$$

for any Borel measurable set $E \subset \mathbb{R} \times (0,\infty)$.

**Theorem 4.4.5.** *Let $u_0$ be a bounded measurable function, then the limit function $R(x,t) = \lim_{\epsilon \to 0} \int_0^x \rho(y,t,\epsilon)dy = R_0(y(t,x))$ satisfies the equation*

$$R_t + uR_x = 0$$

*in the sense of Volpert and* $\lim_{t\to 0} R(x,t) = R_0(x)$, *a.e.* $x \in \mathbb{R}$.

*Proof.* From the formula, it is clear that $u$ is a function of bounded variation and it satisfies

$$u_t + uu_x = \Gamma(x,t)$$

in the region $x \in \mathbb{R},\ t > 0$. Now we show that $R(x,t)$ satisfies the equation $R_t + uR_x = 0$ in the volpert sense. We divide our proof into four steps.

**Step** 1. Let us introduce the equation

$$\begin{aligned} (u_n)_t + u_n(u_n)_x &= \Gamma(x,t) \\ u_n(x,0) &= u_{0,n}(x). \end{aligned} \tag{4.4.14}$$

Take a partition $-R-LT = x_1 < x_2... < x_n = R+LT$, for some $T > 0$ and $L = ||u_0||_\infty$. Here the initial data $u_{0,n}(x)$ is a piecewise constant function

$$u_{0,n}(x) = \begin{cases} u_1, & \text{if} x < x_1 \\ u_i, & \text{if} x_i \le x \le x_{i+1} \\ u_n, & \text{if} x > x_n, \end{cases}$$

and $\int_{|x|\le R+LT} \Big| u_0(x) - u_{0,n}(x) \Big| dx \to 0$.

Then by $L^1$- contraction (see [2]), we have,

$$\int_0^T \int_{|x|\le R} \Big| u(x,t) - u_n(x,t) \Big| dxdt \to 0 \;\text{ as }\; n \to \infty.$$

So, $y_n(x,t) \to y(x,t)$ in $L^1([-R,R]\times[\delta,T])$ for some $\delta > 0$, as $n \to \infty$, where $y_n(x,t)$ is the minimizer corresponding to the initial data $u_{0,n}$.

**Step** 2. We claim that the set of all discontinuous points of $u_n(x,t)$ of (4.4.14) can be written as the union of finite numbers of Lipschitz continuous curves.

Observe that for $(x,t) \in [-R,R]\times[\delta,T]$ is a point of discontinuity, the minimizer is not unique. Let the left most minimizer is $(y_n)_-(x,t)$ and the right most minimizer is $(y_n)_+(x,t)$. We see that,

(i) If $(y_n)_-(x,t),\;(y_n)_+(x,t) \in [x_i, x_{i+1}]$, then $(y_n)_-(x,t) = x_i$ and $(y_n)_+(x,t) = x_{i+1}$. This is because $\int_0^x u_{0n}(z)dz$ is an affine function on $(x_i, x_{i+1})$.

(ii) If $(y_n)_-(x,t) \in (x_i, x_{i+1})$ and $(y_n)_+(x,t) \in (x_j, x_{j+1})$ $(i < j)$, then

$$\frac{l(t)x - (y_n)_-(x,t)}{m(t)} = u_{0n}((y_n)_-(x,t)) = u_i$$

and

$$\frac{l(t)x - (y_n)_+(x,t)}{m(t)} = u_{0n}((y_n)_+(x,t)) = u_j$$

Therefore from (i) and (ii), we have

$$|(y_n)_-(x,t) - (y_n)_+(x,t)| > \delta_1 |u_i - u_j|,$$

for $t > \delta$. Choose $\delta_2 = \min\{\delta_1|u_i - u_j|, i \neq j \;\; \forall i, j\}$. If $(x,t)$ is a point of discontinuity, then $((y_n)_+(x,t) - (y_n)_-(x,t)) > \delta_1$. Now $D(u_n)$ be the set of all discontinuous points of $u_n$. Let $t_1 = \inf\limits_{(x,t)\in D(u_n)}\{t\}$. Pick $x_1$ such that $(x_1,t_1) \in D(u_n)$. Consider the discontinuous curve $\gamma_1$ starting at $(x_1,t_1)$ which is possible because any discontinuous point is part of a discontinuous curve. Now let $t_2 = \inf\limits_{(x,t)\in D\setminus\gamma_1^*}\{t\}$ where $\gamma_1^*$ denotes the range of $\gamma_1$. Similarly one can choose $x_2$ such that $(x_2,t_2) \in D(u_n)$. Again consider the discontinuous curve $\gamma_2$ starting from $(x_2,t_2)$. Continuing this way we will find discontinuous curves $\gamma_1, \gamma_2, \gamma_3....$ with starting points $(x_1,t_1), (x_2,t_2), (x_3,t_3).....$ respectively and this process will eventually stop after a finite number of steps as the characteristic triangles constructed from the points $(x_i,t_i)$ will never intersect and the length of the base of the characteristic triangles is atleast $\delta_1$ when $t > \delta$. So this implies in the compact set $[-R,R]\times[\delta,T]$, $\gamma_i$'s are finite. This completes the proof of our claim.

**Step** 3. We show that $R_n = R_0(y_n(x,t))$ satisfies the equation $(R_n)_t + \bar{u_n}(R_n)_x = 0$ in the sense of Volpert. Since there are only finite number of discontinuity curves and in the complement solution is smooth. Let $\{x = \chi_j(t)\}_{j=1}^n$ be the discontinuous curves. Now we write

$$(R_n)_t = (R_n^c)_t + \big(R_n(\chi_j(t)-,t) - R_n(\chi_j(t)+,t)(-\dot{\chi_j}(t))\delta_{x=\chi_j(t)}$$

$$(R_n)_x = (R_n^c)_x + \big(R_n(\chi_j(t)-,t) - R_n(\chi_j(t)+,t)\delta_{x=\chi_j(t)}$$

Where $R_n^c$ denote the continuous part of $R_n$ and using (4.4.12), we have

$$\overline{u_n}(x,t) = \begin{cases} u_n(x,t) \quad \text{if } (x,t) \in \Omega - \{(\chi_j(t),t), j = 1...n\} \\ \frac{u_n(\chi_j(t)-,t)+u_n(\chi_j(t)+,t)}{2} \quad \text{if } (x,t) \in \{(\chi_j(t),t), j = 1...n\}. \end{cases}$$

Now using R-H condition, we get,

$$(R_n)_t + \overline{u_n}(R_n)_x = (R_n^c)_t + u_n(R_n^c)_x.$$

If $(x,t)$ is a point of continuity of $u_n$, it is enough to show:

$$(R_n^c)_t + u_n(R_n^c)_x = 0.$$

Let $(x,t) \in S_c$, i.e, the point of continuity of $u_n$ and we know that

$$u_n(x,t) = n(t)x + \frac{l(t)}{m(t)}[l(t)x - y_n(x,t)].$$

Then

$$(u_n)_t = x\frac{d}{dt}\Big[n(t) + \frac{l^2(t)}{m(t)}\Big] - y_n(x,t)\frac{d}{dt}\Big(\frac{l(t)}{m(t)}\Big) - \frac{l(t)}{m(t)}(y_n)_t(x,t),$$
$$(u_n)_x = \Big[n(t) + \frac{l^2(t)}{m(t)}\Big] - \frac{l(t)}{m(t)}(y_n)_x(x,t).$$

We calculate

$$\begin{aligned}(u_n)_t + u_n(u_n)_x =& x\frac{d}{dt}\Big[n(t) + \frac{l^2(t)}{m(t)}\Big] - y_n(x,t)\frac{d}{dt}\Big(\frac{l(t)}{m(t)}\Big) - \frac{l(t)}{m(t)}(y_n)_t(x,t)\\ &+ \Big([n(t) + \frac{l^2(t)}{m(t)}]x - \frac{l(t)}{m(t)}y_n(x,t)\Big)\Big[n(t) + \frac{l^2(t)}{m(t)} - \frac{l(t)}{m(t)}(y_n)_x(x,t)\Big]\\ =& x\Big[\frac{d}{dt}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) + \Big(n(t) + \frac{l^2(t)}{m(t)}\Big)^2\Big]\\ &- \Big[\frac{d}{dt}\Big(\frac{l(t)}{m(t)}\Big) + \frac{l(t)}{m(t)}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big)\Big]y_n\\ &- \frac{l(t)}{m(t)}((y_n)_t + u(y_n)_x).\end{aligned} \tag{4.4.15}$$

Now,

$$\begin{aligned}&\frac{d}{dt}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) + \Big(n(t) + \frac{l^2(t)}{m(t)}\Big)^2\\ &= \frac{d}{dt}n(t) + n^2(t) + \frac{d}{dt}\Big(\frac{l^2(t)}{m(t)}\Big) + \Big(\frac{l^2(t)}{m(t)}\Big)^2 + 2\frac{l^2(t)n(t)}{m(t)}\\ &= \frac{h''(t)}{h(t)} + \frac{d}{dt}\Big(\frac{m'(t)}{m(t)}\Big) + \Big(\frac{m'(t)}{m(t)}\Big)^2 + 2\frac{m'(t)}{m(t)}n(t)\\ &= \frac{h''(t)}{h(t)} + \frac{m''(t)}{m(t)} + 2\frac{m'(t)}{m(t)}n(t)\\ &= \frac{h''(t)}{h(t)} + \frac{2l(t)\Big(l'(t) + l(t)n(t)\Big)}{m(t)} = \frac{h''(t)}{h(t)}.\end{aligned}$$

In the above calculation we used the relation (4.2.9). A similar calculation reveals that the coefficient of $y_n$ in the equation (4.4.15), that is, $\frac{d}{dt}\big(\frac{l(t)}{m(t)}\big) + \frac{l(t)}{m(t)}\big(n(t) + \frac{l^2(t)}{m(t)}\big) = 0$. Since $u_n$ satisfies $(u_n)_t + u_n(u_n)_x = \Gamma(x,t)$, we get

$$(y_n)_t + u_n(y_n)_x = 0.$$

Therefore

$$(R_n^c)_t + u_n(R_n^c)_x = \rho_0(y_n(x,t))[(y_n)_t + u(y_n)_x] = 0.$$

**Step** 4. From step 2, we have $(R_n)_t + \bar{u_n}(R_n)_x = 0$. This can be rewritten by using the chain rule for Volpert product(see [34])

$$(R_n)_t + (u_n R_n)_x - \bar{R_n}(u_n)_x = 0. \tag{4.4.16}$$

We calculate,

$$\begin{aligned} \bar{R_n}(u_n)_x &= \overline{R_0(y_n(x,t))}\Big(n(t)x + \frac{l(t)}{m(t)}[l(t)x - y_n(x,t)]\Big)_x \\ &= \overline{R_0(y_n(x,t))}\Big[\big(n(t) + \frac{l^2(t)}{m(t)}\big) - \frac{l(t)}{m(t)}(y_n(x,t))_x\Big] \\ &= \overline{R_0(y_n(x,t))}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) - \frac{l(t)}{m(t)}\Big(M_0(y_n(x,t))\Big)_x, \end{aligned} \tag{4.4.17}$$

where $R_0(y) = M_0'(y)$. Since $(R_0, M_0) \in W^{1,\infty}_{loc}(\mathbb{R})$, step 1 gives,

$$R_0(y_n(x,t)) \to R_0(y(x,t)) \quad \text{in} \quad L^1([-R,R]\times[\delta,T]),$$

$$M_0(y_n(x,t)) \to M_0(y(x,t)) \quad \text{in} \quad L^1([-R,R]\times[\delta,T]).$$

Hence,

$$\begin{aligned} &\overline{R_0(y_n(x,t))}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) - \frac{l(t)}{m(t)}\Big(M_0(y_n(x,t))\Big)_x \\ &\to \overline{R_0(y(x,t))}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) - \frac{l(t)}{m(t)}\Big(M_0(y(x,t))\Big)_x \end{aligned} \tag{4.4.18}$$

in the sense of distribution.

So from equations (4.4.16)-(4.4.18), we get

$$(R_n)_t + (u_n R_n)_x - \bar{R_n}(u_n)_x \to$$

$$(R)_t + (uR)_x - \overline{R_0(y(x,t))}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) + \frac{l(t)}{m(t)}\Big(m_0(y(x,t))\Big)_x$$

in the sense of distribution.

Finally one has,

$$R_t + (uR)_x - \overline{R_0(y(x,t))}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) + \frac{l(t)}{m(t)}\Big(m_0(y(x,t))\Big)_x = 0,$$

i,e.

$$R_t + (uR)_x - \overline{R_0(y(x,t))}\Big(n(t) + \frac{l^2(t)}{m(t)}\Big) + \frac{l(t)}{m(t)}\overline{R_0(y(x,t))}y_x(x,t) = 0.$$

This implies,

$$\begin{aligned} &R_t + (uR)_x - \overline{R_0(y(x,t))}\Big(n(t)x + \frac{l(t)}{m(t)}[l(t)x - y(x,t)]\Big)_x \\ &\implies R_t + (uR)_x - \bar{R}u_x \implies R_t + \overline{u}R_x = 0. \end{aligned}$$

This completes the proof of the theorem. □

### 4.4.2 The Riemann problem

The following theorem describes an explicit solution for (4.1.1) when the initial data is of Riemann type, i.e,

$$\begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_L \\ \rho_L \end{pmatrix}, & \text{if} \quad x < 0 \\ \begin{pmatrix} u_R \\ \rho_R \end{pmatrix}, & \text{if} \quad x > 0. \end{cases}$$

Using explicit formula (4.1.10), one can easily deduce the solutions for Riemann type initial data.

**Theorem 4.4.6.** *The solution for the problem*

$$\begin{aligned} &u_t + uu_x = \Gamma(x,t) \;\; x \in \mathbb{R}, \;\; t > 0 \\ &\rho_t + (u\rho)_x = 0, \;\; x \in \mathbb{R}, \;\; t > 0, \end{aligned}$$

*with initial data*

$$(u,\rho)^T(x,0) = \begin{cases} (u_L, \rho_L)^T & \textit{if}\, x < 0, \\ (u_R, \rho_R)^T & \textit{if}\, x > 0, \end{cases}$$

*where* $(.,.)^T$ *denotes the transpose, case by case are given below.*

*Case:* $u_L > u_R$*:*

$$u(x,t) = \begin{cases} n(t)x + l(t)u_L & \text{if } x < g(t), \\ \left[\frac{n(t)m(t)}{l(t)} + l(t)\right]\left(\frac{u_L+u_R}{2}\right) & \text{if } x = g(t), \\ n(t)x + l(t)u_R & \text{if } x > g(t), \end{cases}$$

*and*

$$\rho(x,t) = \begin{cases} l(t)\rho_L & \text{if } x < g(t), \\ m(t)(\rho_L+\rho_R)\left(\frac{u_L-u_R}{2}\right)\delta_{x=g(t)} & \text{if } x = g(t). \\ l(t)\rho_R & \text{if } x > g(t), \end{cases}$$

*where*

$$g(t) = \frac{m(t)}{l(t)}\left(\frac{u_L+u_R}{2}\right)$$

*Case:* $u_L \leq u_R$*:*

$$u(x,t) = \begin{cases} n(t)x + l(t)u_L & \text{if } x < \frac{m(t)}{l(t)}u_L, \\ \left(n(t) + \frac{l^2(t)}{m(t)}\right)x & \text{if } \frac{m(t)}{l(t)}u_R < x < \frac{m(t)}{l(t)}u_L, \\ n(t)x + l(t)u_R & \text{if } x > \frac{m(t)}{l(t)}u_R \end{cases}$$

*and*

$$\rho(x,t) = \begin{cases} l(t)\rho_L & \text{if } x < \frac{m(t)}{l(t)}u_L, \\ 0 & \text{if } \frac{m(t)}{l(t)}u_R < x < \frac{m(t)}{l(t)}u_L, \\ l(t)\rho_R & \text{if } x > \frac{m(t)}{l(t)}u_R, \end{cases}$$

*where* $n(t)$*,* $l(t)$*,* $m(t)$ *are defined as in* (4.2.11).

*Proof.* **Case I:** $u_L > u_R$ : For $y < 0$, the minimizer of the functional $G(y,x,t)$ is $y = l(t)x - u_L m(t)$ and the minimum value is $u_L\big(l(t)x - \frac{u_L}{2}m(t)\big)$.

For $y > 0$, the minimizer of the functional $G(y,x,t)$ is $y = l(t)x - u_R m(t)$ and the minimum value is $u_R\big(l(t)x - \frac{u_R}{2}m(t)\big)$.

For $y = 0$, the value of $G(y, x, t)$ is $\frac{l^2(t)x^2}{2m(t)}$.

A straightforward calculation gives,

$$u_L\big(l(t)x - \frac{u_L}{2}m(t)\big) < u_R\big(l(t)x - \frac{u_R}{2}m(t)\big)$$

iff

$$x < \frac{m(t)}{l(t)}\Big(\frac{u_L + u_R}{2}\Big) = g(t).$$

Now observing the fact

$$u_L\big(l(t)x - \frac{u_L}{2}m(t)\big) \leq \frac{l^2(t)x^2}{2m(t)}$$
$$u_R\big(l(t)x - \frac{u_R}{2}m(t)\big) \leq \frac{l^2(t)x^2}{2m(t)},$$

we conclude that, in the region $x < g(t)$, the minimizer of the functional $G(y, x, t)$ is given by $y(x, t) = l(t)x - u_L m(t)$. Now recalling the formula

$$u(x, t) = n(t)x + \frac{l(t)}{m(t)}\big(l(t)x - y(x, t)\big),$$
$$R(x, t) = R_0(y(x, t)),$$

we obtain

$$u(x, t) = n(t)x + l(t)u_L, \;\; R(x, t) = \rho_L\big(l(t)x - u_L m(t)\big).$$

Similarly, in the region $x > g(t)$, the minimizer of the functional $G(y, x, t)$ $G(y, x, t)$ is given by $y(x, t) = l(t)x - u_R m(t)$, we get

$$u(x, t) = n(t)x + l(t)u_R, \;\; R(x, t) = \rho_R\big(l(t)x - u_R m(t)\big).$$

Since $\rho = R_x$ in the sense of distribution, we get the above formula.

**Case II:** $u_L < u_R$ : In the region $x < \frac{m(t)}{l(t)}u_L$, the minimizer of the functional $G(y, x, t)$ is $y(x, t) = l(t)x - u_L m(t)$. So the from the explicit formula for $u(x, t)$ and $R(x, t)$, we get

$$u(x, t) = n(t)x + l(t)u_L$$

$$R(x, t) = \rho_L(l(t)x - u_L m(t)).$$

Similarly, in the region $x > \frac{m(t)}{l(t)} u_R$, the minimizer of the functional $G(y,x,t)$ is $y(x,t) = l(t)x - u_R m(t)$. So the from the explicit formula for $u(x,t)$ and $R(x,t)$, we get

$$
\begin{aligned}
u(x,t) &= n(t)x + l(t)u_R \\
R(x,t) &= \rho_R(l(t)x - u_R m(t)).
\end{aligned}
$$

In the region $\frac{m(t)}{l(t)} u_L < x < \frac{m(t)}{l(t)} u_R$, $G(y,x,t)$ attains the minimum at $y = 0$, hence the solution is given by

$$
\begin{aligned}
u(x,t) &= \big(n(t) + \frac{l^2(t)}{m(t)}\big)x \\
R(x,t) &= 0.
\end{aligned}
$$

This completes the proof of the theorem. □

## 4.5 Space independent non-homogeneous system

In this short section, we are interested in the hyperbolic system with non-homogeneous terms depending only on $t$. We follow Lax [47] to show the existence of a solution for this system of balance laws. Let us consider the following system

$$
\begin{cases} u_t + uu_x = h(t), \;\; x \in \mathbb{R}, \;\; t > 0, \\ \rho_t + (u\rho)_x = 0, \;\; x \in \mathbb{R}, \;\; t > 0, \end{cases} \tag{4.5.1}
$$

where $h(t)$ is a bounded measurable function on $[0, \infty)$. For simplicity, we take the initial data for $u$ and $\rho$ as bounded measurable function.

$$
u(x,0) = u_0(x), \rho(x,0) = \rho_0(x). \tag{4.5.2}
$$

Define the following functions

$$
\begin{aligned}
f_1(t) &= -\int_0^t \Big(\frac{1}{w^2}\int_0^w sh(s)ds\Big)dw, \\
f_2(t) &= -\frac{1}{t}\int_0^t sh(s)ds, \\
f_3(t) &= -\int_0^t f_2^2(s)ds.
\end{aligned}
$$

Note that $f_1(t)$, $f_2(t)$ and $f_3(t)$ satisfies the following differential equations.

$$\begin{aligned} {f_1}'(t) &= \frac{f_2(t)}{t}, \\ {f_2}'(t) &= -h(t) - \frac{f_2(t)}{t}, \\ f_3'(t) &= -f_2^2(t). \end{aligned} \tag{4.5.3}$$

Define

$$U(x,y,t) = \min_{y\in\mathbb{R}}\Big\{\frac{(x-y)^2}{2t} + f_1(t)y + f_2(t)x + f_3(t) + \int_0^y u_0(z)dz\Big\}. \tag{4.5.4}$$

Almost repeating the proof of lemmas (4.1) and (4.2), one can get

**Lemma 4.5.1.** *Suppose $U(x,y,t)$ attains minimum at $y_1$ for fixed $(x,t)$. Then $U(x',y_1,t) < U(x',y,t)$ holds for $y < y_1$ and $x < x'$.*

**Lemma 4.5.2.** *For fixed t, $y_-(x,t)$ and $y_+(x,t)$ are monotonic increasing function in the variable $x$. Hence there is a unique minimizer of $U(x,y,t)$ for a.e. $(x,t) \in \mathbb{R}\times(0,\infty)$.*

**Theorem 4.5.3.** *For a.e. $(x,t) \in \mathbb{R}\times(0,\infty)$, there exists a unique minimizer of* (4.5.4) *and the solution for the equation* (4.5.1)-(4.5.2) *is given below.*

$$\begin{aligned} u(x,t) &= \frac{x-y(x,t)}{t} + f_2(t) \\ \rho(x,t) &= \frac{\partial}{\partial x}R_0(y(x,t)), \end{aligned} \tag{4.5.5}$$

*$\frac{\partial}{\partial x}$ is understood as the distributional derivative with respect to $x$.*

*Proof.* Denote

$$G(x,y,t) = \frac{(x-y(x,t))^2}{2t} + f_1(t)y(x,t) + f_2(t)x + f_3(t) + \int_0^{y(x,t)} u_0(z)dz.$$

Following Lax [47], we define

$$U^N(x,t) = -\frac{1}{N}\log\Big[\int_{-\infty}^{\infty} e^{-NG(x,y,t)}dy\Big].$$

Then

$$U_x^N = \frac{\int_{-\infty}^{\infty}\Big(\frac{x-y}{t}+f_2(t)\Big)e^{-NG(x,y,t)}dy}{\int_{-\infty}^{\infty}e^{-NG(x,y,t)}dy}$$

$$U_t^N = \frac{\int_{-\infty}^{\infty}\Big(-\frac{(x-y)^2}{2t^2}+f_1'(t)y+f_2'(t)x+f_3'(t)\Big)e^{-NG(x,y,t)}dy}{\int_{-\infty}^{\infty}e^{-NG(x,y,t)}dy}$$

Because of the relations (4.5.3),

$$-\frac{(x-y)^2}{2t^2}+f_1'(t)y+f_2'(t)x+f_3'(t) = -\frac{\Big(\frac{x-y}{t}+f_2(t)\Big)^2}{2}-h(t)x.$$

By lemma (5.2) and using the method of proof of the previous result in section 4, one can also show for a fixed $t$, the functions $U_t^N$ and $U_x^N$ converges a.e. $x\in\mathbb{R}$ to the following limit.

$$\begin{aligned}\lim_{N\to\infty}U_x^N(x,t) &= \frac{x-y(x,t)}{t}+f_2(t) = u(x,t)\\ \lim_{N\to\infty}U_t^N(x,t) &= -\frac{(x-y(x,t))^2}{2t^2}+f_1'(t)y(x,t)+f_2'(t)x+f_3'(t)\\ &= -\frac{u(x,t)^2}{2}-h(t)x.\end{aligned}$$

So the above limit holds for a.e. $(x,t)\in\mathbb{R}\times(0,\infty)$.

Since $U_{xt}^N = U_{tx}^N$, multiplying with test function and integrating by parts on the domain $\mathbb{R}\times[t_1\infty)$, we get,

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}U_x^N\phi_t(x,t)dxdt-\int_{t_1}^{\infty}\int_{-\infty}^{\infty}U_t^N\phi_x(x,t)dxdt+\int_{-\infty}^{\infty}U_x^N\phi(x,t_1)dx = 0.$$

Using integration by parts,

$$\int_{t_1}^{\infty}\int_{-\infty}^{\infty}U_x^N\phi_t(x,t)dxdt-\int_{t_1}^{\infty}\int_{-\infty}^{\infty}U_t^N\phi_x(x,t)dxdt-\int_{-\infty}^{\infty}U^N\phi_x(x,t_1)dx = 0.$$

Now passing to the limit as $t_1\to 0$, we get

$$\int_0^{\infty}\int_{-\infty}^{\infty}U_x^N\phi_t(x,t)dxdt-\int_0^{\infty}\int_{-\infty}^{\infty}U_t^N\phi_x(x,t)dxdt-\int_{-\infty}^{\infty}\Big(\int_0^x u_0(z)dz\Big)\phi_x(x,0)dx = 0.$$

Now passing to the limit as $N \to \infty$, we get

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t dxdt + \int_0^\infty \int_{-\infty}^\infty (\frac{u^2}{2} + h(t)x)\phi_x dxdt - \int_{-\infty}^\infty \Big(\int_0^x u_0(z)dz\Big)\phi_x(x,0)dx = 0.$$

Applying integration by parts, we get

$$\int_0^\infty \int_{-\infty}^\infty \big(u\phi_t + \frac{u^2}{2}\phi_x\big)dxdt - \int_0^\infty \int_{-\infty}^\infty h(t)\phi(x,t)dxdt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0.$$

The proof that $R(x,t) = R_0(y(x,t))$ satisfies the equation

$$R_t + uR_x = 0,$$

with initial data $R(x,0) = R_0(x)$ is some what similar to the proof of Theorem 4.6 and details are omitted. Therefore $\rho = R_x$ satisfies

$$\rho_t + (u\rho)_x = 0,$$

with initial data $\rho(x,0) = \rho_0(x)$. □

**Remark 4.5.4.** *We showed that the formula* (4.5.5) *satisfies the equation* (4.5.1)-(4.5.2) *when the initial data $u_0(x)$ and $\rho_0(x)$ are bounded measurable functions. But this formula* (4.5.5) *still satisfies the equation when $\int_0^x u_0(z)dz = o(x^2)$ and $\int_0^x \rho_0(z)dz = O(|x|^\beta)$, for any $\beta \in \mathbb{N}$.*

# Chapter 5

# Initial-boundary value problem for 1D pressureless gas dynamics

## 5.1 Introduction

This chapter addresses the solvability of the initial-boundary value problem for the system of pressureless gas dynamics

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2)_x = 0 \end{aligned} \tag{5.1.1}$$

in one space dimension. Here $\rho$ denotes the density and $u$ the velocity. The two lines in equation (5.1.1) express conservation of mass and momentum, respectively. We adjoin initial data

$$\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \quad x > 0, \tag{5.1.2}$$

and ask under what conditions and in what sense boundary data

$$u(0,t) = u_b(t), \quad (\rho u)(0,t) = (\rho_b u_b)(t), \quad t > 0, \tag{5.1.3}$$

can be prescribed. It is assumed that the data $u_0$ and $u_b$ are bounded measurable functions with $u_b > 0$. Further, $\rho_0$ and $\rho_b$ are positive locally bounded measurable functions.

The initial value problem (5.1.1), (5.1.2) has been intensively studied in the literature. The key issue is that, in general, $\rho$ is no longer a function, but a measure. This led to the introduction of various strongly related notions of weak solutions, such as measure solutions [66], duality solutions [67] (based on [68]), duality solutions obtained by vanishing viscosity [69], mass and momentum potentials [70, 29, 71], together with generalized characteristics [72], generalized potentials and variational principles [14, 73, 74]. In this paper, we shall extend the approach of [73, 74] to the boundary value problem.

Let us begin by discussing what is meant by a generalized solution to the system of differential equations (5.1.1). We shall construct locally bounded measurable functions $m(x,t)$, $u(x,t)$ such that for almost all $t$, $m(x,t)$ is of locally bounded variation with respect to $x$. Thus, for almost all $t$, the distributional derivative $m_x$ defines a Radon measure $\rho$. In addition, $u$ is measurable with respect to $\rho$. Following [74, Definition 1.1], the pair $(\rho, u)$ is viewed as a generalized solution to (5.1.1), if

$$\begin{aligned} &\iint \varphi_t m \, dx \, dt - \iint \varphi u \, dm \, dt = 0 \\ &\iint \left(\psi_t u + \psi_x u^2\right) dm \, dt = 0 \end{aligned} \tag{5.1.4}$$

for all test functions $\varphi, \psi \in \mathcal{D}(\mathbb{R}^2_+)$. The construction of the solution $(m,u)$ to (5.1.4) will be based on the method of generalized potentials and characteristic triangles [74]. However, differently from [74], we will need two types of generalized potentials (initial and boundary potential) and their relation, as well as different types of characteristic triangles, depending on the location of their apex.

More precisely, the initial and boundary potentials are defined by

$$F(y,x,t) = \int_0^y [tu_0(\eta) + \eta - x]\rho_0(\eta) d\eta, \tag{5.1.5}$$

$$G(\tau,x,t) = \int_0^\tau [x - u_b(\eta)(t-\eta)]\rho_b(\eta) u_b(\eta) d\eta\,. \tag{5.1.6}$$

Further,

$$F(x,t) = \min_{y\in[0,\infty)} F(y,x,t), \tag{5.1.7}$$

$$G(x,t) = \min_{\tau\in[0,\infty)} G(\tau,x,t). \tag{5.1.8}$$

The characteristic triangles with apex $(x,t)$ will depend on whether $F(x,t) < G(x,t)$, $F(x,t) > G(x,t)$ or $F(x,t) = G(x,t)$. For $x = 0$, the respective relation between $F(0,t)$ and $G(0,t)$ will also decide about the assumption of the boundary data.

To further clarify the solution concept, we wish to show that $(\rho, u)$ actually is a weak solution to system (5.1.1) in its proper sense. Let us recall the measure-theoretic point of view and the distributional point of view (for simplicity in the one-dimensional case). If $m$ is a function of locally bounded variation, it defines a Lebesgue-Stieltjes measure $dm$. On the other hand, its derivative in the sense of distributions defines a Radon measure $\rho = m_x$. The two objects are the same, identified by the chain of equalities

$$\int_{\mathbb{R}} \varphi(x)m(dx) = \langle \rho, \varphi \rangle = -\langle m, \varphi_x \rangle = -\int_{\mathbb{R}} \varphi_x(x)m(x)dx$$

for $\varphi \in \mathcal{D}(\mathbb{R})$. The first equality can be extended to $\varphi \in \mathcal{C}(\mathbb{R})$ with compact support. What is more, the Lebesgue-Stieltjes integral can be extended to all functions $\varphi$ which are integrable with respect to $\rho$. This a priori makes no sense at the other end of the chain of equalities but allows us to *define* the product of the measure $\rho$ with the bounded, $\rho$-measurable function $u$ as the distribution given by

$$\langle \rho u, \varphi \rangle = \int_{\mathbb{R}} \varphi(x)u(x)m(dx)$$

for $\varphi \in \mathcal{D}(\mathbb{R})$.

Using this identification, the second line in (5.1.4) means

$$0 = \iint \left(\psi_t u + \psi_x u^2\right) dm\, dt = \langle \rho u, \psi_t \rangle + \langle \rho u^2, \psi_x \rangle,$$

which is exactly the distributional meaning of the second line of (5.1.1). To obtain the first line of (5.1.1), one has to insert $\varphi_x$ in place of $\varphi$ in (5.1.4) to obtain

$$0 = \iint \varphi_{xt} m\, dx\, dt - \iint \varphi_x u\, dm\, dt = -\langle \rho, \varphi_t \rangle - \langle \rho u, \varphi_x \rangle.$$

The actual proof of (5.1.4) will be done on yet a higher level. Apart from the mass potential $m(x,t)$, momentum and energy potentials $q(x,t)$ and $E(x,t)$ will be constructed, both bounded measurable functions which are in addition of bounded variation in $x$ for almost all $t$. Further, the Lebesge-Stieltjes measures $dq$ and $dE$ are absolutely continuous with respect to $dm$, namely

$$dq = udm, \quad dE = \tfrac{1}{2}u^2 dm,$$

and they satisfy the system

$$\begin{aligned} m_t + q_x &= 0 \\ q_t + (2E)_x &= 0 \end{aligned} \tag{5.1.9}$$

in the sense of distributions. By similar arguments as above, this system is equivalent to (5.1.4).

What concerns the initial data, we will show that (5.1.2) is satisfied in the sense that $\rho$ and $u$ are continuous functions of time with values in $\mathcal{D}'(\mathbb{R})$. Actually for almoust all $x$ we show $\lim_{t\to 0} u(x,t) = u_0(x)$, and $\lim_{t\to 0} m(x,t) = \int_0^x \rho_0(y)dy$.

We turn to the assumption of the boundary data (5.1.3). As is well known from the theory of conservation laws [75, 76, 77], one cannot arbitrarily prescribe boundary data, because a priori there is no control of the sign of $u(0+,t)$, except in the case when $u_0$ is positive and hence $u(x,t) > 0$ everywhere (recall that $u_b$ was assumed to be positive).

We will show the following: If $t > 0$ is a Lebesgue point of $u_b$ and $\rho_b$ and $F(0,t) > G(0,t)$, then $\lim_{x\to 0+} u(x,t) = u_b(t)$. If in addition $u_b$ is continuously differentiable and $\rho_b$ is locally Lipschitz continuous, then $\lim_{x\to 0+} \rho(x,t)u(x,t) = \rho_b(t)u_b(t)$. If $F(0,t) < G(0,t)$ then $u(0+,t) < 0$ and the boundary condition (5.1.3) cannot be fulfilled. Rather, it may happen that mass accumulates at the boundary in the form of $\delta \cdot (m(0+,t) - m(0,t))$. However, the solution we construct conserves total mass. Momentum is conserved at the points of time $t$ for which $F(0,t) \geq G(0,t)$, while it satisfies an inequality otherwise. For further aspects of boundary conditions for systems involving measure solution, see [78, 79].

The plan of this chapter is as follows: Section 2 is devoted to the construction of the solution. In Section 3 it will be shown that the constructed solution satisfies system (5.1.4), and hence (5.1.1). Section 4 addresses the assumption of initial and boundary values, as well as conservation of mass and momentum. In Section 5, it will be shown that the solution satisfies Oleinik's entropy condition. Finally, Section 6 contains several examples illustrating some of the possibly occurring effects.

## 5.2 Construction of solution

In this section, we construct the solution for the initial-boundary value problem. Recall the definition of the initial and boundary potentials (5.1.5), (5.1.6),

$$F(y,x,t) = \int_0^y [tu_0(\eta) + \eta - x]\rho_0(\eta)d\eta\,,$$
$$G(\tau,x,t) = \int_0^\tau [x - u_b(\eta)(t-\eta)]\rho_b(\eta)u_b(\eta)d\eta\,.$$

Given $(x,t)$, let $\tau^*(x,t)$ and $\tau_*(x,t)$ be the uppermost and lowermost points on the $t$-axis such that

$$\min_{\tau\geq 0} G(\tau,x,t) = G(\tau^*(x,t),x,t) = G(\tau_*(x,t),x,t)\,.$$

Similarly, let $y_*(x,t)$ and $y^*(x,t)$ be the leftmost and rightmost points respectively on the $x$-axis such that

$$\min_{y\geq 0} F(y,x,t) = F(y_*(x,t),x,t) = F(y^*(x,t),x,t)\,.$$

Note that these minima exist as real numbers because $u_0$ is bounded from above and below and $u_b$ is positive. The following lemma collects some properties of the minimizers.

**Lemma 5.2.1.** *With our assumptions on the initial and boundary data we have*

1. *$\tau_*(x,t)$ and $\tau^*(x,t)$ are, for fixed $x$, monotonically increasing in $t$ and for fixed $t$ monotonically decreasing in $x$. Moreover, we have for $t_1 < t_2$ that $\tau^*(x,t_1) \leq \tau_*(x,t_2)$ and for $x_1 < x_2$ that $\tau_*(x_1,t) \geq \tau^*(x_2,t)$.*

2. *$y_*(x,t)$ and $y^*(x,t)$ are, for fixed t, monotonically increasing in $x$ and for $x_1 < x_2$ we have $y^*(x_1,t) \leq y_*(x_2,t)$.*

3. *$\tau_*(0,t) = \tau^*(0,t) = t$.*

4. *$y_*(x,t)$ is lower semicontinuous and $y^*(x,t)$ is upper semicontinuous.*

5. *$\tau_*(x,t)$ is lower semicontinuous and $\tau^*(x,t)$ is upper semicontinuous.*

*Proof.* (1) Let $x,\ t_1,\ t_2 > 0$ be arbitrary but fixed and $\tau_1$ a minimizer of $G(\tau,x,t_1)$ and $\tau_2$ one of $G(\tau,x,t_2)$. Now we have

$$0 \leq G(\tau_2,x,t_1) - G(\tau_1,x,t_1)\,, \text{ and } 0 \leq G(\tau_1,x,t_2) - G(\tau_2,x,t_2)\,.$$

Summing the two inequalities results in

$$0 \leq (t_2 - t_1)\int_{\tau_1}^{\tau_2} \rho(\eta)u_b(\eta)^2 d\eta\,.$$

Since the term in the integral is positive by our assumptions we conclude that the minimizers have to be increasing in $t$.

Now on the other hand fixing $t,\ x_1,\ x_2$ and denoting by $\tau_1$ a minimizer of $G(\tau,x_1,t)$ and by $\tau_2$ one of $G(\tau,x_2,t)$ we derive in the same way

$$0 \leq (x_1 - x_2)\int_{\tau_1}^{\tau_2} \rho(\eta)u_b(\eta) d\eta\,.$$

From this one can conclude that the minimizers are decreasing in $x$.

(2) is Lemma 2.1 in [74], from which also the proof of (1) is adopted. (3) is obvious, (4) see Lemma 2.2 in [74]. (5) is proved along the lines of (4). □

**Remark 5.2.2.** *If* $\min_{\tau\geq 0} G(\tau,x,t)$ *is constant on an interval* $[x_1,x_2]\times\{t\}$ *one can argue similarly to the proof of (1) above:*

$$0 \leq G(\tau_2,x_1,t) - G(\tau_1,x_1,t) = G(\tau_2,x_1,t) - G(\tau_2,x_2,t) = (x_1 - x_2)\int_0^{\tau_2} \rho_b(\eta)u_b(\eta)d\eta$$

*Now since $\rho_b$ and $u_b$ are assumed to be strictly positive we conclude that any minimizer $\tau_2$ of $G(\tau, x_2, t)$ has to be zero and thus $\min_{\tau \geq 0} G(\tau, x_2, t) = 0$. Since $G$ is constant on the interval it has to be equal to zero on the whole interval. Note also that the minimizers on the whole interval have to be zero (uniquely) because one can replace $x_2$ by any point in the interval in the estimate above.*
*This situation will correspond to the case when the solution contains a rarefaction wave starting at the origin.*

We first quote the following result, which was established by Wang, Huang, and Ding [74] in their study of the initial value problem. This will be central also in our work for parts of the solution depending only on the initial data.

**Lemma 5.2.3.** *For fixed $(x, t)$, let the minimum $\min_{y \in [0,\infty)} F(y, x, t)$ be attained at $y(x, t)$. Then for any given point $(x', t')$ on the line segment joining $(y(x, t), 0)$ and $(x, t)$, we have $F(y, x', t') > F(y(x, t), x', t')$ for $y \neq y(x, t)$.*

*Proof.* The proof follows directly from the proof of Lemma 2.3. in [74]. It also follows from the poof of Lemma 2.4 in [73], noting that we assumed $\rho_0$ to be strictly positive (at least in the $L_\infty$-sense). □

Now we establish a similar result for the part of the solution depending on the boundary data.

**Lemma 5.2.4.** *For fixed $(x, t)$, $x, t > 0$, let $\tau = \tau_1$ be a point which minimizes the functional $G(\tau, x, t)$. Let $(\bar{x}, \bar{t}) \neq (x, t)$ be any point on the line segment joining $(x, t)$ and $(0, t_1)$. Then the minimizer of $G(\tau, \bar{x}, \bar{t})$ is unique and is $\tau_1$.*

*Proof.* We want to show that for $\tau \neq \tau_1$:

$$G(\tau, \bar{x}, \bar{t}) - G(\tau_1, \bar{x}, \bar{t}) > 0\,.$$

By definition we have

$$\begin{aligned}G(\tau,\bar{x},\bar{t})-G(\tau_1,\bar{x},\bar{t}) &= \int_{\tau_1}^{\tau}\left[\bar{x}-u_b(\eta)(\bar{t}-\eta)\right]\rho_b(\eta)u_b(\eta)d\eta = \\ &= \bar{x}\int_{\tau_1}^{\tau}\left[1-u_b(\eta)\frac{\bar{t}-\eta}{\bar{x}}\right]\rho_b(\eta)u_b(\eta)d\eta = \\ &= \bar{x}\int_{\tau_1}^{\tau}\left[1-u_b(\eta)\frac{\bar{t}-\tau_1}{\bar{x}}-u_b(\eta)\frac{\tau_1-\eta}{\bar{x}}\right]\rho_b(\eta)u_b(\eta)d\eta\,.\end{aligned}$$

Since $(\bar{x},\bar{t})$ lies on the line connecting $(x,t)$ and $(0,\tau_1)$ we conclude

$$\begin{aligned}G(\tau,\bar{x},\bar{t})-G(\tau_1,\bar{x},\bar{t}) &= \bar{x}\int_{\tau_1}^{\tau}\left[1-u_b(\eta)\frac{t-\tau_1}{x}-u_b(\eta)\frac{\tau_1-\eta}{\bar{x}}\right]\rho_b(\eta)u_b(\eta)d\eta = \\ = \bar{x}\int_{\tau_1}^{\tau}\left[1-u_b(\eta)\frac{t-\eta}{x}\right]\rho_b(\eta)u_b(\eta)d\eta &+ \bar{x}\int_{\tau_1}^{\tau}u_b^2(\eta)\rho_b(\eta)(\tau_1-\eta)\left[\frac{1}{x}-\frac{1}{\bar{x}}\right]d\eta\,.\end{aligned}$$

Now the first term in the sum is $\frac{\bar{x}}{x}\left[G(\tau,x,t)-G(\tau_1,x,t)\right]$, which is non-negative by assumption. For the second term observe that $\bar{x}<x$. Thus it is strictly positive if $\tau_1<\tau$ but (considering the direction of integration) also if $\tau_1>\tau$. □

The minima of the initial and boundary potentials, respectively, were introduced in (5.1.7) and (5.1.8) as

$$\begin{aligned}F(x,t) &= \min_{y\in[0,\infty)} F(y,x,t)\,, \\ G(x,t) &= \min_{\tau\in[0,\infty)} G(\tau,x,t)\,.\end{aligned}$$

Observe that for a fixed $t>0$ the function $F(x,t)$ is monotonically decreasing in $x$ while $G(x,t)$ is monotonically increasing.

**Lemma 5.2.5.** *The function* $[0,\infty[\times[0,\infty[\to\mathbb{R}\colon (x,t)\mapsto F(x,t)$ *is locally Lipschitz continuous and the same holds for* $G(x,t)$.

*Proof.* Let $U$ be a bounded open subset of $[0,\infty[\times[0,\infty[$. Since $y_*(x,t)$ is locally bounded in $[0,\infty[\times[0,\infty[$, there exists an $M>0$ such that

$$\Big|\int_0^{y_*(x,t)}\rho_0(\eta)d\eta\Big|<M$$

for all $(x,t) \in U$. For $(x_1,t),(x_2,t) \in U$ we have

$$F(x_1,t) - F(x_2,t) = F(y_*(x_1,t),x_1,t) - F(y_*(x_2,t),x_2,t) =$$
$$= [F(y_*(x_1,t),x_1,t) - F(y_*(x_1,t),x_2,t)] + [F(y_*(x_1,t),x_2,t) - F(y_*(x_2,t),x_2,t)]\,.$$

Since the second term is non-negative we infer that

$$F(x_1,t) - F(x_2,t) \geq (x_2 - x_1)\int_0^{y_*(x_1,t)} \rho_0(\eta)d\eta\,.$$

Similarly, we get

$$F(x_1,t) - F(x_2,t) \leq (x_2 - x_1)\int_0^{y_*(x_2,t)} \rho_0(\eta)d\eta\,.$$

Combining the inequalities above results in

$$|F(x_1,t) - F(x_2,t)| \leq M|x_1 - x_2|\,.$$

On the other hand varying $t$ we obtain in a similar manner for $(x,t_1),(x,t_2) \in U$,

$$|F(x,t_1) - F(x,t_2)| \leq M|t_1 - t_2|\,.$$

Therefore for $(x_1,t_1),(x_2,t_2) \in U$ we conclude

$$|F(x_1,t_1) - F(x_2,t_2)| \leq M\big(|x_1 - x_2| + |t_1 - t_2|\big)\,.$$

Lipschitz continuity of $G$ can be checked similarly. □

Next, we define the characteristic triangle associated to a point $(x,t)$. We will later show that this triangle contains all the initial or boundary information, respectively, necessary to give the solution at point $(x,t)$.

**Definition 5.2.6.** *Let $x \geq 0$, $t > 0$ and $F(x,t)$, $G(x,t)$ be given by equations* (5.1.7)*,* (5.1.8).

1. *For* $F(x,t) < G(x,t)$ *and* $x > 0$ *we define the characteristic triangle at the point* $(x,t)$ *as the convex hull generated by the points* $(x,t), (y_*(x,t),0)$ *and* $(y^*(x,t),0)$.

2. *For* $F(x,t) > G(x,t)$ *we define the characteristic triangle at the point* $(x,t)$ *as the convex hull generated by the points* $(x,t), (0,\tau_*(x,t)$, *and* $(0,\tau^*(x,t))$.

3. *For* $F(x,t) = G(x,t)$ *we define the characteristic triangle at the point* $(x,t)$ *as the convex hull generated by the points* $(x,t)$, $(y^*(x,t),0)$, $(0,\tau^*(x,t))$ *and* $(0,0)$.

4. *For* $x = 0$ *and* $F(0,t) < G(0,t)$ *we define the characteristic triangle as the convex hull generated by the points* $(0,t)$, $(0,0)$ *and* $(y^*(x,t),0)$.

*We denote the characteristic triangle associated with the point* $(x,t)$ *by* $\Delta(x,t)$.

Note that the characteristic triangle may collapse to a line segment or even to a single point (Case 2 with $x = 0$). Figure 5.1 serves as an illustration of possible cases for characteristic triangles.

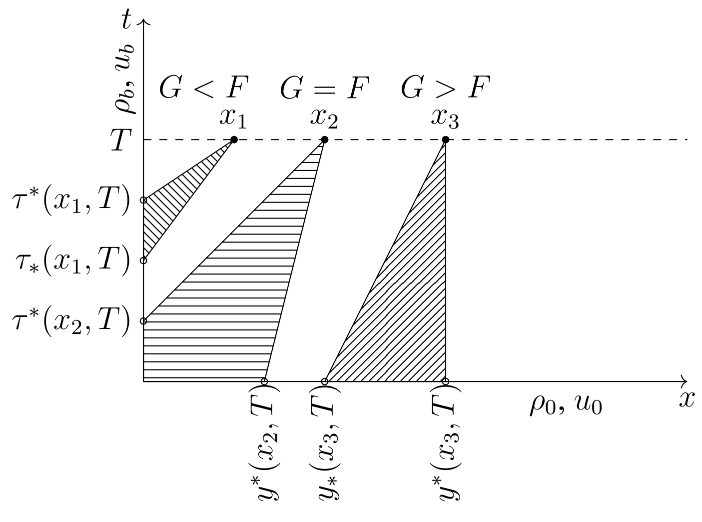


Figure 5.1: Illustration of characteristic triangles

Note that $G = F$ can happen on an interval in $x$ for fixed $t$. Since however $F$ is decaying in $x$ and $G$ is increasing in $x$ this can only happen if both, $F$ and $G$, are constant. We denote this closed interval by

$$I(t) = \{x|F(x,t) = G(x,t)\} = [l(t), r(t)]\,.$$

The following Lemma gives a characterization of the set where $F = G$.

**Lemma 5.2.7.** *With the notation as above, let $t$ be such that $\overset{\circ}{I}(t) \neq \emptyset$. Then:*

1. *For all $x \in I(t)$ it holds that $F(x,t) = G(x,t) = 0$.*
2. *$\tau_*(x,t) = \tau^*(x,t) = 0$ on $]l(t), r(t)]$ and $\tau_*(l(t),t) = 0$.*
3. *$y_*(x,t) = y^*(x,t) = 0$ on $[l(t), r(t)[$ and $y_*(r(t),t) = 0$.*
4. *For all $t' < t$ we have $\overset{\circ}{I}(t') \neq 0$.*
5. *The set $\bigcup_{0 \leq t' \leq t} I(t)$ is star-shaped with respect to $(0,0)$.*

*Proof.* (1) and (2) are direct consequences of Remark 5.2.2.
For the proof of (3) note that, since $F(x,t) = 0$, clearly $y = 0$ is a minimizer of $F(y,x,t)$ and thus $y_*(x,t) = 0$ on $I(t)$. The statement for $y^*$ then follows from the second point in Lemma 5.2.1.
To prove (4) let $x_1 < x_2 \in \overset{\circ}{I}(t)$ and consider the line segments joining $(x_i,t)$ and $(0,0)$. Now denote the points on these line segments at time $t' < t$ by $(x_1', t')$ and $(x_2', t')$. Then from Lemma 5.2.3 and Lemma 5.2.4

$$\tau_*(x_1', t') = \tau_*(x_2', t') = y_*(x_1', t') = y_*(x_2', t') = 0\,,$$

and thus $\forall x \in [x_1', x_2']\colon F(x,t') = G(x,t')$, leading to $\overset{\circ}{I}(t') \neq \emptyset$.
(5) follows from the proof of (4) immediately. □

**Corollary 5.2.8.** *If for some $t' > 0$ we have that $l(t') = r(t')$, then*

$$\forall t > t' \colon l(t) = r(t)\,.$$

Figure 5.2 illustrates the proof of Lemma 5.2.7 as well as the set G = F, including the characteristic triangles, in the situation of Corollary 5.2.8

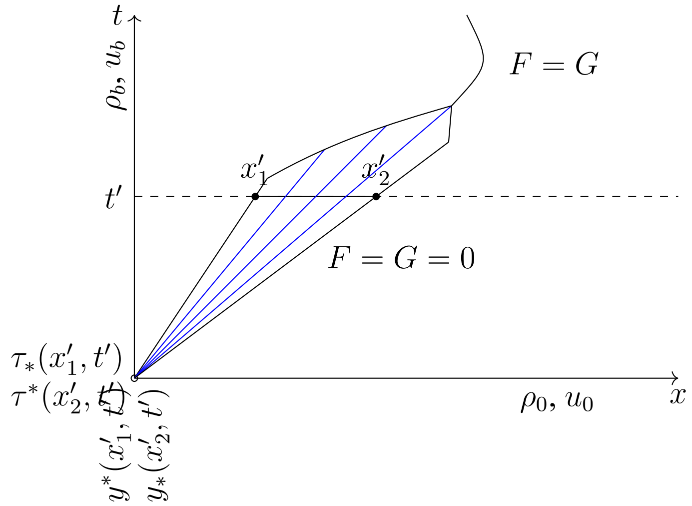


Figure 5.2: Characteristic triangles (blue lines) corresponding to a rarefaction wave emanating from the origin

Our next goal is to show that, as anticipated in Figure 5.1, characteristic triangles associated with different positions at the same time do not intersect. For two triangles reaching only the initial data, this is clear from Lemma 5.2.3. The same holds true if both triangles only reach the boundary data by Lemma 5.2.4. We still need to study the case when one of the triangles is as defined in points (3) or (4) of Definition 5.2.6. For this purpose, we prove the following lemma.

**Lemma 5.2.9.** *Let $t > 0$ be fixed, and $x_1,\, x_2 > 0$, $x_1 \neq x_2$ but arbitrary. Then the characteristic triangles associated to $(x_1, t)$ and $(x_2, t)$ do not intersect in the interior of $\mathbb{R}^2_+$.*

*Proof.* Let $t$ be fixed and assume w.l.o.g. that $x_2 > x_1$. Since $F(x, t)$ is decreasing and $G(x, t)$ is increasing in $x$ we only have the following cases:

Case 1, $G(x_2,t) < F(x_2,t)$: Then automatically also $G(x_1,t) < F(x_1,t)$. Now assume that the two characteristic triangles intersect. Since the point of intersection $p = (x_p, t_p)$

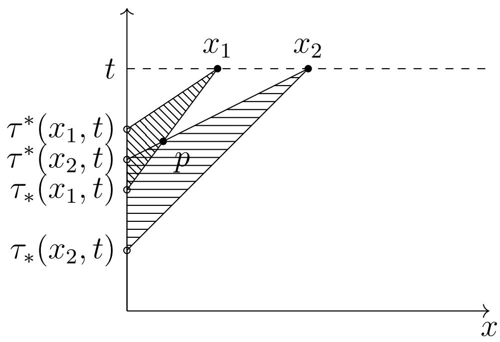


Figure 5.3: Intersecting triangles for boundary potential

lies on the line segment joining $(x_1,t)$ to $(0,\tau_*(x_1,t))$ we know from Lemma 5.2.4, that $G(\tau, x_p, t_p)$ attains its minimum for $\tau = \tau_*(x_1,t)$. Since $p$ is also on the line segment joining $(x_2,t)$ to $(0,\tau^*(x_2,t))$ we have that $G(\tau^*(x_2,t), x_p, t_p) = G(\tau_*(x_1,t), x_p, t_p)$. This however contradicts Lemma 5.2.4, because the minimizer is not unique.

Case 2, $G(x_2,t) = F(x_2,t)$: If also $G(x_1,t) = F(x_1,t)$ then on an interval the characteristic triangles are lines and do not intersect (see Figure 5.2). Otherwise, if $G(x_1,t) \neq F(x_1,t)$, denote by $y_2$ a minimizer $F(x_2,t) = F(y_2,x_2,t)$ and by $\tau_2$ one with $G(x_2,t) = G(\tau_2,x_2,t)$. Then we have for $y > 0$ arbitrary

$$F(y,x_1,t) > F(y,x_2,t) \geq F(y_2,x_2,t) = G(\tau_2,x_2,t) > G(\tau_2,x_1,t)\,,$$

and we conclude $G(x_1,t) < F(x_1,t)$. Thus we are in the situation depicted in Figure 5.4. Again we can conclude as in Case 1 that an intersection is impossible.

Case 3, $G(x_2,t) > F(x_2,t)$: Here we distinguish three more cases, namely

1. $G(x_1,t) > F(x_1,t)$. This case is the same as Case 1 but using Lemma 5.2.3.

2. $G(x_1,t) = F(x_1,t)$. The argument follows along the lines of Case 2, using Lemma 5.2.3.

3. $G(x_1,t) < F(x_1,t)$. Here intersection is not possible by definition.

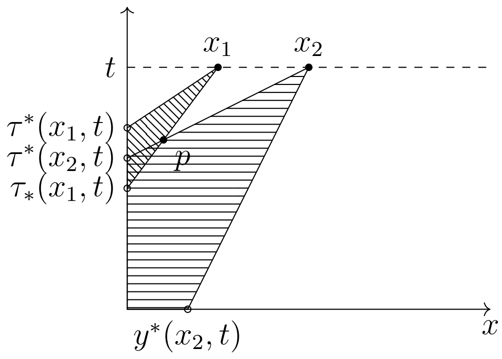


Figure 5.4: Intersecting triangles

□

**Lemma 5.2.10.** *Let $t > 0$ be fixed, and $x_1 > 0$. Then the characteristic triangles associated to $(0,t)$ and $(x_1,t)$ do not intersect in the interior of $\mathbb{R}^2_+$.*

*Proof.* The proof is trivial if $F(0,t) \geq G(0,t)$. Thus let us assume that $F(0,t) < G(0,t)$. Then, by monotonicity, we have $F(x,t) < G(x,t)$ for all $x > 0$. Thus the characteristic triangle at $(x_1,t)$ is the convex hull of $(x_1,t)$, $(y_*(x_1,t),0)$ and $(y^*(x_1,t),0)$. Since by point (2) of Lemma 5.2.1 we have $y^*(0,t) \leq y_*(x_1,t)$ the assertion follows. □

Now we are in the position to state the first theorem that combines the results for fixed $t$ and the lemmas before.

**Theorem 5.2.11.** *If two characteristic triangles intersect in $\mathbb{R}^2_+$, then one is contained in the other. Moreover, if they intersect on the boundary at more than one point, then also one has to be contained in the other.*

*Proof.* The result follows from the direct application of Lemmas 5.2.3–5.2.10. □

The properties of characteristic triangles established so far allow us to derive additional properties of $\tau_*, \tau^*, y_*, y^*$, which will be used frequently later. We collect them in a remark.

**Remark 5.2.12.** *(a) At fixed* $t$*, the function* $x \to \tau_*(x,t)$ *is right continuous and* $x \to \tau^*(x,t)$ *is left continuous. Further,* $\tau_*(x+,t) = \tau_*(x,t) = \tau^*(x+,t)$ *for all* $x \geq 0$ *and* $\tau_*(x-,t) = \tau^*(x,t) = \tau^*(x-,t)$ *for* $x > 0$*.*

*(b) At fixed* $t$*, the function* $x \to y_*(x,t)$ *is left continuous and* $x \to y^*(x,t)$ *is right continuous. Further,* $y_*(x+,t) = y^*(x,t) = y^*(x+,t)$ *for all* $x \geq 0$*.*

*Indeed, to verify (a), combine the semicontinuity properties stated in Lemma 5.2.1 with the fact that* $\tau(x,t)$ *is decreasing in* $x$*. In particular,* $\tau_*(x,t) = \tau_*(x+,t)$*. By Lemma 5.2.9,* $\tau^*(x+,t) \leq \tau_*(x,t)$*. On the other hand,* $\tau_*(x+,t) \leq \tau^*(x+,t)$*. Combining the inequalities leads to* $\tau_*(x+,t) = \tau_*(x,t) = \tau^*(x+,t)$*. The second assertion and item (b) is proved in the same way.*

The following lemma states that the domain of interest is indeed covered by characteristic triangles.

**Lemma 5.2.13.** *For any time* $t_0 > 0$ *we have*

$$\bigcup_{x\in[0,\infty)} \Delta(x,t_0) = \left\{(x,t) | x \in [0,\infty), 0 \leq t \leq t_0\right\}.$$

*Proof.* Case $I(t_0) \neq \emptyset$: Assume first that $I(t_0)$ consists of the single point $x_0 = l(t_0) = r(t_0)$. Let $(x,t)$ be a point which lies left of $\Delta(x_0,t_0)$ (see Figure 5.5). Consider points $(z,t_0)$ on the horizontal line segment joining $(0,t_0)$ with $(x_0,t_0)$. As $z$ decreases to $0$, both $\tau_*(z,t_0)$ and $\tau^*(z,t_0)$ converge to $(0,t_0)$ (Remark 5.2.12). Let

$$x_1 = \inf\{z : \tau^*(z,t_0) \leq \tau_*(x,t)\}.$$

Then $\tau^*(x_1+,t_0) \leq \tau_*(x,t)$, and by Remark 5.2.12, $\tau^*(x_1+,t_0) = \tau_*(x_1,t_0)$, so that $\tau_*(x_1,t_0) \leq \tau_*(x,t)$. Whenever $z < x_1$, we have $\tau^*(z,t_0) > \tau_*(x,t)$ and consequently, by the non-intersection property, $\tau^*(z,t_0) \geq \tau^*(x,t)$ as well. Again by Remark 5.2.12, $\tau^*(x_1,t_0) = \tau^*(x_1-,t_0) \geq \tau^*(x,t)$. Thus $\Delta(x,t) \subset \Delta(x_1,t_0)$, as desired.

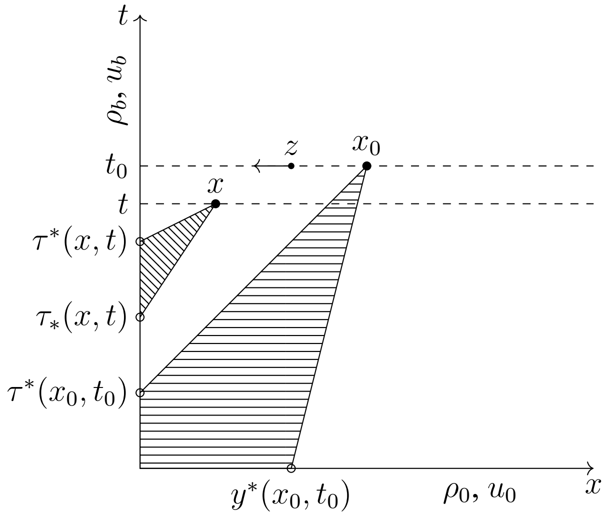


Figure 5.5: Illustration of Lemma 5.2.13

Second, if $l(t_0) \neq r(t_0)$, then $\tau_*(l(t_0), t_0) = 0$ and $y_*(r(t_0), t_0) = 0$ by Lemma 5.2.7. We may apply the same arguments to points $(x, t)$ lying to the left of the segment joining $(0, \tau^*(x_0, t_0))$ with $(l(t_0), t_0)$ or to the right of the segment joining $(y^*(x_0, t_0))$ with $(r(t_0), t_0)$, respectively. If $(x, t)$ lies between those segments, then $x \in I(t)$ and Figure 5.2 applies.

Case $I(t_0) = \emptyset$: Then $\Delta(0, t_0)$ contains the line segment connecting $(0, t_0)$ and $(y^*(0, t_0), 0)$ as well as all points to the left of it. For points to the right of the line segment the same argument as in the first case ensures that they are contained in the characteristic triangle of a point $(z, t_0)$. □

**Lemma 5.2.14.** *Let $t_1$ be strictly positive. Each point $(x_1, t_1)$ uniquely determines a curve $x = X(t)$, for $t \geq t_1$, with $x_1 = X(t_1)$ such that the characteristic triangles associated to points on the curve form an increasing family of sets. This curve is Lipschitz continuous as a function of $t \in [t_1, \infty[$. At every $t \geq t_1$, and $(x, t)$ on the curve we have the following:*

*(i) If* $F(x,t) < G(x,t)$ *then*

$$\lim_{t'',t'\searrow t} \frac{X(t'') - X(t')}{t'' - t'} = \begin{cases} \dfrac{x - y_*(x,t)}{t} & \text{if} \quad y_*(x,t) = y^*(x,t) \\ \dfrac{\int_{y_*(x,t)}^{y^*(x,t)} \rho_0 u_0}{\int_{y_*(x,t)}^{y^*(x,t)} \rho_0} & \text{if} \quad y_*(x,t) < y^*(x,t)\,. \end{cases} \tag{5.2.1}$$

*(ii) If* $F(x,t) > G(x,t)$ *then*

$$\lim_{t'',t'\searrow t} \frac{X(t'') - X(t')}{t'' - t'} = \begin{cases} \dfrac{x}{t - \tau_*(x,t)} & \text{if} \quad \tau_*(x,t) = \tau^*(x,t) \\ \dfrac{\int_{\tau_*(x,t)}^{\tau^*(x,t)} \rho_b u_b^2}{\int_{\tau_*(x,t)}^{\tau^*(x,t)} u_b \rho_b} & \text{if} \quad \tau_*(x,t) < \tau^*(x,t)\,. \end{cases} \tag{5.2.2}$$

*(iii) If* $F(x,t) = G(x,t)$ *and* $y^*(x,t) \neq 0$ *or* $\tau^*(x,t) \neq 0$ *then*

$$\lim_{t'',t'\searrow t} \frac{X(t'') - X(t')}{t'' - t'} = \frac{\int_0^{y^*(x,t)} \rho_0 u_0 + \int_0^{\tau^*(x,t)} \rho_b u_b^2}{\int_0^{y^*(x,t)} \rho_0 + \int_0^{\tau^*(x,t)} u_b \rho_b}\,. \tag{5.2.3}$$

*(iv) If* $F(x,t) = G(x,t)$ *and* $y^*(x,t) = \tau^*(x,t) = 0$*, then*

$$\lim_{t'',t'\searrow t} \frac{X(t'') - X(t')}{t'' - t'} = \frac{x}{t}$$

*(v) If* $x = 0$*,* $t > 0$ *and* $F(0,t) < G(0,t)$*, then*

$$\lim_{t'',t'\searrow t} \frac{X(t'') - X(t')}{t'' - t'} = 0$$

*Proof.* The statement (i) corresponds to Lemma 2.4 in [74] and the proof can be found there.

For the proof of (ii) let $t'' > t' > t$ and $X(t'') = x''$, $X(t') = x'$. The non-intersecting property of characteristic triangles implies the chain of inequalities

$$\tau_*(x'',t'') \le \tau_*(x',t') \le \tau_*(x,t) \le \tau^*(x,t) \le \tau^*(x',t') \le \tau^*(x'',t''),$$

and the semicontinuity then gives that $\tau_*(x'',t'') \to \tau_*(x,t)$ and $\tau^*(x'',t'') \to \tau^*(x,t)$ as $t'' \to t$. Consider the case when $\tau^*(x,t) = \tau_*(x,t)$. From Figure 5.6 it is straightforward to see the following inequality on inclinations:

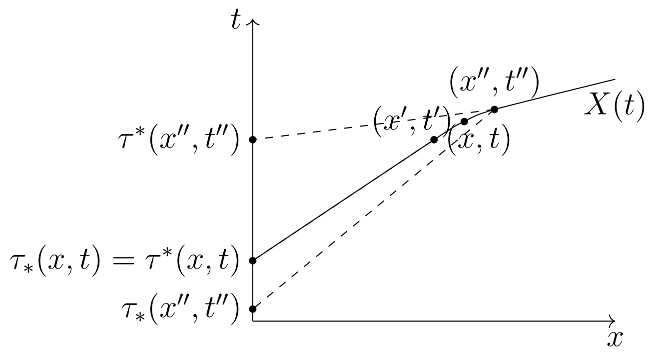


Figure 5.6: Bounds on slope

$$\frac{x''}{t''-\tau^*(x'',t'')} \geq \frac{x''-x'}{t''-t'} \geq \frac{x''}{t''-\tau_*(x'',t'')}. \tag{5.2.4}$$

Passing to the limit as $t'', t' \searrow t$ leads to the first identity of equation (5.2.2).

Now consider $\tau_*(x,t) < \tau^*(x,t)$. From the definitions of the boundary functional $G(\tau, y, x, t)$, $\tau_*$ and $\tau^*$, we have

$$\begin{aligned} G\big(\tau^*(x'',t''),x'',t''\big) - G\big(\tau_*(x',t'),x'',t''\big) &\leq 0 \leq \\ &\leq G\big(\tau^*(x'',t''),x',t'\big) - G\big(\tau_*(x',t'),x',t'\big). \end{aligned} \tag{5.2.5}$$

After simplification inequality (5.2.5) yields

$$\int_{\tau_*(x',t')}^{\tau^*(x'',t'')} [x'' - u_b(\eta)(t''-\eta)]\rho_b(\eta)u_b(\eta)d\eta \leq \int_{\tau_*(x',t')}^{\tau^*(x'',t'')} [x' - u_b(\eta)(t'-\eta)]\rho_b(\eta)u_b(\eta)d\eta\,.$$

Thus we can conclude

$$\frac{x''-x'}{t''-t'} \leq \frac{\int_{\tau_*(x',t')}^{\tau^*(x'',t'')} u_b^2(\eta)\rho_b(\eta)d\eta}{\int_{\tau_*(x',t')}^{\tau^*(x'',t'')} u_b(\eta)\rho_b(\eta)d\eta}\,. \tag{5.2.6}$$

On the other hand, considering the inequality

$$\begin{aligned} G\big(\tau_*(x'',t''),x'',t''\big) - G\big(\tau^*(x',t'),x'',t''\big) &\leq 0 \leq \\ &\leq G\big(\tau_*(x'',t''),x',t'\big) - G\big(\tau^*(x',t'),x',t'\big), \end{aligned}$$

we get, using $\tau^*(x',t') \geq \tau_*(x'',t'')$ that

$$\frac{x''-x'}{t''-t'} \geq \frac{\int_{\tau_*(x'',t'')}^{\tau^*(x',t')} u_b^2(\eta)\rho_b(\eta)d\eta}{\int_{\tau_*(x'',t'')}^{\tau^*(x',t')} u_b(\eta)\rho_b(\eta)d\eta}\,. \tag{5.2.7}$$

Now passing to the limit as $t'', t' \searrow t$ in equations (5.2.6) and (5.2.7), we proved the second identity of (5.2.2).

Now to verify the statement of (iii) we assume $F(x,t) = G(x,t)$. Then by definition of the curve and the characteristic triangles, $F(X(t'),t') = G(X(t'),t')$ for all $t' \geq t$. Using the minimizing properties we have the following inequality.

$$\begin{aligned}
&F\big(y^*(x'',t''),x'',t''\big) - G\big(\tau^*(x',t'),x'',t''\big) = \\
&= G\big(\tau^*(x'',t''),x'',t''\big) - G\big(\tau^*(x',t'),x'',t''\big) \leq 0 \leq \\
&\leq F\big(y^*(x'',t''),x',t'\big) - F\big(y^*(x',t'),x',t'\big) = \\
&= F\big(y^*(x'',t''),x',t'\big) - G\big(\tau^*(x',t'),x',t'\big)\,.
\end{aligned}$$

This inequality implies

$$\begin{aligned}
&\int_0^{y^*(x'',t'')} \rho_0(\eta)u_0(\eta)(t''-t') + \rho_0(\eta)(x'-x'')d\eta \leq \\
&\qquad \leq \int_0^{\tau^*(x',t')} (x''-x')\rho_b(\eta)u_b(\eta) + u_b^2(\eta)\rho_b(\eta)(t'-t'')d\eta\,,
\end{aligned}$$

and simplification leads to

$$\frac{\int_0^{y^*(x'',t'')} \rho_0 u_0 + \int_0^{\tau^*(x',t')} \rho_b u_b^2}{\int_0^{y^*(x'',t'')} \rho_0 + \int_0^{\tau^*(x',t')} u_b \rho_b} \leq \frac{x''-x'}{t''-t'}\,. \tag{5.2.8}$$

Now in the same way starting from the inequality

$$\begin{aligned}
&G\big(\tau^*(x'',t''),x'',t''\big) - F\big(y^*(x',t'),x'',t''\big) \leq \\
&\qquad \leq G\big(\tau^*(x'',t''),x',t'\big) - F\big(y^*(x',t'),x',t'\big)
\end{aligned}$$

and simplifying as before we get

$$\frac{\int_0^{y^*(x',t')} \rho_0 u_0 + \int_0^{\tau^*(x'',t'')} \rho_b u_b^2}{\int_0^{y^*(x',t')} \rho_0 + \int_0^{\tau^*(x'',t'')} u_b \rho_b} \geq \frac{x''-x'}{t''-t'}\,. \tag{5.2.9}$$

Passing to the limit as $t'', t' \searrow t$ in (5.2.8) and (5.2.9) completes the proof of (iii).

To prove (iv) observe that in this case the curve $X(t')$ for $t' \leq t$ is just a straight line with inclination $x/t$.

In case (v) the statement is obvious by the definition of characteristic triangles.

Finally, the Lipschitz continuity follows from the fact that, whenever $t'' > t' > t$, the differences $X(t'') - X(t')$ are bounded by a constant times $t'' - t'$, according to (5.2.4), (5.2.6), (5.2.7), (5.2.8) and (5.2.9). □

**Remark 5.2.15.** *Note that from (v) it follows that when such a curve $X(t)$ reaches (with increasing time) $x = 0$ it stays there until $F$ and $G$ become equal and then leaves $x = 0$ according to (iii).*

We conclude this sections by *defining* the functions $u(x,t)$ and $m(x,t)$. In the next section we will prove that these are indeed solutions of system (5.1.4).

**Definition 5.2.16.** *For $x, t > 0$ we define the real valued function $u(x,t)$ by*

$$u(x,t) = \begin{cases} \dfrac{x - y_*(x,t)}{t} & \text{if } F(x,t) < G(x,t) \text{ and } y_*(x,t) = y^*(x,t) \\ \dfrac{\int_{y_*(x,t)}^{y^*(x,t)} \rho_0 u_0}{\int_{y_*(x,t)}^{y^*(x,t)} \rho_0} & \text{if } F(x,t) < G(x,t) \text{ and } y_*(x,t) < y^*(x,t) \\ \dfrac{x}{t - \tau_*(x,t)} & \text{if } F(x,t) > G(x,t) \text{ and } \tau_*(x,t) = \tau^*(x,t) \\ \dfrac{\int_{\tau_*(x,t)}^{\tau^*(x,t)} \rho_b u_b^2}{\int_{\tau_*(x,t)}^{\tau^*(x,t)} \rho_b u_b} & \text{if } F(x,t) > G(x,t) \text{ and } \tau_*(x,t) < \tau^*(x,t) \\ \dfrac{\int_0^{\tau^*(x,t)} \rho_b u_b^2 + \int_0^{y^*(x,t)} \rho_0 u_0}{\int_0^{\tau^*(x,t)} \rho_b u_b + \int_0^{y^*(x,t)} \rho_0} & \text{if } F(x,t) = G(x,t) \text{ and } \begin{cases} y^*(x,t) \neq 0 \text{ or} \\ \tau^*(x,t) \neq 0 \end{cases} \\ \dfrac{x}{t} & \text{if } F(x,t) = G(x,t) \text{ and } \begin{cases} y^*(x,t) = 0 \text{ and} \\ \tau^*(x,t) = 0\,. \end{cases} \end{cases}$$

*For $x = 0$ and $t > 0$ we define $u = u_b$ if $F(0,t) > G(0,t)$ and $u = 0$ if $F(0,t) < G(0,t)$. For $F(0,t) = G(0,t)$ we use the same definition as for $x > 0$.*

More specifically, when $x = 0$ and $F(0,t) = G(0,t)$, the second to last formula applies because $\tau^*(0,t) = t$ always.

**Definition 5.2.17.** *For $t > 0$ and $x \geq 0$ we define the real valued function $m(x,t)$ by*

$$m(x,t) = \begin{cases} \displaystyle\int_0^{y_*(x,t)} \rho_0(\eta)d\eta & \text{if } F(x,t) \leq G(x,t) \text{ and } x > 0 \\ \displaystyle -\int_0^{\tau_*(x,t)} \rho_b(\eta)u_b(\eta)d\eta & \text{if } F(x,t) > G(x,t) \text{ or } x = 0\,. \end{cases}$$

**Remark 5.2.18.** *Since only (weak) derivatives of $m$ are of interest later, the definition at isolated points is not important apart from $x = 0$, where we will use the height of the jump from $m(0,t)$ to $\lim_{x\searrow 0} m(x,t)$ times delta as the mass concentrated at $x = 0$. If we have a whole area with $F = G$ then $m$ will be zero, since in this case $F = G = 0$ and $y^* = \tau^* = 0$ (c.f. Lemma 5.2.7 ). This is also consistent with tracing back along the triangles (lines) in the rarefaction wave.*

Note that in areas along the $t$-axis $x = 0$ where $F \leq G$ we can *not* ask to fulfill the boundary conditions (5.1.3) locally. For all other regions on the boundary, the boundary conditions will be shown to hold, at least under some mild regularity conditions on the boundary data (cf. Section 5.4).

## 5.3 Existence of generalized solution

In this section, we are going to show that $(u,m)$ as described in definitions 5.2.16 and 5.2.17 satisfy the system of equations (5.1.4). For that purpose, we first show how to extend the curves defined in Lemma 5.2.14 to the initial- or boundary manifold.

**Lemma 5.3.1.** *There is a countable set $S$ of points on the $x$- and $t$-axis with the following properties.*

*(i) For all $(\eta,0) \notin S$ there is a unique Lipschitz continuous curve $x = X(\eta,t)$, $t \geq 0$, such that $X(\eta,0) = \eta$ and the characteristic triangles associated to points on the curve form an increasing family of sets.*

*(ii) For all* $(0,\eta) \notin S$ *there is a unique Lipschitz continuous curve* $x = Y(\eta,t)$*,* $t \geq \eta$*, such that* $Y(\eta,\eta) = 0$ *and the characteristic triangles associated to points on the curve form an increasing family of sets.*

*Further, for all* $\eta > 0$ *such that* $(\eta,0)$ *and* $(0,\eta)$ *does not belong to* $S$*,*

$$\frac{\partial}{\partial t} X(\eta,t) = u(X(\eta,t),t) \quad \textit{for almost all } t > 0$$

$$\frac{\partial}{\partial t} Y(\eta,t) = u(Y(\eta,t),t) \quad \textit{for almost all } t > \eta\,,$$

*where the right hand side is a measurable function.*

*Proof.* Our proof follows the arguments found in [73] and [74] extends them to include the boundary points. We introduce $a^{\pm}(\eta,t)$ and $b^{\pm}(\xi,t)$ to consider all the cases simultaneously. For a fixed point $(0,\xi)$ on the $t$-axis and $t > 0$ we define

$$B^-(\xi,t) = \{x \in [0,\infty[\colon F(x,t) \geq G(x,t) \text{ and } \tau_*(x,t) < \xi\}$$

$$B^+(\xi,t) = \{x \in [0,\infty[\colon F(x,t) \geq G(x,t) \text{ and } \tau_*(x,t) > \xi\}\,,$$

and

$$b^-(\xi,t) = \begin{cases} \sup B^-(\xi,t)\,, & B^-(\xi,t) \neq \emptyset \\ 0\,, & B^-(\xi,t) = \emptyset\,, \end{cases}$$

as well as

$$b^+(\xi,t) = \begin{cases} \sup B^+(\xi,t)\,, & B^+(\xi,t) \neq \emptyset \\ 0, & B^+(\xi,t) = \emptyset\,. \end{cases}$$

Now for a fixed point $(\eta,0)$ on x-axis and $t > 0$ we define similarly

$$A^-(\eta,t) = \{x \in ]0,\infty[\colon F(x,t) \leq G(x,t) \text{ and } y_*(x,t) < \eta\}$$

$$A^+(\eta,t) = \{x \in ]0,\infty[\colon F(x,t) \leq G(x,t) \text{ and } y_*(x,t) > \eta\}\,,$$

and

$$a^-(\eta,t) = \sup A^-(\eta,t)\,, \quad a^+(\eta,t) = \inf A^+(\eta,t)\,.$$

Let us denote

$$S_b(t) = \{(0,\xi)\colon \xi \in (0,\infty), b^-(\xi,t) \neq b^+(\xi,t)\}$$

$$S_a(t) = \{(\eta,0)\colon \eta \in (0,\infty), a^-(\eta,t) \neq a^+(\eta,t)\}\,.$$

We define the set $S(t)$ by

$$S(t) = S_a(t) \cup S_b(t) \,.$$

Then for any fixed $t > 0$ the set $S(t)$ is countable. This follows from the fact that the intervals $[a^-(\eta,t), a^+(\eta,t)]$ and $[a^-(\eta',t), a^+(\eta',t)]$, can not intersect for $\eta \neq \eta'$ except at the endpoints, and the same holds true for $[b^-(\xi,t), b^+(\xi,t)]$ and $[b^-(\xi',t), b^+(\xi',t)]$ for $\xi \neq \xi'$. Indeed, let $\eta' > \eta$ and assume that $a^-(\eta',t) < a^+(\eta,t)$. This means that

$$\sup\{x \in \mathbb{R} \colon F(x,t) \leq G(x,t) \text{ and } y_*(x,t) < \eta'\} < a^+(\eta,t) \,.$$

Therefore, for all $x$ between $a^-(\eta',t)$ and $a^+(\eta,t)$ we have $y_*(x,t) \geq \eta' > \eta$, contradicting the definition of $a^+(\eta,t)$. A similar proof works for the other intervals.
Observe that for decreasing $t$ the sets $S_a(t)$ form an increasing family of sets. To see this let $0 < t' < t$ and $\eta \in S_a(t)$. Then necessarily $y_*(x,t) = \eta$ for all $x \in [a^-(\eta,t), a^+(\eta,t)]$. Let $L(x,t)$ be the line connecting $(x,t)$ and $(\eta,0)$ (for such $x$). By Lemma 2.3 of [74] we have that $y_* = \eta$ along this line. In particular, the line segment cut out by the bounding lines $L(a^-(\eta,t),t)$ and $L(a^+(\eta,t),t)$ at height $t'$ is contained in $[a^-(\eta,t'), a^+(\eta,t')]$. Thus $\eta \in S_a(t')$. In particular, for every $t$ there is $n \in \mathbb{N}$ such that $S_a(t) \subset S_a(\frac{1}{n})$. A similar reasoning can be applied to $S_b(t)$. Therefore, the set

$$S = \bigcup_{t>0} S(t) = \bigcup_{n \in \mathbb{N}} S\big(\tfrac{1}{n}\big)$$

is countable.

Next, we discuss the definition of generalized characteristics according to [73]. For $(\eta,0) \notin S$, the generalized characteristic is defined by

$$X(\eta,t) = a^+(\eta,t) \,, \; t > 0 \,,$$

with $X(\eta,0) = \eta$. Since $(\eta,0) \notin S$, $a^-(\eta,t) = a^+(\eta,t)$ for all $t > 0$.

Let $x = X(\eta, t)$ be the generalized characteristic. We claim that the characteristic triangles at the point $(X(\eta, t), t)$ form an increasing family of sets, that is if $(x_1, t_1)$ and $(x_2, t_2)$ are two points on the curve with $t_1 < t_2$, then $\Delta(x_1, t_1) \subset \Delta(x_2, t_2)$. Indeed, in the case when both characteristic triangles are given by $(x_i, t_i)$, $(y_*(x_i, t_i), 0)$, $(y^*(x_i, t_i), 0)$, consider the line segment joining the point $(x_2, t_2)$ to $(y^*(x_2, t_2), 0)$ and assume that segment intersects $t = t_1$ at the point $(x_3, t_1)$. Then from Lemma 5.2.3 we have $y^*(x_2, t_2) = y_*(x_3, t_1)$. Then by definition of $X(\eta, t)$, we find

$$X(\eta, t_1) \leq x_3 .$$

On the other hand, assume that the the line segment joining the points $(x_2, t_2)$ and $(y_*(x_2, t_2), 0)$ intersects the line $t = t_1$ at $(\bar{x}_3, t_1)$. Then we claim

$$\bar{x}_3 \leq X(\eta, t_1) .$$

Indeed, the map $x \to y_*(x, t_2)$ is lower semicontinuous and increasing, hence left continuous. Further, $x_2 = \sup\{x : F(x, t_2) \leq G(x, t_2)$ and $y_*(x, t_2) < \eta\}$. Thus $y_*(x_2, t_2) = \lim_{x \to x_2 -} y_*(x, t_2) \leq \eta$, and consequently $\bar{x}_3 \leq \eta$ as well. Combining the two arguments shows that $(x_1, t_1) \in \Delta(x_2, t_2)$ and hence using the non-intersection property of characteristic triangles we have

$$\Delta(x_1, t_1) \subset \Delta(x_2, t_2) .$$

In the case when either of the characteristic triangles is given by the edges $(x_i, t_i)$, $(0, \tau^*(x_2, t_2))$, $(y^*(x_i, t_i), 0)$, the inclusion $\Delta(x_1, t_1) \subset \Delta(x_2, t_2)$ follows directly from the first argument above and the non-intersection property.

Now we turn to the time derivative of the curves $X(\eta, t)$. It is obvious that $\eta \to X(\eta, t)$ is an increasing function at fixed $t > 0$. Therefore, it is Borel (and Lebesgue) measurable. Lemma 5.2.14 together with Definition 5.2.16 mean that

$$\frac{\partial}{\partial t} X(\eta, t) = u(X(\eta, t), t) \tag{5.3.1}$$

in the sense of a right derivative, at least for $(\eta, 0) \notin S$. Since $X(\eta, t)$ is Lipschitz continuous, it is differentiable almost everywhere and its derivative satisfies (5.3.1). Further, $u(X(\eta, t), t)$ is a limit of difference quotients of measurable functions, and thus measurable.

Similarly, for $(0, \xi) \notin S$ the generalized characteristic is defined by

$$Y(\xi, t) = b^{+}(\xi, t), t > 0\,.$$

An analogous assertion as above is true for the characteristic triangles, and

$$\frac{\partial}{\partial t} Y(\xi, t) = u(Y(\xi, t), t)$$

holds for almost all $t > \xi$. □

**Remark 5.3.2.** *The exceptional set $S$ corresponds to points on the $x$- or $t$-axis from which a rarefaction wave starts. Indeed, if $(\eta, 0) \notin S_a(t)$ and $t > 0$, then $y_*(x, t) = \eta = y^*(x, t)$ whenever $x \in [a_-(\eta, t), a_+(\eta, t)]$. One could define $X(\eta, t)$ by $X(\eta-, t)$ as is done in [72] or by $X(\eta+, t)$ (to include $\eta = 0$). However, to state and prove the results of the present paper, it is not required to assign a value to $X(\eta, t)$ at the exceptional points.*

**Definition 5.3.3.** *We define, for $x, t > 0$ the momentum and the kinetic energy associated to equation 5.1.4 by*

$$q(x,t) = \begin{cases} \displaystyle\int_0^{y_*(x,t)} \rho_0(\eta) u_0(\eta) d\eta\,, & \text{if } F(x,t) \le G(x,t) \\ \displaystyle -\int_0^{\tau_*(x,t)} \rho_b(\eta) u_b^2(\eta) d\eta\,, & \text{if } F(x,t) > G(x,t)\,, \end{cases} \tag{5.3.2}$$

*and*

$$E(x,t) = \begin{cases} \displaystyle \tfrac{1}{2}\int_0^{y_*(x,t)} \rho_0(\eta) u_0(\eta) u(X(\eta,t),t) d\eta\,, & \text{if } F(x,t) \le G(x,t) \\ \displaystyle -\tfrac{1}{2}\int_0^{\tau_*(x,t)} \rho_b(\eta) u_b^2(\eta) u(Y(\eta,t),t) d\eta\,, & \text{if } F(x,t) > G(x,t)\,. \end{cases} \tag{5.3.3}$$

The following lemma will make our physical interpretation more precise.

**Lemma 5.3.4.** *In the sense of Radon-Nikodym derivatives in $x$, the following holds in the interior of $\mathbb{R}^2_+$: (i) $dq = udm$, (ii) $dE = \frac{1}{2}u^2dm$.*

*Proof.* If $F(x,t) < G(x,t)$, then the result is Lemma 2.8. in [73].

Let $(x,t)$ be a point where $G(x,t) < F(x,t)$. If $\tau_*$ is constant in some neighborhood of $(x,t)$, then the above quantities are constant and the lemma holds trivially. Now suppose $\tau_*(x,t)$ is not constant in a neighborhood of $(x,t)$ and assume $\tau_*(x,t) = \tau^*(x,t) = \tau(x,t)$. Let $x_1 < x < x_2$, then by definition

$$
\begin{aligned}
G(\tau_*(x_1,t),x_1,t) &= \int_0^{\tau_*(x_1,t)} [x_1 - (t-\eta)u_b(\eta)]\rho_b(\eta)u_b(\eta)d\eta \\
G(\tau_*(x_2,t),x_1,t) &= \int_0^{\tau_*(x_2,t)} [x_1 - (t-\eta)u_b(\eta)]\rho_b(\eta)u_b(\eta)d\eta \,.
\end{aligned}
$$

By the minimizing properties we have we have

$$G(\tau_*(x_1,t),x_1,t) \le G(\tau_*(x_2,t),x_1,t)\,,$$

and using the definitions leads to

$$\frac{x_1}{t-\tau_*(x_2,t)} \ge \frac{\int_{\tau_*(x_1,t)}^{\tau_*(x_2,t)} \left(\frac{t-\eta}{t-\tau_*(x_2,t)}\right) u_b^2(\eta)\rho_b(\eta)d\eta}{\int_{\tau_*(x_1,t)}^{\tau_*(x_2,t)} \rho_b(\eta)u_b(\eta)d\eta}\,. \tag{5.3.4}$$

Now since

$$\frac{t-\tau_*(x_1,t)}{t-\tau_*(x_2,t)} \le \frac{t-\eta}{t-\tau_*(x_2,t)} \le 1\,,$$

and since $\tau_*(x,t) = \tau^*(x,t) = \tau(x,t)$ is continuous at $(x,t)$ we can take the limits $x_1 \nearrow x$ and $x_2 \searrow x$ in (5.3.4) to derive

$$\frac{x}{t-\tau(x,t)} \ge \lim_{x_2,x_1\to x} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)}\,. \tag{5.3.5}$$

Similarly considering the inequality

$$G(\tau_*(x_2,t),x_2,t) \le G(\tau_*(x_1,t),x_2,t)$$

and following the analysis as above, we get

$$\frac{x}{t-\tau(x,t)} \leq \lim_{x_2,x_1\to x} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)}\,. \tag{5.3.6}$$

From equations (5.3.5) and (5.3.6) and Definition (5.2.16), we conclude $dq = udm$.

If $\tau_*(x,t) < \tau^*(x,t)$, then

$$\lim_{x_2,x_1\to x} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} = \lim_{x_2,x_1\to x} \frac{\int_{\tau_*(x_2,t)}^{\tau_*(x_1,t)} \rho_b u_b^2}{\int_{\tau_*(x_2,t)}^{\tau_*(x_1,t)} \rho_b u_b} = \frac{\int_{\tau_*(x,t)}^{\tau^*(x,t)} \rho_b u_b^2}{\int_{\tau_*(x,t)}^{\tau^*(x,t)} \rho_b u_b}\,.$$

Here we used Remark 5.2.12. Now we consider the remaining case, where $(x,t)$ is a point with $F(x,t) = G(x,t)$. If this happens in an isolated point then we have, for $x_1 < x < x_2$

$$\lim_{x_2,x_1\to x} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} =$$

$$= \lim_{x_2,x_1\to x} \frac{\int_0^{\tau_*(x_1,t)} \rho_b u_b^2 + \int_0^{y_*(x_2,t)} \rho_0 u_0}{\int_0^{\tau_*(x_1,t)} \rho_b u_b + \int_0^{y_*(x_2,t)} \rho_0} = \frac{\int_0^{\tau^*(x,t)} \rho_b u_b^2 + \int_0^{y^*(x,t)} \rho_0 u_0}{\int_0^{\tau^*(x,t)} \rho_b u_b + \int_0^{y^*(x,t)} \rho_0}\,,$$

again using Remark 5.2.12. If on the other hand $F(x,t) = G(x,t)$ in a whole neighborhood of $(x,t)$ then we are in a rarefaction wave emanating from zero and thus $\tau^*(x,t) = y^*(x,t) = 0$ in a whole neighborhood. Then $m$, $q$, and $E$ are zero by definition and the proof is finished. In the boundary points of the region $F(x,t) = G(x,t)$, the same proof as in the case $F < G$ or $F > G$ works, on the right and left boundary, respectively. Thus in all possible cases, we derived that $dq = udm$ in the sense of Radon-Nikodym derivative.

Now we turn our attention to the proof of $dE = \frac{1}{2}u^2 dm$. First let $(x,t)$ again be a point where $G(x,t) < F(x,t)$ and $\tau_*(x,t) = \tau^*(x,t)$. Then

$$E(x_2,t) - E(x_1,t) = \tfrac{1}{2} \int_{\tau_*(x_2,t)}^{\tau_*(x_1,t)} \rho_b(\eta) u_b^2(\eta) u(Y(\eta,t),t)\, d\eta\,. \tag{5.3.7}$$

Note that for $\tau_*(x_2,t) \leq \eta \leq \tau_*(x_1,t)$ we have

$$\frac{x}{t-\tau_*(x_2,t)} \leq u(Y(\eta,t),t) \leq \frac{x}{t-\tau_*(x_1,t)}\,. \tag{5.3.8}$$

Hence from equation (5.3.7) and (5.3.8) we derive

$$\frac{1}{2}\frac{x}{t-\tau_*(x_2,t)} \leq \frac{E(x_2,t)-E(x_1,t)}{q(x_1,t)-q(x_2,t)} \leq \frac{1}{2}\frac{x}{t-\tau_*(x_1,t)}\,.$$

In the limit $x_1 \nearrow x \swarrow x_2$, we have $\frac{E(x_2,t)-E(x_1,t)}{q(x_1,t)-q(x_2,t)} \to \frac{1}{2}u$ and we know $\frac{dq}{dm} = u$. Combining these two, we get $dE = \frac{1}{2}u^2 dm$.

For the final case assume $F(x,t) = G(x,t)$ in an isolated point. Then noting that $(x_1,t)$ lies left of $(x,t)$ and thus $F(x_1,t) > G(x_1,t)$, and the opposite holds true for $(x_2,t)$, we have

$$\lim_{x_1,x_2\to x} \frac{E(x_2,t)-E(x_1,t)}{q(x_2,t)-q(x_1,t)} =$$
$$= \frac{\frac{1}{2}\int_0^{y^*(x,t)} \rho_0(\eta)u_0(\eta)u(X(\eta,t),t)d\eta + \frac{1}{2}\int_0^{\tau^*(x,t)} \rho_b(\eta)u_b^2(\eta)u(Y(\eta,t),t)d\eta}{\frac{1}{2}\int_0^{y^*(x,t)} \rho_0(\eta)u_0(\eta)d\eta + \frac{1}{2}\int_0^{\tau^*(x,t)} \rho_b(\eta)u_b^2(\eta)d\eta}$$

For $\eta \in [0, y^*(x,t))]$, we have $u(X(\eta,t),t) = u(x,t)$ and similarly for $\eta \in [0,\tau^*(x,t))]$ we know that $u(Y(\eta,t),t) = u(x,t)$.

Thus from the equation above, we have

$$\lim_{x_1,x_2\to x} \frac{E(x_2,t)-E(x_1,t)}{q(x_2,t)-q(x_1,t)} = \frac{1}{2}u(x,t)\,.$$

Since $dq = udm$, we again conclude $dE = \frac{1}{2}u^2$. This completes the proof. □

**Lemma 5.3.5.** *Define*

$$\mu(x,t) = \min(F(x,t), G(x,t))\,,$$

*then the following holds for* $x_1, x_2, t > 0$:

$$\int_{x_1}^{x_2} m(x,t)dx = \mu(x_1,t) - \mu(x_2,t)\,. \tag{5.3.9}$$

$$\int_{t_1}^{t_2} q(x,t)dt = \mu(x,t_2) - \mu(x,t_1)\,. \tag{5.3.10}$$

*Proof.* We start with proving the first equality. For that purpose let $t > 0$ be fixed and pick any two points $x, x' \in [x_1, x_2]$, $x < x'$. We claim that

$$(x-x')m(x',t) \leq \mu(x',t) - \mu(x,t) \leq (x-x')m(x,t)\,. \tag{5.3.11}$$

For $\mu(x,t)$ and $\mu(x',t)$ depending upon the minimization the possible cases are:

$$\begin{aligned}
&(a) \quad \mu(x,t) = F(x,t)\,, \quad \mu(x',t) = F(x',t)\\
&(b) \quad \mu(x,t) = G(x,t)\,, \quad \mu(x',t) = F(x',t)\\
&(c) \quad \mu(x,t) = G(x,t)\,, \quad \mu(x',t) = G(x',t)
\end{aligned} \tag{5.3.12}$$

The proof of the inequality (5.3.11) for the case $(a)$ in (5.3.12) can be found in Lemma 2.9. in [73]. In case (b) we have

$$\begin{aligned}
&\mu(x',t) - \mu(x,t) = F(y_*(x',t),x',t) - G(\tau_*(x,t),x,t) = \\
&\quad = [F(y_*(x',t),x',t) - F(y_*(x',t),x,t)] + [F(y_*(x',t),x,t) - G(\tau_*(x,t),x,t)]\,.
\end{aligned}$$

Since the term in the second bracket is positive, we get

$$\mu(x',t) - \mu(x,t) \geq F(y_*(x',t),x',t) - F(y_*(x',t),x,t). \tag{5.3.13}$$

One can also write

$$\begin{aligned}
&\mu(x',t) - \mu(x,t) = F(y_*(x',t),x',t) - G(\tau_*(x,t),x,t) = \\
&\quad = [F(y_*(x',t),x',t) - G(\tau_*(x,t),x',t)] + [G(\tau_*(x,t),x',t) - G(\tau_*(x,t),x,t)]\,.
\end{aligned}$$

Since the term in the first bracket is negative, we get

$$\mu(x',t) - \mu(x,t) \leq G(\tau_*(x,t),x',t) - G(\tau_*(x,t),x,t)\,. \tag{5.3.14}$$

Combining (5.3.13)-(5.3.14) and using the definition of $m$, $F$ and $G$, we conclude (5.3.11).

In case(c) we have

$$\begin{aligned}
&\mu(x',t) - \mu(x,t) = G(\tau_*(x',t),x',t) - G(\tau_*(x,t),x,t)\\
&\quad = [G(\tau_*(x',t),x',t) - G(\tau_*(x,t),x',t)] + [G(\tau_*(x,t),x',t) - G(\tau_*(x,t),x,t))]\,.
\end{aligned}$$

Since the first bracket of the above expression is negative, we get

$$\mu(x',t) - \mu(x,t) \leq G(\tau_*(x,t),x',t) - G(\tau_*(x,t),x,t)\,. \tag{5.3.15}$$

On the other hand, we write

$$\begin{aligned}\mu(x',t) - \mu(x,t) &= G(\tau_*(x',t),x',t) - G(\tau_*(x,t),x,t)\\ &= [G(\tau_*(x',t),x',t) - G(\tau_*(x',t),x,t)] + [G(\tau_*(x',t),x,t) - G(\tau_*(x,t),x,t))]\,.\end{aligned}$$

Now the second bracket of the expression is positive and thus

$$\mu(x',t) - \mu(x,t) \geq G(\tau_*(x',t),x',t) - G(\tau_*(x',t),x,t)\,. \tag{5.3.16}$$

Again, combining (5.3.15), (5.3.16) and using the definitions of $G$ and $m$ we have (5.3.11).

Since $m$ is monotonous in $x$, it is also Riemann integrable. Taking Riemann sums and using (5.3.11), we get

$$\int_{x_1}^{x_2} m(x,t)dx = \mu(x_1,t) - \mu(x_2,t)\,,$$

finishing the proof of (5.3.9).

For the proof of (5.3.10) let $x > 0$ be fixed and pick $t < t'$ in $[t_1, t_2]$. We claim that

$$(t'-t)q(x,t') \leq \mu(x,t') - \mu(x,t) \leq (t'-t)q(x,t)\,. \tag{5.3.17}$$

To verify this, we distinguish the following possibilities for $\mu(x,t)$ and $\mu(x,t')$:

$$(a)\quad \mu(x,t) = F(x,t)\,,\quad \mu(x,t') = F(x,t')$$

$$(b)\quad \mu(x,t) = G(x,t)\,,\quad \mu(x,t') = G(x,t')$$

$$(c)\quad \mu(x,t) = F(x,t)\,,\quad \mu(x,t') = G(x,t')$$

$$(d)\quad \mu(x,t) = G(x,t)\,,\quad \mu(x,t') = F(x,t')$$

Note that again the case $(a)$ on $\{x\} \times [t_1, t_2]$ is covered by Lemma 2.9. in [73]. Since the proofs in all cases are rather similar, we only present the proof of case $(b)$ explicitly. We start by observing that

$$\begin{aligned}\mu(x,t') - \mu(x,t) &= G(\tau_*(x,t'),x,t') - G(\tau_*(x,t),x,t) = \\ &= [G(\tau_*(x,t'),x,t') - G(\tau_*(x,t),x,t')] + [G(\tau_*(x,t),x,t') - G(\tau_*(x,t),x,t))] \leq \\ &\leq G(\tau_*(x,t),x,t') - G(\tau_*(x,t),x,t))\,,\end{aligned}$$

and

$$\begin{aligned}\mu(x,t') - \mu(x,t) &= G(\tau_*(x,t'),x,t') - G(\tau_*(x,t),x,t) = \\ &= [G(\tau_*(x,t'),x,t') - G(\tau_*(x,t'),x,t)] + [G(\tau_*(x,t'),x,t) - G(\tau_*(x,t),x,t))] \geq \\ &\geq G(\tau_*(x,t'),x,t') - G(\tau_*(x,t'),x,t)\,.\end{aligned}$$

Those two inequalities combined imply (5.3.17).

For a fixed $x$ the function $y_*(x,t)$ is monotone in the interval $[t_1, t_2]$ and thus $q(x,t)$ is a function of bounded variation, hence Riemann integrable. Now following a similar argument as before, identity (5.3.10) follows from (5.3.17). □

From the previous lemma we have $\mu_x = -m$ and $\mu_t = q$, and thus we verified the first equation of system (5.1.9). As anticipated in the introduction, for a test function $\varphi$ with compact support in $]0,\infty[^2$ we infer using Lemma 5.3.4:

$$\begin{aligned}0 &= \iint [-\varphi_t \mu_x(x,t) + \varphi_x \mu_t(x,t)]dxdt = \\ &= \iint [\varphi_t(x,t)m(x,t) + \varphi_x(x,t)q(x,t)]dxdt = \\ &= \iint \varphi_t(x,t)m(x,t)dxdt - \iint \varphi(x,t)u(x,t)m(dx,t)dt\,. \end{aligned} \tag{5.3.18}$$

This identity proves that $(u, m)$ satisfies the first equation of the system (5.1.4).

To prove the second equation, we use the following notation.

**Lemma 5.3.6.** *For $x, t > 0$, let us denote*

$$H(x,t) = \begin{cases} H_1(x,t) = \int_0^{y_*(x,t)} \rho_0(\eta)u_0(\eta)(X(\eta,t)-x)d\eta\,, & \text{if } F(x,t) \le G(x,t) \\ H_2(x,t) = -\int_0^{\tau_*(x,t)} \rho_b(\eta)u_b^2(\eta)(Y(\eta,t)-x)d\eta\,, & \text{if } F(x,t) > G(x,t)\,. \end{cases}$$

*Then we have $H_x = -q$ and $H_t = 2E$ in the weak sense.*

*Proof.* We will first show that

$$-\int_{x_1}^{x_2} q(x,t)dx = H(x_2,t) - H(x_1 r,t)\,. \tag{5.3.19}$$

Let $t$ be fixed and $[x_1, x_2]$ be an interval. Assume that $x$ and $x'$ are any two points in $[x_1, x_2]$ with $x < x'$. We will again argue by taking Riemann sums. First, depending on the minimization we distinguish the cases

$$\begin{aligned} &(a) \quad H(x_1,t) = H_1(x_1,t)\,, \quad H(x_2,t) = H_1(x_2,t) \\ &(b) \quad H(x_1,t) = H_2(x_1,t)\,, \quad H(x_2,t) = H_2(x_2,t) \\ &(c) \quad H(x_1,t) = H_2(x_1,t)\,, \quad H(x_2,t) = H_1(x_2,t)\,. \end{aligned}$$

For case $(a)$ it is shown in [74] (what we denote by $H_1$ is denoted by $\theta$ in [74]), that (5.3.19) holds.

For studying case $(b)$, we define $H_2(\tau,x,t) = -\int_0^\tau \rho_b(\eta)u_b^2(\eta)(Y(\eta,t)-x)d\eta$. Since $Y(\eta,t)$ is decreasing in $\eta$ (by the non-intersecting property of the characteristic triangles) and $Y(\eta,t) = x$ for $\tau_*(x,t) < \eta < \tau^*(x,t)$, we have

$$H_2(x,t) = \min_{\tau \ge 0} H_2(\tau,x,t)\,. \tag{5.3.20}$$

In particular, $H_2(\cdot,t)$ is upper semicontinuous. Note that $f(x,t) > G(x,t)$ and hence $H(x,t) = H_2(x,t)$ for all $x$ in the interval $[x_1, x_2]$. The rest of the proof is similar to the one of Lemma 5.3.5. We have, denoting minimizers in the usual way,

$$\begin{aligned} H(x,t)-H(x',t) &= H_2(x,t)-H_2(x',t) = H_2(\tau_*(x,t),x,t)-H_2(\tau_*(x',t),x',t) = \\ &= [H_2(\tau_*(x,t),x,t)-H_2(\tau_*(x',t),x,t)] + [H_2(\tau_*(x',t),x,t)-H_2(\tau_*(x',t),x',t)] \le \\ &\le H_2(\tau_*(x',t),x,t)-H_2(\tau_*(x',t),x',t) = -(x-x')q(x',t)\,. \end{aligned}$$

On the other hand

$$\begin{aligned} H(x,t)-H(x',t) &= H_2(x,t)-H_2(x',t) = H_2(\tau_*(x,t),x,t)-H_2(\tau_*(x',t),x',t) = \\ &= [H_2(\tau_*(x,t),x,t)-H_2(\tau_*(x,t),x',t)] + [H_2(\tau_*(x,t),x',t)-H_2(\tau_*(x',t),x',t)] \ge \\ &\ge H_2(\tau_*(x,t),x,t)-H_2(\tau_*(x,t),x',t) = -(x-x')q(x,t)\,. \end{aligned}$$

Combining those two inequalities establishes

$$-(x-x')q(x,t) \le H(x,t)-H(x',t) \le -(x-x')q(x',t)\,. \qquad (5.3.21)$$

Now taking the supremum of the Riemann sums over all partitions of the interval $[x_1,x_2]$, we deduce (5.3.19) on all intervals $[x_1,x_2]$ with $F>G$.

The case $F(x,t)=G(x,t)$ requires special consideration. Recall from Lemma 5.2.7 that this happens on an interval $I(t)=[l(t),r(t)]$. If $l(t)=r(t)=x$, then $\Delta(x,t)$ contains $[0,y^*(x,t)]\times\{0\}$ and $\{0\}\times[0,\tau^*(x,t)]$. Thus by the non-intersecting property, we have $X(\eta,t)=x$ for $\eta\in[0,y^*(x,t)]$ and $Y(\eta,t)=x$ for $\eta\in[0,\tau^*(x,t))]$. This implies $H_1(x,t)=0=H_2(x,t)$. Note that this is also true if $(x,t)$ lies in a rarefaction wave emanating from 0. In this case we have $y^*(x,t)=\tau^*(x,t)=0$ for all points in the interior of $I(t)$ and thus $H$ as well as $q$ is zero in these points. Further, $\Delta(l(t),t)$ contains $\{0\}\times[0,\tau^*(l(t),t)]$ and $\Delta(r(t),t)$ contains $[0,y^*(r(t),t)]\times\{0\}$, so $H$ is also zero in the boundary points of $I(t)$.

The proof in case $(c)$ then follows from these facts. Since $F(x_1,t)>G(x_1,t)$ and $F(x_2,t)\le$

$G(x_2,t)$, we have $I(t) \subset ]x_1,x_2]$. Write $x_3 = l(t)$, $x_4 = r(t)$. We split the integral in three (possibly only two) parts and use the results of cases $(a)$ and $(b)$ accordingly,

$$-\int_{x_1}^{x_2} q(x,t)dx = -\int_{[x_1,x_3[} q(x,t)dx - \int_{[x_3,x_4]} q(x,t)dx - \int_{]x_4,x_2]} q(x,t)dx =$$
$$= H_2(x_3-,t) - H_2(x_1,t) + 0 + H_1(x_2,t) - H_1(x_4+,t) = H(x_2,t) - H(x_1,t)\,.$$

Here we used the assertions of Remark 5.2.12, namely $\tau_*(x_3-,t) = \tau^*(x_3,t)$ and $y_*(x_4+,t) = y^*(x_4,t)$. The consideration above then allows us to conclude that $H_2(x_3-,t) = H_1(x_4+,t) = 0$.

Collecting all cases proves that $H_x = -q$ weakly for fixed $t$. Similar arguments can be used to show $H_t = 2E$. □

Now we conclude that again for a test function with compact support in $]0,\infty[^2$

$$\begin{aligned}0 = \iint H(x,t)(-\psi_{xtx} + \psi_{txx})dxdt &= \iint [H_x\psi_{tx}(x,t) - H_t\psi_{xx}(x,t)]dxdt = \\ &= \iint [-q(x,t)\psi_{tx}(x,t) - 2E\psi_{xx}(x,t)]dxdt = \\ &= \iint u(x,t)\psi_t(x,t)dmdt + \iint u^2(x,t)\psi_x(x,t)dmdt\,. \quad (5.3.22)\end{aligned}$$

Identity (5.3.22) proves that $(u,m)$ satisfies the second equation of the system (5.1.4). Combining (5.3.18) and (5.3.22) we proved the following theorem:

**Theorem 5.3.7.** *The functions $u$ and $m$ as given in Definiton 5.2.16 and 5.2.17respectively, are global solutions of* (5.1.4) *in the sense specified in the introduction. The functions $m$, $q$ and $E$ given in Definition 5.2.17 and Definition 5.3.3 are global weak solutions of system* (5.1.9) *on* $\mathbb{R}^2_+$.

*Proof.* The statement about solutions of (5.1.9) is clear from Lemma 5.3.5 and 5.3.6. Now by Lemma 5.3.4 these functions and $u$ are related in the correct way as Radon-Nikodym derivatives. Thus following the discussion in the introduction $u$ and $m$ are solutions of (5.1.4). □

Note that we did not discuss initial and boundary data yet. In the next section we will show that $\rho$, as the derivative of $m$, and $u$ satisfy the boundary and initial conditions (5.1.2), (5.1.3) in an appropriate sense.

## 5.4 Verification of initial and boundary condition

Now we turn our attention to the initial and boundary conditions. For that purpose, we define, as already discussed in the introduction, the Radon measure $\rho$ as the derivative of $m$. We also explicitly define the mass at $x = 0$ as the one-sided distributional derivative of $m$ and will show later in this chapter that this leads to conservation of mass.

**Definition 5.4.1.** *Let $m$ be as in Definition 5.2.17. Then we define*

$$\rho(x,t) = \begin{cases} \partial_x m & \text{for } x,t > 0 \\ \lim\limits_{x \searrow 0} \rho(x,t) & \text{for } x = 0 \text{ and } F(0,t) > G(0,t) \\ \delta \cdot \lim\limits_{x \searrow 0} (m(x,t) - m(0,t)) & \text{for } x = 0 \text{ and } F(0,t) \leq G(0,t)\,. \end{cases} \tag{5.4.1}$$

*Here $\delta$ is the Dirac measure, $\partial_x$ is the distributional derivative and $\rho$ is interpreted as a measure.*

**Lemma 5.4.2.** *For $u$ according to Definition 5.2.16 and for all $t > 0$ we have*

1. $F(0,t) < G(0,t) \Rightarrow u(0+,t) < 0$
2. $u(0+,t) < 0 \Rightarrow F(0,t) \leq G(0,t)\,.$

*Proof.* We start by proving the first implication. Since $F$ and $G$ are continuous in $(x,t)$, the statement $F(x,t) < G(x,t)$ holds true in a whole neighborhood of the point $(0,t)$. Now following Definition 5.2.16 in this neighborhood $u$ is defined as

$$u(x,t) = \begin{cases} \dfrac{x - y_*(x,t)}{t} & \text{if } y_*(x,t) = y^*(x,t) \\ \dfrac{\int_{y_*(x,t)}^{y^*(x,t)} \rho_0 u_0}{\int_{y_*(x,t)}^{y^*(x,t)} \rho_0} & \text{if } y_*(x,t) < y^*(x,t)\,. \end{cases}$$

In the first case we see that for $x \searrow 0$ the velocity $u$ becomes negative, apart from the case when $\lim_{x \searrow 0} y_*(x,t) = 0$. This is however impossible because that would lead to $F(y_*(0,t),0,t) = 0$ and thus $F(0,t) = 0 \geq G(0,t)$ contradicting the assumption.

In the second case, observing that $F(y_*(x,t),x,t) = F(y^*(x,t),x,t)$ and simplifying we get

$$\int_{y_*(x,t)}^{y^*(x,t)} (tu_0(\eta) + \eta - x)\rho_0(\eta)d\eta = 0.$$

This implies

$$tu(x,t) - x = -\frac{\int_{y_*(x,t)}^{y^*(x,t)} \eta\rho_0(\eta)d\eta}{\int_{y_*(x,t)}^{y^*(x,t)} \rho_0(\eta)d\eta} \leq -y_*(x,t).$$

Now passing to the limit as $x \searrow 0$, we obtain $tu(0+,t) \leq -y_*(0+,t) = y^*(0,t)$ by Remark 5.2.12. Since $G(0,t) \leq 0$ always and $F(0,t) < G(0,t)$ by assumption, we must have $y^*(0,t) > 0$ and hence $u(0+,t) < 0$.

In order to prove (2) first note that, for $x > 0$ we have that $\tau^*(x,t) < t$. This is due to the fact that at fixed $(x,t)$ the quantity $G(\tau,x,t)$ becomes increasing for $\tau > t - x/\|u_b\|_\infty$, as seen from the definition of $G$ and remembering that we assumed $\rho_b$ and $u_b$ to be positive. Thus the only cases in Definition 5.2.16 that can lead to negative $u(0+,t)$ are cases with $F < G$ or $F \leq G$. □

**Theorem 5.4.3.** *The pair* $(\rho, u)$ *as defined above solves equation* (5.1.1) *in* $\mathbb{R}^2_+$.

*The initial conditions are satisfied in the sense that for almost all* $x$ *we have* $\lim_{t \searrow 0} u(x,t) = u_0(x)$ *and* $\rho = \partial_x m$ *with* $\lim_{t \searrow 0} m(x,t) = \int_0^x \rho_0(y)dy$.

*The boundary condition is satisfied in regions where* $F(0,t) > G(0,t)$ *in the following sense: For almost all t we have* $\lim_{x \searrow 0} u(x,t) = u_b(t)$.

*If in addition* $u_b$ *is continuously differentiable and* $\rho_b$ *is locally Lipschitz continuous, then* $\lim_{x \searrow 0} \rho(x,t)u(x,t) = \rho_b(t)u_b(t)$.

*Proof.* The fact that $(\rho, u)$ is a solution is clear from the discussion in the introduction since they are derived from a solution $(m, u)$ of (5.1.4).

To prove the validity of the initial conditions observe that

$$\lim_{t\searrow 0} F(y,x,t) = \int_0^y (\eta - x)\rho_0(\eta)d\eta \tag{5.4.2}$$

and

$$\lim_{t\searrow 0} G(x,t) = 0. \tag{5.4.3}$$

The assumption $\rho_0 > 0$ implies that $F(x,t) < G(x,t)$ for small values of $t$. So the initial condition holds for $u$ by the arguments of Wang [72]. Since $y_*(x,t), y^*(x,t) \to x$ as $t \searrow 0$ , we get

$$\lim_{t\searrow 0} m(x,t) = \int_0^x \rho_0(y)dy.$$

Next we verify the boundary conditions. From our assumptions we have $F(0,t) > G(0,t)$ and by continuity, for $x$ close enough to zero, $G(x,t) < F(x,t)$. Let $t_0$ be a Lebesgue point of $u_b$ and $\rho_b$. We start by showing that the boundary conditions for $u$ hold, i.e. $\lim_{x\searrow 0} u(x,t_0) = u_b(t_0)$. Let $(x_n,t_0)$ be sequence converging to $(0,t_0)$. First assume that $\tau_*(x_n,t_0) = \tau^*(x_n,t_0) = \tau(x,t_0)$. In this case, the minimum is unique and thus for $h > 0$ (in fact, any $h \neq 0$) we have

$$G(\tau(x_n,t_0), x_n, t_0) < G(\tau(x_n,t_0) + hx_n, x_n, t_0).$$

This implies

$$\begin{aligned} x_n \int_{\tau(x_n,t_0)}^{\tau(x_n,t_0)+hx_n} \rho_b(\eta)u_b(\eta)d\eta &\geq \int_{\tau(x_n,t_0)}^{\tau(x_n,t_0)+hx_n} u_b^2(\eta)(t_0-\eta)\rho_b(\eta)d\eta \\ &\geq (t_0 - \tau(x_n,t_0) - hx_n) \int_{\tau(x_n,t)}^{\tau(x_n,t_0)+hx_n} u_b^2(\eta)\rho_b(\eta)d\eta. \end{aligned} \tag{5.4.4}$$

Simplifying (5.4.4) and using Definition5.2.16, we get

$$\Big(\frac{1}{u(x_n,t_0)} - h\Big)\frac{\int_{\tau(x_n,t_0)}^{\tau(x_n,t_0)+hx_n} u_b^2(\eta)\rho_b(\eta)d\eta}{\int_{\tau(x_n,t_0)}^{\tau(x_n,t_0)+hx_n} \rho_b(\eta)u_b(\eta)d\eta} \leq 1\,,$$

and as $n \to \infty$, we have

$$\limsup_{n\to\infty} \Big(\frac{1}{u(x_n,t_0)} - h\Big) u_b(t_0) \le 1\,. \tag{5.4.5}$$

Similarly considering the inequality

$$G(\tau(x_n,t_0), y, x_n, t_0) < G(\tau(x_n,t_0) - hx_n, y, x_n, t_0),$$

and following the same analysis as above, we find

$$\liminf_{n\to\infty} \Big(\frac{1}{u(x_n,t_0)} + h\Big) u_b(t_0) \ge 1. \tag{5.4.6}$$

Since $h$ is arbitrary positive number, inequalities (5.4.5) and (5.4.6) result in

$$\lim_{n\to\infty} \frac{1}{u(x_n,t_0)} u_b(t_0) = 1\,.$$

Thus we proved that $\lim_{n\to\infty} u(x_n,t_0) = u_b(t_0)$ implying continuity of $u$ up to the boundary and validity of the boundary condition in this case.

If $\tau_*(x_n,t_0) < \tau^*(x_n,t_0)$, then by Definition 5.2.16 we have

$$u(x_n,t_0) = \frac{\int_{\tau_*(x_n,t_0)}^{\tau^*(x_n,t_0)} \rho_b u_b^2}{\int_{\tau_*(x,t)}^{\tau^*(x,t)} \rho_b u_b}$$

So if $t_0$ is a Lebesgue point of $\rho_b u_b$ and $\rho_b u_b^2$, as is true for almost all $t_0$, we get $\lim_{n\to\infty} u(x_n,t_0) = u_b(t_0)$ again.

It remains to show that the boundary condition for $\rho$ holds. In this case, if the boundary potential

$$G(\tau,x,t) = \int_0^\tau [x - u_b(\eta)(t-\eta)]\rho_b(\eta)u_b(\eta)d\eta$$

attains the minimum in $(0,\infty)$ at a point $\bar\tau$, then $\frac{\partial G}{\partial \tau}|_{\tau=\bar\tau} = 0$. That is

$$\frac{\partial G}{\partial \tau}|_{\tau=\bar\tau} = (x - u_b(\bar\tau)(t-\bar\tau))\,\rho_b(\bar\tau)u_b(\bar\tau) = 0\,.$$

Since $\tau_*(x,t), \tau^*(x,t) \nearrow t$ as $x \searrow 0$, for $x$ close enough to $0$, the minimizing point lies in $(0,\infty)$. Consider the function $f : \mathbb{R}^+ \times [0,\infty) \to \mathbb{R}$ where $f$ is defined as

$$f(x,\tau) = x - u_b(\tau)(t-\tau)\,.$$

Then $\frac{\partial f}{\partial \tau} = -(t-\tau)u_b'(\tau) + u_b(\tau)$, $f(0,t) = 0$ and $\frac{\partial f}{\partial \tau}(0,t) > 0$. Thus by the implicit function theorem there exists a neighborhood of $(0,t)$ where $f(x,\tau) = 0$ has a unique solution. That gives the unique minimizer of the boundary potential $G(\tau,x,t)$ and thus $\tau_*(x,t)$ is a continuously differentiable function of $x$ in some neighborhood of $0$. Now from the relation $u(x,t) = \frac{x}{t-\tau_*(x,t)}$, we see that $\lim_{x\searrow 0} \frac{\partial}{\partial x}\tau_*(x,t) = -\frac{1}{u_b(t)}$. Consequently,

$$\lim_{x\searrow 0} \rho(x,t) = \lim_{x\searrow 0} \frac{\partial}{\partial x} m(x,t) = -\lim_{x\searrow 0} \rho_b(\tau_*(x,t))u_b(\tau_*(x,t))\frac{\partial}{\partial x}\tau_*(x,t) = \rho_b(t)\,.$$

This completes the verification of the initial and boundary condition for $(u,\rho)$. □

The next theorem concerns the global conservation of mass and momentum for our solutions.

**Theorem 5.4.4.** *Let $\rho_0 \in L^1([0,\infty))$, then the solution given in Definition 5.4.1 conserves the total mass, i.e.*

$$\forall t \geq 0\colon \int_0^\infty \rho(dx,t) = \int_0^\infty \rho_0(x)dx + \int_0^t \rho_b(\eta)u_b(\eta)d\eta\,.$$

*Moreover, the total momentum is conserved* but only for times without influx to the boundary. *More precisely we have for $t > 0$:*

$$\int_0^\infty u(x,t)\rho(dx,t) \begin{cases} = \displaystyle\int_0^\infty \rho_0(x)u_0(x)dx + \int_0^t \rho_b(\eta)u_b^2(\eta)d\eta & \text{if } F(0,t) \geq G(0,t) \\ \geq \displaystyle\int_0^\infty \rho_0(x)u_0(x)dx - \int_0^t \rho_b(\eta)u_b^2(\eta)d\eta & \text{if } F(0,t) < G(0,t)\,. \end{cases}$$

*Proof.* First we prove the mass conservation property. We start in the situation, when $F(0,t) > G(0,t)$. Since $F$ and $G$ are continuous functions, in a neighborhood of $(0,t)$

we have $F(x,t) > G(x,t)$ and thus $m(x,t) = -\int_0^{\tau_*(x,t)} u_b(\eta)\rho_b(\eta)d\eta$. Note that $\tau_*(x,t)$ is decreasing in $x$ and $m(x,t)$ is increasing in $x$. Moreover, $m(x,t)$ is upper semicontinuous in this case as we shall prove now.

For this purpose consider a sequence $z_n = (x_n, t_n) \to z = (x,t)$. Then,

$$\begin{aligned}\limsup_{n\to\infty} m(z_n) &= -\lim_{n\to\infty}\sup_{n\geq k}\int_0^{\tau_*(z_n)} \rho_b(\eta)u_b(\eta)d\eta = \\ &= -\lim_{n\to\infty}\int_0^{\sup_{n\geq k}\tau_*(z_n)} \rho_b(\eta)u_b(\eta)d\eta = -\int_0^{\lim_{n\to\infty}\sup_{n\geq k}\tau_*(z_n)} \rho_b(\eta)u_b(\eta)d\eta\,.\end{aligned}$$

By lower semicontinuity we have $\tau_*(z) \leq \liminf_{n\to\infty}\tau_*(z_n) \leq \limsup_{n\to\infty}\tau_*(z_n)$. Thus we conclude

$$\limsup_{n\to\infty} m(z_n) = -\int_0^{\lim_{n\to\infty}\sup_{n\geq k}\tau_*(z_n)} \rho_b(\eta)u_b(\eta)d\eta \leq -\int_0^{\tau_*(z)} \rho_b(\eta)u_b(\eta)d\eta = m(z)\,.$$

Now $m(x,t)$ is right continuous in $x$ since it is increasing and upper semicontinuous. This leads to the desired mass conservation for $F(0,t) > G(0,t)$,

$$\int_0^\infty \rho(dx,t) = \int_{x\in(0,\infty)} m(dx,t) = m(\infty,t) - m(0,t) = \int_0^\infty \rho_0(x)dx + \int_0^t u_b(\eta)\rho_b(\eta)d\eta\,.$$

For the other case when $F(0,t) \leq G(0,t)$ we decompose the integral in the following manner:

$$\begin{aligned}\int_0^\infty \rho(dx,t) &= m(0+,t) - m(0,t) + \int_{x\in(0,\infty)} m(dx,t) = m(\infty,t) - m(0,t) = \\ &= \int_0^\infty \rho_0(x)dx + \int_0^t u_b(\eta)\rho_b(\eta)d\eta\,.\end{aligned}$$

This finishes the proof of mass conservation.

Now we show momentum conservation. The momentum $\rho.u$ is understood as

$$\rho(x,t)u(x,t) = \begin{cases} um_x = q_x & \text{for } x,t>0 \\ \rho_b(t)u_b(t) & \text{for } x=0 \text{ and } F(0,t)>G(0,t) \\ 0 & \text{for } x=0 \text{ and } F(0,t)<G(0,t) \\ \rho(0,t)u(0,t) & \text{for } x=0 \text{ and } F(0,t)=G(0,t)\,. \end{cases} \tag{5.4.7}$$

Note that in the last case this is the Dirac mass at zero, multiplied by the non-zero velocity expected at this point. This is consistent with the point-wise interpretation as $\rho u$ up to the boundary if $u$ is as in Definition 5.2.16 and $\rho$ as in Definition 5.4.1. When $F(0,t) > G(0,t)$, by similar argument as in the proof of mass conservation one can show that $q(x,t)$ is right continuous in $x$. Hence we find

$$\begin{aligned}\int_0^\infty u\rho(dx,t) = \int_{x\in(0,\infty)} q(dx,t) = q(\infty,t) - q(0,t) = \\ = \int_0^\infty \rho_0(x)u_0(x)dx + \int_0^t \rho_b(\eta)u_b(\eta)d\eta\,.\end{aligned}$$

Now we consider the case $F(0,t) < G(0,t)$. In this case

$$\begin{aligned}\int_0^\infty u\rho(dx,t) = \int_{x\in(0,\infty)} q(dx,t) + 0 = q(\infty,t) - q(0+,t) = \\ = \int_0^\infty \rho_0(x)u_0(x)dx - \int_0^{y^*(0,t)} \rho_0(\eta)u_0(\eta)d\eta\,.\end{aligned}$$

For the last equality, Remark 5.2.12 was used. Since we are in the case $F(0,t) < G(0,t)$, we have

$$\int_0^{y_*(0,t)} (tu_0(\eta)+\eta)\rho_0(\eta)d\eta \leq -\int_0^t (t-\eta)u_b^2(\eta)\rho_b(\eta)d\eta\,,$$

leading to

$$\begin{aligned}0 \geq -t\int_0^t u_b^2(\eta)\rho_b(\eta)d\eta - \int_0^{y_*(0,t)} \eta\rho_0(\eta)d\eta \geq \\ \geq \int_0^{y_*(0,t)} tu_0(\eta)\rho_0(\eta)d\eta - \int_0^t \eta u_b^2(\eta)\rho_b(\eta)d\eta \geq \\ \geq t\left(\int_0^{y_*(0,t)} u_0(\eta)\rho_0(\eta)d\eta - \int_0^t \rho_b(\eta)u_b^2(\eta)d\eta\right)\,.\end{aligned}$$

This implies

$$\int_0^\infty \rho(x,t)u(x,t)dx \geq \int_0^\infty \rho_0(x)u_0(x)dx - \int_0^t \rho_b(\eta)u_b^2(\eta)d\eta\,.$$

Finally we have to consider the case $F(0,t) = G(0,t)$. First note that we have $y_*(0+,t) =$

$y^*(0,t)$, again by Remark 5.2.12. Now for $F(0,t) = G(0,t)$ we have

$$\int_0^\infty u\rho(dx,t) =$$
$$= \int_{x\in(0,\infty)} q(dx,t) + \lim_{x\searrow 0} \big(m(x,t) - m(0,t)\big) \frac{\int_0^{\tau^*(0,t)} \rho_b u_b^2 + \int_0^{y^*(0,t)} \rho_0 u_0}{\int_0^{\tau^*(0,t)} \rho_b u_b + \int_0^{y^*(0,t)} \rho_0} =$$
$$= q(\infty,t) - q(0+,t) + \left(\int_0^{y_*(0+,t)} \rho_0 d\eta + \int_0^t \rho_b u_b d\eta\right) \frac{\int_0^t \rho_b u_b^2 + \int_0^{y^*(0,t)} \rho_0 u_0}{\int_0^t \rho_b u_b + \int_0^{y^*(0,t)} \rho_0} =$$
$$= \int_0^\infty \rho_0(x) u_0(x) dx - \int_0^{y_*(0+,t)} \rho_0(\eta) u_0(\eta) d\eta + \int_0^t \rho_b u_b^2 d\eta + \int_0^{y^*(0,t)} \rho_0 u_0 d\eta =$$
$$= \int_0^\infty \rho_0(x) u_0(x) dx + \int_0^t \rho_b u_b^2 d\eta \,,$$

completing the proof. □

# 5.5 Entropy condition

In this section, we will show that the solution we constructed is an entropy solution up to the boundary.

**Theorem 5.5.1.** *Let $t > 0$ and $x > 0$. If $x$ is a point of discontinuity of $u(\cdot,t)$, then we have*

$$u(x-,t) > u(x,t) > u(x+,t)\,.$$

*Moreover, for almost all $t > 0$ we have that $u(\cdot,t)$ is either right continuous at $x = 0$ or*

$$u(0,t) > u(0+,t)\,.$$

*Proof.* First let $x > 0$, $t > 0$. If $(x,t)$ is a point where $F(x,t) < G(x,t)$ and $y_*(x,t) = y^*(x,t)$ then $u(x,t) = (x - y_*(x,t))/t$ is continuous at $(x,t)$. On the other hand if we have $y_*(x,t) < y^*(x,t)$ singularities can form and this case is considered in [72].
Now we look at points $(x,t)$, where $F(x,t) > G(x,t)$ and we have $\tau_*(x,t) < \tau^*(x,t)$. Using again Remark 5.2.12 we have that $u(x-,t) = \frac{x}{t-\tau^*(x,t)}$ and $u(x+,t) = \frac{x}{t-\tau_*(x,t)}$.

From

$$G(\tau_*(x,t),x,t) = G(\tau^*(x,t),x,t)\,,$$

we know that

$$\int_{\tau_*(x,t)}^{\tau^*(x,t)} [x - u_b(\eta)(t-\eta)]\rho_b(\eta)u_b(\eta)d\eta = 0\,.$$

This implies

$$\frac{x}{t-\tau_*(x,t)}\int_{\tau_*(x,t)}^{\tau^*(x,t)} \rho_b(\eta)u_b(\eta)d\eta = \int_{\tau_*(x,t)}^{\tau^*(x,t)} \frac{(t-\eta)}{t-\tau_*(x,t)}u_b^2(\eta)\rho_b(\eta)d\eta\,,$$

from which we conclude $u(x+,t) < u(x,t)$. The other inequality also follows easily if one divides the above equality by $(t-\tau^*(x,t))$.

The case $F(x,t) = G(x,t)$ is slightly more complicated. If $F(x,t) = G(x,t)$ in an interval then we are in the case of the rarefaction wave emanating from zero and the solution is continuous. Thus we can assume that the equality holds in an isolated point. As already observed above, we have that $u(x-,t) = \frac{x}{t-\tau^*(x,t)}$. Note that

$$F(y^*(x,t),x,t) - G(\tau^*(x,t),x,t) < F(y^*(x,t),0,\tau^*(x,t)) - G(\tau^*(x,t),0,\tau^*(x,t))\,,$$

since the term on the left side is zero. The inequality above yields

$$\int_0^{y^*(x,t)}\Big(\big(t-\tau^*(x,t)\big)u_0(\eta) - x\Big)\rho_0(\eta)d\eta < \\ < \int_0^{\tau^*(x,t)}\Big(x - \big(t-\tau^*(x,t)\big)u_b(\eta)\Big)\rho_b(\eta)u_b(\eta)d\eta$$

and dividing by $t-\tau^*(x,t)$ we get $u(x,t) < \frac{x}{t-\tau^*(x,t)}$, using the definition of $u(x,t)$.

To derive the inequality for $u(x+,t) = \frac{x-y^*(x,t)}{t}$ we proceed in a similar way, starting from

$$G(\tau^*(x,t),x,t) - F(y^*(x,t),x,t) < G(\tau^*(x,t),y^*(x,t),0) - F(y^*(x,t),y^*(x,t),0)\,.$$

This completes the proof in the interior.

It remains to consider the boundary $x = 0$. For almost all $t > 0$ such that $F(0,t) > G(0,t)$

we know from Theorem 5.4.3 that $\lim_{x\searrow 0} u(x,t) = u_b(t)$, that is $u(\cdot,t)$ is right continuous. For $F(0,t) < G(0,t)$, Lemma 5.4.2 implies that $u(0+,t) < 0$ while $u(0,t) = 0$ according to Definition 5.2.16. For the only remaining case $F(0,t) = G(0,t)$, the entropy condition follows from the same argument as in the interior, since $u(0,t)$ is defined in the same manner in this situation. □

## 5.6 Some explicit examples

First, we present an example that includes a rarefaction wave emerging from zero and a delta mass forming from the initial data due to a downward jump in the velocity. To make the construction of the solution easier to follow we present $\mu = \min(F,G)$ in the left half of Figure 5.7. The right half is the solution that we get from the derivatives of $\mu$. The boundary conditions $\rho_b = 1$, $u_b = 1$ are satisfied up to $t = 16/3$. At this time the $\delta$ hits the boundary and stays there, increasing its mass, as influx from the right continues. We also indicated the values of $\tau$ and $y$ in the right picture.

The next example, presented in Figure 5.8, is interesting since it is a case where a delta at zero forms driven by influx from the initial data. Moreover, the example is constructed in such a way that the influx from the initial manifold stops after some time while the boundary data continues to gives a positive momentum influx. Thus the delta indeed leaves zero when the influx of momentum from the boundary is sufficient. Note that the delta does not leave $x = 0$ with zero velocity. Thus our solution can be interpreted as one for a sticky boundary. Moreover, it can be seen from this that the solution does not satisfy a semi-group property. It could be restored by keeping the negative momentum in such cases, which however would lead to the momentum no longer being the product of mass and velocity. We choose not to do that for physical reasons - even if the momentum could be considered as a kind of dummy variable in this case. Note also that strictly speaking the situation depicted in Figure 5.8 is

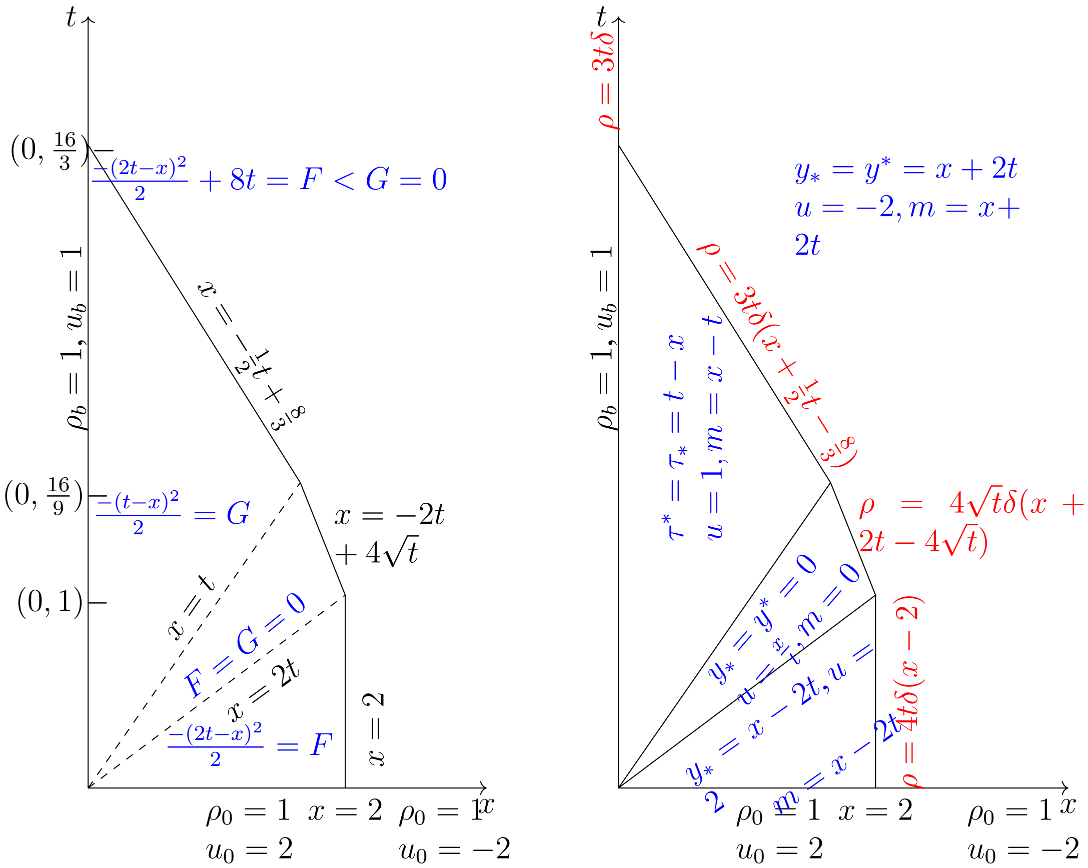


Figure 5.7: Rarefaction absorbed by shock generated from initial data and meets the boundary at finite time.

not covered by our theory since $\rho_0 = 0$ for $x > 2$ but the solution is similar if $\rho$ is very small for $x > 2$.

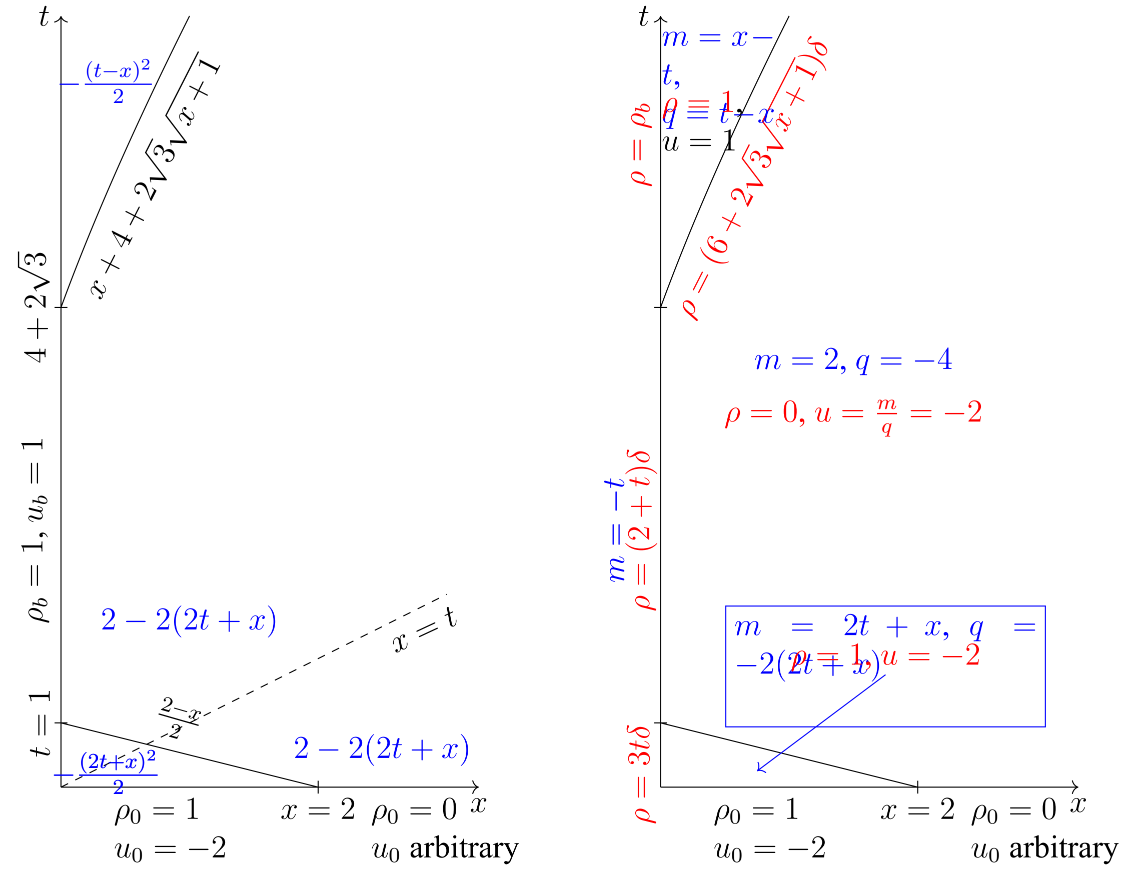


Figure 5.8: Left: Potential $\mu = \min\{F, G\}$ in blue. Right: Solution, calculated from potential according to Lemma 5.3.5 and Definition 5.4.1

In Figure 5.9 we give an example where a delta forms due to a jump up in the boundary velocity. Moreover, we included a jump down in the initial data and the two discontinuities merge into a single Dirac-mass. For the values, we choose the delta that does not reach the boundary again.

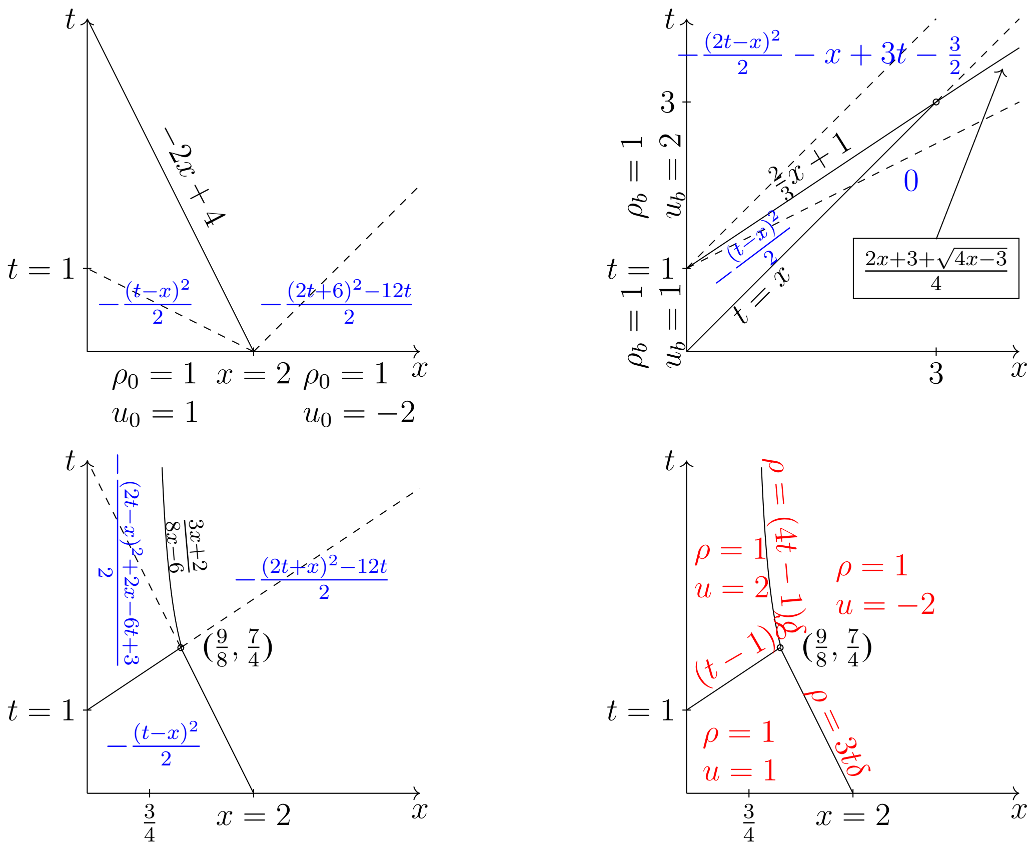


Figure 5.9: upper left: $F$, upper right: $G$, lower left: $\mu = \min\{F, G\}$, lower right: Solution satisfying initial and boundary conditions. The delta approaches $x = 3/4$ asymptotically for a large time.

Finally in figure 5.10 we show that rarefaction can generate from the initial and boundary values. The initial and boundary data are the following: initial data $\rho_0 = 1, u_0 = 3$ for $x < 1$ and $\rho_0 = 1, u_0 = 4$ for $x > 1$ and boundary data $\rho_b = 1, u_b = 2$ for $t < 1$ and $\rho_b = 1, u_b = 2$ for $t > 1$. Rarefactions are generated from the points $(x,t) = (1,0)$, $(x,t) = (0,0)$ and $(x,t) = (0,1)$.

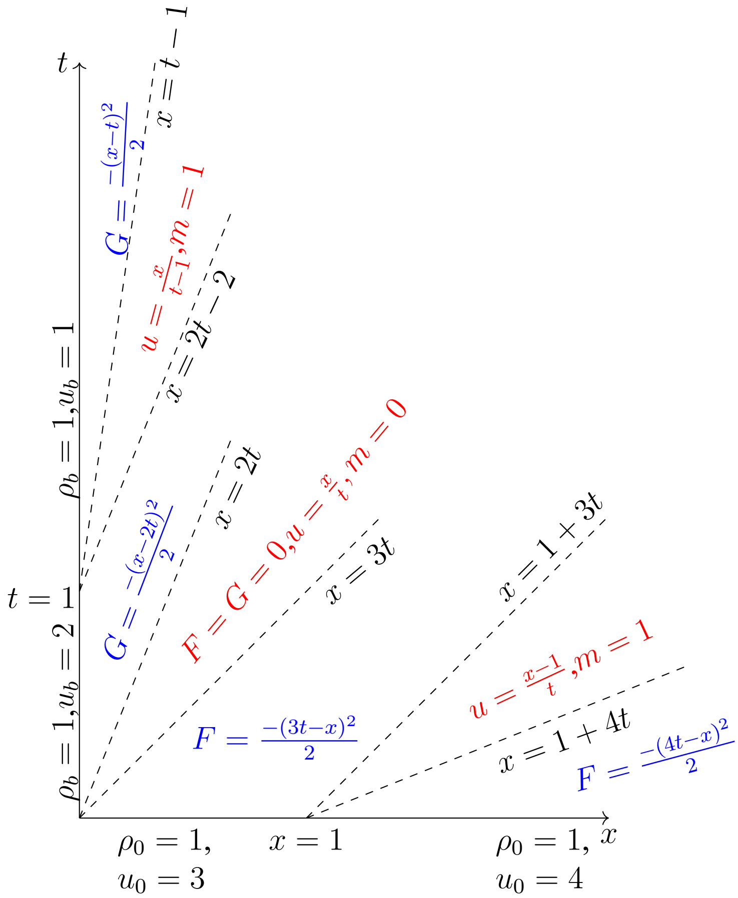


Figure 5.10: Rarefaction generated from initial and boundary data.

# Chapter 6

# Solutions with concentration for conservation laws with discontinuous flux and its applications to numerical schemes for hyperbolic systems

## 6.1 Introduction

The aim of this study is two-fold:

1. To look for solutions with concentration for the following scalar conservation law with discontinuous flux:

$$u_t + (F(x,u))_x = 0, \ \ (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{6.1.1}$$

$$u(x,0) = u_0(x), \ \ x \in \mathbb{R}, \tag{6.1.2}$$

where $F(x,u) = H(x)f(u) + (1 - H(x))g(u)$, $u_0$ is a bounded measurable function, $H$ is the Heaviside function, $g$ and $f$ are locally Lipschitz continuous functions on $\mathbb{R}$.

2. To use the above and propose convergent, conservative finite volume numerical schemes for the general class of hyperbolic systems of the following form:

$$v_t + (k(v))_x = 0, \tag{6.1.3}$$

$$u_t + (l(u)k^{'}(v))_x = 0, \tag{6.1.4}$$

which may not admit bounded weak solutions depending on the nature of the functions $k$ and $l$.

The system of the form (6.1.3)-(6.1.4) is physically important, for example, the Augmented Burgers system finds applications in cosmology and is closely linked to Zeldovich approximation, see [26]. According to this model, the evolution in the last stage of the expansion of the universe, the matter is described as cold dust moving under gravity alone and the laws are governed by the system (6.1.3)–(6.1.4) with $k(v) = \frac{v^2}{2}$ and $l(u) = u$ in one dimension. This system was studied extensively in the literature, see for example, [37] and the references therein.

For a hyperbolic system of $n$ equations of conservation laws given by

$$\begin{aligned} \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x &= 0, \quad x \in \mathbb{R}, t > 0, \\ \mathbf{u}(x, 0) &= \begin{cases} \mathbf{u}_L, & \text{if } x < 0, \\ \mathbf{u}_R, & \text{if } x > 0, \end{cases} \end{aligned}$$

it can be seen that in the absence of strict hyperbolicity and small total variation of the initial data, the system may not admit bounded solutions, for example, see Keyfitz-Kranzer system, [80], Augmented Burgers system [16], the system of Pressureless gas dynamics[51], the system of Pressureless gas dynamics with Coulomb friction [81] etc.

It is to be noted that since the system (6.1.3)-(6.1.4) admits $\delta$-shocks, see[16], exact or standard approximate Riemann solvers cannot be used. To this end, we aim to approximate this system using a finite volume type scheme, by viewing the system equation by equation. It was shown in the literature that this system admits $\delta$- shock whenever the first equation, i.e. the Burgers equation admits a classical shock. Since the first equation (6.1.3) of the system is a scalar nonlinear conservation law, any standard 3- point scheme such as Godunov scheme can be used for (6.1.3). However, the second equation is of the form

$$u_t + (k'(v(x,t))u)_x = 0, \;\; (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

if we assume that $v(x,t)$ is known at the point $(x,t)$. This is a scalar conservation law with linear flux $k'(v(x,t))u$ that has a variable coefficient $k'(v(x,t))$. It will be shown in this chapter that it can be solved numerically by considering a local Riemann problem at each numerical interface of the form (6.1.1)-(6.1.2). Depending on the nature of $g, f$, the IVP (6.1.1)-(6.1.2) has numerous physical applications in different fields such as two-phase flow in a discontinuous porous medium [82], sedimentation procedure [83], traffic flow on highways with different flow density [84], opinion dynamics [85], blood flow, the flow of gas in a non-constant channel, fabrication of semiconductor device [86] etc. The IVP (6.1.1)-(6.1.2) has been studied in the literature extensively. Various methods have established the existence of solutions, for example, vanishing viscosity method [87, 88, 89], front tracking method [90, 91], convergent numerical schemes [92, 93, 94, 95] and the references therein. In the context of the existence and uniqueness of entropy solutions, it has been observed in the literature that the geometry of flux functions $g$ and $f$ play an important role. In fact, the IVP (6.1.1)-(6.1.2) has been shown to have unique bounded solutions when $g$ and $f$ are functions with one critical point or they are monotone with the same monotonicity, see [63, 96, 97]. In fact, Hopf-Lax Type formulae were derived in some of these studies. However, the existence of bounded weak solution of the IVP (6.1.1)-(6.1.2) for the *overcomperessive flux pair* $(f, g)$ with $g$ increasing and $f$ decreasing remains currently unsettled. For example, with $g(u) = au, f(u) = bu,$ the existence of the solutions for the Riemann problem for the PDE (6.1.1)-(6.1.2) with the Riemann initial data

$$u_0(x) = u_l\chi_{\{x<0\}} + u_r\chi_{\{x>0\}},$$

can be handled using the existing literature of discontinuous flux, except for the case $a \geq 0, b \leq 0$. The case $a \geq 0, b \leq 0$, does not fit into the existing theory of conservation laws with discontinuous flux as, $g$ and $f$ do not have decreasing and increasing parts respectively to look for the intermediate connection values $A$ and $B$ such that $g(A) = f(B), g'(A) \leq 0, f'(B) \geq 0$, see [94] for details. As characteristics overlap each other at the interface $x = 0$, cases may arise when there may not exist a weak solution for the IVP (6.1.1)-(6.1.2) that satisfy the following weak formulation:

$$\int_0^\infty \int_{\mathbf{R}} (u(x)\phi_t(x,t) + F(x,u(x,t))\phi_x(x,t)dxdt + \int_{\mathbf{R}} u_0(x)\phi(x,0)dx = 0, \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}\times(0,\infty)). \tag{6.1.5}$$

In [95], the authors relaxed the R-H condition at the interface $x = 0$ to seek weak solution and proposed a non–conservative bounded entropy solution. The linear transport equation with variable coefficient has been studied in some earlier papers, for example, [68] for one-dimensional case and [98] for multi-dimensions but this case of overcompressive flux pair has not been dealt with. In view of mass conservation, a natural choice to define a solution, in this case, could be to look for a solution that conserves the mass but it may not be bounded. At this point, we propose a different concept of solution, the so-called $\delta$-shock type solution for the IVP (6.1.1)-(6.1.2). This paper proves that this proposed solution is a vanishing viscosity limit of an approximate parabolic PDE, that approximates (6.1.1)-(6.1.2). Moreover, it also shows a surprising connection between fractional differential equation and the IVP (6.1.1)-(6.1.2). Asymptotic behavior of this parabolic approximation is also studied in §6.4. The concept of solution with $\delta$-measure mainly arises in the existence theory of non-strictly hyperbolic systems. This type of unbounded solution was first introduced by Korchinski in his Ph.D. thesis[9], post which it was explored extensively in the literature. In this regard, we mention the vanishing viscosity method [12, 16, 27], weak asymptotic method [33, 99, 100], Colombeau generalized functions[36], shadow wave approach [18] and more recently, singular flux function limit [46, 58]. Numerical schemes for capturing $\delta$-shocks were proposed in, see for example, [101, 51, 102, 103, 104] and references therein. Also, it is to be noted that the numerical schemes for the hyperbolic systems admitting $\delta$-shocks fail to converge to the expected solution if the numerical schemes are constructed using bounded solutions proposed in [94]. It is interesting to note that the hyperbolic systems of Euler gas dynamics also admit $\delta$-shocks for suitable pressure $P$, [51, 17] which are of the form

$$\rho_t + (\rho u)_x = 0, \tag{6.1.6}$$

$$(\rho u)_t + (\rho u^2 + P(\rho))_x = 0. \tag{6.1.7}$$

These models work as a suitable mathematical approximation to calculate the lifting force on a wing of an airplane in aerodynamics, find presence in cosmology, and are used as possible models for dark energy. It is to be noted that the first equation of the system (6.1.6) is also of the form (6.1.1).

In this chapter, a numerical scheme for the system (6.1.3)-(6.1.4) will be proposed by proposing a scheme for the IVP (6.1.1)-(6.1.2). Since (6.1.1)-(6.1.2) is a type of linear transport equation with a discontinuous coefficient, for which the existence theory of solutions is not completely settled, this paper will also establish the notion of the solution along with its existence using the vanishing viscosity approach.

The rest of the chapter is organized as follows: In §6.2, we give the notion of *generalized* weak solution for conservation laws with discontinuous flux and discuss the uniqueness of the solutions. In §6.3 and §6.4, we derive the solution for linear overcompressive flux pair and show that this solution is obtained as a distributional limit of viscous approximation of the problem (6.1.1)-(6.1.2). Also, the asymptotic behavior of this viscous approximation is studied. In §6.5, we derive the Lax–Oleinik type formulae for the $\delta$- shock type solutions of (6.1.1)-(6.1.2) with non-linear overcompressive flux pair $(f, g)$. In §6.6, upwind numerical scheme for (6.1.1)-(6.1.2) is proposed and shown to converge analytically to the *generalized* weak solution. Some additional properties of the scheme are also established and the scheme is also extended to the system (6.1.3)-(6.1.4) and the convergence of the numerical scheme to the solution is established. The scheme is also extended to balance laws, i.e., (6.1.1)-(6.1.2) with source terms, both real-valued and measure-valued. Finally, in §6.7, numerical results are presented to exhibit the performance of the schemes.

## 6.2 Preliminaries

In this section, we collect some preliminary results and basic facts.

**Definition 6.2.1.** *(Interior Entropy Condition) For $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ and $f, g$ Lipschitz continuous, the interior entropy condition is given by,*

$$\begin{aligned} \partial_t|u-k| + \partial_x\left[sgn(u-k)(g(u)-g(k))\right] \le 0, \quad &for \quad x<0, \\ \partial_t|u-k| + \partial_x\left[sgn(u-k)(f(u)-f(k))\right] \le 0, \quad &for \quad x>0, \end{aligned} \tag{6.2.1}$$

*for all $k \in \mathbb{R}$, in the sense of distribution.*

**Lemma 6.2.2.** *Let $u, v \in L^\infty(\mathbb{R} \times [0,\infty)) \cap C([0,\infty); L^1_{loc}(\mathbb{R}))$ satisfying the interior entropy condition* (6.2.1) *and $u(x+,t), v(x+,t), u(x-,t)$ and $v(x-,t)$ exist, for almost every $t > 0$, where $u(x\pm,t) = \lim_{x\to 0^\pm} u(x,t)$. then, we have the following:*

$$\int_0^\infty \phi'(t)\left(\int_{a-Mt}^{b+Mt} |u(x,t)-v(x,t)|dx\right)dt \ge \int_0^\infty \Big(\tilde{g}(u^-(t),v^-(t)) - \tilde{f}(u^+(t),v^+(t))\Big)\phi(t)dt,$$

*where $\tilde{h}(a,b) = sgn(a-b)\big(h(a)-h(b)\big), h = g, f$.*

The above lemma is true for any Lipschitz continuous flux pair $(f,g)$ and is a consequence of integration by parts followed by a doubling of the variable technique. Proof can be found in, for example, [63].

**Theorem 6.2.3.** *The IVP* (6.1.1)-(6.1.2) *with an overcomperessive flux pair $(f,g)$ admits at most one weak solution* (6.1.5) *satisfying the hypothesis of the Lemma6.2.2.*

*Proof.* For $f$ decreasing and $g$ increasing, $\tilde{f}(a,b) \le 0$ and $\tilde{g}(a,b) \ge 0$. Suppose $u$ and $v$ are two weak solutions, then from Lemma 6.2.2, we have the following,

$$\int_0^\infty \phi'(t)\left(\int_{a-Mt}^{b+Mt} |u(x,t)-v(x,t)|dx\right)dt \ge 0,$$

which implies $t \mapsto \int\limits_{a-Mt}^{b+Mt} |u(x,t) - v(x,t)|dx$ is non increasing and leads to the $L^1$ contraction inequality,

$$\int\limits_{a-Mt}^{b+Mt} |u(x,t) - v(x,t)|dx \leq \int\limits_{a}^{b} |u_0(x) - v_0(x)|dx,$$

which implies the $u = v$ a.e. □

The above theorem talks about uniqueness of the solutions of (6.1.1)-(6.1.2) given that the weak solutions exist. However, bounded weak solution satisfying (6.1.5) does not exist in general for $u_0 \in L^\infty(\mathbb{R})$. For example, with $g(u) = u, f(u) = -u, u_0(x) = k \neq 0$, (6.1.1)-(6.1.2) does not have any bounded weak solution satisfying (6.1.5). In this paper, we look for measure-valued solutions which are solutions of the IVP (6.1.1)-(6.1.2) in the following sense:

**Definition 6.2.4.** *A measure valued function $u(x,t) = \overline{u}(x,t) + w(t)\delta_{\{x=0\}}$ is said to be a generalized weak solution of the IVP* (6.1.1)-(6.1.2) *if the following holds:*

$$\int_0^\infty \int_{\mathbf{R}} (\overline{u}(x)\phi_t(x,t) + F(x,\overline{u}(x,t))\phi_x(x,t)dxdt + \int_0^\infty w(t)\phi_t(0,t)dt$$
$$+ \int_{\mathbf{R}} u_0(x)\phi(x,0)dx = 0, \forall \phi \in C_c^\infty(\mathbb{R} \times (0,\infty)),$$

*where $\overline{u}$ and $w$ are locally bounded.*

**Theorem 6.2.5.** *If $u_1(x,t) = \overline{u}_1(x,t) + w_1(t)\delta_{\{x=0\}}$ and $u_2(x,t) = \overline{u}_2(x,t) + w_2(t)\delta_{\{x=0\}}$ are two generalized weak solutions of* (6.1.1)-(6.1.2)*, in the sense of* (6.2.4)*. Suppose $\overline{u}_1$ and $\overline{u}_2$ satisfy the hypothesis of the Lemma 6.2.2 and $w_1, w_2 \in C([0,T];\mathbb{R})$, then $\overline{u}_1 = \overline{u}_2$ a.e. and $w_1 = w_2$.*

*Proof.* Note that $\bar{u}_1$ and $\bar{u}_2$ are bounded solutions of the conservation laws away from the interface. Thus $\bar{u}_1$ and $\bar{u}_2$ satisfies interior entropy condition and a proof can be found in

[63]. Hence by Lemma 6.2.2 implies that $\overline{u}_1 = \overline{u}_2$. From Definition 6.2.4, we have

$$\int_0^\infty (w_1(t) - w_2(t))\, \phi_t(0,t)dt = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R} \times (0,\infty)),$$

which implies $w_1(t) - w_2(t) = C$, C, some constant. Since $w_1(0) = w_2(0) = 0$, we have $w_1 = w_2$. □

We now give the generalized weak solution for (6.1.1)-(6.1.2) with overcompressive flux pair $(f, g)$ and start with the case when $g$ and $f$ are linear.

## 6.3 Generalized weak solution with linear overcomperessive flux pair

Let $g(u) = au, f(u) = bu, a \geq 0, b \leq 0$. To propose the generalized weak solution for (6.1.1)-(6.1.2), we consider the following nonlinear version of (6.1.1)-(6.1.2):
For an $\epsilon > 0$, consider

$$\begin{aligned} u_t + (F_\epsilon(x,u))_x &= 0, \ \ (x,t) \in \mathbb{R} \times \mathbb{R}^+, && (6.3.1)\\ F_\epsilon(x,u) &= H(x)f_\epsilon(u) + (1-H(x))g_\epsilon(u), && (6.3.2)\\ g_\epsilon(u) &= au + \frac{\epsilon}{2}(au)^2, && (6.3.3)\\ f_\epsilon(u) &= bu + \frac{\epsilon}{2}(bu)^2, && (6.3.4)\\ u_0(x) &= u_l\chi_{\{x<0\}} + u_r\chi_{\{x>0\}}. && (6.3.5) \end{aligned}$$

It is clear that $f_\epsilon$ and $g_\epsilon$ are convex functions and the above PDE has a unique entropic solution with any Riemann data $(u_l, u_r)$ owing to the theory of discontinuous flux of [94]. For example, in the case where $g_\epsilon(u_l) \geq f_\epsilon(u_r)$:

$$u^\epsilon(x,t) = \begin{cases} u_l, & \text{if } \ x < 0, \\ \dfrac{-1}{b}\left(\dfrac{2}{\epsilon} + au_l\right), & \text{if } \ 0 < x < s_\epsilon t, \\ u_r, & \text{if } \ x > 0, \end{cases},$$

where

$$s_\epsilon = \frac{-\epsilon b}{2}(au_l - bu_r).$$

For any time $t > 0$, we have

$$\int_0^{s_\epsilon t} u^\epsilon(x,t)dx = (au_l - bu_r)(1 + au_l\frac{\epsilon}{2}),$$

which as $\epsilon \to 0$, converges to $au_l - bu_r$. Also, note that the pointwise limit

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = \overline{u}(x,t) := u_l\chi_{\{x<0\}} + u_r\chi_{\{x>0\}}, \tag{6.3.6}$$

is, in fact, the solution proposed in [94]. It is worthwhile noting that this pointwise limit in (6.3.6) does not respect the conservation of mass in the interval $[\alpha,\beta], \alpha < 0 < \beta$ as

$$0 = \frac{d}{dt}\int_\alpha^\beta u(x,t)dx \neq -bu_r + au_l,$$

instead, the weak convergence of $\{u_\epsilon\}_{\{\epsilon>0\}} \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R})$ in the space of signed Radon measures gives

$$u(x,t) := \overline{u}(x,t) + t(au_l - bu_r)\delta_{\{x=0\}}(x,t),$$

takes care of the missing mass $u_r$ by concentrating it at the point $x = 0$, through the term $t(au_l - bu_r)\delta_{\{x=0\}}(x)$. Since we are looking for solutions which satisfies the conservation principle for (6.1.1)-(6.1.2), it motivates to look for solutions of the type

$$\overline{u}(x,t) + w(t)\delta_{\{x=0\}}(x) \tag{6.3.7}$$

so that the missing mass $(au_l - bu_r)t$ is represented by the time-dependent function $w$. The weak limit stated in the equation (6.3.7) solves the problem (6.1.1)-(6.1.2) in the sense of Definition 6.2.4. It is important to note that the solutions of the problem (6.1.1)-(6.1.2) with any other sign of $a, b$ can also be seen as vanishing $\epsilon-$ limit of the solutions of the same nonlinear version (6.3.1)-(6.3.5).

Similar calculations allow us to consider the IVP (6.1.1)-(6.1.2) with a source term, namely

$$u_t + (bH(x)u + a(1 - H(x))u)_x = k(t)u.$$

For example, for $a = 1, b = -1$, $V(x,t) = \exp\left(-\int_0^t k(s)ds\right), u(x,t)$ satisfies the homogeneous equation

$$V_t + (-H(x)V + (1 - H(x))V)_x = 0, \tag{6.3.8}$$

which gives,

$$u(x,t) = \begin{cases} \exp\left(\int_0^t k(s)ds\right) u_0(x+t), & \text{if } \; x > 0, t > 0, \\ \exp\left(\int_0^t k(s)ds\right) [U_0(t) - U_0(-t)]\delta_{\{x=0\}}, & \text{if } \; x = 0, \\ \exp\left(\int_0^t k(s)ds\right) u_0(x-t) & \text{if } \; x < 0, t > 0. \end{cases}$$

We can also consider (6.1.1)-(6.1.2) with point source,

$$u_t + F(x,u)_x = S(t)\delta_{\{x=0\}},$$

for which the weak formulation is given by

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}} (u(x,t)\phi_t(x,t) + F(x,u(x,t))\phi_x(x,t)dxdt + \int_0^t w(t)\phi_t(0,t)dt \\ &+ \int_{\mathbf{R}} u_0(x)\phi(x,0)dx = \int_0^t S(t)\phi(0,t)dt, \quad \forall \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+ \cup \{0\}). \end{aligned} \tag{6.3.9}$$

If $\overline{u}(x,t) + w(t)\delta_{\{x=0\}}$ is the solution of the problem

$$u_t + F(x,u)_x = 0,$$

subjected to the initial data $u_0 \in L^\infty(\mathbb{R})$, then the problem

$$u_t + F(x,u)_x = -\dot{w}(t)\delta_{\{x=0\}}(x),$$

has a bounded solution for the same initial data. Physically, this can be interpreted as the mass concentration at $x = 0$ can be avoided by keeping a negative point source (sink) of a suitable strength at $x = 0$.

In this section, we show that the previously stated solution can be obtained as a limit of the vanishing viscosity of the solutions of viscous formulation of the equation (6.1.1)-(6.1.2).

## 6.4 Distributional Solution of Linear Transport Equation as a Vanishing Viscosity Limit

In this section, we obtain the previously stated solution as a limit of the vanishing viscosity of the solutions of viscous formulation of the equation (6.1.1)-(6.1.2)

$$\begin{aligned} &u_t + F(x,u)_x = \epsilon u_{xx}, \\ &u(x,0) = u_0(x). \end{aligned} \tag{6.4.1}$$

In the section §6.4.1, we obtain the solution of (6.4.1), which is the parabolic approximation of (6.1.1)-(6.1.2) and in the section §6.4.2, we show that the vanishing viscosity limit of these solutions is the distributional solution of (6.1.1)-(6.1.2).

### 6.4.1 Continuous Solution of the Parabolic Approximation (6.4.1)

The weak formulation of the problem (6.4.1) is given as follows:

**Definition 6.4.1.** *A function $u \in \mathbf{C^0}(\mathbb{R} \times \mathbb{R}^+; \mathbb{R})$ is a weak solution of the equation* (6.4.1) *if the following integral identity holds*

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty (u\phi_t - \epsilon u_x \phi_x) dx dt + \int_0^\infty \int_{-\infty}^\infty F(x,u)\phi_x dx dt \\ &\quad + \int_0^\infty [f(u(0,t) - g(u(0,t))]\phi(0,t) dt + \int_{-\infty}^\infty u_0(x)\phi(x,0) dx = 0, \end{aligned} \tag{6.4.2}$$

*for all $\phi \in \mathbf{C_c^\infty}(\mathbb{R} \times [0,\infty))$.*

For simplicity, assume $a = 1, b = -1$. In this section, the explicit solution for the problem (6.4.1) will be derived in the above sense and in the next section, we show that as $\epsilon \to 0$, the limit is a measure and satisfies the weak formulation stated in Definition6.2.4. The next theorem gives us an explicit formula for the solution for the parabolic approximation of discontinuous flux problem. We consider the problem in two separate quarter planes with a boundary function $\beta(t)$. Then the idea is to study two boundary value problems separately and in the end we found an explicit formulation for the boundary function $\beta(t)$ which will be useful to study asymptotic behaviour later.

**Theorem 6.4.2.** *The explicit solution of the equation* (6.4.1) *in the sense of the Definition* (6.4.2) *is given by,*

$$u(\epsilon, x, t) = u^1(\epsilon, x, t)\chi_{\{x>0,t>0\}} + u^2(\epsilon, x, t)\chi_{\{x<0,t>0\}}, \tag{6.4.3}$$

*with*

$$\begin{aligned} &u^1(\epsilon, x, t) \\ &= \frac{1}{2\sqrt{\pi\epsilon t}} \int_0^\infty u_0(\xi)\Big[\exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t}\right) - \exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t} - \frac{x}{\epsilon}\right)\Big]d\xi \\ &\quad + \frac{1}{2\sqrt{\pi\epsilon}} \int_0^t \frac{\beta(\tau)x}{(t-\tau)^{3/2}} \exp\left(-\frac{(x+t-\tau)^2}{4\epsilon(t-\tau)}\right) d\tau \end{aligned} \tag{6.4.4}$$

*and*

$$\begin{aligned} &u^2(\epsilon, x, t) \\ &= \frac{1}{2\sqrt{\pi\epsilon t}} \int_0^\infty u_0(-\xi)\Big[\exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t}\right) - \exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t} + \frac{x}{\epsilon}\right)\Big]d\xi \\ &\quad - \frac{1}{2\sqrt{\pi\epsilon}} \int_0^t \frac{\beta(\tau)x}{(t-\tau)^{3/2}} \exp\left(-\frac{(x-t+\tau)^2}{4\epsilon(t-\tau)}\right) d\tau, \end{aligned} \tag{6.4.5}$$

*where*

$$\begin{aligned} \beta(t) = \frac{1}{2}\int_0^t (t-\tau)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}\Big(-\frac{1}{2}\sqrt{\frac{t-\tau}{\epsilon}}\Big)\Big[\sqrt{\epsilon}\exp\left(-\frac{t-\tau}{4\epsilon}\right) F(\tau) d\tau \\ - \int_0^t \frac{2\beta(0)}{\sqrt{\pi}}\frac{1}{\sqrt{\tau}}\exp\left(-\frac{t}{4\epsilon}\right)\Big]d\tau, \end{aligned} \tag{6.4.6}$$

*with $E_{\alpha,\tilde{\alpha}}$ being the Mittag-Leffler function [105] with $\alpha, \tilde{\alpha} > 0$ and $F(.)$ be defined in* (6.4.24).

*Proof.* The above problem (6.4.1) can be split into the following two initial-boundary value problems:

$$u^1_t - u^1_x = \epsilon u^1_{xx}, x > 0, t > 0, \qquad u^1(x,0) = u_0(x), u^1(0,t) = \beta(t), \tag{6.4.7}$$

and

$$u^2_t + u^2_x = \epsilon u^2_{xx}, \;\; x < 0, t > 0, \qquad u^2(x,0) = u_0(x), u^2(0,t) = \beta(t). \tag{6.4.8}$$

The problem (6.4.7) has the explicit formula[106, sec. 1.1.4]:

$$u^1(\epsilon, x, t) = \int_0^\infty u_0(\xi) G(x,\xi,t) d\xi + \epsilon \int_0^t \beta(\tau)\lambda(x, t-\tau) d\tau \tag{6.4.9}$$

with

$$\begin{aligned} G(x,\xi,t) = &\; \frac{1}{2\sqrt{\pi\epsilon t}} \exp\left(\frac{\xi - x}{2\epsilon} - \frac{t}{4\epsilon}\right)\left[\exp\left(\frac{-(x-\xi)^2}{4\epsilon t}\right) - \exp\left(\frac{-(x+\xi)^2}{4\epsilon t}\right)\right] \\ = &\; \frac{1}{2\sqrt{\pi\epsilon t}}\Big[\exp\left(-\frac{(\xi - (x+t))^2}{4\epsilon t}\right) - \exp\left(-\frac{(\xi + (x-t))^2}{4\epsilon t} - \frac{x}{\epsilon}\right)\Big] \end{aligned} \tag{6.4.10}$$

and

$$\lambda(x,t) = \frac{x}{2\sqrt{\pi}(\epsilon t)^{3/2}} \exp\left(-\Big(\frac{x}{2\epsilon} + \frac{t}{4\epsilon} + \frac{x^2}{4\epsilon t}\Big)\right) = \frac{x}{2\sqrt{\pi}(\epsilon t)^{3/2}} \exp\left(-\frac{(x+t)^2}{4\epsilon t}\right). \tag{6.4.11}$$

Now substituting $G(x,\xi,t)$ and $\lambda(x,t-\tau)$ in (6.4.9), we get the required expression (6.4.4). Now, simplifying the second integral involving $\beta(\tau)$) from (6.4.4), we have

$$\frac{1}{2\sqrt{\pi\epsilon}} \int_0^t \frac{\beta(\tau)x}{(t-\tau)^{3/2}} \exp\left(-\frac{(x+t-\tau)^2}{4\epsilon(t-\tau)}\right) d\tau = \frac{1}{\sqrt{\pi}} \int_0^t \beta(\tau)\frac{dL_1}{d\tau} d\tau, \tag{6.4.12}$$

where

$$L_1 = \mathbf{erf}\left(\frac{x+t-\tau}{2\sqrt{\epsilon(t-\tau)}}\right) + \mathbf{erf}\Big(\frac{x-(t-\tau)}{2\sqrt{\epsilon(t-\tau)}}\Big) \;\exp\left(-\frac{x}{\epsilon}\right),$$

where

$$\mathbf{erf}(z) = \int_0^z \exp\left(-s^2\right) ds, \mathbf{erf}(\infty) = \lim_{z\to\infty} \mathbf{erf}(z).$$

Now integrating (6.4.12) by parts, we have,

$$\begin{aligned}
&\frac{1}{2\sqrt{\pi\epsilon}}\int_0^t \frac{\beta(\tau)x}{(t-\tau)^{3/2}}\exp\Big(-\frac{(x+t-\tau)^2}{4\epsilon(t-\tau)}\Big)\,d\tau\\
&=\frac{-1}{\sqrt{\pi}}\int_0^t \beta'(\tau)\Big[\mathbf{erf}\Big(\frac{x+t-\tau}{2\sqrt{\epsilon(t-\tau)}}\Big)+\mathbf{erf}\Big(\frac{x-(t-\tau)}{2\sqrt{\epsilon(t-\tau)}}\Big)\exp\Big(-\frac{x}{\epsilon}\Big)\Big]d\tau\\
&\quad+\frac{\beta(t)}{\sqrt{\pi}}\Big[\mathbf{erf}(\infty)+\mathbf{erf}(\infty)\exp\Big(-\frac{x}{\epsilon}\Big)\Big]\\
&\quad-\frac{\beta(0)}{\sqrt{\pi}}\Big[\mathbf{erf}\Big(\frac{x+t}{2\sqrt{\epsilon t}}\Big)+\mathbf{erf}\Big(\frac{x-t}{2\sqrt{\epsilon t}}\Big)\exp\Big(-\frac{x}{\epsilon}\Big)\Big]
\end{aligned}\tag{6.4.13}$$

Therefore, $u^1(\epsilon,x,t)=$

$$\begin{aligned}
&\frac{1}{2\sqrt{\pi\epsilon t}}\times\int_0^\infty u_0(\xi)\Big[\exp\Big(-\frac{(\xi-(x+t))^2}{4\epsilon t}\Big)-\exp\Big(-\frac{(\xi+(x-t))^2}{4\epsilon t}-\frac{x}{\epsilon}\Big)\Big]d\xi\\
&\frac{-1}{\sqrt{\pi}}\int_0^t \beta'(\tau)\Big[\mathbf{erf}\Big(\frac{x+t-\tau}{2\sqrt{\epsilon(t-\tau)}}\Big)+\mathbf{erf}\Big(\frac{x-(t-\tau)}{2\sqrt{\epsilon(t-\tau)}}\Big)\exp\Big(-\frac{x}{\epsilon}\Big)\Big]d\tau\\
&+\frac{1}{\sqrt{\pi}}\beta(t)\Big[\mathbf{erf}(\infty)+\mathbf{erf}(\infty)\exp\Big(-\frac{x}{\epsilon}\Big)\Big]\\
&-\frac{\beta(0)}{\sqrt{\pi}}\Big[\mathbf{erf}\Big(\frac{x+t}{2\sqrt{\epsilon t}}\Big)+\mathbf{erf}\Big(\frac{x-t}{2\sqrt{\epsilon t}}\Big)\exp\Big(-\frac{x}{\epsilon}\Big)\Big].
\end{aligned}\tag{6.4.14}$$

Differentiating $u^1$ with respect to $x$, we get

$$\begin{aligned}
u_x^1(\epsilon,x,t)=&\frac{1}{2\sqrt{\pi\epsilon t}}\int_0^\infty u_0(\xi)\Big[\exp\Big(-\frac{(\xi-(x+t))^2}{4\epsilon t}\Big)\Big(\frac{\xi-(x+t)}{2\epsilon t}\Big)\\
&+\exp\Big(-\frac{(\xi+(x-t))^2}{4\epsilon t}-\frac{x}{\epsilon}\Big)\Big(\frac{\xi+x-t}{2\epsilon t}+\frac{1}{\epsilon}\Big)\Big]d\xi\\
&-\frac{1}{\sqrt{\pi}}\int_0^t \beta'(\tau)\Big[\exp\Big(-\frac{(x+t-\tau)^2}{4\epsilon(t-\tau)}\Big)\frac{1}{\sqrt{\epsilon(t-\tau)}}\\
&-\frac{1}{\epsilon}\mathbf{erf}\Big(\frac{x-(t-\tau)}{2\sqrt{\epsilon(t-\tau)}}\Big)\exp\Big(-\frac{x}{\epsilon}\Big)\Big]d\tau-\frac{\beta(t)}{\sqrt{\pi\epsilon}}\mathbf{erf}(\infty)\exp\Big(-\frac{x}{\epsilon}\Big)\\
&-\frac{\beta(0)}{\sqrt{\epsilon\pi t}}\Big[\exp\Big(-\frac{(x+t)^2}{4\epsilon t}\Big)-\sqrt{\frac{t}{\epsilon}}\mathbf{erf}\Big(\frac{x-t}{2\sqrt{\epsilon t}}\Big)\exp\Big(\frac{-x}{\epsilon}\Big)\Big].
\end{aligned}\tag{6.4.15}$$

Similarly, let us now consider the problem (6.4.8). Using the transformation $v(\epsilon,x,t)=u^2(\epsilon,-x,t)$, for $x>0$, we observe from (6.4.8) that $v(x,t)$ satisfies

$$\begin{aligned}
&v_t-v_x=\epsilon v_{xx},\ \ x>0,\ \ t>0,\\
&v(x,0)=u_0(-x),\ \ x>0,\ \ v(0,t)=\beta(t),\ \ t>0.
\end{aligned}\tag{6.4.16}$$

The explicit formula for equation (6.4.16) is then given by[106, sec. 1.1.4]:

$$v(x,t) = \int_0^\infty v_0(\xi)G(x,\xi,t)d\xi + \epsilon\int_0^t \beta(\tau)\lambda(x,t-\tau)d\tau. \tag{6.4.17}$$

where

$$\begin{aligned} G(x,\xi,t) = & \frac{1}{2\sqrt{\pi\epsilon t}}\exp\left(\frac{\xi-x}{2\epsilon}-\frac{t}{4\epsilon}\right)\left[\exp\left(\frac{-(x-\xi)^2}{4\epsilon t}\right)-\exp\left(\frac{-(x+\xi)^2}{4\epsilon t}\right)\right] \\ = & \frac{1}{2\sqrt{\pi\epsilon t}}\Big[\exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t}\right)-\exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t}-\frac{x}{\epsilon}\right)\Big] \end{aligned} \tag{6.4.18}$$

and

$$\lambda(x,t) = \frac{x(\epsilon t)^{-3/2}}{2\sqrt{\pi}}\exp\left(-\Big(\frac{x}{2\epsilon}+\frac{t}{4\epsilon}+\frac{x^2}{4\epsilon t}\Big)\right) = \frac{x(\epsilon t)^{-3/2}}{2\sqrt{\pi}}\exp\left(-\frac{(x+t)^2}{4\epsilon t}\right). \tag{6.4.19}$$

Now substituting $G(x,\xi,t)$ and $\lambda(x,t-\tau)$ in (6.4.17), we get $v(\epsilon,x,t) =$

$$\begin{aligned} &\frac{1}{2\sqrt{\pi\epsilon t}}\times\int_0^\infty v_0(\xi)\Big[\exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t}\right)-\exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t}-\frac{x}{\epsilon}\right)\Big]d\xi \\ &\quad+\frac{1}{2\sqrt{\pi\epsilon}}\int_0^t\frac{\beta(\tau)x}{(t-\tau)^{3/2}}\exp\left(-\frac{(x+t-\tau)^2}{4\epsilon(t-\tau)}\right)d\tau. \end{aligned} \tag{6.4.20}$$

Replacing $x$ by $-x$, we get (6.4.5) and further simplification gives $u^2(\epsilon,x,t) =$

$$\begin{aligned} &\frac{1}{2\sqrt{\pi\epsilon t}}\times\int_0^\infty u_0(-\xi)\Big[\exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t}\right)-\exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t}+\frac{x}{\epsilon}\right)\Big]d\xi \\ &\quad\frac{-1}{\sqrt{\pi}}\int_0^t\beta'(\tau)\Big[\mathbf{erf}\Big(\frac{-x+t-\tau}{2\sqrt{\epsilon(t-\tau)}}\Big)+\mathbf{erf}\Big(\frac{-x-(t-\tau)}{2\sqrt{\epsilon(t-\tau)}}\Big)\exp\left(\frac{x}{\epsilon}\right)\Big]d\tau \\ &\quad+\frac{1}{\sqrt{\pi}}\beta(t)\Big[\mathbf{erf}(\infty)+\mathbf{erf}(\infty)\exp\left(\frac{x}{\epsilon}\right)\Big] \\ &\quad-\frac{\beta(0)}{\sqrt{\pi}}\Big[\mathbf{erf}\Big(\frac{-x+t}{2\sqrt{\epsilon t}}\Big)+\mathbf{erf}\Big(\frac{-x-t}{2\sqrt{\epsilon t}}\Big)\exp\left(\frac{x}{\epsilon}\right)\Big]. \end{aligned} \tag{6.4.21}$$

Differentiating $u^2$ in (6.4.21) with respect to $x$ and following a similar analysis as before, we get

$$
\begin{aligned}
-u_x^2(\epsilon,x,t) = \frac{1}{2\sqrt{\pi\epsilon t}}\int_0^\infty u_0(-\xi)\Big[\exp\Big(-\frac{(\xi+(x-t))^2}{4\epsilon t}\Big)\Big(\frac{(\xi+(x-t))}{2\epsilon t}\Big)\\
+\exp\Big(-\frac{(\xi-(x+t))^2}{4\epsilon t}+\frac{x}{\epsilon}\Big)\Big(\frac{\xi-x-t}{2\epsilon t}+\frac{1}{\epsilon}\Big)\Big]d\xi\\
-\frac{1}{\sqrt{\pi}}\int_0^t \beta'(\tau)\Big[\exp\Big(-\frac{(x-(t-\tau))^2}{4\epsilon(t-\tau)}\Big)\Big(\frac{1}{\sqrt{\epsilon(t-\tau)}}\Big)\\
+\frac{1}{\epsilon}\mathbf{erf}\Big(\frac{x+(t-\tau)}{2\sqrt{\epsilon(t-\tau)}}\Big)\exp\Big(\frac{x}{\epsilon}\Big)\Big]d\tau-\frac{\beta(t)}{\sqrt{\pi\epsilon}}\mathbf{erf}(\infty)\exp\Big(\frac{x}{\epsilon}\Big)\\
-\frac{\beta(0)}{\sqrt{\epsilon\pi t}}\Big[\exp\Big(-\frac{(-x+t)^2}{4\epsilon t}\Big)-\sqrt{\frac{t}{\epsilon}}\mathbf{erf}\Big(\frac{-x-t}{2\sqrt{\epsilon t}}\Big)\exp\Big(\frac{x}{\epsilon}\Big)\Big].
\end{aligned}
\tag{6.4.22}
$$

Now, we aim to characterize $\beta(t)$ such that the solution of the problem (6.4.1) is differentiable and hence we must have

$$
u_x^1(\epsilon,0+,t)=u_x^2(\epsilon,0-,t).
$$

Now passing through the limit as $x\to 0^+$ in equation (6.4.15), $x\to 0^-$ in equation (6.4.22) and equalizing we get

$$
\begin{aligned}
&\frac{1}{2\sqrt{\pi\epsilon t}}\int_0^\infty u_0(\xi)\Big[2\exp\Big(-\frac{(\xi-t)^2}{4\epsilon t}\Big)\Big(\frac{\xi-t}{2\epsilon t}\Big)+\frac{1}{\epsilon}\exp\Big(-\frac{(\xi-t)^2}{4\epsilon t}\Big)\Big]d\xi\\
&-\frac{1}{\sqrt{\pi}}\int_0^t\beta'(\tau)\Big[\exp\Big(-\frac{t-\tau}{4\epsilon}\Big)\frac{1}{\sqrt{\epsilon(t-\tau)}}+\frac{1}{\epsilon}\mathbf{erf}\Big(\sqrt{\frac{t-\tau}{4\epsilon}}\Big)\Big]d\tau-\frac{\beta(t)\mathbf{erf}(\infty)}{\epsilon\sqrt{\pi}}\\
&-\frac{\beta(0)}{\sqrt{\epsilon\pi t}}\Big[\exp\Big(-\frac{t}{4\epsilon}\Big)-\sqrt{\frac{t}{\epsilon}}\mathbf{erf}\Big(-\sqrt{\frac{t}{4\epsilon}}\Big)\Big]\\
&=\frac{-1}{2\sqrt{\pi\epsilon t}}\int_0^\infty u_0(-\xi)\Big[2\exp\Big(-\frac{(\xi-t)^2}{4\epsilon t}\Big)\Big(\frac{\xi-t}{2\epsilon t}\Big)+\frac{1}{\epsilon}\exp\Big(-\frac{(\xi-t)^2}{4\epsilon t}\Big)\Big]d\xi\\
&\quad+\frac{1}{\sqrt{\pi}}\int_0^t\beta'(\tau)\Big[\exp\Big(-\frac{t-\tau}{4\epsilon}\Big)\frac{1}{\sqrt{\epsilon(t-\tau)}}+\frac{1}{\epsilon}\mathbf{erf}\Big(\sqrt{\frac{t-\tau}{4\epsilon}}\Big)\Big]d\tau+\frac{\beta(t)\mathbf{erf}(\infty)}{\epsilon\sqrt{\pi}}\\
&\quad+\frac{\beta(0)}{\sqrt{\epsilon\pi t}}\Big[\exp\Big(-\frac{t}{4\epsilon}\Big)-\sqrt{\frac{t}{\epsilon}}\mathbf{erf}\Big(-\sqrt{\frac{t}{4\epsilon}}\Big)\Big].
\end{aligned}
\tag{6.4.23}
$$

This implies

$$\begin{aligned}
&\frac{2}{\sqrt{\pi}}\int_0^t \beta'(\tau)\Big[\frac{1}{\epsilon}\textbf{erf}\Big(\frac{\sqrt{t-\tau}}{2\sqrt{\epsilon}}\Big)+\frac{1}{\sqrt{\epsilon(t-\tau)}}\exp\left(-\frac{t-\tau}{4\epsilon}\right)\Big]d\tau+\frac{\beta(t)}{\epsilon}\\
&+\frac{2\beta(0)}{\sqrt{\epsilon\pi t}}\exp\left(-\frac{t}{4\epsilon}\right)+\frac{2\beta(0)}{\sqrt{\pi}\epsilon}\textbf{erf}\left(\sqrt{\frac{t}{4\epsilon}}\right)\\
&=\frac{1}{2\sqrt{\pi\epsilon t}}\int_0^\infty\Big[u_0(\xi)+u_0(-\xi)\Big]\Big[2\exp\left(-\frac{(\xi-t)^2}{4\epsilon t}\right)\left(\frac{\xi-t}{2\epsilon t}\right)\\
&\quad+\frac{1}{\epsilon}\exp\left(-\frac{(\xi-t)^2}{4\epsilon t}\right)\Big]d\xi=F(t),\quad [\text{since } \textbf{erf}(x)=-\textbf{erf}(-x)]
\end{aligned} \tag{6.4.24}$$

Using integration by parts, we calculate the first term in LHS of (6.4.24)

$$\begin{aligned}
&\frac{2}{\sqrt{\pi}}\int_0^t \beta'(\tau)\Big[\frac{1}{\epsilon}\textbf{erf}\Big(\frac{\sqrt{t-\tau}}{2\sqrt{\epsilon}}\Big)+\frac{1}{\sqrt{\epsilon(t-\tau)}}\exp\left(-\frac{t-\tau}{4\epsilon}\right)\Big]d\tau\\
&=\frac{2}{\sqrt{\pi}}\int_0^t \beta(\tau)\Big[\frac{1}{4\epsilon\sqrt{\epsilon(t-\tau)}}\exp\left(-\frac{t-\tau}{4\epsilon}\right)\Big]+\beta'(\tau)\frac{1}{\sqrt{\epsilon(t-\tau)}}\exp\left(-\frac{t-\tau}{4\epsilon}\right)d\tau\\
&-\frac{2\beta(0)}{\sqrt{\pi}\epsilon}\textbf{erf}\left(\sqrt{\frac{t}{4\epsilon}}\right)\\
&=\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{\epsilon}}\exp\left(-\frac{t}{4\epsilon}\right)\int_0^t\frac{\frac{d}{d\tau}\left(\beta(\tau)\exp\left(\frac{\tau}{4\epsilon}\right)\right)}{\sqrt{(t-\tau)}}d\tau-\frac{2\beta(0)}{\sqrt{\pi}\epsilon}\textbf{erf}\left(\sqrt{\frac{t}{4\epsilon}}\right).
\end{aligned} \tag{6.4.25}$$

From equations (6.4.23)-(6.4.25), we get

$$\frac{1}{\sqrt{\pi}}\int_0^t\frac{\frac{d}{d\tau}\left(\beta(\tau)\exp\left(\frac{\tau}{4\epsilon}\right)\right)}{\sqrt{t-\tau}}d\tau+\frac{1}{2\sqrt{\epsilon}}\exp\left(\frac{t}{4\epsilon}\right)\beta(t)=\frac{\sqrt{\epsilon}}{2}\exp\left(\frac{t}{4\epsilon}\right)F(t)-\frac{\beta(0)}{\sqrt{\pi t}}. \tag{6.4.26}$$

The above equation is in the form of a non-homogeneous fractional differential equation

$${}_0D_t^{\frac{1}{2}}y(t)-\lambda y(t)=h(t), \tag{6.4.27}$$

where ${}_0D_t^{\frac{1}{2}}y(t)$ is defined as

$${}_0D_t^{\frac{1}{2}}y(t)=\frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty(t-\tau)^{-\frac{1}{2}}y'(t)d\tau.$$

Solution of the above problem, see [105, p. 140], is given by

$$\beta(t)\exp\left(\frac{t}{4\epsilon}\right)=\frac{1}{2}\int_0^t(t-\tau)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\Big(-\frac{1}{2}\sqrt{\frac{t-\tau}{\epsilon}}\Big)\Big[\sqrt{\epsilon}\exp\left(\frac{\tau}{4\epsilon}\right)F(\tau)-\frac{2\beta(0)}{\sqrt{\pi\tau}}\Big]d\tau, \tag{6.4.28}$$

where $E_{\alpha,\tilde{\alpha}}(.)$ is Mittag-Leffler function defined as

$$E_{\alpha,\tilde{\alpha}}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\tilde{\alpha})}, \qquad (\alpha>0,\tilde{\alpha}>0,z\in\mathbb{C}).$$

From equation (6.4.28), we get (6.4.6). Now using integration by parts, one can easily check that $u$ given in (6.4.3) satisfies weak formulation (6.4.2). □

**Proposition 6.4.3.** *The boundary data $\beta(t)$ is bounded, independent of $\epsilon$ and $t$.*

*Proof.* Let us recall that the explicit formula for $\beta(t)$ is the following:

$$\begin{aligned}\beta(t)&=I_1+I_2\\&=\frac{1}{2}\int_0^t(t-\tau)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\Big(-\frac{1}{2}\sqrt{\frac{t-\tau}{\epsilon}}\Big)\Big[\sqrt{\epsilon}\exp\left(-\frac{t-\tau}{4\epsilon}\right)F(\tau)d\tau\\&\quad-\int_0^t\frac{2\beta(0)}{\sqrt{\pi}}\frac{1}{\sqrt{\tau}}\exp\left(-\frac{t}{4\epsilon}\right)\Big]d\tau,\end{aligned} \tag{6.4.29}$$

where,

$$I_1=\frac{1}{2}\int_0^t(t-\tau)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\Big(-\frac{1}{2}\sqrt{\frac{t-\tau}{\epsilon}}\Big)\sqrt{\epsilon}\exp\left(-\frac{t-\tau}{4\epsilon}\right)F(\tau)d\tau.$$

and

$$I_2=-\int_0^t(t-\tau)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\Big(-\frac{1}{2}\sqrt{\frac{t-\tau}{\epsilon}}\Big)\frac{\beta(0)}{\sqrt{\pi}}\frac{1}{\sqrt{\tau}}\exp\left(-\frac{t}{4\epsilon}\right).$$

Now, we consider

$$\begin{aligned}&\sqrt{\pi}F(t)\\&=\frac{1}{2\sqrt{\epsilon t}}\int_0^{\infty}(u_0(\xi)+u_0(-\xi))\Big[2\exp\left(\frac{-(\xi-t)^2}{4\epsilon t}\right)\left(\frac{\xi-t}{2\epsilon t}\right)+\frac{1}{\epsilon}\exp\left(\frac{-(\xi-t)^2}{4\epsilon t}\right)\Big]d\xi\\&=\int_{\frac{-t}{\sqrt{4\epsilon t}}}^{\infty}\Big[u_0(\sqrt{4\epsilon t}z+t)+u_0(-\sqrt{4\epsilon t}z-t)\Big]\Big[\exp\left(-z^2\right)\left(\frac{2z}{\sqrt{\epsilon t}}+\frac{1}{\epsilon}\right)\Big]dz,\end{aligned} \tag{6.4.30}$$

by a simple change of variable formula. Since $u_0$ is bounded measurable function, clearly $|F(t)| \leq K\Big[\frac{1}{\sqrt{\pi\epsilon t}}\exp\left(-\frac{t}{4\epsilon}\right)+\frac{1}{\epsilon\sqrt{\pi}}\mathbf{erfc}\left(-\sqrt{\frac{t}{4\epsilon}}\right)\Big]$, where $K$ is independent of $\epsilon$ and $t$. Now using a change of variable in $I_1$, one gets

$$\begin{aligned}|I_1| &\leq \epsilon K\int_0^{\sqrt{\frac{t}{4\epsilon}}}|E_{\frac{1}{2},\frac{1}{2}}(-z)|\exp\{-z^2\}\frac{1}{\sqrt{\pi\epsilon(t-4\epsilon z^2)}}\exp\left(-\frac{(t-4\epsilon z^2)}{4\epsilon}\right)dz\\ &\quad+\epsilon K\int_0^{\sqrt{\frac{t}{4\epsilon}}}|E_{\frac{1}{2},\frac{1}{2}}(-z)|\exp\{-z^2\}\frac{1}{\epsilon\sqrt{\pi}}\mathbf{erfc}(-\sqrt{\frac{(t-4\epsilon z^2)}{4\epsilon}})dz\\ &= I_{11}+I_{12}.\end{aligned} \tag{6.4.31}$$

Using the change of variable formula, one can prove that $I_{12}$ is locally bounded independent of $\epsilon$ in $\mathbb{R}^+\cup\{0\}$. Now,

$$|I_{11}| \leq \sqrt{\frac{\epsilon}{\pi}}K\exp\left(-\frac{t}{4\epsilon}\right)\int_0^{\sqrt{\frac{t}{4\epsilon}}}|E_{\frac{1}{2},\frac{1}{2}}(-z)|\frac{1}{\sqrt{(t-4\epsilon z^2)}}dz.$$

Now using the recursive relation $E_{\alpha,\tilde{\alpha}}(z)=zE_{\alpha,\alpha+\tilde{\alpha}}(z)+\frac{1}{\Gamma(\tilde{\alpha})}$, we get

$$E_{\frac{1}{2},\frac{1}{2}}(-z) = -z\exp\{z^2\}\mathbf{erfc}(z)+\frac{2}{\sqrt{\pi}}, \tag{6.4.32}$$

where $E_{\frac{1}{2},1}(z)=\exp\{z^2\}\mathbf{erfc}(-z)$ and $\mathbf{erfc}(z)$ is the complementary error function defined as

$$\mathbf{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty \exp\left(-s^2\right)ds.$$

Now from (6.4.31) and (6.4.32), we have

$$\begin{aligned}|I_{11}| &\leq \sqrt{\frac{\epsilon}{\pi}}K\exp\left(-\frac{t}{4\epsilon}\right)\int_0^{\sqrt{\frac{t}{4\epsilon}}}\Big|-z\ \mathbf{erfc}(z)\exp\left(z^2\right)+\frac{2}{\sqrt{\pi}}\Big|\frac{1}{\sqrt{(t-4\epsilon z^2)}}dz\\ &\leq \sqrt{\frac{\epsilon}{\pi}}K_1\int_0^{\sqrt{\frac{t}{4\epsilon}}}\frac{1}{\sqrt{(t-4\epsilon z^2)}}dz.\ \Big(\text{since } -z\ \mathbf{erfc}(z)\exp\left(z^2\right)+\frac{2}{\sqrt{\pi}} \text{ is bounded}\Big)\end{aligned}$$

Again using the same change of variable in $I_2$, we get

$$|I_2| \leq \frac{4K_2|\beta(0)|\sqrt{\epsilon}}{\sqrt{\pi}}\int_0^{\sqrt{\frac{t}{4\epsilon}}}\frac{1}{\sqrt{t-4\epsilon z^2}}dz.$$

Therefore, $\beta(t)$ is bounded independent $t$ and $\epsilon$. □

Now, we obtain the asymptotic behavior of the solution of the problem (6.4.1). Let us assume

$$u_0(\pm\infty) = \lim_{x\to\pm\infty} u_0(x)$$

**Theorem 6.4.4.** *Let* $\lim_{x\to\pm\infty} u_0(x)$ *exists.Then*

$$\lim_{t\to\infty} \beta(t) = \frac{3}{2}[u_0(\infty) + u_0(-\infty)]$$

*and the solution of the equation* (6.4.1) *approaches to the steady state solution:*

$$\lim_{t\to\infty} u(\epsilon, x, t) \quad = \quad \begin{cases} u_0(\infty) + \frac{1}{2}\big[u_0(\infty) + 3u_0(-\infty)\big] \exp\left(-\dfrac{x}{\epsilon}\right), & \text{if } x > 0,\ t > 0, \\ u_0(-\infty) + \frac{1}{2}\big[u_0(\infty) + 3u_0(-\infty)\big] \exp\left(\dfrac{x}{\epsilon}\right), & \text{if } x < 0,\ t > 0. \end{cases}$$

*The above convergence is uniform in compact sets.*

*Proof.* First, let us consider the region $x > 0, t > 0$: From the expression of $F(t)$ in (6.4.30), it is straightforward to see that

$$\lim_{t\to\infty} F(t) = \frac{1}{\epsilon}[u_0(\infty) + u_0(-\infty)]. \tag{6.4.33}$$

Now, the task is to find the asymptotic behavior of $\beta(t)$, when $t \to \infty$, where

$$\beta(t) = \frac{1}{2}\int_0^t (t-\tau)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}\Big(-\frac{1}{2}\sqrt{\frac{t-\tau}{\epsilon}}\Big)\Big[\sqrt{\epsilon}\exp\left(-\frac{t-\tau}{4\epsilon}\right)F(\tau) - \frac{2\beta(0)}{\sqrt{\pi\tau}}\exp\left(-\frac{t}{4\epsilon}\right)\Big]d\tau. \tag{6.4.34}$$

After a change of variables, in the above equation, we get,

$$\beta(t) = 2\sqrt{\epsilon}\int_0^{\sqrt{\frac{t}{4\epsilon}}} E_{\frac{1}{2},\frac{1}{2}}(-z)\Big[\sqrt{\epsilon}\exp\left(-z^2\right)F(t-4\epsilon z^2) - \frac{2\beta(0)}{\sqrt{\pi}}\frac{\exp\left(-\frac{t}{4\epsilon}\right)}{\sqrt{t-4\epsilon z^2}}\Big]dz. \tag{6.4.35}$$

As $t \to \infty$,

$$\begin{aligned} \lim_{t\to\infty} \beta(t) &= 2[u_0(+\infty) + u_0(-\infty)]\int_0^\infty E_{\frac{1}{2},\frac{1}{2}}(-z)\exp\left(-z^2\right)dz. \\ &= \frac{3}{2}[u_0(+\infty) + u_0(-\infty)]. \end{aligned} \tag{6.4.36}$$

Using a change of variables, we get

$$\frac{-(x+t)}{\sqrt{4\epsilon t}} = p, \frac{x-t}{\sqrt{4\epsilon t}} = q$$

$$\begin{aligned} u^1(\epsilon, x, t) =& \frac{1}{\sqrt{\pi}}\Big[\int_p^\infty u_0(\sqrt{4\epsilon t}z + (x+t)) \exp\left(-z^2\right) dz \\ &- \exp\big(-\frac{x}{\epsilon}\big) \int_q^\infty u_0(\sqrt{4\epsilon t}z + t - x)) \exp\left(-z^2\right) dz\Big] \\ &+ \frac{1}{2\sqrt{\pi\epsilon}} \int_0^t \frac{\beta(t-z)x}{z^{\frac{3}{2}}} \exp\left(-\frac{(x+z)^2}{4\epsilon z}\right) dz. \end{aligned} \tag{6.4.37}$$

Clearly the first term of the above equation approaches to $u_0(+\infty)\Big(1 - \exp\left(-\frac{x}{\epsilon}\right)\Big)$ as $t \to \infty$. In the view of (6.4.12), the second term of the above equation can be written as,

$$\frac{1}{2\sqrt{\pi\epsilon}} \int_0^t \frac{\beta(t-z)x}{z^{\frac{3}{2}}} \exp\left(-\frac{(x+z)^2}{4\epsilon z}\right) dz = -\frac{1}{\sqrt{\pi}} \int_0^t \beta(t-z)\frac{dL}{dz}dz \tag{6.4.38}$$

where

$$L = \mathbf{erf}\Big(\frac{x+z}{2\sqrt{\epsilon z}}\Big) + \mathbf{erf}\Big(\frac{x-z}{2\sqrt{\epsilon z}}\Big) \exp\left(-\frac{x}{\epsilon}\right).$$

Now passing to the limit as $t \to \infty$ in (6.4.38), we have

$$-\frac{3[u_0(+\infty) + u_0(-\infty)]}{2\sqrt{\pi}} \int_0^\infty \frac{dL}{dz}dz = \frac{3}{2}\big[u_0(+\infty) + u_0(-\infty)\big] \exp\left(-\frac{x}{\epsilon}\right).$$

From the above analysis, it can be easily seen that that

$$\lim_{t\to\infty} u^1(\epsilon, x, t) = u_0(+\infty) + \frac{1}{2}\big[u_0(+\infty) + 3u_0(-\infty)\big] \exp\left(-\frac{x}{\epsilon}\right),$$

which is clearly the steady state solution. The case $x < 0, t > 0$ can be similarly handled.

□

### 6.4.2 Vanishing viscosity limit

In this section, we obtain the vanishing viscosity limit for the problem (6.4.1). Let

$$U_0(x) = \int_0^x u_0(z)dz.$$

Then, we have the following theorem:

**Theorem 6.4.5.** *The distributional limit of* $u(\epsilon, x, t)$ *as* $\epsilon \to 0$ *is* $u(x,t)$ *and is given by*

$$u(x,t) = \begin{cases} u_0(x+t), & \text{if } x > 0, t > 0, \\ [U_0(t) - U_0(-t)]\delta_{\{x=0\}}, & \text{if } x = 0, \\ u_0(x-t), & \text{if } x < 0, t > 0. \end{cases} \tag{6.4.39}$$

*Proof.* Let us recall the expression of $u^1(\epsilon, x, t)$.

$$\begin{aligned} &u^1(\epsilon, x, t) \\ &= \frac{1}{2\sqrt{\pi\epsilon t}} \int_0^\infty u_0(\xi)\Big[\exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t}\right) - \exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t} - \frac{x}{\epsilon}\right)\Big]d\xi \\ &\quad + \frac{1}{2\sqrt{\pi\epsilon}} \int_0^t \frac{\beta(\tau)x}{(t-\tau)^{3/2}} \exp\left(-\frac{(x+t-\tau)^2}{4\epsilon(t-\tau)}\right) d\tau \\ &= \frac{\partial}{\partial x}\big[A_1(\epsilon,x,t) + B_1(\epsilon,x,t) - B_1(\epsilon,0,t)\big], \end{aligned}$$

where

$$\begin{aligned} &A_1(\epsilon, x, t) \\ &= \frac{1}{2\sqrt{\pi\epsilon t}} \int_0^\infty U_0(\xi) \exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t}\right) d\xi \\ &\quad + \frac{\exp\left(-\frac{x}{\epsilon}\right)}{2\sqrt{\pi\epsilon t}} \int_0^\infty U_0(\xi) \exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t}\right) d\xi \\ &\quad + \frac{1}{\sqrt{\pi\epsilon}} \int_0^\infty U_0(\xi)\mathbf{erf}\Big(\frac{\xi+x+t}{\sqrt{4\epsilon t}}\Big) \exp\left(-\frac{\xi}{\epsilon}\right) d\xi \end{aligned}$$

and

$$B_1(\epsilon,x,t) = \sqrt{\frac{\epsilon}{\pi}} \int_0^t \frac{\beta(\tau)}{\sqrt{t-\tau}}\Big[-\exp\left(-\frac{(x+t-\tau)^2}{4\epsilon(t-\tau)}\right) - \sqrt{\frac{t-\tau}{\epsilon}}\mathbf{erf}\Big(\frac{x+(t-\tau)}{\sqrt{4\epsilon(t-\tau)}}\Big)\Big]d\tau.$$

Similarly, one deduces

$$\begin{aligned} &u^2(\epsilon, x, t) \\ &= \frac{1}{2\sqrt{\pi\epsilon t}} \int_0^\infty u_0(-\xi)\Big[\exp\left(-\frac{(\xi+(x-t))^2}{4\epsilon t}\right) - \exp\left(-\frac{(\xi-(x+t))^2}{4\epsilon t} + \frac{x}{\epsilon}\right)\Big]d\xi \\ &\quad - \frac{1}{2\sqrt{\pi\epsilon}} \int_0^t \frac{\beta(\tau)x}{(t-\tau)^{3/2}} \exp\left(-\frac{(x-t+\tau)^2}{4\epsilon(t-\tau)}\right) d\tau \\ &= \frac{\partial}{\partial x}\big[A_2(\epsilon,x,t) + B_2(\epsilon,x,t) - B_2(\epsilon,0,t)\big]. \end{aligned}$$

where

$$
\begin{aligned}
&A_2(\epsilon, x, t)\\
&= \frac{1}{2\sqrt{\pi\epsilon t}} \int_0^\infty U_0(-\xi) \exp\left(-\frac{(\xi + (x-t))^2}{4\epsilon t}\right) d\xi\\
&\quad + \frac{\exp\left(\frac{x}{\epsilon}\right)}{2\sqrt{\pi\epsilon t}} \int_0^\infty U_0(-\xi) \exp\left(-\frac{(\xi - (x+t))^2}{4\epsilon t}\right) d\xi\\
&\quad + \frac{1}{\sqrt{\pi}\epsilon} \int_0^\infty U_0(-\xi) \mathbf{erf}\Big(\frac{\xi - x + t}{\sqrt{4\epsilon t}}\Big) \exp\left(-\frac{\xi}{\epsilon}\right) d\xi
\end{aligned}
$$

and

$$
B_2(\epsilon, x, t) = \sqrt{\frac{\epsilon}{\pi}} \int_0^t \frac{\beta(\tau)}{\sqrt{t-\tau}} \Big[ \exp\left(-\frac{(x-t+\tau)^2}{4\epsilon(t-\tau)}\right) + \sqrt{\frac{t-\tau}{\epsilon}} \mathbf{erf}\Big(\frac{-x + (t-\tau)}{\sqrt{4\epsilon(t-\tau)}}\Big)\Big] d\tau.
$$

Now define

$$
U(\epsilon, x, t) = \begin{cases} A_1(\epsilon, x, t) + B_1(\epsilon, x, t) - B_1(\epsilon, 0, t), & \text{if } x > 0, t > 0,\\ A_2(\epsilon, x, t) + B_2(\epsilon, x, t) - B_2(\epsilon, 0, t), & \text{if } x < 0, t > 0. \end{cases}
$$

Then, $U$ is $C^2$ function and $U_x = u$. $U(\epsilon, x, t)$ is a locally bounded function, therefore the point wise limit is the distributional limit. Using change of variable, $A_1(\epsilon, x, t) =$

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}} \int_{-\frac{x+t}{\sqrt{4\epsilon t}}}^\infty U_0(\sqrt{4\epsilon t} z + x + t) \exp\left(-z^2\right) dz\\
&+ \frac{\exp\left(-\frac{x}{\epsilon}\right)}{\sqrt{\pi}} \int_{-\frac{x+t}{\sqrt{4\epsilon t}}}^\infty U_0(\sqrt{4\epsilon t} z - x + t) \exp\left(-z^2\right) dz\\
&+ \frac{1}{\sqrt{\pi}} \int_0^\infty U_0(\epsilon z) \mathbf{erf}\Big(\frac{\epsilon z + x + t}{\sqrt{4\epsilon t}}\Big) \exp\left(-z\right) dz
\end{aligned}
$$

The pointwise limit of the first term, second term, third term and fourth term of $A_1(\epsilon, x, t)$ respectively are

$$
U_0(x+t), \quad 0, \; U_0(0) = 0,
$$

and the point wise limit for $B_1(\epsilon, x, t) = -\frac{1}{2}\int_0^t \beta(\tau) d\tau$. Hence, the point–wise limit of $A_1(\epsilon, x, t) + B_1(\epsilon, x, t) - B_1(\epsilon, 0, t)$ is $U_0(x+t)$. Similarly the point-wise limit of $A_2(\epsilon, x, t) + B_2(\epsilon, x, t) - B_2(\epsilon, 0, t)$ is $U_0(x-t)$. Therefore, the distributional limit of

$U(\epsilon, x, t)$ is

$$\lim_{\epsilon\to 0} U(\epsilon, x, t) = U(x, t) = \begin{cases} U_0(x+t), & \text{if } x > 0, t > 0, \\ U_0(x-t), & \text{if } x < 0, t > 0. \end{cases} \tag{6.4.40}$$

Since $u(\epsilon, x, t) = U_x(\epsilon, x, t)$, the distributional limit of $u(\epsilon, x, t)$ is equal to the distributional derivative of $U(x, t)$ and is given by

$$u(x, t) = \begin{cases} u_0(x+t), & \text{if } x > 0, t > 0, \\ \Big[U_0(t) - U_0(-t)\Big]\delta_{\{x=0\}}, & \text{if } x = 0, \\ u_0(x-t), & \text{if } x < 0, t > 0. \end{cases} \tag{6.4.41}$$

□

If we assume that

$$u_0(\pm\infty) = \lim_{x\to\pm\infty} u_0(x),$$

then $u_0(\pm\infty) = 0$ as $u_0 \in L^1(\mathbb{R})$. Therefore, from (6.3.8) the asymptotic behavior of the solutions of (6.1.1)-(6.1.2) can be given as follows:

$$\lim_{t\to\infty} u(x, t) = \Big(\int_{\mathbb{R}} u_0(x)dx\Big)\delta_{\{x=0\}}$$

The above limit is understood in the sense of distribution. Now, recalling the asymptotic behavior of the viscous equation given in Theorem 6.4.4, we see that $\lim_{t\to\infty} u(\epsilon, x, t) = 0$. Therefore, $\lim_{\epsilon\to 0}\lim_{t\to\infty} u(\epsilon, x, t) \neq \lim_{t\to\infty}\lim_{\epsilon\to 0} u(\epsilon, x, t)$, i.e., the vanishing viscosity and large time limits do not commute. In [107], this property of viscosity solution $u(\epsilon, x, t)$ was established for Burgers equation. To this end, we observed that (6.1.1)-(6.1.2) is an example of a scalar conservation laws with discontinuous fluxes where this property holds.

**Remark 6.4.6.** *One can also consider the multi-dimensional transport equation with a discontinuous coefficient, that is*

$$u_t + \sum_{i=1}^{n} a_i u_{x_i} = 0, \text{ if } x' \in \mathbb{R}^{n-1}, x_n < 0,$$
$$u_t + \sum_{i=1}^{n} b_i u_{x_i} = 0, \text{ if } x' \in \mathbb{R}^{n-1}, x_n > 0.$$

*and* $a_n > 0$*,* $b_n < 0$*. A similar analysis as previous gives the solution,*

$$u(x,t) = \begin{cases} u_0(x_1 - b_1 t, x_2 - b_2 t, .., x_n - b_n t), & \textit{if } x_n > 0, t > 0, \\ [U_0(x_1 - b_1 t, x_2 - b_2 t, .., -b_n t) & \\ -U_0(x_1 - a_1 t, x_2 - a_2 t, .. - a_n t)]\delta_{\{x_n=0\}}, & \textit{if } x_n = 0, \\ u_0(x_1 - a_1 t, x_2 - a_2 t, .., x_n - a_n t), & \textit{if } x_n < 0, t > 0, \end{cases}$$

Now, we move on to providing explicit formulae for generalized weak solutions for nonlinear overcompressive flux pairs.

## 6.5 Explicit Formulae for generalized weak solution

This section concentrates on deriving explicit formulae for the solutions for (6.1.1)-(6.1.2) for overcompressive flux pair $(f, g)$, where $f$ and $g$ are either strictly convex or concave. We derive the formulae for the case when $f$ and $g$ are both strictly convex, the other three cases can be handled similarly.

**Lemma 6.5.1.** *Let* $v$ *be Lipschitz continuous in* $\mathbb{R}/\{0\} \times (0, \infty)$*, which is the viscosity solution of*

$$\begin{aligned} v_t + g(v_x) &= 0, \quad x < 0, t > 0, \\ v_t + f(v_x) &= 0, \quad x > 0, t > 0. \end{aligned}$$

*then, its distributional derivative satisfies* (6.1.1)-(6.1.2) *in the sense of Definition 6.2.4.*

*Proof.* The proof is motivated by the proof in [Chapter 3, Theorem 2, [1]]. Since $v$ is Lipschitz in $\mathbb{R}/\{0\} \times (0, \infty)$, it is differentiable a.e. in $\mathbb{R} \times \mathbb{R}^+$ and satisfies

$$v_t + (H(-x)g(v_x) + H(x)f(v_x)) = 0 \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}^+,$$

thus, for every $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$, we have

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} v_t \phi_x dx dt + (H(-x)g(v_x) + H(x)f(v_x))\phi_x dx dt = 0. \tag{6.5.1}$$

Now note that

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{0}^{\infty} v_t\phi_x dtdx &= -\int_{-\infty}^{\infty}\int_{0}^{\infty} v\phi_{xt} dtdx - \int_{-\infty}^{\infty} v(x,0)\phi_x dx \\
&= -\int_{0}^{\infty}\int_{0}^{\infty} v\phi_{tx} dxdt - \int_{0}^{\infty}\int_{-\infty}^{0} v\phi_{tx} dxdt + \int_{-\infty}^{\infty} v_x(x,0)\phi dx \\
&= \int_{0}^{\infty}\int_{-\infty}^{\infty} v_x\phi_t dxdt + \int_{-\infty}^{\infty} \left[v(0+,t) - v(0-,t)\right]\phi_t dt + \int_{-\infty}^{\infty} v_x(x,0)\phi dx.
\end{aligned}
$$

Substituting this in (6.5.1) and setting $\overline{u} = v_x$, we get,

$$
\int_{-\infty}^{\infty}\int_{0}^{\infty} \left[\overline{u}\phi_t + (H(-x)g(\overline{u}) + H(x)f(\overline{u}))\phi_x\right] dxdt \quad + \int_{-\infty}^{\infty} \left[v(0+,t) - v(0-,t)\right]\phi_t dt + \int_{-\infty}^{\infty} \overline{u}(x,0)\phi(x)dx = 0,
$$

i.e.

$$
\int_{-\infty}^{\infty}\int_{0}^{\infty} \left[\overline{u}\phi_t + (H(-x)g(\overline{u}) + H(x)f(\overline{u}))\phi_x\right] dxdt + \int_{-\infty}^{\infty} w(t)\phi_t dt + \int_{-\infty}^{\infty} u_0(x)\phi(x)dx = 0,
$$

where $w(t) = v(0+,t) - v(0-,t)$. □

Before proceeding further, we recall some preliminaries in the following lemma:

**Lemma 6.5.2.** *Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function with superlinear growth .i.e. $\lim\limits_{|u|\to\infty} \dfrac{h(u)}{|u|} = \infty$. then the following properties hold:*

*(i) Define the Legendre transform by $h^*(u) := \sup\limits_{p\in\mathbb{R}} \{pu - h(p)\}$, thenh$^*$ : $\mathbb{R} \to \mathbb{R}$ is a convex function with super-linear growth.*

*(ii) $h^{**}(u) := \sup\limits_{p\in\mathbb{R}} \{pu - h^*(p)\} = h(u)$.*

*(iii) In addition if $h \in C^1(\mathbb{R})$ and strictly convex, then the following holds true.*

*(a)* $h^{*\prime}(u) = (h')^{-1}(u)$.

*(b)* $h^*(h'(u)) = uh'(u) - h(u)$.

*(c)* $h(h^{*\prime}(u)) = uh^{*\prime}(u) - h^*(u)$.

Consider the overcompressive flux pair $(f, g)$ such that, $f, g \in C^1(\mathbb{R})$ are strictly convex and suppose initial data $u_0 \in [m, M]$, define $\tilde{f}, \tilde{g} \in C^1(\mathbb{R})$ which are strictly convex functions with superlinear growth such that $\tilde{f} = f$ and $\tilde{g} = g$ on $[m, M]$. Then, we have the following explicit formula.

**Theorem 6.5.3.** *Let* $u_0 \in [m, M]$, $v_0(x) = \int\limits_0^x u_0(s)ds$. *Define the following cost functional,*

$$v_1(x,t) = \min_{y_1 \le 0} \left\{ v_0(y_1) + t\tilde{g}^* \left( \frac{x - y_1}{t} \right) \right\}, \quad x \le 0, t > 0, \tag{6.5.2}$$

$$v_2(x,t) = \min_{y_2 \ge 0} \left\{ v_0(y_2) + t\tilde{f}^* \left( \frac{x - y_2}{t} \right) \right\}, \quad x \ge 0, t > 0. \tag{6.5.3}$$

*If* $u(x,t) = \overline{u}(x,t) + w(t)\delta_0$ *is the solution of* (6.1.1)-(6.1.2)*, then the Lax-Oleinik type formula for the solution of* (6.1.1)-(6.1.2) *is given by,*

$$\overline{u}(x,t) = \begin{cases} (\tilde{g}')^{-1}\left(\frac{x - y_1(x,t)}{t}\right), & \text{if} \quad x < 0, t > 0, \\ (\tilde{f}')^{-1}\left(\frac{x - y_2(x,t)}{t}\right), & \text{if} \quad x > 0, t > 0, \end{cases}$$

*with*

$$w(t) = v_2(0,t) - v_1(0,t), \quad t > 0,$$

*where* $y_1(x,t)$ *and* $y_2(x,t)$ *are the minimizers of* $v_1$ *and* $v_2$ *respectively at the point (x, t) as defined in* (6.5.2)-(6.5.3)*. [see [1]].*

*Proof.* Following the proofs in [1, 63] it follows that, $v_1(\cdot,\cdot)$ and $v_2(\cdot,\cdot)$ defined in (6.5.2)-(6.5.3) are Lipschitz continuous in $\mathbb{R}^- \times \mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^+$, respectively and satisfy the following ,

(i) $m \leq (v_1)_x, (v_2)_x \leq M$.

(ii) $v_1$ and $v_2$ is the viscosity solution to the following problem:

$$(v_1)_t + \tilde{g}((v_1)_x) = 0, \quad x < 0, t > 0,$$
$$(v_2)_t + \tilde{f}((v_2)_x) = 0, \quad x > 0, t > 0.$$

(iii) $(v_1)_x$ and $(v_2)_x$ are given by,

$$(v_1)_x(x,t) = (\tilde{g}')^{-1}\left(\frac{x - y_1(x,t)}{t}\right), \quad \text{if} \quad x < 0, t > 0,$$
$$(v_2)_x(x,t) = (\tilde{f}')^{-1}\left(\frac{x - y_2(x,t)}{t}\right), \quad \text{if} \quad x < 0, t > 0.$$

From Lemma 6.5.1,

$$\overline{u}(x,t) = \begin{cases} (v_1)_x(x,t), & \text{if} \quad x < 0, t > 0, \\ (v_1)_x(x,t), & \text{if} \quad x > 0, t > 0, \end{cases}$$

with

$$w(t) = v_2(0,t) - v_1(0,t),$$

satisfies Definition 6.2.4 with $F(x,u) = H(-x)\tilde{g}(u) + H(x)\tilde{f}(u)$. Due to (i) and the Definition of $\tilde{f}$ and $\tilde{g}$, for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ we have,

$$H(-x)\tilde{g}(\overline{u}(x,t)) + H(x)\tilde{f}(\overline{u}(x,t)) = H(-x)g(\overline{u}(x,t)) + H(x)f(\overline{u}(x,t)).$$

Thus we have $u$ satisfies (6.2.4) with $F(x,u) = H(-x)g(u) + H(x)f(u)$. □

**Remark 6.5.4.** *The formula for $v(x,t)$ is independent of the choice of the extensions $\tilde{f}$ and $\tilde{g}$, of $f$ and $g$. $\overline{u}$ satisfies interior entropy condition* (6.2.1).

In the next section, we propose a finite volume numerical scheme approximation (6.1.1)-(6.1.2).

## 6.6 Numerical Scheme

### 6.6.1 Numerical Scheme for (6.1.1)-(6.1.2)

In this section, we consider the Riemann problem for (6.1.1)-(6.1.2) with overcomperessive flux pair $(f, g)$. We start with proposing the scheme for the linear case. The scheme is based on the fact that the Riemann problem for (6.1.1)-(6.1.2) with $f(u) = bu$ and $g(u) = au$ can be shown to have the solutions using the vanishing $\epsilon$ limit of solutions of a nonlinearification of the conservation law given by (6.3.1)-(6.3.5). It can be observed that

$$\lim_{\epsilon\to 0} g_\epsilon(u) = au, \lim_{\epsilon\to 0} f_\epsilon(u) = bu,$$

and

$$u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t).$$

The Riemann Problem solution of (6.3.1)-(6.3.5) is known in each case and Rankine Hugoniot condition is satisfied, hence, the flux at the interface $x = 0$ is explicitly known and is given by

$$F_{\epsilon,0}(a,b,u_l,u_r) := g_\epsilon(u^-) = f_\epsilon(u^+),$$

where $u^- = \lim_{x\to 0^-} u_\epsilon(x,t), u^+ = \lim_{x\to 0^+} u_\epsilon(x,t)$ are obtained from the solutions of the Riemann problem of (6.3.1)-(6.3.5). Since $g_\epsilon$ and $f_\epsilon$ are non-linear convex functions, the flux at the interface $x = 0$ can be obtained using the theory of discontinuous flux [94] and is given by:

$$F_{\epsilon,0}(a,b,u_l,u_r) = \max\left(g_\epsilon\left(\max\left(u_l, \frac{-a}{\epsilon}\right)\right), f_\epsilon\left(\min\left(u_r, \frac{-b}{\epsilon}\right)\right)\right). \tag{6.6.1}$$

Owing to the behavior of the solutions and the fluxes as $\epsilon \to 0$, we define the flux at the interface $x = 0$ for the problem (6.1.1)-(6.1.2) with $g(u) = au, f(u) = bu$ as

$$F_0(a,b,u_l,u_r) := \lim_{\epsilon\to 0} g_\epsilon(u^-) = \lim_{\epsilon\to 0} f_\epsilon(u^+) = \lim_{\epsilon\to 0} F_{\epsilon,0}(a,b,u_l,u_r),$$

which implies that

$$F_0(a,b,u_l,u_r) = \begin{cases} au_l & \text{if} \quad a \geq 0, b > 0, \\ bu_r & \text{if} \quad a < 0, b \leq 0, \\ 0 & \text{if} \quad a < 0, b > 0, \\ \max(au_l, bu_r) & \text{if} \quad a \geq 0, b \leq 0. \end{cases} \tag{6.6.2}$$

The above defines the flux at the interface at $x = 0$ for the problem (6.1.1)-(6.1.2). We now present the numerical scheme. Let $h > 0$ and $x_{i+\frac{1}{2}} = ih, i \in \mathbf{Z}$ with $x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$ such that $x_{\frac{1}{2}} = 0$, the location of the $\delta$- shock. For $\Delta t > 0$, define the time discretization points $t_n = n\Delta t$ for non-negative integer $n$, and $\lambda = \Delta t/h$ is fixed. Define

$$u_i^n = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)dx,$$

as the approximation for $u$ in the cell $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ at time $t_n$. Then the finite volume scheme is given by

$$u_i^{n+1} = u_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{6.6.3}$$

where $F_{i+\frac{1}{2}}^n$ is the numerical flux associated with the flux $F(x,u)$ at the interface $x = x_{i+\frac{1}{2}}$ and defined as follows,

$$F_{i+\frac{1}{2}}^n = \begin{cases} au_l & \text{if} \quad i < 0, \\ bu_r & \text{if} \quad i > 0, \\ F_0(a,b,u_l,u_r) & \text{if} \quad i = 0, \end{cases} \tag{6.6.4}$$

as away from the interface $x_{\frac{1}{2}}$, the fluxes are $au$ and $bu$ with $a \geq 0, b \leq 0$ and hence take backward and forward numerical fluxes respectively. For a nonlinear overcomperessive flux pair $(f,g)$,

$$F_{i+\frac{1}{2}}^n = \begin{cases} g(u_l) & \text{if} \quad i < 0, \\ f(u_r) & \text{if} \quad i > 0, \\ \max(g(u_l), f(u_r)) & \text{if} \quad i = 0, \end{cases} \tag{6.6.5}$$

as away from the interface $x_{\frac{1}{2}} = 0$, the fluxes are $g$ and $f$ which are increasing and decreasing respectively and hence take backward and forward numerical fluxes respectively.

**Lemma 6.6.1.** *Under the CFL condition,*

$$1 - \lambda \sup_{u \in \mathbb{R}} (|g'(u)|, |f'(u)|) \geq 0,$$

*the following properties hold true:*

*(a)*

$$\sum_{i \in \mathbb{Z}} u_i^{n+1} = \sum_{i \in \mathbb{Z}} u_i^n.$$

*(b) Let*

$$u_i^{n+1} = H(u_{i-1}^n, u_i^n, u_{i+1}^n),$$

*Then, the function $H$ is monotone in each of the arguments.*

*(c) Let $(u_i^n)_{\{i \in \mathbb{Z}\}}$ denote the sequence of approximation obtained by the numerical scheme. Define a new sequence $z_n = (u_i^n)_{\{i \in \mathbb{Z}/\{\pm 1\}\}}$. Then*

$$||z_{n+1}||_\infty \leq ||z_n||_\infty.$$

*(d)*

$$h \sum_{i \in \mathbb{Z}} |u_i^{n+1}| \leq h \sum_{i \in \mathbb{Z}} |u_i^n|.$$

*(e) If $u_0 \in \mathbb{R}^+ \cup \{0\}$, then $u_i^n \in \mathbb{R}^+ \cup \{0\},\ \forall i, n.$*

*(f) If $u_0(x) \in \mathbb{L}^1(\mathbb{R})$, then the following result holds,*

$$\sum_{i \in \mathbb{Z}} \left(u_i^{n+1} - u_i^n\right) = \sum_{i \in \mathbb{Z}} \lambda \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) = 0.$$

Proofs can be done on the same lines as in [94, 108]. We now proceed to prove the existence of the solution proposed in §6.2 via the convergence of the numerical scheme (6.6.3). To this end, we assume that initial data lies in $\mathbb{L}_{\mathbb{1}} \cap \mathbf{BV}(\mathbb{R})$. Let $\dfrac{\Delta t_N}{h_N} = \lambda$. We then have the following theorem:

**Theorem 6.6.2.** *Let $u_h$ and $w_h$ be the numerical approximations obtained from the scheme* (6.6.3). *Assume that there exists a sequence $h_k$ which tends to* $0$ *as $k \to \infty$ such that if we set $\Delta_k t = \lambda h_k, w_h^i(t) = hu_i^n, t \in [n\Delta t, (n+1)\Delta t), i = 0, 1,$ with*

*(a)* $\left\|w_{h_k}^i\right\|_{\mathbb{L}^\infty(C_0\times\mathbb{R}^+)} \leq K, \left\|\left(1-\chi_{[-\frac{h_k}{2},\frac{h_k}{2}]}\right)u_{h_k}\right\|_{\mathbb{L}^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq K$ *for some* $K > 0$

*(b)* $\left(1-\chi_{[-\frac{h_k}{2},\frac{h_k}{2}]}\right)u_{h_k}$ *converges in* $\mathbf{L^1_{loc}}(\mathbb{R}/\{0\}\times\mathbb{R}^+)$ *and a.e to a function* $\overline{u}$ *and* $w_{h_k}$ *converges in* $\mathbf{L^1_{loc}}(\mathbb{R}^+)$ *and a.e to a function* $w$,

*Then,* $\overline{u} + w\delta_{\{x=0\}}$ *is the weak solution of* (6.1.1)-(6.1.2).

*Proof.* The proof follows on the same lines as in [94, 108] except for the cells adjacent to the interface $\{x = 0\}$. Let $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$ and $\phi_i^n = \phi(x_i, t_n)$. Define

$$\begin{array}{rll} \phi_h(x,t) = \phi_i^n & \text{if} & (x,t) \in C_i \times [n\Delta t, (n+1)\Delta t)\,, \\ u_h(x,t) = u_i^n & \text{if} & (x,t) \in C_i \times [n\Delta t, (n+1)\Delta t) \quad i \neq 0, 1, \\ F_h(x,t) = F_{i+\frac{1}{2}}^n & \text{if} & (x,t) \in (x_i, x_{i+1}) \times [n\Delta t, (n+1)\Delta t)\,, \\ w_h^i(t) = hu_i^n & \text{if} & t \in [n\Delta t, (n+1)\Delta t) \quad i = 0, 1. \end{array} \tag{6.6.6}$$

Multiplying (6.6.3) by $\phi_i^n$ for each $i \in \mathbb{Z}, n \geq 0$ and summing them,

$$h\sum_{\mathbf{R}}\sum_{n=0}^{\infty}\left(u_i^{n+1}-u_i^n\right)\phi_i^n + \Delta t\sum_{i=2}^{\infty}\sum_{n=0}^{\infty}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right)\phi_i^n$$

$$+\Delta t\sum_{-\infty}^{i=-1}\sum_{n=0}^{\infty}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right)\phi_i^n + \Delta t\sum_{n=0}^{\infty}\left(F_{\frac{3}{2}}^n - F_{\frac{1}{2}}^n\right)\phi_1^n + \Delta t\sum_{n=0}^{\infty}\left(F_{\frac{1}{2}}^n - F_{-\frac{1}{2}}^n\right)\phi_0^n = 0$$

Summation by parts and rearranging the terms we get,

$$0 = h\sum_{-\infty}^{\infty}\sum_{n=1}^{\infty}\left(\phi_i^{n+1}-\phi_i^n\right)u_i^n - \Delta t\sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty}F_{i+\frac{1}{2}}^n\left(\phi_{i+1}^n - \phi_i^n\right) + h\sum_{\mathbf{R}}u_i^0\phi_i^0$$

Converting the above summations into integrals, we have the following,

$$
\begin{aligned}
0 = &\int_{\mathbb{R}/[-h,h]} \int_{\Delta t}^{\infty} u_h(x,t)\frac{\phi_h(x,t)-\phi_h(x,t-\Delta t)}{\Delta t}dxdt+ \\
&\int_{\mathbf{R}} \int_{0}^{\infty} F_h(x,t)\frac{\phi_h(x+h,t)-\phi_h(x,t)}{\Delta x}dxdt \\
&+\int_R u_h(x,0)\phi_h(x,0)dx + \int_{\Delta t}^{\infty} w_h^0(t)\frac{\phi_h(\frac{-h}{2},t)-\phi_h(0,t-\Delta t)}{\Delta t}dt \\
&+\int_{\Delta t}^{\infty} w_h^1(t)\frac{\phi_h(\frac{h}{2},t)-\phi_h(0,t-\Delta t)}{\Delta t}dt
\end{aligned}
$$

Applying Lebesgue dominated convergence theorem and passing to the limit,

$$
\begin{aligned}
\int_{\mathbb{R}\times\mathbb{R}^+} \left(\overline{u}\phi_t + F(x,\overline{u})\phi_x\right)dxdt + \int_{\mathbb{R}} u_0(x)\phi(x,0)dx + \int_0^{\infty} w^1(t)\phi_t(0,t)dt \\
+\int_0^{\infty} w^2(t)\phi_t(0,t)dt = 0.
\end{aligned}
$$

Now, setting $w(t) = w^1(t) + w^2(t)$ gives the desired weak formulation

$$
\int_{\mathbb{R}\times\mathbb{R}^+} \left(\overline{u}\phi_t + F(x,\overline{u})\phi_x\right)dxdt + \int_{\mathbb{R}} u_0(x)\phi(x,0)dx + \int_0^{\infty} w(t)\phi_t(0,t)dt = 0.
$$

Substituting the above limits, we get,

$$
\int_{\mathbb{R}\times\mathbb{R}^+} \left(\overline{u}\phi_t + F(x,\overline{u})\phi_x\right)dxdt + \int_{\mathbb{R}} u_0(x)\phi(x,0)dx + \int_0^{\infty} w(t)\phi_t(0,t)dt = 0.
$$

Since $\phi \in C_c^{\infty}(\mathbb{R}\times\mathbb{R}^+)$ is arbitrary, $\overline{u} + w\delta_{\{x=0\}}$ is a weak solution of (6.1.1)-(6.1.2). □

### 6.6.2 Extension to Systems

We extend the above ideas to the system (6.1.3)-(6.1.4), which admits $\delta$-shocks, see[16], exact or standard approximate Riemann solvers cannot be used. The idea is to avoid creating a Riemann solver based on the eigenstructure of the system and instead treat each equation of the system separately, assuming that the flux of the equation is a function of the remaining state variable at the previous time step. As discussed in the introduction, this system admits $\delta$- shock whenever the first equation admits a classical shock. Since the first equation (6.1.3)-(6.1.4) of the system is a hyperbolic conservation law, any standard $3-$ point scheme such as Godunov scheme can be used for (6.1.3)-(6.1.4). However, the second equation is of the form

$$u_t + (k'(v(x,t))u)_x = 0, \;\; (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{6.6.7}$$

if we assume that $l(u) = u$ and $v(x,t)$ is known at the point $(x,t)$ from the first equation. This is a scalar conservation law with a variable coefficient, which can be discontinuous. Note that $k'(v(x,t))$ is always bounded.

Since the previous section only dealt with a single discontinuity of the flux of the linear transport equation, to propose the scheme for (6.6.7), we need to generalize the scheme for any general linear transport equation with variable, possibly discontinuous coefficient, both in space and time, in particular,

$$u_t + (a(x,t)u)_x = 0, \;\; (x,t) \in \mathbb{R} \times \mathbb{R}^+, u(x,0) = u_0(x), \;\; x \in \mathbb{R}. \tag{6.6.8}$$

For $h > 0$, let the space grid points as $x_{i+\frac{1}{2}} = ih, i \in \mathbb{Z}$. Let $x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$. For $\Delta t > 0$, define the time discretization points $t_n = n\Delta t$ for non-negative integer $n$, and $\lambda = \Delta t/h$. Define

$$a_i^n = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} a(x,t^n)dx, u_i^n = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)dx,$$

as the approximation for $a$ and $u$ in the cell $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ at time $t_n$. Then the finite volume scheme for the system (6.6.8) is given by

$$u_i^{n+1} = u_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{6.6.9}$$

where $F^n_{i+\frac{1}{2}}$ is the numerical flux associated with the flux $F(x,t,u) = a(x,t)u$ at $x_{i+\frac{1}{2}}$ at time $t^n$. They are the functions of left and right values of $a$ and $u$ at $x_{i+\frac{1}{2}}$ at time $t^n$ with

$$F^n_{i+\frac{1}{2}} = F(a^n_i, a^n_{i+1}, u^n_i, u^n_{i+1}).$$

Denote

$$F = a(x,t)u,$$

where $a(x,t)$ is known function at time $t^n$ which is allowed to be discontinuous at the space discretization points. Therefore on each $C_i \times (t^n, t^{n+1})$, we look at the conservation law,

$$v_t + (F(a^n_i, u))_x = 0, \tag{6.6.10}$$

where $F(a^n_i, u) = a^n_i u$, with $a(x,t^n) = a^n_i$ for $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. Now, the corresponding local Riemann problem reduces to

$$u_t + l(x,a)_x = 0 \quad \text{ in } (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (t^n, t^{n+1}), \tag{6.6.11}$$

where

$$l(x,u) = \begin{cases} F(a^n_i, u) & \text{if} \quad x < x_{i+\frac{1}{2}}, \\ F(a^n_{i+1}, u) & \text{if} \quad x > x_{i+\frac{1}{2}}, \end{cases}$$

with the initial data

$$u(x,t^n) = \begin{cases} u^n_i & \text{if} \quad x < x_{i+\frac{1}{2}}, \\ u^n_{i+1} & \text{if} \quad x > x_{i+\frac{1}{2}}. \end{cases}$$

The objective is to construct an upwind numerical flux $F_{i+\frac{1}{2}}$ for the flux function $l(x,u)$ at the cell interface $x_{i+\frac{1}{2}}$. Note that each $i^{th}$ conservation law is of the type (6.1.1)-(6.1.2) and hence considering the different signs of $a^n_i$ , we can define the numerical flux $F^n_{i+\frac{1}{2}}$ by

$$F^n_{i+\frac{1}{2}} = F_0(a^n_i, a^n_{i+1}, u^n_i, u^n_{i+1}), \tag{6.6.12}$$

where $F_0(a,b,u_l,u_r)$ is given by (6.6.2).

Using the scheme (6.6.9) with $a(x,t) = k'(v(x,t))$ for (6.1.4) and Godunov scheme for (6.1.3) gives the scheme for the system (6.1.3)-(6.1.4). In particular, the scheme is given by:

$$v_i^{n+1} = v_i^n - \lambda\left(F^{v,n}_{i+\frac{1}{2}} - F^{v,n}_{i-\frac{1}{2}}\right)$$
$$u_i^{n+1} = u_i^n - \lambda\left(F^{u,n}_{i+\frac{1}{2}} - F^{u,n}_{i-\frac{1}{2}}\right)$$

where $F^{v,n}_{i+\frac{1}{2}} = F_{God}(v_i^n, v_{i+1}^n)$ and $F^{n}_{i+\frac{1}{2}} = F_0(a_i^n, a_{i+1}^n, u_i^n, u_{i+1}^n)$, with $a(x,t) = k'(v(x,t))$. Let us define the piecewise constant approximate solution to the system (6.1.3)-(6.1.4) $U_h(x,t) = \begin{pmatrix} v_h \\ u_h \end{pmatrix}$. We then have the following theorem showing that the above scheme converges to a weak solution of the system.

**Theorem 6.6.3.** *For every* $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$, *we have*

$$\lim_{h\to 0} \int\limits_{\mathbb{R}\times\mathbb{R}^+} u_h\phi_t + k'(v_h)u_h\phi_x = 0, \tag{6.6.13}$$

*Proof.* Fix $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$. Let $i \in \mathbf{Z}$, $n \in \mathbf{N}$, $\phi_i^n = \phi(x_i, t_n)$ and $\phi^n_{x,i} = \dfrac{\phi(x_i,t^n) - \phi(x_{i-1},t^n)}{h}$, $\forall i, n$. To prove the theorem, it is enough to show that

$$\lim_{h\to 0} A(h) = 0, \tag{6.6.14}$$

where

$$A(h) = -h \sum_{i\in\mathbb{Z}, n\in\mathbb{N}} \left[u_i^{n+1} - u_i^n + \lambda((k'(v)u)_i^n - (k'(v)u)_{i-1}^n)\right]\phi_i^n.$$

By the Definition of the scheme,

$$A(h) = -h\sum_{i,n}\left[-\lambda(F^{u,n}_{i+\frac{1}{2}} - F^{u,n}_{i-\frac{1}{2}}) + \lambda((k'(v)u)_i^n - (k'(v)u)_{i-1}^n)\right]\phi_i^n,$$

which, on rearranging the terms and applying summation by parts, gives

$$A(h) = -\lambda h^2 \sum_{i,n}\left[F^{u,n}_{i-\frac{1}{2}} - (k'(v)u)_{i-1}^n\right]\phi^n_{x,i}. \tag{6.6.15}$$

The possible choices for $F^{u,n}_{i-\frac{1}{2}}$ are $k'(v^n_{i-1})u^n_{i-1}, k'(v^n_{i-1})u^n_{i-1}$ and $0$.

Let $i_0 \in \mathbb{Z}$. For a fixed $n$, the only term containing $k'(v^n_{i_0})u^n_{i_0}$ is

$$-\lambda h^2 (k'(v)u)^n_{i_0} \left[\phi^n_{x,i_0} - \phi^n_{x,i_0+1}\right] \leq K_\phi \lambda h^2 \|k'(v_0)u_0\|_{\mathbb{L}^1}, \tag{6.6.16}$$

using the $L_1$ stability of $k'(v)u$, where $K_\phi = 2\|\phi_x\|_{\mathbb{L}^\infty(\mathbb{R}\times\mathbb{R}^+)}$. Summing over $n \in \mathbb{N}$, we get

$$-\lambda h^2 \sum_{i,n} \left[F^{u,n}_{i-\frac{1}{2}} - (k'(v)u)^n_{i-1}\right] \phi^n_{x,i} \leq \sum_n K_\phi \lambda h^2 \|k'(v_0)u_0\|_{\mathbb{L}^1} = K_\phi T h \|k'(v_0)u_0\|_{\mathbb{L}^1 1}, \tag{6.6.17}$$

where $T$ is the final time. This shows that (6.6.15) and equivalently $A(h) = \mathcal{O}(h)$ as $h \to 0$. This proves the claim. □

### 6.6.3 Numerical Scheme for Balance Laws

This section aims to extend the numerical scheme described in the previous section for the equations of the type

$$u_t + F(x,u)_x = s(x,u,t),$$

where $s(x,t)$ is a real valued function. We will also extend the numerical scheme for singular source term. The idea will be to ”locally” modify the balance law as a conservation law with a space-time dependent discontinuous coefficient

$$u_t + \tilde{F}(x,u,t)_x = 0,$$

where $\tilde{F}$ is the flux function locally modified by the source. This strategy has been earlier used in literature, see for example, [109] for balance laws with continuous convex flux functions and [110, 111] for non-strictly hyperbolic systems with source terms. For this, we modify the flux function $F(x,u)$ in each cell $C_i$ by including with it, an approximate divergence form of source term $s(x,t,u)$. To this end, we define

$$B^n(x) = \sum_i B^n_i \chi_{C_i}(x,t), B^n_i = \int_{x_{\frac{1}{2}}}^{x_i} s(x,u,t)dx,$$

and can be calculated using an approximate integration rule. As earlier, the discretization is done in such a way that the location of the $\delta$-shock lies in the cell $C_0$. Then, for $[t^n, t^{n+1})$, we solve the following conservation law with discontinuous coefficient,

$$u_t + (F(x,u) - B^n(x))_x = 0, u(x,t^n) = u^n(x),$$

then define

$$u_i^{n+1} = u_i^n - \lambda(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}), \tag{6.6.18}$$

where $F^n_{i+\frac{1}{2}}$ is the numerical flux at the interface $x = x_{i+\frac{1}{2}}$ and defined as follows,

$$F^n_{i+\frac{1}{2}} = \begin{cases} bu^n_{i+1} - B^n_{i+1} & \text{if} \quad i \geq 0, \\ au^n_i - B^n_i & \text{if} \quad i < 0, \end{cases}$$

Now, let us extend the numerical scheme for the case when $s(x,t,u) = k\delta_{\{x=0\}}$. These kind of equations have been of interest, for example, [112, 83]. Since the distributional derivative of

$$(kH)'(x) = k\delta_{\{x=0\}},$$

the balance law can be rewritten as

$$u_t + \tilde{F}(x,u) = 0, \ \tilde{F}(x,u) = H(x)au + (1 - H(x))(bu - k),$$

for which schemes have already been discussed before.

## 6.7 Numerical Results

This section displays the performance of the scheme proposed in §6.6 for various sets of Riemann data in capturing both $\delta$-shocks and classical shock solutions efficiently for the system (6.1.3)-(6.1.4) and (6.1.1)-(6.1.2) with and without source terms. The domain for the system (6.1.3)-(6.1.4) is $[-1,1]$ and for (6.1.1)-(6.1.2) is $[-0.2, 0.2]$, with $M = 200$, with $a = 1, b = -1, T = 1.008$, $k = 1$. The final time $T$ is taken small enough so that the solution does not reach the boundary of the numerical domain. Hence, we can take boundary conditions to be $u_l$ and $u_r$ respectively.

### 6.7.1 Simulations for the equation (6.1.1)-(6.1.2)

Example 1: $s(x,u) = 0$:

(a) Classical and $\delta$- shock solution of (6.1.1)-(6.1.2)

We take two cases here, first where there is only classical solution and second where there is a concentration at the interface $x = 0$.

i. $au_l - bu_r = 0, u_l = 1, u_r = -1$ : It can be seen in Figure 6.1 that there is no $\delta$- shock as the Rankine-Hugoniot condition is satisfied at $x = 0$. It coincides with the exact solution

$$u(x,t) = \begin{cases} u_l, & \text{if } x < 0, \\ u_r, & \text{if } x > 0. \end{cases}$$

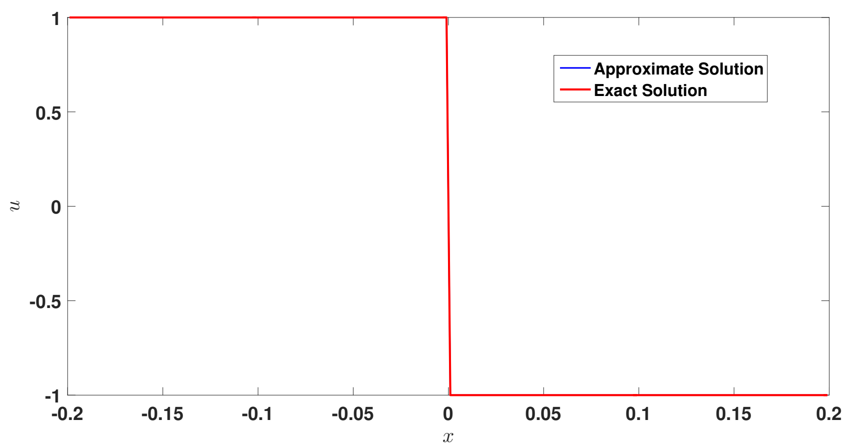


Figure 6.1: No $\delta$-shock with $au_l - bu_r = 0$

ii. $au_l - bu_r \neq 0, u_l = 1, u_r = 2$ : To show the efficiency of the scheme to capture the weight of $\delta$- shock, primitive of the approximate solution is calculated at final time $T$. At the location of $\delta$- shock, the graph of the primitive will be seen to have a sudden jump, equal to the weight of the $\delta$-shock. It can be seen in Figure 6.2 that there is a $\delta$- shock with numerical weight $3.0024$ which is the exact weight of the $\delta$-shock at time $T$, i.e. $(au_l - bu_r)T = 3.0024$.

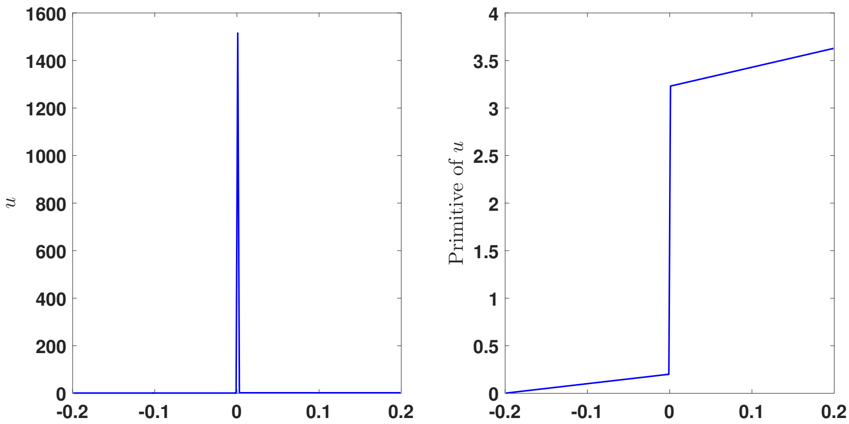


Figure 6.2: $\delta$-shock at $x = 0$ with weight $3.0024$

It can be also be seen that the approximate solution coincides with the exact solution

$$u(x,t) = \begin{cases} u_l, & \text{if } x < 0, \\ (au_l - bu_r)\delta_{\{x=0\}}, & \text{if } x = 0, \\ u_r, & \text{if } x > 0. \end{cases}$$

away from the interface.

iii. Comparison with schemes of [113]: To show the efficiency of the scheme to capture the weight of $\delta$- shock, we compare the results of our scheme with the results presented in [113]. The flux is assumed to be $F(x,u) = -H(x-.5) + (1-H(x-.5))1$ with $u_0(x) = 0H(x-.5) + (1-H(x-.5))1$. Let the domain be $[0.44, 0.56], T = 0.5, h = 0.002$. The solution is given by $u(x, 0.5) = 0.5(1 - H(x-.5))$. The numerical weight is computed as earlier and it can be seen in Figure 6.3 that our scheme does not have diffusion and captures the exact weight $0.5$. Also, on comparison with Figures $1$ and $2$ of [113], it can be noticed that the location of $\delta$-shock is little shifted to the left of the exact $\delta$-shock location, $x = 0.5,$ while the scheme presented in this paper captures the $\delta$-shock precisely at the location.

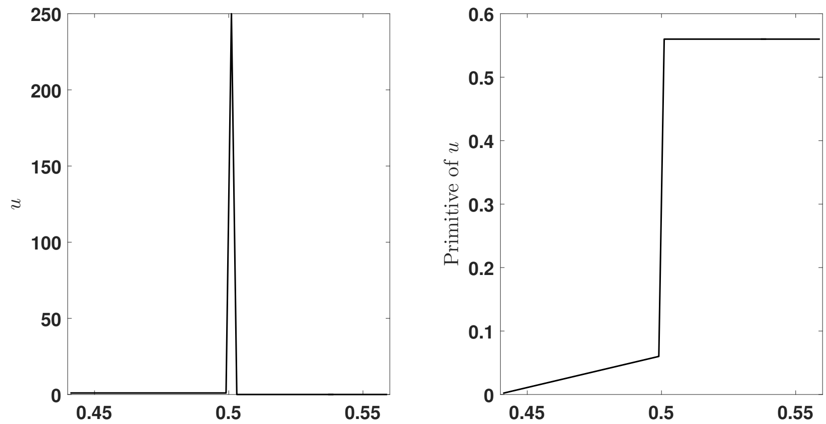


Figure 6.3: $\delta$-shock at $x = 0$ with weight $0.5$

(b) (6.1.1)-(6.1.2) as a vanishing $\epsilon-$ limit of (6.3.1):

This experiment is to show that the non–linear approximation (6.3.1) is a good approximation to (6.1.1)-(6.1.2). Since (6.3.1) is a conservation law with discontinuous flux considered in [114], with both functions convex, the finite volume scheme for (6.3.1) is given by:

$$u_i^{n+1} = u_i^n - \lambda(F^n_{\epsilon,i+\frac{1}{2}} - F^n_{\epsilon,i-\frac{1}{2}}),$$

$$F^n_{\epsilon,i+\frac{1}{2}} = \max\left(g_\epsilon\left(\max\left(u_i^n, \tfrac{-1}{v_i\epsilon}\right)\right), f_\epsilon\left(\min(u^n_{i+1}, \tfrac{-1}{v_{i+1}\epsilon})\right)\right),$$

$$v_i = a\chi_{\{i<0\}} + b\chi_{\{i>0\}}.$$

It can be seen in Figure (6.4) that the solutions of (6.1.1)-(6.1.2) and (6.3.1) are centered around the $x = 0$ with the width of the shock increasing with increasing

$\epsilon$. Also, as $\epsilon$ is tending towards zero, the height of the shock is approaching the height achieved by the solution of (6.1.1)-(6.1.2). It is evident that with decreasing $\epsilon$, the solution of (6.3.1) are having more height and less width and hence converging to $\delta$-shock solution of (6.1.1)-(6.1.2).

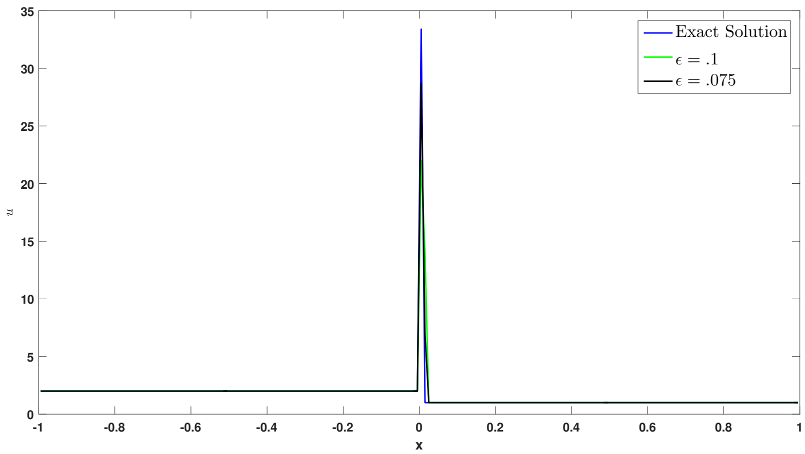


Figure 6.4: Non-linear approximation for small $\epsilon$

Example 2: $s(x,u) = ku, k \neq 0$: Now, we present the numerical results with a linear non-zero source term

(a) $au_l - bu_r = 0, u_l = 1, u_r = -1$ : It can be seen in Figure 6.5 that there is no $\delta$-shock. It coincides with the exact solution

$$u(x,t) = \exp(T) \begin{cases} u_l, & \text{if } x > 0, \\ u_r, & \text{if } x < 0. \end{cases}$$

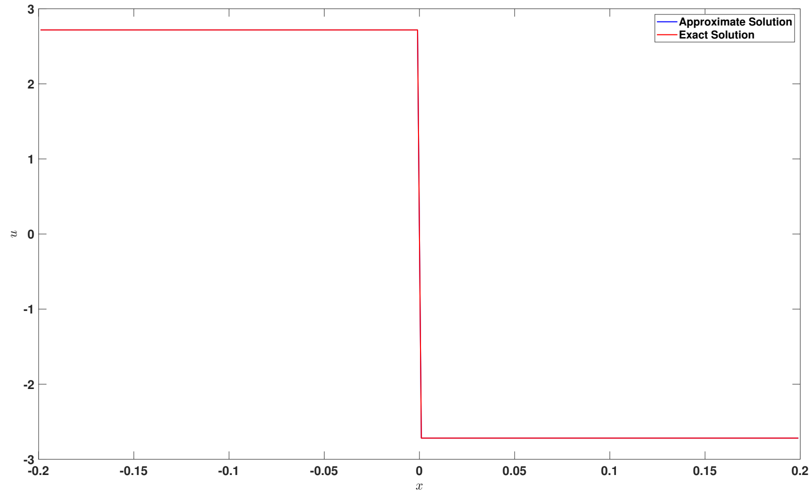


Figure 6.5: No $\delta$-shock when $au_l - bu_r = 0$

(b) $\underline{au_l - bu_r \neq 0, u_l = 1, u_r = 2}$ : It can be noted that the numerical solution obtained from either of the schemes is same as the exact solution

$$u(x,T) = \exp(kT) \begin{cases} u_r, & \text{if} \quad x > 0, \\ (au_l - bu_r)\delta_{\{x=0\}}, & \text{if} \quad x = 0, \\ u_l, & \text{if} \quad x < 0, \end{cases}$$

away from the interface. We compare the performance of our scheme with the following standard scheme for source term

$$u_i^{n+1} = u_i^n - \lambda(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}) + k\Delta t u_i^n,$$

which we call **Scheme 2**. The difference between weight captured by our scheme and the exact weight $3T\exp(T) = 8.1679$ is $8.06e-4$, while by **Scheme 2** is $9e-3$. In the Figure 6.6, our scheme is shown in —— and results by **Scheme 2** is given in —— while the exact solution away from interface is given by ——.

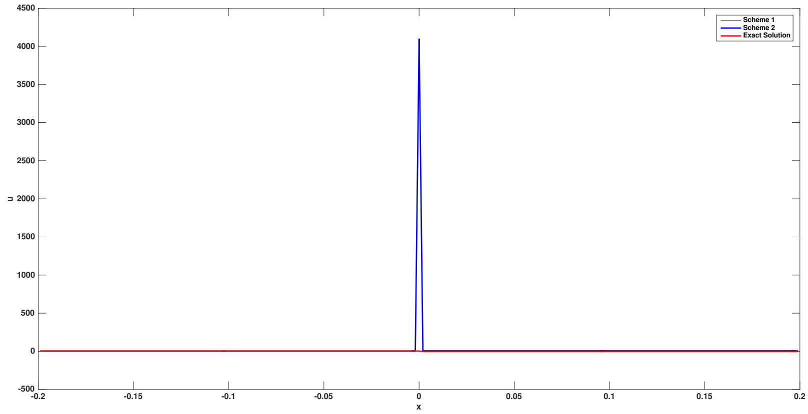


Figure 6.6: Comparison of schemes in presence of source terms for capturing $\delta$-shocks

Example 3: $\underline{s(x,u) = k\delta_{\{x=0\}}}$ : Let us start with $a = 0 = u_l$ and $b = 1, u_r = -1$ so that there is $a = u_l = 0, b = u_r = -1, k = 0$ $\delta$-shock with weight $-T$. Now, a singular source term of the same weight is added to the right-hand side of the homogeneous conservation laws so that the previously occurring $\delta$-shock in "nullified" and concentration will be absorbed. Results can be seen Figure 6.7.

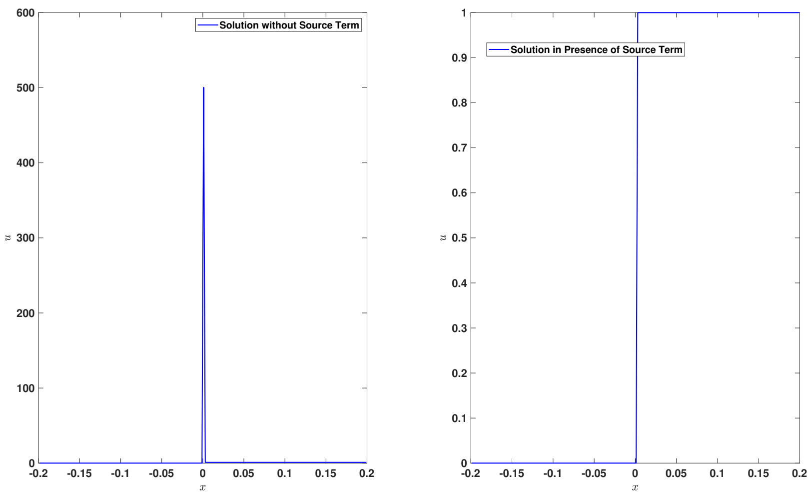


Figure 6.7: $\delta$-shock with weight $-T$ being absorbed by a point source term

Now, let's start with the datum $a = u_l = 1, b = u_r = -1$ so that there is no $\delta$-shock. Then, by adding a point source term $-\delta_{\{x=0\}}$, a negative delta shock of weight $-T$ is obtained. The figure 6.8 displays that the negative $\delta$-shock can be nullified by appropriately modify the flux $F(x, u)$ to $a = 2, b = -1$.

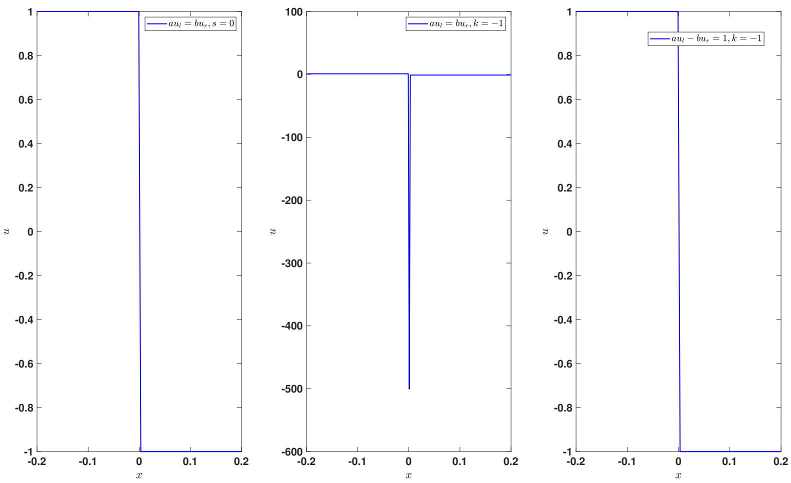


Figure 6.8: Source-Sink Phenomenon of $\delta$-shock

Example 4: Non–linear Overcompressive flux Pair:

Here we illustrate the performance of our schemes for overcompressive flux pairs.

In particular, let

$$g(u) = \frac{u^3}{3}, f(u) = \frac{-u^5}{5}, T = .05, M = 101,$$

with the domain $[-1, 1]$. Then the problem (6.1.1)-(6.1.2) admits $\delta$-shocks at the interface $x = 0$ with $u_0(x) = \begin{cases} 1, & \text{if } \ x < -0.5, \\ 1.5, & \text{if } \ -0.5 < x < 0, \\ 2, & \text{if } \ 0 < x < 0.5, \\ 0, & \text{else} . \end{cases}$

It can be seen in the Figure 6.9that the scheme is able to capture the $\delta$- shock at the interface effectively.

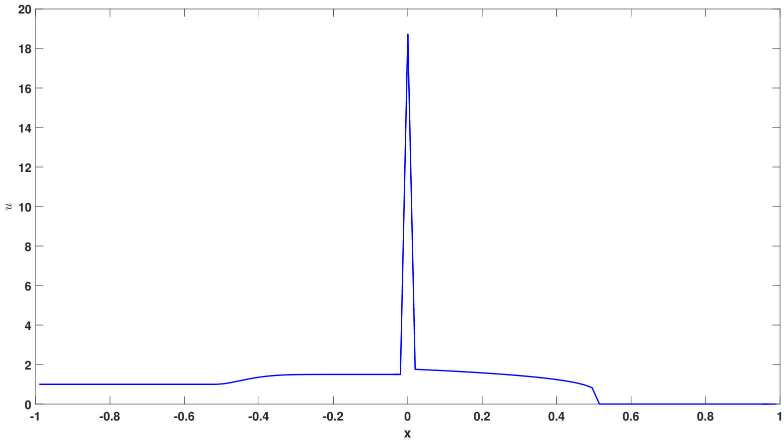


Figure 6.9: Non–linear Monotone fluxes

### 6.7.2 Simulations for the system (6.1.3)-(6.1.4)

This section displays the performance of the scheme proposed in §6.6.2 for various Riemann data in capturing both $\delta$-shocks and classical shock solutions efficiently for Augmented Burgers system (6.1.3)-(6.1.4). The numerical solutions have been compared with the solutions established in [37, 16] and references therein.

Example 1: Classical Shock for $u$:

Let $v_l = v_r = 1, u_l = 1, u_r = 2$. The solution is given by: $v(x,t) = v_0(x)$ and

$$u(x,t) = \begin{cases} u_l, & \text{if } \; x < v_0 t, \\ u_r, & \text{if } \; x > v_0 t. \end{cases}.$$

The figure (6.10) shows that the right solution has been captured by the scheme.

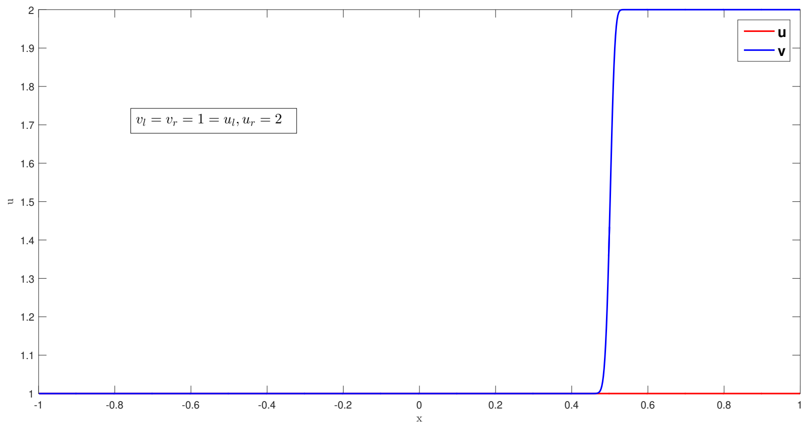


Figure 6.10: No $\delta$-shock with initial data as constant $v$

Example 2: Moving Classical Shocks:

Let $v_l = 3 = 3v_r, u_l = -1 = -u_r$. The solution is given by:

$$v(x,t) = \begin{cases} v_l, & \text{if } \; x < st, \\ v_r, & \text{if } \; x > st. \end{cases}, s = .5(v_l + v_r),$$

and

$$u(x,t) = \begin{cases} u_l, & \text{if } \; x < v_0 t, \\ u_r, & \text{if } \; x > v_0 t. \end{cases}, v_0 = \frac{v_l u_l - v_r u_r}{v_l - v_r}.$$

The figure (6.11) shows that the right solution has been captured by the scheme.

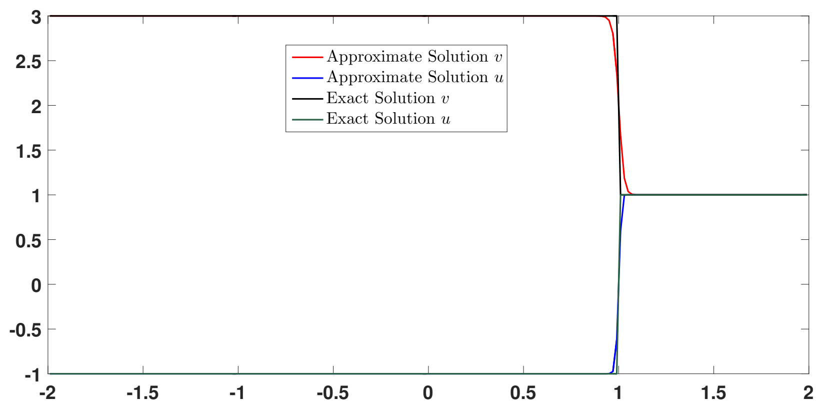


Figure 6.11: Classical Moving Shocks for $u, v$

Example 3: Vacuum Solution:

Let $-2v_l = v_r = 1 = u_l = 2u_r$. The solution is given by:
$v(x,t) = \begin{cases} v_l, & \text{if } \; x < v_l t, \\ \frac{x}{t}, & \text{if } \; v_l t < x < v_r t, \\ v_r, & \text{if } \; x > v_r t, \end{cases}$ and $u(x,t) = \begin{cases} u_l, & \text{if } \; x < v_l t, \\ 0, & \text{if } \; v_l t < x < v_r t, \\ u_r, & \text{if } \; x > v_r t. \end{cases}$ The figure (6.12) shows that the right solution has been captured by the scheme.

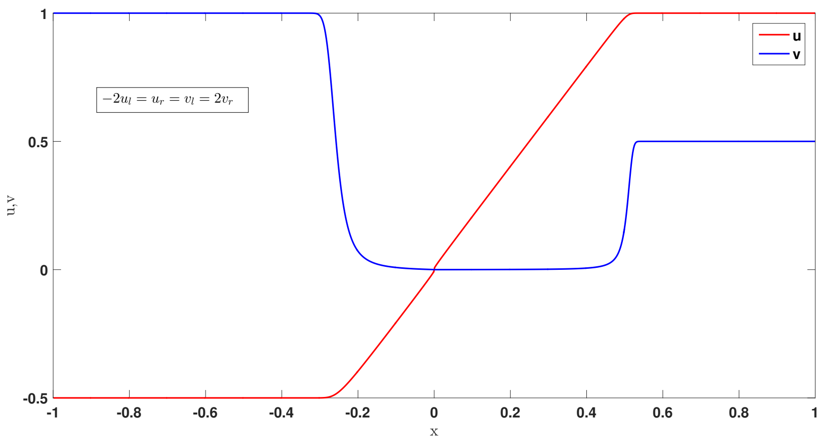


Figure 6.12: Vacuum solution for $u$ with initial data as Riemann Data $v_l < v_r$

Example 4: Stationary $\delta$-shock for $u$:

Let $v_l = -v_r = 1 = u_r = 2u_l$. The solution is given by:
$v(x,t) = \begin{cases} v_l, & \text{if } \; x < 0, \\ v_r, & \text{if } \; x > 0, \end{cases}$ and $u(x,t) = \begin{cases} u_l, & \text{if } \; x < 0, \\ (u_l - u_r)st\delta_{\{x=0\}}, & \text{if } \; x = 0, \\ u_r, & \text{if } \; x > 0, \end{cases}$ where $s = \dfrac{v_l + v_r}{2}$. The figure (6.13) shows that the right solution has been captured by the scheme.

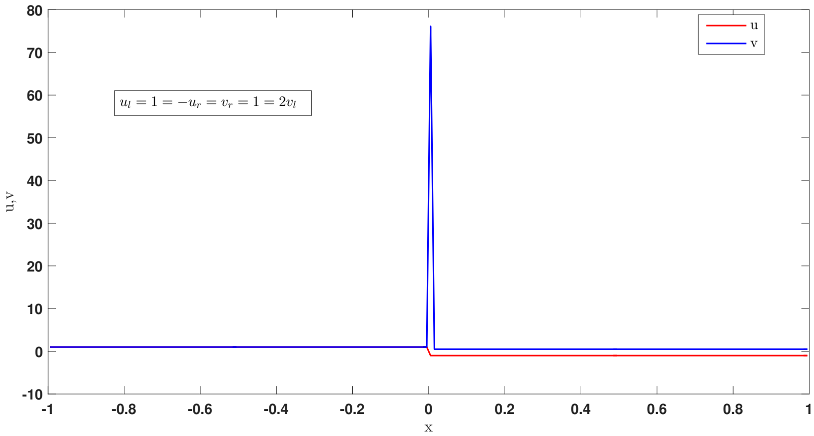


Figure 6.13: Stationary $\delta$-shock for $u$ with weight $v_l = -v_r > 0$

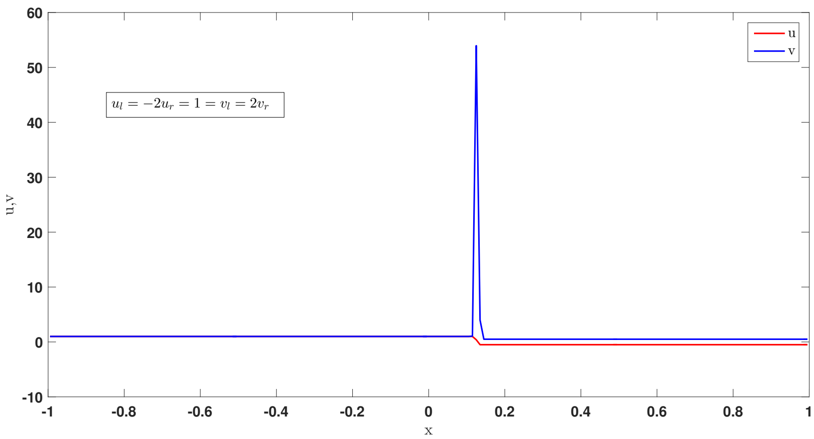


Figure 6.14: Moving $\delta$-shock at $x = st$ with weight $v_l > v_r$, where $s = .5(v_l + v_r)$

Example 5: Moving $\delta$-shock for $u$: Let $v_l = -v_r = 1 = u_r = 2u_l$. Let $s = \dfrac{v_l + v_r}{2}$. The solution is given by:

$$v(x,t) = \begin{cases} v_l, & \text{if } \; x < st, \\ v_r, & \text{if } \; x > st, \end{cases} \text{ and } u(x,t) = \begin{cases} u_l, & \text{if } \; x < st, \\ (u_l - u_r)st\delta_{\{x=0\}} & \text{if } \; x = st, \\ u_r, & \text{if } \; x > st. \end{cases}$$

The figure (6.14) shows that the right solution has been captured by the scheme.

# Chapter 7

# Conclusions and future research

## 7.1 Conclusions

The main focus of this thesis is to study the non-classical ( measured valued ) solutions for system of conservation laws and scalar conservation laws with discontinuous flux function with initial and boundary values. For system of conservation laws, my study revolves within the $2 \times 2$ systems arising from various physical models. The main tools, we used to study the systems are, a generalised Hopf-Cole transformation, vanishing viscosity method, volpert's product, singular flux function limit or vanishing pressure limit, shadow waves approach and analysis on characteristic curves. The detailed chapterwise conclusions are given below.

### 7.1.1 Vanishing pressure limit for some strictly hyperbolic systems

In the second and third chapter of my thesis, we studied the limiting behavior of some strictly hyperbolic system of conservation laws. The second chapter considers the following strictly hyperbolic system of conservation laws known as Euler equation of compressible fluid flow,

$$u_t + (\frac{u^2}{2} + P(\rho))_x = 0;$$
$$\rho_t + (\rho u)_x = 0$$

The limiting behavior of solutions for the above system with Riemann type initial data is studied when the pressure term $P(\rho)$ approaches zero. For that purpose we take the scalar function $P$ not only as a function of density $\rho$ but also a small parameter $\epsilon > 0$ satisfying $\lim_{\epsilon \to 0} P(\rho, \epsilon) = 0$. More precisely, we take $P(\rho, \epsilon) = \epsilon p(\rho)$ where $p(\rho)$ is a twice differentiable function. One can readily observe that when $\epsilon$ goes to zero, the above system formally goes to the one-dimensional model for the large-scale structure formation of the universe. This is an example of a non-strictly hyperbolic system that has no classical solution and a solution containing concentration ($\delta$-shocks) had introduced by Korchinski [9]. In the presence of pressure ($\epsilon > 0$), there will be no concentration of mass and the solution will lie in the space of bounded variation. Therefore Lax's theory can be employed to construct the solution. Then the distributional limit as $\epsilon$ tends to zero of the solution for this strictly hyperbolic system converges to the $\delta$-shock solution of the non-strictly hyperbolic system[11]. This kind of method can be thought of as an alternative to the vanishing viscosity method.

In the second part of the second chapter, we studied another strictly hyperbolic system which is a perturbed version of the model for large scale structure formation of universe, namely,

$$u_t + (\frac{(u+\epsilon)^2}{2})_x = 0;$$
$$\rho_t + (\rho u)_x = 0$$

When $\epsilon > 0$, the above system is strictly hyperbolic and can be solved by Lax's theory only for close by Riemann type initial data. It is observed that if $u_- - u_+ \geq 2\epsilon$, one can not get Lax type solution containing shock and rarefaction waves. Therefore to handle the case of large Riemann type data, the Shadow wave approach is chosen. This system also provides an example where a smallness condition is required on the initial data to get Lax type solution.

The third chapter is a continuation of the first one, where I dealt with a more general system. The general equation of compressible fluid flow reads

$$u_t + (\frac{u^2}{2} + F(\rho,\epsilon))_x = 0;$$
$$\rho_t + (\rho u + G(\rho,\epsilon))_x = 0$$

where $F(\rho,\epsilon) = \epsilon f(\rho)$ and $G(\rho,\epsilon) = \epsilon g(\rho)$ and $f$ is $C^2$, increasing and convex function where as $g$ is any differentiable linear decreasing function. It is to be noted that when $\epsilon = 1$, $f(\rho) = \frac{\rho^2}{2}$ and $g(\rho) = -\rho$ the system takes the form of well known *Brio system* which admits a $\delta$-shock solution. The above system is strictly hyperbolic when $\epsilon > 0$. To deal with this problem we used a similar techniques from the previous work mentioned above, with some additional difficulties and showed that the distributional limit (as $\epsilon$ tends to zero) of the solution for strictly hyperbolic system converges to the solution of non-strictly hyperbolic system which contains $\delta$ measure. In the same project we also deal with some non linear $g$, for example $g(\rho) = -\rho^2$ and $f(\rho) = \frac{\rho^2}{2}$.

### 7.1.2 Vanishing viscosity approach for some non-strictly hyperbolic systems

The fourth chapter of the thesis is devoted to study a non-strictly system of balance laws, namely,

$$u_t + uu_x = \Gamma(x,t);$$

$$\rho_t + (\rho u)_x = 0$$

with the initial data $u(x, 0) = u_0(x)$, $\rho(x, 0) = \rho_0(x)$, where $u_0$ and $\rho_0$ are locally integrable functions with certain growth condition. This kind of balance laws arises form 1-D Saint-Vanent model which is a type of shallow water equation, modeling incompressible fluid flow in an open channel of an arbitrary cross-section. This chapter deals with the kinematic case of the Saint-Vanent model. Besides this, the problem is of mathematical interest also. Indeed, the vanishing viscosity limit for the Burgers equation was studied by E.Hopf[59] and in 2003 Ding xiaqi and Ding Yi[61] extended the idea of Hopf for Burgers equation with a nonhomogeneous term. The important point is to note here that the nonhomogeneous term considered by Ding is unbounded in the space variable. In this work, we addressed the above-mentioned problem coupled with a continuity equation. Unlike Ding et.al, we consider a system of conservation laws when the nonhomogeneous term depends both on time and space variables. The vanishing viscosity method is used to find an explicit formula for that inhomogeneous system with a general type of initial data. Along the way, we contributed some developments in the vanishing viscosity method, also by localization of initial data, we showed that the vanishing viscosity limit for the first equation ( inhomogeneous Burgers equation ) satisfies the equation. Now the question is, in what sense $\rho$ satisfies the equation as $u\rho$ is not well defined in general. To handle this situation, we use non-conservative product [35], for example Volpert product[34]. There are also certain difficulties depending upon the geometry of $u$. We resolve the problems as follows: if $u$ is piecewise smooth then we showed that $\rho$ satisfies the second equation in the classical sense in the continuous region of $u$. Along the discontinuity curves we showed that the measure $R_t + uR_x = 0, (\rho = R_x)$ where $uR_x$ is understood as Volpert product. If $u$ is not piecewise smooth then by using a limiting approximation argument and the properties of Volpert product we showed that $\rho$ satisfies the equation. The techniques used here are significantly different from Ding and may be applied to other systems.

### 7.1.3 Measure valued solutions for conservation laws with discontinuous flux

In the fifth chapter, we propose the measured valued weak solution for scalar conservation laws with discontinuous flux and provide an explicit formula for the same. The scalar conservation laws with discontinuous flux read

$$\begin{aligned} &u_t + (F(x,u))_x = 0, \\ &u(x,0) = u_0(x) \end{aligned} \tag{7.1.1}$$

where $F(x,u) = H(x)f(u) + (1 - H(x))g(u)$, $H$ is a Heaviside function, $u_0$ is bounded measurable function and $f$,$g$ are locally Lipschitz function in general. Depending on the nature of $f$ and $g$ the above equation has various physical applications in different fields such as two-phase flow in the discontinuous porous medium, traffic flow on highways with different densities, etc. The above initial value problem has been shown to have a unique bounded solution when $f$ and $g$ are functions with one critical point or they are monotone with the same monotonicity[96, 63]. In fact, [63] provides a Lax-Oleinik type formula when $f, g \in C^1(\mathbb{R})$ and strictly convex with superlinear growth, i.e $\lim\limits_{|u|\to\infty} \frac{f(u)}{|u|} = \infty$. However, the existence of bounded weak solution, when $f$ and $g$ are strictly convex(or strictly concave) with $g$ increasing and $f$ decreasing remained unsettled. It is important to note that when $f, g$ are decreasing and increasing respectively, in whole $\mathbb{R}$, one already loses the superlinearity condition. Also, in this case, as the characteristics overlap each other at the interface $x = 0$, cases may arise when there is no bounded weak solution exists. To settle this question, first we looked for the linear case, that is $g(u) = au$ and $f(u) = bu$, with $a \geq 0$, $b \leq 0$. Observe that the linear fluxes are not strictly convex and therefore need to be treated separately. We use the vanishing viscosity method and end up with a solution containing $\delta$-measure. For any general $f, g \in C^1(\mathbb{R})$ strictly convex with $f' \leq 0$ and $g' \geq 0$, we are also able to provide a Lax-Oleinik type formula with a $\delta$-measure along the interface. Furthermore, convergent, conservative finite volume numerical schemes are proposed to capture $\delta$-shock efficiently. The numerical schemes are also extended to the hyperbolic system which does not pose a bounded weak solution.

### 7.1.4 Initial-boundary value problem for 1D pressureless gas dynamics

The fifth chapter of the thesis deals with the pressureless gas dynamics system in the quarter plane. More precisely, the question is to give an explicit entropy solution for pressureless gas flow with initial-boundary values, i.e

$$\begin{aligned} \rho_t + (\rho u)_x &= 0; \\ (\rho u)_t + (\rho u^2)_x &= 0. \end{aligned} \tag{7.1.2}$$

with the initial data $(\rho(x,0), u(x,0)) = (\rho_0, u_0)$ and the boundary data $(\rho(0,t), \rho(0,t)u(0,t)) = (\rho_b, \rho_b u_b)$, where the initial and boundary data are bounded measurable function. Also we ask under what conditions and in what sense boundary data can be prescribed. The initial value problem for (7.1.2) has been intensively studied in the literature. The key issue is that, in general, $\rho$ is no longer a function but a measure. This led to the introduction of various strongly related notions of weak solutions such as measure solutions, duality solutions, duality solutions based on vanishing viscosity etc. On the other hand explicit formula for the above system with initial value was given by Rykov, Sinai and Weinan[14] via generalized variational principle and these formulae are extended by Wang,Huang and Ding[74] when $u_0$ is not continuous. Their key idea is to introduce a generalized potential,

$$F(y,x,t) = \int_0^y \rho_0(\eta)(tu_0(\eta) + \eta - x)d\eta.$$

The explicit formula for the Burgers equation in the quarter plane via vanishing viscosity was given by Joseph [115]. Then Joseph and Gowda extended their previous work to provide an explicit formula for scalar conservation laws when the flux is strictly convex with superlinear growth [76]. To the best of our knowledge, no attempts have been made in the literature so far to solve the initial-boundary value problem for the pressureless gas dynamics model. The reason for this neglect may be that it is not clear in what sense boundary data can be prescribed.

In this chapter, we succeeded in giving a physically meaningful answer to this question, in one space dimension. we extend the method of Huang and Wang [73] by introducing a second type of potential – boundary potentials,

$$G(\tau,x,t) = \int_0^\tau [x - (t-\eta)u_b(\eta)]u_b(\eta)\rho_b(\eta)d\eta.$$

The locus at which the new boundary potentials and the previously established initial potentials coincide plays a crucial role as a new ingredient in constructing the solution.

The possibility of prescribing boundary data is shown to depend on the behavior of the generalized potentials at the boundary. We show that the constructed solution satisfies an entropy condition and it conserves mass, whereby mass may accumulate at the boundary. Conservation of momentum again depends on the behavior of the generalized boundary potentials.

## 7.2 Future Research

The future problems/ proposals are mostly on (i) vanishing viscosity approximation of conservation laws and (ii) constructing explicit formula for gas dynamics equation and related models.

### 7.2.1 Vanishing viscosity approximation for scalar conservation laws with discontinuous flux

The study of the conservation laws with discontinuous flux and related parabolic problems has been an important area of interest during the last decade. Even in one dimension, the problems are quite interesting because of the possibility to give different non-equivalent generalizations of Kruzkov's entropy solution. Moreover, in the literature there are several concepts of entropy solutions for conservation laws with discontinuous flux, see equation. Adimurthi et.al[94] studied the concept of $AB$-entropy solution when both the fluxes are either convex or concave. For linear fluxes, they used convex(or concave) approximations to find $AB$ connections. However, the additional conditions $f' \leq 0$ and $g' \geq 0$ gives rise to a problem of finding $AB$ connections, i.e it may happen that there doesn't exist any $(A, B)$ connection which can produce a $AB$-entropy solution. In this case, there is a concentration($\delta$-wave) along the discontinuity which is successfully solved in chapter 6. We used vanishing viscosity method for linear fluxes $f(u) = bu$ and $g(u) = au$ with $a \geq 0$, $b \leq 0$ and for general convex fluxes $f$,$g$ with the conditions $f' \leq 0$ and $g' \geq 0$, a Lax-Oleinik type formula is given. Our next aim is to use the vanishing viscosity approach for general fluxes $f, g$ with the above-mentioned conditions (we call it *overcompressive flux pair*). The introduction of a viscous term in the right-hand side of (7.1.1) makes the equation parabolic type and it reads

$$u_t + F(x, u)_x = \epsilon u_{xx}, \tag{7.2.1}$$

where

$$F(x,u) = H(x)f(u) + (1 - H(x))g(u).$$

In Chapter 6, the weak formulation for the above viscous equation is given by

**Definition 7.2.1.** *A function* $u \in \mathbb{C}^0(\mathbb{R} \times \mathbb{R}^+; \mathbb{R})$ *is a weak solution of the equation* (7.2.1) *if the following integral identity holds*

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty (u\phi_t - \epsilon u_x \phi_x) dx dt + \int_0^\infty \int_{-\infty}^\infty F(x,u)\phi_x dx dt \\ &+ \int_0^\infty [f(u(0,t) - g(u(0,t))]\phi(0,t)dt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0, \end{aligned} \tag{7.2.2}$$

*for all* $\phi \in \mathbb{C}_c^\infty(\mathbb{R} \times [0,\infty))$.

We shall study the problem (7.2.1) and find out an appropriate entropy condition for the limit solution. To carry out the analysis, our plan is to study the viscous equation separately in two quarter planes putting boundary condition $u(0,t) = \beta(t)$. That is we shall study two parabolic boundary value problems such as

$$\begin{aligned} &u_t + f(u)_x = \epsilon u_{xx}, \;\; x > 0 \\ &u(x,0) = u_0(x), u(0,t) = \beta(t) \end{aligned} \tag{7.2.3}$$

and

$$\begin{aligned} &u_t + g(u)_x = \epsilon u_{xx}, \;\; x < 0 \\ &u(x,0) = u_0(x), u(0,t) = \beta(t) \end{aligned} \tag{7.2.4}$$

and to show the existence of $C^1$ (with respect to the space variable) solution $u^\epsilon(\epsilon > 0)$ in the half plane. Then one has to obtain proper estimates on the boundary function $\beta(t)$ and this in turn will give estimates on $u^\epsilon$ for passing to the limit as $\epsilon$ tends to zero.

In Chapter 6, we obtained the asymptotic behavior for the solution of (7.2.1) and showed that the solution converges to a *steady state solution* as $t \to \infty$ when fluxes are linear, i.e $f(u) = bu$ and $g(u) = au$ with $a \geq 0$, $b \leq 0$. Now we propose to study the asymptotic behavior for more general *overcompressive flux pair.* Also we aim to construct a numerical scheme in the presence of the viscous term (i.e for the parabolic equation).

### 7.2.2 Vanishing viscosity limit for adhesion model

The next proposal is to study the vanishing viscosity limit for the adhesion model [116]

$$u_t + (u \cdot \nabla)u = \epsilon \Delta u$$

$$\rho_t + \nabla \cdot (u\rho) = 0.$$

If $u = \nabla\phi$, the vanishing viscosity limit for the first equation is determined in [116]. The limit for $\rho^\epsilon$ is expected to be a measure known for the one-dimensional case. We aim to determine an explicit representation of the second component $\rho^\epsilon$ as $\epsilon$ tends to zero for multi-dimension. There are certain difficulties to find the limit for $\rho^\epsilon$ in the multidimensional case. Indeed, the explicit formula of $\rho^\epsilon(\epsilon > 0)$ is in the non-conservative form. Even the limits for the individual terms are known, we cannot find the limit as a whole in the space of distribution. A variant of compensated compactness argument may be required to find the limit. The next difficulty is to develop a notion of solution such that the limit satisfies the second equation. This is because the limit of $\rho^\epsilon$ is a measure in general, the limit of $u^\epsilon$ is a function of bounded variation and the product does not make sense in general in the space of distributions.

### 7.2.3 Initial-boundary value problem for 1D pressureless flow with measure data

As an extension of Chapter $5$, next we want to construct an explicit entropy solution for the initial-boundary value problem (7.1.2) when datum are non-negative Radon measures. More precisely, $\rho_0$ and $\rho_b$ are non-negative Radon measures and $u_0, u_b$ are bounded measurable (with respect to $\rho_0$ and $\rho_b$ respectively) functions. The choice of initial-boundary data as non-negative measures is natural since the solution for the pressureless model turns out to be measure or produce a vacuum. For example, in the shock case the solution for the Riemann type data contains a $\delta-$ measure in the second component $\rho$, and in the rarefaction case $\rho$ contains a vacuum. Huang and Wang [73] obtained the explicit formula for purely initial value problem by introducing the initial potential. In order to construct a solution for the initial boundary value problem, we need a new boundary potential and a proper notion of boundary condition.

Another interesting direction is to study the explicit formula for various systems related to the pressureless gas model (e.g pressureless gas model with a time-dependent source, Eulerian droplet model, multidimensional gas model [79], gas flow in a general nozzle [117] ) accompanied with initial-boundary value by determining the appropriate generalized potentials and studying generalized characteristics.

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