Quantization of $A_0(K)$-spaces and M-ideals in matrix ordered spaces

By

ANINDYA GHATAK

MATH11201304001

**National Institute of Science Education and Research,
Bhubaneswar**

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

April, 2019

Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by **ANINDYA GHATAK** entitled **``Quantization of $A_0(K)$-spaces and M-ideals in matrix ordered spaces"** and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman - **Prof. V. Muruganandam**		**Date:** 16/April/19
Guide - **Dr. Anil Kumar Karn**		**Date:** 16-4-2019
Examiner - **Prof. T. S. S. R. K. Rao**		**Date:** 16/4/19
Member 1- **Prof. P. Bandyopadhyay**		**Date:** 16/4/19
Member 2- **Dr. Sanjay Parui**		**Date:** 16/4/19
Member 3- **Dr. Sutanu Roy**		**Date:** 16/4/19

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: 16-04-2019

Place: NISER, Bhubaneswar

(Dr. Anil Kumar Karn)

Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Anindya Ghatak

(Anindya Ghatak)

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Anindya Ghatak

(Anindya Ghatak)

List of Publications arising from the thesis

Accepted

1. "*M-ideals and split faces in a non-unital ordered Banach spaces*", Anindya Ghatak and Anil K. Karn, Positivity, DOI: 10.1007/s11117-018-0614-1, To appear.

Others

1. "*Quantization of $A_0(K)$-spaces*", Anindya Ghatak and Anil K. Karn, Communicated.

2. "*Cones, ideals, and their duality in order smooth p-normed spaces*", Anindya Ghatak, Communicated.

3. "*CM-ideals in ordered operator spaces*", Anindya Ghatak and Anil K. Karn, Communicated.

Anindya Ghatak

(Anindya Ghatak)

Dedicated to

MY PARENTS

&

my beloved niece

Nandita

ACKNOWLEDGEMENTS

First and foremost, I would like to express my deepest gratitude to my advisor Dr. Anil Kumar Karn for all his guidance, constant encouragement, and continuous support throughout my stay at NISER. Without his careful guidance, it would have been impossible to complete my Ph.D.

I would like to thank all the members of my doctoral committee, Prof. P Bandyopadhyay, Prof. V. Muruganandam for the many constructive suggestions they had given during the doctoral committee meetings. I wish to thank all faculty members of NISER for the courses they gave me. I am thankful to the administrative staff and other members of NISER for their cooperation and for making my stay at NISER comfortable. Also, I wish to thank DAE for providing the fellowship during my Ph.D. program.

I would like to thank my dear friends Bikram, Anoop, Moni, Abhash, and many others. The journey would have never been so pleasant and delightfully without them.

I am deeply indebted to my teachers who have taught me at different levels and especially Prof. T. K. Dutta and Dr. Sandip Jana for encouraging me to pursue higher studies in mathematics. I am also thankful to my post-graduate teacher Prof. S. K. Acharya for the excellent course he offered in Measure theory.

I am blessed to have unconditional love and encouragement all these years with support from my father, and mother, and one and only one aunty late Sabita Ghatak, and others family members. My father and mother will always be an inspiration to me throughout my life. I would like to dedicate this thesis to them.

Index of Notations

$W^{\perp}$ .... Annihilator of W

V_1 .... Closed unit ball of normed space V

K .... Compact convex set

$co(A)$ .... Convex hull of A

u, v, w .... Elements of V

f, g, h .... Elements of V^*

$\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ .... Elements of V^{**}

H .... Hilbert space

E .... Locally convex space

$\mathcal{B}(H)$ .... Set of all bounded linear operator on Hilbert space H

$A(K)$ .... Set of all continuous affine functions on K

$A_0(K)$ .... Set of all continuous affine functions on K vanishing at 0

$\text{ext}(K)$ .... Set of all extreme point of K

$cone(F)$ .... Smallest cone containing set F

^{T} .... Transpose operation

Contents

Synopsis

Homi Bhabha National Institute

Synopsis of Ph.D. Thesis

1. Name of the Student:	**Anindya Ghatak**
2. Name of the Constituent Institution:	**National Institute of Science Education and Research, Bhubaneswar**
3. Enrolment No. :	**MATH11201304001**
4. Title of the Thesis:	**Quantization of $A_0(K)$ spaces and M-ideals in matrix ordered spaces**
5. Board of Studies:	**Mathematical Sciences**

1 Introduction

In this thesis, we work in two directions in the area of order theoretic functional analysis (commutative and non-commutative). On the one hand, we concentrate on the representations of C*-ordered operator spaces and operator system using continuous affine functions. More preciously, if $\{K_n\}$ is an L^1-matrix convex set, then by giving proper bi-module action and linear and order structure on $\{A_0(K_n\})$, we show that every C^*-ordered operator space can be characterized as $\{A_0(K_n)\}$ for some suitable L^1-matrix convex set $\{K_n\}$. This is a generalization of a program initiated by Kadison [1951] and later independently by Asimov [1968], Choi-Effros [1977], Ruan [1988], Blecher-Ruan-Sinclair [1990], Webster-Winkler [1999] and Karn [2010].

On the other hand, we study the order theoretic properties of M-ideals in non-unital ordered Banach spaces as well as CM-ideals in (non-unital) ordered operator spaces. Note that in a non-unital ordered Banach space, the state space may not be compact and convex. However, we know that the quasi state space is compact and convex. Keeping this in mind, we introduce the notion of split faces of the quasi state space, and that of L^1-matricial split faces in the matricial version. We characterize M-ideals and in terms of split faces of the quasi state space and similarly for CM-ideals in terms of L^1-matricial split faces of the quasi state space. Note that the notion of an M-ideal is compatible with order smooth ∞-normed spaces and that of an L-ideal is compatible with order smooth 1-normed spaces. We generalize these notions to smooth p-order ideals in order smooth p-normed spaces. We study their duality. It may further be noted that the notion of smooth ∞-order ideals may be also seen as a generalization of the Archimedean ideals studied by Størmer [1968].

There are six chapters in this thesis. Chapter 1 is the introduction of the thesis. In Chapter 2, we recall some basic definitions and properties of order normed spaces, affine function spaces, M-ideals and concept of ordered operator spaces etc. This is needed in the rest of the chapters. Now we discuss the contents of the other four chapters in the following sections.

2 Quantization of $A_0(K)$-space

In 2001, Karn [40] proved that an (abstract) C*-ordered operator space is precisely an (abstract) $*$-operator space which can be "order embedded" in a C*-algebra. We prove a 'quantized' functional representation of C*-ordered operator spaces. Main definitions and results in this context are the following:

Definition 2.1 **(C*-ordered operator spaces)** *[40] A matrix ordered space $(V, \{M_n(V)^+\})$ together with a matrix norm $\{\|\cdot\|_n\}$ is said to be a* C*-ordered *operator space if $(V, \{\|\cdot\|_n\})$ is an abstract operator space and V^+ is proper such that for each $n \in \mathbb{N}$, the following conditions hold:*

1. *$*$ is isometry on $M_n(V)$;*
2. *$M_n(V)^+$ is closed;*
3. *If $f, g, h \in M_n(V)_{sa}$ such that $f \leq g \leq h$, then $\|g\|_n \leq \max\{\|f\|_n, \|h\|_n\}$, (In other words, $M_n(V)_{sa}$ satisfies $(O.\infty.1)$ property).*

Let K be a compact convex set in a locally convex set E such that $0 \in ext(K)$. An element $k \in K$ is called a *lead point* of K ($k \in lead(K)$) if $k = \alpha k_1$ for some $k_1 \in K$ with $\alpha \in [0, 1]$, then $\alpha = 1$.

Definition 2.2 **(L^1-matrix convex set)** *Let V be a $*$-locally convex space. Let $\{K_n\}$ be a collection of compact convex sets $K_n \subseteq M_n(V)_{sa}$ such that*

$0 \in ext(K_n)$ *for all* n*. Then the collection* $\{K_n\}$ *of sets is called an* L^1-matrix convex set *if the following conditions hold:*

L_1 *If* $u \in K_n$ *and* $\gamma_i \in \mathbb{M}_{n,n_i}$ *such that* $\sum_{i=1}^k \gamma_i\gamma_i^* \leq I_n$*, then* $\oplus_{i=1}^k \gamma_i^* u \gamma_i \in K_{\sum_{i=1}^k n_i}$.

L_2 *If* $u \in K_{2n}$ *so that* $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ *for some* $u_{11}, u_{22} \in K_n$ *and* $u_{12} \in M_n(V)$*, then* $u_{12} + u_{12}^* \in co(K_n \cup -K_n)$.

L_3 *Let* $u \in K_{m+n}$ *with* $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ *so that* $u_{11} \in K_m, u_{22} \in K_n$ *and* $u_{12} \in M_{m,n}(V)$ *and if* $u_{11} = \alpha_1 \widehat{u_{11}}, u_{22} = \alpha_{22}\widehat{u_{22}}$ *with* $\widehat{u_{11}} \in lead(K_m), \widehat{u_{22}} \in lead(K_n)$*, then* $\alpha_1 + \alpha_2 \leq 1$.

With the help of this definition, we arrive the following characterization.

Theorem 2.3 $(A_0(K_1, V), \{M_n(A_0(K_1, V))^+\}, \{\|\cdot\|_n\})$ *is a* C**-ordered operator space.*

We introduce the concept of regular embedding in the L^1-matrix convex set to characterize the operator systems among the C*-ordered operator spaces (see Theorem 2.5).

Definition 2.4 *Let* $\{K_n\}$ *be an* L^1*-matrix convex set in a* $*$*-locally convex space* V*. Then* $\{K_n\}$ *is called* regularly embedded *in* V *if* L_1 *is regularly embedded in* V_{sa}*. In other words,*

1. L_1 *is compact and convex; and*

2. $\chi : V_{sa} \mapsto (A(L_1)_{sa}^*)_{w*}$ *is a linear homeomorphism.*

Here $\chi(w)(a) = \lambda a(u) - \mu a(v)$ for all for all $a \in A(L_1)_{sa}$ if $w = \lambda u - \mu v$ for some $u, v \in L_1$ and $\lambda, \mu \in \mathbb{R}^+$.

Theorem 2.5 *Let $\{K_n\}$ be a regularly embedded, L^1-matricial cap in V. Then $A_0(K_1, V)$ has an order unit, say e so that $(A_0(K_1, V), e)\}$ is a matrix order unit space.*

3 M-ideals in non-unital ordered Banach spaces

We recall that closed subspace W of a real Banach space V is said to be an *L-summand* if there exists a unique closed subspace W' of V such that

$$V = W \oplus_1 W'.$$

A closed subspace W of a real Banach space V is said to be an M-ideal if $W^\perp$ (the annihilator of W) is an L-summand of V^*.

Definition 3.1 *Let V be a normed space and let K be a non-empty, closed, convex set in V. A proper face F of K is said to be a split face of K if F_K^C is a proper face of K such that $K = F \oplus_c F_K^C$. Here $F_K^C = \cup\{face_K(v) : v \in K \text{ and } face_K(v) \cap F = \emptyset\}$ and by $K = F \oplus_c F_K^C$, we mean that for each $v \in K$ there exist unique $u \in F, w \in F_K^C$ and $\lambda \in [0, 1]$ such that $v = \lambda u + (1 - \lambda)w$.*

Theorem 3.2 *Let V be a complete order smooth ∞-normed space and let W be a closed subspace of V. Then W is an M-ideal in V if and only if $W^{\perp'+}$ is convex and $V^{*+} = W^{\perp+} \oplus_1 W^{\perp'+}$.*

Proposition 3.3 *Let V be a complete order smooth ∞-normed space and let W be a closed subspace of V. Then W is an M-ideal in V if and only if $W^\perp \cap Q(V)$*

is a split face of $Q(V)$.

3.1 M-ideals and adjoining of an order unit

Let V be an order smooth ∞-normed space. Consider $\tilde{V} = V \oplus \mathbb{R}$. If we define $\tilde{V}^+ = \{(v, \alpha) : l_V(v) \leq \alpha\}$ where $l_V(v) = \inf\{\|u\| : u, u+v \in V^+\}$, then $(\tilde{V}, \tilde{V}^+)$ becomes a real order unit space.

Theorem 3.4 *Let V be a complete order smooth ∞-normed space. Then V is an M-ideal in $\tilde{V}$ if and only if V is an approximate order unit space.*

4 CM-ideals in Ordered operator spaces

Let $1 \leq p \leq \infty$. An L^p-matrically normed matrix ordered space $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$ is said to be *matricially order smooth p-normed space*, if $\|\cdot\|_n$ satisfies $(O.p.1)$ and $(O.p.2)$ conditions on $M_n(V)_{sa}$ for each $n \in \mathbb{N}$.

A projection P of an operator space V is called a *CM-projection* if $\|v\|_n = \max\{\|P_n(v)\|_n, \|(I-P)_n(v)\|_n\}$ for all $v \in M_n(V)$.

Let V be an operator space and let W be a closed subspace of V. Then W is called a *CM-summand* if $W = P(V)$ for some CM-projection P of V. Let V be an operator space and let W be a closed subspace of V. Then W is called a *CM-ideal* in V if $W^{\perp\perp}$ is a CM-summand in V^{**}.

4.1 Characterization of CM-ideals

Let V be an operator space and P be a projection of V^*. We call P as *CL-projection* if $\|f\|_n = \|P_n(f)\|_n + \|(I-P)_n(f)\|_n$ for all $f \in M_n(V^*)$. Let W be a subspace of V. Then $W^\perp$ is called *CL-summand* of V^* if there is a CL-projection P of V^* such that $P(V^*) = W^\perp$.

Proposition 4.1 *Let $(V, \{\| \cdot \|_n\})$ be an operator space. Let P be a CM-projection of V^{**}. Then there exists a unique CL-projection L of V^* such that $L_n^* = P_n$ for all $n \in \mathbb{N}$.*

Corollary 4.2 *Let V be an operator space and W be a closed subspace of V. If W is a CM-ideal in V, then there exists CL projection L of V^* onto $W^\perp$ and $W^\perp$ is an CL-summand of V^*.*

Proposition 4.3 *Let V be a matricially order smooth ∞-normed space and let W be a self-adjoint subspace of V. Let P be the CL-projection of V^* onto $W^\perp$. Then $P_n(f^*) = P_n(f)^*$ for all $f \in M_n(V^*)$.*

Theorem 4.4 *Let V be a matricially order smooth ∞-normed space and let W be a closed self-adjoint subspace of V. Then W is a CM-ideal in V if and only if $M_n(W)_{sa}$ is an M-ideal in $M_n(V)_{sa}$ for each $n \in \mathbb{N}$.*

We assume that V is a matricially order smooth ∞-normed space and we denote $K_n = M_n(V^*)_{sa} \cap M_n(V^*)_1$ for each $n \in \mathbb{N}$.

Proposition 4.5 *Let V be an matricially order smooth ∞-normed space. If $f \in K_n$ and $\gamma_i \in \mathbb{M}_{n,n_i}$ such that $\sum_{i=1}^k \gamma_i \gamma_i^* \leq I_n$, then $\oplus_{i=1}^k \gamma^* f \gamma_i \in K_{\sum_{i=1}^k n_i}$.*

Theorem 4.6 *Let V be a matricially order smooth ∞-normed space and W be a self-adjoint subspace of V. If L is an CL-projection of V^* onto $W^\perp$. Then L is a CP-map.*

Definition 4.7 *[32] Let V be a matricially order smooth ∞-normed space. Then a collection $\{D_n\}$ of sets with $D_n \subset M_n(V^*)_{sa}$ and $0 \in \partial_e(D_n)$ is called an* L^1-matrix convex set *if the following conditions hold:*

1. *If $f \in D_n$ and $\gamma_i \in M_{n,n_i}$ such that $\sum_{i=1}^k \gamma_i \gamma_i^* \leq I_n$, then $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in D_{\sum_{i=1}^k n_i}$;*

2. *If* $f \in D_{2n}$ *so that* $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$ *for some* $f_{11}, f_{22} \in D_n$ *and* $f_{12} \in M_n(V^*)$*, then then* $f_{12} + f_{12}^* \in co(D_n \cup -D_n)$*;*

3. *Let* $f \in D_{m+n}$ *with* $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$ *so that* $f_{11} \in D_m$ *and* $f_{22} \in M_{m,n}(V^*)$ *and if* $f_{11} = \alpha_1 \widehat{f_{11}}$ *and* $f_{22} = \alpha_2 \widehat{f_{22}}$ *with* $\widehat{f_{11}} \in lead(D_m)$ *and* $\widehat{f_{22}} \in lead(D_n)$*, then we have* $\alpha_1 + \alpha_2 \leq 1$*.*

We note that if V is a matricially order smooth ∞-normed space. Then $\{Q_n(V)\}$ is an L^1-matrix convex set. Let V be a matricially order smooth ∞-normed space. Then an L^1-matricial convex set $\{D_n\}$ of V^* such that $D_n \subset Q_n(V)$ is called an L^1*-matricial split face* of $\{Q_n(V)\}$ if for each n, D_n is a split face of $Q_n(V)$.

Theorem 4.8 *Let* V *be a matricially order smooth* ∞*-normed space and* W *be a self adjoint subspace of* V*. Then* W *is a* CM*-ideal of* V *if and only if* $\{M_n(W^\perp) \cap Q_n(V)\}$ *is an* L^1*-matricial split face of* $\{Q_n(V)\}$*.*

5 Smooth p-order ideals

Theorem 5.1 *Let* $(V, V^+, \|.\|)$ *be an order smooth* p*-normed space and* W *be a subspace of* V*. Let* $\varphi_W : V \mapsto V/W$ *and* $\varphi_{W^\perp}^* : V^* \mapsto V^*/W^\perp$ *be the natural homomorphisms. Then we have the following duality:*

1. $(W, W^+, \|.\|)$ *is an order smooth* p*-normed space iff* $(V^*/W^\perp, (V^*/W^\perp)^+, \|.\|)$ *is an order smooth* p'*-normed space satisfying* $(OS.p'.2)$*.*

2. $(V/W, (V/W)^+, \|.\|)$ *is an order smooth* p*-normed space if and only if* $(W^\perp, W^{\perp +}, \|.\|)$ *is an order smooth* p'*-normed space satisfying* $(OS.p'.2)$*.*

Theorem 5.2 *Let* $(V, V^+, \|.\|)$ *be an order smooth* p*-normed space and* W *be a subspace of* V*. Let* $\varphi_W : V \mapsto V/W$ *and* $\varphi_{W^\perp} : V^* \mapsto V^*/W^\perp$ *be the natural homomorphisms. Then we have the following duality:*

1. $(W^\perp, W^{\perp+}, \|.\|)$ *is an order smooth* p'*-normed space if and only if*

$$(V^{**}/W^{\perp\perp}, (V^{**}/W^{\perp\perp})^+, \|.\|)$$

   is an order smooth p*-normed space satisfying* $(OS.p.2)$*;*

2. *If* $(V^*/W^\perp, (V^*/W^\perp)^+, \|.\|)$ *is an order smooth* p'*-normed space, then* $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ *is an order smooth* p*-normed space satisfying* $(OS.p.2)$*;*

3. *Assume that* $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$*. If* $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ *is an order smooth* p*-normed space, then* $(V^*/W^\perp, (V^*/W^\perp)^+, \|.\|)$ *is an order smooth* p'*-normed space.*

Definition 5.3 *If* $(V, V^+, \|.\|)$ *is an order smooth* p*-normed space. Then a subspace* W *is called* smooth p-order ideal *in* V *if* W *satisfies the following conditions:*

1. $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$;

2. $(W, W^+, \|.\|)$ *is an order smooth* p*-normed space;*

3. $(V/W, (V/W)^+, \|.\|)$ *is an order smooth* p*-normed space.*

5.1 Smooth ∞-order ideals

Theorem 5.4 *Let* $(V, V^+, \|.\|)$ *be an order smooth* ∞*-normed space and* W *be a subspace of* V*. Then the following are equivalent:*

1. $((V/W), (V/W)^+, \|.\|)$ *is an order smooth* ∞*-normed space;*

2. $(W^{\perp}, W^{\perp+}, \|.\|)$ *satisfying* $(OS.1.2)$;

3. $\|v + W\| = \sup\{|f(v)| : f \in (W^{\perp})_1 \cap W^{\perp+}\}$;

4. $\|\mathfrak{f} + W^{\perp\perp}\| = \sup\{|\mathfrak{f}(f)| : f \in (W^{\perp})_1 \cap W^{\perp+}\}$;

5. $((V^{**}/W^{\perp\perp}), (V^{**}/W^{\perp\perp})^{+}, \|.\|)$ *is an order smooth* ∞*-normed space.*

Proposition 5.5 *Let* $(V, V^{+}, \|.\|)$ *be an order smooth* ∞*-normed space and let* W *be a subspace of* V*. If* W *is an* M*-ideal, then* $((V/W), (V/W)^{+}, \|.\|)$ *is an order smooth* ∞*-normed space.*

Theorem 5.6 *Let* $(V, V^{+}, \|.\|)$ *be an order smooth* ∞*-normed space and* W *be a subspace of* V*. If* W *is an* M*-ideal, then following are equivalent:*

1. $(W, W^{+}, \|.\|)$ *is an order smooth* ∞*-normed space;*

2. *if* $f \in W^{*+}$*, then there is a* $g \in V^{*+}$ *such that* $g_{|_W} = f$;

3. *if* $\varphi_{W^{\perp}}(V^{*+}) = \overline{\varphi_{W^{\perp}}(V^{*+})}^{w^*}$;

4. $\|f\| = \sup\{f(w) : w \in W^{+} \cap W_1\} \ \forall \ f \in W^{*+}$.

Theorem 5.7 *Let* $V^{+}, \|.\|)$ *be an order smooth* 1*-normed space, satisfying* $(OS.1.2)$ *and let* W *be a subspace of* V*. If* W *is an* L*-summand, then* W *is an smooth* 1*-order ideal in* V*.*

CHAPTER 1

Introduction

Order structure is an important aspect of functional analysis. Its roots can be traced in late 30's in the work of Kantarovich [45]. In 1941, using lattice structure, Kakutani characterized $C_{\mathbb{R}}(K)$ (K is a compact and Hausdorff space) as AM spaces [46]. Therefore, following Gelfand-Naimark Theorem [31], the self-adjoint part of every commutative C*-algebra is a Banach lattice. In 1951, Kadison proved a representation theorem for the self-adjoint part of an arbitrary unital C*-algebra $\mathcal{A}$ as the space of continuous real valued affine functions on the state space of $\mathcal{A}$ [39]. This appears to be one of the early corner-stone in the order-theoretic (non-commutative) functional analysis. In this seminal paper, he observed that the same result holds for the self-adjoint part of any unital self-adjoint subspace of $\mathcal{A}$ (that is, a concrete operator system in $\mathcal{A}$). Let K be a compact and convex set in a locally convex space E, and let $A(K)$ be the space of all real valued continuous affine functions on K. Then $A(K)$ is an order unit space. In particular, the self-adjoint part of an operator system is an order unit space.

A nice duality theory of ordered Banach spaces was developed during 1950's and 60's in the works of Bonsall, Edwards, Ellis, Asimov and Ng and many others [6, 7, 15, 16, 30]. For example, in 1964, D. A. Edwards introduced the notion of base normed spaces [23] and A. J. Ellis studied duality between order unit spaces and base normed spaces [30]. For more details one may refer to [2] and [37] and references therein.

In 1968, Asimov introduced the notion of a universal cap (say, K) of a cone in a real ordered vector space and studied $A_0(K)$ as a non-unital prototype of $A(K)$ (also see, [48]). However, the functional representation theorem of Kadison (and the work that followed) was limited to self-adjoint elements only. Subsequently, the order theoretic functional analysis was limited to only real scalars.

After a long gap, in 1977, Effros [29] observed the following relation between the norm of an arbitrary element of a C*-algebra $\mathcal{A}$ and the order structure in $M_2(\mathcal{A})$:

$$\|a\| \leq 1 \text{ if and only if } \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \geq 0.$$

Following this, in 1977, Choi and Effros introduced matrix ordered spaces and proved a generalization of Kadison's order unit spaces [18]. More precisely, they proved that every (concrete) operator system is exactly a matrix order unit space. This theory is also known as a beginning of quantization of functional analysis. In this sense, the Choi-Effros realization of an operator system as a matrix order unit space is a quantization of order unit space.

On the other hand, an emerging area of the theory of operator space was conceived in Ruan's Thesis in 1988 and was initially nurtured by Effros and Ruan and also by Blecher and Paulsen besides many others. The completely bounded maps of C^*-algebras, studied by W. B. Arveson in the late 1960s, are the proper morphism in the category of operator spaces (see e.g [5]). This can be described as a non-commutative generalization of Banach spaces. To be precise, Ruan studied L^p-matricially normed spaces ($1 \leq p \leq \infty$). He proved that L^∞-matrically normed spaces characterize operator spaces [54], also (see e.g. [27]). A characterization of (concrete) operator algebras as L^∞-matricially normed algebra was given by Ruan, Blecher and Sinclair in 1992 [14]. For more properties of L^p-matricially normed space one can see [28, 61, 62]. For tensor products and duality of operator space one can see work of Blecher and Paulsen [12, 13]. The non-commutative Hahn Banach Theorem was given by G. Wittstock in 1981 [64] (also see e.g. [28]). A quantization of $A(K)$-space appeared in the work of Webster and Winkler [60] in 1999. They proved an operator space version of the Krein-Milman theorem. For more literature on operator space theory (see also, [24, 25, 63]).

The non-unital matrix ordered spaces were studied by Schreiner in 1998 as "matrix regular operator space" [55] and independently by Karn and Vasudevan as "matricially Riesz normed spaces" [43, 44]. In 2007, Blecher and Neal studied ordered aspect of TROs [10] (see also, [9, 11]). Also one can see the works of Paulsen, Todorov and Tomforde in 2011 [49].

In 2007, Blecher and Neal [10] showed that the operator space dual of a C*-algebra can not be order embedded in any C*-algebra. In 2011, Karn [40] showed further that if a matrix ordered space is order embedded in a C*-algebra, then its operator space dual can not be order embedded in any C*-algebra. Thus the operator space duality fails to work in the context of ordered operator spaces. In 2010, Karn proposed a pair of axioms $(O.p.1)$ and $(O.p.2)$ for $1 \leq p \leq \infty$ and renewed the study of a matrix ordered space with a (matrix) norm, in which the matrix norm is related to the (matrix) order. He called it a (matricially) order smooth p-normed space. The advantage of studying these spaces over L^p-matricially normed spaces is that every matricially order smooth ∞-normed space can be order embedded in some C*-algebra. Here, he also showed that if V is a matricially order ∞-normed space, then an order unit can be adjoined to it so that the resulting space $\tilde{V}$ is an operator system (of co-dimension one). (A similar theory can be found in the work of Werner [62]. The two approaches are independent and lead to different directions.) This theory goes naturally with the matrix duality and thus extends "Choi Effros-Ruan"-program of matrix ordered matricially normed spaces [18, 54].

In 1957, J. Dixmier [21] characterized the closed two sided ideals of a von Neumann algebra as unitary invariant order ideals of the algebra. In 1963, E. G. Effros [22] showed that order ideals of a C*-algebra may be characterized as one sided ideals of it. More preciously, if $\mathcal{A}$ is a C*-algebra and $\mathcal{I}$ is a norm closed subspace of $\mathcal{A}$, then the following forms are equivalent:

1. $\mathcal{I}^+$ is an order ideal of $\mathcal{A}$;

2. $\mathcal{I}$ is a left ideal of $\mathcal{A}$;

3. left invariant subspace of its dual $\mathcal{A}^*$.

Following the representation of self-adjoint part of a unital C*-algebra $\mathcal{A}$ as affine function space $A(K)$ by Kadison [39] and study of order ideals in C*-algebra by Effros [22], many mathematicians got interested in the study of ideals in partially order vector spaces and in affine function spaces of compact convex sets during 1960s and 1970s. The order theoretic properties of ideals and its connection with faces of compact convex sets was one of the main interest for them. For example, in 1954, F. Bonsall [16] studied sub-linear functionals and generalized the Krein Milman's Theorem. Further, in 1956 he studied [15] regular ideals in ordered normed space and proved certain types of monotone extension theorem. Also, one can see work of Asimov [6, 7] in this direction.

In 1966, E. Størmer [57] studied the Archimedean ordered vector spaces which have a strong order unit and are complete in the order norm. He introduced the notion of Archimedean ideals and Archimedean faces of a compact convex set. A norm closed order ideal W of V is an *Archimedean ideal* if W is positively generated and V/W is Archimedean. He proves that if $\mathcal{I}$ is a closed subspace of C^*-algebra, then $\mathcal{I}_{sa}$ is an Archimedean ideal if and only if $\mathcal{I}$ is a two sided ideal.

In 1970, E. M. Alfsen and T. B. Andersen studied the split faces of compact convex sets (see e.g. [3]). In this paper, they discussed the extension properties of split faces of compact convex sets. An Archimedean ideal W of $A(K)$ is said to be 'near lattice ideal' if the corresponding (quotient) homomorphism $\varphi : A(K) \to A(K)/W$ satisfies the following property: For every $\epsilon > 0$ and $a_1, a_2 \in A(K)^+$, one has

$$[0, \varphi(a_1)] \cap [0, \varphi(a_2)] \subset \varphi([0, a_1 + \epsilon] \cap [0, a_2 + \epsilon]).$$

They showed that W is a near lattice ideal in $A(K)$ if and only if $W^\perp \cap K$ is a split face of K. In 1971, T. B. Anderson studied the order bounded extension properties of continuous affine functions of split faces of compact convex sets [4]. For extensive literature see [2].

In 1972, E. M. Alfsen with E. G. Effros wrote twin papers " Structure in real Banach spaces I and II" (see e.g. [1]). The central theme of the paper was the investigation of certain subspaces of V called "M-ideals", which are analogous to the self-adjoint parts of closed two sided ideals in a C*-algebra. Also, they prove in particular that W is an M-ideal in $A(K)$ if and only if $W^\perp \cap K$ is closed split face of K. This way, the notion of lattice ideal in $A(K)$ was generalized as M-ideal in Banach space context.

In this thesis, we work in two directions in the area of order-theoretic functional analysis (commutative and non-commutative). On the one hand, we concentrate on the representations of C*-ordered operator spaces and operator systems using continuous affine functions. More precisely, we introduce the notion of an L^1-matrix convex set in a $*$-locally convex space E. We show that if $\{K_n\}$ is an L^1-matrix convex set, then by defining appropriate proper bi-module action and linear and order structure on $\{A_0(K_n)\}$, every C*-ordered operator space can be characterized as $(A_0(K_1, E), \{M_n(A_0(K_1, E))^+\}, \{\| \cdot \|_n\})$ for some suitable L^1-matrix convex set $\{K_n\}$. This is a generalization as well as a quantization of a the functional representation of operator systems by Kadison in 1951.

On the other hand, we study the order theoretic properties of M-ideals in non-unital ordered Banach spaces as well as CM-ideals in (non-unital) ordered operator spaces. Note that in a non-unital ordered Banach space, the state space may not be compact and convex. However, we know that the quasi state space is compact and convex. Keeping this in mind, we introduce the notion of split faces of the quasi state space and that of L^1-matricial split faces in the matricial version. We characterize M-ideals in terms of split faces of the quasi state space and similarly for CM-ideals in terms of L^1-matricial split faces of the matricial quasi state space. Next, we generalize the notion of M-ideals by smooth p-order ideals in order smooth p-normed spaces. We study their duality relation. It may be noted that the notion of smooth ∞-order ideal may be seen as a generalization of the Archimedean ideals studied by Størmer in 1968 [57].

1.1 Arrangement of the remaining chapters

In the second chapter, we recall the basic definitions and properties of ordered normed spaces, affine function spaces, M-ideals, and some other notions related to ordered operator spaces.

In the third chapter, we prove a 'quantized' functional representation of C*-ordered operator spaces. The quantized functional representation of abstract operator systems was given by Webster and Winkler (see e.g. [60]). They rely on matrix convex sets. However, we consider matrix (Choi-Effros) duality and introduce the notion of L^1-matrix convex sets. We show that if V is a C*-ordered operator space and $Q_n(V) = \{f \in M_n(V^*)^+ : \|f\| \leq 1\}$ (in the matrix duality), then $\{Q_n(V)\}$ is an L^1-matrix convex set. We show in Theorem 3.1.2 that if V is a C^*-ordered operator space, then V is complete isometric, completely order isomorphic to $(A_0(Q(V), V^*), \{M_n(A_0(Q(V), V^*))^+\}, \{\|.\|_n\})$. Conversely, we show in Theorem 3.2.5 that if $\{K_n\}$ is an L^1-matrix convex set in a $*$-locally convex space E, then $(A_0(K_1, E), \{M_n(A_0(K_1, E))^+\}, \{\| \cdot \|_n\})$ is a C*-ordered operator space. Further, we introduce the matricial version of regularity embedding property of an L^1-matrix convex set $\{K_n\}$ and the matricial version of universal cap of an L^1-matrix convex set. Using the concepts of a universal cap and regular embedding property of an L^1-matrix convex set, in Theorem 3.3.4, we give a characterization of all abstract operator systems among all C*-ordered operator spaces.

In the fourth chapter, we study order theoretic properties of M-ideals in order smooth ∞-normed space. We obtain an order-theoretic version of the 'Alfsen-Effros' cone decomposition theorem [1, Theorem 2.9] for order smooth 1-normed spaces satisfying condition $(OS.1.2)$. As an immediate application of this result, we sharpen a result on the extension of bounded positive linear functionals on subspaces of order smooth ∞-normed spaces [41, Theorem 4.3]. In Proposition 4.3.13, we give a characterization of M-ideals in order smooth ∞-normed spaces by extending the notion of split faces of the state space to those of the quasi-state space. This result is a generalization of its counterpart for order unit spaces studied by Alfsen and Effros [1]. At the end of the chapter, we prove (in Theorem 4.4.7) that an order smooth ∞-normed spaces V is an M-ideal in $\tilde{V}$ if and only if it is an approximate order unit space. Here $\tilde{V}$ is the order unit space obtained by adjoining an order unit to V.

In the fifth chapter, we discuss some of the order theoretic properties of a CM-ideals in matricially ordered smooth ∞-normed spaces. We prove the duality between CM-ideals and CL-summands in the matrix duality set up. This result was proved for operator spaces in the operator space duality setup by Poon and Ruan in [52]. To be more specific, in Corollary 5.2.2, we show that if W is closed subspace of matricially order smooth ∞-normed space, then W is a CM-ideal in V if and only if $W^{\perp}$ is a CL-summand of V^*, where V^* is the matricial dual V. In 1994, Effros and Ruan proved that W is a CM-ideal in V if and only if $M_n(W)$ is an M-ideal in $M_n(V)$ for each n [26]. Thus, our result is the counterpart of this result in self-adjoint case. We show in Theorem 5.2.2 that if W is closed self-adjoint subspace of matricially order smooth ∞-normed space, then W is a CM-ideal in V if and only if $M_n(W^{\perp})$ is a CL-summand of $M_n(V^*)$ for each n. We introduce the notion of an L^1-matricial split face of matrical dual of the matricially order smooth ∞-normed spaces. We characterize CM-ideals in terms of L^1-matricial split faces of matricially ordered smooth ∞ normed spaces in Theorem 5.4.4. This result is the non-commutative non-unital generalization of the result that W is an M-ideals in $A(K)$ space if and only if

$W^{\perp} \cap K$ is split face of K. Also, this result extends to all (abstract) operator systems.

In the last chapter, we introduce the notion of smooth p-order ideals in order smooth p-normed spaces for $1 \leq p \leq \infty$. We show in Theorem 6.1.5 and Theorem 6.1.8 that smooth p-order ideals respect duality. In Proposition 6.2.7 and Theorem 6.2.8, we show that in order smooth ∞-normed space, every smooth ∞-order ideal is an M-ideal under certain condition. It may be noted that every M-ideal in complete order unit space is an order smooth ∞-order ideal. In Theorem 6.2.9, we show that every smooth order 1-order ideal in order smooth 1-normed space is L-summand. Thus the smooth p-order ideals may be seen as the (possible) interpolation spaces (ideals) of M-ideals and L-summands.

CHAPTER 2

Preliminaries

In this chapter, we recall some of the basic concepts of ordered normed spaces, ordered operator spaces, and M-ideal theory which is useful to understand the subsequent chapters. In the first section, we discuss the notions of M-ideals in Banach spaces. In the second section, we discuss some basic definitions and some properties of ordered normed spaces such as order unit spaces, approximate order unit spaces, base normed spaces, and their duality. Also, we discuss the representation of order unit spaces. In the third section, we describe the notions of (abstract) operator systems, (abstract) operator spaces. We discuss their matricial dual, and operator space dual and representation theorems.

2.1 M-ideals in Banach spaces

Let V be a Banach space. A projection P on V is called an *M-projection* if

$$\|v\| = \max\{\|P(v)\|, \|v - P(v)\|\}$$

for all $v \in V$. A projection P on V is called an *L-projection* if

$$\|v\| = \|P(v)\| + \|v - P(v)\|$$

for all $v \in V$. Any two M-projections (L-projections) on V commute with each other. Let W be a closed subspace of a Banach space V. If W is the range of an L-projection, it is called an *L-summand*; if W is the range of M-projection, it is called an *M-summand*; and if $W^{\perp}$ is an L-summand of V^*, then W is called an *M-ideal* in V. Note that W is an M-ideal in V if and only if there exists a unique closed subspace $W^{\perp'}$ of V^* such that

$$V^* = W^{\perp} \oplus_1 W^{\perp'}.$$

For more details on L-summands and M-ideals see [1, 20]. For extensive literature on M-ideals and its properties, please refer to [36] (also see e.g. [59]).

A non-empty convex subset F of a convex set K in a real vector space V is called a *face* of K if for any $u, v \in K$ with

$$\lambda u + (1 - \lambda)v \in F$$

and some $0 < \lambda < 1$, we have $u, v \in F$. A subset C of a real vector space V is called a *cone* if

$$\alpha u + v \in C \text{ whenever } u, v \in C \text{ and } \alpha \in \mathbb{R}^+.$$

If K is a convex subset of V, then

$$cone(K) := \cup_{\lambda \geq 0} \lambda K$$

is the smallest cone containing K.

Let S be a non-empty subset of a convex set K. Then $face_K(S)$ is the smallest face of K containing S. Thus

$$face_K(S) = \cap\{F : F \text{ is a face of } K \text{ containing } S\}.$$

If $S = \{v\}$, we write, $face_K(v)$ for $face_K(S)$.

Now, let V be a (real) normed space and let V_1 be the closed unit ball of V. We say that a cone C in V is *facial* if $C = \{0\}$ or $C = cone(F)$ for some proper face F of V_1. Note that any facial cone is a proper. If $v \in V$ with $v \neq 0$, then

$$w \in face_{V_1}\left(\frac{v}{\|v\|}\right)$$

if and only if

$$\frac{v}{\|v\|} = \lambda w + (1-\lambda)u$$

for some $\lambda \in (0,1)$ and $u \in V_1$. For $v \neq 0$, we define

$$C(v) := cone(face_{V_1}\left(\frac{v}{\|v\|}\right)) \tag{2.1.1}$$

for the smallest facial cone containing v. We define $C(0) = \{0\}$.

For a cone C in V, we write

$$C' = \{v \in V : C \cap C(v) = \{0\}\}. \tag{2.1.2}$$

It may be noted that C' may not be convex in general. Let $u, v \in V$. Then we say $u \prec v$ if

$$\|v\| = \|u\| + \|v-u\|.$$

These notions and facts can be found with details in [1, Part I, Section 2].

Lemma 2.1.1 *[1, Lemma 2.3] Let V be a normed space and let $v_1, \cdots, v_n \in V$. Then the following facts are equivalent:*

1. $v_1, \cdots, v_n \in C(v_1 + \cdots + v_n)$;
2. $\|\Sigma_{1=1}^n v_i\| = \Sigma_{i=1}^n \|v_i\|$.

Theorem 2.1.2 *[1, Part I, Theorem 2.9] Let V be a Banach space and let $C \subset V$ be a norm closed cone. Then each $u \in V$ admits a decomposition*

$$u = v + w, \text{ and } \|u\| = \|v\| + \|w\|,$$

where $v \in C$, and $w \in C'$. Given u_0 with $u_0 \prec u$, one can choose $u_0 \prec v$.

Theorem 2.1.3 *[1, Part I, Proposition 3.1] Let W be a closed subspace of a Banach space V. Then W is an M-ideal in V if and only if $W^{\perp'}$ is convex.*

2.2 Ordered vector spaces

Let V^+ be a cone in a real vector space V. We define an order relation $\leq$ in V by $u \leq v$ if and only if $v - u \in V^+$. If $v \in V^+$, we say v is positive. We note that $\leq$ is reflexive and transitive. Further, if $u \leq v$ for some $u, v \in V$, then $\lambda u \leq \lambda v$ and $u + w \leq v + w$ for all $w \in V$ and $\lambda > 0$. Conversely, if V is a real vector space and if $\leq$ possesses these properties, then $V^+ = \{v \in V : v \geq 0\}$ is a cone in V. A real vector space V together with a cone V^+ is called an *ordered vector space* and is denoted by (V, V^+). In what follows, in an ordered vector space, its cone and the corresponding order structure is identified with each other.

The cone V^+ of an ordered vector space (V, V^+) is *proper*, if $V^+ \cap -V^+ = \{0\}$. We say that V^+ is *generating*, if $V = V^+ - V^+$. We note that V^+ is proper if and only if the corresponding vector order $\leq$ is anti-symmetric.

Let (V, V^+) be an ordered vector space and $W \subseteq V$. Then W is called *directed upward* if for any two elements $u_1, u_2 \in W$, there is an element $u_3 \in W$ such that $u_1, u_2 \leq u_3$.

Let (V_i, V_i^+) be the ordered vector spaces for $i = 1, 2$ and let $\phi : V_1 \to V_2$ be a linear map. We say that ϕ is *positive* if

$$\phi(V_1^+) \subseteq V_2^+.$$

Moreover, ϕ is called an *order isomorphism* if ϕ is a linear isomorphism and ϕ, ϕ^{-1} are both positive. Let (V, V^+) be an ordered vector space and let W be a subspace of V. Then W is called an *order ideal* if $u, v \in W$ and $w \in V$ with $u \leq w \leq v$ implies $w \in W$. For more details of ordered vector space see [37].

An *ordered topological vector space* is a triple $(V, V^+, \mathcal{P})$ such that V is a topological vector space with respect to topology $\mathcal{P}$ and V is an ordered vector space with respect to the cone V^+. It may be noted that there may not be any relation between the cone V^+ and the topology $\mathcal{P}$. An ordered topological vector space $(V, V^+, \mathcal{P})$ is called an *ordered convex space* if V is a locally convex space with respect to topology $\mathcal{P}$. An ordered topological vector space V is an *ordered normed space*, if the topology is given by a norm on V.

Let $(V, V^+, \|.\|)$ be an ordered normed space. Then its Banach dual V^* is also an ordered normed space with the dual cone

$$V^{*+} = \{f \in V^* : f(v) \geq 0 \ \forall \ v \in V^+\}.$$

Let (V, V^+) be an ordered vector space. Then V^+ is called *Archimedean* if

$$nu \geq v, \ \forall n \in \mathbb{N} \text{ for some } v \in V \text{ implies } u \geq 0.$$

Let (V, V^+) be an ordered vector space. Then $e \in V^+$ is called *order unit* if for any $u \in V$, there is a $\lambda \in \mathbb{R}^+$ such that

$$-\lambda e \leq u \leq \lambda e.$$

An ordered vector space (V, V^+) with an order unit e is called an *order unit space* if V^+ is proper and Archimedean. An order unit space (V, V^+, e) admits a norm

$$\|v\| = \inf\{r \geq 0 : -re \leq v \leq re\} \tag{2.2.1}$$

satisfying

$$-\|v\|e \leq v \leq \|v\|e. \tag{2.2.2}$$

Let (V_i, V_i^+, e_i) be the order unit spaces for $i = 1, 2$, and let $\phi : V_1 \to V_2$ be a linear map such that $\phi(e_1) = e_2$. Then ϕ is positive if and only if ϕ is bounded and $\|\phi\| = 1$. Let $(V, V^+, \|.\|)$ be an ordered normed space. Then a linear functional $f : V \to \mathbb{R}$ is called a *state* if f is positive and $\|f\| = 1$. The set of all state of V is called as *state space* of V and denoted by $S(V)$. If (V, V^+, e) is an order unit space, then a linear map $f : V \to \mathbb{R}$ is a state of V if and only if f is positive and $f(e) = 1$. If $f, g : V \to \mathbb{R}$ are two positive linear maps, where V is an order unit space, then

$$\|f + g\| = \|f\| + \|g\| = f(e) + g(e).$$

If (V, V^+, e) is an order unit space, then $S(V)$ is convex.

Let (V, V^+) be an ordered vector spaces. Then a net $\{e_\lambda : \lambda \in \Lambda\}$ in V^+ is called *approximate order unit* if

$$\lambda_1 \leq \lambda_2 \implies e_{\lambda_1} \leq e_{\lambda_2},$$

and for any $v \in V,$ there are $r \in \mathbb{R}^+$ and $\lambda \in \Lambda$ such that

$$-re_\lambda \leq v \leq re_\lambda.$$

Let $(V, V^+, \|.\|)$ be an ordered normed space such that V^+ is norm closed. Then V is called an *approximate order unit space* if there is an approximate order unit $\{e_\lambda : \lambda \in \Lambda\}$ in V^+ such that

$$\|v\| = \inf\{|r| : -re_\lambda \leq v \leq re_\lambda\}$$

for all $v \in V$.

Now, we describe another class of ordered Banach spaces which generally occur as the dual of order unit spaces and that of approximate order unit spaces. Let (V, V^+) be an ordered vector space such that V^+ is generating. Then a convex subset B of a vector space V is *radially compact* if $B \cap L$ is closed and bounded for every line segment L passing through origin of V. A convex subset B of a hyperplane H not passing through the origin of an ordered vector space (V, V^+) is called *base* for the cone V^+ if for each $v \in V^+, v \neq 0$, there exists a unique $k > 0$ and $u \in B$ such that $v = ku$.

Let (V, V^+) be an ordered vector space such that V^+ is generating. Let B be a base for a cone V^+ such that $co(B \cup -B)$ is radially compact, then V is an ordered normed space with norm

$$\|v\| = \inf\{r \geq 0 : v \in rco(B \cup -B)\}. \tag{2.2.3}$$

In this case, V is called a based normed space and we denote it by (V, B). If $co(B \cap -B)$ is compact with respect to some Hausdorff topology $\mathcal{P}$, then V is $\|.\|$-complete (see e.g. [2, p.-76]).

Let (V, B) be a base normed space. Then there is a linear functional

$$e_B : V \to \mathbb{R}$$

such that

$$e_B(u) = 1$$

for all $u \in B$. Also the linear functional e_B has property that

$$e_B(u) = \|u\|$$

for all $u \in V^+$. Further, for any $u \in V$, there are $v, w \in V^+$ such that

$$u = v - w, \qquad \|u\| = \|v\| + \|w\|.$$

If (V, V^+, e) is an order unit space, then $(V^*, S(V))$ is a base normed space, where the base norm is the usual norm of V^* considered as Banach dual of V with the order unit norm. Conversely if (V, B) be a base normed space, then (V^*, e_B) is an order unit space, where order unit norm is the usual norm of V^* considered as Banach dual of V with the base norm (see e.g. [2, 30]).

Theorem 2.2.1 *[65, p.-94] Let* $(V, V^+, \|.\|)$ *be an ordered Banach space and let*

$(V^*, V^{*+}\|.\|)$ be the Banach dual. Then following are equivalent:

1. *$(V, V^+, \|.\|)$ is an approximate order unit space;*

2. *$(V^*, V^*, \|.\|)$ is a base normed space.*

2.2.1 Functional representations of ordered Banach spaces

Let K be a compact convex set in a locally convex space E. A function $a : K \to \mathbb{R}$ is called *affine* if

$$a(\lambda u + (1 - \lambda)v) = \lambda a(u) + (1 - \lambda)a(v)$$

for all $u, v \in K$ and $0 \leq \lambda \leq 1$. We define

$$A(K) = \{a : K \to \mathbb{R} \mid a \text{ is continuous and affine}\}.$$

Let K be a compact convex set containing 0. We define

$$A_0(K) = \{a \in A(K) : a(0) = 0\}.$$

Let (V, V^+, e) be an order unit spaec. If $\|.\|$ is complete order unit norm, then (V, V^+, e) is isometrically order isomorphic to $A(S(V))$, where $S(V)$ is the state space of V.

Let $(V, V^+, \mathcal{P})$ be an order convex space. A compact convex set $K \subset V^+$ is called a *cap* of V^+ if $V^+ \setminus K$ is convex. A cap K of V^+ is called a *universal cap* if $V^+ = \cup_{\lambda \geq 0} \lambda K$ (see e.g. [8, 65]).

Theorem 2.2.2 *[65, p.-98] Let $(V, V^+, \|.\|)$ be an ordered Banach space with a closed cone V^+. Then the following are equivalent:*

1. $(V, V^+, \|.\|)$ *is an approximate order unit space;*

2. *there exist a universal cap* K *of the cone* V^{*+} *such that* $(V, V^+, \|.\|)$ *is isometrically order isomorphic to* $A_0(K)$.

Let K be a compact convex set in a locally convex space E. Let H be a hyperplane containing K such that $Span(K) = E$ and $0 \notin K$. Then for each point u in K determines a unique linear functional $\chi(u)$ on $A(K)$ by defining

$$\chi(u)(a) = \lambda a(v) - \mu a(w)$$

for all $u = \lambda v - \mu w$ for some $v, w \in K$ and $\lambda, \mu \in \mathbb{R}^+$.

It can be easily checked that the map $u \mapsto \chi(u)$ is linear. In general, $\chi(u)$ may not be continuous on $A(K)$. We say that K is *regularly embedded in* E if the map $u \mapsto \chi(u)$ is a topological isomorphism (see e.g. [2, p. 80]).

Let K be a compact convex set in locally convex space E. Let $A(K, E)$ be the vector space of all real valued continuous affine functions on K which has a continuous affine extension to E.

Proposition 2.2.3 *[2, Corollary II.2.3] Let* K *be a compact convex set in a locally convex space* E*. If* K *is regularly embedded* E*, then* $A(K, E) = A(K)$.

Most of the ideas of this section has been taken from [65] (also see [2, 37]).

2.2.2 Order smooth p-normed spaces

Now, we recall some definitions and facts about (non-unital) ordered normed spaces studied in [41].

Definition 2.2.4 *[41] Let* (V, V^+) *be a real ordered vector space such that the cone* V^+ *is proper and generating. Let* $\|.\|$ *be a norm on* V *such that* V^+ *is* *closed. For a fixed real number $p, 1 \leq p < \infty$, consider the following geometric conditions on V:*

$(O.p.1)$ *For u, v, w in V with $u \leq v \leq w$, we have*

$$\|v\| \leq (\|u\|^p + \|w\|^p)^{\frac{1}{p}}.$$

$(O.p.2)$ *For $v \in V$ and $\epsilon > 0$, there are $v_1, v_2 \in V^+$ such that*

$$v = v_1 - v_2 \text{ and } (\|v_1\|^p + \|v_2\|^p)^{\frac{1}{p}} < \|v\| + \epsilon.$$

$(OS.p.2)$ *For $v \in V$, there are $v_1, v_2 \in V^+$ such that*

$$v = v_1 - v_2 \text{ and } (\|v_1\|^p + \|v_2\|^p)^{\frac{1}{p}} \leq \|v\|.$$

For $p = \infty$, consider the similar conditions on V:

$(O.\infty.1)$ *For u, v, w in V with $u \leq v \leq w$, we have*

$$\|v\| \leq \max(\|u\|, \|w\|).$$

$(O.\infty.2)$ *For $v \in V$ and $\epsilon > 0$, there exist $v_1, v_2 \in V^+$ such that*

$$v = v_1 - v_2 \text{ and } \max(\|v_1\|, \|v_2\|) < \|v\| + \epsilon.$$

$(OS.\infty.2)$ *For $v \in V$, there are $v_1, v_2 \in V^+$ such that*

$$v = v_1 - v_2 \text{ and } \max(\|v_1\|, \|v_2\|) \leq \|v\|.$$

Definition 2.2.5 *[41] Let (V, V^+) be a real ordered vector space such that the cone V^+ is proper and generating. Let $\|.\|$ be a norm on V such that V^+ is closed. For a fixed p, $1 \leq p \leq \infty$, we say that V is an order smooth p-normed space, if $\|.\|$ satisfies the $(O.p.1)$ and $(O.p.2)$ on V.*

Note that order unit spaces and approximate order unit spaces are order smooth ∞-normed spaces, and base normed spaces are order smooth 1-normed spaces. Moreover,

Proposition 2.2.6 *[65] Let $(V, V^+, \|.\|)$ be an ordered normed space such that V^+ is norm closed and let $U = \{v \in V : \|v\| < 1\}$. Then the following statements are equivalent:*

1. *V is an approximate order unit space;*
2. *V satisfies $(O.\infty.1)$ and U is directed upward;*
3. *U is an order ideal and directed upward in V.*

Now, we consider other types of order smooth p-normed spaces. Let H be a complex Hilbert space. Let $\mathcal{B}(H)$ denote the set of all bounded linear operators on H. An element $T \in \mathcal{B}(H)$ is *self-adjoint* if $T = T^*$. The set of all self-adjoint elements of $\mathcal{B}(H)$ is denoted by $\mathcal{B}(H)_{sa}$. A self-adjoint element T is *positive* if

$$\langle Tx, x\rangle \geq 0 \; \forall \; x \in H.$$

The set of all positive elements of $\mathcal{B}(H)$ is denoted by $\mathcal{B}(H)^+$. It easy to check from the definition that T^*T is always positive for all $T \in \mathcal{B}(H)$. We note that every positive element $T \in B(H)$ has a unique square root i.e. there is a unique $S \in \mathcal{B}(H)^+$ such that $T = S^2$ and we write $T^{\frac{1}{2}} = S$. For all $T \in \mathcal{B}(H)$, the *absolute value* of T is denoted and defined by

$$|T| = (T^*T)^{\frac{1}{2}}.$$

Let H be a Hilbert space and let $\{e_\alpha\}$ be an orthogonal basis of H. If $T \in \mathcal{B}(H)$, then *trace* of T is denoted and defined by

$$tr(T) = \sum_\alpha \langle T(e_\alpha), e_\alpha \rangle.$$

For fixed p $(1 \leq p < \infty)$,

$$\mathcal{T}_p(H)_{sa} = \{T \in \mathcal{B}(H)_{sa} : tr(|T|^p) < \infty\}.$$

is an order smooth p-normed space [41]. Let $\mathcal{K}(H)$ be the set of all compact operators on the Hilbert space H. Then $\mathcal{K}(H)$ is an order smooth ∞-normed space satisfying $(OS.1.2)$. In general, if $\mathcal{A}$ is a C*-algebra, then $\mathcal{A}_{sa}$ is an order smooth ∞-normed space (see e.g. [41]). For more details of positive elements, and positive linear functionals of C*-algebras, one can see [38]. Next, we note that $(O.p.1)$ and $(O.p.2)$ enjoy the following duality.

Theorem 2.2.7 *[41] Let (V, V^+) be a real ordered vector space such that the cone V^+ is proper and generating. Let $\|.\|$ be a norm on V such that V^+ is closed. For each $p, 1 \leq p \leq \infty$, we have*

1. *$\|.\|$ satisfies $(O.p.1)$ condition on V if and only if $\|.\|^*$ satisfies the condition $(OS.p'.2)$ on the Banach dual V^*.*

2. *$\|.\|$ satisfies the condition $(O.p.2)$ on V if and only if $\|.\|^*$ satisfies the condition $(O.p'.1)$ on V^*.*

Theorem 2.2.8 *[41] Let (V, V^+) be a real ordered vector space such that the cone V^+ is proper and generating. Let $\|.\|$ be a norm on V such that V^+ is closed. For a fixed $p, 1 \leq p \leq \infty$, V is an order smooth p-normed space if and only if its Banach dual V^* is an order smooth p'-normed space satisfying the condition $(OS.p'.2)$.*

2.3 Ordered operator spaces

If V is a complex $*$-vector space, we denote V_{sa} to be the set of *self-adjoint* elements of V. We say that (V, V^+) is a *complex ordered vector space* if (V_{sa}, V^+) is a real ordered vector space, that is, V^+ is a cone in V_{sa}. More details of complex ordered vector space can be found on the paper [50].

Let (V_i, V_i^+) be the complex ordered vector spaces for $i = 1, 2$ and let $\phi : V_1 \to V_2$ be a self-adjoint linear map. We recall that ϕ is *positive* if

$$\phi(V_1^+) \subseteq V_2^+.$$

Moreover, ϕ is an *order isomorphism* if ϕ is an isomorphism and ϕ, ϕ^{-1} both are positive. Let V be a complex vector space. If the matrices $\alpha \in \mathbb{M}_{m,p}, [v_{i,j}] \in M_{p,q}(V), \beta \in \mathbb{M}_{q,n}$, then matrix product $\alpha v \beta \in M_{m,n}(V)$ is given by

$$\alpha v \beta = \left[\sum_{k,l} \alpha_{i,k} v_{k,l} \beta_{l,j} \right]_{i \in m, j \in n}.$$

Further, if V is a $*$-vector space, then $M_n(V)$ is also $*$-vector space with

$$[v_{i,j}]^* = [v_{j,i}^*].$$

A complex $*$-vector space V is called *matrix ordered* if

$$M_n(V)^+ \subseteq M_n(V)_{sa}$$

is a cone for each n such that

$$\gamma^* M_m(V)^+ \gamma \subseteq M_n(V)^+$$

whenever $\gamma \in \mathbb{M}_{m,n}$. A matrix ordered space $(V, \{M_n(V)^+\})$ with an order unit e is called a *matrix order unit space* if V^+ is proper and $M_n(V)^+$ is Archimedean for each n [18, Choi, Effros]. Let $\phi : V_1 \to V_2$ be a linear map of complex vector spaces. Then *n-amplification* of ϕ is given by

$$\phi_n([v_{i,j}]) = [\phi(v_{i,j})]$$

for all $[v_{i,j}] \in M_n(V)$. Let V be a linear space and let V^d be the dual space of the linear space V. It may be noted that $M_n(V)$ and $M_n(V^d)$ are also linear spaces. The *scalar pairing* and *matrix pairing* between $M_n(V)$ and $M_n(V^d)$ are denoted and defined by

$$\langle [v_{i,j}], [f_{i,j}] \rangle = \sum_{i,j=1}^{n} \langle v_{i,j}, f_{i,j} \rangle = \sum_{i,j=1}^{n} f_{i,j}(v_{i,j}) \qquad (2.3.1)$$

and

$$\langle\langle [v_{i,j}], [f_{p,q}] \rangle\rangle = [\langle v_{i,j}, f_{p,q} \rangle] \qquad (2.3.2)$$

for all $[v_{i,j}] \in M_n(V), [f_{i,j}] \in M_n(V^d)$.

Let $(V, \{M_n(V)^+\})$ be a matrix ordered space, and let V^d be a self-adjoint dual of V. Then V^d becomes a matrix ordered space under the scalar pairing (see e.g. [18, Lemma 4.2]). The cone in $M_n(V^d)_{sa}$ is given by

$$M_n(V^d)^+ = \{f \in M_n(V^d)_{sa} : f(v) \geq 0 \;\; \forall v \in M_n(V)^+\}.$$

A *concrete operator system* is a unital $*$-subspace of $\mathcal{B}(H)$ for some Hilbert space H.

Theorem 2.3.1 *[18, Choi, Effros] Let $(V, \{M_n(V)^+\}, e)$ be a matrix order unit space. Then there is a Hilbert space H and a concrete operator system $\mathcal{S} \subseteq \mathcal{B}(H)$ and a complete order isomorphism $\varphi : V \to \mathcal{S}$ such that $\varphi(e) = I$, where I is the identity operator on H.*

Due to the above representation theorem, we call a matrix order unit space as an *abstract operator system.* An *L^∞-matricially normed space* $(V, \{\|\cdot\|_n\})$ is a complex vector space V together with a sequence of matrix norms $\{\|\cdot\|_n\}$ such that

1. $(M_n(V), \|\cdot\|_n)$ is a normed space for all n;

2. $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$ for all $v \in M_n(V), w \in M_m(V)$;

3. $\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_n\|\beta\|$ for all $\alpha \in \mathbb{M}_n, v \in M_n(V),$ and $\beta \in \mathbb{M}_n(V)$.

Every abstract operator space is completely isometric to some closed subspace of $\mathcal{B}(H)$(concrete operator space) for some Hilbert space H (see e.g. [54]). An L^∞-matricially normed space is called an *abstract operator space.*

For fixed p $(1 \leq p < \infty)$, an *L^p-matricially normed space* $(V, \{\|\cdot\|_n\})$ is a complex vector space together with a sequence of matrix norms $\{\|\cdot\|_n\}$ such that

1. $(M_n(V), \|\cdot\|_n)$ is a normed space for each n;

2. $\|v \oplus w\|_{n+m}^p = \|v\|_n^p + \|w\|_m^p$ for all $v \in M_n(V), w \in M_m(V)$;

3. $\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_n\|\beta\|$ for all $\alpha \in \mathbb{M}_n, v \in M_n(V),$ and $\beta \in \mathbb{M}_n(V)$.

We know, by [54, Theorem 5.1], that if V is an L^∞-matricially normed space, then its matricial dual V^* is an L^1-matricially normed space under the scaler pairing (see e.g. [54]). The following notion introduced by Karn to study non-commutative order in L^∞-matricilly normed space.

Definition 2.3.2 (C*-ordered operator spaces) *[40] A matrix ordered space $(V, \{M_n(V)^+\})$ together with a matrix norm $\{\|\cdot\|_n\}$ is said to be a C*-ordered operator space if $(V, \{\|\cdot\|_n\})$ is an abstract operator space, and V^+ is proper such that for each $n \in \mathbb{N}$ the following conditions hold:*

1. *$*$ is isometry on $M_n(V)$;*

2. *$M_n(V)^+$ is closed;*

3. *$M_n(V)_{sa}$ satisfies $(O.\infty.1)$ property i.e.*

$$\|f\|_n \leq \max\{\|g\|_n, \|h\|_n\},$$

whenever $f \leq g \leq h$ with $f, g, h \in M_n(V)_{sa}$ for each $n \in \mathbb{N}$.

We know that every abstract operator system is a C*-ordered operator space. Let $\phi : V_1 \to V_2$ be a linear map between two operator spaces. Then ϕ is *completely isometry* if ϕ_n is isometry for each $n \in \mathbb{N}$, where ϕ_n is the n-amplification of ϕ.

Theorem 2.3.3 *[40] Let $(V, \{M_n(V)^+\}, \{\|\cdot\|_n\})$ be a C*-ordered operator space. Then there exists a completely order isometry $\varphi : V \to \mathcal{A}$ for some C*-algebra $\mathcal{A}$.*

For details properties of C*-ordered operator space one can see [40].

CHAPTER 3

Quantization of $A_0(K)$-space

In this chapter, we discuss a quantization of the space of continuous affine functions vanishing at 0. In the first section, we study the matricial dual of C*-ordered operator space. We find some properties of the quasi state spaces. In the second section, we introduce and study L^1-matrix convex sets in $*$-locally convex spaces. We show that every C*-ordered operator space is complete isometrically, completely order isomorphic to $(A_0(K_1, E), \{M_n(A_0(K_1, E))^+\}, \{\|\cdot\|_n\})$ for a suitable L^1-matrix convex set $\{K_n\}$. In the third section, we generalize the notion of regular embedding of a compact convex set to L^1-regular embedding properties of L^1-matrix convex set. Using L^1-regular embedding of L^1-convex set $\{K_n\}$, we find conditions under which $(A_0(K_1, E), \{M_n(A_0(K_1, E))^+\}, \{\|\cdot\|_n\})$ is an abstract operator system.

3.1 Convexity of matricial quasi state spaces

Let $(V, \{\|.\|_n\}, \{M_n(V)^+\})$ be a C*-ordered operator space. Then its matricial dual $(V^*, \{\|.\|_n\}, \{M_n(V^*)^+\})$ is an L^1-matricially normed space with an involution $*$ such that $*$ is isometry on $M_n(V^*)$. Also $(V^*, \{M_n(V^*)^+\})$ is a matrix ordered space such that $M_n(V^*)^+$ is norm closed for each n. We put

$$Q_n(V) = \{f \in M_n(V^*) : f \geq 0, \|f\|_n \leq 1\},$$

and call it the *quasi state space* of $M_n(V)$. We note that $Q_n(V)$ is a compact convex set with respect to w^*-topology (see e.g. [40]). Throughout this chapter, we assume that V is a C*-ordered operator space.

Lemma 3.1.1 $M_n(V^*)_{sa} \cap M_n(V^*)_1 = co(Q_n(V) \cup -Q_n(V))$.

Proof. Let $f \in M_n(V^*)_{sa}$. Since $M_n(V)_{sa}$ satisfies $(0.\infty.1)$, by Theorem 2.2.7, $M_n(V)^*$ satisfies $(OS.1.2)$ on $M_n(V^*)_{sa}$. Thus there are $g, h \in M_n(V^*)^+$ such that

$$f = g - h \text{ and } \|f\|_n = \|g\|_n + \|h\|_n.$$

Therefore $M_n(V^*)_{sa} \cap M_n(V)_1 \subseteq co(Q_n(V) \cup (-Q_n(V)))$. Since $\pm Q_n(V) \subseteq M_n(V^*)_{sa} \cap M_n(V)_1$ and $M_n(V^*)_{sa} \cap M_n(V)_1$ is convex, we have $co(Q_n(V) \cup (-Q_n(V))) \subseteq M_n(V^*)_{sa} \cap M_n(V)_1$. □

Now, we describe a 'quantized' functional representation of a C*-ordered operator space V.

Theorem 3.1.2 *Let $(V, \{M_n(V)^+\}, \{\|\cdot\|_n\})$ be a C*-ordered operator space. For $v \in V$, define $\check{v} : V^* \to \mathbb{C}$ given by $\check{v}(f) = f(v)$ $(f \in V^*)$ and set $\check{v}|_{Q(V)} = \hat{v}$. Then $\hat{v} : Q_1(V) \to \mathbb{C}$ is an affine, w^*-continuous map with $\hat{v}(0) = 0$ such that $\check{v}$ is the unique extension of $\hat{v}$ to V^* as a w^*-continuous linear functional. We write, $A_0(Q_1(V), V^*)$ for the space of all w^*-continuous affine mappings from $Q_1(V)$ into $\mathbb{C}$ vanishing at 0 and having a unique w^*-continuous linear extension to V^*. Thus $v \mapsto \hat{v}$ determines a surjective $*$-isomorphism $\Gamma : V \to A_0(Q_1(V), V^*)$. We can transport a matrix order and a matrix norm on it so that it becomes a C*-*

ordered operator space and Γ *turns out to be a complete isometric, complete order isomorphism.*

Proof. Note that $\check{v}$ is the unique extension of $\hat{v}$ on V^* as a w^*-continuous linear functional for $V^{*+} = \cup_{k\in\mathbb{N}} kQ_1(V)$ and V^{*+} spans V^*. Further, we note that $v \mapsto \hat{v}$ determines a linear $*$-isomorphism from $\Gamma : V \to A_0(Q_1(V), V^*)$. Also, as w^*-dual of V^* is identified with V, we may conclude that Γ is surjective. For $v \in V$, set $(\check{v})^* = (\check{v^*})$ so that

$$(\check{v})^*(f) = f(v^*) = \overline{f^*(v)} = \overline{\check{v}(f^*)}$$

for all $f \in V^*$. In particular for $v \in V_{sa}$ and $f \in V^*_{sa}$, $(\check{v})^*(f) = \check{v}(f) \in \mathbb{R}$. Similarly, if $v \in V^+$ and $f \in V^{*+}$, then $\check{v}(f) \geq 0$. In fact, as $v \in V^+$ if and only if $f(v) \geq 0$ for every $f \in Q(V)$. We may conclude that

$$\begin{aligned}\Gamma(V^+) &= \{\phi \in A_0(Q_1(V), V^*)_{sa} : \phi(f) \geq 0 \ \forall f \in Q_1(V)\} \\ &:= A_0(Q_1(V), V^*)^+.\end{aligned}$$

In other words, Γ is an order isomorphism. Now using matrix duality, we may further conclude that

$$\Gamma_n : M_n(V) \mapsto A_0(Q_n(V), M_n(V^*))$$

given by

$$\Gamma_n([v_{i,j}]) = [\hat{v_{i,j}}], \quad [v_{i,j}] \in M_n(V)$$

is a surjective order isomorphism for each $n \in \mathbb{N}$. Now, if we identify $A_0(Q_n(V), M_n(V^*))$ with $M_n(A_0(Q_1(V), V^*))$ for each $n \in \mathbb{N}$, then $\Gamma : V \mapsto A_0(Q_1(V), V^*)$ is a surjective order isomorphism.

Next, we describe a norm on $A_0(Q_n(V), M_n(V^*))$. Let $F \in A_0(Q_n(V), M_n(V^*))$. Then there is a unique $v \in M_n(V)$ such that $F = \Gamma_n(v) = \widehat{v}$. We define

$$\|F\|_{\infty,n} = \sup\left\{ \left| \begin{bmatrix} 0 & \widehat{v} \\ \widehat{v^*} & 0 \end{bmatrix}(f) \right| : f \in Q_{2n}(V) \right\}. \tag{3.1.1}$$

As $v \in M_n(V)$, we have $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{2n}(V)_{sa}$. Since $*$ is isometry in V, using Lemma 3.1.1, we have

$$\begin{aligned} \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_n &= \sup\left\{ \left| \begin{bmatrix} 0 & \widehat{v} \\ \widehat{v^*} & 0 \end{bmatrix}(f) \right| : f \in M_{2n}(V^*)_{sa} \cap \mathbb{M}_n(V^*)_1 \right\} \\ &= \sup\left\{ \left| \begin{bmatrix} 0 & \widehat{v} \\ \widehat{v^*} & 0 \end{bmatrix}(f) \right| : f \in Q_n(V) \right\}. \end{aligned}$$

Also as $*$ is isometry and $\{\|\cdot\|_n\}$ is an L^∞-matrix norm, we have $\|v\|_n = \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{2n}$ so that $\|v\|_n = \|\widehat{v}\|_{\infty,n}$. □

In what follows, we deduce some of the geometric properties of $\{Q_n(V)\}$ to present an intrinsic version of Theorem 3.1.2.

Definition 3.1.3 *[24] Let W be a vector space. A collection $\{K_n\}$ with $K_n \subseteq M_n(W)$ is called a* matrix convex set *if*

$$\sum_{i=1}^{k} \gamma_i^* w_i \gamma_i \in K_m$$

whenever $w_i \in M_{n_i}(W)$, and $\gamma_i \in \mathbb{M}_{n_i,m} (1 \leq i \leq k)$ satisfy $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_m$.

The notion of matrix convexity introduced by G. Wittstock [63] in 1984. We note that if V is an L^∞- matrically normed space (abstract operator space), then $\{M_n(V)_1\}$ is a matrix convex set (see e.g. [54, p. 101-103)]). Here $M_n(V)_1 := \{v \in M_n(V)_1 : \|v\| \le 1\}$. In particular, if V is a C*-ordered operator space, then $\{M_n(V)_1^+\}$ is a matrix convex set. However, $\{Q_n(V)\}$ is not a matrix convex set. To see this, let $f \in Q(V)$ with $||f|| = 1$. Then $\|f \oplus f\|_2 = 2$ so that $f \oplus f \notin Q_2(V)$. Since in a matrix convex set $\{K_n\}$, we have $K_1 \oplus K_1 \subseteq K_2$, we deduce that $\{Q_n(V)\}$ is not a matrix convex set. Nevertheless, it has some interesting properties which we illustrate in the following result. Put

$$S_n(V) = \{f \in Q_n(V) : \|f\|_n = 1\}.$$

Proposition 3.1.4 *Let V be a* C*-*ordered operator space and let $f \in Q_{m+n}(V)$ so that*

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$$

for some $f_{11} \in M_m(V^*)^+$, $f_{22} \in M_n(V^*)^+$ and $f_{12} \in M_{m,n}(V^*)$. Then

(i) $f_{11} \in Q_m(V)$ *and* $f_{22} \in Q_m(V)$;

(ii) $\begin{bmatrix} f_{11} & e^{i\theta} f_{12} \\ e^{-i\theta} f_{12}^* & f_{22} \end{bmatrix} \in Q_{m+n}(V)$ *for any* $\theta \in \mathbb{R}$;

(iii) $\left\| \begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix} \right\|_{m+n} \le \left\| \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} \right\|_{m+n}, \left\| \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix} \right\|_{m+n} \le \left\| \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} \right\|_{m+n};$

(iv) If $m = n$, then

$$f_{12} + f_{12}^* \in co(Q_n(V) \cup (-Q_n(V))).$$

(v) Let $f \in Q_n(V)$ *and let* $\gamma_i \in \mathbb{M}_{n,n_i}$ *such that* $\sum_{i=1}^k \gamma_i \gamma_i^* \leq I_n$*. Then*

$$\oplus_{i=1}^k \gamma_i^* f \gamma_i \in Q_{\sum_{i=1}^k n_i}(V).$$

(vi) Let $f \in Q_{m+n}(V)$ *with* $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$ *so that* $f_{11} \in Q_m(V), f_{22} \in Q_n(V)$ *and* $f_{12} \in M_n(V)$ *and let* $f_{11} = \alpha_1 \widehat{f_{11}}, f_{22} = \alpha_{22} \widehat{f_{22}}$ *with* $\widehat{f_{11}} \in S_m(V), \widehat{f_{22}} \in S_n(V)$*. Then* $\alpha_1 + \alpha_2 \leq 1$*.*

Proof. We know that if $\alpha \in \mathbb{M}_{m,n}$, $f \in M_n(V^*)$ and V is an abstract operator space, then we have

$$\|\alpha f \alpha^*\|_m \leq \|\alpha\|^2 \|f\|_n.$$

Since V^* is a matrix ordered space, for $f \in M_n(V^*)^+$ and $\alpha \in \mathbb{M}_{m,n}$, we have $\alpha f \alpha^* \in M_m(V^*)^+$ (also see e.g. [18, Lemma 4.2]). Using this argument, we can prove (i) and (ii) if we choose $\alpha = \begin{bmatrix} I_m & 0_{m,n} \end{bmatrix}$, $\alpha = \begin{bmatrix} 0_{n,m} & I_n \end{bmatrix}$ and $\alpha = \begin{bmatrix} e^{i\theta} I_m & 0 \\ 0 & I_n \end{bmatrix}$ respectively. In particular, $\left\| \begin{bmatrix} f_{11} & \pm f_{12} \\ \pm f_{12}^* & f_{22} \end{bmatrix} \right\|_{m+n} \leq 1$. Now

$$2 \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} + \begin{bmatrix} f_{11} & -f_{12} \\ -f_{12}^* & f_{22} \end{bmatrix}$$

and

$$2 \begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix} - \begin{bmatrix} f_{11} & -f_{12} \\ -f_{12}^* & f_{22} \end{bmatrix}.$$

Thus (iii) follows from the triangle inequality. Next, as

$$\left\|\begin{bmatrix} f_{12}^* & 0 \\ 0 & f_{12} \end{bmatrix}\right\|_{2n} = \left\|\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix}\right\|_{2n} \leq \left\|\begin{bmatrix} 0 & f_{12} \\ f_{12}^* & 0 \end{bmatrix}\right\|_{2n} \leq 1,$$

we have

$$\|f_{12} + f_{12}^*\|_n \leq \|f_{12}^*\|_n + \|f_{12}\|_n \leq \left\|\begin{bmatrix} f_{12}^* & 0 \\ 0 & f_{12} \end{bmatrix}\right\|_{2n} \leq 1.$$

Since $f_{12} + f_{12}^* \in M_n(V^*)_{sa}$, by Lemma 3.1.1, we may conclude that

$$f_{12} + f_{12}^* \in co(Q_n(V) \cup (-Q_n(V))).$$

(v) As $f \in Q_n(V) \subset M_n(V^*)^+$ and $\gamma_i \in \mathbb{M}_{n,n_i}$, we have $\gamma_i^* f \gamma_i \in M_{n_i}(V^*)^+$ for $1 \leq i \leq k$. Thus $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in M_{\sum_{i=1}^k n_i}(V^*)^+$. We show that $\| \oplus_{i=1}^k \gamma_i^* f \gamma_i\| \leq 1$. Let $v \in (M_{\sum_{i=1}^k n_i}(V)_{sa})_1$ and say $v = [v_{i,j}]$ where $v_{i,j} \in M_{n_i,n_j}(V)$ and $v_{i,j}^* = v_{j,i}, 1 \leq i,j \leq k$. Then

$$\begin{aligned} |\langle \oplus_{i=1}^k \gamma_i^* f \gamma_i, v\rangle| &= |\sum_{i=1}^n \langle \gamma_i^* f \gamma_i, v_{ii}\rangle| \\ &= |\sum_{i=1}^n \langle f, \gamma_i^{*T} v_{i,i} \gamma_i^*\rangle| \\ &\leq \|\sum_{i=1}^k \gamma_i^{*T} v_{i,i} \gamma_i^T\| \text{ for } f \in Q_n(V). \end{aligned}$$

Since $\sum_{i=1}^k \gamma_i \gamma_i^* \leq I_n$, we have

$$\left\|\sum_{i=1}^k \gamma_i^{*T} \gamma_i^T\right\| = \left\|\left(\sum_{i=1}^k \gamma_i \gamma_i^*\right)^T\right\| = \left\|\sum_{i=1}^k \gamma_i \gamma_i^*\right\| \leq 1.$$

Thus $\sum_i \gamma_i^{*T} \gamma_i^T \leq I_n$. Since $\|v_{i,i}\|_{n_i} \leq \|v\|_{\sum_{i=1}^k n_i} \leq 1$ for $1 \leq i \leq k$ and since

$\{(M_n(V)_{sa})_1\}$ is a matrix convex set, we find that $\| \sum_{i=1}^k \gamma_i^{*T} v_{i,i} \gamma_i^T \| \leq 1$. Thus $\| \oplus_{i=1}^k \gamma_i^* f \gamma_i \| \leq 1$ so that $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in Q_{\sum_{i=1}^k n_i}(V)$.

(vi) Let $f = \begin{bmatrix} f_{11} & f_{12} \\ F_{12} & f_{22} \end{bmatrix} \in Q_{m+n}(V)$. Then by (iii), $f_{11} \in M_m(V^*)^+$ and $f_{22} \in M_n(V^*)^+$ and we have $\|f_{11}\|_m + \|f_{22}\|_n \leq 1$. Find $\widehat{f_{11}} \in S_m(V)$, $\widehat{f_{22}} \in S_n(V)$ such that $f_{11} = \|f_{11}\|_m \widehat{f_{11}}$ and $f_{22} = \|f_{22}\|_n \widehat{f_{22}}$. Thus (vi) holds. □

3.2 A quantized $A_0(K)$-space

3.2.1 L^1-matrix convex sets.

Definition 3.2.1 *Let K be a compact convex set in a locally convex set E such that $0 \in ext(K)$. An element $k \in K$ is called a* lead point *of K ($k \in lead(K)$) if $k = \alpha k_1$ for some $k_1 \in K$ with $\alpha \in [0,1]$, then $\alpha = 1$.*

We observe that $ext(K) \setminus \{0\} \subseteq lead(K)$.

Proposition 3.2.2 *For each $k \in K \setminus \{0\}$. There are unique $\alpha \in (0,1]$ and $\widehat{k} \in lead(K)$ such that $k = \alpha \widehat{k}$.*

Proof. Without any loss of generality, we may assume that $k \in K \setminus lead(K)$. Then by the definition of lead, there is an $\alpha \in (0,1]$ and $k \in K$ such that $k = \alpha k_1$. Thus the set $R_K = \{\beta \geq 1 : \beta k \in K\}$ is non-empty. As K is a compact, R_K is bounded and we have $\beta_0 = \sup R_K \in R_K$. Let $k_0 = \beta_0 k \in K$ so that $k = \beta_0^{-1} k_0$. We show that $k_0 \in lead(K)$. If possible, assume that $k_0 \notin lead(K)$. Then by the definition of lead, there is a $\beta \in (0,1)$ and $k' \in K$ such that $k_0 = \beta k'$. But, then $\beta^{-1}\beta_0 k \in K$, where $\beta^{-1}\beta_0 > \beta$, which contradict $\beta_0 = \sup R_K$. Thus $k_0 \in lead(K)$. Next, we prove the uniqueness of k_0. Let $k = \alpha_1 k_1$ for some

$k_1 \in lead(K)$ and $\alpha_1 \in (0,1]$. We see that $k_1 = \alpha_1^{-1}\beta_0 k_0$. Thus $\alpha_1^{-1}\beta_0 = 1$ and hence $\alpha_1 = \beta_0, k_1 = k_0$. □

By a $*$-*locally convex space*, we mean a locally convex space E together with an involution $*$ which is a homeomorphism. In this case, $M_n(E)$ is also a $*$-locally convex space with respect to the product topology.

Definition 3.2.3 (L^1-matrix convex set) *Let E be a $*$-locally convex space. Let $\{K_n\}$ be a collection of compact convex sets $K_n \subseteq M_n(E)_{sa}$ with $0 \in ext(K_n)$ for all n. Then $\{K_n\}$ is called an L^1-*matrix convex set *if the following conditions hold:*

L_1 *If $u \in K_n$ and $\gamma_i \in \mathbb{M}_{n,n_i}$ with $\sum_{i=1}^k \gamma_i\gamma_i^* \leq I_n$, then*

$$\oplus_{i=1}^k \gamma_i^* u \gamma_i \in K_{\sum_{i=1}^k n_i}.$$

L_2 *If $u \in K_{2n}$ so that $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ for some $u_{11}, u_{22} \in K_n$ and $u_{12} \in M_n(E)$, then*

$$u_{12} + u_{12}^* \in co(K_n \cup -K_n).$$

L_3 *Let $u \in K_{m+n}$ with $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ so that $u_{11} \in K_m, u_{22} \in K_n$ and $u_{12} \in M_{m,n}(E)$ and if $u_{11} = \alpha_1\widehat{u_{11}}, u_{22} = \alpha_{22}\widehat{u_{22}}$ with $\widehat{u_{11}} \in lead(K_m), \widehat{u_{22}} \in lead(K_n)$, then $\alpha_1 + \alpha_2 \leq 1$.*

Remark 3.2.4 *Let V be a C^*-ordered operator space. Then by Proposition 3.1.4, $\{Q_n(V)\}$ is an L^1-matrix convex set with $lead(Q_n(V)) = S_n(V)$. In particular, $M_n(\mathcal{T}(H))_1^+$ is an L^1-matrix convex set.*

3.2.1.1 Quantized $A_0(K)$-spaces

Now, we describe the converse of Theorem 3.1.2. Let E be a $*$-locally convex space and let $\{K_n\}$ be an L^1- matrix convex set in E. We assume that $M_n(E)^+ := \cup_{r=1}^{\infty} rK_n$ is a cone in $M_n(E)_{sa}$ for all n. Using $\mathbf{L_1}$, we get that $(E, \{M_n(E)^+\})$ is a matrix ordered space such that E^+ is proper. We further assume that E^+ is generating too. For each n, we define

$$A_0(K_n, M_n(E)) := \{a : K_n \mapsto \mathbb{C} | a \text{ is continuous and affine}; a(0) = 0; \text{and}$$
$$a \text{ extends to a continuous linear functional } \tilde{a} : M_n(E) \mapsto \mathbb{C}\}.$$

Let $a \in A_0(K_n, M_n(E))$. Since $\{K_n\}$ is an L_1-matrix convex set and since K_n spans $M_n(E)$, for $v \in M_n(E)$, we have $v = \sum_{j=1}^{r} \lambda_j v_j + i \sum_{k=1}^{r} \lambda'_k v'_k$ where $v_j, v'_j \in K_n$ and $\lambda_i, \lambda'_j \in \mathbb{R}$. Thus $\tilde{a}(v) = \sum_{j=1}^{r} \lambda_j a(v_j) + i \sum_{k=1}^{r} \lambda'_k a(v'_k)$. Therefore, such an extension is always unique.

We consider the following algebraic operations:

1. For $\alpha \in \mathbb{M}_{m,n}$, $\beta \in \mathbb{M}_{n,m}$ and $a \in A_0(K_n, M_n(E))$, we define

$$\alpha a \beta(v) := \tilde{a}(\alpha^T v \beta^T) \text{ for all } v \in K_m.$$

   Then $\alpha a \beta \in A_0(K_m, M_m(E))$. In fact, the map $v \mapsto \alpha^T v \beta^T$ from $M_m(E)$ to $M_n(E)$ is continuous so that the map $v \mapsto \tilde{a}(\alpha^T v \beta^T)$ from $M_m(E)$ into $\mathbb{C}$ is also continuous. Thus $\widetilde{\alpha v \beta} : M_n(E) \mapsto \mathbb{C}$ is continuous and hence $\alpha a \beta \in A_0(K_n, M_n(E))$.

2. For $a \in A_0(K_n, M_n(E))$ and $b \in A_0(K_m, M_m(E))$, we define

$$(a \oplus b)(v) := a(v_{11}) + b(v_{22})$$

for all $v \in K_{n+m}$ where $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$ with $v_{11} \in K_n, v_{22} \in K_m, v_{12} \in M_{n,m}(E)$. Then $a \oplus b \in A_0(K_{n+m}, M_{n+m}(E))$. In fact, the maps $v \mapsto v_{11}$ from K_{m+n} into K_n and $v \mapsto v_{22}$ from K_{m+n} into K_m are continuous so that $v \mapsto a(v_{11}) + b(v_{22})$ is also continuous. As $\widetilde{a \oplus b} = \tilde{a} \oplus \tilde{b}$, we see that $\widetilde{a \oplus b}$ is also continuous from $M_{m+n}(E) \mapsto \mathbb{C}$. Therefore, $a \oplus b \in A_0(K_{m+n}, M_{m+n}(E))$.

For $a \in A_0(K_n, M_n(E))$, we define $a^*(u) = \overline{a(u)}$ for all $u \in K_n$ so that $\tilde{a^*}(u) = \overline{\tilde{a}(u^*)}$ for all $u \in M_n(E)$. Then $a \mapsto a^*$ is an involution. We set

$$A_0(K_n, M_n(E))_{sa} = \{a \in A_0(K_n, M_n(E)) : a^* = a\}.$$

We put

$$A_0(K_n, M_n(E))^+ := \{a \in A_0(K_n, M_n(E))_{sa} : a(f) \geq 0 \ \ \forall f \in K_n\}.$$

Next, for $a \in A_0(K_n, M_n(E))$, we define

$$\|a\|_{\infty,n} := \sup \left\{ \left| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} (u) \right| : u \in K_{2n} \right\} \text{ for } \ a \in A_0(K_n, M_n(E)).$$

Finally, for each $n \in \mathbb{N}$, we define $\Phi_n : M_n(A_0(K_1, E)) \to A_0(K_n, M_n(E))$ as follows: Let $a_{ij} \in A_0(K_1, E)$ for $1 \leq i, j \leq n$. Define

$$\Phi_n([a_{ij}]) : K_n \to \mathbb{C} \text{ given by } \Phi_n([a_{ij}])([v_{ij}]) = \sum_{i,j=1}^{n} \widetilde{a_{ij}}(v_{ij}) \text{ for all } [v_{ij}] \in K_n.$$

Now, it is routine to show that $\Phi_n([a_{ij}]) \in A_0(K_n, M_n(E))$. (Note that Φ_n is an amplification of Φ_1. That is, $\Phi_n([a_{ij}]) = [\Phi_1(a_{ij})]$, if $[a_{ij}] \in M_n(A_0(K_1, E))$.) In this identification, we note that $[a_{i,j}]^* = [a^*_{j,i}]$ is an involution in $M_n(A_0(K_1,E))$ so that Φ_1 is a $*$-isomorphism.

For each $n \in \mathbb{N}$, we set

$$M_n(A_0(K_1,E))^+ := \left\{[a_{ij}] \in M_n(A_0(K_1,E))_{sa} : \sum_{i,j=1}^{n} \widetilde{a_{i,j}}(v_{i,j}) \geq 0 \text{ for all } [v_{i,j}] \in K_n\right\}$$

and transport the norm

$$\|[a_{i,j}]\|_n := \|\Phi_n([a_{i,j}])\|_{\infty,n}$$

for all $[a_{i,j}] \in M_n(A_0(K_1,E))$. Under these notions, we have

Theorem 3.2.5 *$(A_0(K_1,E), \{M_n(A_0(K_1,E))^+\}, \{\|\cdot\|_n\})$ is a* C**-ordered operator space.*

Proof. We prove the theorem in several steps.

It is easy to deduce from the definition that $(\alpha a\beta)^* = \beta^* a^* \alpha^*$ and that $(a \oplus b)^* = a^* \oplus b^*$.

1. Let $\alpha \in \mathbb{M}_{m,n}$, $a \in A_0(K_n, M_n(E))^+$ and let $v \in K_n$. Without any loss of generality, we may assume that $\|\alpha\| \leq 1$. Then, by the definition of an L^1-matrix convex set, we have $\alpha^{T^*} v \alpha^T \in K_n$. Thus $\alpha^* a \alpha(v) = a(\alpha^{T^*} v \alpha^T) \geq 0$ so that $\alpha^* a \alpha \in A_0(K_n, M_n(E))^+$.

2. Let $a \in A_0(K_m, M_m(E))^+, b \in A_0(K_n, M_n(E))^+$ and let $u \in K_{m+n}$ with $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$, for some $u_{11} \in K_m, u_{22} \in K_n$ and $u_{12} \in M_{m,n}(E)$. Then

$$(a \oplus b)(u) = a(u_{11}) + b(u_{22}) \geq 0$$

so that $a \oplus b \in A_0(K_{m+n}, M_{m+n}(E))^+$.

Now, it follows from (1) and (2) and the construction of $M_n(A_0(K_1, E))$ that the sequence of cones $\{M_n(A_0(K_1, E))\}$ is a matrix order on $A_0(K_1, E)$. Also, it is easy to verify that $A_0(K_1, E)^+$ is proper.

3. It is routine to verify that $\|\cdot\|_{\infty,n}$ is a semi-norm on $A_0(K_n, M_n(E))$. We show that it is a norm. Let $a \in A_0(K_n, M_n(E))$ such that $\|a\|_n = 0$. Let $u \in K_n$ and $\alpha = [\frac{1}{\sqrt{2}}I_n, \frac{1}{\sqrt{2}}I_n]$. Then$\alpha^*\alpha \leq I_{2n}$ and therefore, $\alpha^* u \alpha = \begin{bmatrix} \frac{u}{2} & \frac{u}{2} \\ \frac{u}{2} & \frac{u}{2} \end{bmatrix} \in K_{2n}$. Also, then $\begin{bmatrix} \frac{u}{2} & i\frac{u}{2} \\ -i\frac{u}{2} & \frac{u}{2} \end{bmatrix} \in K_{2n}$. Thus, as $\|a\|_{\infty,n} = 0$, we get

$$0 = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{u}{2} & i\frac{u}{2} \\ -i\frac{u}{2} & \frac{u}{2} \end{bmatrix} \right) = \tilde{a}(\frac{iu}{2}) + \tilde{a^*}(\frac{-iu}{2}) = \frac{i}{2}a(u) + \frac{-i}{2}\overline{a(u)}.$$

Similarly,

$$0 = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{u}{2} & \frac{u}{2} \\ \frac{u}{2} & \frac{u}{2} \end{bmatrix} \right) = \frac{a(u)}{2} + \frac{\overline{a(u)}}{2}.$$

Therefore $a(u) \pm \overline{a(u)} = 0$ for all $u \in K_n$ and consequently $a(u) = 0$ for all $u \in K_n$. Hence $a = 0$.

4. Further, note that $\begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix} \in K_{2n}$ if and only if $\begin{bmatrix} v_{11} & v_{12}^* \\ v_{12} & v_{22} \end{bmatrix} \in K_{2n}$ and that

$$\begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \left(\begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix} \right) = \begin{bmatrix} 0 & a^* \\ a & 0 \end{bmatrix} \left(\begin{bmatrix} v_{11} & v_{12}^* \\ v_{12} & v_{22} \end{bmatrix} \right)$$

for $a \in A_0(K_n, M_n(E))$. Thus $\|a^*\|_{\infty,n} = \|a\|_{\infty,n}$ for all $a \in A_0(K_n, M_n(E))$.

5. Next, we show that if $a \in A_0(K_n, M_n(E))_{sa}$, then

$$\|a\|_{\infty,n} = \sup\{|a(v)| : v \in K_n\}.$$

In particular, we have

$$\|a\|_{\infty,n} = \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \right\|_{\infty,2n}$$

for every $a \in A_0(K_n, M_n(E))$.

To see this, we put $r_n(a) = \sup\{|a(v)| : v \in K_n\}$. Since K_{2n} is a compact set, we have $\|a\|_n = \left| \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} (v) \right|$ for some $v \in K_{2n}$. Let $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$. Since $\{K_n\}$ is an L^1-matrix convex set, we have $v_{12}+v_{12}^* \in co(K_n \cup (-K_n))$. As K_n is convex, there are $v, w \in K_n$ and $\lambda \in [0,1]$ such that $v_{12} + v_{12}^* = \lambda u - (1-\lambda)w$. Thus

$$\begin{aligned} \|a\|_{\infty,n} &= |\tilde{a}(v_{12}) + \tilde{a}(v_{12}^*)| = |\tilde{a}(v_{12} + v_{12}^*)| \\ &= |\tilde{a}(\lambda u - (1-\lambda)w)| = |\lambda a(u) - (1-\lambda)a(w)| \\ &\leq \lambda r_n(a) + (1-\lambda) r_n(a) = r_n(a) \end{aligned}$$

Again as K_n is a compact convex set, we have $r_n(a) = |a(v)|$ for some $v \in K_n$. Since $\{K_n\}$ is an L^1-matrix convex set, we have $\begin{bmatrix} \frac{v}{2} & \frac{v}{2} \\ \frac{v}{2} & \frac{v}{2} \end{bmatrix} \in K_{2n}$. Therefore,

$$r_n(a) = \left| \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{v}{2} & \frac{v}{2} \\ \frac{v}{2} & \frac{v}{2} \end{bmatrix} \right) \right| \leq \|a\|_{\infty,n}.$$

6. In particular, for $a \leq b \leq c$ in $A_0(K_n, M_n(E))_{sa}$, we have

$$\|b\|_{\infty,n} \leq \max\{\|a\|_{\infty,n}, \|c\|_{\infty,n}\}.$$

To prove this, let $a \leq b \leq c$ in $A_0(K_n, M_n(E))_{sa}$. Then $a(u) \leq b(u) \leq c(u)$ for all $u \in K_n$ so that $|b(u)| \leq \max\{|a(u)|, |c(u)|\}$. Thus by (5), we get $|b(u)| \leq \max\{\|a\|_{\infty,n}, \|b\|_{\infty,n}\}$ for all $u \in K_n$ so that $\|b\|_{\infty,n} \leq \max\{\|a\|_{\infty,n}, \|c\|_{\infty,n}\}$.

7. Now, we prove that $\|a \oplus b\|_{\infty,m+n} = \max\{\|a\|_{\infty,m}, \|b\|_{\infty,n}\}$ for all $a \in A_0(K_m, M_m(E))_{sa}$ and $b \in A_0(K_n, M_n(E))_{sa}$.

Let $a \in A_0(K_m, M_m(E))_{sa}$ and $b \in A_0(K_n, M_n(E))_{sa}$. Now for every $v \in K_m$, we have

$$|a(v)| = |(a \oplus b)(v \oplus 0)|.$$

Since $\{K_n\}$ is an L^1-matrix convex set, we have $v \oplus 0 \in K_{m+n}$ whenever $v \in K_m$. Therefore by (5), we may conclude that $\|a\|_{\infty,m} \leq \|a \oplus b\|_{\infty,m+n}$. Similarly, we can show that $\|b\|_{\infty,n} \leq \|a \oplus b\|_{\infty,m+n}$.

Conversely, let $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix} \in K_{m+n}$. Then there exist $\widehat{v_{11}} \in lead(K_m), \widehat{v_{22}} \in lead(K_n)$ and $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 \leq 1$ such that $v_{11} = \alpha_1 \widehat{v_{11}}, v_{22} = \alpha_2 \widehat{v_{22}}$. Thus

$$\begin{aligned} |(a \oplus b)(v)| &= |a(v_{11}) + b(v_{22})| \\ &= |\alpha_1 a(\widehat{v_{11}}) + \alpha_2 b(\widehat{v_{22}})| \\ &\leq \alpha_1 \|a\|_{\infty,m} + \alpha_2 \|b\|_{\infty,n} \\ &\leq \max\{\|a\|_{\infty,m}, \|b\|_{\infty,n}\}. \end{aligned}$$

Therefore $\|a \oplus b\|_{\infty,m+n} = \max\{\|a\|_{\infty,m}\|b\|_{\infty,n}\}$.

8. Next, we prove that for $a \in A_0(K_n, M_n(E))_{sa}$ and $\alpha \in \mathbb{M}_{m,n}$, we have $\|\alpha^* a\alpha\|_{\infty,n} \leq \|\alpha\|^2\|a\|_{\infty,n}$.

   Let $a \in A_0(K_m, M_m(E))_{sa}$ and $\alpha \in \mathbb{M}_{m,n}$ such that $\|\alpha\| \leq 1$ and let $v \in K_n$. Since $\{K_n\}$ is an L^1-matrix convex set and $\alpha^{*T}\alpha^T \leq I_m$, we have $\alpha^{T^*} v\alpha^T \in K_m$. Also, we know that

$$|(\alpha^* a\alpha)(v)| = |a(\alpha^{T^*} v\alpha^T)|.$$

   Since a is self-adjoint, by (5), we have $\|\alpha^* a\alpha\|_{\infty,n} \leq \|a\|_{\infty,n}$ for $a = a^*$. In particular, if $m = n$ and if $\alpha \in \mathbb{M}_m$ is unitary, then $\|\alpha^* a\alpha\|_{\infty,m} = \|a\|_{\infty,m}$. Also, in general, for $a \in A_0(K_n, M_n(E))_{sa}$ and $\alpha \in \mathbb{M}_{m,n}$, we have

$$\|\alpha^* a\alpha\|_{\infty,n} \leq \|\alpha\|^2\|a\|_{\infty,n}.$$

9. Let $a \in A_0(K_m, M_m(E))$ and $b \in A_0(K_n, M_n(E))$. Put $\gamma = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}$.

   Then $\gamma \in \mathbb{M}_{2m+2n}$ is a unitary and

$$\gamma^* \begin{bmatrix} 0 & a \oplus b \\ a^* \oplus b^* & 0 \end{bmatrix} \gamma = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix}$$

   so that

$$\left\| \begin{bmatrix} 0 & a \oplus b \\ a^* \oplus b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n} = \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n}$$

by (8). Thus by (5), we have

$$\begin{aligned}
\|a \oplus b\|_{m+n} &= \left\| \begin{bmatrix} 0 & a \oplus b \\ a^* \oplus b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n} \\
&= \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \right\|_{\infty,2m+2n} \\
&= \max \left\{ \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \right\|_{\infty,2m}, \left\| \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \right\|_{\infty,2n} \right\} \\
&= \max\{\|a\|_{\infty,m}, \|b\|_{\infty,n}\}.
\end{aligned}$$

10. Let $\alpha \in \mathbb{M}_{m,n}, a \in A_0(K_n, M_n(E))$ and $\beta \in \mathbb{M}_{n,m}$. Then by (5), we have

$$\|\alpha a \beta\|_{\infty,m} = \left\| \begin{bmatrix} 0 & \alpha a \beta \\ \beta^* a^* \alpha & 0 \end{bmatrix} \right\|_{\infty,2m}$$

For $t \in \mathbb{R}^+ \setminus \{0\}$, we have

$$\begin{bmatrix} t\alpha & 0 \\ 0 & \frac{1}{t}\beta^* \end{bmatrix} \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \begin{bmatrix} t\alpha^* & 0 \\ 0 & \frac{1}{t}\beta \end{bmatrix} = \begin{bmatrix} 0 & \alpha a \beta \\ \beta^* a^* \alpha & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\|\alpha a \beta\|_{\infty,m} &\leq \left\| \begin{bmatrix} t\alpha & 0 \\ 0 & \frac{1}{t}\beta^* \end{bmatrix} \right\| \left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \right\|_{\infty,2n} \left\| \begin{bmatrix} t\alpha^* & 0 \\ 0 & \frac{1}{t}\beta \end{bmatrix} \right\| \\
&\leq \max\{\|t\alpha\|, \|\frac{1}{t}\beta^*\|\}^2 \|a\|_{\infty,n} \\
&= \max\{t^2\|\alpha\|^2, \frac{1}{t^2}\|\beta\|^2\}\|a\|_{\infty,n}.
\end{aligned}$$

Taking infimum over $t \in \mathbb{R}^+ \setminus \{0\}$, we may conclude that $\|\alpha a \beta\|_{\infty,m} \leq \|\alpha\| \|a\|_{\infty,n} \|\beta\|$.

This completes the proof. □

Remark 3.2.6 *Let $\{K_n\}$ be an L^1-matrix convex set of E. Then by [40, Theorem 1.7] there is a complete order isometry $\phi : A_0(K_1, E) \to \mathcal{A}$ for some* C*-*algebra* $\mathcal{A}$.

3.3 Completely regularaity

In this section, we propose a matricial version of regular embedding of L^1-matrix convex sets and the notion of L^1-matricial caps. We prove that if $\{K_n\}$ is a regularly embedded, L^1-matricial cap in a $*$-locally convex space E, then $A_0(K_1, E)$ has an order unit so that $(A_0(K_1, E), e)$ is a matrix order unit space.

Definition 3.3.1 *Let $\{K_n\}$ be an L^1-matrix convex set in a $*$-locally convex space E and let L_n be the lead of K_n for each n. We call $\{L_n\}$ the* matricial lead *of $\{K_n\}$. We also assume that $M_n(E)^+ = \cup_{r=1}^{\infty} r K_n$ is a cone in $M_n(E)_{sa}$ for all n (so that $(E, \{M_n(E)^+\})$ is a matrix ordered space) such that E^+ is proper and generating. We call $\{K_n\}$ an L^1-*matricial cap *of E if*

(1) L_1 is convex; and

(2) if $v \in L_{m+n}$ with $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}$ for some $v_{11} \in K_m, v_{22} \in K_n$ and $v_{12} \in M_{m,n}(E)$ and if $v_{11} = \alpha_1 \widehat{v_1}, v_{22} = \alpha_{22} \widehat{v_2}$ for some $\widehat{v_1} \in L_m, \widehat{v_2} \in L_n$ and $\alpha_1, \alpha_2 \in [0, 1]$, then $\alpha_1 + \alpha_2 = 1$.

Theorem 3.3.2 *Let $\{K_n\}$ be an L^1-matricial cap of E. Then L_n is convex for every n.*

Proof. We prove this result in several steps.

Step I. L_2 is convex.

Let $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix}, w = \begin{bmatrix} w_{11} & w_{12} \\ w_{12}^* & w_{22} \end{bmatrix} \in L_2$ and let $\lambda \in [0,1]$. Then by Definition 3.3.1(2), we have $v_{11} = \alpha_1 \widehat{v_1}, v_{22} = \alpha_2 \widehat{v_2}$ with $\alpha_1 + \alpha_2 = 1$, for some $\widehat{v_1}, \widehat{v_2} \in L_1$, and $w_{11} = \beta_1 \widehat{w_1}, w_{22} = \beta_2 \widehat{w_2}$ with $\beta_1 + \beta_2 = 1$, for some $\widehat{w_1}, \widehat{w_2} \in L_1$. Now

$$u := \lambda v + (1-\lambda) w = \begin{bmatrix} \lambda v_{11} + (1-\lambda) w_{11} & \lambda v_{12} + (1-\lambda) w_{12} \\ \lambda v_{12}^* + (1-\lambda) w_{12}^* & \lambda v_{22} + (1-\lambda) w_{22} \end{bmatrix} \in K_2.$$

Let $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{22} \end{bmatrix}$ so that $u_{11} = \lambda v_{11} + (1-\lambda) w_{11} = \lambda \alpha_1 \widehat{v_1} + (1-\lambda) \beta_1 \widehat{w_1}$ and $u_{22} = \lambda v_{22} + (1-\lambda) w_{22} = \lambda \alpha_2 \widehat{v_2} + (1-\lambda) \beta_2 \widehat{w_2}$. Since L_1 is convex, we get $\widehat{u_1} := (\lambda \alpha_1 + (1-\lambda)\beta_1)^{-1} u_{11} \in L_1$ and $\widehat{u_2} := (\lambda \alpha_2 + (1-\lambda)\beta_2)^{-1} u_{22} \in L_1$. Put $(\lambda \alpha_1 + (1-\lambda)\beta_1) = \gamma_1$ and $(\lambda \alpha_2 + (1-\lambda)\beta_2) = \gamma_2$, then $u = \begin{bmatrix} \gamma_1 \widehat{u_1} & u_{12} \\ u_{12}^* & \gamma_2 \widehat{u_2} \end{bmatrix}$ and $\gamma_1 + \gamma_2 = \lambda(\alpha_1 + \alpha_2) + (1-\lambda)(\beta_1 + \beta_2) = \lambda + (1-\lambda) = 1$. Let $u = \gamma \widehat{u}$, where $\widehat{u} \in L_2$ and $\gamma \in [0,1]$. We show that $\gamma = 1$. Let $\widehat{u} = \begin{bmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{bmatrix}$. Then $x_{11}, x_{22} \in K_1$ with $\gamma x_{11} = u_{11}, \gamma x_{22} = u_{22}$. Thus $x_{11} = \gamma^{-1} \gamma_1 \widehat{u_1}$ and $x_{22} = \gamma^{-1} \gamma_2 \widehat{u_2}$. Now by Definition 3.3.1(2), we get $1 = \gamma^{-1} \gamma_1 + \gamma^{-1} \gamma_2 = \gamma^{-1}$ so that $\gamma = 1$. Thus $u \in L_2$, and hence L_2 is convex.

Now, by induction, we may deduce that L_{2^n} is convex for every n.

Step II. For $m, n \in \mathbb{N}$, we have L_m is convex if L_{m+n} is convex.

First, we show that $v \mapsto v \oplus 0$ maps L_m into L_{m+n}. Let $v \in L_m$. Then

$v \oplus 0 \in K_{m+n}$ so that $v \oplus 0 = \alpha \widehat{w}$ for some $\widehat{w} \in L_{m+n}$ and $\alpha \in [0,1]$. Thus

$$v = \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix} (v \oplus 0) \begin{bmatrix} I_n \\ 0_{m,n} \end{bmatrix} = \alpha \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix} \widehat{w} \begin{bmatrix} I_n \\ 0_{m,n} \end{bmatrix} = \alpha w_1.$$

where $w_1 = \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix} \widehat{w} \begin{bmatrix} I_n \\ 0_{m,n} \end{bmatrix} \in K_m$. Now, as L_m is the lead of K_m, we have $\alpha = 1$ and $w_1 = v$. Thus $v \oplus 0 = \widehat{w} \in L_{m+n}$.

Fix $m \in \mathbb{N}$. Let $v, w \in L_m$ and $\alpha \in (0,1)$. As L_{2^m} is convex, we get

$$(\alpha v + (1-\alpha)w) \oplus 0 = \alpha(v \oplus 0) + (1-\alpha)(w \oplus 0) \in L_{2^m}.$$

Put $u = \alpha v \oplus (1-\alpha)w$. Then $u \in K_m$ so that $u = \lambda \widehat{u}$ for some $\widehat{u} \in L_m$ and $\lambda \in [0,1]$. As $\widehat{u} \in L_m$, we get that $\widehat{u} \oplus 0 \in L_{2^m}$. Now $\lambda(\widehat{u} \oplus 0) = u \oplus 0 \in L_{2^m}$ so that $\lambda = 1$ and $u = \widehat{u} \in L_m$. Thus L_m is also convex. □

When L_1 is compact and convex, by $A(L_1)$ we denote the set of all complex valued continuous affine functions on L_1. Then $A(L_1)_{sa}$ is an order unit space so that $A(L_1)^*_{sa}$, the ordered Banach dual of $A(L_1)_{sa}$, is a base normed space (see e.g. [2, 37]).

Definition 3.3.3 *Let $\{K_n\}$ be an L^1-matrix convex set in a $*$-locally convex space E. Then $\{K_n\}$ is called* regularly embedded *in E if L_1 is regularly embedded in E_{sa}. In other words,*

1. *L_1 is compact and convex; and*

2. *$\chi : E_{sa} \mapsto (A(L_1)^*_{sa})_{w*}$ is a linear homeomorphism.*

Here $\chi(w)(a) = \lambda a(u) - \mu a(v)$ for all $a \in A(L_1)_{sa}$ with $w = \lambda u - \mu v$ for some $u, v \in L_1$ and $\lambda, \mu \in \mathbb{R}^+$.

We note that $\chi(w)$ is well defined. To see this, let $w = \lambda_1 u_1 - \mu_1 v_1 = \lambda_2 u_2 - \mu_2 v_2$ for some $u_i, v_i \in L_1$ and $\lambda_i, \mu_i \in \mathbb{R}^+$ for $i = 1, 2$. As L_1 is convex and

$$\frac{\lambda_1 + \mu_2}{\lambda_2 + \mu_1}\left(\frac{\lambda_1 u_1 + \mu_2 v_1}{\lambda_1 + \mu_2}\right) = \frac{\lambda_2 u_2 + \mu_1 v_1}{\lambda_2 + \mu_1},$$

by Proposition 3.2.2, we have $\lambda_1 + \mu_2 = \lambda_2 + \mu_1$. So if a is an affine function on L_1, then

$$\frac{\lambda_1 a(u_1) + \mu_2 a(v_2)}{\lambda_1 + \mu_2} = a\left(\frac{\lambda_1 u_1 + \mu_2 v_2}{\lambda_1 + \mu_2}\right) = a\left(\frac{\lambda_2 u_2 + \mu_1 v_1}{\lambda_2 + \mu_1}\right) = \frac{\lambda_2 a(u_2) + \mu_1 a(v_1)}{\lambda_2 + \mu_1}.$$

Thus $\lambda_1 a(u_1) - \mu_1 a(v_1) = \lambda_2 a(u_2) - \mu_2 a(v_2)$ so that $\chi(w)$ is well defined linear functional on $A(L_1)_{sa}$ for all $u, v \in L_n$ and $\lambda, \mu \in \mathbb{R}^+$.

Theorem 3.3.4 *Let $\{K_n\}$ be a regularly embedded, L^1-matricial cap in E. Then $A_0(K_1, E)$ has an order unit, say e, so that $(A_0(K_1, E), e)\}$ is a matrix order unit space.*

Proof. As L_1 is the lead of K_1, there exists a mapping $e : K_1 \setminus \{0\} \to (0, 1]$ given by $e(k) = \alpha$ if $k = \alpha\widehat{k}$ for some $\widehat{k} \in L_1$ and $\alpha \in (0, 1]$. Since α and $\widehat{k}$ are uniquely determined by $k \in K_1 \setminus \{0\}$, e is well defined. We extend e to K by putting $e(0) = 0$. Since L_1 is convex, we may conclude that $e : K_1 \to [0, 1]$ is affine. Again since K_1 spans E, we can extend e to a self-adjoint linear functional $\tilde{e} : E \to \mathbb{C}$. Following this way, for each $n \in \mathbb{N}$, we can construct a self-adjoint linear functional $\widetilde{e_n} : M_n(E) \to \mathbb{C}$ such that $\widetilde{e_n}(v) = 1$ for all $v \in L_n$. (We write e_n for $\widetilde{e_n}|_{L_n}$.)

We show that $\tilde{e}$ is continuous. It suffices to show that $\tilde{e}|_{V_{sa}}$ is continuous at 0. Let $\{\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha\}$ be a net in E_{sa} for some $u_\alpha, v_\alpha \in L_1$ and $\lambda_\alpha, \mu_\alpha \in \mathbb{R}^+$ such that $\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha \to 0$. Since $\{K_n\}$ is L^1-regularly embedded in E, we get

$\chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \to 0$ in $(A(L_1)^*_{sa})_{w*}$. Let I_{L_1} be the constant map on L_1 such that $I_{L_1}(v) = 1$ for all $v \in L_1$. Then $I_{L_1} \in A(L_1)_{sa}$. Thus $\chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha)(I_{L_1}) \longrightarrow 0$ so that $\tilde{e}(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \longrightarrow 0$. Now it follows that $e \in A_0(K_1, E)$.

Next, fix $n \in \mathbb{N}$ and consider $e^n \in M_n(A_0(K_1, E))$ so that by Theorem 3.2.5, $e_0^n := \Phi_n(e^n) \in A_0(K_n, M_n(E))$. We show that $e_0^n = e_n$. Let $[v_{i,j}] \in L_n$ so that $v_{i,i} \in K_1$ for $i = 1, \dots, n$. Let $v_{ii} = \alpha_i \widehat{v_i}$ for some $\alpha_i \in [0,1]$ and $\widehat{v_i} \in L_n$. Since $\{K_n\}$ is an L^1-matricial cap, we have $\sum_{i=1}^n \alpha_i = 1$. Thus

$$e_0(v) = \sum_{i=1}^n e_i(v_{i,i}) = \sum_{i=1}^n \alpha_i e_i(\widehat{v_i}) = \sum_{i=1}^n \alpha_i = 1$$

so that $e_0(v) = e_n(v)$ for all $v \in L_n$. Since L_n is the lead of K_n and since K_n spans $M_n(E)$, it follows that $\widetilde{e_n} = e_0$ and that $e_n \in A_0(K_n, M_n(E))$.

Note that $\|e\|_{\infty,1} = 1$. We show that e is an order unit for $A_0(K_1, E)_{sa}$. To see this, let $a \in A_0(K_1, E)_{sa}$. Then $|a(k)| \leq \|a\|_{\infty,1}$ for all $k \in K_1$. Let $k \in K_1$. If $k = 0$, then $a(0) = 0$ so that

$$-\|a\|_{\infty,1} e(0) = 0 = \|a\|_{\infty,1} e(0).$$

Let $k \neq 0$. Then there exists a unique $\widehat{k} \in L_1$ and $\alpha \in (0,1]$ such that $k = \alpha \widehat{k}$. Now

$$-\|a\|_{\infty,1} e(\hat{k}) = -\|a\|_{\infty,1} \leq a(\hat{k}) \leq \|a\|_{\infty,1} = \|a\|_{\infty,1} e(\hat{k}).$$

so that

$$-\|a\|_{\infty,1} e(k) \leq a(k) \leq \|a\|_{\infty,1} e(k)$$

for all $k \in K$. Thus we have $-\|a\|_{\infty,1} e \leq a \leq \|a\|_{\infty,1} e$ for all $a \in A_0(K_1, E)_{sa}$. In other words, e is an order unit for $A_0(K_1, E)_{sa}$ which determines $\|\cdot\|_{\infty,1}$ as an order unit norm on it. Similarly, we can show that for each $n \in \mathbb{N}$, e_n is an order unit for $A_0(K_n, M_n(E))_{sa}$ which determines $\|\cdot\|_{\infty,n}$ as an order unit norm on it. Again, being function space, $A_0(K_n, M_n(E))$ is Archimedean for every n. Hence $(A_0(K_1, E), e)$ is a matrix order unit space. Next, we prove the completeness of $(A_0(K_1, E), e)$.

Proposition 3.3.5 *Let $\{K_n\}$ be an L^1-matrix convex set in a $*$-locally convex space E. Then $\overline{A_0(K_n, M_n(E))_{sa}} = A_0(K_n)_{sa}$ for every $n \in \mathbb{N}$.*

Proof. By the definition, $A_0(K_n, M_n(E))_{sa} \subset A_0(K_n)_{sa}$. Also, since $A_0(K_n)_{sa}$ is norm complete, we get $\overline{A_0(K_n, M_n(E))_{sa}} \subset A_0(K_n)_{sa}$. Conversely, let $a \in A_0(K_n)_{sa}$ and $\epsilon > 0$. Then $G_{K_n}(a)$ and $G_{K_n}(a + \epsilon)$ are compact convex set in $M_n(E)_{sa} \times \mathbb{R}$, where

$$G_{K_n}(b + \lambda) := \{(k, b(k) + \lambda) : k \in K_n\}$$

with $b \in A_0(K_n)_{sa}$ and $\lambda \in [0, \infty)$. Thus $G_{K_n}(a) \cap G_{K_n}(a+\epsilon) = \emptyset$. Therefore, by the Hahn Banach separation theorem, there are $f \in (M_n(E)_{sa})^*(= (M_n(E)^*)_{sa})$ and $\lambda \in \mathbb{R}$ such that

$$(f, \lambda)(u, a(u)) < (f, \lambda)(v, a(v) + \epsilon) \; \forall u, v \in K_n.$$

Simplifying this, we get

$$f(u) + \lambda a(u) < f(v) + \lambda(a(v) + \epsilon) \; \forall u, v \in K_n.$$

In particular, when $u = v = 0$, we get $\lambda > 0$. Similarly, for $u = 0$ and $v = 0$ separately, we have

$$\lambda^{-1} f(u) + a(u) < \epsilon \text{ and } \lambda^{-1} f(v) + a(v) > -\epsilon \; \forall u, v \in K_n.$$

Let us put $a_1 = -\lambda^{-1}f$, then $a_1 \in A_0(K_n, M_n(E))_{sa}$ and $|a_1(u) - a(u)| < \epsilon$ for all $u \in K_n$. Now, by (5) of the proof of Theorem 3.2.5, we have $\|a_1 - a\|_{\infty,n} \leq \epsilon$. This completes the proof.

Proposition 3.3.6 *Under the assumptions of Theorem 3.3.4, $A_0(K_n, M_n(E)) = A_0(K_n)$ for each $n \in \mathbb{N}$.*

Proof. We know that $A_0(K_1, E) \subseteq A_0(K_1)$. Let $a \in A_0(K_1)$ so that $a = a_1 + ia_2$ for some $a_1, a_2 \in A_0(K_1)_{sa}$ and let $\{\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha\}$ be a net in E_{sa} for some $u_\alpha, v_\alpha \in L_1$ and $\lambda_\alpha, \mu_\alpha \geq 0$ such that $\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha \longrightarrow 0$. Since K_1 generates E, a_i has a unique linear extension $\widetilde{a_i}$ for $i = 1, 2$. Since $\{K_n\}$ is L^1-regularly embedded in E, $\chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \longrightarrow 0$ in $(A(L_1)^*_{sa})_{w*}$. Thus

$$\begin{aligned}\widetilde{a_i}(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) &= \lambda_\alpha a_i(u_\alpha) - \mu_\alpha a_i(v_\alpha)\\ &= \lambda_\alpha a_i|_{L_1}(u_\alpha) - \mu_\alpha a_i|_{L_1}(v_\alpha)\\ &= \chi(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha)(a_i|_{L_1}) \to 0\end{aligned}$$

Put $\widetilde{a} = \widetilde{a_1} + i\widetilde{a_2}$. Then $\widetilde{a}|_{K_1} = a$ and $\widetilde{a}(\lambda_\alpha u_\alpha - \mu_\alpha v_\alpha) \longrightarrow 0$. Thus $\widetilde{a}$ is continuous on E and consequently, $a \in A_0(K_1, E)$. Therefore we have $A_0(K_1) = A_0(K_1, E)$. It follows that $A_0(K_1, E)$ is $\|\cdot\|_1$-complete so that $(A_0(K_n, M_n(E))$ is $\|\cdot\|_{\infty,n}$-complete. Since $\overline{A_0(K_n, M_n(E))_{sa}} = A_0(K_n)_{sa}$, by Proposition 3.3.5, we may conclude that

$$A_0(K_n) = \overline{A_0(K_n, M_n(E))} = A_0(K_n, M_n(E))$$

for $A_0(K_n, M_n(E))$ is $\|\cdot\|_{\infty,n}$-complete.

Remark 3.3.7 *Under the assumptions of Theorem 3.3.4, L_n is compact for each $n \in \mathbb{N}$. To see this, let $\{u_\alpha\}$ be a net in L_n. Since $L_n \subseteq K_n$ and K_n is* *compact, u_α has a subnet $\{u_\beta\}$ that convergent $u_0 \in K_n$. Since $e_n \in A_0(K_n)$. Therefore $1 = e_n(u_\beta) \longrightarrow e_n(u_0)$ so that $e_n(u_0) = 1$. Hence $u_0 \in L_n$.*

Proposition 3.3.8 *$A_0(K_n)$ is order isomorphic to $A(L_n)$.*

Proof. It suffices to prove that the map $a \mapsto a|_{L_n}$ from $A_0(K_n)$ into $A(L_n)$ is surjective. Let $a \in A(L_n)$. Since L_n is convex, there is an affine map b on K_n such that $b|_{L_n} = a$ and $b(0) = 0$. Let u_α be a net in K_n such that $u_\alpha \longrightarrow u_0$ in K_n. Since $e_n \in A_0(K_n)$, $e_n(u_\alpha) \longrightarrow e_n(u_0)$, by Proposition 3.2.2, we have $u_\alpha = \lambda_\alpha \widehat{u_\alpha}$ for some $\widehat{u_\alpha} \in L_n$ and $\lambda_\alpha \in [0, 1]$. If $u_0 = 0$, then $\lambda_\alpha = \lambda_\alpha e_n(\widehat{u_\alpha}) = e_n(u_\alpha) \longrightarrow e(0) = 0$. Therefore, $b(u_\alpha) = \lambda_\alpha a(\widehat{u_\alpha}) \longrightarrow 0 = b(0)$. Again if $u_0 \neq 0$, then by Proposition 3.2.2, we have $u_0 = \lambda_0 \widehat{u_0}$ for some $\lambda_0 \in (0, 1]$ and $\widehat{u_0} \in L_n$. Thus $\lambda_\alpha = \lambda_\alpha e_n(\widehat{u_\alpha}) = e_n(u_\alpha) \longrightarrow e_n(u_0) = \lambda_0$ so that $\widehat{u_\alpha} \longrightarrow \widehat{u_0}$. Since $b(u_\alpha) = \lambda_\alpha a(\widehat{u_\alpha})$, we have $b(u_\alpha) \longrightarrow \lambda_0 a(u_0) = b(u_0)$.

CHAPTER 4

M-ideals in non-unital ordered Banach spaces

In this chapter, we investigate order theoretic properties of M-ideals in order smooth ∞-normed spaces. In the first section, we recall the notion of M-ideals and L-summands. We characterize approximate order unit spaces among order smooth ∞-normed spaces. In the second section, we prove the cone decomposition properties which is beneficial for the rest part of the chapter. Also, as an application of cone decomposition theorem, we prove positive norm preserving extension theorem of all bounded positive functionals of a certain class of subspaces of order smooth ∞-normed spaces. In the third section, we characterize the M-ideals of an order smooth ∞-normed space in terms of split faces by extending the notion of split faces of the state space to those of the quasi-state space. In the last section, we characterize approximate order unit spaces as those order smooth ∞-normed spaces V which are M-ideals in $\tilde{V}$. Here $\tilde{V}$ is the order unit space obtained by adjoining an order unit to V.

4.1 Introduction

Let us recall that the closed subspace W of a real Banach space V is said to be an *L-summand* if there exists a unique closed subspace W' of V such that

$$V = W \oplus_1 W'.$$

A closed subspace W of a real Banach space V is said to be an M-ideal if $W^{\perp}$ (the annihilator of W) is an L-summand of V^*. The following proposition characterizes an approximate order unit space among order smooth ∞-normed spaces.

Proposition 4.1.1 *[65, Proposition 9.5] Let $(V, V^+, \|.\|)$ be an ordered normed space such that $\|.\|$ is additive on V^+ and V satisfies $(OS.1.2)$. Then V is a base normed space.*

Proposition 4.1.2 *Let V be an order smooth ∞-normed space. Then V is an approximate order unit space if and only if $S(V)$ is convex.*

Proof. If $(V, \{e_\lambda\})$ is an approximate order unit space, then $S(V)$ is convex. In fact, for any $f \in V^{*+}$ we have $\|f\| = \sup_\lambda \{f(e_\lambda)\}$. Conversely, let V be an order smooth ∞-normed space for which $S(V)$ is convex. Then the norm is additive on V^{*+}. Notice that if $f, g \in V^{*+} \setminus \{0\}$, then $f_0 = \|f\|^{-1} f, g_0 = \|g\|^{-1} g \in S(V)$. Now, by the convexity, $(\|f\| + \|g\|)^{-1}(\|f\| f_0 + \|g\| g_0) \in S(V)$. Thus

$$\|f + g\| = \|(\|f\| f_0 + \|g\| g_0)\| = \|f\| + \|g\|.$$

Next, as V is an order smooth ∞-normed space, by Theorem 2.2.8, V^* satisfies $(OS.1.2)$. It follows from Proposition 4.1.1 that V^* is a base normed space.

Now, by Theorem 2.2.1, V is an approximate order unit space. □

4.2 Cone-decomposition property

In this section, we prove an order-theoretic version of (the 'Alfsen-Effros' cone decomposition) Theorem 2.1.2 for order smooth 1-normed spaces satisfying $(OS.1.2)$. We recall that if W is a subset of an ordered vector space (V, V^+), we write $W^+ = W \cap V^+$.

Theorem 4.2.1 *Let V be a complete order smooth 1-normed space satisfying $(OS.1.2)$ and let W be a closed cone in V. Then for any $v \in V^+$, there are $w \in W^+$ and $w' \in W'^+$ such that*

$$v = w + w' \quad \textit{and} \quad \|v\| = \|w\| + \|w'\|.$$

We use the following fact to prove Theorem 4.2.1.

Lemma 4.2.2 *Let V be an order smooth 1-normed space satisfying $(OS.1.2)$. If $u \geq 0$, then $face_{V_1}(\frac{u}{\|u\|}) \subseteq V^+$.*

Proof. Let $u \in V^+$. Without any loss of generality, we may assume that $\|u\| = 1$. Let $v \in face_{V_1}(u)$. Then by the definition of $face_{V_1}(u)$, there exists $w \in V_1$ such that

$$u = \lambda v + (1 - \lambda)w$$

for some $\lambda \in (0, 1)$. Since $\|u\| = 1$, we have $\|v\| = 1 = \|w\|$. Also, as V satisfies $(OS.1.2)$, there exist $v_1, v_2, w_1, w_2 \in V^+$ such that $v = v_1 - v_2$ and $w = w_1 - w_2$ with $\|v\| = \|v_1\| + \|v_2\|$ and $\|w\| = \|w_1\| + \|w_2\|$. Thus $u = u_1 - u_2$ where $u_i = \lambda v_i + (1 - \lambda)w_i$ for $i = 1, 2$. Since $0 \leq u \leq u_1$ and since V is an order smooth 1-normed space, we have

$$\begin{aligned} 1 &= \|u\| \leq \|u_1\| \\ &\leq \|\lambda v_1 + (1-\lambda) w_1\| \\ &\leq \lambda \|v_1\| + (1-\lambda)\|w_1\| \\ &\leq \lambda(\|v_1\| + \|v_2\|) + (1-\lambda)(\|w_1\| + \|w_2\|) \\ &\leq \lambda\|v\| + (1-\lambda)\|w\| = 1. \end{aligned}$$

Thus $v_2 = 0 = w_2$ so that $v, w \in V^+$. □

Proof of Theorem 4.2.1. Let W be a closed cone of V and $u \in V^+$. Then by Theorem 2.1.2, we have $u = v + w$ with $\|u\| = \|v\| + \|w\|$ for some $v \in W$ and $w \in W'$. Now, by Lemmas 2.1.1 and 4.2.2, we conclude that v and $w \in V^+$. □

A quick consequence of Lemma 4.2.2 is the following:

Corollary 4.2.3 *Let (V, V^+) be a complete order smooth 1-normed space satisfying $(OS.1.2)$. Then*

$$face_{V_1}\left(\frac{u}{\|u\|}\right) = face_{V_1^+}\left(\frac{u}{\|u\|}\right)$$

and $C(u) \subseteq V^+$ whenever $u \in V^+$. Here $V_1^+ = V_1 \cap V^+$.

We also have the following:

Corollary 4.2.4 *Let (V, V^+) be a complete order smooth 1-normed space satisfying $(OS.1.2)$. Then we have $(-V^+)' = V^+, (V^+)' = -V^+$.*

Proof. We only prove $(-V^+)' = V^+$, as similar arguments are valid for the other case. Put $C = -V^+$ and let $u \in V^+$. By Theorem 4.2.1, we have, $u = v + w$ with $\|u\| = \|v\| + \|w\|$ for some $v \in C^+, w \in C'^+$. But $C^+ = C \cap V^+ = -V^+ \cap V^+ = \{0\}$ so that $u = w \in C'^+$.

Conversely, let $v \in C' := (-V^+)'$. Then by the definition, $C(v) \cap (-V^+) = \{0\}$. Since V satisfies $(OS.1.2)$, there are $v_1, v_2 \in V^+$ such that $v = v_1 - v_2$ and $\|v\| = \|v_1\| + \|v_2\|$. By Lemma 2.1.1, $-v_2 \in C(v)$. But $C(v) \cap (-V^+) = \{0\}$ so that $v = v_1 \in V^+$. $\square$

We apply Theorem 4.2.1 to sharpen [41, Theorem 4.3]. Actually, we prove positive and norm preserving extensions of positive bounded linear functionals without the assumption that the order smooth subspace be 'strong' ([41, Definition 3.4]).

Theorem 4.2.5 *Let W be an order smooth subspace of an order smooth ∞-normed space $(V, V^+, \|.\|)$. Then every positive bounded linear functional on W has a positive norm preserving extension on V.*

Here by an *order smooth subspace* W of an order smooth p-normed space V, we mean that W is also an order smooth p-normed space when the order and the norm of V is restricted to W.

Proof. Let f be a positive bounded linear functional on W. By the Hahn Banach theorem there exists $\tilde{f} \in V^*$ such that $\|\tilde{f}\| = \|f\|$. We prove that $\tilde{f}$ is positive. Since V^* satisfy $(OS.1.2)$, by Theorem 2.2.8, there are $\tilde{f}_1, \tilde{f}_2 \in V^{*+}$ such that $\tilde{f} = \tilde{f}_1 - \tilde{f}_2$ with $\|\tilde{f}\| = \|\tilde{f}_1\| + \|\tilde{f}_2\|$. Since $\tilde{f}_1, \tilde{f}_2 \in V^{*+}$ and V^* is complete, by Theorem 4.2.1, there are $\tilde{f}_{11}, \tilde{f}_{21} \in W^{\perp +}$ and $\tilde{f}_{12}, \tilde{f}_{22} \in W^{\perp '+}$ such that $\tilde{f}_1 = \tilde{f}_{11} + \tilde{f}_{12}$ with $\|\tilde{f}_1\| = \|\tilde{f}_{11}\| + \|\tilde{f}_{12}\|$ and $\tilde{f}_2 = \tilde{f}_{21} + \tilde{f}_{22}$ with $\|\tilde{f}_2\| = \|\tilde{f}_{21}\| + \|\tilde{f}_{22}\|$. Now $\tilde{f} = \tilde{f}_{11} - \tilde{f}_{21} + \tilde{f}_{12} - \tilde{f}_{22}$, where $\tilde{f}_{11}, \tilde{f}_{21} \in W^{\perp +}$ and $\tilde{f}_{12}, \tilde{f}_{22} \in W^{\perp '+}$ such that $\|\tilde{f}\| = \|\tilde{f}_{11}\| + \|\tilde{f}_{21}\| + \|\tilde{f}_{12}\| + \|\tilde{f}_{22}\|$. If $f_{ij} = \tilde{f}_{ij}\big|_W$ for all $i, j \in \{1, 2\}$.

Then $f_{11} = f_{21} = 0$ so that $f = f_{12} - f_{22}$. Further, as f is positive, we have $0 \le f \le f_{12}$. Thus by $(O.1.1)$ property of V^*, we get $\|f\| \le \|f_{12}\|$. Therefore,

$$\begin{aligned} \|f\| &\le \|f_{12}\| \\ &\le \|\tilde{f}_{11}\| + \|\tilde{f}_{21}\| + \|\tilde{f}_{12}\| + \|\tilde{f}_{22}\| \\ &= \|\tilde{f}\| = \|f\| \end{aligned}$$

and consequently, $\tilde{f}_{11} = \tilde{f}_{21} = \tilde{f}_{22} = 0$. Hence $\tilde{f} = \tilde{f}_{12} \in V^{*+}$. □

4.3 M-ideals in order smooth ∞-normed spaces

We begin with a characterization of M-ideals in a complete approximate order unit space due to Alfsen and Effros (see e.g. [1]). First, we recall the following notion.

Definition 4.3.1 *Let V be a normed space. Let K be a non-empty, closed and convex set in V. A proper face F of K is said to be a split face of K if F_K^C is a proper face of K such that $K = F \oplus_c F_K^C$. Here*

$$F_K^C = \cup\{face_K(v) : v \in K \text{ and } face_K(v) \cap F = \emptyset\}$$

and by $K = F \oplus_c F_K^C$, we mean that for each $v \in K$ there exist unique $u \in F, w \in F_K^C$ and $\lambda \in [0,1]$ such that

$$v = \lambda u + (1 - \lambda) w.$$

Theorem 4.3.2 *[1, Corollary 5.9, Part II] Let $(V, V^+, \{e_\lambda\})$ be a complete approximate order unit space and let W be a closed subspace of V. Then W is an*

M-ideal in V if and only if $W^{\perp} \cap S(V)$ is a closed split face of the state space $S(V)$.

In this section, we prove an analogue of this result for complete order smooth ∞-normed spaces. We noted in Theorem 4.1.2 that in general, in an order smooth ∞-normed space V, the state space $S(V)$ may not be convex. To overcome this situation, we present an alternative form of Theorem 4.3.2. For brevity, we adopt the following convention: Let V be an order smooth 1-normed space and let C and D be subsets of V^+. We write

$$V^+ = C \oplus_1 D,$$

if for $v \in C$ and $w \in D$, we have

$$\|v + w\| = \|v\| + \|w\|$$

and if every element u of V^+ can be written uniquely as

$$u = v + w$$

with $v \in C$ and $w \in D$.

Proposition 4.3.3 *Let V be an approximate order unit space and let W be a closed subspace of V. Then $W^{\perp} \cap S(V)$ is a split face of $S(V)$ if and only if the following conditions hold:*

1 *$W^{\perp\prime +}$ is convex;*

2 *$V^{*+} = W^{\perp +} \oplus_1 W^{\perp\prime +}$.*

Proof. Let us observe that

$$(W^{\perp +} \cap S(V))^C_{S(V)} = W^{\perp' +} \cap S(V). \qquad (4.3.1)$$

To see this, we let $f \in W^{\perp' +} \cap S(V)$. Then $C(f) \cap W^{\perp} = \{0\}$ with $\|f\| = 1$. Then by the definition of $C(f)$ (also see equation (2.1.1)), we have $face_{V_1}(\frac{f}{\|f\|}) \cap W^{\perp} = \emptyset$. Then by Corollary 4.2.3, we may deduce that $face_{S(V)}(f) \cap W^{\perp +} = \emptyset$. Thus $f \in (W^{\perp +} \cap S(V))^C_{S(V)}$. Now tracing back the proof, we may conclude that (4.3.1) holds.

Now first, we assume that $W^{\perp} \cap S(V)$ is a split face of $S(V)$. We show that conditions (1) and (2) hold. Now, we prove (1). For this let $f, g \in W^{\perp' +}$ and $\alpha \in (0,1)$. Then $\|f\|^{-1} f, \|g\|^{-1} g \in W^{\perp' +} \cap S(V)$. Thus by the convexity of $W^{\perp' +} \cap S(V) = (W^{\perp +} \cap S(V))^C_{S(V)}$, we get

$$(\alpha\|f\| + (1-\alpha)\|g\|)^{-1}\{\alpha\|f\|(\|f\|^{-1} f) + (1-\alpha)\|g\|(\|g\|^{-1} g)\} \in W^{\perp' +} \cap S(V).$$

Therefore, $\alpha\|f\|(\|f\|^{-1} f) + (1-\alpha)\|g\|(\|g\|^{-1} g) \in W^{\perp' +}$ so that (1) holds.

To prove (2), let $f \in V^{*+} \setminus \{0\}$. Then $\|f\|^{-1} f \in S(V)$. Since $W^{\perp +} \cap S(V)$ is a split face of $S(V)$, we have

$$S(V) = W^{\perp +} \cap S(V) \oplus_c (W^{\perp +} \cap S(V))^C_{S(V)} = W^{\perp +} \cap S(V) \oplus_c (W^{\perp' +} \cap S(V).$$

Thus there exist a unique $g_0 \in W^{\perp +} \cap S(V)$ and $h_0 \in W^{\perp' +} \cap S(V)$ and $\lambda \in [0,1]$ such that $\|f\|^{-1} f = \lambda g_0 + (1-\lambda) h_0$. Then $f = g + h$ where $g = \lambda \|f\| g_0 \in W^{\perp +}$ and $h = \lambda \|f\| h_0 \in W^{\perp' +}$.

Next, assume that conditions (1) and (2) hold. We show that $F = W^{\perp +} \cap S(V)$ is a split face of $S(V)$. Put $G = W^{\perp' +} \cap S(V)$. Since $W^{\perp +}, W^{\perp' +}$ are faces of V^{*+}, by Proposition 4.3.6, we get that F and G are faces of $S(V)$. Also, since $W^{\perp} \cap W^{\perp'} = \{0\}$, we may conclude that $F \cap G = \emptyset$. We prove that $S(V) = F \oplus_c G$. It suffices to show that $S(V) \subseteq F \oplus_c G$. Let $f \in S(V) \setminus F \cup G \subseteq V^{*+}$. By (2), there exist unique $g_0 \in W^{\perp +}$ and $h_0 \in W^{\perp' +}$ such that $f = g_0 + h_0$. Thus $g = \frac{g_0}{\|g_0\|} \in F, h = \frac{h_0}{\|h_0\|} \in G$. Also $\|g_0\| + \|h_0\| = 1$ so that $\|g_0\| g + \|h_0\| h \in F \oplus_c G$. Therefore, $S(V) \subseteq F \oplus_c G = F \oplus_c F^C_{S(V)}$ by (4.3.1). □

Remark 4.3.4 *Let V be a complete approximate order unit space and let W be a closed subspace of V. Then W is an M-ideal in V if and only if W satisfies the following conditions:*

(i) $W^{\perp' +}$ is convex.

(ii) $V^{*+} = W^{\perp +} \oplus_1 W^{\perp' +}$.

Now, we prove the main result of this section.

Theorem 4.3.5 *Let V be a complete order smooth ∞-normed space and W be a closed subspace of V. Then W is an M-ideal in V if and only if W satisfies the following conditions.*

(i) $W^{\perp' +}$ is convex.

(ii) $V^{*+} = W^{\perp +} \oplus_1 W^{\perp' +}$.

We use the following results to prove Theorem 4.3.5. We begin with the following observation.

Proposition 4.3.6 *Let V be an order smooth ∞-normed space and let W be a closed subspace of V such that following conditions hold:*

(i) $W^{\perp' +}$ is convex;

(ii) $V^{*+} = W^{\perp +} \oplus_1 W^{\perp' +}$.

Then $W^{\perp +}$ *and* $W^{\perp' +}$ *are faces of* V^{*+}.

Proof. Let $f_1, f_2 \in V^{*+}$ with $f = \alpha f_1 + (1-\alpha) f_2 \in W^{\perp +}$ for some $\alpha \in (0,1)$. By assumption (ii), we have $f_1 = g_1 + h_1$ and $f_2 = g_2 + h_2$ for some unique $g_1, g_2 \in W^{\perp +}$ and $h_1, h_2 \in W^{\perp' +}$. Put $g = \alpha g_1 + (1-\alpha) g_2$ and $h = \alpha h_1 + (1-\alpha) h_2$. Since $W^{\perp +}$ and $W^{\perp' +}$ are convex, we have $g \in W^{\perp +}$ and $h \in W^{\perp' +}$. Then $f = g + h$ is a decomposition of f in $W^{\perp +} \oplus_1 W^{\perp' +}$. As $f \in W^{\perp +}$, by the uniqueness of decomposition, we may conclude that $h = 0$. Thus $h_1 = 0 = h_2$ so that $W^{\perp +}$ is a face of V^{*+}. Now, by symmetry, $W^{\perp' +}$ is also a face of V^{*+}. □

For the next result, we use the following notion defined in [1]. Let V be a normed space. For $u, v \in V$ we define $u \prec v$, if

$$\|v\| = \|u\| + \|v - u\|.$$

Form Lemma 2.1.1, we can write $u \prec v$ if $u \in C(v)$. A subspace W of V is said to be *hereditary* if $v \in W$ with $u \prec v$ implies $u \in W$. This relation is transitive.

Proposition 4.3.7 *(i) Let* (V, B) *be a complete base normed space and let* W *be a closed subspace of* V*. If* W *is hereditary, then* $W \cap B$ *is a face of* B*, or equivalently,* $W \cap V_1^+$ *is a face of* V_1^+*.*

(ii) Let V *be a complete order smooth* 1*-normed space and let* W *be a closed subspace of* V*. If* $W \cap V_1^+$ *is a face in* V_1^+*. Then* W *is hereditary.*

(iii) Let W *be a hereditary subspace of a complete order smooth* 1*-normed space* V*. If* V *satisfies* $(OS.1.2)$*, then so does* W*.*

Proof. (i): Let $u_1, u_2 \in B$ such that $u = \alpha u_1 + (1-\alpha) u_2 \in W \cap B$ for some $\alpha \in (0,1)$. Then $\alpha u_1, (1-\alpha) u_2 \leq u$. Since V is a base normed space, we get

$\|u\| = \|\alpha u_1\| + \|(1-\alpha)u_2\|$. Thus $\alpha u_1 \prec u, (1-\alpha)u_2 \prec u$. Since W is hereditary, we have $\alpha u_1, (1-\alpha)u_2 \in W$. Thus $u_1, u_2 \in W$ so that $u_1, u_2 \in W \cap B$. Hence $W \cap B$ is a face of B.

(ii): Let $v \in W$ and $u \in C(v)$. We prove that $u \in W$. By $(OS.1.2)$ property of V, we have $u = u_1 - u_2$ with $\|u\| = \|u_1\| + \|u_2\|$ for some $u_1, u_2 \in V^+$. Thus $u_1 \prec u \prec v$. Since W^+ is a cone in V, by Theorem 2.1.2 , we have $v = v_1 + v_2$, with $\|v\| = \|v_1\| + \|v_2\|$ and $u_1 \prec v_1$ where $v_1 \in W^+$ and $v_2 \in W^{+'}$. Since $u_1 \prec v_1$, we also have $v_1 - u_1 \prec v_1$. Then $v_1 - u_1 \in C(v_1)$. By Corollary 4.2.3, we have $C(v_1) \subseteq V^+$ so that $v_1 - u_1 \in V^+$. Since $\|v_1\| = \|u_1\| + \|v_1 - u_1\|$, the right hand side of the expression

$$\frac{v_1}{\|v_1\|} = \frac{\|u_1\|}{\|v_1\|}\left(\frac{u_1}{\|u_1\|}\right) + \frac{\|v_1 - u_1\|}{\|v_1\|}\left(\frac{v_1 - u_1}{\|v_1 - u_1\|}\right)$$

is a convex combination of $\frac{u_1}{\|u_1\|}, \frac{v_1 - v_1}{\|v_1 - u_1\|}$ in V_1^+. Since $W \cap V_1^+$ is a face of V_1^+ and since $\frac{u_1}{\|u_1\|} \in W \cap V_1^+$, we have $u_1 \in W^+$. By a similar argument, we can show that $u_2 \in W^+$. Hence $u \in W$.

(iii): Let $w \in W$, then by $(OS.1.2)$ property of V, there are $u, v \in V^+$ such that $w = u - v$ and $\|w\| = \|u\| + \|v\|$. Therefore $u, -v \prec w$, so by definition of hereditary subspaces, $u, -v \in W$. Thus $u, v \in W^+$ so that W also satisfies $(OS.1.2)$. □

Proof of Theorem 4.3.5.

Let W be an M-ideal in V. Then $W^\perp$ is an L-summand of V^* so that $W^{\perp'}$ is also an L-summand of V^* with $V^* = W^\perp \oplus_1 W^{\perp'}$. Thus $W^{\perp'+} = W^{\perp'} \cap V^{*+}$ is convex. Also, by the order cone decomposition Theorem 4.2.1, condition (ii) holds.

Conversely, assume that conditions (i) and (ii) hold. Let $f \in V^{*+}$. Then by condition (ii), there exist unique $g \in W^{\perp +}$ and $h \in W^{\perp' +}$ such that $f = g + h$ with $\|f\| = \|g\| + \|h\|$. Let us write $L_0(f) = g$. Then by the uniqueness of decomposition $L_0 : V^+ \to V^+$ is well defined and $L_0(\alpha f) = \alpha L_0(f)$ for all $\alpha \geq 0$. Now, let $f_1, f_2 \in V^+$. Again applying condition (ii), we can find unique $g_1, g_2 \in W^{\perp +}$ and $h_1, h_2 \in W^{\perp' +}$ such that $f_i = g_i + h_i$ and $\|f_i\| = \|g_i\| + \|h_i\|$ for $i = 1, 2$. Then $f_1 + f_2 = (g_1 + g_2) + (h_1 + h_2)$, where $g_1 + g_2 \in W^{\perp +}$ and $h_1 + h_2 \in W^{\perp' +}$ by the condition (i). Thus by the condition (ii), we have $\|f_1 + f_2\| = \|(g_1 + g_2)\| + \|(h_1 + h_2)\|$ so that $L_0(f_1 + f_2) = L_0(f_1) + L_0(f_2)$. Now, let $f \in V^*$. By the condition $(OS.1.2)$ in V^*, there are $f_1, f_2 \in V^{*+}$ such that $f = f_1 - f_2$ with $\|f\| = \|f_1\| + \|f_2\|$. Let us write $L(f) = L_0(f_1) - L_0(f_2)$. As L_0 is additive on V^{*+}, it is routine to check that $L : V^* \to V^*$ is a well defined, positive linear mapping with $L(V^*) \subset W^\perp$. We prove that L is an *L-projection* onto $W^\perp$. Let $f \in V^*$, then by $(OS.1.2)$ in V^*, there are $g, h \in V^{*+}$ such that $f = g - h$ with $\|f\| = \|g\| + \||h\|$. Now

$$\begin{aligned}
\|f\| &\leq \|L(f)\| + \|f - L(f)\| \\
&= \|L(g) - L(h)\| + \|g - h - L(g) + L(h)\| \\
&\leq \|L(g)\| + \|L(h)\| + \|g - L(g)\| + \|h - L(h)\| \\
&= (\|L_0(g)\| + \|g - L_0(g)\|) + (\|L_0(h)\| + \|h - L_0(h)\|) \\
&= \|g\| + \|h\| = \|f\|
\end{aligned}$$

so that $\|L(f)\| + \|f - L(f)\| = \|f\|$ for all $f \in V^*$. Next, we show that $L(f) = f$ for all $f \in W^\perp$. To see this, let $f \in W^\perp$. Since by Proposition 4.3.7, $W^\perp$ satisfies $(OS.1.2)$, there are $f_1, f_2 \in W^{\perp +}$ such that $f = f_1 - f_2$ with $\|f\| = \|f_1\| + \|f_2\|$. Now, by Theorem 4.2.1, $f_i = g_i + h_i$, where $g_i \in W^{\perp +}$ and $h_i \in W^{\perp' +}$ for $i = 1, 2$. As $f_i, g_i \in W^{\perp +}$, we have $h_i \in W^\perp \cap W^{\perp' +} = \{0\}$. Thus by the construction,

$g_1 = L_0(f_1)$ and $g_2 = L_0(f_2)$ so that $L(f) = L_0(f_1) - L_0(f_2) = g_1 - g_2 = f$ if $f \in W^\perp$. Thus for any $f \in V^*$, we have $L^2(f) = L(L(f)) = L(f)$. Hence L is an L-projection of V^* onto $W^\perp$ and therefore W is an M-ideal in V. □

If we attempt to get a prototype of Theorem 4.3.2 for order smooth ∞-normed spaces, we need to replace the "state spaces" by the "quasi-state spaces" as the quasi-state space of an order smooth ∞-normed space is always convex. Accordingly, we need to 'adjust' the definition of split faces as well.

First, let us note that if V is an approximate order unit space, then there is a bijective correspondence between the class of faces of $S(V)$ and the class of non-zero faces of $Q(V)$ containing zero. These correspondences are given by

$$F \subseteq S(V) \mapsto co(F \cup \{0\}) \subseteq Q(V)$$

and

$$G \subseteq Q(V) \mapsto G \cap S(V) \subseteq S(V).$$

Lemma 4.3.8 *Let V be an approximate order unit space and let F be a face of $S(V)$. Then we have*

$$S(V) \cap (cone(co(F \cup \{0\})))' = F^C_{S(V)}.$$

Proof. Let $f \in S(V)$. Then

$$\begin{aligned} f \in (cone(co(F \cup \{0\})))' &\Leftrightarrow C(f) \cap co(F \cup \{0\}) = \{0\} \qquad (4.3.2) \\ &\Leftrightarrow co(face_{S(V)}(f) \cup \{0\}) \cap co(F \cup \{0\}) = \{0\}. \end{aligned}$$

Now as, $face_{S(V)}(f) \subseteq S(V)$ and $F \subseteq S(V)$, we get that $face_{S(V)}(f) \cap F = \emptyset$. Thus $f \in F^C_{S(V)}$. Therefore we have $S(V) \cap (cone(co(F \cup \{0\})))' \subseteq F^C_{S(V)}$.

Conversely, let $f \in F^C_{S(V)}$. Then $f \in S(V)$ and $face_{S(V)}(f) \cap F = \emptyset$. By equation (4.3.2), it suffices to show that $co(face_{S(V)}(f) \cup \{0\}) \cap co(F \cup \{0\}) = \{0\}$. Let $g \in co(face_{S(V)}(f) \cup \{0\}) \cap co(F \cup \{0\})$. Then there exist $g_1 \in face_{S(V)}(f)$, $g_2 \in F$ and $\lambda, \mu \in [0,1]$ such that $g = \lambda g_1 = \mu g_2$. As $g_1, g_2 \in S(V)$, we get $\lambda = \mu$. Now, if $\lambda \neq 0$, then $g_1 = g_2 \in face_{S(V)}(f) \cap F = \emptyset$ so that $\lambda = 0 = \mu$ and consequently, $g = 0$. Hence $S(V) \cap (cone(co(F \cup \{0\})))' = F^C_{S(V)}$. □

Definition 4.3.9 *Let V be an order smooth ∞-normed space. Let G and H be any two faces of $Q(V)$ containing zero such that $G \cap H = \{0\}$. We define*

$$G \oplus_{c,1} H = \{\lambda g + (1-\lambda)h : g \in G, h \in H, \|g\| = \|h\|, \lambda \in [0,1]\}.$$

For a face G of $Q(V)$ containing zero, we say that G is a split face *of $Q(V)$, if $G'_{Q(V)} := (cone(G))' \cap Q(V)$ is also a face of $Q(V)$ (containing zero) and if every element in $Q(V)$ has a unique representation in $G \oplus_{c,1} G'_{Q(V)}$.*

Remark 4.3.10 *By the definition of a split face, we get that $\|f\| \leq \|g\| = \|h\|$. But we can show that these norms are equal. To see this, let $f \in Q(V) \setminus \{0\}$, Then $f_1 = \|f\|^{-1} f \in Q(V)$. Thus there exist unique $g_1 \in W^\perp \cap Q(V), h_1 \in (W^\perp \cap Q(V))'_{Q(V)}$ with $\|g_1\| = \|h_1\|$ such that $f_1 = \lambda g_1 + (1-\lambda) h_1$ for some $\lambda \in [0,1]$. Now*

$$1 = \|f_1\| \leq \lambda \|g_1\| + (1-\lambda)\|h_1\| \leq 1.$$

Thus we have $f = \lambda g + (1-\lambda)h$, where $g = \|f\| g_1 \in W^\perp \cap Q(V)$ and $h = \|f\| h_1 \in (W^\perp \cap Q(V))'_{Q(V)}$ with $\|g\| = \|f\| = \|h\|$.

We show that this notion is an extension of a split face of $S(V)$ as follows:

Theorem 4.3.11 *Let V be an approximate order unit space and let F be a face of V. Then F is a split face of $S(V)$ if and only if $co(F \cup \{0\})$ is a split face of $Q(V)$ that is*

$$Q(V) = co(F \cup \{0\}) \oplus_{c,1} co(F \cup \{0\})'_{Q(V)}.$$

Proof. Let F be split face of $S(V)$. Since $co(F \cup \{0\})'_{Q(V)} = co(F^C_{S(V)} \cup \{0\})$ and since $F^C_{S(V)}$ is a face of $S(V)$, we conclude that $co(F \cup \{0\})'_{Q(V)}$ is a face of $Q(V)$. Let $f \in Q(V) \setminus \{0\}$. Then $\|f\|^{-1} f \in S(V)$. Since F is a split face of $S(V)$. There exist an unique element $g_0 \in F, h_0 \in F^C_{S(V)}$ such that $\|f\|^{-1} f = \lambda g_0 + (1-\lambda) h_0$ for some $\lambda \in [0,1]$. Put $g = \|f\| g_0, h = \|f\| h_0$. Then $f = \lambda g + (1-\lambda) h$ where $\|g\| = \|h\| = \|f\|$ and $g \in co(F \cup \{0\})$ and $h \in co(F^C_{S(V)} \cup \{0\}) = co(F \cup \{0\})'_{Q(V)}$. To prove uniqueness, let $g_1 \in co(F^C_{S(V)} \cup \{0\})$ and $h_1 \in co(F \cup \{0\})'_{Q(V)}$ such that $\|g_1\| = \|h_1\| = \|f\|$ and $f = \mu g_1 + (1-\mu) h_1$ for some $\mu \in [0,1]$. Then we have $\|f\|^{-1} f = \mu \|f\|^{-1} g_1 + (1-\mu) \|f\|^{-1} \|h_1\|$. Now by uniqueness of split face we have $\|f\|^{-1} g_1 = g_0 = \|f\|^{-1} g$. Thus we have $g_1 = g$ and similarly, we have $h_1 = h$. Hence $co(F \cup \{0\})$ is a split face of $Q(V)$.

Conversely, let $co(F \cup \{0\})$ is a split face of $Q(V)$. We have to show that F is a split face of $S(V)$. Since $co(F \cup \{0\})$ is a split face of $Q(V)$, $co(F \cup \{0\})'_{Q(V)}$ is also a face of $Q(V)$. But $co(F \cup \{0\})'_{Q(V)} = co(F^C_{S(V)} \cup \{0\})$. Thus $F^C_{S(V)}$ is a face of $S(V)$ as well. Let $f \in S(V) \subseteq Q(V) = co(F \cup \{0\}) \oplus_{c,1} co(F \cup \{0\})'_{Q(V)}$. Then there exist unique pair $g \in co(F \cup \{0\}), h \in co(F^C_{S(V)} \cup \{0\})$ with $\|g\| = \|h\| = \|f\| = 1$ (so that $g, h \in S(V)$) such that $f = \lambda g + (1-\lambda) h$ for some $\lambda \in [0,1]$. Thus $g \in co(F \cup \{0\}) \cap S(V) = F$ and $h \in co(F^C_{S(V)} \cup \{0\}) \cap S(V) = F^C_{S(V)}$. Thus $f \in F \oplus_c F^C_{S(V)}$. □

Lemma 4.3.12 *Let V be a complete order smooth ∞-normed space and let W be a closed subspace of V. Then $W^{\perp +'} \cap Q(V) = W^{\perp'} \cap Q(V)$.*

Proof. Since $cone(W^\perp \cap Q(V)) = W^{\perp+}$, we have

$$(W^\perp \cap Q(V))'_{Q(V)} = (cone(W^\perp \cap Q(V)))' \cap Q(V) = W^{\perp+'} \cap Q(V).$$

We show that $W^{\perp+'} \cap Q(V) = W^{\perp'} \cap Q(V)$. If $f \in Q(V)$, then $C(f) \subseteq V^{*+}$, by Corollary 4.2.3. Thus $C(f) \cap W^{\perp+} = C(f) \cap W^\perp$. Since $W^{\perp'} = \{f \in V^* : C(f) \cap W = \{0\}\}$, it follows that $W^{\perp+'} \cap Q(V) = W^{\perp'} \cap Q(V)$. □

Proposition 4.3.13 *Let V be a complete order smooth ∞-normed space and let W be a closed subspace of V. Then W is an M-ideal in V if and only if $W^\perp \cap Q(V)$ is a split face of $Q(V)$.*

Proof. First, let us assume that W is an M-ideal in V. Then by Theorem 4.3.5, $W^{\perp'+}$ is convex and $V^{*+} = W^{\perp+} \oplus_1 W^{\perp'+}$. Also, by Lemma 4.3.12, we have $(W^\perp \cap Q(V))'_{Q(V)} = W^{\perp'} \cap Q(V)$. We show that $W^{\perp'} \cap Q(V)$ is a face of $Q(V)$. Let $f_1, f_2 \in Q(V)$ be such that $f = \alpha f_1 + (1-\alpha) f_2 \in W^{\perp'} \cap Q(V)$ for some $\alpha \in (0,1)$. As $V^{*+} = W^{\perp+} \oplus_1 W^{\perp'+}$, there are unique $g_1, g_2 \in W^{\perp+}, h_1, h_2 \in W^{\perp'+}$ such that $f_i = g_i + h_i$, with $\|f_i\| = \|g_i\| + \|h_i\|$, for $i = 1, 2$. Then $f = (\alpha g_1 + (1-\alpha) g_2) + (\alpha h_1 + (1-\alpha) h_2) = g + h$, where $g = \alpha g_1 + (1-\alpha) g_2$ and $h = \alpha h_1 + (1-\alpha) h_2$. As $W^{\perp+}$ and $W^{\perp'+}$ are convex, we get that $g \in W^{\perp+}, h \in W^{\perp'+}$. Next, as W is an M-ideal in V, $W^{\perp'}$ is an L-summand of V^* so that $W^{\perp'+} - W^{\perp'+} \subseteq W^{\perp'}$. Thus $g = f - h \in W^{\perp'+} - W^{\perp'+} \subseteq W^{\perp'}$ so that $g \in W^{\perp'} \cap W^\perp = \{0\}$. Therefore, $g_1 = g_2 = 0$ and consequently, $f_1 = h_1, f_2 = h_2 \in W^{\perp'+} \cap Q(V)$ so that $(W \cap Q(V))'_{Q(V)}$ is a face of $Q(V)$. Similarly, we can prove that $W^\perp \cap Q(V)$ is also a face of $Q(V)$.

Now, we show that $W^\perp \cap Q(V)$ is a split face of $Q(V)$. For this, let $f \in Q(V)$. Since $Q(V) \subseteq V^{*+} = W^{\perp+} \oplus_1 W^{\perp'+}$, there are unique $g_0 \in W^{\perp+}, h_0 \in W^{\perp'+}$ such that $f = g_0 + h_0$ and $\|f\| = \|g_0\| + \|h_0\|$. Put $g = \|f\| \|g_0\|^{-1} g_0, h =$

$\|f\|\|h_0\|^{-1}h_0$. Then $g \in W^\perp \cap Q(V)$ and $h \in W^{\perp'+} \cap Q(V)) = (W^\perp \cap Q(V))'_{Q(V)}$ and consequently,

$$f = (\|g_0\|\|f\|^{-1})g + (\|h_0\|\|f\|^{-1})h \in W^\perp \cap Q(V) \oplus_{c,1} (W^\perp \cap Q(V))'_{Q(V)}.$$

Hence $W^\perp \cap Q(V)$ is a split face of $Q(V)$.

Conversely, assume that $W^\perp \cap Q(V)$ is a split face of $Q(V)$. We show that $W^{\perp'+}$ is convex and that $V^{*+} = W^{\perp+} \oplus_1 W^{\perp'+}$. Let $f, g \in W^{\perp'+}$ and $\alpha \in (0,1)$. Put $h = \alpha f + (1-\alpha)g$. If we put $\lambda = \max\{\|f\|, \|g\|\}$, $h_0 = \lambda^{-1}h$, $f_0 = \lambda^{-1}f$ and $g_0 = \lambda^{-1}g$, then $f_0, g_0 \in W^{\perp'} \cap Q(V)$ with $h_0 = \alpha f_0 + (1-\alpha)g_0$. Since $W^\perp \cap Q(V)$ is a split face of $Q(V)$, $W^{\perp'} \cap Q(V) = (W^\perp \cap Q(V))'_{Q(V)}$ is convex. Thus $h_0 \in W^{\perp'} \cap Q(V)$ and consequently, $h \in W^{\perp'+}$. Therefore, $W^{\perp'+}$ is convex.

Finally, let $f \in V^{*+} \setminus \{0\}$. Then $f_1 = \|f\|^{-1}f \in Q(V) = W^\perp \cap Q(V) \oplus_{c,1} W^{\perp'} \cap Q(V)$. Thus there exist unique $g_1 \in W^\perp \cap Q(V)$, $h_1 \in W^{\perp'} \cap Q(V)$ and $\alpha \in [0,1]$ such that $f_1 = \alpha g_1 + (1-\alpha)h_1$ with $\|g_1\| = \|h_1\| = \|f_1\| = 1$. Then $f = g + h$, where $g = \alpha\|f\|g_1 \in W^{\perp+}$ and $(1-\alpha)\|f\|h_1 \in W^{\perp'+}$. Also

$$\|f\| = \|g + h\| \leq \|g\| + \|h\| \leq \alpha\|f\| + (1-\alpha)\|f\| = \|f\|$$

so that $\|f\| = \|g\| + \|h\|$. This completes the proof. □

4.4 M-ideals and adjoining of an order unit

Let V be an order smooth ∞-normed space. Consider $\tilde{V} = V \oplus \mathbb{R}$. If we define

$$\tilde{V}^+ = \{(v, \alpha) : l_V(v) \leq \alpha\}$$

where

$$l_V(v) = \inf\{\|u\| : u, u + v \in V^+\},$$

then $(\tilde{V}, \tilde{V}^+)$ becomes a real ordered vector space. In this case, $(0, 1)$ acts as an order unit and $\tilde{V}^+$ is Archimedean so that $(\tilde{V}, (0, 1))$ is an order unit space. Moreover, $v \mapsto (v, 0)$ is an isometric order isomorphic embedding of V in $(\tilde{V}, (0, 1))$. Further, $(\tilde{V}, (0, 1))$ is determined uniquely by V upto a unital order isomorphism in such a way that V is a normed closed order ideal of $(\tilde{V}, (0, 1))$ with co-dimension 1. For a detailed information, one can see [41, Section 4]. In this section, we obtain the conditions under which V is an M-ideal in $\tilde{V}$. The following result (due to Alfsen and Effros) is used for this purpose. Throughout this section, we assume that all order normed spaces are (norm) complete.

Theorem 4.4.1 *[1, Theorem 6.10] Let (V, e) be an order unit space and let W be a closed subspace of V. Then following sets of statements are equivalent:*

1. *W is an M-ideal.*
2. *W satisfies each of the following conditions:*
   (a) *W is positively generated;*
   (b) *W is an order ideal;*
   (c) *$(V/W, \varphi(e))$ is an order unit space;*
   (d) *Given $v, w \in V^+$ and $\epsilon > 0$, one has*

$$\varphi([0, v]) \cap \varphi([0, w]) \subset \varphi([0, v + \epsilon e] \cap [0, w + \epsilon e])$$

where $[u, v] := \{w \in V : u \leq w \leq v\}$ for $u \leq v$ in V.

Here $\varphi : V \to V/W$ is the canonical quotient mapping.

We apply this result to characterize approximate order unit spaces as those order smooth ∞-normed spaces which are M-ideals in order unit spaces obtained by adjoining order units to these spaces. First, we prove the sufficient condition.

Theorem 4.4.2 *Let $(V, V^+, \{e_\lambda\}_{\lambda \in D})$ be an approximate order unit space and let $(\tilde{V}, \tilde{V}^+)$ be the order unit space obtained by adjoining an order unit to V. Then V is an M-ideal in $\tilde{V}$.*

We prove this result in several steps.

Proposition 4.4.3 *Let V be an order smooth ∞-normed space and let $\tilde{V}$ be the order unit space obtained by adjoining an order unit to V. Then*

(i) V is positively generated.

(ii) V is an order ideal in $\tilde{V}$.

(iii) $(\tilde{V}/V, \tilde{\varphi}((0,1)))$ is an order unit space.

Here $\tilde{\varphi} : \tilde{V} \to \tilde{V}/V$ is the natural quotient mapping.

Proof. Condition (i) follows from the definition of V and condition (ii) follows from the construction of $\tilde{V}$ (see e.g. [41, Theorem 4.1]). To prove (iii), first note that the natural quotient map $\tilde{\varphi} : \tilde{V} \to \tilde{V}/V$ is positive and that $\tilde{\varphi}(0,1)$ is an order unit for $\tilde{V}/V$. We show that $(\tilde{V}/V)^+$ is Archimedean. Let $\tilde{\varphi}(u, \alpha) \in \tilde{V}/V$ such that $\tilde{\varphi}(u, \alpha) \leq \frac{1}{n}\tilde{\varphi}(0,1)$ for all $n \in \mathbb{N}$. Then $\tilde{\varphi}(0, \frac{1}{n} - \alpha) = \tilde{\varphi}(-u, \frac{1}{n} - \alpha) \geq 0$ so that $\frac{1}{n} - \alpha \geq 0$ for all $n \in \mathbb{N}$. Consequently, $\tilde{\varphi}(u, \alpha) = \tilde{\varphi}(0, \alpha) \leq 0$. □

Lemma 4.4.4 *Let V be an order smooth ∞-normed space and let $\tilde{V}$ be the order unit space obtained by an order unit to V. If $\tilde{\varphi} : \tilde{V} \to \tilde{V}/V$ is the natural quotient map, then for all $(u, \lambda) \in \tilde{V}^+$, we have*

$$\tilde{\varphi}[(0,0),(u,\lambda)] = \{\tilde{\varphi}(0,\mu) : 0 \leq \mu \leq \lambda\}.$$

Proof. Let us consider the order interval $[(0,0),(u,\lambda)]$, where $(u,\lambda)\in\tilde{V}^+$. Now for any $\mu\in[0,\lambda]$, we have $0\le(\frac{\mu}{\lambda}u,\mu)\le(u,\lambda)$. Thus

$$\tilde{\varphi}(0,\mu)=\tilde{\varphi}(\frac{\mu}{\lambda}u,\mu)\in\tilde{\varphi}[(0,0),(u,\lambda)].$$

Conversely, let (x,μ) in $\tilde{V}$ such that $0\le(x,\mu)\le(u,\lambda)$ in $(\tilde{V},\tilde{V}^+)$. Then we have $0\le\mu\le\lambda$. Now the observation $\tilde{\varphi}(x,\mu)=\tilde{\varphi}(0,\mu)$ completes the proof. □

Proof of Theorem 4.4.2.

Let $(u_i,\gamma_i)\in\tilde{V}^+$ for $i=1,2$ and let $\epsilon>0$. We show that

$$\begin{aligned}&\tilde{\varphi}([(0,0),(u_1,\gamma_1)])\cap\tilde{\varphi}([(0,0),(u_2,\gamma_2)])\\&\qquad\subset\tilde{\varphi}([(0,0),(u_1,\gamma_1+\epsilon)]\cap[(0,0),(u_2,\gamma_2+\epsilon)]).\end{aligned}$$

Since $(u_i,\gamma_i)\in\tilde{V}^+$, we have $l_V(u_i)\le\gamma_i$ for $i=1,2$. Thus as $\epsilon>0$ and as $\{e_\lambda\}$ is an approximate order unit for V, there exist λ such that $u_i+(\gamma_i+\epsilon)e_{\lambda_i}\in V^+$ for $i=1,2$. Put $\gamma=\min\{\gamma_1,\gamma_2\}$. Then for $i=1,2$ we have

$$u_i+\gamma e_\lambda+(\gamma_i-\gamma+\epsilon)e_\lambda=u_i+(\gamma_i+\epsilon)e_\lambda\in V^+$$

so that $l_V(u_i+\gamma e_\lambda)\le\gamma_i-\gamma+\epsilon$, or equivalently, $(u_i+\gamma e_\lambda,\gamma_i-\gamma+\epsilon)\in\tilde{V}^+$. Thus $(-\gamma e_\lambda,\gamma)\le(u_i,\gamma_i+\epsilon)$ for $i=1,2$. As $\|e_\lambda\|\le1$, we have $0\le(e_\lambda,0)\le(0,1)$ so that $(-\gamma e_\lambda,\gamma)\in\tilde{V}^+$. Hence $(-\gamma e_\lambda,\gamma)\in[(0,0),(u_1,\gamma_1+\epsilon)]\cap[(0,0),(u_2,\gamma_2+\epsilon)]$. Now the result follows from Proposition 4.4.3. □

Remark 4.4.5 *Let X be a Banach space and let Y be a closed subspace of X such that $Y\ne0$. It follows from [1, Proposition 2.2] that the M-ideals of Y are precisely the M-ideals of X contained in Y. Thus each M-ideal in an approximate* *order unit space* $(V, \{e_\lambda\})$ *is an* M*-ideal in* $(\tilde{V}, \tilde{V}^+)$.

Now, we proceed to prove the converse of Theorem 4.4.2. More precisely, we aim to prove that a complete order smooth ∞-normed space V is an M-ideal in $\tilde{V}$, only if V is an approximate order unit space.

Lemma 4.4.6 *Let* V *be an order smooth* ∞*-normed space and consider* $\tilde{V}$*, the order unit space obtained by adjoining an order unit to* V*. Then* $S(\tilde{V})$ *is affinely homeomorphic to* $Q(V)$.

Proof. Let $g \in Q(V)$. Define $\tilde{g}(v, \alpha) = g(v) + \alpha$ for $(v, \alpha) \in \tilde{V}$. If $(v, \alpha) \in \tilde{V}^+$, then for $\epsilon > 0$, there exist $u \in V^+$ such that $u + v \geq 0$ and $\|u\| < \alpha + \epsilon$. Thus

$$\tilde{g}(v, \alpha) = g(v) + \alpha \geq -g(u) + \alpha \geq -\|u\| + \alpha > -\epsilon.$$

Since ϵ is independent of g, we see that $\tilde{g}$ is a positive linear map on $\tilde{V}$ with $\tilde{g}(0, 1) = 1$. So $\tilde{g}$ in $S(\tilde{V})$. Further, if $h \in S(\tilde{V})$ is any extension of g, then $h(0, 1) = 1 = \tilde{g}(0, 1)$ so that $\tilde{g} = h$. Thus each $g \in Q(V)$ has a unique extension $\tilde{g} \in S(\tilde{V})$ and consequently, we obtain a well defined and bijective map $\phi : Q(V) \to S(\tilde{V})$ by $\phi(f) = \tilde{f}$, where $\tilde{f}(v, \alpha) = f(v) + \alpha$. Now, it is routine to check that ϕ is affine as well as w^*-w^* homeomorphism. □

Theorem 4.4.7 *Let* V *be a complete order smooth* ∞*-normed space. Then* V *is an* M*-ideal in* $\tilde{V}$ *if and only if* V *is an approximate order unit space.*

Proof. If V is an approximate order unit space, then by Theorem 4.4.2, V is an M-ideal in $\tilde{V}$. Conversely, assume that V be an order smooth ∞-normed space such that V is an M-ideal in $\tilde{V}$. Note that

$$V^\perp = \{f \in (\tilde{V})^* : f(v, 0) = 0 \text{ forall } v \in V\} = \mathbb{R}\tilde{0}$$

where $\tilde{0} \in S(\tilde{V})$ with $\tilde{0}(v, \alpha) = \alpha$ (as described in the proof of Lemma 4.4.6). Thus $V^{\perp} \cap S(\tilde{V}) = \{\tilde{0}\}$. Since V is an M-ideal in $\tilde{V}$, by Theorem 4.3.2, we may conclude that $\{\tilde{0}\}$ is a split face of $S(\tilde{V})$.

Next, we show that $\{0\}^{C}_{S(\tilde{V})} = \phi(S(V))$ so that $\phi(S(V))$ is also a (split) face of $S(\tilde{V})$ where $\phi : Q(V) \to S(\tilde{V})$ is the affine homeomorphism described in the proof of Lemma 4.4.6. In fact

$$\begin{aligned}
\{\tilde{0}\}^{C}_{S(\tilde{V})} &= \{g \in S(\tilde{V}) : face_{S(\tilde{V})}(g) \cap \{\tilde{0}\} = \emptyset\} \\
&= \{g \in S(\tilde{V}) : \tilde{0} \notin face_{S(\tilde{V})}(g)\} \\
&= \{\phi(f) : f \in Q(V), 0 \notin face_{Q(V)}(f)\} \\
&= \{\phi(f) : f \in S(V)\} \\
&= \phi(S(V)).
\end{aligned}$$

Thus $S(V)$ is convex. Now by Proposition 4.1.2, V is an approximate order unit space. □

CHAPTER 5

CM-ideals in ordered operator spaces

In this chapter, we investigate the order theoretic properties of CM-ideals in ordered operator spaces. In the first section, we discuss the notion of CM-ideals in operator spaces and matricially order smooth p-normed spaces. In the second section, we characterize the notion of CM-ideals in an operator space in terms of CL-projections on the matricial dual of the space. In other words, we investigate the notion of CL-projections in matricially order smooth 1-normed spaces. We characterize the notion of CM-ideals in terms of M-ideals in the self-adjoint part of each level of the given operator spaces which is one of the main theorem of this chapter. In the last section, we introduce the notion of an L^1-matricial split face. We show that W is a CM-ideal in V if and only if $\{W^{\perp} \cap Q_n(V)\}$ is an L^1-matricial split face of $\{Q_n(V)\}$.

5.1 Some basic facts

A projection P on an operator space $(V, \{\|.\|_n\})$ is called a *CM-projection* if

$$\|v\|_n = \max\{\|P_n(v)\|_n, \|(I-P)_n(v)\|_n\}$$

for all $v \in M_n(V)$, where P_n is the n-amplification of P. Let W be a closed subspace of V. Then W is called a *CM-summand* if $W = P(V)$ for some CM-projection P on V.

If V is a matricially order smooth ∞-normed space, then by Theorem 5.1.4, we see that V^{**} is also a matricially order smooth ∞-normed space. Thus V^{**} is an operator space.

Definition 5.1.1 *Let V be an operator space and let W be a closed subspace of V. Then W is called a CM-*ideal *in V if $W^{\perp\perp}$ is a CM-summand in V^{**}.*

We recall that if $T : V_1 \to V_2$ be a linear map of complex vector spaces, then *n-amplification* of T is a linear map

$$T_n : M_n(V_1) \to M_n(V_2)$$

defined by

$$T_n([v_{i,j}]) = [T(v_{i,j})] \tag{5.1.1}$$

for all $[v_{i,j}] \in M_n(V_1)$.

Definition 5.1.2 *[41] Let $1 \leq p \leq \infty$. An L^p-matricially normed matrix ordered space $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$ is said to be* matricially order smooth p-

normed space, *if* $\| \cdot \|_n$ *satisfies* $(O.p.1)$ *and* $(O.p.2)$ *conditions on* $M_n(V)_{sa}$ *for each* $n \in \mathbb{N}$.

Example 5.1.3 1. *Every* C*-*algebra is a matricially order smooth* ∞-*normed space [41].*

2. *Every matrix order unit space is a matricially order smooth* ∞-*normed space [41].*

3. *Every approximate matrix order unit space* $(V, \{M_n(V)^+\}, \{e_\lambda\})$ *is a matricially order smooth* ∞-*normed space [41].*

4. *Every matricially base normed space is a matricially order smooth* 1-*normed space [41].*

Theorem 5.1.4 *[41] Let* $1 \leq p \leq \infty$. *Then an* L^p-*matricially normed matrix ordered space* $(V, \{\| \cdot \|_n\}, \{M_n(V)^+\})$ *is a matricially ordered smooth* p-*normed space if and only if* $(V^*, \{\| \cdot \|_n\}, \{M_n(V^*)^+\})$ *is a matricially order smooth* p'-*normed space, where* $\frac{1}{p} + \frac{1}{p'} = 1$.

5.2 Characterization of CM-ideals

Let V be an operator space and consider its matricial dual V^*. For a projection P on V^*, we call P a *CL-projection* if

$$\|f\|_n = \|P_n(f)\|_n + \|(I - P)_n(f)\|_n$$

for all $f \in M_n(V^*)$. Let W be a closed subspace of V. Then $W^\perp$ is called *CL-summand* of V^* if there is a CL-projection P on V^* such that $P(V^*) = W^\perp$.

Ruan and Poon [52, Theorem 5.1] characterize the CM-ideals in a given operator space V in terms of CL-projections on operator space dual. In the next proposition, we characterize the notion of CM-ideal of a given operator space V in terms of CL-projection on matricial dual V^*.

Proposition 5.2.1 *Let $(V, \{\|\cdot\|_n\})$ be an operator space. Let P be a CM-projection on V^{**}. Then there exists a unique CL-projection L on V^* such that $L_n^* = P_n$ for all $n \in \mathbb{N}$.*

Proof. Let P be a CM-projection on V^{**}. Since P_n is an M-projection on $M_n(V^{**})$, P_n is a w^*-continuous linear projection (see e.g. [36, Theorem 1.9]). For $f \in V^*$, we define $\phi_f : V^{**} \to \mathbb{C}$ by letting

$$\phi_f(\mathfrak{g}) = P(\mathfrak{g})(f)$$

for all $\mathfrak{g} \in V^{**}$. Let $\{\mathfrak{g}_\alpha\}$ be a net in V^{**} such that $\mathfrak{g}_\alpha \to \mathfrak{g}$ in w^*-topology for some $\mathfrak{g} \in V^{**}$. Since P is a w^*-continuous linear projection, $P(\mathfrak{g}_\alpha) \to P(\mathfrak{g})$ in w^*-topology. Thus $\phi_f(\mathfrak{g}_\alpha) \to \phi_f(\mathfrak{g})$ whenever $\mathfrak{g}_\alpha \to \mathfrak{g}$ in w^*-topology. Therefore ϕ_f is a w^*-continuous linear functional on V^{**}, thus $\phi_f \in V^*$ for all $f \in V^*$. Now we define a map $Q : V^* \to V^*$ by $Q(f) = \phi_f$ for all $f \in V^*$. To show that Q is linear, let $f_1, f_2 \in V^*$ and $\lambda \in \mathbb{C}$. Then we have

$$\begin{aligned}
\mathfrak{g}(Q(\lambda f_1 + f_2)) &= P(\mathfrak{g})(\lambda f_1 + f_2) \\
&= \lambda P(\mathfrak{g})(f_1) + P(\mathfrak{g})(f_2) \\
&= \lambda \mathfrak{g}(Q(f_1)) + \mathfrak{g}(Q(f_2)) \\
&= \mathfrak{g}(\lambda Q(f_1) + Q(f_2))
\end{aligned}$$

for all $\mathfrak{g} \in V^{**}$. Thus we have $Q(\lambda f_1 + f_2) = \lambda Q(f_1) + Q(f_2)$ for all $f_1, f_2 \in V^*$

and $\lambda \in \mathbb{C}$. Now, let $f \in V^*$, then

$$\begin{aligned} \|\mathfrak{g}(Q(f))\| &= \|P(\mathfrak{g})(f)\| \\ &\leq \|P(\mathfrak{g})\|\|f\| \\ &\leq \|\mathfrak{g}\|\|f\| \\ &\leq \|f\| \end{aligned}$$

for all $\mathfrak{g} \in M_n(V^{**})_1$. Thus $\|Q(f)\| \leq \|f\|$ for all $f \in M_n(V^*)$. Therefore $Q : V^* \to V^*$ is a bounded linear operator. Similarly for each n, we can construct bounded projection $Q^n : M_n(V^*) \to M_n(V^*)$ given by

$$\mathfrak{g}(Q^n(f)) = P_n(\mathfrak{g})(f)$$

for all $\mathfrak{g} \in M_n(V^{**})$ and $f \in M_n(V^*)$. We claim that Q^n is the n-amplification of Q. Let Q_n be the n-amplification of Q. We show that $Q_n = Q^n$ for all $n \in \mathbb{N}$. Let $[f_{i,j}] \in M_n(V^*)$. For $[\mathfrak{g}_{i,j}] \in M_n(V^{**})$, we have

$$\begin{aligned} [\mathfrak{g}_{i,j}](Q^n([f_{i,j}])) &= (P_n[\mathfrak{g}_{i,j}])([f_{i,j}]) \\ &= [P(\mathfrak{g}_{i,j})]([f_{i,j}]) \\ &= \sum_{i,j=1}^{n} P(\mathfrak{g}_{i,j})(f_{i,j}) \\ &= \sum_{i,j=1}^{n} \mathfrak{g}_{i,j}(Q(f_{i,j})) \\ &= [\mathfrak{g}_{i,j}]([Q(f_{i,j})]) \\ &= [\mathfrak{g}_{i,j}](Q_n([f_{i,j}])). \end{aligned}$$

Therefore $Q^n = Q_n$. Let $f \in M_n(V^*)$. We show that

$$\|f\|_n = \|Q_n(f)\|_n + \|(1-Q)_n(f)\|_n.$$

Let $\epsilon > 0$, then there exist $\mathfrak{g}_1, \mathfrak{g}_2 \in M_n(V^{**})_1$ such that $\|Q_n(f)\|_n - \epsilon < Q_n(f)(\mathfrak{g}_1)$ and $\|(1-Q)_n(f)\|_n - \epsilon < (1-Q)_n(f)(\mathfrak{g}_2)$. Thus we have

$$\begin{aligned} P_n(\mathfrak{g}_1)(f) + (1-P)_n(\mathfrak{g}_2)(f) &= \mathfrak{g}(Q_n(f)) + \mathfrak{g}((1-Q)_n(f)) \\ &> \|Q_n(f)\|_n + \|(1-Q)_n(f)\|_n - 2\epsilon \end{aligned}$$

and

$$\begin{aligned} P_n(\mathfrak{g}_1)(f) + (1-P)_n(\mathfrak{g}_2)(f) &\le \|P_n(\mathfrak{g}_1) + (1-P)_n(\mathfrak{g}_2)\|_n \|f\|_n \\ &\le \max\{\|P_n(\mathfrak{g}_1)\|_n, \|(1-P)_n(\mathfrak{g}_2)\|_n\} \|f\|_n \\ &\le \|f\|_n. \end{aligned}$$

Therefore $\|Q_n(f)\|_n + \|(1-Q)_n(f)\|_n - 2\epsilon < \|f\|_n$. Since ϵ is arbitrary, we have $\|Q_n(f)\|_n + \|(1-Q)_n(f)\|_n \le \|f\|_n$. Therefore by the virtue of the triangle inequality, we get $\|Q_n(f)\|_n + \|(1-Q)_n(f)\|_n = \|f\|_n$. □

Corollary 5.2.2 *Let V be an operator space and let W be a closed subspace of V. If W is a CM-ideal in V, then there exists a CL projection L from V^* onto $W^\perp$ and $W^\perp$ is a CL-summand of V^*.*

Proposition 5.2.3 *Let V be a matricially order smooth ∞-normed space and let W be a self-adjoint subspace of V. Let P be the CL-projection from V^* onto $W^\perp$. Then $P_n(f^*) = P_n(f)^*$ for all $f \in M_n(V^*)$.*

Proof. Let P be the CL-projection from V^* onto $W^\perp$. Let $L : V^* \to V^*$ be a map defined by $L(f) = P(f^*)^*$ for all $f \in V^*$. Then L is a linear map such that

$$L^2(f) = L(P(f^*)^*) = (P^2(f^*))^* = P(f^*)^* = L(f)$$

for all $f \in V^*$. Thus L is a projection. Since W is self-adjoint, $W^\perp$ is also self-adjoint. We show that $ran(L) = ran(P)(= W^\perp)$. Let $f \in V^*$, then $L(f) = P(f^*)^* \in W^\perp$. Thus we have $ran(L) \subseteq ran(P)$. Conversely, let $f \in ran(P)$. Since $W^\perp$ is self-adjoint, $f^* \in W^\perp$ so that $P(f^*) = f^*$. Thus we have $L(f) = P(f^*)^* = f$ and $W^\perp \subseteq ran(L)$. Therefore we have $ran(P) = ran(L)$.

Let $f \in V^*$. Then we have

$$\begin{aligned}\|L(f)\| + \|f - L(f)\| &= \|P(f^*)^*\| + \|f - P(f^*)^*\| \\ &= \|P(f^*)\| + \|f^* - P(f^*)\| \\ &= \|f^*\| = \|f\|.\end{aligned}$$

Thus L is an L-projection on V^* such that $ran(P) = ran(L)(= W^\perp)$. By the uniqueness of an L-projection, we have $L = P$ so that $P(f^*) = P(f)^*$ for all $f \in V^*$.

Since $W^\perp$ is self-adjoint, $M_n(W^\perp)$ is also self-adjoint. Since P_n is an L-projection from $M_n(V^*)$ onto $M_n(W^\perp)$, we have $P_n(f^*) = P_n(f)^*$ for all $f \in M_n(V^*)$. □

Theorem 5.2.4 *Let V be a matricially order smooth ∞-normed space and let W be a closed self-adjoint subspace of V. Then W is a CM-ideal in V if and only if $M_n(W)_{sa}$ is an M-ideal in $M_n(V)_{sa}$ for each $n \in \mathbb{N}$.*

We need the following results before proving this theorem.

Lemma 5.2.5 *Let $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$ be a matricially order smooth ∞-*

normed space and let W be a self-adjoint subspace of V. If L^n is an L-projection from $M_n(V^*)_{sa}$ onto $M_n(W^\perp)_{sa}$ for each n, then we have the following properties:

(i) $L^n(\alpha^* f \alpha) = \alpha^* L^n(f)\alpha$ for all $f \in M_n(V^*)_{sa}$ and for all unitary matrix $\alpha \in \mathbb{M}_n$;

(ii) $L^{2n}(f_{11} \oplus f_{22}) = L^n(f_{11}) \oplus L^n(f_{22})$ for all $f_{11}, f_{22} \in M_n(V^*)_{sa}$.

Proof. (i) Let $\alpha \in \mathbb{M}_n$ be a unitary. Let us define $P^n(f) = \alpha L^n(\alpha^* f \alpha)\alpha^*$ for all $f \in M_n(V)_{sa}$. Then $P^n : M_n(V^*)_{sa} \to M_n(V^*)_{sa}$ is a linear map such that for $f \in M_n(V^*)_{sa}$, we have

$$\begin{aligned}
(P^n)^2(f) &= P^n(\alpha L^n(\alpha^* f \alpha)\alpha^*) \\
&= \alpha L^n(\alpha^* \alpha L^n(\alpha^* f \alpha)\alpha^* \alpha)\alpha^* \\
&= \alpha (L^n)^2(\alpha^* f \alpha)\alpha^* \\
&= \alpha L^n(\alpha^* f \alpha)\alpha^* \\
&= P^n(f).
\end{aligned}$$

Thus P^n is a projection. We claim that $ran(P^n) = ran(L^n)(= M_n(W^\perp)_{sa})$. Let $f \in M_n(V^*)_{sa}$. Then $L^n(\alpha^* f \alpha) \in M_n(W^\perp)_{sa}$. Thus we have $P^n(f) = \alpha L_n(\alpha^* f \alpha)\alpha^* \in M_n(W^\perp)_{sa}$ so that $ran(P^n) \subset M_n(W^\perp) = ran(L^n)$. Conversely, let $f \in M_n(W^\perp)_{sa}$, then $P^n(f) = \alpha L^n(\alpha^* f \alpha)\alpha^* = \alpha^* \alpha f \alpha \alpha^* = f$. Thus we have $M_n(W^\perp) \subset ran(P^n)$, and therefore $ran(P^n) = ran(L^n)$. Now we show that P^n is an L-projection. Let $f \in M_n(V^*)_{sa}$. Since L^n is an L-projection on $M_n(V^*)_{sa}$. Therefore we have

$$\|\alpha^* f \alpha\|_n = \|L^n(\alpha^* f \alpha)\|_n + \|\alpha^* f \alpha - L^n(\alpha^* f \alpha)\|_n.$$

Since α is unitary matrix, we have

$$\begin{aligned}\|f\|_n &= \|\alpha L^n(\alpha^* f\alpha)\alpha^*\|_n + \|f - \alpha L^n(\alpha^* f\alpha)\alpha^*\|_n \\ &= \|P_n(f)\|_n + \|f - P_n(f)\|_n.\end{aligned}$$

Thus P^n, L^n are two L-projections on $M_n(V^*)$ such that $ran(P^n) = ran(L^n)$. Therefore by the uniqueness of L-projection, we have $P^n = L^n$. Hence $L^n(\alpha^* f\alpha) = \alpha^* L^n(f)\alpha$ for all $f \in M_n(V)_{sa}$ and for all unitary $\alpha \in \mathbb{M}_n$.

(ii) Let $f_{11} \in M_m(V^*)_{sa}$ and $f_{22} \in M_n(V^*)_{sa}$. Since $f_{11} \oplus f_{22} \in M_{m+n}(V^*)_{sa}$ and since L^{m+n} is an L-projection from $M_{m+n}(V^*)_{sa}$ onto $M_{m+n}(W^\perp)_{sa}$, we have $L^{m+n}(f_{11} \oplus f_{22}) \in M_{m+n}(W^\perp)_{sa}$. Let $L^{m+n}(f_{11} \oplus f_{22}) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12}^* & g_{22} \end{bmatrix}$ for some $g_{11} \in M_m(W^\perp)_{sa}$ and $g_{22} \in M_n(W^\perp)_{sa}$ and $g_{12} \in M_{m,n}(W^\perp)$. Put $\alpha = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}$. Then α is a unitary matrix such that $\alpha^*(f_{11} \oplus f_{22})\alpha = f_{11} \oplus f_{22}$. Thus by (i), we have

$$L^{2n}(f_{11} \oplus f_{22}) = \alpha^* \begin{bmatrix} g_{11} & g_{12} \\ g_{12}^* & g_{22} \end{bmatrix} \alpha = \begin{bmatrix} g_{11} & -g_{12} \\ -g_{12}^* & g_{22} \end{bmatrix}.$$

Thus $g_{12} = 0$ so that $L^{2n}(f_{11} \oplus f_{22}) = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix}$. Since V^* is a matricially order smooth ∞-normed space and since

$$\|f_{11} \oplus f_{22}\|_{m+n} = \|g_{11} \oplus g_{22}\|_{m+n} + \|(f_{11} - g_{11}) \oplus (f_{22} - g_{22})\|_{m+n},$$

we have

$$\|f_{11}\|_m + \|f_{22}\|_n = \|g_{11}\|_m + \|g_{22}\|_n + \|(f_{11} - g_{11})\|_m + \|(f_{22} - g_{22})\|_n.$$

Therefore we have $\|f_{11}\|_m = \|g_{11}\|_m + \|(f_{11} - g_{11})\|_m$ and $\|f_{22}\|_n = \|g_{22}\|_n + \|(f_{22} - g_{22})\|_n$, where $g_{11} \in M_m(W^{\perp})_{sa}, g_{22} \in M_n(W^{\perp})_{sa}$. Since L^m is an L-projections from $M_m(V^*)_{sa}$ onto $M_m(W^{\perp})_{sa}$, we get $L^m(f_{11}) = g_{11}$. Similarly $L^n(f_{22}) = g_{22}$. Hence we have $L^{m+n}(f_{11} \oplus f_{22}) = L^m(f_{11}) \oplus L^n(f_{22})$. □

Let $L^n : M_n(V^*)_{sa} \to M_n(W^{\perp})_{sa}$ be a linear map. We extend L^n to $\widehat{L^n} : M_n(V^*) \to M_n(W^{\perp})_{sa}$ by

$$\widehat{L^n}(f) = L^n(g) + iL^n(h)$$

whenever $f = g + ih$ and $g, h \in M_n(V^*)_{sa}$. It is customary to check that $\widehat{L^n}$ is a linear map and $\widehat{L^n}(f^*) = \widehat{L^n}(f^*)$ for all $f \in M_n(V^*)$.

Lemma 5.2.6 *Let $(V, \{\|.\|_n\}, \{M_n(V)^+\})$ be a matricially order smooth ∞-normed space and let W be a self-adjoint subspace of V. If L^n is an L-projection from $M_n(V^*)_{sa}$ onto $M_n(W^{\perp})_{sa}$ for each n, then we have the following properties:*

(i) $\widehat{L^n}(\alpha^* f \alpha) = \alpha^* \widehat{L^n}(f)\alpha$ *for all* $f \in M_n(V)$ *and* $\alpha \in \mathbb{M}_n$;

(ii) $\widehat{L^{m+n}}\left(\begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix}\right) = \begin{bmatrix} \widehat{L^m}(f_{11}) & 0 \\ 0 & \widehat{L^n(f_{22})} \end{bmatrix}$ *for all* $f_{11} \in M_m(V), f_{22} \in M_n(V)$.

Proof. (i) Let $f = g + ih$ for some $g, h \in M_n(V^*)_{sa}$. Also, let $\alpha \in \mathbb{M}_n$ be a unitary matrix. Then we have,

$$\begin{aligned}\widehat{L^n}(\alpha^* f\alpha) &= \widehat{L^n}(\alpha^* g\alpha + i\alpha^* h\alpha)\\ &= L^n(\alpha^* g\alpha) + iL^n(\alpha^* h\alpha)\\ &= \alpha^* L^n(g)\alpha + i\alpha^* L^n(h)\alpha\\ &= \alpha^*(L^n(g) + iL^n(h))\alpha\\ &= \alpha^*\widehat{L^n}(f)\alpha.\end{aligned}$$

(ii) Let $f_{11} = g_{11} + ih_{11}$ with $g_{11}, h_{11} \in M_m(V^*)_{sa}$ and $f_{22} = g_{22} + ih_{22}$ where $g_{22}, h_{22} \in M_n(V^*)_{sa}$. Then

$$\begin{aligned}\widehat{L^{m+n}}\left(\begin{bmatrix} f_{11} & 0\\ 0 & f_{22}\end{bmatrix}\right) &= \widehat{L^{m+n}}\left(\begin{bmatrix} g_{11}+ih_{11} & 0\\ 0 & g_{22}+ih_{22}\end{bmatrix}\right)\\ &= L^{m+n}\left(\begin{bmatrix} g_{11} & 0\\ 0 & g_{22}\end{bmatrix}\right) + iL^{m+n}\left(\begin{bmatrix} h_{11} & 0\\ 0 & h_{22}\end{bmatrix}\right)\\ &= \begin{bmatrix} L^m(g_{11}) & 0\\ 0 & L^n(g_{11})\end{bmatrix} + i\begin{bmatrix} L^m(h_{11}) & 0\\ 0 & L^n(h_{22})\end{bmatrix}\\ &= \begin{bmatrix} \widehat{L^m}(g_{11}+ih_{11}) & 0\\ 0 & \widehat{L^n}(g_{22}+ih_{22})\end{bmatrix}\\ &= \begin{bmatrix} \widehat{L^m}(f_{11}) & 0\\ 0 & \widehat{L^n}(f_{22})\end{bmatrix}.\end{aligned}$$

□

Lemma 5.2.7 *Let V be a matricially order smooth ∞-normed space and let W be a self-adjoint subspace of V. If L^n is an L-projection from $M_n(V^*)_{sa}$ onto*

$M_n(W^\perp)_{sa}$ *for each* n*, then*

$$\widehat{L^{2n}}\left(\begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 0 & \widehat{L^n}(f) \\ \widehat{L^n}(f)^* & 0 \end{bmatrix}\right) \text{ for all } f \in M_n(V^*).$$

Proof. Let $f = g + ih \in M_n(V^*)$ where $g, h \in M_n(V^*)_{sa}$. Since L^n is an L-projection on $M_n(V^*)_{sa}$, we have

$$\|g\|_n = \|L^n(g)\|_n + \|g - L^n(g)\|_n.$$

Since V^* is a matricially ordered smooth 1-normed space, thus

$$\begin{aligned} \left\| \begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix} \right\|_{2n} &= 2\|g\|_n \\ &= 2(\|L^n(g)\|_n + \|g - L^n(g)\|_n) \\ &= \left\| \begin{bmatrix} 0 & L^n(g) \\ L^n(g) & 0 \end{bmatrix} \right\|_{2n} + \left\| \begin{bmatrix} 0 & g - L^n(g) \\ g - L^n(g) & 0 \end{bmatrix} \right\|_{2n}. \end{aligned}$$

Now we have

$$\begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix} = \begin{bmatrix} 0 & L^n(g) \\ L^n(g) & 0 \end{bmatrix} + \begin{bmatrix} 0 & g - L^n(g) \\ g - L^n(g) & 0 \end{bmatrix}$$

where $\begin{bmatrix} 0 & L^n(g) \\ L^n(g) & 0 \end{bmatrix} \in M_{2n}(W^\perp)_{sa}$.

So by the uniqueness of the decomposition of the L-projection, we have

$$L^{2n}\left(\begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & L^n(g) \\ L^n(g) & 0 \end{bmatrix}.$$

Again since $h \in M_n(V^*)_{sa}$ and since L_n is an L-projection, we have

$$\|h\|_n = 2(\|L^n(h)\|_n + \|h - L^n(h)\|_n).$$

Since V^* is a matricially order smooth 1-normed space,

$$\begin{aligned}\left\|\begin{bmatrix} 0 & ih \\ -ih & 0 \end{bmatrix}\right\|_{2n} &= \|ih\|_n + \|-ih\|_n \\ &= \|iL^n(h)\|_n + \|i(h - L^n(h))\|_n + \|-iL^n(h)\|_n + \|-i(h - L^n(h))\|_n \\ &= \left\|\begin{bmatrix} 0 & iL^n(h) \\ -iL^n(h) & 0 \end{bmatrix}\right\|_{2n} + \left\|\begin{bmatrix} 0 & i(h - L^n(h)) \\ -i(h - L^n(h)) & 0 \end{bmatrix}\right\|_{2n}.\end{aligned}$$

As $\begin{bmatrix} 0 & iL^n(h) \\ -iL^n(h) & 0 \end{bmatrix} \in M_{2n}(W^\perp)_{sa}$, we have

$$L_{2n}\left(\begin{bmatrix} 0 & ih \\ -ih & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & iL^n(h) \\ -iL^n(h) & 0 \end{bmatrix}.$$

Therefore

$$\widehat{L^{2n}}\left(\begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix}\right) = L^{2n}\left(\begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix}\right) + iL^{2n}\left(\begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & L^n(g) \\ L^n(g) & 0 \end{bmatrix} + i \begin{bmatrix} 0 & L^n(h) \\ -L^n(h) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \widehat{L^n}(g+ih) \\ \widehat{L^n}(g-ih) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \widehat{L^n}(f) \\ \widehat{L^n}(f^*) & 0 \end{bmatrix}.$$

□

Lemma 5.2.8 *Let V be a matricially order smooth ∞-normed space and let W be a self-adjoint subspace of V. If L^n is an L-projection from $M_n(V^*)_{sa}$ onto $M_n(W^\perp)_{sa}$ for each n, then* $\widehat{L^{2n}}\left(\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}\right) = \begin{bmatrix} \widehat{L^n}(f_{11}) & \widehat{L^n}(f_{12}) \\ \widehat{L^n}(f_{21}) & \widehat{L^n}(f_{22}) \end{bmatrix}$.

Proof. Let $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in M_{2n}(V^*)$ where $f_{i,j} \in M_n(V^*)$ for $i,j = 1,2$. Now we can write

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix} + \begin{bmatrix} 0 & \frac{f_{12}+f_{21}^*}{2} \\ \frac{f_{21}+f_{12}^*}{2} & 0 \end{bmatrix} + i \begin{bmatrix} 0 & \frac{f_{12}-f_{21}^*}{2i} \\ \frac{f_{21}-f_{12}^*}{2i} & 0 \end{bmatrix} \qquad (5.2.1)$$

Thus by using Lemma 5.2.6 and Lemma 5.2.7, we have

$$\widehat{L^{2n}}\left(\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}\right) = \begin{bmatrix} \widehat{L^n}(f_{11}) & \widehat{L^n}(f_{12}) \\ \widehat{L^n}(f_{21}) & \widehat{L^n}(f_{22}) \end{bmatrix}.$$

□

Lemma 5.2.9 *Let V be a matricially order smooth 1-normed space and let W*

be a self-adjoint subspace of V. If L^n is an L-projection from $M_n(V^*)_{sa}$ onto $M_n(W^\perp)_{sa}$ for each n, then $\widehat{L^n} = \widehat{L^1}_n$ ($\widehat{L^1}_n$ is the n-amplification of $\widehat{L^1}$).

Proof. Using induction on Lemma 5.2.8, we have $\widehat{L^{2^n}} = \widehat{L^1}_{2^n}$ for every $n \in \mathbb{N}$. Now, let $n \in \mathbb{N}$ and $f \in M_n(V^*)$. Then we have

$$\begin{aligned} \widehat{L^n}(f) \oplus 0 &= \widehat{L^{2^n}}(f \oplus 0)(\text{ Lemma } 5.2.6) \\ &= \widehat{L^1}_{2^n}(f \oplus 0) \\ &= \widehat{L^1}_n(f) \oplus 0. \end{aligned}$$

Therefore, we have $\widehat{L^n} = \widehat{L^1}_n$ for every n. □

Proof of theorem 5.2.4. Let V be a matricially order smooth ∞-normed space and let W be a closed self-adjoint subspace. Now if W is a CM-ideal in V. Then there is a CL-projection L from V^* onto $W^\perp$. Since Banach dual of $M_n(V)_{sa}$ is $M_n(V^*)_{sa}$. Therefore it is sufficient to show that $M_n(W^\perp)_{sa}$ is an L-summand of $M_n(V^*)_{sa}$. Now by Proposition 5.2.3, we have $L_n(M_n(V^*)_{sa}) \subset M_n(W^\perp)_{sa}$. Since L_n is a projection from $M_n(V^*)$ onto $M_n(W^\perp)$, thus for all $f \in M_n(W^\perp)_{sa}$, we have $f = L_n(f)$. Therefore $L_n(M_n(V^*)_{sa}) = M_n(W^\perp)_{sa}$. Since L_n is an L-projection, $L_n|_{(M_n(V^*)_{sa})}$ is the L-projection from $M_n(V^*)_{sa}$ onto $M_n(W^\perp)_{sa}$.

Conversely, let $M_n(W)_{sa}$ be an M-ideal in $M_n(V)_{sa}$ for all $n \in \mathbb{N}$. Thus let L^n be an L-projection from $M_n(V^*)_{sa}$ onto $M_n(W^\perp)_{sa}$ for each n. Now let $\widehat{L^n}$ be its linear extension to $M_n(V^*)$. Now by the construction of $\widehat{L^n}$, we have $ran(\widehat{L^n}) \subset M_n(W^\perp)$. Also if $f \in M_n(W^\perp)$, then there exist $g, h \in M_n(W^\perp)_{sa}$ such that $f = g + ih$. Now $\widehat{L^n}(f) = L^n(g) + iL^n(h) = g + ih = f$. Therefore

$ran(\widehat{L^n}) = M_n(W^\perp)$. Again,

$$
\begin{aligned}
2\|f\|_n &= \left\| \begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix} \right\|_{2n} \\
&= \left\| L^{2n}\left(\begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix} \right) \right\|_{2n} + \left\| \begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix} - L^{2n}\left(\begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix} \right) \right\|_{2n} \\
&= \left\| \begin{bmatrix} 0 & \widehat{L^n}(f) \\ \widehat{L^n}(f^*) & 0 \end{bmatrix} \right\|_{2n} + \left\| \begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & \widehat{L^n}(f) \\ \widehat{L^n}(f^*) & 0 \end{bmatrix} \right\|_{2n} \\
&= \|\widehat{L^n}(f)\|_n + \|\widehat{L^n}(f^*)\|_n + \|f - \widehat{L^n}(f)\|_n + \|f^* - \widehat{L^n}(f^*)\|_n.
\end{aligned}
$$

Thus we have

$$\|f\|_n = \|\widehat{L^n}(f)\|_n + \|f - \widehat{L^n}(f)\|_n$$

for all $f \in M_n(V^*)$. Now by Lemma 5.2.9, we know that $\widehat{L^n}$ is the n-amplification of $\widehat{L^1}$, therefore $\widehat{L^1}$ is the CL-projection from V^* onto $W^\perp$. Let P be the adjoint map of $\widehat{L^1}$. We can prove that $\widehat{P}$ is the CM-projection on V^{**}. Therefore $1 - P$ is a CM-projection on V^{**} such that

$$ran(I - P) = \ker(P) = ran\widehat{L^1}^{\perp} = W^{\perp\perp}.$$

Hence W is a CM-ideal in V. □

5.3 CL-projections and CP-maps

In the rest of the sections of this chapter, we assume that V is a matricially order smooth ∞-normed space and we write

$$K_n = M_n(V^*)_{sa} \cap M_n(V^*)_1$$

for each $n \in \mathbb{N}$. Then K_n is a compact convex set in w^*-topology. It follows from the Proposition 3.1.4 (v) that $\{K_n\}$ is an L^1-matrix convex set in V^*. Note that K_n is the closed unit ball of the real Banach space $M_n(V^*)_{sa}$ for each n. Thus the following discussions and the statements are restated from the paper [1, Part I, Section 2]. Hence we omit the proof.

Let D_n be a subset of K_n for each n, we define $face_{K_n}(D_n)$ is the smallest face containing D_n. Therefore

$$face_{K_n}(D_n) = \cap\{F_n : F_n \text{ is a face of } K_n, D_n \subset F_n\}.$$

In particular, if D_n is a convex subset of K_n, we can prove that

$$face_{K_n}(D_n) = \{g \in K_n : \lambda g + (1-\lambda)h \in D_n \text{ for some } h \in K_n, \lambda \in (0,1)\}.$$

If $f \in K_n$, we write $face_{K_n}(f)$ for $face_{K_n}(\{f\})$. Let

$$\mathcal{F}_n = \{F : F \text{ is a face of } K_n \text{ and } 0 \notin F\}.$$

Then $\mathcal{F}_n$ has a maximal element. If $F \in \mathcal{F}_n$ is a maximal element, then F is closed. Note that if $F \in \mathcal{F}_n$, then $f \in F$ implies $\|f\|_n = 1$. Also for $f \in K_n \setminus \{0\}$, we have $face_{K_n}(\frac{f}{\|f\|_n}) \subset \mathcal{F}_n$. We define

$$C_n(f) = \cup_{\lambda \geq 0} \lambda face_{K_n}(\frac{f}{\|f\|}).$$

A cone $C \subseteq M_n(V^*)_{sa}$ is called *facial cone* if $C = cone(F)$ for some face $F \in \mathcal{F}_n$ or $C = \{0\}$ where $cone(F) = \cup_{\lambda \geq 0} \lambda F$.

Lemma 5.3.1 *Let V be a matricially order smooth ∞-normed space. Let $f_1, f_2, \cdots, f_r \in M_n(V^*)_{sa}$. Then the following are equivalent:*

(i) $f_1, f_2, \cdots, f_r \in cone(F)$ for some $F \in \mathcal{F}_n$;

(ii) $f_1, f_2, \cdots, f_r \in C_n(f_1 + f_2 + \cdots + f_r)$;

(iii) $\|f_1 + f_2 + \cdots + f_r\|_n = \|f_1\|_n + \|f_2\|_n + \cdots + \|f_r\|_n$.

If $f, g \in M_n(V^*)_{sa}$, we say that f and g are *co-directional* (we write $f \mid g$) if $\|f + g\|_n = \|f\|_n + \|g\|_n$. More generally $f_1, f_2, \cdots, f_r \in M_n(V^*)_{sa}$ are *co-directional* if $\|f_1 + \cdots + f_r\|_n = \|f_1\|_n + \cdots + \|f_r\|_n$. If $f, g \in M_n(V^*)_{sa}$, we write $f \prec g$ if $\|g\| = \|f\| = \|g - f\|$.

Proposition 5.3.2 *Let $f_1, f_2, \cdots f_r$ and $g_1, g_2, \cdots g_r \in M_n(V^*)_{sa}$ such that $f_i \prec g_i$ for $i = 1, 2, \cdots, r$ and $g_1, \cdots g_r$ are co-directional, then $f_1, f_2, \cdots, f_r$ are co-directional and $f_1 + \cdots + f_r \prec g_1 + \cdots + g_r$.*

Proposition 5.3.3 *Let V be an order smooth ∞-normed space and let $f \in M_n(V^*)_{sa}$. Then*

$$C_n(f) = \{g \in M_n(V^*)_{sa} : g \prec \alpha f \text{ for some } \alpha > 0\}.$$

Let V be a matricially order smooth ∞-normed space and let C_n be a cone in $M_n(V^*)_{sa}$ for each $n \in \mathbb{N}$. Then the *complementary set* C_n' of C_n is the set defined by

$$C_n' = \{f \in M_n(V^*)_{sa} : C_n(f) \cap C_n = \{0\}\}.$$

Note that C_n' may not be a cone in general.

Theorem 5.3.4 *Let V be a matricially order smooth-∞-normed space and let C_n be a cone in $M_n(V^*)_{sa}$. Then for each $f \in M_n(V^*)_{sa}$, there exist $g \in C_n$ and $h \in C_n'$ such that*

$$f = g + h \text{ and } \|f\|_n = \|g\|_n + \|h\|_n.$$

Theorem 5.3.5 *Let V be a matricially order smooth ∞-normed space and let C_n be a closed cone of $M_n(V^*)_{sa}$. Then for any $f \in M_n(V^*)^+$, there are $g \in C_n^+$ and $h \in C_n'^+$ such that*

$$f = g + h, \text{ and } \|f\|_n = \|g\|_n + \|h\|_n.$$

Proof. Since V is a matricially order smooth ∞-normed space, $M_n(V^*)_{sa}$ is an order smooth 1-normed space satisfying $(OS.1.2)$ for each n. Then the result follows from Theorem 4.2.1. □

Lemma 5.3.6 *Let V be a matricially order smooth ∞-normed space. Then*

$$face_{K_n}(\|f\|_n^{-1} f) \subset M_n(V^*)^+$$

for all $f \in M_n(V^*)^+ \setminus \{0\}$.

Proof. As $M_n(V^*)_{sa}$ is an order smooth 1-normed space satisfying $(OS.1.2)$ and K_n is the closed unit ball of $M_n(V^*)$, the result follows from Lemma 4.2.2. □

Theorem 5.3.7 *Let V be a matricially order smooth ∞-normed space and let W be a self-adjoint subspace of V. If L is a CL-projection from V^* onto $W^\perp$. Then L is a CP-map.*

Proof. Let $f \in M_n(V^*)^+$. Since P is a CL-projection, therefore $\|f\|_n = \|L_n(f)\|_n + \|f - L_n(f)\|_n$. Thus by Lemma 5.3.1, we have $L_n(f), f - L_n(f) \in C_n(f)$. Since $f \in M_n(V^*)^+$, we have $C_n(f) \subset M_n(V^*)^+$. Hence $L_n(f) \geq 0$. □

5.4 L^1-matricial split face and CM-ideal

Let K be a compact convex set in a locally convex set V such that $0 \in ext(K)$. An element $k \in K$ is called a *lead point* of K ($k \in lead(K)$) if $k = \alpha k_1$ for some $k_1 \in K$ with $\alpha \in [0,1]$, then $\alpha = 1$. We observe that $ext(K) \setminus \{0\} \subseteq lead(K)$. For each $k \in K \setminus \{0\}$, there is a unique $\alpha \in (0,1]$ and $k_1 \in lead(K)$ such that $k = \alpha k_1$.

The notion of an L^1-matrix convex set has been discussed in Chapter 3. We recall the notion for a quick reference.

Definition 5.4.1 *Let V be a matricially order smooth ∞-normed space. Then a collection $\{D_n\}$ of sets with $D_n \subset M_n(V^*)_{sa}$ and $0 \in ext(D_n)$ is called an* L^1-matrix convex set *if the following conditions hold:*

(a) If $f \in D_n$ and $\gamma_i \in \mathbb{M}_{n,n_i}$ such that $\sum_{i=1}^k \gamma_i \gamma_i^* \leq I_n$, then $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in D_{\sum_{i=1}^k n_i}$;

(b) If $f \in D_{2n}$ so that $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$ for some $f_{11}, f_{22} \in D_n$ and $f_{12} \in M_n(V^*)$, then $f_{12} + f_{12}^* \in co(D_n \cup -D_n)$;

(c) *Let* $f \in D_{m+n}$ *with* $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$ *so that* $f_{11} \in D_m$ *and* $f_{22} \in M_{m,n}(V^*)$ *and if* $f_{11} = \alpha_1 \widehat{f_{11}}$ *and* $f_{22} = \alpha_2 \widehat{f_{22}}$ *with* $\widehat{f_{11}} \in lead(D_m)$ *and* $\widehat{f_{22}} \in lead(D_n)$*, then we have* $\alpha_1 + \alpha_2 \leq 1$.

Notice that if V is a matricially order smooth ∞-normed space, then by Remark 3.2.4, $\{Q_n(V)\}$ is an L^1-matrix convex set. We introduce the notion of split faces of the L^1-matrix convex set $\{Q_n(V)\}$.

Definition 5.4.2 *Let* V *be a matricially order smooth* ∞*-normed space. Then an* L^1*-matricial convex set* $\{D_n\}$ *of* V^* *such that* $D_n \subset Q_n(V)$ *is called an* L^1-matricial split face *of* $\{Q_n(V)\}$ *if for each* n*,* D_n *is a split face of* $Q_n(V)$*.*

Note that the above definition may be stated for general L^1 -matrix convex sets.

Lemma 5.4.3 *Let* V *be a matricially order smooth* ∞*-normed space and let* W *be a self-adjoint subspace of* V*. If* W *is a* CM*-ideals in* V*, then* $C_n(f) \subset M_n(W^\perp)$ *whenever* $f \in M_n(W^\perp)_{sa}$*.*

Proof. Let $f \in M_n(W^\perp)_{sa}$. Without loss of generality, we assume that $\|f\|_n = 1$. Let $g \in face_{K_n}(f)$. Then there exist $h \in K_n$ and $\lambda \in (0,1)$ such that $f = \lambda g + (1-\lambda)h$. Since $\|g\|, \|h\| \leq 1$, it is follows from the triangle inequality that $\|g\| = 1 = \|h\|$. Also, by Theorem 5.3.4, there are $g_1, h_1 \in M_n(W^\perp)_{sa}$ and $g_2, h_2 \in M_n(W^\perp)'_{sa}$ such that

$$g = g_1 + g_2 \qquad \|g\|_n = \|g_1\|_n + \|g_2\|_n$$
$$h = h_1 + h_2 \qquad \|h\|_n = \|h_1\|_n + \|h_2\|_n.$$

We can write as $-\lambda g_2 = (f - (\lambda g_1 + (1-\lambda)h_1)) + (1-\lambda)h_2$ where $f - (\lambda g_1 + (1-\lambda)h_1) \in M_n(W^\perp)_{sa}$ and $(1-\lambda)h_2 \in M_n(W^\perp)'_{sa}$. We know from Theorem

5.2.4 that $M_n(W^{\perp})_{sa}$ is an L-summand of $M_n(V^*)_{sa}$. It follows that $\|-\lambda g_2\|_n = \|(f-(\lambda g_1+(1-\lambda)h_1))\|_n + \|(1-\lambda)h_2\|_n$. Similarly, we can show that $\|(1-\lambda)h_2\|_n = \|(f-(\lambda g_1+(1-\lambda)h_1))\|_n+\|\lambda g_2\|_n$. Consequently, $f=\lambda g_1+(1-\lambda)h_1$. Since $\|g_1\|_n, \|h_1\|_n \leq 1$. Thus by virtue of the triangle inequality, we have $\|g_1\|_n = \|h_1\|_n = 1$, and therefore $g_2 = 0 = h_2$. Hence $face_{K_n}(f) \subset M_n(W^{\perp})_{sa}$ and $C_n(f) \subset M_n(W^{\perp})_{sa}$. □

Now, we characterize CM-ideals in terms of L^1-matricial split faces.

Theorem 5.4.4 *Let V be a matricially order smooth ∞-normed space and let W be a self-adjoint subspace of V. Then W is a CM-ideal in V if and only if $\{M_n(W^{\perp}) \cap Q_n(V)\}$ is an L^1-matricial split face of $\{Q_n(V)\}$.*

Proof. We show that the conditions (a), (b) and (c) of Definition 5.4.1 hold.

(a) Let $f \in M_n(W^{\perp}) \cap Q_n(V)$ and $\gamma_i \in \mathbb{M}_{n,n_i}$ such that $\sum_{i=1}^k \gamma_i \gamma_i^* \leq I_n$. Then $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in M_{\sum_{i=1}^k}(W^{\perp})$. Since $\{Q_n(V)\}$ is an L^1-matrix convex set, $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in Q_{\sum_{i=1}^k n_i}(V)$. Therefore we have $\oplus_{i=1}^k \gamma_i^* f \gamma_i \in M_{\sum_{i=1}^k}(W^{\perp}) \cap Q_{\sum_{i=1}^k n_i}(V)$.

(b) Let $f \in M_{2n}(W^{\perp}) \cap Q_{2n}(V)$ and $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{bmatrix}$. Since $\{Q_n(V)\}$ is an L^1-matrix convex set, thus $f_{12}+f_{12}^* \in co(Q_n(V) \cup -Q_n(V))$. Thus we have $\|f_{12}+f_{12}^*\|_n \leq 1$. Since $M_n(V^*)_{sa}$ satisfies $(OS.1.2)$, there are $g_1, g_2 \in M_n(V^*)_{sa}$ such that $f_{12}+f_{12}^* = g_1 - g_2$ and $\|f_{12}+f_{12}^*\|_n = \|g_1\|_n + \|g_2\|_n$. Thus by Lemma 5.3.1, we have $g_1, -g_2 \in C_n(f_{12}+f_{12}^*)$. Also by Lemma 5.4.3, we have $g_1, g_2 \in M_n(W^{\perp})$. Hence $f_{12}+f_{12}^* \in co(M_n(W^{\perp}) \cap Q_n(V) \cup -M_n(W^{\perp}) \cap Q_n(V))$.

(c) Since $Q_n(V)$ is an L^1-matrix convex set and $lead(M_n(W^{\perp}) \cap Q_n(V)) = M_n(W^{\perp}) \cap S_n(V)$. Thus $M_n(W^{\perp}) \cap Q_n(V))$ is an L^1-matrix convex set.

Since W is a CM-ideal in V, it follows from Theorem 5.2.4 that $M_n(W^\perp)_{sa}$ is an M-ideal in $M_n(V^*)_{sa}$. Then by applying Proposition 4.3.13, we may conclude that $M_n(W^\perp)_{sa} \cap Q_n(V)$ is a split face of $Q_n(V)$ for each n. This completes the proof. $\square$

CHAPTER 6

Smooth p-order ideals

In this final chapter, we discuss the notion "smooth p-ordered ideals" in order smooth p-normed spaces which generalizes the notion of M-ideals in order smooth ∞-normed spaces. In the first section, we discuss the order structure of subspaces, and of quotient spaces of ordered normed spaces. In the second section, we show that given an order smooth p-normed space V, and its closed subspace W, we have W is a smooth p-ordered ideal in V if and only if $W^{\perp}$ is a smooth p'-order ideal in an order smooth p'-normed space V^* if and only if $W^{\perp\perp}$ is a p-order ideal in order smooth p-normed space V^{**}. In the last section, we show that if W is an M-ideal in order smooth ∞-normed space V, then W is a smooth ∞-order ideal in V under certain condition. If W is an L-summand of an order smooth 1-normed space, then W is a smooth 1-order ideal.

6.1 Smooth p-order ideals

Let (V, V^+) be a real ordered vector space, and V^* be a dual of V. We recall that V^* is an ordered vector space with the cone

$$V^{*+} = \{f \in V^* : f(v) \geq 0 \ \forall v \in V^+\}.$$

If W is a subspace of an ordered vector space V, then W is also ordered vector space together with cone $W^+ = W \cap V^+$. Let $\varphi_W : V \to V/W$ be the canonical homomorphism. Then V/W is also an ordered vector space with the cone $\varphi_W(V^+)$.

We note that V^{*+} is a w^*-closed set in V^*. In particular, if V^+ is a cone in ordered normed space V, then V^{*+} is a norm closed cone in V^*. Let $(V, V^+, \|.\|)$ be an ordered normed space. We define another cone V_+ on V by

$$V_+ = \{v \in V : f(v) \geq 0 \ \forall f \in V^{*+}\}.$$

Similarly, we define cones for V_+^* and V_+^{**}. The following result connects these cones.

Proposition 6.1.1 *Let $(V, V^+, \|.\|)$ be an ordered normed space. Then we have the following:*

(i) if V^+ is norm closed, then $V^+ = V_+$;

(ii) $V^{*+} = V_+^*$;

(ii) $V^{**+} = V_+^{**}$.

Proof. By the definition, $V^+ \subseteq V_+$. If possible, let $v \in V_+ \setminus V^+$. Then by the Hahn Banach Separation theorem, there is a $f \in V^*$ such that $f(v) < 0$ and

$f(w) \geq 0$ for all $w \in V^+$ so that $f \in V^{*+}$. Since $v \in V_+$, and $f \in V^{*+}$, we have $f(v) \geq 0$ which is a contradiction. Hence $V^+ = V_+$.

Now (ii) and (iii) follow from (i). □

Let V be a Banach space, and let V^* be its Banach dual. If W be a closed subspace of V, then we have the following Banach space isometries:

(i) $(V/W)^* \cong W^\perp$;

(ii) $W^* \cong V^*/W^\perp$;

(iii) $W^{**} = (V^*/W^\perp)^* \cong W^{\perp\perp}$;

(iv) $(V/W)^{**} \cong (W^\perp)^* \cong V^{**}/W^{\perp\perp}$.

Let V be an ordered normed space with closed cone V^+ and let W be a closed subspace of V. Then $W, W^\perp$, and $W^{\perp\perp}$ are also ordered normed spaces with closed cone given by

$$W^+ = W \cap V^+, \quad W^{\perp+} = W^\perp \cap V^{*+}$$

and

$$W^{\perp\perp+} = W^{\perp\perp} \cap V^{**+}$$

respectively. Let $\varphi_W : V \to V/W$, $\varphi_{W^\perp} : V^* \to V^*/W^\perp$ and $\varphi_{W^{\perp\perp}} : V^{**} \to V^{**}/W^{\perp\perp}$ be the natural homomorphisms. Notice that $\varphi_W(V^+)$ may not be a closed cone in V/W. Similarly, $\varphi_{W^\perp}(V^{*+})$ and $\varphi_{W^{\perp\perp}}(V^{**+})$ may not be closed in $V^*/W^\perp$ and $V^{**}/W^{\perp\perp}$ respectively. In this direction, we have the following result.

Proposition 6.1.2 *Let V be an ordered normed space and let W be a closed subspace of V. Let $\varphi_W, \varphi_{W^\perp}$ and $\varphi_{W^{\perp\perp}}$ be the natural homomorphisms. Then* *we have the following cone relations:*

(i) $\overline{\varphi_W(V^+)}^{\|\cdot\|} = \overline{\varphi_W(V^+)}^{w}$;

(ii) $\overline{\varphi_{W^\perp}(V^{*+})}^{\|\cdot\|} = \overline{\varphi_{W^\perp}(V^{*+})}^{w}$; *and*

(iii) $\overline{\varphi_{W^{\perp\perp}}(V^{**+})}^{\|\cdot\|} = \overline{\varphi_{W^{\perp\perp}}(V^{**+})}^{w}$.

Proof. It is sufficient to prove (i), for similar arguments may be used to prove $(ii),(iii)$. Let $v + W \in \overline{\varphi_W(V^+)}^{\|\cdot\|}$. Then there exists a sequence $\{v_n\} \in V^+$ such that $v_n + W$ convergent to $v + W$ in the norm. so that $v_n + W \longrightarrow v + W$ in w-topology. Therefore $\overline{\varphi_W(V^+)}^{\|\cdot\|} \subset \overline{\varphi_W(V^+)}^{w}$.

Conversely, if possible let $v+W \in \overline{\varphi_W(V^+)}^{w} \setminus \overline{\varphi_W(V^+)}^{\|\cdot\|}$. Then by the Hahn Banach separation theorem, there is a $f \in W^\perp$ such that $f(v) < 0$ and $f(u) \geq 0$ for all $u \in V^+$. Thus $f \in V^{*+}$. Since $v + W \in \overline{\varphi_W(V^{*+})}^{w}$, thus there exists a net $\{v_\alpha + W\}$, where $v_\alpha \in V^+$ such that $v_\alpha + W \longrightarrow v + W$ in w-topology. Thus we have $f(v_\alpha) \longrightarrow f(v)$. Since $f(v_\alpha) \geq 0$, we have $f(v) \geq 0$, which is a contradiction. Hence $\overline{\varphi_W(V^+)}^{\|\cdot\|} = \overline{\varphi_W(V^+)}^{w}$. □

However, the cones in the dual spaces are expected to be w^*-closed. So, we adopt the following definition.

Definition 6.1.3 *Let V be an ordered normed space, and let W be a closed subspace of V. Let $\varphi_W : V \to V/W$, $\varphi_{W^\perp} : V^* \to V^*/W^\perp$ and $\varphi_{W^{\perp\perp}} : V^{**} \to V^{**}/W^{\perp\perp}$ be the natural homomorphisms. Then we define order structure on $V/W, V^*/W^\perp$ and $V^{**}/W^{\perp\perp}$ as*

(i) $(V/W)^+ := \overline{\varphi_W(V^+)}^{\|\cdot\|}$;

(ii) $(V^*/W^\perp)^+ := \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$;

(iii) $(V^{**}/W^{\perp\perp})^+ := \overline{\varphi_{W^{\perp\perp}}(V^{**+})}^{w^*}$.

Now onwards, we assume that all order normed space V are norm complete, and W is a closed subspace of V.

Lemma 6.1.4 *Let $(V, V^+, \|.\|)$ be an order smooth p-normed space and let W be a subspace of V. Let $\varphi_W : V \to V/W$ and $\varphi^*_{W^\perp} : V^* \to V^*/W^\perp$ be the natural homomorphisms. Then*

(i) $\{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\} = (V^*/W^\perp)^+$*;*

(ii) $\{f \in W^\perp : f(v) \geq 0 \ \forall v + W \in (V/W)^+\} = W^{\perp +}$.

Proof. Let (V, V^+) be an order smooth p-normed space, and W be a subspace of V.

(i) We note that Banach dual of W can be identified with $V^*/W^\perp$. We claim that $\{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\}$ is a w^*-closed set. Let $\{f_\alpha + W^\perp\}$ be a net in $\{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\}$ such that $f_\alpha + W^\perp \longrightarrow f + W^\perp$ for some $f \in V^*$ in w^*-topology. Since W is a predual of $V^*/W^\perp$, we have $f_\alpha(w) \longrightarrow f(w)$ for all $w \in W^+$. Now $f_\alpha(w) \geq 0$ for all $w \in W^+$, so that $f(w) \geq 0$ for all $w \in W^+$. Hence $\{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\}$ is a w^*-closed set. From the definition we note that $(V^*/W^\perp)^+ = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$. Let $f \in V^{*+}$. Since $f(w) \geq 0$ for all $w \in W^+$. Thus $f + W^\perp \in \{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\}$. Thus we have $\overline{\varphi_{W^\perp}(V^{*+})}^{w^*} \subset \{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\}$.

Conversely, if possible let $f + W^\perp \in \{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\} \setminus \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$. Then by the Hahn Banach separation theorem, there is a $w \in W$ such that $f(w) < 0$ and $g(w) \geq 0$ for all $g \in V^{*+}$. Thus $w \in V^+$ so that $v \in W^+$. Therefore $f(w) \geq 0$, which is a contradiction. Hence $\{f + W^\perp : f(w) \geq 0 \ \forall w \in W^+\} = (V^*/W^\perp)^+ \subset \{f \in W^\perp : f(v) \geq 0 \ \forall v + W \in (V/W)^+\}$.

(ii) Let $f \in W^{\perp +}$. Then $f \in W^\perp$ and $f \in V^{*+}$. Thus $f(v) \geq 0$ for all $v \in V^+$ so that $f(v) \geq 0$ for all $v + W \in \varphi_W(V^+)$. We show that $f(v) \geq 0$

for all $v+W \in \overline{\varphi_W(V^+)}^{\|.\|}$. Let $v+W \in \overline{\varphi_W(V^+)}^{\|.\|}$. Then there is a sequence $v_n+W \in \varphi_W(V^+)$ such that $v_n+W \longrightarrow v+W$ in the norm. Since $f \in W^\perp$, we have $f(v_n) \longrightarrow f(v)$. But $f(v_n) \geq 0$ for all $n \in \mathbb{N}$. Therefore we have $f(v) \geq 0$. We know from the definition that $(V/W)^+ = \overline{\varphi_W(V^+)}^{\|.\|}$. Thus $W^{\perp+} \subset \{f \in W^\perp : f(v) \geq 0,\ \forall v+W \in (V/W)^+\}$.

Conversely, let $f \in \{f \in W^\perp : f(v) \geq 0,\ \forall v+W \in (V/W)^+\}$. Let $v \in V^+$. Then $v+W \in (V/W)^+$. Thus $f(v) \geq 0$ so that $f \in V^{*+}$ and $f \in W^{\perp+}$. □

Theorem 6.1.5 *Let $(V, V^+, \|.\|)$ be a order smooth p-normed space and let W be a subspace of V. Let $\varphi_W : V \to V/W$ and $\varphi^*_{W^\perp} : V^* \to V^*/W^\perp$ be the natural homomorphisms. Then:*

(i) $(W, W^+, \|.\|)$ is an order smooth p-normed space if and only if

$$(V^*/W^\perp, (V^*/W^\perp)^+, \|.\|)$$

is an order smooth p'-normed space satisfying $(OS.p'.2)$.

(ii) $(V/W, (V/W)^+, \|.\|)$ is an order smooth p-normed space if and only if $(W^\perp, W^{\perp+}, \|.\|)$ is an order smooth p'-normed space satisfying $(OS.p'.2)$.

Proof. (i) The Banach dual of W is identified with $V^*/W^\perp$ and from (i) of Lemma 6.1.4, we know that $\{f+W^\perp : f(w) \geq 0\ \forall w \in W^+\} = (V^*/W^\perp)^+$. Thus by applying Theorem 2.2.7 between W and $V^*/W^\perp$, we may conclude that $(W, W^+, \|.\|)$ is an order smooth p-normed space if and only if $(V^*/W^\perp, (V^*/W^\perp)^+, \|.\|)$ is an order smooth p'-normed space satisfying $(OS.p'.2)$.

(ii) The Banach dual of V/W is identified with $W^\perp$ and from Lemma 6.1.4, we know that $\{f \in W^\perp : f(v) \geq 0\ \forall v+W \in (V/W)^+\} = W^{\perp+}$. Thus by applying Theorem 2.2.7 between V/W and $W^\perp$, we may conclude that

$(V/W, (V/W)^+, \|.\|)$ is an order smooth p-normed space if and only if $(W^\perp, W^{\perp+}, \|.\|)$ is an order smooth p'-normed space satisfying $(OS.p'.2)$. $\square$

Proposition 6.1.6 *Let $(V, V^+, \|.\|)$ be an order smooth p-normed space and let W be a subspace of V. Then $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$ if and only if $f \in W^{*+}$ implies there is a $g \in V^{*+}$ such that $g_{|W} = f$.*

Proof. First, assume that $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$. Let $f : W \to \mathbb{R}$ be a bounded linear functional such that $f(w) \geq 0$ for all $w \in W^+$. Then by the Hahn Banach separation theorem, there exists a $f_1 : V \to \mathbb{R}$, a bounded linear functional, such that $f_{1_{|W}} = f$ and $\|f_1\| = \|f\|$. Now by Lemma 6.1.4, $f_1 + W^\perp \in \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$. Thus by assumption, there is a $g \in V^{*+}$ such that $f_1 + W^\perp = g + W^\perp$. Therefore $f_{1_{|W}} = g_{|W}$.

Conversely, assume that if $f \in W^{*+}$, then there is a $g \in V^{*+}$ such that $g_{|W} = f$. Let $f + W^\perp \in \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$. Then there exists a net $\{g_\alpha\}$ in V^{*+} such that $g_\alpha + W^\perp \longrightarrow f + W^\perp$ in w^*-topology. Thus $g_\alpha(w) \longrightarrow f(w)$ for all $w \in W^+$. So by assumption, there exists a $g \in V^*$ such that $g_{|W} = f_{|W}$ and $g \in V^{*+}$. Therefore $g + W = f + W$ so that $f + W^\perp \in \varphi_{W^\perp}(V^{*+})$. $\square$

Lemma 6.1.7 *Let $(V, V^+, \|.\|)$ be an order smooth p-normed space and let W be a subspace of V. Then we have the following:*

$$\{\mathfrak{f} + W^{\perp\perp} : \mathfrak{f}(f) \geq 0 \ \forall f \in W^{\perp+}\} = (V^{**}/W^{\perp\perp})^+.$$

Proof. We know from Definition 6.1.3, that $(V^{**}/W^{\perp\perp})^+ = \overline{\varphi_{W^{\perp\perp}}(V^{**+})}^{w^*}$. It is clear from the definition that $\varphi_{W^{\perp\perp}}(V^{**+}) \subset \{F + W^{\perp\perp} : F(f) \geq 0 \ \forall f \in W^{\perp+}\}$. Since $\{\mathfrak{f} + W^{\perp\perp} : \mathfrak{f}(f) \geq 0 \ \forall f \in W^{\perp+}\}$ is a w^*-closed, we get

$$\overline{\varphi(V^{**+})}^{w^*} \subset \{\mathfrak{f} + W^{\perp\perp} : \mathfrak{f}(f) \geq 0 \ \forall f \in W^{\perp+}\}.$$

If possible, let $\mathfrak{g} + W^{\perp\perp} \in \{\mathfrak{f} + W^{\perp\perp} : \mathfrak{F}(f) \geq 0 \; \forall f \in W^{\perp+}\} \setminus \overline{\varphi_{W^{\perp\perp}}(V^{**+})}^{w^*}$. Since $W^{\perp}$ is a predual of $V^{**}/W^{\perp\perp}$. Thus by the Hahn Banach separation theorem, there exists a $g \in W^{\perp}$ such that $\mathfrak{g}(g) < 0$ and $\mathfrak{f}(g) \geq 0$ for all $\mathfrak{f}+W^{\perp\perp} \in \overline{\varphi_{W^{\perp\perp}}(V^{**+})}^{w^{**}}$. Since $\mathfrak{f}(g) \geq 0$ for all $\mathfrak{f} \in V^{**+}$, from Proposition 6.1.1, we have $g \in W^{\perp+}$. Then $\mathfrak{g}(g) \geq 0$, which is a contradiction. Hence we have $\{\mathfrak{f} + W^{\perp\perp} : \mathfrak{f}(f) \geq 0 \; \forall f \in W^{\perp+}\} = (V^{**}/W^{\perp\perp})^{+}$. □

Theorem 6.1.8 *Let $(V, V^{+}, \|.\|)$ be an order smooth p-normed space and let W be a subspace of V. Let $\varphi_W : V \to V/W$ and $\varphi_{W^{\perp}} : V^{*} \to V^{*}/W^{\perp}$ be the natural homomorphisms. Then we have the following duality:*

(i) $(W^{\perp}, W^{\perp+}, \|.\|)$ is an order smooth p'-normed space if and only if

$$(V^{**}/W^{\perp\perp}, (V^{**}/W^{\perp\perp})^{+}, \|.\|)$$

is an order smooth p-normed space satisfying $(OS.p.2)$;

(ii) If $(V^{*}/W^{\perp}, (V^{*}/W^{\perp})^{+}, \|.\|)$ is an order smooth p'-normed space, then $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ is an order smooth p-normed space satisfying $(OS.p.2)$;

(iii) If $\varphi_{W^{\perp}}(V^{*+}) = \overline{\varphi_{W^{\perp}}(V^{*+})}^{w^*}$ and $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ is an order smooth p-normed space, then $(V^{*}/W^{\perp}, (V^{*}/W^{\perp})^{+}, \|.\|)$ is an order smooth p'-normed space.

Proof. (i) The Banach dual of $W^{\perp}$ is identified with $V^{**}/W^{\perp\perp}$. We know from Lemma 6.1.7 that $\{\mathfrak{f} + W^{\perp\perp} : \mathfrak{f}(f) \geq 0 \; \forall f \in W^{\perp+}\} = (V^{**}/W^{\perp\perp})^{+}$. Therefore from Theorem 2.2.7, we conclude that $(W^{\perp}, W^{\perp+}, \|.\|)$ is an order smooth p'-normed space if and only if $(V^{**}/W^{\perp\perp}, (V^{**}/W^{\perp\perp})^{+}, \|.\|)$ is an order smooth p-normed space satisfying $(OS.p.2)$.

(ii) We have the following cone relation on $W^{\perp\perp}$.

$$\begin{aligned} &\{\mathfrak{f} \in W^{\perp\perp} : \mathfrak{f}(f) \geq 0,\ \forall f + W \in (V^*/W^{\perp})^+\} \\ =&\{\mathfrak{f} \in W^{\perp\perp} : \mathfrak{f}(f) \geq 0\ \forall f + W^{\perp} \in \overline{\varphi_{W^{\perp}}(V^{*+})}^{w^*}\} \\ \subset&\{\mathfrak{f} \in W^{\perp\perp} : \mathfrak{f}(f) \geq 0\ \forall f + W^{\perp} \in \varphi_{W^{\perp}}(V^{*+})\} \\ =&\{\mathfrak{f} \in W^{\perp\perp} : \mathfrak{f}(f) \geq 0\ \forall f \in V^{*+}\} \\ =&W^{\perp\perp} \cap V^{**+} = W^{\perp\perp+}. \end{aligned} \tag{6.1.1}$$

We know that $W^{\perp\perp+}$ is proper and closed. Since $(V^*/W, (V^*/W)^+, \|.\|)$ is an order smooth p'-normed space, by Theorem 2.2.7, $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ is an order smooth p'-normed space satisfying $(OS.p.2)$ with respect to the cone $\{\mathfrak{f} \in W^{\perp\perp} : \mathfrak{f}(f) \geq 0,\ \forall f + W \in (V^*/W^{\perp})^+\}$. Since $\{\mathfrak{f} \in W^{\perp\perp} : \mathfrak{f}(f) \geq 0,\ \forall f + W \in (V^*/W^{\perp})^+\} \subset W^{\perp\perp+}$ is a proper closed cone, thus $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ is also an order smooth p-normed space satisfying $(OS.p.2)$.

(iii) Let $\varphi_{W^{\perp}}(V^{*+}) = \overline{\varphi_{W^{\perp}}(V^{*+})}^{w^*}$. Then by equation 6.1.1, we can easily check that $\{\mathfrak{f} \in W^{\perp\perp} : \mathfrak{f}(f) \geq 0,\ \forall f + W \in (V^*/W^{\perp})^+\} = W^{\perp\perp+}$. Therefore if $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ is an order smooth p-normed space, then its predual $(V^*/W^{\perp}, (V^*/W^{\perp})^+, \|.\|)$ is an order smooth p'-normed space. □

Corollary 6.1.9 *Let $(V, V^+, \|.\|)$ be an order smooth p-normed space and let W be a subspace of V. Then*

(i) $(W, W^+, \|.\|)$ is an order smooth p-normed space if and only if $(W^{\perp\perp}, W^{\perp\perp+}, \|.\|)$ is an order smooth p-normed space of V^{**} satisfying $(OS.p.2)$.

(ii) If $\varphi_{W^{\perp}}(V^{*+}) = \overline{\varphi_{W^{\perp}}(V^{*+})}^{w^*}$, then $(V/W, (V/W)^+, \|.\|)$ is an order smooth p-normed space if and only if $(V^{**}/W^{\perp\perp}, (V^{**}/W^{\perp\perp})^+, \|.\|)$ is an order smooth p-normed space satisfying $(OS.1.2)$.

We summarize this observations in the form of the following notion.

Definition 6.1.10 *If $(V, V^+, \|.\|)$ is an order smooth p-normed space. Then a subspace W is called* smooth p-order ideal *in V if W satisfies following conditions:*

(i) $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$;

(ii) $(W, W^+, \|.\|)$ *is an order smooth p-normed space;*

(iii) $(V/W, (V/W)^+, \|.\|)$ *is an order smooth p-normed space.*

Remark 6.1.11 *If W is smooth p-order ideal, then $W, W^\perp$, and $W^{\perp\perp}$ are order ideals.*

6.2 Smooth ∞-order ideals

Let W be an order smooth ∞-normed space of an order smooth ∞-normed space $(V, V^+, \|.\|)$. By Theorem 4.2.5, for every $f \in W^{*+}$, there is a $g \in V^{*+}$ such that $g_{|W} = f$. Thus by Proposition 6.1.6, we may conclude that $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$. Hence, we write a quick corollary for the notion of a smooth ∞-order ideal.

Corollary 6.2.1 *Let $(V, V^+, \|.\|)$ be an order smooth ∞-normed space, and let W be a subspace of V. Then W is a smooth ∞-order ideal if and only if W satisfies following conditions:*

(i) $(W, W^+, \|.\|)$ *is an order smooth ∞-normed space;*

(ii) $(V/W, (V/W)^+, \|.\|)$ *is an order smooth ∞-normed space.*

Theorem 6.2.2 *Let $(V, V^+, \|.\|)$ be an order smooth ∞-normed space and let W be a subspace of V. Then the following conditions are equivalent:*

(i) $((V/W), (V/W)^+, \|.\|)$ *is an order smooth ∞-normed space;*

(ii) $(W^\perp, W^{\perp+}, \|.\|)$ *satisfying* $(OS.1.2)$*;*

(iii) $\|v + W\| = \sup\{|f(v)| : f \in (W^\perp)_1 \cap W^{\perp+}\}$*;*

(iv) $\|\mathfrak{f} + W^{\perp\perp}\| = \sup\{|\mathfrak{f}(f)| : f \in (W^\perp)_1 \cap W^{\perp+}\}$*;*

(v) $((V^{**}/W^{\perp\perp}), (V^{**}/W^{\perp\perp})^+, \|.\|)$ *is an order smooth ∞-normed space.*

Proof. It is clear that (i), (ii), (v) are equivalent and (iv) implies (iii). Thus it is sufficient to prove that (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iv).

(ii) $\Longrightarrow$ (iii): Let $v \in V$, then we have

$$\begin{aligned}\|v + W\| &= \sup\{|f(v)| : f \in (W^\perp)_1\} \\ &= \sup\{|f(v)| : f \in co((W^{\perp+} \cap\ (W^\perp)_1 \cup -(W^{\perp+} \cap\ (W^\perp)_1)\} \\ &= \sup\{|f(v)| : f \in (W^\perp)_1 \cap W^{\perp+}\}.\end{aligned}$$

(iii) $\Longrightarrow$ (i): Let $(W^\perp, W^{\perp+}, \|.\|)$ satisfy $(OS.1.2)$. Since $(V, V^+, \|.\|)$ is an order smooth ∞-normed space. Thus by Theorem 2.2.7, $(V^*, V^{*+}, \|.\|)$ is an order smooth 1-normed space satisfying $(OS.1.2)$. Since $W^\perp \subset V^*$, thus $W^\perp$ satisfying $(O.1.1)$. Since $W^\perp$ satisfies $(OS.1.2)$, thus $(W^\perp, W^{\perp+}, \|.\|)$ is an order smooth 1-normed space satisfying $(OS.1.2)$. Therefore by Theorem 6.1.5, $(V/W, (V/W)^+, \|.\|)$ is an order smooth ∞-normed space.

(ii) $\Longrightarrow$ (iv): Let $F \in V^{**}$, then we have

$$\|F + W\| = \sup\{|F(f)| : f \in (W^\perp)_1\}$$

$$= \sup\{|F(f)| : f \in co((W^{\perp +} \cap \ (W^{\perp}))_1 \cup -(W^{\perp +} \cap \ (W^{\perp})_1)\}$$
$$= \sup\{|F(f)| : f \in (W^{\perp})_1 \cap W^{\perp +}\}.$$

□

6.2.1 M-ideals and smooth ∞-order ideals

Lemma 6.2.3 *Let $(V, V^+, \|.\|)$ be an order smooth 1-normed space satisfying $(OS.1.2)$ and let W be an L-summand in V. If $u \in W$, then $C(u) \subseteq W$.*

Proof. Let $w \in W \setminus \{0\}$. Let $u \in C(w)$ and without loss of generality we may assume that $\|u\| = 1$. Then by definition of $C(w)$, there is a $v \in V_1$ such that $\lambda u + (1-\lambda)v = \frac{w}{\|w\|}$ for some $\lambda \in (0,1)$. By the triangle inequality, $\|v\| = 1$. Since $u, v \in V^*$ and let W is a subspace of a complete normed space, there are $u_1, v_1 \in W$ and $u_2, v_2 \in W'$ such that

$$u = u_1 + u_2 \qquad \|u\| = \|u_1\| + \|u_2\|,$$
$$v = v_1 + v_2 \qquad \|v\| = \|v_1\| + \|v_2\|.$$

Now, $\frac{w}{\|w\|} = \lambda u_1 + (1-\lambda)v_1 + \lambda u_2 + (1-\lambda)v_2$. We can rewrite it as

$$(\lambda u_1 + (1-\lambda)v_1 - \frac{w}{\|w\|}) + \lambda u_2 = -(1-\lambda)v_2,$$
$$(\lambda u_1 + (1-\lambda)v_1 - \frac{w}{\|w\|}) + (1-\lambda)v_2 = -\lambda u_2.$$

Since $\lambda u_1 + (1-\lambda)v_1 - \frac{w}{\|w\|} \in W$ and $u_2, v_2 \in W'$, and W is an L-summand, from last two equations, we get the following norm equalities:

$$\|\lambda u_1 + (1-\lambda)v_1 - \frac{w}{\|w\|}\| + \lambda\|u_2\| = (1-\lambda)\|v_2\|,$$
$$\|\lambda u_1 + (1-\lambda)v_1 - \frac{w}{\|w\|}\| + (1-\lambda)\|v_2\| = \lambda\|v_2\|,$$

Now, it follows that $\lambda u_1 + (1-\lambda)v_1 = \frac{w}{\|w\|}$. Since $\|u_1\|, \|v_1\| \leq 1$, by the triangle inequality, $\|u_1\| = 1 = \|u_2\|$ so that $u_2 = 0 = v_2$. Hence $C(w) \subset W$. □

The following result can be proved on the lines of Theorem 5.3.7.

Lemma 6.2.4 *Let* $(V, V^+, \|.\|)$ *be an order smooth* 1*-normed space satisfying* $(OS.1.2)$. *If* L *is an* L*-projection of* V^*, *then* L *is a positive linear map.*

Lemma 6.2.5 *Let* $(V, V^+, \|.\|)$ *be an order smooth* ∞*-normed space and let* W *be an* M*-ideal in* V *so that* $V^* = W^\perp \oplus_1 W^{\perp'}$, *where* $W^{\perp'}$ *is the complemented subspace of* $W^\perp$. *If* $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$, *then* $(V^*/W^\perp, (V^*/W^\perp)^+, \|.\|)$ *is isometrically order isomorphic to* $(W^{\perp'}, W^{\perp'+}, \|.\|)$.

Proof. Let P be the L-projection of V^* onto $W^{\perp'}$. We define a map $\varphi : V^*/W^\perp \to W^{\perp'+}$ by

$$\varphi(f + W^\perp) = P(f)$$

for all $f \in V^*$. Let $f \in V^*$ such that $P(f) = 0$. Then $f \in W^\perp$ so that $f + W^\perp = W^\perp$. Hence φ is well defined. Let $f \in V^*$. We show that $\|f + W^\perp\| = \|P(f)\|$. Since $f - P(f) \in W^\perp$, we have $f + W^\perp = P(f) + W^\perp$. Since P is an L-projection on $W^{\perp'}$, thus $\|P(f) + g\| = \|P(f)\| + \|g\|$ for all $g \in W^\perp$. Thus $\|P(f)\| = \|f + W^\perp\|$, so that φ is isometry onto $W^{\perp'}$. By assumption, $(V^*/W^\perp)^+ = \varphi_{W^\perp}(V^{*+})$. Let $f \in V^{*+}$. Since P is an L-projection, by Lemma 6.2.4, P is a positive map so that $P(f) \in W^{\perp'+}$. Hence φ is a positive map. Conversely, if $g \in W^{\perp'+}$, then $g \in V^{*+}$ and $\varphi(g + W^\perp) = P(g) = g$. Thus

φ^{-1} is also a positive map. Hence $\varphi : V^*/W^\perp \to W^{\perp'}$ is an isometrical order isomorphism. □

Lemma 6.2.6 *Let $(V, V^+, \|.\|)$ be an order smooth ∞-normed space and let W be an M-ideal in V so that $V^* = W^\perp \oplus_1 W^{\perp'}$, where $W^{\perp'}$ is the complemented subspace of $W^\perp$. Then $(W^{\perp'}, W^{\perp'+}, \|.\|)$ is an order smooth 1-normed space.*

Proof. Since $W^{\perp'} \subset V^{*+}$, $W^{\perp'}$ satisfies $(O.1.1)$. We prove that $W^{\perp'}$ satisfies $(OS.1.2)$. Let $f \in W^{\perp'}$. Since $W^{\perp'}$ is an L-summand of an order smooth 1-normed space V^* satisfying $(OS.1.2)$, by Lemma 6.2.3, we may conclude that $C(f) \in W^{\perp'}$. Since $f \in V^*$ and V^* satisfies $(OS.1.2)$, there are $g, h \in V^{*+}$, such that $f = g - h$ and $\|f\| = \|g\| + \|h\|$. By [1, Lemma 2.3, part I], we have $g, -h \in C(f)$ so that $g, h \in W^{\perp'+}$. Thus $W^{\perp'}$ is an order smooth 1-normed space satisfies $(OS.1.2)$. □

Proposition 6.2.7 *Let $(V, V^+, \|.\|)$ be an order smooth ∞-normed space and let W be a subspace of V. If W is an M-ideal, then $((V/W), (V/W)^+, \|.\|)$ is an order smooth ∞-normed space.*

Proof. Let $f \in W^\perp$. Since V^* satisfies $(OS.1.2)$, there are $g, h \in V^{*+}$ such that $f = g - h$, and $\|f\| = \|g\| + \|h\|$. Thus by Lemma 2.1.1. we have $g, -h \in C(f)$. Since $W^\perp$ is an L-summand, and $f \in W^\perp$, we have $g, -h \in C(f)$ so that $g, h \in W^{\perp+}$. Hence $W^\perp$ satisfies $(OS.1.2)$. Therefore by Theorem 6.2.2, $(V/W, (V/W)^+, \|.\|)$ is an order smooth ∞-normed space.

□

Theorem 6.2.8 *Let $(V, V^+, \|.\|)$ be an order smooth ∞-normed space and let W be a subspace of V. If W is an M-ideal, then following are equivalent:*

(i) $(W, W^+, \|.\|)$ *is an order smooth* ∞*-normed space;*

(ii) For $f \in W^{*+}$*, there is a* $g \in V^{*+}$ *such that* $g_{|W} = f$*;*

(iii) $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$*;*

(iv) $\|f\| = \sup\{f(w) : w \in W^+ \cap W_1\}$ *for all* $f \in W^{*+}$*.*

Proof. We note the following:

(i) $\implies$ (ii): It follows from Theorem 4.2.5.

(ii) $\implies$ (iii): It follows from 6.1.6.

(iii) $\implies$ (i): Since W is an M-ideal, by Lemma 6.2.6, $(W^{\perp'}, W^{\perp'}+, \|.\|)$ is an order smooth 1-normed space satisfying $(OS.1.2)$. As $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$, thus by Lemma 6.2.5, we have $(V^*/W^\perp, (V^*/W^\perp)^+, \|.\|)$ is an order smooth 1-normed space satisfies $(OS.1.2)$. Hence by Theorem 6.1.5, $(W, W^+, \|.\|)$ is an order smooth ∞-normed space.

(iv) $\implies$ (ii): Let f be a positive bounded linear functional on W. By the Hahn Banach Theorem, there exists a $g \in V^*$ such that $g_{|W} = f$, and $\|g\| = \|f\|$. We claim that g is positive. Since V^* satisfy $(OS.1.2)$, there are $g_1, g_2 \in V^{*+}$ such that

$$g = g_1 - g_2 \text{ with } \|g\| = \|g_1\| + \|g_2\|.$$

Since $g_1, g_2 \in V^{*+}$, and V^* is complete, by Theorem 4.2.1, there are $g_{11}, g_{21} \in W^{\perp+}$ and $g_{12}, g_{22} \in W^{\perp'+}$ such that $g_1 = g_{11} + g_{12}$ with $\|g_1\| = \|g_{11}\| + \|g_{12}\|$ and $g_2 = g_{21} + g_{22}$ with $\|g_2\| = \|g_{21}\| + \|g_{22}\|$. Now $g = g_{11} - g_{21} + g_{12} - g_{22}$, where $g_{11}, g_{21} \in W^{\perp+}$ and $g_{12}, g_{22} \in W^{\perp'+}$ such that $\|g\| = \|g_{11}\| + \|g_{21}\| + \|g_{12}\| + \|g_{22}\|$. If $f_{ij} = g_{ij}|_W$ for all $i, j \in \{1, 2\}$. Then $f_{11} = f_{21} = 0$ so that $f = f_{12} - f_{22}$. Further, as f is positive, we have $0 \le f \le f_{12}$. Let $\epsilon > 0$, then by assumption, there exist $w \in W^+ \cap W_1$ such that $\|f\| - \epsilon < f(w)$. Since $0 \le f \le f_1$, thus we have $0 \leq f(w) \leq f_{12}(w)$. Now $\|f\| - \epsilon \leq \|f_{12}\|$ and ϵ is arbitrary, we have $\|f\| \leq \|f_{12}\|$. Therefore,

$$\begin{aligned} \|f\| &\leq \|f_{12}\| \\ &\leq \|g_{11}\| + \|g_{21}\| + \|g_{12}\| + \|g_{22}\| \\ &= \|g\| = \|f\| \end{aligned}$$

and consequently, $g_{11} = g_{21} = g_{22} = 0$. Hence $g = g_{12} \in V^{*+}$.

(i) $\Longrightarrow$ (iv): Let $f \in W^{*+}$. Let $\epsilon > 0$, then there exist $w \in W$ and $\|w\| < 1$ such that $\|f\| - \epsilon < f(w)$. Since W is an order smooth ∞-normed space, there exist $w_1, w_2 \in W^+$ such that $w = w_1 - w_2$ and $\max\{\|w_1\|, \|w_2\|\} < 1$. Since $w_1, w_2 \geq 0$, we have $f(w_1), f(w_2) \geq 0$. Now we have $\|f\| - \epsilon < f(w) \leq f(w_1) \leq \sup\{f(w) : w \in W^+ \cap W_1\}$. Since $\epsilon > 0$ ia an arbitrary, we have $\|f\| = \sup\{f(w) : w \in W^+ \cap W_1\}$ for all $f \in W^{*+}$. □

Theorem 6.2.9 *Let $(V, V^+, \|.\|)$ be an order smooth 1-normed space satisfying $(OS.1.2)$ and let W be a subspace of V. If W is an L-summand, then W is a smooth 1-order ideal in V.*

Proof. To prove W is an order smooth 1-normed space, it suffices to show that W satisfies $(OS.1.2)$. Let $w \in W$. Since V satisfies $(OS.1.2)$, there are $u, v \in V^+$ such that $w = u - v$ and $\|w\| = \|u\| + \|v\|$. By Lemma 2.1.1, we have $u, -v \in C(w)$. Also by Lemma 6.2.3, we have $v, -w \in W$. Thus W satisfying $(OS.1.2)$.

We claim that $\varphi_{W^\perp}(V^{*+}) = \overline{\varphi_{W^\perp}(V^{*+})}^{w^*}$. Let $f : W \to \mathbb{R}$ be a positive linear functional. Let P be the L-projection of V onto W. Then by Lemma 6.2.4, P is a positive linear map. Let $g(v) = f(P(v))$ for all $v \in V$. Then $g : V \to \mathbb{R}$ is a positive linear map such that for all $w \in W$, we have $g(w) = f(P(w)) = f(w)$. Hence by proposition 6.1.6, we have $\varphi_{W^{\perp}}(V^{*+}) = \overline{\varphi_{W^{\perp}}(V^{*+})}^{w^*}$.

Since W is an L-summand, there is a unique subspace W' of V such that $V = W \oplus_1 W'$. Since W' is also an L-summand of V, $(W', W'^{+}, \|.\|)$ is an order smooth 1-normed space satisfying $(OS.1.2)$. We define a map $\varphi : V/W \to W'$ by

$$\varphi(v + W) = Q(v)$$

for all $v \in V$, where Q is the L-projection of V onto W'. It is easy to check that φ is an isometry onto W'. We claim that $\varphi_W(V^+) = \overline{\varphi_W(V^+)}^{\|.\|}$. So let $v_n \in V^+$ such that $v_n + W \to v + W$ for some $v \in V$. Since Q is an L-projection, by Lemma 6.2.4, Q is a positive linear map. Thus $Q(v_n) \in V^+$ is positive. Since $\|Q(v_n) - Q(v)\| = \|v_n - v + W\| \longrightarrow 0$, we have $Q(v) \geq 0$. Since $v - Q(v) \in \ker(Q)(= W)$, we have $v + W = Q(v) + W$. Therefore $\varphi_W(V^+) = \overline{\varphi_W(V^+)}^{\|.\|}(= (V/W)^+)$. Let $v + W \in (V/W)^+$. Since $(V/W)^+ = \varphi_W(V^+)$, with out loss of generality, we may assume $v \in V^+$. Since Q is an L-projection, Q is a positive map. Since $v \in V^+$, we have $\varphi(v + W) = Q(v) \geq 0$. Therefore φ is a positive map.

Conversely, let $v \in W^+$. Now $\varphi(v + W) = Q(v) = v$, φ^{-1} is also a positive linear map. Since $\varphi : V/W \to W'$ is an isometrical order isomorphism, and $(W', W'^{+}, \|.\|)$ is an order smooth 1-normed space satisfying $(OS.1.2)$, thus $((V/W), (V/W)^+, \|.\|)$ is an order smooth 1-normed space which satisfies $(OS.1.2)$ property. □

Bibliography

[1] E. M. Alfsen, E. G. Effros, *Structure in real Banach spaces I and II*, Ann. of Math. **96** (1972), 98–173.

[2] E. M. Alfsen, *Compact convex sets and bounded integrals*, Springer Verlag, Berlin-Heidelberg-New York, (1971).

[3] E. M. Alfsen, T. B. Andersen, *Split faces of compact convex sets*, Proc. Lond. Math. Soc. **21** (1970), 415–442.

[4] T. B. Anderson, *On dominated extension of continuous affine functions on split faces*, Math. Scand. **29** (1971), 298–306.

[5] W. Arveson, *Subalgebras of* C**-algebras, I*, Acta. Math. **123** (1969), 142–224.

[6] L. Asimow, *Complementary cones in dual Banach spaces*, Illinois J. Math. **18** (1974), 657–668.

[7] L. Asimow, *Monotone extension in ordered Banach spaces and their duals*, J. Lond. Math. Soc. **6** (1973), 563–569.

[8] L. Asimov, *Well-capped convex cones*, Pacific J. Math. **26** (1968), 421–431.

[9] D. P. Blecher, *Positivity in operator algebras and operator spaces*, Positivity, Trends Math., Birkhäuser, Basel, 27–71, (2007).

[10] D. P. Blecher, M. Neal, *Open partial isometries and positivity in operator spaces*, Studia Math. **182** (2007), no. 3, 227–262.

[11] D. P. Blecher, K. Kirkpatrick, M. Neal, W. Werner. *Ordered involutive operator spaces*, Positivity **11** (2007), 497–510.

[12] D. P. Blecher, V. I. Paulsen, *Tensor products of operator spaces*, J. Funct. Anal. **99** (1991), 262–292.

[13] D. P. Blecher, *Tensor products of operator spaces II*, Canad. J. Math. **44** (1992), 75–90.

[14] D. P. Blecher, Z. J. Ruan, Allan M. Sinclair *A characterization of operator algebras*, J. Funct. Anal. **89** (1990), 188–201.

[15] F. Bonsal, *Regular ideals of partially ordered vector spaces*, Proc. Lond. Math. Soc. (1956), 626–640.

[16] F. Bonsal, *Sublinear functional and ideals in a partially ordered vector spaces*, Proc. Lond. Math. Soc. **3** (1954), 402–418.

[17] J. Bunce, *A note on two sided ideals in* C^**-algebras*, Proc. Amer. Soc. **28** (1971), 635.

[18] M. D. Choi, E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209.

[19] M. D. Choi, E. G. Effros, *Lifting problems and the cohomology of* C^**-algebras*, Canad. J. of Math. (1977), 1092–1111.

[20] F. Cunningham Jr., *L-structure in L-spaces*, Trans. Amer. Math. Soc. **95** (1960), 275–299.

[21] J. Diximier, *Les algbrès d'operateurs dans l'espace Hilbertien*, Paris, (1957).

[22] E. G. Effros, *Order ideals in a* C*-*algebra*, Duke Math. J. **30** (1963), 391–411.

[23] D. A. Edwards, *The homeomorphic affine embedding of a locally compact cone into a Banach dual space endowed with the vague topology*, Proc. Lond. Math. Soc. **14** (1964), 399–414.

[24] E. G. Effros, Z. J. Ruan, *Operator spaces*, Oxford University press, New York, (2000).

[25] E. G. Effros, S. Winkler, *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, J. Funct. Anal. **144** (1996), 117–152.

[26] E.G. Edward. Z. J. Ruan, *Mapping spaces and liftings for operator spaces*, Proc. Lond. Math. Soc. **69** (1994), 171–197.

[27] E. G. Effros, Z. J. Ruan, *On the abstract characterization of operator spaces*, Proc. Amer. Math. Soc. **119** (1993), 579–584.

[28] E. G. Effors, Z. J. Ruan, *On matricially normed spaces*, Pacific J. Math. **132** (1988), 243–299.

[29] E. G. Effros, *Aspects of non commutative order in* C*-*algebras and applications to physics*, (Proc. Second Japan-USA Sem., Los Angeles,

Calif., 1977), **650** (1978), Lecture notes in Math., Springer and Berlin 1–40.

[30] A. J. Ellis, *The duality of partially ordered normed spaces*, J. Lond. Math. Soc. **39** (1964), 730–744.

[31] I. Gelfand, M. Neumark,*On the imbedding of normed rings into the ring of operators in Hilbert space*, Rec. Math. [Mat. Sbornik] N.S. **12** (1943) (54), Number 2, 197–217.

[32] A. Ghatak, A. Karn, *M-ideals and split faces of the quasi state spaces of a non-unital ordered Banach spaces*, To appear in Positivity (DOI: 10.1007/s11117-018-0614-1).

[33] A. Ghatak, *Cones, ideal and their duality in ordered smooth p-normed spaces*, Communicated.

[34] A. Ghatak, A. K. Karn, *Quantization of $A_0(K)$ spaces*, Communicated.

[35] A. Ghatak, A..K. Karn *CM-ideals and L^1-matricial split faces*, Communicated.

[36] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Springer-verlag, Berlin, Lecture Notes in Math. **1547** (1993).

[37] G. Jameson, *Order linear spaces*, Springer-Verlag, Berlin, Lecture Notes No. **141** (1970).

[38] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, G.S.M. Amer. Math. Soc. Vol.1, **15** (1997).

[39] R. V. Kadison, *A representation theory for commutative topological algebras*, Mem. Amer. Math. Soc. **7** (1951).

[40] A. K. Karn, *Order embedding of matrix ordered spaces*, Bull. Aust. Math. Soc. **84** (2011), 10–18.

[41] A. K. Karn, *A p-theory of order normed spaces*, Positivity **14** (2010), 441–458.

[42] A. K. Karn, *Adjoining an order unit in a matrix ordered space*, Positivity **9** (2005), 207–223.

[43] A . K. Karn, R. Vasudevan, *Matrix norms in matrix ordered spaces*, Glas. Mat. Ser. III 32(52) (1997), 87–97.

[44] A. K. Karn, R. Vasudevan, *Approximate matrix order unit spaces*, Yokohama Math. J. **44** (1997), 73–91.

[45] L. Kantorovič, *Lineare ha⊥bgeordenete răume*, Mat. Sbornik **44** (1937), 121–168.

[46] S. Kakutani, *Concrete representation of abstract M-space*, Ann. of Math. **42** (1941), 523–557.

[47] A. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. **227** (1977), 1–62.

[48] K. F. Ng, *The duality of partially ordered Banach spaces*, Proc. Lond. Math. Soc. **3** (1969), 269–288.

[49] V. I. Paulsen, I. G. Todorov, M. Tomforde, *Operator system structures on ordered spaces*, Proc. Lond. Math. Soc. **102** (2011), 25–49.

[50] V. I. Paulsen, M. Tomforde, *Vector spaces with an order unit*, Indiana University Math. J. **58** (2009), 1319–1359.

[51] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics **78** Cambridge University Press, Cambridge (2002).

[52] Y. T. Poon, Z. J. Ruan, *Operator algebras with contractive approximate identities*, Canad. J. Math. **46** (1994), 397–414.

[53] V. I. Paulsen, R. I. Leobel, *Some remarks on* C^**-convexity*, Linear Algebra and its Applications **3** (1981), 63–78.

[54] Z. J. Ruan, *Subspaces of* C^**-algebras*, J. Funct. Anal. **29** (1988), 217–230.

[55] W. J. Schreiner, *Matrix regular operator spaces*, J. Funct. Anal. **152** (1998), 136–175.

[56] W. F. Stinespring, *Positive functions of* C^**-algebras*, Proc. Amer. Soc. **315** (1955), 211–216.

[57] E. Størmer, *On partially ordered vector spaces and their duals, with applications to simplexes and* C^**-algebras*, Proc. Lond. Math. Soc. **18** (1968), 245–265.

[58] E. Størmer, *Two-sided ideals in* C^**-algebras*, Bull. Amer. Math. Soc. **73** (1967), 254–257.

[59] R. R. Smith, J. D. Ward, *M-ideals structure in Banach Algebras*, J. Funct. Anal. **29** (1978), 337–349.

[60] C. Webster, S. Winkler, *Krein Milman theorem for operator convexity*, Trans. Amer. Math. Soc. **351** (1999), 307–322.

[61] W. Werner, *Multipliers on matrix ordered operator spaces and some K-groups*, J. Funct. Anal. **206** (2004), 356–378.

[62] W. Werner, *Subspaces of $L(H)$ that are $*$-invariant*, J. Funct. Anal. **193** (2002), 207–223.

[63] G. Wittstock, *On matrix order and convexity*, Functional Analysis: Survey and Recent Results, Math. studies, North-Holland, Amsterdam, **90** (1984), 175–188.

[64] G. Wittstock, *Ein operator wertiger Hahn-Banach Satz*, J. Funct. Anal. **40** (1981),127–150.

[65] Y. C. Wong, K. F. Ng, *Partially ordered topological vector spaces*, Oxford University Press, (1973).