Lipschitz operators, their generalization and related operator ideals

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ARINDAM MANDAL

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Name	: Anil Kumar Karn
Designation	: Professor
Department/ Centre	: School of Mathematical Sciences
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Arindam Mandal

Arindam Mandal

Dedicated

to

my mother

Smt. Dipali Pal

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Arindam Mandal

Arindam Mandal

ABSTRACT

Over the past few decades, Lipschitz spaces and their structural properties have received significant attention, especially in nonlinear functional analysis. An important result in this direction was obtained in 1964 by Joram Lindenstrauss [42, Theorem 2]. He proved the existence of a continuous linear projection from the Banach space $Lip_0(X, \mathbb{R})$ onto the linear dual space $L(X, \mathbb{R})$.

One of the primary aims of this thesis is to generalize the Lindenstrauss's result (particularly when the codomain is general Banach space Y) both in Lipschitz and two-Lipschitz cases. We study the structure of Lipschitz operator spaces and two-Lipschitz operator spaces, together with their quotient spaces. We also give a necessary and sufficient condition for a Lipschitz map to be linear, and for a two-Lipschitz map to be bilinear.

As an application, we identify ker($\beta_{\mathbb{R}}$) as the predual of the quotient space $L^\infty(\mathbb{R})/\tilde{\mathbb{R}}$ (respectively, $L^\infty(\mathbb{R}^2)/\tilde{\mathbb{R}}$), where $\tilde{\mathbb{R}}$ denotes the space of constant functions.

Motivated by classical results of Lindenstrauss we further investigate quotient spaces of the form $Lip_0(X)/\mathcal{A}$, where $\mathcal{A}$ is a finite-dimensional subspace. We prove that these quotients are dual spaces and provide an explicit description of their preduals.

We then focus on the space $Lip_0^{ph}(X)$ of positively homogeneous real-valued Lipschitz functions. This space satisfies

$$X^* \subsetneq Lip_0^{ph}(X) \subsetneq Lip_0(X).$$

We show that $Lip_0^{ph}(X)$ is a dual space and also the preannihilator of a closed subspace of the Lipschitz-free space. As a consequence, the quotient $Lip_0(X)/Lip_0^{ph}(X)$ is also a dual space. Moreover, with a suitable multiplication, $Lip_0^{ph}(X)$ becomes a Banach algebra.

Finally, we introduce the notion of extensively bounded mappings, which generalize Lipschitz mappings, and study the associated nonlinear operator ideals. With an appropriate modification in the linear setting, we define extensively p-summing mappings. We establish Pietsch-type domination and factorization theorems for these mappings. We also study extensively bounded finite-rank, compact, p-compact mappings, p-integral, p-nuclear operators and describe the corresponding operator ideals.

Contents

Summary

One of the main purposes of this thesis is to explore the position of the space of bounded linear operators $L(X, Y)$ between Banach spaces X and Y inside the larger space of zero-preserving Lipschitz maps $Lip_0(X, Y)$. Since $L(X, Y)$, with the usual operator norm, is a closed subspace of $Lip_0(X, Y)$ endowed with the Lipschitz norm, a natural question arises: is $L(X, Y)$ complemented in $Lip_0(X, Y)$? In Chapter 2, we provide a positive answer to this when Y is a dual space. This extends the Lindenstrauss's result [42, Theorem 2]. However, in one of the examples we see that there exists a norm-one projection from $Lip_0(\ell_1, Y)$ onto $L(\ell_1, Y)$ for any Banach space Y. Hence, while the condition that Y is a dual space is sufficient for $L(X, Y)$ to be complemented in $Lip_0(X, Y)$, it is not necessary. Moreover, we also examine $Lip_0(X, Y) \big/ L(X, Y)$ as an operator space, whenever Y is injective.

While exploring this problem, we became interested in identifying an infinite-dimensional $weak^*$ closed subspace of $Lip_0(X)$ that properly contains X^*. In Chapter 3, we study one such space, namely the space of positive homogeneous Lipschitz maps, and discuss its various interesting properties.

More recently, Lipschitz theory has been extended to the two-variable setting. In addition to the one-variable case, Chapter 4 studies projections and various quotient structures on the space of two-Lipschitz maps that vanish on the coordinate axes.

It is well known that Lipschitz maps can be studied through the notion of the modulus of continuity. We slightly modify the notion of modulus of continuity and introduce the modulus of continuity at a point. In particular, the modulus of continuity at the origin characterizes maps that are Lipschitz at the origin. Such maps provide a natural generalization of Lipschitz maps and extend the idea of bounded linear operators between Banach spaces. Motivated by this, we define a broader class of maps $E(X, Y)$ between the Banach spaces X and Y, called *extensively bounded maps*. Furthermore, we show that $E(X, Y)$ forms a Banach space with respect to a suitable norm $\|\cdot\|_e$ (defined later). In particular, $(E(X, \mathbb{R}), \|\cdot\|_e)$ is linearly isometrically isomorphic to $\ell_\infty(X \setminus \{0\}, \mathbb{R})$ (space of all bounded maps on $X \setminus \{0\}$). Therefore, although $E(X, \mathbb{R})$ forms a Banach algebra under pointwise addition and multiplication, since it is isomorphic to $\ell_\infty(X \setminus \{0\}, \mathbb{R})$ (a well-established structure), this multiplication is not of significant interest.

On the other hand, the space of extensively bounded maps on X, denoted by $E(X, X)$, carries a natural algebraic structure: pointwise addition serves as the additive operation, while composition acts as the multiplication between extensively bounded maps. We develop this structure within the framework of (nonlinear) operator ideal theory and introduce a nonlinear Banach operator ideal theory in this setting. As an application, we study extensively finite rank, extensively compact, extensively p-summing, p-integral, and p-nuclear operator ideals, thereby extending their classical linear counterparts.

Chapter 1

Introduction

In the literature Lipschitz maps between two Banach spaces have been considered as a non-linear generalization of bounded linear operators between two Banach spaces. Mathematicians have been studying these maps from several point of views, for example their operator ideals, their geometry, their relation with group theory, approximation property etc. In this thesis, we also consider Lipschitz maps and focus on two concepts. One, we discuss the existence of projection from the space of (two-)Lipschitz maps onto the space of bounded (bi-)linear maps. Two, we discuss a generalization of Lipschitz maps and study an operator ideal theory corresponding to these maps.

Let (M, d_M) and (N, d_N) be metric spaces. A mapping $f : M \to N$ is said to be *Lipschitz* if there exists a constant $k > 0$ such that

$$d_N\left(f(x_1), f(x_2)\right) \leq k\, d_M(x_1, x_2) \quad \text{for all } x_1, x_2 \in M.$$

The notion of Lipschitz continuity was introduced by Rudolf Lipschitz in the early ninteenth century. It is evident from the definition that the notion of Lipschitz continuity is stronger than that of uniform continuity. In fact, these notions can be described using the *modulus of continuity* which was introduced by Henri Lebesgue in 1910 (see [41]). Let $f : M \to N$ and $t > 0$. Define

$$\omega_f(t) := \sup\{d_N(f(x), f(y)) : d_M(x, y) \leq t\}.$$

The function $\omega_f : (0, \infty) \to [0, \infty]$ is called the modulus of continuity of f and is monotone increasing. A function f is uniformly continuous on M if and only if

$$\lim_{t \to 0^+} \omega_f(t) = 0,$$

whereas f is Lipschitz on M if and only if

$$\sup \left\{ \frac{\omega_f(t)}{t} : t > 0 \right\} < \infty.$$

If X and Y are normed linear spaces, then any bounded linear map $T : X \to Y$ is Lipschitz (with $T(0) = 0$). By $Lip_0(X, Y)$, we denote the space of all Lipschitz maps from X to Y that satisfy $f(0) = 0$. Then $Lip_0(X, Y)$ is a normed linear space when equipped with the Lipschitz norm

$$Lip(f) := \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|}.$$

If $T \in L(X, Y)$, the space of all bounded linear maps from X to Y, then $Lip(T) = \|T\|$. Thus $L(X, Y)$ is a closed subspace of $Lip_0(X, Y)$. $(Lip_0(X, Y), Lip(\cdot))$ is called a Lipschitz space.

Over the past few decades, Lipschitz spaces and their structural properties have received significant attention, especially in nonlinear functional analysis. An important result in this direction was obtained in 1964 by Joram Lindenstrauss [42, Theorem 2]. He proved the existence of a continuous linear projection from the Banach space $Lip_0(X, \mathbb{R})$ onto the linear dual space $L(X, \mathbb{R})$ ($= X^*$). In other words, $L(X, \mathbb{R})$ is complemented in $Lip_0(X, \mathbb{R})$ (also see [11, Proposition 7.5, p 173]). It appears that Lipschitz spaces were being studied even before this result, possibly implicitly. (See, for example, [46], [19].) However, the author has not been able to locate any appropriate source of references that documents such earlier work. For a convenient reference, we refer to the monograph Lipschitz Algebras by Nik Weaver (see [56, 57]).

Note that $L(X, \mathbb{R}) = X^*$ is the Banach dual of the normed linear space X. Interestingly $Lip_0(X, \mathbb{R})$ (also known as Lipschitz dual) is also the Banach dual of another interesting

Banch space which is now popularly known as a Lipschitz-free space. This predual space was first introduced in 1956 by Arens-Eells [9] though without using the phrase ”Lipschitz-free space”. For this purpose, they considered the notion of molecules on a metric space M and introduced a norm on their linear span. They proved that M can be embedded isometrically in the closed linear span of these molecules (which is denoted as $AE(M)$ by Nik Weaver, see [56, Section 2.2]). It was proved in [56, Theorem 2.2.2] that $AE(M)^* \cong Lip_0(M, \mathbb{R})$. This duality has been discussed in several works, for example see [19], [33], [21]. As we don’t use this approach we are not giving the details of this construction.

In 2002 Godefroy and Kalton took a functional approach to introduce the same space what they formally called Lipschitz-free space [27]. Let M be a metric space together with a distinguished point 0. For any $x \in M$, consider $\delta_x : Lip_0(M, \mathbb{R}) \to \mathbb{R}$ given by $\delta_x(f) = f(x)$ for all $f \in Lip_0(M, \mathbb{R})$. Then $\delta_x \in Lip_0(M, \mathbb{R})^*$. The Lipschitz free space over M is defined as

$$F(M) := \overline{\text{span}}^{\|\cdot\|}\{\delta_x : x \in M\} \subseteq \text{Lip}_0(M, \mathbb{R})^*.$$

$F(M)$ is the unique Banach space (upto bi-Lipschitz homeomorphism for separable metric space M) into which M embeds isometrically. We also know that the image $\delta(M \setminus \{0\})$ constitutes a basis of linearly independent vectors for the space (see [45]). Godefroy and Kalton further proved that $Lip_0(M, \mathbb{R})$ is the dual of the Lipschitz-free space $F(M)$. Later on Weaver has shown that for a complete convex metric space M, the spaces $AE(M)$ and $F(M)$ are isomorphic as Banach spaces (see [58]). The papers by Godefroy and Kalton [27] and by Lindenstrauss [42] have been the main motivation for our research.

1.1 Position of a bounded (bi-) linear maps among (two-) Lipschitz maps

In the first part of the thesis we discuss the condition under which $L(X, Y)$ is complemented in $Lip_0(X, Y)$ and extend this problem to two-variables.

Recall that for Banach spaces X and Y, $L(X, Y)$ is a closed subspace of $Lip_0(X, Y)$. So a natural question arises: Is $L(X, Y)$ complemented in $Lip_0(X, Y)$? When $Y = \mathbb{R}$, the answer is affirmative and is given by Lindenstrauss [42, Theorem 2]. We have obtained a sufficient condition on Y under which $L(X, Y)$ is complemented in $Lip_0(X, Y)$. Some of the main results in one variable in this direction are as follows:

- $L(X, Y)$ is complemented in $Lip_0(X, Y)$, whenever Y is a dual space. However, the duality of Y is sufficient but not necessary.
- $Lip_0(X, Y) \big/ L(X, Y)$ is isometrically isomorphic to an operator space, when Y is injective.
- A characterization of linearity of a Lipschitz map.
- A concrete identification of a predual of $L^\infty(\mathbb{R})/\operatorname{span}\{\mathbf{1}\}$.

After discussing the position of $L(X, Y)$ in $Lip_0(X, Y)$, we next move to a bigger class of maps in $Lip_0(X, Y)$ with a similar set of questions. Let $f \in Lip_0(X, Y)$. Then we say f is positive homogeneous if $f(\alpha x) = \alpha f(x)\ \forall\, \alpha \geq 0,\ x \in X$. The space of all positive homogeneous Lipschitz maps is denoted by $Lip_0^{ph}(X, Y)$. We observe that $(Lip_0^{ph}(X, Y), Lip(\cdot))$ is a closed subspace of $Lip_0(X, Y)$ with

$$L(X, Y) \subsetneq Lip_0^{ph}(X, Y) \subsetneq Lip_0(X, Y).$$

We also prove that $Lip_0^{ph}(X, \mathbb{R})$ turns out to be both a dual space as well as the annihilator of a closed subspace of $F(X)$. Here are some main results in this direction:

- $Lip_0^{ph}(X, \mathbb{R})$ is complemented in $Lip_0(X, \mathbb{R})$.
- When $\dim(X) > 1$, we have obtained a predual of $Lip_0^{ph}(X, \mathbb{R})$ that contains a subspace which is linearly isomorphic to $L^1(\mathbb{R})$.
- $Lip_0(X, \mathbb{R}) \Big/ Lip_0^{ph}(X, \mathbb{R})$ is a dual space.

Note that the method used to construct the projection from $Lip_0(X, Y)$ onto $L(X, Y)$, whenever Y is a dual space is different from the one used to project $Lip_0(X, \mathbb{R})$ onto $Lip_0^{ph}(X, \mathbb{R})$. In the first case, we use an invariant mean, whereas in the second case the projection is given by a direct and explicit construction. Moreover, we have defined an appropriate multiplication so that $(Lip_0^{ph}(X, \mathbb{R}), Lip(\cdot))$ becomes a Banach algebra. We proved that for any normed linear space X with $dim(X) = 1$, $Lip_0^{ph}(X, \mathbb{R})$ is a two dimensional space, where as in this case $Lip_0(X, \mathbb{R})$ is linearly isometrically isomorphic to $L^\infty(\mathbb{R})$. But as soon as $\dim(X) > 1$, $Lip_0^{ph}(X, \mathbb{R})$ becomes infinite dimensional.

Now we turn to the two-variable Lipschitz setting.

Let X, Y, and E be real Banach spaces. A map T from $X \times Y$ into E is said to be *two-Lipschitz* if there exists a constant $k > 0$ such that

$$\|T(x, y) - T(x', y) - T(x, y') + T(x', y')\| \leq k\|x - x'\| \cdot \|y - y'\|$$

for all $x, x' \in X$ and $y, y' \in Y$.

For the collection of all such mappings, we write $BLip(X, Y; E)$. Then it is immediate that $BLip(X, Y; E)$ forms a vector space with respect to pointwise addition and scalar multiplication. We consider

$$BLip_0(X, Y; E) = \{T \in BLip(X, Y) : T(x, 0) = 0 = T(0, y) \, \forall x \in X, \forall \, y \in Y\},$$

equipped with the norm $BLip(\cdot)$ given by

$$BLip(T) = \sup \left\{ \frac{\|T(x, y) - T(x', y) - T(x, y') + T(x', y')\|}{\|x - x'\| \cdot \|y - y'\|} : x, x' \in X, y, y' \in Y; x \neq x', y \neq y' \right\},$$

for all $T \in BLip_0(X,Y;E)$ forms a Banach space.

The space $BLip_0(X,Y;E)$ is viewed as a natural nonlinear extension of $Blin(X,Y;E)$, the space of bounded bilinear operators from $X \times Y$ into E. $Blin(X,Y;E)$, equipped with the standard operator norm:

$$\|S\| = \sup\left\{\frac{\|S(x,y)\|}{\|x\|\cdot\|y\|} : x \in X \setminus \{0\} \text{ and } y \in Y \setminus \{0\}\right\}$$

for any $S \in Blin(X,Y;E)$ is a Banach space. Moreover, for $S \in Blin(X,Y;E)$, we have $\|S\| = BLip(S)$ so that $Blin(X,Y;E)$ is a closed subspace of $BLip_0(X,Y;E)$ (see [32, Remark 2.4]).

Several mathematicians have developed the theory of Lipschitz spaces in a two-variable setting using different approaches. Dubei et al. [23] (2009) introduced two-Lipschitz maps on products of pointed metric spaces and showed that such maps admit a continuous bilinearization via suitable Lipschitz-free Banach spaces. Later, Sánchez-Pérez studied these maps under an extra condition that they vanish at coordinate axes [54, Section 3.4]. Hamidi et al. discussed a Banach space valued version of the maps considered by Sánchez-Pérez type two-Lipschitz maps and studied their operator ideals relating them to linear operator ideals in [32].

In this thesis, we follow the definition of two-Lipschitz operators considered by Hamidi et al. and discuss the existence of projection in the two-variable Lipschitz setting. Some of the main results in this framework are as follows:

- $Blin(X,Y;E)$ is complemented in $BLip_0(X,Y;E)$ whenever E is a dual space.
- A vector–valued duality between the space $BLip_0(X,Y;E)$ and the projective tensor product $F(X)\,\widehat{\otimes}_\pi\, F(Y)$ of Lipschitz–free spaces.
- Identification of various quotient spaces of $BLip_0(X,Y;E)$, in particular when E is injective.

- A characterization of bilinearity of a *two-Lipschitz* operator.
- An application identifying a concrete predual of $L^\infty(\mathbb{R}^2)/\operatorname{span}\{1\}$ via the tensor description.

Though similarities appear between these results and those concerning the position of $L(X,Y)$ in $Lip_0(X,Y)$, the techniques used in the proofs are different.

1.2 Generalization of Lipschitz maps

Let us recall that Lipschitz maps can be determined using modulus of continuity. However we notice that the modulus of continuity is uniform in nature and does not describe poitwise behaviour. For example using modulus of continuity we can describe the uniform continuity but not the pointwise continuity.

However with a modification in definition, we can make it point specific. In this way we can describe continuity at a point. Using the similar method we can describe the notion of pointwise Lipschitz maps.

Let us first recall the definition of the modulus of continuity of a function. Let M and N be metric spaces, and let $f : M \to N$. For $t > 0$, define

$$\omega_f(t) := \sup\{d(f(x), f(y)) : d(x,y) \leq t\}.$$

The map $\omega_f : (0,\infty) \to [0,\infty]$ is called the *modulus of continuity* of f. Reall further that f is Lipschitz on M if and only if $\sup\left\{\frac{\omega_f(t)}{t} : t > 0\right\} < \infty$.

To describe pointwise notions, we go for the following modification. Fix a point $x_0 \in M$. For $t > 0$ and define

$$\omega_f^{x_0}(t) := \sup\{d(f(x), f(x_0)) : d(x, x_0) \leq t\}.$$

The map $\omega_f^{x_0} : (0,\infty) \to [0,\infty]$ is again increasing. Now, f is continuous at x_0 if and only if $\lim_{t\to 0^+} \omega_f^{x_0}(t) = 0$. We call $\omega_f^{x_0}$ the *modulus of continuity of f at x_0*. It motivates us to further define

Definition 1.2.1. Let M and N be metric spaces and let $f : M \to N$. For a fixed $x_0 \in M$, we say that f is *Lipschitz at x_0*, if

$$\sup\left\{\frac{\omega_f^{x_0}(t)}{t} : t > 0\right\} < \infty.$$

The set of all functions $f : M \to N$ which are Lipschitz at x_0 is denoted by $Lip^{x_0}(M, N)$.

We show that for any $x_0 \in M$, we have $f \in Lip^{x_0}(M, N)$ if and only if there exists $k > 0$ such that $d(f(x), f(x_0)) \le kd(x, x_0)$ for all $x \in M$ (Proposition 6.1.2). A function f is called *locally Lipschitz at x_0* if there exists a neighbourhood N_{x_0} of x_0 in M such that f is Lipschitz on N_{x_0} (see [14, Chapter 2]). Note that the Lipschitz condition at a point is different from the notion of being locally Lipschitz at a point.

We say f is *pointwise Lipschitz in M*, if $f \in Lip^x(M, N)$ for each $x \in M$. The set of all such mappings is denoted by $Lip^P(M, N)$. Thus, $Lip^P(M, Y) = \bigcap_{x\in M} Lip^x(M, N)$. If we replace N by a (real) normed linear space Y, then $Lip^{x_0}(M, Y)$ is also a (real) vector space under pointwise operations. For $f \in Lip^{x_0}(M, Y)$, we define

$$\|f\|_{x_0} := \sup\left\{\frac{\omega_f^{x_0}(t)}{t} : t > 0\right\}.$$

Then $\|\cdot\|_{x_0}$ is a semi-norm on $Lip^{x_0}(M, Y)$. In fact, we have

$$\begin{aligned} \|f\|_{x_0} &= \inf\{k > 0 : \|f(x) - f(x_0)\| \le kd(x, x_0) \text{ for all } x \in M\} \\ &= \sup\left\{\frac{\|f(x) - f(x_0)\|}{d(x, x_0)} : x \in M, x \neq x_0\right\} \end{aligned}$$

for all $f \in Lip^{x_0}(M, Y)$. Also $\|f\|_{x_0} = 0$ if and only if f is a constant function. Thus for $f \in Lip^{x_0}(M, Y)$ the value

$$\|f\|_{L^{x_0}} := \max\{\|f\|_{x_0}, \|f(x_0)\|\}$$

determines a norm on $Lip^{x_0}(M,Y)$. Moreover, we have been able to show that for any $x \in M$, $(Lip^x(M,Y), \|\cdot\|_{L^x})$ is isometrically isomorphic to $\ell_\infty(M,Y)$ (Proposition 6.1.3). Thus $(Lip^x(M,Y), \|\cdot\|_{L^x})$ is a Banach space, if and only if Y is so. Thus we can conclude that $(Lip^{x_1}(M,Y), \|\cdot\|_{L^{x_1}})$ is isometrically isomorphic to $(Lip^{x_2}(M,Y), \|\cdot\|_{L^{x_2}})$ for all $x_1, x_2 \in M$.

Once again recall that the space $Lip_0(X,Y)$ has been studied as a non-linear generalization of bounded linear maps between normed linear spaces X and Y. In this thesis, we propose a further generalization of $Lip_0(X,Y)$ as below:

$$Lip^x_x(M,Y) := \{f \in Lip^x(M,Y) : f(x) = 0\}.$$

For $f \in Lip^x_x(M,Y)$, we have

$$\|f\|_{L^x} = \|f\|_x = \sup\left\{\frac{\|f(x')\|}{d(x',x)} : x' \in M \setminus \{x\}\right\}.$$

Thus $(Lip^x_x(M,Y), \|\cdot\|_x)$ is isometrically isomorphic to $\ell_\infty(M\setminus\{x\}, Y)$ and $(Lip^x_x(M,Y), \|\cdot\|_x)$ is a Banach space, if and only if Y is so. Note that $f : M \to Y$ is in $Lip^x_x(M,Y)$ if and only if there exists $k > 0$ such that $\|f(x')\| \leq kd(x',x)$ for all $x' \in M$. Thus

$$\|f\|_x = \inf\{k > 0 : \|f(x')\| \leq kd(x',x) \text{ for all } x' \in M\}.$$

Let us fix $x \in M$ and consider the space

$$Lip_x(M,Y) := \{f : M \to Y | f \text{ is Lipschitz in } M \text{ and } f(x) = 0\}$$

together with the norm $Lip(\cdot)$. Then $(Lip_x(M,Y), Lip(\cdot))$ is a normed linear space and is complete whenever Y is so.(see [14, Chapter 8, Theorem 8.1.3]) Thus $(Lip^x_x(M,Y), \|\cdot\|_x)$ is a natural generalization of $(Lip_x(M,Y), Lip(\cdot))$. We re-describe $Lip^x_x(M,Y)$ with a simpler notation. Since for any $x_1, x_2 \in M$, $(Lip^{x_1}_{x_1}(M,Y), \|\cdot\|_{x_1})$ is isometrically isomorphic to $(Lip^{x_2}_{x_2}(M,Y), \|\cdot\|_{x_2})$, there is no loss of generality in fixing a designated point. We generally describe it as 0.

In the setting where $M = X$ and $N = Y$ are normed linear spaces and $x_0 = 0$, the space $Lip_0^0(X, Y)$ consists of what we call extensively bounded maps, denoted by $E(X, Y)$. This space forms a normed linear space with respect to pointwise addition and scalar multiplication under the norm:

$$\|f\|_e = \sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|},$$

further it is a Banach space, whenever Y is a Banach space. The scalar-valued space $E(X, \mathbb{R})$ is linear isometrically isomorphic to $\ell_\infty(X \setminus \{0\})$ (see 6.2), a commutative Banach algebra with pointwise addition and pointwise multiplication. However, the vector-valued space $E(X, X)$ possesses a richer structure. We show that under composition as multiplication and usual pointwise addition, $E(X, X)$ forms a non-commutative Banach algebra. This motivates the development of an (non-linear)operator ideal theory analogous to classical operator ideals but situated within the extensively bounded setting.

The concept of a Banach operator ideal, though simple, plays an important role in structuring the framework of operator theory. Pietsch first highlighted its significance in 1968 (see [49]), and since then it has grown enormously in many directions. Even today it is an active area of research. The classical operator ideal theory is linear in nature. In 2016, Achour et al. initiated a (nonlinear) theory of Lipschitz operator ideals. This theory brings together several classes of Lipschitz mappings and provides a framework to study the approximation property in metric spaces and Lipschitz mappings [2].

In this thesis, we have generalized the nonlinear theory of Lipschitz operator ideals to the notion of extensively bounded operator ideals (or E-operator ideals). These are subclasses of $E(X, Y)$ satisfying axioms parallel to those defining classical operator ideals, with appropriate compatibility under composition with both linear and extensively bounded maps. We define corresponding notions of finite-rank extensively bounded maps and study normed and Banach structures on such ideals. In particular, while discussing the nonlinear operator ideal theory in this new setting 6.2.2, we examine in detail the necessity of composing linear operators from the left. For further details and examples of various Lipschitz operator ideals, we refer the reader to [4], [3], and [53].

Building on this foundation, we define nonlinear analogues of well-known classes of operator ideals: compact, p-summing, p-integral, and p-nuclear operators, within the extensively bounded framework. We investigate how these nonlinear ideals relate to their classical linear counterparts and examine structural properties unique to the extensively bounded setting.

Chapter 2

Preliminary and Notation

The introductory chapter is followed by this preliminary chapter. Here we discuss the necessary background on pointed metric spaces, Lipschitz spaces, Lipschitz free spaces, and classical operator ideals that are crucial to the development of this work. While many of the results generalize naturally to complex Banach spaces, we restrict ourselves to real spaces for simplicity. Let us agree with the following notations, which will be used throughout the thesis.

Notations: In this thesis, all the Banach spaces X will be considered over $\mathbb{R}$ (the set of real numbers). We denote by B_X its closed unit ball and by S_X its closed unit sphere. We also denote by X^* the topological dual of X. Given X and Y Banach spaces, $X \simeq Y$ stands for X is linearly isomorphic to Y. $X \simeq^h Y$ stands for X is linearly homeomorphic to Y. $X \cong Y$ stands for X is linear isometrically isomorphic to Y. $X \overset{c}{\hookrightarrow} Y$ stands for X is linear isomorphic to a complemented subspace of Y. All the other relevant notations will be followed as described in the respective places. We often write $Lip_0(X), Lip_0^{ph}(X)$ instead of writing $Lip_0(X,\mathbb{R}), Lip_0^{ph}(X,\mathbb{R})$, for the space of zero preserving real valued Lipschitz maps and positively homogeneous Lipschitz maps, respectively. For $1 \leq p < \infty$, ℓ_p, ℓ_∞ denote the space of p-summable sequences and bounded sequences on $\mathbb{R}$, respectively.

2.1 Pointed metric space

We will use the symbol d generically to denote distance in any metric space. A pointed metric space is a metric space (M, d) equipped with a distinguished element or 'base point', usually denoted 0_M. Mostly, we consider pointed metric spaces throughout this thesis. Later on, we omit the subscript for the notation of the corresponding distinguished element.

The detailed discussion will be provided later; however, we briefly outline here the relevance of these classes of metric spaces in our context. In particular, the natural morphisms between pointed metric spaces are those functions that preserve the base point, that is, they map the base point of the domain to the base point of the codomain. The additive identity 0 will be the distinguished element for any normed linear space.

2.2 Lipschitz space

In the context of metric spaces, Lipschitz functions arise as a natural class of morphisms. In particular, our focus is on base-point-preserving Lipschitz maps and on maps that are Lipschitz at the distinguished point (Definition 6.1.1).

Definition 2.2.1. Let M and N be metric spaces. A mapping $f : M \to N$ is said to be *Lipschitz*, if there exists $k > 0$ in $\mathbb{R}$ such that $d(f(x_1), f(x_2)) \leq kd(x_1, x_2)$ for all $x_1, x_2 \in M$, and the smallest such k is called the Lipschitz number.

The set of all Lipschitz mappings from M into N is denoted by $Lip(M, N)$. If we replace N by a normed linear space Y, then $Lip(M, Y)$ is a vector space. The natural candidate for norm of such a function is just its Lipschitz number. As the Lipschitz number of any constant function is zero, it cannot serve as a norm for functions that do not vanish at the base point. For Lipschitz functions on unpointed spaces, the definition of the norm becomes slightly more involved. Nevertheless, any set can be transformed into a pointed set by adjoining an extra element and designating it as the base point. In the metric setting, an analogous construction allows, for our purposes, the treatment of unpointed spaces as a special case of pointed spaces. However, we deal with pointed metric spaces only, in this thesis. For details of this construction we refer Weaver's book [56, Chapter 1].

Equipped with the Lipschitz norm

$$Lip(f) := sup\left\{\frac{\|f(x_1) - f(x_2)\|}{d(x_1, x_2)} : x_1 \neq x_2 \in M\right\},$$

the space of Lipschitz functions, vanishing at the origin; $Lip_0(M, Y) := \{f \in Lip(M, Y) : f(0) = 0\}$ forms a normed linear space. In fact, $(Lip_0(M, Y), Lip(\cdot))$ becomes a Banach space whenever Y is a Banach space.

2.2.1 Lipschitz free space

For a pointed metric space M, the Lipschitz free space over M is defined by

$$F(M) := \overline{span\,\{\delta_x :\ x \in M\}}^{\|\cdot\|} \subset Lip_0(M, \mathbb{R})^*,$$

where for any $x \in M$, $\delta_x : Lip_0(M, \mathbb{R}) \to \mathbb{R}$ be the evaluational functional, that is, $\delta_x(f) = f(x)$ for any $f \in Lip_0(M, \mathbb{R})$. The Banach space $F(M)$ is in fact a canonical predual of $Lip_0(M, \mathbb{R})$. Despite the active research on Lipschitz free spaces, their structural properties are still not fully understood. The structure of $F(\mathbb{R}^n)$ is not completely characterized for $n > 1$. Further, it is known that $F(\mathbb{R})$ is isometrically isomorphic to $L^1(\mathbb{R})$. More recently, several scattered results have appeared in this direction; for instance, it has been shown that for any infinite metric space M, the space $F(M)$ contains a copy of ℓ_1. For further details and several other results towards this direction, we refer to [16], [7], [5], [38], [31], [8], [15], [24].

2.2.2 Some general facts about Lipschitz free spaces

First, we give some of the fundamental properties of Lipschitz free spaces. The results presented here are standard and well-known. An important result here is the fundamental linearization property of free spaces: Any Lipschitz map from a metric space M to a Banach space Y extends uniquely into a continuous linear map from $F(M)$ to Y. We list down the following well-established facts below: (for proof and other details see [27], [56], [14]).

Let M, N be a pointed metric spaces.

1. We consider the evaluational operator $\delta_M : M \to Lip_0(M)^*$ given by $\delta_M(m)(f) := f(m)$ for any $m \in M$ and $f \in Lip_0(M)^*$. Then δ_M is a base point preserving non-linear isometric embedding. Note that even if we consider $M = X$ be a normed linear space in this setting, the map δ_X remains non-linear. However, in this case δ_X admits a left inverse: a bounded linear contraction $\beta_X : F(X) \to X$ (also known as the barycentre map).

2. One of the crucial property of the Lipschitz free space is the following universal property: There exists only one (upto isometry) Banach space $F(M)$ such that there is an isometry $\delta_M : M \to F(M)$ such that:

   (a) $\delta_M(M \setminus \{0\})$ is linearly independent with $\overline{span\ \delta_M(M)}^{\|\cdot\|} = F(M)$, and

   (b) for every Banach space Y anf for any $f \in Lip_0(M, Y)$ there exists $T_f \in L(F(M), Y)$ such that $T_f \circ \delta_M = f$ and $\|T_f\| = Lip(f)$. In fact $(Lip_0(M, Y), Lip(\cdot))$ is linear isometrically isomorphic to $L(F(M), Y)$. That is, the following diagram commutes in the above sense:

$$\begin{tikzcd} & F(M) \arrow[dr, "T_f"] & \\ M \arrow[ur, "\delta_M"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

   In particular for Y to be the real vector space $\mathbb{R}$, we have $Lip_0(M, \mathbb{R})$ is dual of $F(M)$. Also, the map T_f is known as the linearization of the Lipschitz map f.

3. Let X and Y be Banach spaces. Then there exists a contractive linear map $\beta_X : F(X) \to X$ such that $\beta_X \delta_X = Id_X$.

4. The adjoint map $(\beta_X)^* : X^* \to F(X)^*$ is an isometry.

5. Let X and Y be any two Banach spaces. Then $F(X)$ is bi-lipschitz isomorphic (i.e- both forward map and its inverse are Lipschitz) to $X \oplus_1 \ker(\beta_X)$, via the map

$$\eta_X : X \oplus_1 \ker(\beta_X) \to F(X) \text{given by } \eta_X(x, \mu) = \delta_x + \mu,$$

for $\mu \in \ker(\beta_X)$ and $x \in X$. It also follows from [27, Theorem 2.12].

Let us mention one more consequence of the Universal property. This is the Lipschitz structure of metric spaces corresponds to the linear structure of the corresponding Lipschitz free spaces.

6. Indeed if $f : M \to N$ is a base point preserving Lipschitz map then it extends to a bounded linear map $C_f : F(M) \to F(N)$ with $\|C_f\| = Lip(f)$. In particular if M and N are Lipschitz isomorphic (that is, there exits a Lipschitz bijection with Lipschitz inverse) then $F(M)$ is linearly isomorphic to $F(N)$ as Banach spaces. That is, the following diagram commutes:

$$\begin{tikzcd}
& F(M) \arrow[rr, "C_f"] & & F(N) \\
M \arrow[ur, "\delta_M"] \arrow[rr, "f"'] & & N \arrow[ur, "\delta_N"'] &
\end{tikzcd}$$

Now we recall some fundamental notions of the theory that will be required in the subsequent discussion. For a detailed account of linear operator ideals and related theory, we refer the reader to [22], [49], [20].

2.3 Operator ideal

Let X_0, X, Y, Y_0 be Banach spaces. B_X denotes the closed unit ball of X. By $F(X, Y), L(X, Y)$ we denote the spaces of finite rank linear operators and bounded linear operators between X and Y, respectively, equipped with the natural operator norm $\|\cdot\|$. In particular when $Y = \mathbb{R}$, we denote the space of bounded linear functionals on X as $X^* = L(X, \mathbb{R})$.

2.3.1 Linear operator ideal

An *operator ideal* $\mathcal{A}$ is a rule that assigns to each pair (X, Y) of Banach spaces a linear subspace $\mathcal{A}(X, Y) \subseteq L(X, Y)$ such that the following conditions hold:

(I1) $x^* \otimes y \in \mathcal{A}(X, Y)$ for any $x^* \in X^*$ and $y \in Y$.

(I2) If X_0 and Y_0 are Banach spaces, then

$$u \circ T \circ w \in \mathcal{A}(X_0, Y_0)$$

whenever $u \in L(Y, Y_0)$, $T \in \mathcal{A}(X, Y)$, and $w \in L(X_0, X)$.

Further, if for each (X, Y), $\mathcal{A}(X, Y)$ is equipped with a norm α such that:

(N1) $\alpha(x^* \otimes y) = \|x^*\| \cdot \|y\|$ for all $x^* \in X^*$ and $y \in Y$,

(N2) $\alpha(u \circ T \circ w) \leq \|u\| \cdot \alpha(T) \cdot \|w\|$ whenever X_0 and Y_0 are Banach spaces and $u \in L(Y, Y_0)$, $T \in \mathcal{A}(X, Y)$, and $w \in L(X_0, X)$,

(N3) $(\mathcal{A}(X, Y), \alpha)$ is a Banach space,

then $(\mathcal{A}, \alpha)$ is called a *Banach operator ideal* (or simply *Banach ideal*).

We say that the Banach ideal $(\mathcal{A}, \alpha)$ is *injective* if for every operator $v \in L(X, Y)$ such that $u \circ v \in \mathcal{A}(X, Y_0)$ for an isometry $u \in L(Y, Y_0)$, then $v \in \mathcal{A}(X, Y)$ and $\alpha(u \circ v) = \alpha(v)$ whenever X, Y, Y_0 are Banach spaces.

Two basic remarks are in order:

1. Regardless of the operator ideal $\mathcal{A}$, we have $F(X, Y) \subseteq \mathcal{A}(X, Y)$ for all Banach spaces X, Y. This follows immediately from (I1) and the linearity of $\mathcal{A}(X, Y)$.

2. Regardless of the Banach ideal $(\mathcal{A}, \alpha)$, we have $\|u\| \leq \alpha(u)$ for any $u \in \mathcal{A}(X, Y)$.

We further present a brief account of some classical Banach operator ideals.

Example 2.3.1. *1. $(F, \|\cdot\|)$ is the smallest operator ideal of finite rank linear operators with the usual operator norm.*

2. *$(L, \|\cdot\|)$, Banach ideal of bounded linear operators with usual operator norm.*

3. ***Compact operator:*** *$T \in L(X, Y)$ is said to be compact operator if $\overline{T(B_X)}$ is compact subset of Y.*

   The space of such compact operators $(K, \|\cdot\|)$ forms a Banach operator ideal with the usual operator norm on it.

4. ***p-summing operator:*** *Let $1 \leq p < \infty$ and let $u : X \to Y$ be a linear operator between Banach spaces. We say that u is p-*summing *if there exists a constant $c \geq 0$ such that, for every natural number m and for every choice of $x_1, \ldots, x_m \in X$, we have*

$$\left(\sum_{i=1}^{m} \|u(x_i)\|^p\right)^{1/p} \leq c \sup_{\varphi \in B_{X^*}} \left(\sum_{i=1}^{m} |\varphi(x_i)|^p\right)^{1/p},$$

   where B_{X^*} denotes the closed unit ball of X^*.

   The least constant c for which the above inequality holds is denoted by $\pi_p(u)$. We write $\Pi_p(X, Y)$ for the set of all p-summing operators from X into Y. Then, $\Pi_p(X, Y)$ is a linear subspace of $L(X, Y)$, the space of all bounded linear operators from X into Y, and that π_p defines a norm on $\Pi_p(X, Y)$, with $\pi_p(u) \geq \|u\|$ for all $u \in \Pi_p(X, Y)$. In fact, $(\Pi_p, \pi_p(\cdot))$ forms a Banach operator ideal.

5. ***p-integral operator:*** *We shall say that a linear mapping $u : X \to Y$ between Banach spaces is a p-*integral operator *$(1 \leq p < \infty)$ if there exist a probability measure μ and bounded linear operators*

$$a : L_p(\mu) \to Y^{**} \quad \textit{and} \quad b : X \to L_\infty(\mu)$$

giving rise to the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "b"'] & Y \arrow[r, "J_Y"] & Y^{**} \\ L_\infty(\mu) \arrow[r, "i_p"] & L_p(\mu) \arrow[ur, "a"'] & \end{tikzcd}$$

Here $i_p : L_\infty(\mu) \to L_p(\mu)$ *is the formal identity, and* $J_Y : Y \to Y^{**}$ *denotes the canonical isometric embedding. The collection of all* p*-integral operators from* X *to* Y *is denoted by* $\mathcal{I}_p(X,Y)$. *For each* $u \in \mathcal{I}_p(X,Y)$ *we associate its* p*-integral norm,*

$$\iota_p(u) = \inf \|a\|\,\|b\|,$$

where the infimum is taken over all probability measures μ *and operators* a, b *as above. Then,* $(\mathcal{I}_p, i_p(\cdot))$ *is a Banach operator ideal.*

6. p***-nuclear operator:*** *A linear map* $u : X \to Y$ *is said to be* p*-nuclear* $(1 \leq p < \infty)$ *if there exist* $a \in L(\ell_p, Y),\ b \in L(X, \ell_\infty),\ \lambda = (\lambda_n) \in \ell_p$ *such that* $u = a \circ M_\lambda \circ b$, *where* $M_\lambda : \ell_\infty \to \ell_p$, *is the multiplication operator* $M_\lambda((x_n)) = (\lambda_n x_n)$.

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[rrr, bend left, "u"] & \ell_\infty \arrow[r, "M_\lambda"] & \ell_p \arrow[r, "a"] & Y \end{tikzcd}$$

$$\nu_p(u) \ := \ \inf\{\, \|a\|\,\|M_\lambda\|\,\|b\| \ : \ u = a\,M_\lambda\,b,\ a \in L(\ell_p, Y),\ b \in L(X, \ell_\infty),\ \lambda \in \ell_p \}\,.$$

Then the collection of p*-nuclear operators* $\mathcal{N}_p$ *forms a Banach operator ideal with respect to the nuclear norm* $\nu_p(\cdot)$.

Chapter 3

Position of $L(X, Y)$ in $Lip_0(X, Y)$

Let X and Y be Banach spaces and we consider the Lipschitz space $Lip_0(X, Y) = \{f \in Lip(X, Y) : f(0) = 0\}$. Then $Lip_0(X, Y)$ is a Banach space with the norm

$$Lip(f) = \sup\left\{\frac{\|f(x) - f(y)\|}{\|x - y\|} : x, y \in X, x \neq y\right\} \text{ for any } f \in Lip_0(X, Y).$$

The space $Lip_0(X, Y)$ is often regarded as natural non-linear generalization of $L(X, Y)$. It is known that $L(X, Y)$ is a subspace of $Lip_0(X, Y)$. The space $L(X, Y)$ carries the usual operator norm,

$$\|T\| = \sup\{\|Tx\| : x \in X \text{ and } \|x\| \leq 1\}$$

for any $T \in L(X, Y)$. We have $\|T\| = Lip(T)$ for all $T \in L(X, Y)$ so that $L(X, Y)$ is a closed subspace of $Lip_0(X, Y)$.

We are interested whether $L(X, Y)$ is complemented in $Lip_0(X, Y)$. This is known to be true when $Y = \mathbb{R}$. Indeed, by a result due to Lindenstrauss ([42, Theorem 2]) we know that there exists a (norm one) projection from $Lip_0(X, \mathbb{R})$ onto $L(X, \mathbb{R})$ (see also [11, Proposition 7.5]). In this chapter, we extend this result to the case where Y is a dual space. Specifically, we construct a surjective linear norm-one projection $P_X^Y : Lip_0(X, Y) \to L(X, Y)$. This leads us to conclude that $L(X, Y)$ is complemented in $Lip_0(X, Y)$, whenever Y is a dual space. More precisely, we prove that $Lip_0(X, Y)$ is equal to $L(X, Y) \oplus_\infty \ker(P_X^Y)$ as vector space and the two norms are equivalent.

Encouraged by the initial set of observations, we proceed to investigate the structure of the quotient space $Lip_0(X, Y) \big/ L(X, Y)$. Since $L(X, Y)$ is a closed subspace of $Lip_0(X, Y)$,

$Lip_0(X,Y) \Big/ L(X,Y)$ itself is a Banach space. Our aim is to explore whether this space can also be described in terms of an operator space. To that end, once again we recall the Lipschitz-free space first and then we set the stage for a more detailed analysis in the following sections.

For each $x \in X$, we define $\delta_x : Lip_0(X,\mathbb{R}) \to \mathbb{R}$ by $\delta_x(f) = f(x)$ for all $f \in Lip_0(X,\mathbb{R})$. Then $\delta_x \in L(Lip_0(X,\mathbb{R}),\mathbb{R})$ with $\|\delta_x\| = \|x\|$ for all $x \in X$. This induces a Lipschitz mapping $\delta_X : X \to L(Lip_0(X,\mathbb{R}),\mathbb{R})$ given by $(x \mapsto \delta_x)$. We consider the closed subspace

$$F(X) := \overline{span\{\delta_x : x \in X\}}^{\|.\|}$$

of $Lip_0(X,\mathbb{R})^*$. We then consider a bounded linear contraction (also known as the barycentre map) $\beta_X : F(X) \to X$ which is a left inverse of the Lipschitz map $\delta_X : X \to F(X)$. We prove that the space $Lip_0(X,Y) \Big/ L(X,Y)$ is isometrically isomorphic to $L(\text{ker}(\beta_X),Y)$ when Y is an injective Banach space.

Next, we examine the relation between P_X^Y and β_X. We note that ker(P_X^Y) is topologically isomorphic to $Lip_0(X,Y) \Big/ L(X,Y)$ which in turn can be proved to be isometrically isomorphic to $L(\text{ker}(\beta_X),Y)$, whenever Y is a dual injective space. Both the results depend on the following observation:

$$\{f \in Lip_0(X,Y) : T_f(\gamma) = 0 \ \ \forall \gamma \in \text{ker}(\beta_X)\} = L(X,Y),$$

where T_f is the linearization of $f \in Lip_0(X,Y)$.

We further prove that $L\left(F(X) \Big/ \text{ker}(\beta_X), Y\right)$ is isometrically isomorphic to $L(X,Y)$, whenever Y is a dual injective space. In fact, we prove more general versions of these results and discuss the embedding of $L(X,Y)$ into $L(F(X),Y)$.

At the end, we present an explicit description of the space ker$(\beta_{\mathbb{R}})$. We prove that it is isometrically isomorphic to $\{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f d\mu = 0\}$, where μ is the Lebesgue measure on $\mathbb{R}$. We deduce that

$$L^\infty(\mathbb{R}) \Big/ \{\text{set of all constant maps on } \mathbb{R}\}$$

is isometrically isomorphic to dual of $\ker(\beta_{\mathbb{R}})$.

Now we will discuss the summary of this chapter. In Section 3.1, we have shown that there is a projection from the Lipschitz space to a linear space using an invariant mean.

In Section 3.2, we recall that $Lip_0(X,Y)$ is isomorphic to $L(F(X),Y)$. This leads us to consider a Y-valued duality between $Lip_0(X,Y)$ and $F(X)$. One of the results here gives a necessary and sufficient condition for a Lipschitz map to be linear.

In Section 3.3, we use the vector-valued duality from the previous section to define some new quotient spaces. In particular, we show that the Lipschitz space modulo the linear space is isomorphic to a certain operator space.

At the end, in Section 3.4, we explicitly describe the kernel of $\beta_{\mathbb{R}}$. Using the results developed in this chapter, we give an alternative proof that

$$L^\infty(\mathbb{R}) \Big/ \{\text{set of all constant maps on } \mathbb{R}\}$$

is a dual space.

3.1 Embedding of $L(X,Y)$ into $Lip_0(X,Y)$

Let X and Y be Banach spaces. We again recall that for $T \in L(X,Y)$, we have $\|T\| = Lip(T)$ so that $L(X,Y)$ is a closed subspace of $Lip_0(X,Y)$. In this section, we prove that $L(X,Y)$ is a complemented subspace of $Lip_0(X,Y)$ whenever Y is a dual Banach space. In fact, we show that in this case, there exists a projection from $Lip_0(X,Y)$ onto $L(X,Y)$. We begin with the following observation.

Let $f : X \to Y$ be a Lipschitz mapping and let $x \in X$. Consider $f_x : X \to Y$ given by $f_x(z) = f(x+z)$ for all $z \in X$. Then f_x is also Lipschitz with $Lip(f_x) = Lip(f)$ for all $f \in Lip(X,Y)$ and $x \in X$. We use this translation to prove the next result.

Proposition 3.1.1. *For any pair of Banach spaces X and Y, $Lip_0(X,Y)$ is linearly isometrically isomorphic to a closed subspace of $Lip_0(X, \ell^\infty(X,Y))$.*

Proof. We fix $f \in Lip_0(X,Y)$ and define $\phi_f : X \to X^Y$ given by

$$\phi_f(x) = f_x - f \text{ for all } x \in X.$$

Let $x, z \in X$. Then

$$\| (\phi_f(x))\,(z)\| = \|(f_x - f)(z)\| = \|f(x+z) - f(z)\| \le Lip(f)\|x\|.$$

Thus $\phi_f(x) \in \ell^\infty(X,Y)$ with $\|\phi_f(x)\|_\infty \le Lip(f)\|x\|$. Now

$$\begin{aligned}
\sup_{x \neq x'} \frac{\|\phi_f(x) - \phi_f(x')\|_\infty}{\|x - x'\|} &= \sup_{x,x',z \in X;\; x \neq x'} \frac{\|\phi_f(x)(z) - \phi_f(x')(z)\|}{\|x - x'\|} \\
&= \sup_{x,x',z \in X;\; x \neq x'} \frac{\|f(x+z) - f(x'+z)\|}{\|(x+z) - (x'+z)\|} \\
&= \sup_{x \neq x'} \frac{\|f(x) - f(x')\|_\infty}{\|x - x'\|} = Lip(f).
\end{aligned}$$

Thus $\phi_f \in Lip_0(X, \ell^\infty(X,Y))$ with $Lip(\phi_f) = Lip(f)$. This induces a linear isometry $\Phi : Lip_0(X,Y) \to Lip_0(X, \ell^\infty(X,Y))$ given by $\Phi(f) = \phi_f$ for all $f \in Lip_0(X,Y)$. $\square$

Remark 3.1.2. For $f \in Lip(X,Y)$, $\phi_f \in Lip(X, \ell^\infty(X,Y))$.

Now we recall the notion of an invariant mean.

Definition 3.1.3. [11] A left *invariant mean* in X is a linear functional M on $\ell^\infty(X)$ (the space of all bounded maps from X to $\mathbb{R}$) such that:

1. $M(\mathbf{1}) = 1$, where $\mathbf{1}(x) = 1$ for all $x \in X$.

2. $M(f) \ge 0$ for all $f \ge 0$.

3. $M(f_x) = M(f)$ for all $x \in X$, where $f_x(z) = f(x+z)$ for all $z \in X$.

Let Y be a dual Banach space. It follows from [11, Corollary C.2] that an invariant mean M in X induces a norm-one linear operator $\mathcal{M}_X^Y : \ell^\infty(X,Y) \to Y$ satisfying:

1. $\mathcal{M}_X^Y(f_x) = \mathcal{M}_X^Y(f)$ for all $f \in \ell^\infty(X,Y)$ and $x \in X$.
2. $\mathcal{M}_X^Y(\hat{y}) = y$ for all $y \in Y$. Here $\hat{y} \in \ell^\infty(X,Y)$ is given by $\hat{y}(x) = y$ for all $x \in X$.

We call $\mathcal{M}_X^Y$ a generalized invariant mean. It is easy to verify that $\mathcal{M}_X^{\mathbb{R}} = M$.

Remark 3.1.4. For $f \in Lip_0(X,Y)$ and $x, x', z \in X$, we have

$$\begin{aligned} \left(\left(\phi_{f_x}\right)(x')\right)(z) &= \left((f_x)_{x'} - f_x\right)(z) = f(x+x'+z) - f(x+z) \\ &= \left(f_{x'} - f\right)(x+z) = \left(\phi_f(x')\right)_x(z). \end{aligned}$$

Thus $\left(\phi_{f_x}\right)(x') = \left(\phi_f(x')\right)_x \in \ell^\infty(X,Y)$ for all $x, x' \in X$. Now it follows that

$$\mathcal{M}_X^Y\left(\left(\phi_{f_x}\right)(x')\right) = \mathcal{M}_X^Y\left(\left(\phi_f(x')\right)_x\right) = \mathcal{M}_X^Y\left(\phi_f(x')\right)$$

for all $f \in Lip_0(X,Y), x, x' \in X$.

Theorem 3.1.5. *Let X and Y be Banach spaces with Y be a dual space. Then $L(X,Y)$ is 1-complemented in $Lip_0(X,Y)$.*

Proof. For $f \in Lip_0(X,)$ and $z \in X$, we set

$$P_X^Y(f)(z) = \mathcal{M}_X^Y(\phi_f(z))$$

where $\mathcal{M}_X^Y : \ell^\infty(X,Y) \to Y$ is a vector-valued generalized invariant mean. Then $P_X^Y(f) : X \to Y$. First, we show that $P_X^Y(f)$ is linear.

Note that for $x, x' \in X$, we have

$$\begin{aligned} \phi_f(x+x') &= f_{x+x'} - f = f_{x+x'} - f_{x'} + f_{x'} - f \\ &= (f_x - f)_{x'} + (f_{x'} - f) \end{aligned}$$

$$= \quad (\phi_f(x))_{x'} + \phi_f(x').$$

Thus as $\mathcal{M}_X^Y$ is translation invariant, we get

$$P_X^Y(f)(x+x') = P_X^Y(f)(x) + P_X^Y(f)(x')$$

for all $x, x' \in X$. Since $\mathcal{M}_X^Y$ and Φ are continuous, so is $P_X^Y(f)$. So, using a standard technique of real analysis, it is routine to prove that $P_X^Y(f)(\alpha x) = \alpha P_X^Y(f)(x)$ for any $x \in X$ and $\alpha \in \mathbb{R}$ whence $P_X^Y(f) \in L(X,Y)$ for all $f \in Lip_0(X,Y)$. Thus $P_X^Y : Lip_0(X,Y) \to L(X,Y)$.

Since $\mathcal{M}_X^Y$ and Φ are linear, so is P_X^Y. Also as

$$\|P_X^Y(f)(z)\| = \|\mathcal{M}_X^Y(\phi_f(z))\| \leq Lip(f)\|z\|,$$

we have $\|P_X^Y(f)\| \leq Lip(f)$ for all $f \in Lip_0(X,Y)$.

Next, we prove that $P_X^Y(T) = T$ for all $T \in L(X,Y)$. Fix $T \in L(X,Y)$ and $z \in X$. Then for any $x \in X$, we have

$$(\phi_T(z))(x) = T(x+z) - T(x) = T(z) \in Y,$$

that is, $\phi_T(z) = \widehat{Tz}$. Thus

$$P_X^Y(T)(z) = \mathcal{M}_X^Y(\phi_T(z)) = \mathcal{M}_X^Y(\widehat{Tz}) = Tz.$$

Therefore, $P_X^Y(T) = T$ for all $T \in L(X,Y)$ so that P_X^Y is a norm one linear projection from $Lip_0(X,Y)$ onto $L(X,Y)$. In other words, $L(X,Y)$ is complemented in $Lip_0(X,Y)$ whenever Y is a dual Banach space. $\square$

Corollary 3.1.6. *Let X and Y be Banach spaces with Y be a dual space. Then $Lip_0(X,Y)$ is topologically isomorphic to $L(X,Y) \oplus_1 \ker(P_X^Y)$.*

Proof. By Theorem 3.1.5, P_X^Y is a (norm one) projection with $P_X^Y(Lip_0(X,Y)) = L(X,Y)$. Thus $Lip_0(X,Y) = L(X,Y) \oplus \ker(P_X^Y)$ as vector spaces. Here the linear isomorphism is given by

$$f \in Lip_0(X,Y) \mapsto (P_X^Y(f), (f - P_X^Y(f))) \in L(X,Y) \oplus \ker(P_X^Y).$$

Moreover, $\|.\|_1$ is equivalent to $Lip(.)$ on $Lip_0(X,Y)$, where

$$\|f\|_1 = \|P_X^Y(f)\| + Lip(f - P_X^Y(f))$$

for all $f \in Lip_0(X,Y)$. In fact,

$$Lip(f) \leq \|P_X^Y(f)\| + Lip(f - P_X^Y(f)) \leq Lip(f) + 2\|P_X^Y(f)\| \leq 3Lip(f),$$

if $f \in Lip_0(X,Y)$. $\square$

Remark 3.1.7. 1. Let us recall the following point. Let $Y \subset X$ be a closed subspace and let $P : X \to Y$ be an onto linear projection with $\|P\| = 1$. For any $x \in X$,

$$d(x, \ker(P)) = \|P(x)\| = \|x - (x - P(x))\|,$$

so that $\ker(P)$ is a proximinal subspace and the quotient space $X/\ker(P)$ is isometric to $P(X)$. Additionally, we have

$$P(X)^* \cong P^*(X^*)$$

isometrically. Now considering this information for our present purpose we have

$$P_X^Y\left(Lip_0(X,Y)\right)^* \cong (P_X^Y)^*\left(Lip_0(X,Y)^*\right).$$

2. We note that $\ker(P_X^Y)$ is topologically isomorphic to $Lip_0(X,Y) \big/ L(X,Y)$. In fact, $Id_{Lip_0(X,Y)} - P_X^Y$ is a bounded linear surjective projection from $Lip_0(X,Y)$ onto $\ker(P_X^Y)$ with kernel $L(X,Y)$.

In Theorem 3.1.5, we showed that the duality of Y ensures the existence of a surjective projection $P_X^Y : Lip_0(X,Y) \to L(X,Y)$. However, the duality of the codomain is not necessary for a projection from $Lip_0(X,Y)$ onto $L(X,Y)$ to exist. When the domain X is finite-dimensional and Y is any Banach space, we can show that there exists a bounded linear projection from $Lip_0(X,Y)$ onto $L(X,Y)$. This observation is subsumed in the next result, where we show that such a projection exists when the domain $X = \ell_1$ and Y is an arbitrary Banach space.

Before stating the next proposition, let us recall the well-known identification

$$L(\ell_1, Y) \cong \left(\bigoplus_{n=1}^{\infty} Y\right)_{\ell_\infty}.$$

Indeed, corresponding to every $T \in L(\ell_1, Y)$, the sequence $(T(e_n))_{n=1}^{\infty}$ belongs to $(\bigoplus_{n=1}^{\infty} Y)_{\ell_\infty}$ and

$$\|T\| = \sup_{n\in\mathbb{N}} \|T(e_n)\|.$$

Conversely, if $(y_n)_{n=1}^{\infty} \in (\bigoplus_{n=1}^{\infty} Y)_{\ell_\infty}$, then the formula

$$T((x_n)) = \sum_{n=1}^{\infty} x_n y_n, \qquad (x_n) \in \ell_1,$$

defines a bounded linear operator $T : \ell_1 \to Y$ with $\|T\| = \sup_n \|y_n\|$. Thus $L(\ell_1, Y)$ is canonically isometrically isomorphic to $(\bigoplus_{n=1}^{\infty} Y)_{\ell_\infty}$.

This observation makes the definition of the projection P more natural, since for each $f \in Lip_0(\ell_1, Y)$, the operator $P(f)$ is precisely the operator associated with the sequence

$$\big(f(e_n)\big)_{n=1}^{\infty} \in \left(\bigoplus_{n=1}^{\infty} Y\right)_{\ell_\infty}.$$

Proposition 3.1.8. *For any Banach space Y, there exists a surjective bounded linear norm one projection from $Lip_0(\ell_1, Y)$ onto $L(\ell_1, Y)$.*

Proof. Let $f, g \in Lip_0(\ell_1, Y)$, $\alpha \in \mathbb{R}$ and $(x_n), (x_n') \in \ell_1$. Then, $\left\| \sum_{n=1}^{\infty} x_n f(e_n) \right\| \leq Lip(f)\|(x_n)\|_1$ so that $\sum_{n=1}^{\infty} x_n f(e_n) \in Y$. Let us put $P(f)((x_n)) = \sum_{n=1}^{\infty} x_n f(e_n)$. Then easily we obtain that $P(f)((\alpha x_n + x'_n)) = \alpha P(f)((x_n)) + P(f)((x'_n))$. Therefore, the map $P : Lip_0(\ell_1, Y) \to L(\ell_1, Y)$ given by $P(f)((x_n)) = \sum\limits_{n=1}^{\infty} x_n f(e_n)$, for all $f \in Lip_0(\ell_1, Y)$ and $(x_n) \in \ell_1$, is well-defined. Again, linearity of P is immediate with $\|P(f)((x_n))\| \leq Lip(f)\|(x_n)\|_1$. Hence, P is a bounded linear contraction. Now, we show that P is also a surjective projection.

Suppose, $T \in L(\ell_1, Y)$. Then

$$P(T)((x_n)) = \sum_{n=1}^{\infty} x_n T(e_n) = T\left(\sum_{n=1}^{\infty} x_n e_n\right) = T((x_n)).$$

Thus $P(T) = T$ for any $T \in L(\ell_1, Y)$. This completes the proof. □

3.2 A vector valued duality (Lipschitz case)

If X and Y are Banach spaces, then $L(X, Y)$ is a closed subspace of the Banach space $Lip_0(X, Y)$ and therefore $Lip_0(X, Y) \big/ L(X, Y)$ is also a Banach space. In the next section, we shall obtain a condition on Y under which the quotient space is seen as an operator space. To facilitate this discussion, in this section let us begin by recalling the following result, from which the subsequent remarks follow immediately. We may recall that the study of the linear structure of Lipschitz-free spaces has been an active field of study. See, for example, [16], [56], [28], [14], [18], [30]. We then introduce an appropriate vector-valued duality, which plays a key role in identifying the corresponding quotient spaces.

Theorem 3.2.1. (2.2.2(2(b))) *Let X and Y be Banach spaces. Then $(Lip_0(X, Y), Lip(\cdot))$ is isometrically isomorphic to $L(F(X), Y)$ via the map $\nabla_X^Y : Lip_0(X, Y) \to L(F(X), Y)$, given by $\nabla_X^Y(f) = T_f$.*

Remark 3.2.2. Let X and Y be Banach spaces.

1. For each $f \in Lip_0(X, Y)$ there exists unique linear map $T_f = \nabla_X^Y(f) \in L(X, Y)$ such that $f = T_f \circ \delta_X$.

2. Let $S \in L(X, Y)$. Then $\nabla_X^Y(S) = S \circ \beta_X$. Thus

$$\nabla_X^Y(L(X, Y)) = \{S \circ \beta_X : S \in L(X, Y)\}$$

is a closed subspace of $L(F(X), Y)$.

3. $\nabla_X^Y(S \circ \delta_X) = S$ for any $S \in L(X, Y)$.

Theorem 3.2.1 encourages us to consider the following 'vector-valued' duality.

Proposition 3.2.3. *For any two Banach spaces X and Y, define*

$$\langle \cdot, \cdot \rangle_Y : Lip_0(X, Y) \times F(X) \to Y$$

given by

$$\langle f, \mu \rangle_Y := T_f(\mu)$$

for all $f \in Lip_0(X, Y)$ and $\mu \in F(X)$ with T_f be the linearization map for each $f \in Lip_0(X, Y)$. Then $\langle \cdot, \cdot \rangle_Y$ is bilinear and non-degenerate.

Proof. It is routine to check that $\langle \cdot, \cdot \rangle_Y$ is bilinear. Linearity in the second co-ordinate follows from the linearity of T_f and the first co-ordinate due to the uniqueness of the linearization map.

Let $f \in Lip_0(X, Y)$ be such that $T_f(\mu) = 0$ for all $\mu \in F(X)$. Then $T_f = 0$, hence $f = 0$, as $\|T_f\| = Lip(f)$. Let $\mu \in F(X)$ with $T_f(\mu) = 0$ for all $f \in Lip_0(X, Y)$. We aim to show $\mu = 0$. Choose $y_0 \in Y \setminus \{0\}$ and for each $g \in Lip_0(X, \mathbb{R})$ we define $h(x) = g(x)y_0$. Then $h \in Lip_0(X, Y)$ and $T_h(\delta_x) = h(x) = g(x)y_0 = T_g(\delta_x)y_0$. Therefore, from linearity and continuity of T_h and T_g we get $T_h(\gamma) = T_g(\gamma)y_0$ for all $\gamma \in F(X)$. Now $T_h(\mu) = 0$ for all $h \in Lip_0(X, Y)$ implies $T_g(\mu)y_0 = 0$ for all $g \in Lip_0(X, \mathbb{R})$. Thus $\mu(g) = 0$ for all $g \in Lip_0(X, \mathbb{R})$. Hence $\mu = 0$. This completes the proof. $\square$

For $\mathcal{A} \subset Lip_0(X,Y)$, we define $\mathcal{A}^{\diamondsuit} := \{\gamma \in F(X) : T_f(\gamma) = 0 \ \ \forall f \in \mathcal{A}\}$. Then $\mathcal{A}^{\diamondsuit} = \bigcap_{f \in \mathcal{A}} \ker(T_f)$ is a closed subspace of $F(X)$.

Similarly, for $\mathcal{D} \subset F(X)$, ${}^{\diamondsuit}\mathcal{D} := \{f \in Lip_0(X,Y) : T_f(\gamma) = 0 \ \ \forall \gamma \in \mathcal{D}\}$ can be shown a closed subspace of $Lip_0(X,Y)$.

3.2.1 Characterization of a Lipschitz map to be linear

Proposition 3.2.4. *Let X and Y be Banach spaces and $f \in Lip_0(X,Y)$. Then the following statements are equivalent.*

1. *f is linear.*
2. *$f \in {}^{\diamondsuit}\ker(\beta_X)$.*

Proof. Let $f \in L(X,Y)$ and $\mu \in \ker(\beta_X)$. Then by Remark 3.2.2 (2), $T_f = f \circ \beta_X$. Therefore, $T_f(\mu) = 0$ and hence $f \in {}^{\diamondsuit}\ker(\beta_X)$.

Let $f \in Lip_0(X,Y)$ such that $T_f(\mu) = 0$ for all $\mu \in \ker(\beta_X)$. We show that f is linear. Let $x_1, x_2 \in X$ and $r \in \mathbb{R}$. Then $-r\delta_{x_1} - \delta_{x_2} + \delta_{rx_1+x_2} \in \ker(\beta_X)$. Thus

$$T_f\left(-r\delta_{x_1} - \delta_{x_2} + \delta_{rx_1+x_2}\right) = 0$$

which amounts to saying

$$-rf(x_1) - f(x_2) + f(rx_1 + x_2) = 0.$$

Therefore, $f(rx_1 + x_2) = rf(x_1) + f(x_2)$ so that f is linear. Now, being Lipschitz, f is uniformly continuous in X so that $f \in L(X,Y)$. This completes the proof. $\square$

Lemma 3.2.5. *Let X and Y be Banach spaces. Then $L(X,Y)^{\diamondsuit} = \ker(\beta_X)$.*

Proof. Let $\mu \in L(X,Y)^{\diamondsuit}$. Then $T_f(\mu) = 0$ for all $f \in L(X,Y)$. That is. $f \circ \beta_X(\mu) = 0$ for all $f \in L(X,Y)$. We show that $\mu \in \ker(\beta_X)$. Put $v = \beta_X(\mu)$. Choose $y_0 \in Y \setminus \{0\}$.

Now for each $\phi \in X^*$ we consider $h_\phi(x) = \phi(x)y_0$ for all $x \in X$. Then $h_\phi \in L(X, Y)$ for all $\phi \in X^*$. Now $f(v) = 0$ for all $f \in L(X, Y)$ implies $h_\phi(v) = \phi(v)y_0 = 0$ for all $\phi \in X^*$. Thus by Hahn-Banach theorem $v = \beta_X(\mu) = 0$. Hence $\mu \in \ker(\beta_X)$.

Again suppose $\mu \in \ker(\beta_X)$. Consequently, we have $T_f(\mu) = f \circ \beta_X(\mu) = 0$ for all $f \in L(X, Y)$. Therefore, $\mu \in L(X, Y)^\diamond$. Hence the proof holds. $\square$

Remark 3.2.6. From the above two results, for any two Banach spaces X and Y we have $\left({}^\diamond \ker(\beta_X)\right)^\diamond = \ker(\beta_X)$ and ${}^\diamond\left(L(X, Y)^\diamond\right) = L(X, Y)$.

3.3 Some quotient spaces

Proposition 3.3.1. *Let $\mathcal{D}$ be a closed subspace of $F(X)$. If Y is an injective Banach space, then $Lip_0(X, Y) \,/{}^\diamond\mathcal{D}$ is linear isometrically isomorphic to $L(\mathcal{D}, Y)$.*

Proof. For a fix $f \in Lip_0(X, Y)$, we define $\Theta : Lip_0(X, Y) \to L(\mathcal{D}, Y)$ given by $\Theta(f) = T_f\big|_{\mathcal{D}}$. It is routine to check that Θ is linear. We show that Θ is surjective also. Let $S \in L(\mathcal{D}, Y)$. Since Y is injective there exists $\widetilde{S} \in L(F(X), Y)$ such that $\widetilde{S}\big|_{\mathcal{D}} = S$ with $\|\widetilde{S}\| = \|S\|$. Thus $\widetilde{S} \circ \delta_X \in Lip_0(X, Y)$. Moreover, this follows directly from linearity, continuity, and the uniqueness of the linearization, that $T_{\widetilde{S}\circ\delta_X} = \widetilde{S}$. Therefore, for any $\gamma \in \mathcal{D}$

$$\Theta(\widetilde{S} \circ \delta_X)(\gamma) = T_{\widetilde{S}\circ\delta_X}(\gamma) = \widetilde{S}(\gamma) = S(\gamma).$$

Thus Θ is surjective.

Also,

$$\begin{aligned} \ker(\Theta) &= \left\{f \in Lip_0(X, Y) : T_f\big|_{\mathcal{D}} = 0\right\} \\ &= \{f \in Lip_0(X, Y) : T_f(\gamma) = 0 \text{ for all } \gamma \in \mathcal{D}\} \\ &= {}^\diamond\mathcal{D}. \end{aligned}$$

So by fundamental theorem of linear algebra Θ induce a linear isomorphism (which we again denote by Θ):

$$\Theta : Lip_0(X,Y) \big/ {}^{\diamond}\mathcal{D} \to L(\mathcal{D}, Y)$$

given by $\Theta(f + {}^{\diamond}\mathcal{D})(\gamma) = T_f(\gamma)$ for all $\gamma \in \mathcal{D}$.

Now, for any $f \in Lip_0(X,Y)$ and $\gamma \in \mathcal{D}$

$$\begin{aligned}
\|\Theta(f + {}^{\diamond}\mathcal{D})(\gamma)\| &= \|T_f(\gamma)\| \\
&= \inf_{g \in {}^{\diamond}\mathcal{D}} \|T_{f+g}(\gamma)\| \\
&\leq \|\gamma\| \inf_{g \in {}^{\diamond}\mathcal{D}} Lip(f+g) \\
&= \|\gamma\| \|f + {}^{\diamond}\mathcal{D}\|.
\end{aligned}$$

Therefore, Θ is a surjective contractive linear isomorphism. So by the open mapping theorem, Θ^{-1} exists and is a bounded linear surjective isomorphism.

Fix $S \in L(\mathcal{D}, Y)$ and let S_1 and S_2 be any two extensions of S in $L(F(X), Y)$. If $\gamma \in \mathcal{D}$, then

$$T_{(S_1 \circ \delta_X - S_2 \circ \delta_X)}(\gamma) = T_{S_1 \circ \delta_X}(\gamma) - T_{S_2 \circ \delta_X}(\gamma) = S(\gamma) - S(\gamma) = 0.$$

Thus $S_1 \circ \delta_X + {}^{\diamond}\mathcal{D} = S_2 \circ \delta_X + {}^{\diamond}\mathcal{D}$ and hence $\widetilde{S} \circ \delta_X + {}^{\diamond}\mathcal{D}$ is uniquely determined by S. Now it follows that $\Theta^{-1}(S) = \widetilde{S} \circ \delta_X + {}^{\diamond}\mathcal{D}$ for all $S \in L(\mathcal{D}, Y)$, where $\widetilde{S} \in L(F(X), Y)$ is a norm preserving extension of S. Since

$$\|\Theta^{-1}(S)\| = \|\widetilde{S} \circ \delta_X + {}^{\diamond}\mathcal{D}\| \leq Lip(\widetilde{S} \circ \delta_X) \leq \|S\|,$$

Θ^{-1} is also a contraction. Now, it is easy to conclude that Θ is surjective linear isometry so that $Lip_0(X,Y) \big/ {}^{\diamond}\mathcal{D}$ is isometrically isomorphic to $L(\mathcal{D}, Y)$. $\square$

Remark 3.3.2. Let X be a Banach space and Y be an injective Banach space. Then it follows from Propositions 3.3.1 and 3.2.4 that the quotient space $Lip_0(X,Y) \big/ L(X,Y)$ is isometrically isomorphic to $L(\ker(\beta_X), Y)$.

Proposition 3.3.3. *Let $\mathcal{D}$ be a closed subspace of $F(X)$, then $L\left(F(X)/_{\mathcal{D}}, Y\right)$ is linear isometrically isomorphic to ${}^{\diamond}\mathcal{D}$.*

Proof. Consider $Q : F(X) \to F(X)/_{\mathcal{D}}$ be the canonical quotient map $Q(\mu) = \mu + \mathcal{D}$.

Also, for any fix $f \in {}^{\diamond}\mathcal{D}$, we consider its linearization map T_f and define the map

$$\Lambda : {}^{\diamond}\mathcal{D} \to L\left(F(X)/_{\mathcal{D}}, Y\right)$$

given by $\Lambda(f)(\gamma + \mathcal{D}) = T_f(\gamma)$ for all $f \in {}^{\diamond}\mathcal{D}$ and $\gamma \in F(X)$. Then Λ is linear. Also

$$\begin{aligned} \|\Lambda(f)(\gamma + \mathcal{D})\| &= \|T_f(\gamma)\| \\ &= \inf_{\mu \in \mathcal{D}} \|T_f(\gamma + \mu)\| \\ &\leq \inf_{\mu \in \mathcal{D}} \|(\gamma + \mu)\| Lip(f) \\ &= Lip(f)\|\gamma + \mathcal{D}\|, \end{aligned}$$

for all $f \in {}^{\diamond}\mathcal{D}$ and $\gamma \in F(X)$. Thus Λ is a contraction. We show that Λ is an isometry.

Let $\epsilon > 0$. Then there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that

$$\frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|} \geq Lip(f) - \epsilon, \text{ that is } \left\| T_f\left(\frac{\delta_{x_1} - \delta_{x_2}}{\|x_1 - x_2\|}\right)\right\| \geq Lip(f) - \epsilon.$$

Therefore,

$$\left\|\Lambda(f)\left(\frac{\delta_{x_1} - \delta_{x_2}}{\|x_1 - x_2\|} + \mathcal{D}\right)\right\| = \frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|} \geq Lip(f) - \epsilon.$$

Since $1 = \left\|\frac{\delta_{x_1} - \delta_{x_2}}{\|x_1 - x_2\|}\right\| \geq \left\|\frac{\delta_{x_1} - \delta_{x_2}}{\|x_1 - x_2\|} + \mathcal{D}\right\|$ and $\epsilon > 0$ is arbitrary, we conclude that $\|\Lambda(f)\| \geq Lip(f)$. Hence Λ is an isometry.

To prove the surjectivity of Λ, let $S \in L\left(F(X)/_{\mathcal{D}}, Y\right)$. Set $f(x) = S(\delta_x + \mathcal{D}) = S \circ Q \circ \delta_X(x)$ for all $x \in X$. Then $f \in Lip_0(X, Y)$ as composition of linear and Lipschitz maps. Now $T_f(\delta_x) = f(x) = S(\delta_x + \mathcal{D}) = S \circ Q(\delta_x)$ for all $x \in X$. Therefore, $T_f = S \circ Q$ as they match on dense subspace. Therefore for all $\gamma \in \mathcal{D}$, $\quad T_f(\gamma) = S(Q(\gamma)) = 0$ so that $f \in {}^{\diamond}\mathcal{D}$. This completes the proof. $\square$

Remark 3.3.4. 1. In particular for $\mathcal{D} = \{0\}$ we have the Theorem 3.2.1.

2. For $\mathcal{D} = \ker(\beta_X)$ we get $L\left(F(X) \big/ \ker(\beta_X), Y\right)$ is linear isometrically isomorphic to $L(X,Y)$.

Remark 3.3.5. 1. If we consider Y to be an injective dual space and $\mathcal{D} = \ker(\beta_X)$, then from the Remark 3.1.7(2), Propositions 3.3.1 and 3.2.4 we have $\ker(P_X^Y)$ is topologically isomorphic to $Lip_0(X,Y) \big/ L(X,Y)$ which is linear isometrically isomorphic to $L(\ker(\beta_X), Y)$.

   In particular for $X = Y = \mathbb{R}$ we avail $(\ker(\beta_{\mathbb{R}}))^*$ is linear isometrically isomorphic to $L^\infty(\mathbb{R}) / \mathbb{R}$. (It is known that $Lip_0(\mathbb{R},\mathbb{R})$ is isometrically isomorphic to $L^\infty(\mathbb{R})$ (see [27, p 128])).

3.4 An example

The following proposition answers that the space $L^\infty(\mathbb{R})$ quotient by the set of all constant functions, that is $span\{\mathbf{1}\}$; is a dual space, in fact, we provide its explicit predual space. In this section, by $\mathcal{S}$ we denote the space of all simple functions on $\mathbb{R}$.

Proposition 3.4.1. $\ker(\beta_{\mathbb{R}})$ *is isometrically isomorphic to* $\{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f dx = 0\}$*, where* μ *is the Lebesgue measure on* $\mathbb{R}$.

Proof. Let us define $\Lambda : L^1(\mathbb{R}) \to \mathbb{R}$ given by

$$\Lambda(f) = \int_{\mathbb{R}} f d\mu.$$

Then, Λ is a surjective linear contraction.

Again recall (for details see [17, p 542]) the linear isometric isomorphism between $F(\mathbb{R})$ and $L^1(\mathbb{R})$ say $\phi : F(\mathbb{R}) \to L^1(\mathbb{R})$ whose action on the spanning elements is as follows:

$$\phi(\delta_x) = \begin{cases} -\chi_{(x,0)} & ;\text{if } x < 0 \\ 0 & ;\text{if } x = 0 \\ \chi_{(0,x)} & ;\text{if } x > 0. \end{cases}$$

In fact, from the proof, it follows that $\mathcal{R} = span\{\delta_x : x \in \mathbb{R}\}$ is linearly isometrically isomorphic to the space of all simple functions $\equiv \mathcal{S}$ $(\subset L^1(\mathbb{R}))$.

Let $x \in \mathbb{R}$. Then

$$\Lambda \circ \phi(\delta_x) = \begin{cases} \Lambda(-\chi_{(x,0)}) = x = \beta_{\mathbb{R}}(\delta_x) & \text{;if } x < 0 \\ 0 & \text{;if } x = 0 \\ \Lambda(\chi_{(0,x)}) = x = \beta_{\mathbb{R}}(\delta_x) & \text{;if } x > 0. \end{cases}$$

Thus $\beta_{\mathbb{R}} = \Lambda \circ \phi$. Further

$$\begin{aligned} \ker(\beta_{\mathbb{R}}) &= \{\gamma \in F(\mathbb{R}) : \beta_{\mathbb{R}}(\gamma) = 0\} \\ &\cong \left\{f \in L^1(\mathbb{R}) : \beta_{\mathbb{R}}\left(\phi^{-1}(f)\right) = 0\right\} \\ &= \left\{f \in L^1(\mathbb{R}) : \Lambda(f) = 0\right\} \\ &= \{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f d\mu = 0\}. \end{aligned}$$

where the symbol $'\cong'$ means linearly isometrically isomorphic. This completes the proof.

$\square$

Corollary 3.4.2. $\ker(\beta_{\mathbb{R}}) \cap \mathcal{R}$ *is linearly isometrically isomorphic to* $\{f \in \mathcal{S} : \int_{\mathbb{R}} f d\mu = 0\}$, *where* $\mathcal{R} = span\,\{\delta_x : x \in \mathbb{R}\}$.

Remark 3.4.3. 1. From the above Corollary we have $\overline{\ker(\beta_{\mathbb{R}}) \cap \mathcal{R}}^{\|\cdot\|}$ is linear isometrically isomorphic to $\overline{\{f \in \mathcal{S} : \int_{\mathbb{R}} f d\mu = 0\}}^{\|\cdot\|_1}$. Further suppose (f_n) be a sequence in $\ker(\beta_{\mathbb{R}}) \cap \mathcal{R}$ such that $f_n \xrightarrow{\|\cdot\|_1} f \in L^1(\mathbb{R})$. Then

$$\begin{aligned} \left|\int_{\mathbb{R}} f d\mu\right| &= \left|\int_{\mathbb{R}} (f - f_n + f_n) d\mu\right| \\ &\leq \int_{\mathbb{R}} |f - f_n| d\mu + \left|\int_{\mathbb{R}} f_n d\mu\right|. \end{aligned}$$

Thus $\int_{\mathbb{R}} f d\mu = 0$ so that $\overline{\{f \in \mathcal{S} : \int_{\mathbb{R}} f d\mu = 0\}}^{\|\cdot\|_1} = \{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f d\mu = 0\}$. Therefore, $\overline{\ker(\beta_{\mathbb{R}}) \cap \mathcal{R}}^{\|\cdot\|} = \ker(\beta_{\mathbb{R}})$. Furthermore, Remark 3.3.2 allows us to further deduce that $\ker(\beta_{\mathbb{R}})^*$ is linear isometrically isomorphic to $Lip_0(\mathbb{R}, \mathbb{R}) \Big/ L(\mathbb{R}, \mathbb{R})$ which is further linear isometrically isomorphic to $L^\infty(\mathbb{R}) \Big/ span\{\mathbf{1}\}$.

2. Further if we consider $X = Y = \mathbb{R}$, $\mathcal{D} = \ker(\beta_{\mathbb{R}})$, closed subspace of $F(\mathbb{R})$; then the Proposition 3.3.3 provides that $L\left(F(\mathbb{R}) \Big/ \ker(\beta_{\mathbb{R}}), \mathbb{R}\right)$ is linear isometrically isomorphic to ${}^{\diamond}\ker(\beta_{\mathbb{R}}) = L(\mathbb{R},\mathbb{R})$, where $L(\mathbb{R},\mathbb{R})$ is identified as a subspace of $L^{\infty}(\mathbb{R},\mathbb{R})$ consisting of all constant functions.

Chapter 4

Positive homogeneous Lipschitz maps

In this chapter we focus on positive homogeneous Lipschitz maps, the space consists of this maps and relevant properties. The motivation comes from the following facts. In chapter 3 we have seen that the space of bounded linear maps $L(X,Y)$ between the Banach spaces X and Y, is 1-complemented in $Lip_0(X,Y)$ (whenever Y is a dual space) and the quotient space $Lip_0(X,Y) \Big/ L(X,Y)$ is linearly isometrically isomorphic to an appropriate space of operators, whenever Y is a dual injective space. In particular, we observe that for any infinite-dimensional normed linear space X, the quotient of its linear dual X^* within the Lipschitz dual $Lip_0(X)$ forms a well-behaved space of operators (indeed, a dual space). Motivated by these results, we were led to investigate the following question:

What can be said about the quotient space $Lip_0(X)/\mathcal{A}$ when $\mathcal{A}$ is a finite-dimensional subspace of $Lip_0(X)$?

In this present chapter in Section 4.1, we prove that this quotient space is also a dual space and provide an explicit characterization of its predual.

While searching for a proper closed subspace of $Lip_0(X)$ that contains X^* properly, our attention naturally turns to $Lip_0^{ph}(X)$, the space of positively homogeneous real-valued Lipschitz functions. Interestingly, $Lip_0^{ph}(X)$ turns out to be both a dual space and the pre-annihilator of a closed subspace of the Lipschitz-free space, satisfying

$$X^* \subsetneq Lip_0^{ph}(X) \subsetneq Lip_0(X).$$

Moreover, we define an appropriate multiplication on it (distinct from the one introduced by the authors in [6]), under which $(Lip_0^{ph}(X), Lip(\cdot))$ also becomes a Banach algebra. It is worth noting that $Lip_0(M)$, equipped with the Lipschitz norm and pointwise multiplication, fails to form a Banach algebra unless M is a bounded metric space. In this sense, the space $Lip_0^{ph}(X)$ offers a slight structural advantage. We now discuss the structure of the article and the key findings as follows:

1. For any Banach space X with dimension greater than 1, we identify a proper closed infinite-dimensional subspace, namely $Lip_0^{ph}(X)$ (Space of positive homogeneous Lipschitz maps) of $Lip_0(X)$ that contains X^* as a proper closed subspace, and we further show that $Lip_0(X) \big/ Lip_0^{ph}(X)$ is a dual space (see Corollary 4.2.4).

2. It is well known that $Lip_0(\mathbb{R}, \mathbb{R})$ is linear isometrically isomorphic to $L^\infty(\mathbb{R})$ (see [17]). In contrast, we see that the space $Lip_0^{ph}(\mathbb{R}, \mathbb{R})$ is two dimensional. However, once we move to $\mathbb{R}^2$, the space $Lip_0^{ph}(\mathbb{R}^2, \mathbb{R})$ becomes infinite-dimensional, as its predual is linearly isomorphic to $L^1(\mathbb{R})$ (see Example 4.2.11(1)). Thus, the imposition of positive homogeneity leads to a remarkable change in the dimensional structure of the corresponding Lipschitz spaces over $\mathbb{R}$. In fact, geometrically, the positively homogeneous Lipschitz maps from $\mathbb{R}$ to $\mathbb{R}$ are precisely the piecewise affine maps that may fail to be differentiable at most at a single point, namely the origin.

3. Analogous to the Lipschitz free space, we identify a canonical predual $F^{ph}(X)$ of $Lip_0^{ph}(X)$ and examine its relationship with the Lipschitz free space over X in Proposition 4.2.9. Furthermore, we show that whenever $\dim(X) > 1$, the space $F^{ph}(X)$ contains a subspace that is linearly isomorphic to $L^1(\mathbb{R})$ (see Proposition 4.3.2). To prove this, we use the McShane-type extension theorem for positive homogeneous Lipschitz maps, proved in Section 4.3.

To move forward, we recall the following results from the theory of Lipschitz functions, which will be used in the subsequent sections. The canonical predual of $Lip_0(M)$; the

Lipschitz free space associated with M is denoted by $F(M)$. It is given by

$$F(M) = \overline{span}^{\|\cdot\|}\{\delta_x : x \in M\} \subset Lip_0(M)^*,$$

where $\delta_x : Lip_0(M) \to \mathbb{R}$ is the linear functional defined by $\delta_x(f) = f(x)$ for all $f \in Lip_0(M)$. Again recall that a key property of Lipschitz free spaces is their universal linearization of Lipschitz functions. Specifically, for any Lipschitz map $f : M \to N$ between pointed metric spaces such that $f(0_M) = 0_N$, there exists a bounded linear operator $C_f : F(M) \to F(N)$ satisfying $\|C_f\| = Lip(f)$, and such that $C_f \circ \delta_M = \delta_N \circ f$, where the map $\delta_M : M \to F(M)$ $(x \mapsto \delta_x)$ is a nonlinear isometric embedding of a metric space into its Lipschitz free space. Another important property is the universal property of Lipschitz maps. Given $f \in Lip_0(M, X)$, there exists a unique bounded linear map $T_f \in L(F(M), X)$ such that

$$T_f \circ \delta_M = f \quad \text{and} \quad Lip(f) = \|T_f\|.$$

The nonlinear isometric embedding δ_X has a linear left inverse $\beta_X : F(X) \to X$, that acts on the basis elements as $\beta_X(\delta_x) = x$. For further details on these concepts, we refer to the important paper by Godefroy and Kalton [27], and the monograph by Weaver [57], which provide a comprehensive study of Lipschitz functions.

Once again, let us recall the following notation. Given X and Y Banach spaces, $X \simeq Y$ stands for X is linearly isomorphic to Y. $X \simeq^h Y$ stands for X is linearly homeomorphic to Y. $X \cong Y$ stands for X is linear isometrically isomorphic to Y. $X \overset{c}{\hookrightarrow} Y$ stands for X is linear isomorphic to a complemented subspace of Y. The closed unit ball and unit sphere of X are denoted by B_X and S_X, respectively.

4.1 Quotient of Finite-Dimensional Spaces in the Lipschitz Dual

Let $\mathcal{A}$ be any n-dimensional subspace of $Lip_0(X, \mathbb{R})$. Therefore, $\mathcal{A} = span\{f_1, f_2, ..., f_n\}$ for some n linearly independent elements $f_1, f_2, ..., f_n \in Lip_0(X)$. The following result provides an explicit identification of the space $Lip_0(X) / \mathcal{A}$.

Proposition 4.1.1. *Let $f_1, f_2, ..., f_n \in Lip_0(X)$ be linearly independent. Then $Lip_0(X) \Big/ span\{f_1, f_2, ..., f_n\}$ is linear isometrically isomorphic to $\left(\bigcap_{i=1}^{n} \ker(T_{f_i})\right)^*$.*

Proof. We consider the closed subspace $M = \bigcap_{i=1}^{n} \ker(T_{f_i})$ of $F(X)$. As $Lip_0(X)$ and $F(X)$ form a dual pair, it is enough to show that $M^{\perp} = span\{f_1, f_2, ..., f_n\}$. Since $F(X)^*$ is linear isometrically isomorphic to $Lip_0(X)$

$$M^{\perp} = \left\{ g \in Lip_0(X) : \gamma(g) = 0 \; \forall\, \gamma \in \bigcap_{i=1}^{n} \ker(T_{f_i}) \right\}.$$

Therefore, $\gamma(g) = 0 \; \forall\, \gamma \in \bigcap_{i=1}^{n} \ker(T_{f_i})$ implies $\bigcap_{i=1}^{n} \ker(T_{f_i}) \subset \ker(T_g)$. Then it follows from a standard technique that $T_g \in span\{T_{f_1}, T_{f_2}, ..., T_{f_n}\}$. That is. there exists $\alpha_i \in \mathbb{R}$ for $i = 1, 2, ..., n$ such that

$$\begin{aligned} T_g = & \sum_{i=1}^{n} \alpha_i T_{f_i} \\ \iff & T_g(\delta_x) = \sum_{i=1}^{n} \alpha_i T_{f_i}(\delta_x) \;\; \forall x \in X. \\ \iff & g \in span\{f_1, f_2, ..., f_n\}. \end{aligned}$$

Hence we get $M^{\perp} = span\{f_1, f_2, ..., f_n\}$ and from [51, Theorem 4.9] the proof follows. □

Remark 4.1.2. 1. Although the above result is proved for Banach spaces, it remains valid when X is replaced by a pointed metric space, and the proof proceeds by the same method.

2. If we consider $X = \mathbb{R}$ and the one dimensional subspace $span\{Id_{\mathbb{R}}\}$, then it follows from the result that $Lip_0(\mathbb{R}) \big/ span\{Id_{\mathbb{R}}\} \cong \ker(\beta_{\mathbb{R}})^*$, which has been shown as a corollary in [37, Remark 5.5].

3. In general, for any finite-dimensional subspace $\mathcal{A}$ of $Lip_0(X,Y)$, it is not known to the author whether the quotient space $Lip_0(X,Y) / \mathcal{A}$ can be identified with a space of linear operators.

4.2 A Proper Closed Subspace of the Lipschitz Dual Properly Containing the Linear Dual

One of our other goal in this chapter is to identify a proper closed subspace of $Lip_0(X)$ that contains X^* as a closed subspace. It is well-known that positive homogeneous uniformly continuous maps between Banach spaces are known to be Lipschitz. In seeking a proper closed subspace of $Lip_0(X)$ containing X^*, the author focused on positively homogeneous Lipschitz maps. In pursuit of this objective, for any Banach spaces X and Y, we consider a subset $S \subset X$ which is closed under nonnegative scalar multiplication (hence $0 \in S$), equipped with the metric induced by the norm on X, that is, $d(x,w) := \|x - w\|$ for all $x, w \in S$. We consider

$$Lip_0^{ph}(S,Y) := \{f \in Lip_0(S,Y) : f(\alpha x) = \alpha f(x) \, \forall \, \alpha \geq 0, \, x \in X\},$$

with the $Lip(\cdot)$ norm. Since any sequence (f_n) in $Lip_0^{ph}(S,Y)$ converges to $f \in Lip_0(S,Y)$ in $Lip(\cdot)$ norm; implies f_n converges to point-wise to f also. It follows that the space $Lip_0^{ph}(S,Y)$ is a closed subspace of $Lip_0(S,Y)$. In particular, $Lip_0^{ph}(X,Y)$ contains $L(X,Y)$ as a closed subspace. Moreover, the inclusions are strict as the map $x \to \|x\|y_0$ is in $Lip_0^{ph}(X,Y) \setminus L(X,Y)$, where as the map $x \to \|x - x_0\|y_0 - \|x_0\|y_0$ is in $Lip_0(X,Y) \setminus Lip_0^{ph}(X,Y)$ for some fixed $x_0 \in X \setminus \{0\}$ and $y_0 \in S_Y$. For $Y = \mathbb{R}$ we denote the space of positive homogeneous real-valued Lipschitz maps on X by $Lip_0^{ph}(X)$.

Moreover, it is known that on the closed unit ball $B_{Lip_0(X)}$, the $weak^*$ topology coincides with the topology of pointwise convergence. It is also straightforward to verify that $B_{Lip_0^{ph}(X)}$ is closed in $B_{Lip_0(X)}$ with respect to this topology. Since $B_{Lip_0(X)}$ is compact under the topology of pointwise convergence, its closed subset $B_{Lip_0^{ph}(X)}$ is likewise compact. Hence, by the Ng–Dixmier theorem [47, Theorem 1], it follows that $Lip_0^{ph}(X)$ is isometrically isomorphic to a dual space. In analogy with the predual of $Lip_0(X)$, we construct the predual of $Lip_0^{ph}(X)$. To proceed in this direction, we first present the following facts. The proofs follow similar lines to the classical case, while the differing parts can be derived from the hints provided herein.

Facts 4.2.1. 1. *Recall the pointwise evaluation functional $\delta_x \in Lip_0(X)^*$. By δ_x^{ph} we denote its restriction to $Lip_0^{ph}(X)$, that is, $\delta_x^{ph} := \delta_x\big|_{Lip_0^{ph}(X)}$. We then define*

$$F^{ph}(X) := \overline{span}^{\|\cdot\|}\left\{\delta_x^{ph} : x \in X\right\},$$

which is a closed subspace of $Lip_0^{ph}(X)^*$.

2. *Analogous to the map $\delta_X : X \to F(X)$ (a nonlinear isometry), the map $\delta_X^{ph} : X \to F^{ph}(X), \quad x \mapsto \delta_x^{ph}$, is also a nonlinear isometry. However, the proof of this isometry differs slightly from the usual one, and we therefore provide its details below.*

   Let $x, y \in X$. Then $\|\delta_x^{ph} - \delta_y^{ph}\| = \sup_{f \in B_{Lip_0^{ph}(X)}} |f(x) - f(y)| \leq \|x - y\|$. For the equality, consider the functional (exists due to Hahn-Banach theorem) $g \in S_{X^*} \left(\subset B_{Lip_0^{ph}(X)}\right)$ such that $g(x-y) = \|x-y\|$. Then $\|\delta_x^{ph} - \delta_y^{ph}\| \geq |g(x) - g(y)| = \|x - y\|$. Hence, the proof follows.

3. *Since δ_X^{ph} is the restriction of δ_X to $Lip_0^{ph}(X)$, it is positive homogeneous—unlike the map δ_X itself.*

4. *For any Banach space X, define*

$$\langle \cdot,\cdot\rangle : Lip_0^{ph}(X) \times F^{ph}(X) \to \mathbb{R}$$

given by

$$\langle f,\mu\rangle := \mu(f)$$

for all $f \in Lip_0^{ph}(X)$ and $\mu \in F^{ph}(X)$. Then $\langle \cdot,\cdot\rangle$ is bilinear and non-degenerate. Also, it can be easily deduced that

$$\sup_{f\in B_{Lip_0^{ph}(X,\mathbb{R})}} |\mu(f)| = \|\mu\| \ \textit{ and } \sup_{\mu\in B_{F^{ph}(X)}} |\mu(f)| = Lip(f).$$

5. *$Lip_0^{ph}(X,Y)$ is linear isometrically isomorphic to $L\left(F^{ph}(X),Y\right)$ via the map $f \mapsto T_f^{ph}$, where f and T_f^{ph} satisfy the following commutative diagram.*

$$\begin{tikzcd} & F^{ph}(X) \arrow[dr, "T_f^{ph}"] & \\ X \arrow[ur, "\delta_X^{ph}"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

The proof follows similar to the classical argument of proving $Lip_0(X,Y)$ is linear isometrically isomorphic to $L(F(X),Y)$, relying on the fact that δ_X^{ph} is positive homogeneous.

6. *Let X and Y be Banach spaces and suppose $f \in Lip_0^{ph}(X,Y)$. Then there exists a unique linear map $C_f^{ph} : F^{ph}(X) \to F^{ph}(Y)$ such that $C_f^{ph} \circ \delta_X^{ph} = \delta_Y^{ph} \circ f$, that is, the following diagram commutes:*

$$\begin{tikzcd} & F^{ph}(X) \arrow[rr, "C_f^{ph}"] & & F^{ph}(Y) \\ X \arrow[ur, "\delta_X^{ph}"] \arrow[rr, "f"'] & & Y \arrow[ur, "\delta_Y^{ph}"'] & \end{tikzcd}.$$

The proof of it follows similar to [27, Lemma 2.2].

4.2.1 $Lip_0^{ph}(X)$ as a quotient space in $Lip_0(X)$

Proposition 4.2.2. *For any Banach space X, consider the closed subspace of $F(X)$*

$$\overline{span}^{\|\cdot\|}\big\{r\delta_x - \delta_{rx} : r \geq 0,\ x \in X\big\}.$$

Then, the pre-annihilator of $Lip_0^{ph}(X)$ is given by ${}^{\perp}Lip_0^{ph}(X) = \overline{span}^{\|\cdot\|}\big\{r\delta_x - \delta_{rx} : r \geq 0,\ x \in X\big\}$.

Proof. Since every $f \in Lip_0^{ph}(X)$ is positive homogeneous and ${}^{\perp}Lip_0^{ph}(X)$ is a closed subspace of $F(X)$, we have $\overline{\text{span}}^{\|\cdot\|}\{r\delta_x - \delta_{rx} : r \geq 0,\ x \in X\} \subset {}^{\perp}Lip_0^{ph}(X)$.

Now, suppose $\gamma \in {}^{\perp}Lip_0^{ph}(X) \setminus \overline{\text{span}}^{\|\cdot\|}\{r\delta_x - \delta_{rx} : r \geq 0,\ x \in X\}$. Then, by the Hahn-Banach separation theorem (see [51, Theorem 3.5]), there exists $g \in Lip_0(X)$ such that $\gamma(g) = 1$ and $\xi(g) = 0$ for all $\xi \in \overline{\text{span}}^{\|\cdot\|}\{r\delta_x - \delta_{rx} : r \geq 0,\ x \in X\}$. In particular, $(r\delta_x - \delta_{rx})(g) = 0$ for all $x \in X$ and $r \geq 0$. Therefore, $g \in Lip_0^{ph}(X)$, which implies $\gamma(g) = 0$, a contradiction. Hence, the claim follows. □

Remark 4.2.3. A similar kind of arguments provides that

$$\overline{\text{span}}^{\|\cdot\|}\{r\delta_x - \delta_{rx} : r \geq 0,\ x \in X\}^{\perp} = Lip_0^{ph}(X).$$

Corollary 4.2.4. *From the above facts we have the following :*

$$Lip_0(X) \Big/ Lip_0^{ph}(X) = \overline{\text{span}}^{\|\cdot\|}\{r\delta_x - \delta_{rx} : r \geq 0,\ x \in X\}^*.$$

Now we see that $Lip_0^{ph}(X)$ is embeded as a complemented subspace in $Lip_0(X)$.

Proposition 4.2.5. *Let X be a Banach space. Then there exists a projection from $Lip_0(X)$ onto $Lip_0^{ph}(X)$.*

Proof. For any $f \in Lip_0(X)$ we consider

$$\tilde{f}(x) = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right), & \text{if } x \neq 0 \\ 0, & \text{if} x = 0. \end{cases}$$

Then $\tilde{f} \in Lip_0^{ph}(X)$ with $Lip(\tilde{f}) \leq 3Lip(f)$. Now we define $P : Lip_0(X) \to Lip_0^{ph}(X)$ given by $P(f) = \tilde{f}$. It is easy to verify that P is bounded linear surjective projection map. $\square$

The next corollary follows from the fact that $\ker(P) = \left\{f \in Lip_0(X) : f\big|_{S_X} = 0\right\}$.

Corollary 4.2.6. *For any Banach space X, $Lip_0(X) = \left\{f \in Lip_0(X) : f\big|_{S_X} = 0\right\} \oplus Lip_0^{ph}(X)$.*

The next result provides a classification of all positive homogeneous real-valued Lipschitz maps on a Banach space by relating them to real-valued Lipschitz maps whose domains are suitably restricted. On $S_X \cup \{0\}$ we consider the restricted metric induced by the norm $\|\cdot\|$ on the Banach space X.

Proposition 4.2.7. *Let X and Z be a Banach spaces. Then $\left(Lip_0^{ph}(X,Z), Lip(\cdot)\right)$ is linearly homeomorphic to $(Lip_0(S_X \cup \{0\}, Z), Lip(\cdot))$.*

Proof. We define $\Lambda : Lip_0^{ph}(X,Z) \to Lip_0(S_X \cup \{0\}, Z)$ given by $\Lambda(f) = f\big|_{S_X \cup \{0\}}$. Clearly Λ is a linear contraction. Using the positive homogeneity of the functions we can prove $ker(\Lambda) = \{0\}$.

Let $g \in Lip_0(S_X \cup \{0\}, Z)$. Put

$$h(x) = \begin{cases} \|x\| g\left(\frac{x}{\|x\|}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then h is positive homogeneous. We show that h is Lipschitz also. Suppose $x, y \in X \setminus \{0\}$. Then

$$\begin{aligned} |h(x) - h(y)| &= \left| \|x\| g\left(\frac{x}{\|x\|}\right) - \|x\| g\left(\frac{y}{\|y\|}\right) + (\|x\| - \|y\|) g\left(\frac{y}{\|y\|}\right) \right| \\ &\leq Lip(g)\|x - y\| + Lip(g)\|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|. \end{aligned}$$

Again, by a standard technique, it follows that $\|x\|\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq 2\|x-y\|$. Therefore, $|h(x)-h(y)| \leq 3Lip(g)\|x-y\|$. The case is considerably simpler when either x or y equals 0. Furthermore, it is straightforward to see that $\Lambda(h) = g$. This completes the proof. □

Remark 4.2.8. For $Z = \mathbb{R}$ with dimension of X greater equal to 2, it follows from [16, Theorem 3.2] and the Proposition 4.2.7 that $\ell_\infty \hookrightarrow Lip_0^{ph}(X)$.

This linear homeomorphism naturally induces a corresponding linear homeomorphism between their predual spaces, which we shall establish in the following proposition. Since the proof follows a similar way using this map Λ, we only provide the outline of it.

Proposition 4.2.9. *For any Banach space X, $F(S_X)$ and $F^{ph}(X)$ are linearly homeomorphic.*

Proof. Let us define the map $\Theta : span\{\delta_x : x \in S_X\} \to F^{ph}(X)$ given by $\Theta\left(\sum\limits_{i=1}^{n} \alpha_i \delta_{x_i}\right) = \sum\limits_{i=1}^{n} \alpha_i \delta_{x_i}^{ph}$. Then the linearity of Θ follows from the definition.

We show Θ is a contraction. For any $\mu = \sum\limits_{i=1}^{n} \alpha_i \delta_{x_i} \in F(S_X)$:

$$\|\Theta(\mu)\|_{F^{ph}(X)} = \sup_{f \in B_{Lip_0^{ph}(X)}} \left|\sum_{i=1}^{n} \alpha_i f(x_i)\right|$$

Since $x_i \in S_X$, the restriction map $\Lambda : Lip_0^{ph}(X) \to Lip_0(S_X \cup \{0\})$ given by $\Lambda(f) = f|_{S_X \cup \{0\}}$ is a contraction (from Proposition 4.2.7). Since every $f \in B_{Lip_0^{ph}(X)}$ restricts to a function in $B_{Lip_0(S_X \cup \{0\})}$, the supremum over the former is bounded by the supremum over the latter:

$$\|\Theta(\mu)\|_{F^{ph}(X)} \leq \sup_{g \in B_{Lip_0(S_X \cup \{0\})}} \left|\sum_{i=1}^{n} \alpha_i g(x_i)\right| = \|\mu\|_{F(S_X)}$$

Thus, $\|\Theta\| \leq 1$ and hence Θ extends uniquely to a continuous linear map $\Theta : F(S_X) \to F^{ph}(X)$.

and boundedness (contraction) holds using the map Λ from the above proposition. Since Θ is continuous and $F^{ph}(X)$ is complete Θ can be extended continuously to $F(S_X)$.

Next, we show that Θ is invertible and we find out its inverse. First, we note that since δ_x^{ph} is positive homogeneous, we have $\delta_x^{ph} = \|x\|\delta_{x/\|x\|}^{ph}$ for any $x \in X \setminus \{0\}$. Thus, $span\{\delta_x^{ph} : x \in X\} = span\{\delta_x^{ph} : x \in S_X\}$.

Let us define $\Phi : span\{\delta_x^{ph} : x \in S_X\} \to F(S_X)$ by:

$$\Phi\left(\sum_{i=1}^{n} \alpha_i \delta_{x_i}^{ph}\right) = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$$

We use the result from Proposition 4.2.7 we have for any $g \in Lip_0(S_X \cup \{0\})$ there exists a unique $\Lambda^{-1}(g) \in Lip_0^{ph}(X)$ such that $Lip\left(\Lambda^{-1}(g)\right) \leq 3Lip(g)$.

$$\left\|\Phi\left(\sum_{i=1}^{n} \alpha_i \delta_{x_i}^{ph}\right)\right\|_{F(S_X)} = \sup_{g \in B_{Lip_0(S_X \cup \{0\})}} \left|\sum_{i=1}^{n} \alpha_i g(x_i)\right|$$

For each $g \in B_{Lip_0(S_X \cup \{0\})}$, let $h = \frac{1}{3}\Lambda^{-1}(g)$. Then $h \in B_{Lip_0^{ph}(X)}$ and $h(x_i) = \frac{1}{3}g(x_i)$ for $x_i \in S_X$.

$$\sup_{g \in B_{Lip_0(S_X \cup \{0\})}} \left|\sum_{i=1}^{n} \alpha_i g(x_i)\right| \leq \sup_{h \in 3B_{Lip_0^{ph}(X)}} \left|\sum_{i=1}^{n} \alpha_i h(x_i)\right| = 3\left\|\sum_{i=1}^{n} \alpha_i \delta_{x_i}^{ph}\right\|_{F^{ph}(X)}$$

Thus, $\|\Phi\| \leq 3$. This map extends continuously to the whole of $F^{ph}(X)$.

Again $\Phi : span\{\delta_x^{ph} : x \in S_X\} \to F(S_X)$ given by $\Phi\left(\sum\limits_{i=1}^{n} \alpha_i \delta_{y_i}^{ph}\right) = \sum\limits_{i=1}^{n} \alpha_i \delta_{y_i}^{ph}$ and sends 0 to 0. Then Φ is linear bounded (by the constant 3), can be extended to $F^{ph}(X)$. Also, immediately it follows that Φ is the inverse of Θ. This completes the proof. $\square$

We work out a couple of examples for a better understanding of the space.

Example 4.2.10. *Consider $X = \mathbb{R}$. Then $F(S_{\mathbb{R}}) = F(\{\pm 1\})$ and $F^{ph}(\mathbb{R})$ are linearly homeomorphic. Therefore, $F^{ph}(\mathbb{R})$ is linearly homeomorphic to $\mathbb{R}^2$. Hence, $Lip_0^{ph}(\mathbb{R}, \mathbb{R})$ is also linearly homeomorphic to $\mathbb{R}^2$. However, independently, we show that $Lip_0^{ph}(\mathbb{R}, \mathbb{R})$ is linear isometrically isomorphic to $(\mathbb{R}^2, \|\cdot\|_\infty)$.*

For any $f \in Lip_0^{ph}(\mathbb{R}, \mathbb{R})$ we have

$$f(x) = \begin{cases} xf(1), & \text{if } x \geq 0 \\ -xf(-1), & \text{if } x < 0. \end{cases}$$

We define the following surjective linear map $\xi : Lip_0^{ph}(\mathbb{R}, \mathbb{R}) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ *given by* $\xi(f) = (f(1), f(-1))$. *Clearly,* $\sup\limits_{x,y\in[0,\infty),x\neq y} \frac{|f(x)-f(y)|}{|x-y|} = |f(1)|$ *and* $\sup\limits_{x,y\in(-\infty,0],x\neq y} \frac{|f(x)-f(y)|}{|x-y|} = |f(-1)|$.

Suppose $x > 0$ *and* $y = -t$ *for some* $t > 0$. *Then* $f(x) - f(y) = xf(1) - tf(-1)$, *and* $|x - y| = x + t > 0$. *Now*

$$\begin{aligned} \sup_{x>0,y<0} \frac{|f(x)-f(y)|}{|x-y|} &= \sup_{x,t>0} \frac{|xf(1)-tf(-1)|}{x+t} \\ &= \sup_{x,t>0} \left| \frac{x}{x+t} f(1) + \frac{t}{x+t}(-f(-1)) \right| \\ &= max\{|f(1)|, |f(-1)|\} \end{aligned}$$

Combining all the cases, we have $Lip(f) = \|(f(1), f(-1))\|_\infty$ *and hence the isometry follows.*

In addition, it can be easily deduced that $Lip_0^{ph}([0,\infty), \mathbb{R})$ is one dimensional.

Example 4.2.11.

1. *From the Proposition 4.2.9 it follows that* $F(S_{\mathbb{R}^2}) \simeq^h F^{ph}(\mathbb{R}^2)$. *Moreover, from [5] and [28] we know that* $F(S_{\mathbb{R}^2}) \simeq F(\mathbb{R}) \cong L^1(\mathbb{R})$. *Therfore, we get* $F^{ph}(\mathbb{R}^2)$ *is linearly isomorphic to* $L^1(\mathbb{R})$ *and hence* $Lip_0^{ph}(\mathbb{R}^2, \mathbb{R})$ *is infinite dimensional.*

2. *In fact, later on we prove that for any normed linear space* X *with* $dim(X) \geq 2$ $('dim'$ *stands for dimension),* $Lip_0^{ph}(X)$ *is of infinite dimension.*

3. *As an application of the previous Corollary 4.2.4 and the above Example 4.2.10 we have* $L^\infty(\mathbb{R}) /_{\mathbb{R}^2} \cong \overline{span}^{\|\cdot\|} \{r\delta_x - \delta_{rx} : r \geq 0,\ x \in \mathbb{R}\}^*$

4.2.2 Characterization of a Lipschitz map to become positive homogeneous

Proposition 4.2.12. *Let* $f \in Lip_0(X, Y)$. *Then* $f \in Lip_0^{ph}(X, Y)$ *if and only if* $\overline{span\, \{\delta_{rx} - r\delta_x : r > 0, x \in X\}}^{\|\cdot\|} \subseteq \ker(T_f)$.

Proof. Let $r > 0$ and $x \in X$. Then $T_f(\delta_{rx} - r\delta_x) = 0$ implies $f(rx) = rf(x)$. Therefore, $f \in Lip_0^{ph}(X, Y)$.

Converse part directly follows from the fact that ker(T_f) is a closed subspace and $\{\delta_{rx} - r\delta_x : r > 0, x \in X\} \subseteq$ ker(T_f). □

4.2.3 $Lip_0^{ph}(X, \mathbb{R})$ as a Banach algebra

Here, we study the Banach space $Lip_0^{ph}(X, \mathbb{R})$ as an algebra over an arbitrary Banach space X. When the pointed metric space $(M, d, 0)$ is bounded, the space Lip$_0(M)$ constitutes a Banach algebra in a weak sense; that is, there exists a constant $C > 1$ such that $Lip(fg) \leq CLip(f)Lip(g)$ for all $f, g \in Lip_0(M)$. In fact, one may choose $C = 2\max\limits_{x\in M} d(x, 0)$. Such structures are referred to as *Gelfand algebras* in [57]. It is worth noting that in the usual definition of a Banach algebra, the above inequality is required to hold with $C = 1$. As discussed in [57, Chapter 7], when M is bounded, one can introduce an equivalent submultiplicative norm on $Lip_0(M)$, making it a genuine Banach algebra. Consequently, since $Lip_0^{ph}(X)$ is a closed subspace of $Lip_0(X)$, it forms a Banach subalgebra. Moreover, we construct another multiplication, obtained by multiplying the usual pointwise product by a factor, defined via the map Λ as in Proposition 4.2.7, with respect to which $(Lip_0^{ph}(X), Lip(\cdot))$ itself becomes a Banach algebra.

Since $S_X \cup \{0\}$ is bounded the space $Lip_0(S_X \cup \{0\}, \mathbb{R})$ is closed under usual pointwise multiplication. Hence, for any $f, g \in Lip_0^{ph}(X)$, it follows that $\Lambda(f)\Lambda(g) \in Lip_0(S_X \cup \{0\}, \mathbb{R})$. We now define

$$\begin{aligned} f \odot g(x) &:= \frac{1}{3}\Lambda^{-1}\left(\Lambda(f)\Lambda(g)\right)(x) \\ &= \begin{cases} \frac{f(x)g(x)}{3\|x\|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases} \end{aligned}$$

Directly, from the definition $f \odot g \in$ Lip$_0^{ph}(X)$. It only remains to show that $Lip(f \odot g) \leq$

$Lip(f)Lip(g)$. Suppose $x, y \in X \setminus \{0\}$. Now

$$\begin{aligned}\left|\frac{f(x)g(x)}{3\|x\|}-\frac{f(y)g(y)}{3\|y\|}\right| &\leq \frac{1}{3}\left\{\left|\frac{1}{\|x\|}f(x)\left(g(x)-g(y)\right)\right|+\left|g(y)\left(\frac{1}{\|x\|}f(x)-\frac{1}{\|y\|}f(y)\right)\right|\right\} \\ &\leq \frac{1}{3}\left\{Lip(f)Lip(g)\|x-y\|+2Lip(f)Lip(g)\|x-y\|\right\} \\ &\leq Lip(f)Lip(g)\|x-y\|.\end{aligned}$$

Further, for $x \neq 0$ and $y = 0$ we get $|f \odot g(x)| \leq Lip(f)Lip(g)\|x\|$. Hence, we have $Lip(f \odot g) \leq Lip(f)Lip(g)$. This recovers the definition of defining multiplication on $Lip_0(X)$, so that it becomes a Banach algebra with respect to $Lip(\cdot)$ norm, by Fernando Albiac *et al.* in [6].

4.2.4 Complementability, and Isomorphism of $F(X)$ and $F^{ph}(X)$

We record at this point that Benyamini and Sternfeld proved in [10] that there exists a Lipschitz retraction, say R, from B_X onto S_X whenever X is an infinite-dimensional normed linear space. Now we consider the following Lipschitz projection $G : X \to B_X$ given by $G(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| \geq 1 \\ x, & \text{if } \|x\| \leq 1. \end{cases}$ Then $R \circ G : X \to S_X$ is a Lipschitz retraction. Note that $R \circ G(0) = R(0) \in S_X$ and hence $R \circ R(0) = R(0)$. Now, from the remark just after [27, Lemma 2.3] we have that $F(S_X)$ is complemented in $F(X)$. Then by Proposition 4.2.9 we get $F^{ph}(X)$ sits as a complemented subspace in $F(X)$.

In [13], Leandro Candido and Pedro L. Kaufmann established a key result that connects $F(X)$ and $F(S_X)$ as isomorphic spaces. They proved that if a Banach space X is isomorphic to its hyperplanes, then the spaces $F(X)$ and $F(S_X)$ are also isomorphic. In view of their result and the Proposition 4.2.9 we also have the following remark.

Remark 4.2.13. Let X be a Banach space such that X is isomorphic to its hyperplanes. Then $F(X)$ and $F^{ph}(X)$ are isomorphic.

4.3 McShane's type extension theorem for positive homogeneous real-valued Lipschitz functions

E. J. McShane, in [43, Theorem 1], proved that any real-valued function f defined on a subset E of a metric space M can be extended to the entire space M while preserving the same Lipschitz constant. In the next proposition, we establish an analogous extension theorem for a positive homogeneous Lipschitz map. The proof is obtained by making a slight modification to the classical one. For further details on McShane's extension theorem, we also refer the reader to [14, Theorem 4.1.1]. The necessary modification is presented in the following proposition, while the remaining arguments follow from the cited proof.

Proposition 4.3.1. *Let S be a subset of the Banach space X, which is closed under non-negative scalar multiplication, and $f \in Lip_0^{ph}(S)$. Then f can be extended to X preserving the same Lipschitz constant.*

Proof. Let us define the functions $F, G : X \to \mathbb{R}$, defined for $x \in X$ by

$$F(x) := \sup_{y \in S} \left(f(y) - Lip(f)\|x - y\| \right)$$

and

$$G(x) := \inf_{y \in S} \left(f(y) + Lip(f)\|x - y\| \right).$$

We show that F, G are positive homogeneous Lipschitz extensions of f preserving the same Lipschitz constant $Lip(f)$, and any other positive homogeneous Lipschitz extension H of f with $Lip(H) = Lip(f)$ satisfies the inequalities $F \leq H \leq G$. We only show that F and G are positive homogeneous, and the rest of the proof follows the same as [14, Theorem 4.1.1]. Since $0 \in S$, $F(0) = G(0) = 0$. Let $t > 0$ and $x \in X$. Because S is closed under positive scalar multiplication, the map $y \mapsto z := y/t$ is a bijection of S onto itself. Therefore,

$$F(tx) \quad = \quad \sup_{y \in S} \left(f(y) - Lip(f)\|tx - y\| \right)$$

$$
\begin{aligned}
&= \sup_{z\in S}\left(f(tz) - Lip(f)\|tx - tz\|\right) \\
&= \sup_{z\in S}\left(tf(z) - Lip(f)t\|x - z\|\right) \\
&= t\sup_{z\in S}\left(f(z) - Lip(f)\|x - z\|\right) \\
&= tF(x).
\end{aligned}
$$

Hence, $F \in Lip_0^{ph}(X)$ and similarly we can prove the same for G. □

We use this extension result to prove that $Lip_0^{ph}(X)$ is infinite dimensional for $dim(X) > 1$.

Proposition 4.3.2. *Let X be a Banach space with $dim(X) \geq 2$. Then $F^{ph}(X)$ contains a copy of $L^1(\mathbb{R})$ as a subspace, and therefore $Lip_0^{ph}(X)$ is infinite dimensional.*

Proof. Let x_1, x_2 be two linearly independent vectors in X. We consider the norm on $\mathbb{R}^2$ induced by the norm on X, that is $\|(a,b)\|_i = \|ax_1 + bx_2\|$ for all $(a,b) \in \mathbb{R}^2$. Then it follows that $\|\cdot\|_i$ is a norm on $\mathbb{R}^2$ and hence the mapping $\mathfrak{i} : (\mathbb{R}^2, \|\cdot\|_i) \to X$ given by $\mathfrak{i}(a,b) = ax_1 + bx_2$ is a linear isometry. Put $Z = \mathfrak{i}(\mathbb{R}^2)$. then the inclusion map $i : Z \to X$ is the canonical embedding. Now recall that the classical linearization map of $\delta_X \circ i$ in the Lipschitz, namely $\widehat{\delta_X \circ i} : F(Z) \to F(X)$ becomes an isometric embedding. The proof relies on the fact that any $h \in Lip_0(Z)$ can be extended to $\hat{h} \in Lip_0(X)$ without altering the Lipschitz norm. Similarly, from Proposition 4.3.1, any $h \in Lip_0^{ph}(Z)$ can be extended to $\hat{h} \in Lip_0^{ph}(X)$ without changing the Lipschitz norm. Hence, in this setup, $F^{ph}(\mathbb{R}^2)$ appears as a closed subspace of $F^{ph}(X)$. Therefore, by Example 4.2.11, $F^{ph}(X)$ contains a copy of $L^1(\mathbb{R})$ as a subspace, and consequently, $Lip_0^{ph}(X)$ is infinite dimensional. □

Chapter 5

Position of $Blin(X,Y;E)$ in $BLip_0(X,Y;E)$

In this present chapter we focus on studying the Banach space $BLip_0(X,Y;E)$ of two–Lipschitz maps $T\colon X\times Y\to E$ vanishing on the coordinate axes, endowed with the usual two–variable Lipschitz seminorm (which becomes a norm on $BLip_0$). The main themes are:

- Complementation of the bilinear subspace $Blin(X,Y;E)$ inside $BLip_0(X,Y;E)$ under suitable assumptions on the range space E.
- A vector–valued duality between the space $BLip_0(X,Y;E)$ and the projective tensor product $F(X)\,\widehat{\otimes}_\pi\,F(Y)$ of Lipschitz–free spaces.
- Identification of various quotient spaces of $BLip_0(X,Y;E)$, in particular when E is injective.
- A characterisation of when a two–Lipschitz operator is actually bilinear, phrased in terms of annihilating a subspace generated by $\ker(\beta_X)$ and $\ker(\beta_Y)$, where $\beta_X\colon F(X)\to X$ is the barycentre map.
- An application identifying a concrete predual of $L^\infty(\mathbb{R}^2)/\operatorname{span}\{\mathbf{1}\}$ via the tensor description.

To begin with, to address one of the questions considered in this chapter, we recall the discussion (1.1) from Chapter 1. Motivated by Lindenstrauss's result, this discussion leads us to the following question:

Question 5.0.1. *Is $Blin(X, Y; E)$ complemented in $BLip_0(X, Y; E)$?*

We show that there is a surjective linear projection $P : BLip_0(X, Y; E) \to Blin(X, Y; E)$ with norm one, whenever E is a dual space. The proof technique relies on the existence of a vector-valued invariant mean (a vector-valued bounded linear map from the space of bounded functions on a normed space; details have been discussed later), which is guaranteed whenever the codomain space E is a dual space. Given a *two-Lipschitz* map $T \in BLip_0(X, Y; E)$, we construct a suitable function in $\ell^\infty(X \times Y, E)$ and then apply the vector-valued invariant mean to the bounded function derived from T. The existence of such a mean enables us to affirmatively answer Question 5.0.1 in the case where E is a dual space. However, the general case remains open. In summary, when E is a dual space, $Blin(X, Y; E)$ is a complemented subspace of $BLip_0(X, Y; E)$, since $BLip_0(X, Y; E) = Blin(X, Y; E) \oplus_1 \ker(P)$ with equivalent norms, implying that $BLip_0(X, Y; E) \big/ Blin(X, Y; E)$ is topologically isomorphic to $\ker(P)$.

5.1 General quotient structure of *two-Lipschitz* space

However, the author has been further interested in addressing the following question:

Question 5.1.1. *Is this quotient space not only topologically but also isometrically isomorphic to a concrete space of operators?*

In response, we show that when E is an injective Banach space; the quotient is indeed isometrically isomorphic to a well-defined operator space. Moreover, we identify additional quotient spaces of $BLip_0(X, Y; E)$ that admit isometric isomorphisms with certain space of operators. To proceed further, let us recall the construction of the Lipschitz free space $F(X)$ (introduced by Godefroy and Kalton in [27]). For any $x \in X$, we denote by δ_x the evaluation functional, i.e. $\delta_x(f) = f(x)$ for $f \in Lip_0(X, \mathbb{R})$. It is easy to see that $\delta_X : X \to span\{\delta_x : x \in X\}; x \mapsto \delta_x$ is an isometric (nonlinear) embedding of X into

$Lip_0(X,\mathbb{R})^*$. The space $F(X)$ is defined to be the closed linear span of $\{\delta_x : x \in X\}$ with the dual space norm denoted simply by $\|\cdot\|$. It is known that $F(\mathbb{R})$ is isometrically isomorphic to $L^1(\mathbb{R})$, we denote it as $F(\mathbb{R}) \cong L^1(\mathbb{R})$ (see [27, p 128]). Then we consider a linear contraction $\beta_X : F(X) \to X$, a left inverse of the Lipschitz map δ_X. For details and additional properties, see [16], [56]. Furthermore, we establish the following more general result, which provides a partial answer to the above-posed Question 5.1.1:

Result 5.1.2. *For any closed subspace $\mathcal{D}$ of $F(X)\hat{\otimes}_\pi F(Y)$ and E is an injective Banach space,*

$$BLip_0(X,Y;E) \,/ {}^\diamond\mathcal{D} \cong L(\mathcal{D},E).$$

Where $F(X)\hat{\otimes}_\pi F(Y)$ be the complete projective tensor product of $F(X)$ and $F(Y)$ (with the projective norm denoted as $\|\cdot\|_\pi$) and ${}^\diamond\mathcal{D} := \{T \in BLip_0(X,Y;E) : T_L(\xi) = 0\ \forall \xi \in \mathcal{D}\}$ (T_L be unique linearization map corresponding to $T \in BLip_0(X,Y;E)$). In the special case for X, Y and E to be $\mathbb{R}$, using this result, we obtain an alternative proof of the following consequence:

$$L^\infty(\mathbb{R}\times\mathbb{R}) \Big/ \{\text{set of all constant maps on } \mathbb{R}\times\mathbb{R}\}$$

is isometrically isomorphic to a dual space. Moreover, we explicitly identify its predual towards the end of this article. For details on bilinear maps and tensor products, we refer to [39]. Through the study of particular aspects of $BLip_0(X,Y;E)$ and $F(X)\hat{\otimes}_\pi F(Y)$, we also provide a necessary and sufficient condition under which a *two-Lipschitz* operator is bilinear.

The rest of the chapter is organized as follows. In Section 5.2, we provide the proof of one of the main results, Theorem 5.2.3, which offers a partial answer to Question 5.0.1. In Section 5.3 we study an E-valued dual action between $BLip_0(X,Y;E)$ and $F(X)\hat{\otimes}_\pi F(Y)$. Section 5.4 presents several general results concerning the quotients of *two-Lipschitz* operator spaces. Also, Remark 5.4.4(2) answers the Question 5.1.1 whenever E is an injective Banach space. Section 5.5 provides some nontrivial examples of the quotient space.

5.2 Embedding of $Blin(X, Y; E)$ into $BLip_0(X, Y; E)$

Let X, Y, and E be real Banach spaces. We again recall that for $T \in Blin(X, Y; E)$, we have $\|T\| = BLip(T)$ so that $Blin(X, Y)$ is a closed subspace of $BLip_0(X, Y; E)$. This section proves that $Blin(X, Y; E)$ is a complemented subspace of $BLip_0(X, Y; E)$ whenever E is a dual Banach space. In this case, we show that a projection exists from $BLip_0(X, Y; E)$ onto $Blin(X, Y; E)$.

Given a map $f : X \times Y \to E$, we define its translation by a point $(p, q) \in X \times Y$ as the map $f_{p,q} : X \times Y \to E$, where

$$f_{p,q}(x, y) := f(p + x, q + y)$$

for all $(x, y) \in X \times Y$. We begin with the following result.

Lemma 5.2.1. *Let $T \in BLip_0(X, Y; E)$, $w, w' \in X$ and $z, z' \in Y$. Consider $T_{p,q}$; translation of T by any $(p, q) \in X \times Y$; that is $T_{p,q}(x, y) = T(p + x, q + y)$ for all $(x, y) \in X \times Y$. Then the following hold:*

1. *$\phi_T^{w,z}$ defined by $\phi_T^{w,z} = T_{w,z} - T_{w,0} - T_{0,z} + T$ belongs to $\ell^\infty(X \times Y; E)$.*

2. *$\phi_T^{w+w',z} = (\phi_T^{w,z})_{w',0} + \phi_T^{w',z}$ and $\phi_T^{w,z+z'} = (\phi_T^{w,z})_{0,z'} + \phi_T^{w,z'}$.*

3. *$\left(\phi_T^{-w,z}\right)_{w,0} = -\phi_T^{w,z}$.*

4. *$\phi_T^{2w,z} = (\phi_T^{w,z})_{w,0} + \phi_T^{w,z}$ and $\phi_T^{w,2z} = (\phi_T^{w,z})_{0,z} + \phi_T^{w,z}$.*

Proof. Let $T \in BLip_0(X, Y; E)$ and fix $(w, z) \in X \times Y$. Then for any $(x, y) \in X \times Y$

$$\|\phi_T^{w,z}(x, y)\| = \|T_{w,z}(x, y) - T_{w,0}(x, y) - T_{0,z}(x, y) + T(x, y)\|$$

$$\begin{aligned} &= \|T(w+x,z+y)-T(w+x,y)-T(x,z+y)+T(x,y)\| \\ &\leq BLip(T)\|w\|\|z\|. \end{aligned}$$

Therefore, $\phi_T^{w,z} \in \ell^\infty(X \times Y; E)$ with $\|\phi_T^{w,z}\|_\infty \leq BLip(T)\|w\|\|z\|$.

Let $w, w' \in X$ and $z, z' \in Y$. Then for any $(x,y) \in X \times Y$

$$\begin{aligned} (\phi_T^{w,z})_{w',0}(x,y) &= (T_{w,z}-T_{w,0}-T_{0,z}+T)_{w',0}(x,y) \\ &= T(w+w'+x,z+y)-T(w+w'+x,y)-T(w'+x,z+y)+T(w'+x,y) \\ &= (T_{w+w',z}-T_{w+w',0}-T_{w',z}+T_{w',0})(x,y). \end{aligned}$$

This implies that $(\phi_T^{w,z})_{w',0} = T_{w+w',z}-T_{w+w',0}-T_{w',z}+T_{w',0}$. Further

$$\begin{aligned} \phi_T^{w+w',z} &= T_{w+w',z}-T_{w+w',0}-T_{0,z}+T \\ &= \underbrace{T_{w+w',z}-T_{w+w',0}-T_{w',z}+T_{w',0}}+\underbrace{T_{w',z}-T_{w',0}-T_{0,z}+T} \\ &= (\phi_T^{w,z})_{w',0}+\phi_T^{w',z}. \end{aligned}$$

In a similar manner we have $\phi_T^{w,z+z'} = (\phi_T^{w,z})_{0,z'}+\phi_T^{w,z'}$.

Again for any $(x,y) \in X \times Y$

$$\begin{aligned} \left(\phi_T^{-w,z}\right)_{w,0}(x,y) &= (T_{-w,z}-T_{-w,0}-T_{0,z}+T)_{w,0}(x,y) \\ &= T(x,y+z)-T(x,y)-T(w+x,y+z)+T(w+x,y) \\ &= -\left((T_{w,z}-T_{w,0}-T_{0,z}+T)\right)(x,y) \\ &= -\phi_T^{w,z}(x,y). \end{aligned}$$

Hence $\left(\phi_T^{-w,z}\right)_{w,0} = -\phi_T^{w,z}$. Also,

$$\begin{aligned} (\phi_T^{w,z})_{w,0}(x,y) &= (T_{w,z}-T_{w,0}-T_{0,z}+T)_{w,0}(x,y) \\ &= T(2w+x,y+z)-T(2w+x,y)-T(w+x,y+z)+T(w+x,y) \\ &= (T_{2w,z}-T_{2w,0}-T_{w,z}+T_{w,0})(x,y) \end{aligned}$$

Further

$$\begin{aligned}
\left(\phi_T^{2w,z}\right)(x, y) &= (T_{2w,z} - T_{2w,0} - T_{0,z} + T)(x, y) \\
&= \Big(\underbrace{T_{2w,z} - T_{2w,0} - T_{w,z} + T_{w,0}} + \underbrace{T_{w,z} - T_{w,0} - T_{0,z} + T}\Big)(x, y) \\
&= \left(\phi_T^{w,z}\right)_{w,0}(x, y) + \left(\phi_T^{w,z}\right)(x, y)
\end{aligned}$$

The other part can be proved similarly. This completes the proof. □

Next, we recall the notion of an invariant mean.

Definition 5.2.2. [11] A left *invariant mean* in X is a linear functional M on $\ell^\infty(X)$ (the space of all bounded maps from X to $\mathbb{R}$) such that:

1. $M(\mathbf{1}) = 1$, where $\mathbf{1}$ be the constant function on X that takes value 1.
2. $M(f) \geq 0$ for all $f \geq 0$.
3. $M(f_x) = M(f)$ for all $x \in X$, where $f_x(z) = f(x + z)$ for all $z \in X$.

Let Y be a dual Banach space. It follows from [11, p. 417] that an invariant mean M in X induces a norm-one linear operator $\mathcal{M} : \ell^\infty(X, Y) \to Y$ satisfying:

1. $\mathcal{M}(f_x) = \mathcal{M}(f)$ for all $f \in \ell^\infty(X, Y)$ and $x \in X$.
2. $\mathcal{M}(\hat{y}) = y$ for all $y \in Y$. Here $\hat{y} \in \ell^\infty(X, Y)$ is given by $\hat{y}(x) = y$ for all $x \in X$.

For the proof and other details of the existence of such an invariant mean, we refer [11, p. 417], [12], [35] , [29]. We call this $\mathcal{M}$ as a generalized invariant mean. Further, it is easy to verify that, in particular for $Y = \mathbb{R}$, $\mathcal{M} = M$, the invariant mean.

For any two normed linear spaces X and Y, the product space $X \times Y$ is equipped with the norm

$$\|(x, y)\|_{X\times Y} = \|x\|_X + \|y\|_Y.$$

In fact $(X \times Y, \|.\|_{X\times Y})$ becomes a Banach space whenever X and Y are so. We will omit the subscript in the norm notation from this point onward. The norm should be understood in the context of the respective spaces. Now, we are in a position to state our main result of this section.

Theorem 5.2.3. *Let X, Y be normed linear spaces and E be a dual space. Then $Blin(X, Y; E)$ is* 1- *complemented in $BLip_0(X, Y; E)$.*

Proof. Let X, Y be normed linear spaces and E be a dual space. Then $X \times Y$ is a normed linear space equipped with the norm described above. Consequently, by 5.2.2, there exists a generalized invariant mean $\mathcal{M}$ on $\ell^\infty(X \times Y, E)$.

We now define $P(T)(w, z) := \mathcal{M}\left(\phi_T^{w,z}\right)$ for each $T \in BLip_0(X, Y; E)$ and $(w, z) \in X \times Y$, where $\phi_T^{w,z}$ is defined in the preceding lemma.

Suppose $w, w' \in X$ and $z, z' \in Y$. Then by Lemma 5.2.1, we obtain

$$\begin{aligned}
P(T)(w + w', z) &= \mathcal{M}\left(\phi_T^{w+w',z}\right) \\
&= \mathcal{M}\left(\left(\phi_T^{w,z}\right)_{w',0} + \phi_T^{w',z}\right) \\
&= \mathcal{M}\left(\left(\phi_T^{w,z}\right)_{w',0}\right) + \mathcal{M}\left(\phi_T^{w',z}\right) \quad (\text{ as } \mathcal{M} \text{ is linear}) \\
&= \mathcal{M}\left(\phi_T^{w,z}\right) + \mathcal{M}\left(\phi_T^{w',z}\right) \quad (\text{ as } \mathcal{M} \text{ is translation invariant}) \\
&= P(T)(w, z) + P(T)(w', z).
\end{aligned}$$

Using similar argument we further conclude that

$$P(T)(w, z + z') = P(T)(w, z) + P(T)(w, z').$$

Thus, $P(T)$ is additive in each variable. Now, using a standard technique of real analysis, we show that $P(T)$ is linear in each argument.

From Lemma 5.2.1(2) for any $n \in \mathbb{N}$ we have $\phi_T^{nw,z} = \left(\phi_T^{(n-1)w,z}\right)_{w,0} + \phi_T^{w,z}$. Using translation invariance and the linearity of $\mathcal{M}$ then gives

$$\mathcal{M}\left(\phi_T^{nw,z}\right) = \mathcal{M}\left(\left(\phi_T^{(n-1)w,z}\right)_{w,0}\right) + \mathcal{M}(\phi_T^{w,z}),$$

by induction. At this stage, utilizing the translation invariance of $\mathcal{M}$ repeatedly, we derive

$$\mathcal{M}\left(\phi_T^{nw,z}\right) = n\mathcal{M}\left(\phi_T^{w,z}\right).$$

Applying Lemma 5.2.1(3) we deduce $\mathcal{M}\left(\phi_T^{nw,z}\right) = n\mathcal{M}\left(\phi_T^{w,z}\right)$ for any $n \in \mathbb{Z}$, that is. $P(T)(nw,z) = nP(T)(w,z)$. Let $\alpha = \frac{m}{n} \in \mathbb{Q}$. Then

$$mP(T)(w,z) = P(T)(mw,z) = P(T)(n\alpha w,z) = nP(T)(\alpha w,z).$$

Thus we have $P(T)(\alpha w,z) = \alpha P(T)(w,z)$ for any $\alpha \in \mathbb{Q}$. Again suppose $r \in \mathbb{R}$, then choose $(r_n) \subset \mathbb{Q}$ such that r_n converges to r. Now we note that the map $u \mapsto \phi_T^{u,z}$ is Lipschitz from X to $\ell^\infty(X \times Y)$ in the $\|\cdot\|_\infty$ norm as:

$$\begin{aligned}\|\phi_T^{u,z} - \phi_T^{v,z}\|_\infty &= \sup_{x\in X, y\in Y} \|T(u+x,z+y) - T(u+x,y) - T(v+x,z+y) + T(v+x,y)\| \\ &\leq BLip(T)\|u-v\|\|z\|.\end{aligned}$$

This gives uniform convergence $\phi_T^{r_n w,z} \to \phi_T^{rw,z}$ in ℓ^∞ and then continuity of $\mathcal{M}$ yields

$$\mathcal{M}\left(\phi_T^{rw,z}\right) = \lim_{n\to\infty} \mathcal{M}\left(\phi_T^{r_n w,z}\right) = \lim_{n\to\infty} r_n\mathcal{M}\left(\phi_T^{w,z}\right) = r\mathcal{M}\left(\phi_T^{w,z}\right).$$

Therefore, $P(T)$ is linear in the first variable. Likewise, it can also be shown to be linear in the second variable. Thus the map

$$P : (BLip_0(X,Y;E), BLip(\cdot)) \to (Blin(X,Y;E), \|\cdot\|)$$

given by

$$P(T)(w,z) = \mathcal{M}\left(\phi_T^{w,z}\right)$$

for each $T \in BLip_0(X,Y;E)$ and $(w,z) \in X \times Y$; is well defined with

$$\|P(T)(w,z)\| = \|\mathcal{M}\left(\phi_T^{w,z}\right)\| \leq \|\phi_T^{w,z}\|_\infty \leq BLip(T)\|w\|\|z\|.$$

This shows that $\|P(T)\| \leq BLip(T)$. Again, from the linearity of $\mathcal{M}$, it easily follows that P is linear. We show that $P(T) = T$ for any $T \in Blin(X,Y;E)$. Suppose $T \in Blin(X,Y;E)$. Then for any $(x,y) \in X \times Y$

$$\phi_T^{w,z}(x,y) = T(w+x, z+y) - T(w+x, y) - T(x, z+y) + T(x,y) = T(w,z).$$

Moreover $\phi_T^{w,z} = \widehat{T(w,z)}$ and hence $P(T)(w,z) = \mathcal{M}(\phi_T^{w,z}) = \mathcal{M}(\widehat{T(w,z)}) = T(w,z)$.

Therefore, $P(T) = T$ for all $T \in Blin(X,Y;E)$ so that P is a norm one linear projection from $BLip_0(X,Y;E)$ onto $Blin(X,Y;E)$. In other words, $Blin(X,Y;E)$ is complemented in $BLip_0(X,Y;E)$ whenever E is a dual Banach space. $\square$

Remark 5.2.4. We note that ker(P) is topologically isomorphic to $BLip_0(X,Y;E) \big/ Blin(X,Y;E)$. In fact, $Id_{BLip_0(X,Y;E)} - P$ is a bounded linear surjective projection form $BLip_0(X,Y;E)$ onto ker(P) with kernel $Blin(X,Y;E)$.

5.3 Vector valued dual action

Here, we define an E-valued dual action between $BLip_0(X,Y;E)$ and $F(X)\hat{\otimes}_\pi F(Y)$, which will serve as a foundational setup for the results presented in the following sections. Let us first recall the following two theorems:

Theorem 5.3.1. *[32, Theorem 2.6] For every* two-Lipschitz *operator* $T \in BLip_0(X,Y;E)$ *there exists a unique bilinear mapping* $T_B : F(X) \times F(Y) \to E$ *satisfying* $T_B(\delta_x, \delta_y) = T(x,y)$ *and* $T = T_B \circ (\delta_X, \delta_Y) : X \times Y \xrightarrow{(\delta_X,\delta_Y)} F(X) \times F(Y) \xrightarrow{T_B} E$. *Furthermore* $BLip(T) = \|T_B\|$. *The bilinear mapping* T_B *is called bi-linearization of the* two-Lipschitz *operator* T.

This result implies that the space $(BLip_0(X,Y;E), BLip(\cdot))$ *is isometrically isomorphic to* $(Blin(F(X), F(Y); E), \|\cdot\|)$.

Theorem 5.3.2. *([52, Theorem 2.9], [32, Remark 2.7]) The bilinear operator T_B admits a linearization $(T_B)_L : F(X)\hat{\otimes}_\pi F(Y) \to E$ satisfies $T = T_B \circ (\delta_X, \delta_Y) = (T_B)_L \circ \sigma_2 \circ (\delta_X, \delta_Y)$, where $\sigma_2 : F(X) \times F(Y) \to F(X)\hat{\otimes}_\pi F(Y)$ is the canonical bilinear operator defined by $\sigma_2(\mu, \nu) = \mu \otimes \nu$. In addition, we have $BLip(T) = \|T_B\| = \|(T_B)_L\|$. The linear operator $(T_B)_L$ is referred to as the linearization of the* two-Lipschitz *operator T. For the simplification, we write T_L instead of $(T_B)_L$.*

By combining these two results, we conclude that the space $(BLip_0(X,Y;E), BLip(\cdot))$ is isometrically isomorphic to $L\left(F(X)\hat{\otimes}_\pi F(Y), E\right)$ via the correspondence $T \mapsto T_L$. This encourages us to consider the following vector-valued duality, which will play a crucial role in the results of this section.

Proposition 5.3.3. *For any Banach spaces X, Y and E, define*

$$\langle \cdot, \cdot \rangle_E : BLip_0(X,Y;E) \times F(X)\hat{\otimes}_\pi F(Y) \to E$$

given by

$$\langle T, \xi \rangle_E := T_L(\xi)$$

for all $T \in BLip_0(X,Y;E)$ and $\xi \in F(X)\hat{\otimes}_\pi F(Y)$. Then $\langle \cdot, \cdot \rangle_E$ is bilinear and non-degenerate.

Proof. The existence of a unique map T_L, corresponding to $T \in BLip_0(X,Y;E)$, provides that $\langle \cdot, \cdot \rangle_E$ is linear in the first coordinate, and the linearity of T_L gives the linearity in the second coordinate of $\langle \cdot, \cdot \rangle_E$. Also the following norm conditions hold:

$$\sup_{\xi \in F(X)\hat{\otimes}_\pi F(Y), \|\xi\| \le 1} \|T_L(\xi)\| = \|T_L\| = BLip(T) \text{ and } \sup_{BLip(T) \le 1} \|T_L(\xi)\| = \sup_{\|T_L\| \le 1} \|T_L(\xi)\| = \|\xi\|.$$

Now we show that $\langle \cdot, \cdot \rangle_E$ is non-degenerate. Let $T \in BLip_0(X,Y;E)$ and $T_L(\xi) = 0$ for all $\xi \in F(X)\hat{\otimes}_\pi F(Y)$. This gives $T_L(\delta_x \otimes \delta_y) = 0$ for all $x \in X$ and $y \in Y$. That is, $T(x,y) = 0$ for all $x \in X$ and $y \in Y$. Hence $T = 0$.

Again let $\xi \in F(X)\hat{\otimes}_\pi F(Y)$ be such that $T_L(\xi) = 0$ for all $T \in BLip_0(X,Y;E)$. That is $T_L(\xi) = 0$ for all $T_L \in L\left(F(X)\hat{\otimes}_\pi F(Y), E\right)$. Then by Hahn-Banach theorem there exists $S \in L\left(F(X)\hat{\otimes}_\pi F(Y), \mathbb{R}\right)$ such that $S(\xi) = \|\xi\|$. Now define $\tilde{S}(\gamma) = S(\gamma)e$ for all $\gamma \in F(X)\hat{\otimes}_\pi F(Y)$ and some unit vector $e \in E$. Then $\tilde{S} \in L\left(F(X)\hat{\otimes}_\pi F(Y), E\right)$ and hence it follows that $\tilde{S}(\xi) = 0$. That is $\|\xi\| = 0$. Thus, $\xi = 0$. This completes the proof. □

For $\mathcal{A} \subset BLip_0(X,Y;E)$ we define $\mathcal{A}^\diamondsuit := \{\xi \in F(X)\hat{\otimes}_\pi F(Y) : T_L(\xi) = 0 \ \ \forall\, T \in \mathcal{A}\}$. Then $\mathcal{A}^\diamondsuit = \bigcap\limits_{T\in\mathcal{A}} \ker(T_L)$ is a closed subspace of $F(X)\hat{\otimes}_\pi F(Y)$.

Further, for $\mathcal{D} \subset F(X)\hat{\otimes}_\pi F(Y)$ we consider the space

$$^\diamondsuit\mathcal{D} := \{T \in BLip_0(X,Y;E) : T_L(\xi) = 0\ \forall \xi \in \mathcal{D}\}.$$

Using the fact $BLip(T) = \|T_L\|$, we can easily verify that it is also a closed subspace of $BLip_0(X,Y;E)$.

5.4 Some quotient space

Building on the dual action established in the preceding section, we present some of the main results, including identifying the quotient space in this context. Additionally, we provide a necessary and sufficient condition for a *two-Lipschitz* map to be bilinear.

Proposition 5.4.1. *Let $\mathcal{D}$ be a closed subspace of $F(X)\hat{\otimes}_\pi F(Y)$ and E be an injective Banach space. Then ${}^{BLip_0(X,Y;E)}/_{^\diamondsuit\mathcal{D}} \cong L(\mathcal{D}, E)$.*

Proof. For a fix $T \in BLip_0(X,Y;E)$, we define $\theta_T : \mathcal{D} \to E$ given by $\theta_T(\xi) = T_L(\xi)$, for all $\xi \in \mathcal{D}$. Then, θ_T is linear. Also, for any $\xi \in \mathcal{D}$, we have

$$\|\theta_T(\xi)\| = \|T_L(\xi)\| \leq \|T_L\|\|\xi\| = BLip(T)\|\xi\|.$$

Thus $\theta_T \in L(\mathcal{D}, E)$ for any $T \in BLip_0(X,Y;E)$.

We define $\Theta : BLip_0(X,Y;E) \to L(\mathcal{D},E)$ given by $\Theta(T) = \theta_T$. It is routine to check that Θ is linear. We show that Θ is also surjective. Let $S \in L(\mathcal{D},E)$. Since E is injective there exists $S_e \in L(F(X)\hat{\otimes}_\pi F(Y),E)$ such that $S_e\big|_{\mathcal{D}} = S$ with $\|S_e\| = \|S\|$. Thus $S_e \circ \sigma_2 \circ (\delta_X, \delta_Y) \in BLip_0(X,Y;E)$. Furthermore, the existence of a unique linearisation map corresponding to T implies that $T_L = S_e$. Thus, Θ is surjective.

Also,

$$\begin{aligned}
\ker(\Theta) &= \{T \in BLip_0(X,Y;E) : \theta_T = 0\} \\
&= \{T \in BLip_0(X,Y;E) : 0 = \theta_T(\xi) = T_L(\xi) \text{ for all } \xi \in \mathcal{D}\} \\
&= {}^{\diamond}\mathcal{D}.
\end{aligned}$$

So by fundamental theorem of linear algebra Θ induce a linear isomorphism (which we again denote by Θ):

$$\Theta : BLip_0(X,Y;E) \,/\, {}^{\diamond}\mathcal{D} \to L(\mathcal{D},E)$$

given by $\Theta(T + {}^{\diamond}\mathcal{D})(\xi) = T_L(\xi)$ for all $\xi \in \mathcal{D}$.

Now, for any $T \in BLip_0(X,Y;E)$ and $\xi \in \mathcal{D}$

$$\begin{aligned}
\|\Theta(T + {}^{\diamond}\mathcal{D})(\xi)\| &= \|T_L(\xi)\| \\
&= \inf_{U \in {}^{\diamond}\mathcal{D}} \|(T+U)_L(\xi)\| \\
&\leq \|\xi\| \inf_{U \in {}^{\diamond}\mathcal{D}} BLip(T+U) \\
&= \|\xi\| \|T + {}^{\diamond}\mathcal{D}\|.
\end{aligned}$$

Therefore, Θ is a surjective contractive linear isomorphism. So, by the open mapping theorem, Θ^{-1} exists and is a bounded linear surjective isomorphism.

Fix $S \in L(\mathcal{D},E)$ and let S_1 and S_2 be any two extensions of S in $L(F(X)\hat{\otimes}_\pi F(Y),E)$. If $\xi \in \mathcal{D}$, then

$$\left(\Theta(S_1 \circ \sigma_2 \circ (\delta_X, \delta_Y)) - \Theta(S_2 \circ \sigma_2 \circ (\delta_X, \delta_Y))\right)(\xi) = (S_1)_L(\xi) - (S_2)_L(\xi) = S(\xi) - S(\xi) = 0.$$

Thus $S_1 \circ \sigma_2 \circ (\delta_X, \delta_Y) + {}^{\Diamond}\mathcal{D} = S_2 \circ \sigma_2 \circ (\delta_X, \delta_Y) + {}^{\Diamond}\mathcal{D}$ and hence $S_e \circ \sigma_2 \circ (\delta_X, \delta_Y) + {}^{\Diamond}\mathcal{D}$ is uniquely determined by S. Now it follows that $\Theta^{-1}(T) = \widetilde{T} \circ \sigma_2 \circ (\delta_X, \delta_Y) + {}^{\Diamond}\mathcal{D}$ for all $T \in L(\mathcal{D}, E)$, where $\widetilde{T} \in L(F(X)\hat{\otimes}_\pi F(Y), E)$ is a norm preserving extension of T. Since

$$\|\Theta^{-1}(T)\| = \|\widetilde{T} \circ \sigma_2 \circ (\delta_X, \delta_Y) + {}^{\Diamond}\mathcal{D}\| \leq BLip(\widetilde{T} \circ \sigma_2 \circ (\delta_X, \delta_Y)) \leq \|T\|,$$

Θ^{-1} is also a contraction. Now, it is easy to conclude that Θ is surjective linear isometry so that $BLip_0(X, Y; E) /{}^{\Diamond}\mathcal{D}$ is isometrically isomorphic to $L(\mathcal{D}, E)$. □

Remark 5.4.2. In case if we consider $\mathcal{D} = F(X)\hat{\otimes}_\pi F(Y)$, we get back the result

$$(BLip_0(X, Y; E), BLip(.)) \cong L\left(F(X)\hat{\otimes}_\pi F(Y), E\right).$$

The following result is key to characterize the quotient $BLip_0(X, Y; E) \Big/ Blin(X, Y; E)$.

5.4.1 A necessary sufficient condition for a *two-Lipschitz* map to be bilinear

Proposition 5.4.3. *Let $T \in BLip_0(X, Y; E)$. Then T is bilinear if and only if $T_L(\xi) = 0$ for all $\xi \in \ker(\beta_X)\hat{\otimes}_\pi F(Y) + F(X)\hat{\otimes}_\pi \ker(\beta_Y)$.*

Proof. Put $Q = \ker(\beta_X)\hat{\otimes}_\pi F(Y) + F(X)\hat{\otimes}_\pi \ker(\beta_Y)$. Let $T \in BLip_0(X, Y; E)$ such that $T_L(\xi) = 0$ for all $\xi \in Q$. We show that T is bilinear. Suppose, $r \in \mathbb{R}$, $x, x' \in X$ and $y \in Y$. Then

$$(\delta_{rx+x'} - (r\delta_x + \delta_{x'})) \otimes \delta_y, \delta_x \otimes (\delta_{ry+y'} - (r\delta_y + \delta_{y'}) \in Q.$$

Therefore, $T_L\left((\delta_{rx+x'} - (r\delta_x + \delta_{x'})) \otimes \delta_y\right) = 0$ implies $T_L(\delta_{rx+x'} \otimes \delta_y) = rT_L(\delta_x \otimes \delta_y) + T_L(\delta_{x'} \otimes \delta_y)$. Hence $T(rx+x', y) = rT(x, y) + T(x', y)$. Similarly, we get $T(x, ry+y') = rT(x, y) + T(x, y')$. Therefore, T is bilinear.

Conversely, let $T \in Blin(X, Y; E) \subset BLip_0(X, Y; E)$. Then its bilinearisation T_B satisfies $T_B(\mu, \nu) = T(\beta_X(\mu), \beta_Y(\nu))$ for all $\mu \in F(X)$ and $\nu \in F(Y)$. We again consider the linearisation $\hat{T} \in L(X\hat{\otimes}_\pi Y, E) \cong Blin(X,Y;E)$ of the map $T \in Blin(X,Y;E)$. Consequently, the linearisation of the two-Lipschitz map T (which is actually bilinear) satisfies

$$T_L = \hat{T} \circ (\beta_X \otimes \beta_Y) : F(X)\hat{\otimes}_\pi F(Y) \to X\hat{\otimes}_\pi Y \to E.$$

Therefore, $T_L(\xi) = 0$ for all $\xi \in \ker(\beta_X)\hat{\otimes}_\pi F(Y) + F(X)\hat{\otimes}_\pi \ker(\beta_Y)$, as this subspace lies in $\ker(\beta_X \otimes \beta_Y)$. This completes the proof. □

Remark 5.4.4. Though $\ker(\beta_X)\hat{\otimes}_\pi F(Y)$ and $F(X)\hat{\otimes}_\pi \ker(\beta_Y)$ are closed subspaces, their sum need not be closed a priori, so we consider $\mathcal{D} = \overline{\ker(\beta_X)\hat{\otimes}_\pi F(Y) + F(X)\hat{\otimes}_\pi \ker(\beta_Y)}^{\|\cdot\|_\pi}$. Then the above proposition concludes that ${}^{\diamond}\mathcal{D} = Blin(X,Y;E)$.

From the above Propositions 5.4.1 and 5.4.3 we get

$$BLip_0(X,Y;E) \Big/ Blin(X,Y;E) \cong L\left(\overline{\ker(\beta_X)\hat{\otimes}_\pi F(Y) + F(X)\hat{\otimes}_\pi \ker(\beta_Y)}^{\|\cdot\|_\pi}, E\right),$$

whenever E is an injective Banach space.

Next, we see an auxiliary result related to the quotient using the duality notion.

Proposition 5.4.5. *Let $\mathcal{D}$ be a closed subspace of $F(X)\hat{\otimes}_\pi F(Y)$, then $L\left(F(X)\hat{\otimes}_\pi F(Y)/\mathcal{D}, E\right)$ is isometrically isomorphic to ${}^{\diamond}\mathcal{D}$.*

Proof. For a fix $T \in {}^{\diamond}\mathcal{D}$, we define $\Lambda_T : F(X)\hat{\otimes}_\pi F(Y) \to E$ given by $\Lambda_T(\xi) = T_L(\xi)$ for all $\xi \in F(X)\hat{\otimes}_\pi F(Y)$. Then, easily, we can verify that Λ_T is a linear map with $\|\Lambda_T\| \leq BLip(T)$. Further $\ker(\Lambda_T) = \left\{\xi \in F(X)\hat{\otimes}_\pi F(Y) : T_L(\xi) = 0\right\} \supset \mathcal{D}$. Therefore Λ_T induces a bounded linear map (again denoted by Λ_T) from $F(X)\hat{\otimes}_\pi F(Y)/\mathcal{D}$ into E. This induces a map

$$\Lambda : {}^{\diamond}\mathcal{D} \to L\left(F(X)\hat{\otimes}_\pi F(Y)/\mathcal{D}, E\right)$$

given by $\Lambda(T)(\xi + \mathcal{D}) = T_L(\xi)$ for all $T \in {}^{\diamond}\mathcal{D}$ and $\xi \in F(X)\hat{\otimes}_\pi F(Y)$. Then, Λ is linear. Also

$$\|\Lambda(T)(\xi + \mathcal{D})\| = \|T_L(\xi)\|$$

$$
\begin{aligned}
&= \inf_{\mu \in \mathcal{D}} \|T_L(\xi+\mu)\| \\
&\leq \inf_{\mu \in \mathcal{D}} \|(\xi+\mu)\| BLip(T) \\
&= BLip(T)\|\xi+\mathcal{D}\|,
\end{aligned}
$$

for all $T \in {}^{\diamond}\mathcal{D}$ and $\xi \in F(X)\hat{\otimes}_\pi F(Y)$. Thus, Λ is a contraction. Further, we show that Λ is an isometry.

We observe that the induced map Λ can be written explicitly using the natural quotient map $q : F(X)\hat{\otimes}_\pi F(Y) \to {F(X)\hat{\otimes}_\pi F(Y)} / {\mathcal{D}}$ given by $\Lambda(T) \circ q(\xi) = T_L(\xi)$ for all $T \in {}^{\diamond}\mathcal{D}$ and $\xi \in F(X)\hat{\otimes}_\pi F(Y)$. Then $\Lambda(T) \circ q = T_L$ and hence we have

$$
BLip(T) = \|T_L\| = \|\Lambda(T) \circ q\| \leq \|\Lambda(T)\|.
$$

Thus, Λ is an isometry.

To prove the surjectivity of Λ, let $S \in L\left({F(X)\hat{\otimes}_\pi F(Y)} / {\mathcal{D}}, E\right)$. Set $T(x,y) = S(\delta_x \otimes \delta_y + \mathcal{D})$ for all $x \in X$ and $y \in Y$. Then it can be easily shown that $T \in BLip_0(X,Y;E)$ with $T_L = S \circ q$ and $\Lambda(T) = S$, where $q : F(X)\hat{\otimes}_\pi F(Y) \to {F(X)\hat{\otimes}_\pi F(Y)} / {\mathcal{D}}$ be the natural quotient map. Further for any $\xi \in \mathcal{D}, T_L(\xi) = S(\xi+\mathcal{D}) = 0$. Hence $T \in {}^{\diamond}\mathcal{D}$. This completes the proof. $\square$

Remark 5.4.6. 1. Considering $\mathcal{D} = \overline{\ker(\beta_X)\hat{\otimes}_\pi F(Y) + F(X)\hat{\otimes}_\pi \ker(\beta_Y)}^{\|\cdot\|_\pi}$, using the Proposition 5.4.5 and Remark 5.4.4 we get

$$
Blin(X,Y;E) \cong L\left({F(X)\hat{\otimes}_\pi F(Y)} \Big/ {\overline{\ker(\beta_X)\hat{\otimes}_\pi F(Y) + F(X)\hat{\otimes}_\pi \ker(\beta_Y)}^{\|\cdot\|_\pi}}, E \right),
$$

whenever E is an injective Banach space.

2. In particular for $\mathcal{D} = \{0\}$ we recover $(BLip_0(X,Y;E), BLip(\cdot))$ is isometrically isomorphic to $L\left(F(X)\hat{\otimes}_\pi F(Y), E\right)$.

5.5 Example

In this section, we aim to provide a nontrivial example of the quotient space, specifically

$$BLip_0(\mathbb{R},\mathbb{R};\mathbb{R}) \Big/ Blin(\mathbb{R},\mathbb{R};\mathbb{R}).$$

We determine it's predual explicitly using the already established results and the following Lemmas. Let us recall the following result 3.4.1.

Lemma 5.5.1. $\ker(\beta_{\mathbb{R}})$ *is isometrically isomorphic to* $\{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f dx = 0\}$*, where* dx *is the Lebesgue measure on* $\mathbb{R}$*.*

In case of $X = Y = \mathbb{R}$, $F(\mathbb{R})$ is isometrically isomorphic to $L^1(\mathbb{R})$(see [27, p 128]), and $\ker(\beta_{\mathbb{R}}) \cong \left\{ f \in L^1(\mathbb{R}) : \int\limits_{\mathbb{R}} f(x)dx = 0 \right\}$, is a closed subspace of $F(\mathbb{R})$. Also $\ker(\beta_{\mathbb{R}})\hat{\otimes}_\pi F(\mathbb{R}) + F(\mathbb{R})\hat{\otimes}_\pi \ker(\beta_{\mathbb{R}})$ is a subspace of $F(\mathbb{R})\hat{\otimes}_\pi F(\mathbb{R})$.

Put $\mathcal{D} = \overline{\ker(\beta_{\mathbb{R}})\hat{\otimes}_\pi F(\mathbb{R}) + F(\mathbb{R})\hat{\otimes}_\pi \ker(\beta_{\mathbb{R}})}^{\|\cdot\|_\pi}$. Then from the Proposition 5.4.1 it follows

$$\begin{aligned} BLip_0(\mathbb{R},\mathbb{R};\mathbb{R}) \,/\diamond_{\mathcal{D}} &\cong L(\mathcal{D},\mathbb{R}), \text{ that is} \\ BLip_0(\mathbb{R},\mathbb{R};\mathbb{R}) \Big/ Blin(\mathbb{R},\mathbb{R};\mathbb{R}) &\cong L\left(\overline{\ker(\beta_{\mathbb{R}})\hat{\otimes}_\pi F(\mathbb{R}) + F(\mathbb{R})\hat{\otimes}_\pi \ker(\beta_{\mathbb{R}})}^{\|\cdot\|_\pi}, \mathbb{R} \right). \end{aligned}$$

Consider the correspondence given by the unique isometrical isomorphism:

$$L^1(\mu)\hat{\otimes}_\pi L^1(\nu) \cong L^1(\mu \times \nu)$$

which is the continuous extension of the canonical linear mapping:

$$\begin{aligned} \theta : L^1(\mu) \otimes L^1(\nu) &\to L^1(\mu \times \nu) \\ f \otimes g &\mapsto f(x)g(y). \end{aligned}$$

Now using the this correspondence and the fact that $F(\mathbb{R}) \cong L^1(\mathbb{R})$ we have,

$$BLip_0(\mathbb{R},\mathbb{R};\mathbb{R}) \cong L(F(\mathbb{R})\hat{\otimes}_\pi F(\mathbb{R}),\mathbb{R}) \cong L^\infty(\mathbb{R}\times\mathbb{R}),$$

where the first isomorphism relation holds due to [52, Theorem 2.9] and the second one follows from [52, Exercise 2.8, p 43]), where on $\mathbb{R}\times\mathbb{R}$ we consider the product measure. Hence we further deduce that

$$L^{\infty}(\mathbb{R}\times\mathbb{R}) \Big/ span\{\mathbf{1}\} \cong L\left(\overline{\ker(\beta_{\mathbb{R}})\hat{\otimes}_{\pi}F(\mathbb{R})+F(\mathbb{R})\hat{\otimes}_{\pi}\ker(\beta_{\mathbb{R}})}^{\|\cdot\|_{\pi}},\mathbb{R}\right).$$

By a standard measure-theoretic argument, it is known that

$$L^{\infty}(\mathbb{R}\times\mathbb{R}) \Big/ span\{\mathbf{1}\} \cong \left\{f\in L^{1}(\mathbb{R}^{2}) : \int\limits_{\mathbb{R}^{2}} f dxdy = 0\right\}^{*},$$

where $dxdy$ is the product measure on $\mathbb{R}^2$. Moreover we showed that $\ker(\beta_{\mathbb{R}})\hat{\otimes}_{\pi}F(\mathbb{R})+F(\mathbb{R})\hat{\otimes}_{\pi}\ker(\beta_{\mathbb{R}})$ is another predual, explicitly described in the following.

Lemma 5.5.2. $\ker(\beta_{\mathbb{R}})\hat{\otimes}_{\pi}F(\mathbb{R})$ *is isometrically isomorphic to* $\left\{f\in L^{1}(\mathbb{R}^{2}) : \int\limits_{\mathbb{R}} f(x,y)dx = 0,\ a.e\ y\right\}$.

Proof. Let us denote $L^1_0(\mathbb{R}) := \left\{f\in L^{1}(\mathbb{R}) : \int\limits_{\mathbb{R}} f(x)dx = 0\right\} \cong \ker(\beta_{\mathbb{R}})$. Since $F(\mathbb{R})\hat{\otimes}_{\pi}F(\mathbb{R}) \cong L^1(\mathbb{R})\hat{\otimes}_{\pi}L^1(\mathbb{R})$, we consider the canonical isometric isomorphism (see Theorem [26, p 230]).

$$\Phi : L^1(\mathbb{R})\hat{\otimes}_{\pi}L^1(\mathbb{R}) \to L^1(\mathbb{R}^2)$$

given by $\Phi(f\otimes g)(x,y) = f(x)g(y)$ for all $f,g\in L^1(\mathbb{R})$ and $x,y\in\mathbb{R}$. Therefore, $\ker(\beta_{\mathbb{R}})\hat{\otimes}_{\pi}F(\mathbb{R}) \cong L^1_0(\mathbb{R})\hat{\otimes}_{\pi}L^1(\mathbb{R})$. So it is enough to show that $L^1_0(\mathbb{R})\hat{\otimes}_{\pi}L^1(\mathbb{R})$ is isometrically isomorphic to $\left\{f\in L^{1}(\mathbb{R}^{2}) : \int\limits_{\mathbb{R}} f(x,y)dx = 0,\ \text{a.e}\ y\right\}$.

Put $V = \Phi\left(L^1_0(\mathbb{R})\hat{\otimes}_{\pi}L^1(\mathbb{R})\right) = \overline{span\{f(x)g(y) : f\in L^1_0(\mathbb{R}), g\in L^1(\mathbb{R})\}}^{\|\cdot\|_1} \subset L^1(\mathbb{R}^2)$, and $W = \left\{f\in L^{1}(\mathbb{R}^{2}) : \int\limits_{\mathbb{R}} f(x,y)dx = 0,\ \text{a.e}\ y\right\}$. It is immediate that $V\subset W$. Suppose $F\in W$. Then there exists $(F_n)\subset L^1(\mathbb{R}^2)$, a sequence of finite sums of elementary $f\otimes g$; approximating F in L^1 norm. Choose $\psi\in C_c(\mathbb{R})$ (space of all compactly supported continuous functions) with $\int\limits_{\mathbb{R}}\psi dx = 1$ and consider

$$\tilde{F_n}(x,y) = F_n(x,y) - \left(\int\limits_{\mathbb{R}} F_n(z,y)dz\right)\psi(x).$$

Then $\int\limits_{\mathbb{R}} \tilde{F_n}(x,y)dx = 0$. Now we show that $\tilde{F_n} \in span\{f(x)g(y) : f \in L^1_0(\mathbb{R}), g \in L^1(\mathbb{R})\}$.

Set $c_n(y) = \int\limits_{\mathbb{R}} F_n(z,y)dz = \sum\limits_{i=1}^{k} \left(\int\limits_{\mathbb{R}} f_i(z)dz \right) g_i(y) \in L^1(\mathbb{R})$. Therefore,

$$\begin{aligned}
\tilde{F_n}(x,y) &= \sum_{i=1}^{k} f_i(x)g_i(y) - c_n(y)\psi(x) \\
&= \sum_{i=1}^{k} f_i(x)g_i(y) - \psi(x)\sum_{i=1}^{k} \left(\int\limits_{\mathbb{R}} f_i(z)dz \right) g_i(y) \\
&= \sum_{i=1}^{k} \underbrace{\left(f_i(x) - \left(\int\limits_{\mathbb{R}} f_i(z)dz \right) \psi(x) \right)} g_i(y) \in span\left\{ f(x)g(y) : f \in L^1_0(\mathbb{R}), g \in L^1(\mathbb{R}) \right\}.
\end{aligned}$$

So it remains to prove that $\|\tilde{F_n} - F\|_{L^1(\mathbb{R}^2)} \to 0$.

We estimate

$$\|\tilde{F_n} - F\|_{L^1(\mathbb{R}^2)} = \|F_n - c_n(y)\psi(x) - F\|_{L^1(\mathbb{R}^2)} \leq \|F_n - F\|_{L^1(\mathbb{R}^2)} + \|c_n(y)\psi(x)\|_{L^1(\mathbb{R}^2)}.$$

Since $F_n \to F$ in L^1 norm, we have $\|F_n - F\|_{L^1(\mathbb{R}^2)}$ converges to 0 as $n \to \infty$. Now

$$\|c_n(y)\psi(x)\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{\mathbb{R}} |c_n(y)\psi(x)|\, dx\, dy = \|\psi\|_{L^1(\mathbb{R})} \cdot \|c_n\|_{L^1(\mathbb{R})}.$$

Further we estimate $\|c_n\|_{L^1(\mathbb{R})}$. Using the assumption that $F \in W$ that is $\int F(x,y)\, dx = 0$, we obtain

$$|c_n(y)| = \left| \int_{\mathbb{R}} F_n(x,y)\, dx \right| = \left| \int_{\mathbb{R}} (F_n(x,y) - F(x,y))\ dx \right| \leq \int_{\mathbb{R}} |F_n(x,y) - F(x,y)|\, dx.$$

Now, integrating over y, we get

$$\|c_n\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |c_n(y)|\, dy \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |F_n(x,y) - F(x,y)|\, dx\, dy = \|F_n - F\|_{L^1(\mathbb{R}^2)}.$$

Thus $\|c_n(y)\psi(x)\|_{L^1} = \|\psi\|_{L^1(\mathbb{R})}\|c_n\|_{L^1(\mathbb{R})}$ converges to 0. Hence, the proof follows. $\square$

Remark 5.5.3. Similarly, we can prove that $F(\mathbb{R})\hat{\otimes}_\pi \ker(\beta_{\mathbb{R}})$ is isometrically isomorphic to

$$\left\{ f \in L^1(\mathbb{R}^2) : \int\limits_{\mathbb{R}} f(x,y)dy = 0, \text{ a.e } x \right\}.$$

Hence we get

$$\overline{\ker(\beta_{\mathbb{R}})\hat{\otimes}_\pi F(\mathbb{R}) + F(\mathbb{R})\hat{\otimes}_\pi \ker(\beta_{\mathbb{R}})}^{\|\cdot\|_1}$$
$$\cong \left\{ f \in L^1(\mathbb{R}^2) : \int\limits_{\mathbb{R}} f(x,y)dx = 0, \text{ a.e } y \right\} + \left\{ f \in L^1(\mathbb{R}^2) : \int\limits_{\mathbb{R}} f(x,y)dy = 0, \text{ a.e } x \right\}.$$

Chapter 6

A generalization of Lipschitz mapping

In this chapter we shift our attention to extensively bounded maps, we propose to expand the scope and develop a similar theory as of Lipschitz functions in a broader setup. The main tool for this generalization is the notion of *modulus of continuity* (see [11]) of a mapping between metric spaces. This notion was formally introduced by H. Lebesgue in 1910 for the functions of real variables. Using the modulus of continuity, we can determine Lipschitz mappings. If we consider a modification in the modulus of continuity to make it point specific, it leads to a generalization of Lipschitz mappings making them point specific as well. If the metric spaces have fixed designated points and if a pointwise Lipschitz mapping takes the designated point in the domain to the designated point of the codomain, we call it *extensively bounded*.

It may be recalled that the space of Lipschitz mappings between metric spaces (which take the designated point in the domain to the designated point of the codomain) do not have any metric defined in it. However, in the space of extensively bounded mappings, we have been able to introduce a metric. Besides, the set of extensively mappings from a metric space into a normed linear space again form a vector space and we can introduce a norm in it as well. Moreover, if we consider the composition of two suitable extensively bounded mappings, it results in another extensively bounded mapping. In this sense we obtain a 'multiplication' between two extensively bounded mappings defined on a normed space. Even though this multiplication is not bilinear, still we get an operator algebra or more generally, an operator ideal like structure. In this sense, we have introduced a category bigger than that of the spaces of Lipschitz mappings between pairs of Banach spaces. From this paper onwards, we shall initiate a study of such mappings and spaces from various points of view.

Other than introductory part above, the chapter is divided in remaining four more sections. In Section 6.1, we consider the notion of *modulus of continuity at a point* of a mapping between metric spaces. As a consequence, we introduce the notion of *Lipschitz condition at a point* and obtain its equivalence in the usual sense of Lipschitz condition. Next, we introduce a semi-norm and consequently, a norm on the space of all the mappings from a metric space into a normed linear space which are Lipschitz at a designated point in the domain. We prove that this space is linearly isometric to an ℓ_∞-space. We characterize a bounded Lipschitz mapping in terms of pointwise Lipschitz mappings.

In Section 6.2, we discuss a generalization of the space of all Lipschitz mappings between two metric spaces with designated points which maps the designated point to the designated point and introduce the notion of *extensively bounded mappings* between two metric spaces with designated points which again maps the designated point to the designated point. We study a metric and completeness in the space of extensively bounded mappings. We show that when the codomain is replaced by a normed linear space, the space of extensively bounded mappings also becomes so. We also discuss its relation with the Lipschitz mappings that vanish at the designated point. Further, we describe a Banach space $F_e(M)$ which is a predual of the Banach space M^e of real-valued extensively bounded mappings on a metric space M. This structure enable us to discuss a linearization of normed linear space valued extensively bounded mappings. Next, we describe a (non-bilinear) multiplication between a pair of extensively bounded mappings between suitable normed linear spaces and a corresponding (non-linear) morphism.

6.1 Modulus of continuity at a point

Let M and N be metric spaces and let $f : M \to N$. For $t \in \mathbb{R}$ with $t > 0$, we define

$$\omega_f(t) := \sup\{d(f(x), f(y)) : d(x, y) \leq t\}.$$

Then $\omega_f : (0, \infty) \to [0, \infty]$ and is called *modulus of continuity of f*. Note that ω_f is an increasing function. It is also well known (see, for example, [11, chapter 1]) that

- f is uniformly continuous in M if and only if $\lim_{t \to 0^+} \omega_f(t) = 0$; and
- f is Lipschitz in M if and only if $\sup\{\frac{\omega_f(t)}{t} : t > 0\} < \infty$.

The modulus of continuity does not describe non-uniformly continuous mappings directly. But it can be done with a slight modification. We fix $x_0 \in M$ and for $t \in \mathbb{R}$ with $t > 0$, we define

$$\omega_f^{x_0}(t) := \sup\{d(f(x), f(x_0)) : d(x, x_0) \leq t\}.$$

Then $\omega_f^{x_0} : (0, \infty) \to [0, \infty]$ again is an increasing function and f is continuous at x_0 if and only if $\lim_{t \to 0^+} \omega_f^{x_0}(t) = 0$. We shall call $\omega_f^{x_0}$ the *modulus of continuity of f at x_0*.

Imitating the modulus of continuity, we consider the following notion introduced by Messerschmidt in [44, Definition 2.6].

Definition 6.1.1. Let M and N be metric spaces and let $f : M \to N$. For a fixed $x_0 \in M$, we say that f is *Lipschitz at x_0*, if

$$\sup\left\{\frac{\omega_f^{x_0}(t)}{t} : t > 0\right\} < \infty.$$

The set of all functions $f : M \to N$ which are Lipschitz at x_0 is denoted by $Lip^{x_0}(M, N)$.

Proposition 6.1.2. *Let M and N be metric spaces and let $f : M \to N$. Then for any $x_0 \in M$, we have $f \in Lip^{x_0}(M, N)$ if and only if there exists $k > 0$ such that $d(f(x), f(x_0)) \leq kd(x, x_0)$ for all $x \in M$.*

Proof. First, we assume that f is Lipschitz at x_0 for some $x_0 \in M$. Then $k := \sup\left\{\frac{\omega_f^{x_0}(t)}{t} : t > 0\right\} < \infty$. Let $x \in M \setminus \{x_0\}$ and put $t = d(x, x_0) > 0$. Then by definition, $d(f(x), f(x_0)) \leq \omega_f^{x_0}(t)$ so that

$$\frac{d(f(x), f(x_0))}{d(x, x_0)} \leq \frac{\omega_f^{x_0}(t)}{t} \leq k.$$

Thus $d(f(x), f(x_0)) \leq kd(x, x_0)$ for all $x \in M$.

Conversely, we assume that $d(f(x), f(x_0)) \leq kd(x, x_0)$ for all $x \in M$. We show that $\sup\left\{\frac{\omega_f^{x_0}(t)}{t} : t > 0\right\} \leq k$. Let $t > 0$ and $\epsilon > 0$. Then we can find $x \in M$, $d(x, x_0) \leq t$ such that $d(f(x), f(x_0)) > \omega_f^{x_0}(t) - \epsilon t$. Thus

$$\omega_f^{x_0}(t) - \epsilon t < d(f(x), f(x_0)) \leq kd(x, x_0) \leq kt.$$

Now it follows that $\frac{\omega_f^{x_0}(t)}{t} - \epsilon < k$ for all $t > 0$ and $\epsilon > 0$. Thus $\sup\left\{\frac{\omega_f^{x_0}(t)}{t} : t > 0\right\} \leq k < \infty$. □

The above result establishes that the two notions, namely pointwise α-Lipschitz and strongly pointwise α-Lipschitz, considered by Messerschmidt, Miek in [44, Definition 2.6] are actually equivalent.

We observe that the notion of the Lipschitz condition at a point is different from that of *local Lipschitz at a point*. (f is said to be *locally Lipschitz at a point* x_0, if there exists a neighbourhood N_{x_0} of x_0 in M such that f is Lipschitz in N_{x_0} (see, for example, [14, Chapter 2]).)

We say that f is *pointwise Lipschitz in* M, if $f \in Lip^x(M, N)$ for each $x \in M$. The set of all such mappings is denoted by $Lip^P(M, Y)$. Thus $Lip^P(M, Y) = \bigcap_{x \in M} Lip^x(M, N)$.

If we replace N by a (real) normed linear space Y, then $Lip^{x_0}(M, Y)$ is also a (real) vector space under pointwise operations. Moreover, the value $\|f\|_{x_0} := \sup\left\{\frac{\omega_f^{x_0}(t)}{t} : t > 0\right\}$ for all $f \in Lip^{x_0}(M, Y)$ determines a semi-norm $\|\cdot\|_{x_0}$ on $Lip^{x_0}(M, Y)$. In fact, we have

$$\|f\|_{x_0} = \inf\{k > 0 : \|f(x) - f(x_0)\| \leq kd(x, x_0) \text{ for all } x \in M\}$$

$$= \sup\left\{\frac{\|f(x)-f(x_0)\|}{d(x,x_0)} : x \in M, x \neq x_0\right\}$$

for all $f \in Lip^{x_0}(M,Y)$. Also $\|f\|_{x_0} = 0$ if and only if f is a constant function. Thus for $f \in Lip^{x_0}(M,Y)$ the value

$$\|f\|_{L^{x_0}} := \max\{\|f\|_{x_0}, \|f(x_0)\|\}$$

determines a norm on $Lip^{x_0}(M,Y)$. We record this fact and some more in the next result.

Proposition 6.1.3. *Let M be a metric space and Y a normed linear space. Then $(Lip^x(M,Y), \|.\|_{L^x})$ is isometrically isomorphic to $\ell_\infty(M,Y)$. Thus $(Lip^x(M,Y), \|.\|_{L^x})$ is a Banach space, if and only if Y is so.*

Proof. Let $f \in \ell_\infty(M,Y)$. Define $\Phi_x(f) : M \to Y$ given by

$$\Phi_x(f)(x') = d(x',x)f(x') + f(x)$$

for all $x' \in M$. Then

$$\|\Phi_x(f)(x') - \Phi_x(f)(x)\| = \|f(x')\|d(x',x) \leq \|f\|_\infty d(x',x)$$

for all $x' \in M$. Thus $\Phi_x(f) \in Lip^x(M,Y)$ for all $f \in \ell_\infty(M,Y)$. Now, it follows that $\Phi_x : \ell_\infty(M,Y) \to (Lip^x(M,Y), \|.\|_{L^x})$ is a linear mapping. Further

$$\begin{aligned}
\|\Phi_x(f)\|_{L^x} &= \sup\left\{\frac{\|\Phi_x(f)(x') - \Phi_x(f)(x)\|}{d(x',x)}, \|f(x)\| : x' \in M \setminus \{x\}\right\} \\
&= \sup\{\|f(x')\| : x' \in M\} \\
&= \|f\|_\infty
\end{aligned}$$

for all $f \in \ell_\infty(M,Y)$ so that Φ_x is a linear isometry. Finally, if $g \in Lip^x(M,Y)$, we set

$$h(x') = \begin{cases} \frac{g(x')-g(x)}{d(x',x)}, & \text{if } x' \neq x \\ g(x), & \text{if } x' = x. \end{cases}$$

As $g \in Lip^x(M,Y)$, there exists $k > 0$ such that $\|g(x') - g(x)\| \leq kd(x',x)$ for all $x' \in M$. Thus $\sup\{\|h(x')\| : x' \in M\} \leq k$ so that $h \in \ell_\infty(M,Y)$. Also, then $\Phi_x(h) = g$ so that Φ_x is the required isometry. □

Let $Lip(M,Y)$ denote the set of all Lipschitz mappings from M into Y where M is a metric space and Y is a normed linear space. Also then $Lip^b(M,N)$ denotes the set of bounded mappings in $Lip(M,N)$. Let us recall that for $f \in Lip^b(M,Y)$ the value

$$\|f\|_L := \sup\left\{\frac{\|f(x_1)-f(x_2)\|}{d(x_1,x_2)}, \|f\|_\infty : x_1,x_2 \in M \text{ with } x_1 \neq x_2\right\}$$

determines a norm $\|\cdot\|_L$ on $Lip^b(M,Y)$.(see [56, Chapter 2, p 35]) Further, $(Lip^b(M,Y), \|\cdot\|_L)$ is a Banach space whenever Y is so. We note that $(Lip^x(M,Y), \|\cdot\|_{L^x})$ generalizes $(Lip^b(M,Y), \|\cdot\|_L)$ in the following sense.

Consider $f \in \bigcap\limits_{x\in M} Lip^x(M,Y)$. Then

$$\begin{aligned}\sup_{x\in M}\|f\|_{L^x} &= \sup_{x\in M}\sup\left\{\frac{\|f(x')-f(x)\|}{d(x',x)}, \|f(x)\| : x' \in M\setminus\{x\}\right\}\\ &= \max\left\{\sup_{x,x'\in M; x\neq x'}\left(\frac{\|f(x')-f(x)\|}{d(x',x)}\right), \sup_{x\in M}\|f(x)\|\right\}\end{aligned}$$

Thus, if f is bounded in M with $\sup\limits_{x\in M}\|f\|_x < \infty$, then $f \in Lip^b(M,Y)$. The converse may be verified in a routine way. We summarize the observation in the following result.

Proposition 6.1.4. *Let M be a metric space M and Y a normed linear space. Then $f \in Lip^b(M,Y)$ if and only if f is bounded in M and $f \in \bigcap\limits_{x\in M} Lip^x(M,Y)$ with $\sup\limits_{x\in M}\|f\|_x < \infty$.*

Remark 6.1.5. Since $(Lip^x(M,Y), \|\cdot\|_{L^x})$ is isometrically isomorphic to $\ell_\infty(M,Y)$ for any $x \in M$, we get that $(Lip^{x_1}(M,Y), \|\cdot\|_{L^{x_1}})$ is isometrically isomorphic to $(Lip^{x_2}(M,Y), \|\cdot\|_{L^{x_2}})$ for all $x_1, x_2 \in M$.

Proposition 6.1.6. *Let X and Y be normed linear spaces. Assume that $x_1, x_2 \in X$ with $x_1 \neq x_2$ and consider the surjective isometry*

$$\Phi_{x_2}\circ\Phi_{x_1}^{-1} : (Lip^{x_1}(M,Y), \|\cdot\|_{L^{x_1}}) \to (Lip^{x_2}(M,Y), \|\cdot\|_{L^{x_2}}).$$

If $f \in Lip^{x_1}(X,Y)\bigcap Lip^{x_2}(X,Y)$ with $\Phi_{x_2}\circ\Phi_{x_1}^{-1}(f) = f$, then $f(x) = 0$ for all $x \in X$ with $\|x-x_1\| \neq \|x-x_2\|$. In particular, $f(x_1) = 0 = f(x_2)$.

Proof. First we note that

$$\Phi_{x_2} \circ \Phi_{x_1}^{-1}(f)(x) = \begin{cases} \frac{\|x-x_2\|}{\|x-x_1\|}(f(x) - f(x_1)) + \frac{f(x_2)-f(x_1)}{\|x_2-x_1\|}, & \text{if } x \neq x_1 \\ \|x_1 - x_2\| f(x_1) + \frac{f(x_2)-f(x_1)}{\|x_2-x_1\|}, & \text{if } x = x_1. \end{cases}$$

Thus as $f(x) = \Phi_{x_2} \circ \Phi_{x_1}^{-1}(f)(x)$, we get $f(x_2) = \frac{f(x_2)-f(x_1)}{\|x_1-x_2\|}$ and hence $f(x_2) = \|x_1 - x_2\| f(x_2) + f(x_1)$. Also $f(x_1) = \|x_1 - x_2\| f(x_1) + f(x_2)$. Now it follows that

$$f(x_1) = -f(x_2) = \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|}.$$

Also, on simplifying, we get

$$\|x - x_1\|(f(x) + f(x_1)) = \|x - x_2\|(f(x) - f(x_1))$$

or, equivalently,

$$(\|x - x_2\| - \|x - x_1\|)f(x) = (\|x - x_2\| + \|x - x_1\|)f(x_1)$$

for all $x \in X$. Put $x_0 = \frac{1}{2}(x_1 + x_2)$. Then $\|x_0 - x_1\| = \|x_0 - x_2\| = \frac{1}{2}\|x_1 - x_2\| > 0$. Thus $f(x_1) = 0$ as $\|x_0 - x_2\| + \|x_0 - x_1\| = \|x_1 - x_2\| > 0$ and consequently, $f(x_2) = 0$ too. Therefore, $(\|x - x_2\| - \|x - x_1\|)f(x) = 0$ for all $x \in X$. Hence the result follows. □

6.2 Extensively bounded mappings

Let X and Y be normed linear spaces and consider

$$Lip_0(X, Y) := \{f \in Lip(X, Y) : f(0) = 0\}.$$

We recall that this space has been studied in the literature (see, for example, [56, Chapter 1]) as a non-linear generalization of bounded linear mappings between the normed linear spaces. We propose a further generalization and consider a particular (closed) subspace of $Lip^x(M, Y)$ as under:

$$Lip^x_x(M, Y) := \{f \in Lip^x(M, Y) : f(x) = 0\}.$$

For $f \in Lip_x^x(M, Y)$, we have

$$\|f\|_{L^x} = \|f\|_x = \sup\left\{\frac{\|f(x')\|}{d(x', x)} : x' \in M \setminus \{x\}\right\}.$$

Thus $(Lip_x^x(M, Y), \|\cdot\|_x)$ is isometrically isomorphic to $\ell_\infty(M\setminus\{x\}, Y)$ and $(Lip_x^x(M, Y), \|\cdot\|_x)$ is a Banach space, if and only if Y is so. Note that $f : M \to Y$ is in $Lip_x^x(M, Y)$ if and only if there exists $k > 0$ such that $\|f(x')\| \leq kd(x', x)$ for all $x' \in M$. Thus

$$\|f\|_x = \inf\{k > 0 : \|f(x')\| \leq kd(x', x) \text{ for all } x' \in M\}.$$

Let us fix $x \in M$ and consider the space

$$Lip_x(M, Y) := \{f : M \to Y | f \text{ is Lipschitz in } M \text{ and } f(x) = 0\}$$

together with the norm $Lip(\cdot)$ given by

$$\begin{aligned} Lip(f) &:= \inf\{k > 0 : \|f(x_1) - f(x_2)\| \leq kd(x_1, x_2) \text{ for all } x_1, x_2 \in M\} \\ &= \sup\left\{\frac{\|f(x_1) - f(x_2)\|}{d(x_1, x_2)} : x_1, x_2 \in M \text{ with } x_1 \neq x_2\right\} \end{aligned}$$

for all $f \in Lip_x(M, Y)$. Then $(Lip_x(M, Y), Lip(\cdot))$ is a normed linear space and is complete whenever Y is so.(see [14, Chapter 8, Theorem 8.1.3]) Thus $(Lip_x^x(M, Y), \|\cdot\|_x)$ is a natural generalization of $(Lip_x(M, Y), Lip(\cdot))$.

Being isometric to $\ell_\infty(M\setminus\{x\}, Y)$, $(Lip_x^x(M, Y), \|\cdot\|_x)$ may appear in a poor light. But we shall show in another paper that when M is also replaced by a normed linear space X, $(Lip_x^x(X, Y), \|\cdot\|_x)$ assumes a non-commutative structure which is not known in ℓ_∞ spaces. Further, this facet leads us to an operator ideal theory generalizing that of the Lipschitz operators. Moreover, we shall later infuse continuity to make it less general.

Towards this goal, we re-describe $(Lip_x^x(M, \mathbb{R}))$ with a simpler notation. Since for any $x_1, x_2 \in M$, $(Lip_{x_1}^{x_1}(M, Y), \|\cdot\|_{x_1})$ is isometrically isomorphic to $(Lip_{x_2}^{x_2}(M, Y), \|\cdot\|_{x_2})$, there is no loss of generality in fixing a designated point. We generally describe it as 0.

Definition 6.2.1. Let M, N be metric spaces with 0 as the distinguished point. A mapping $f : M \to N$ is said to be *extensively bounded*, if there exists $k > 0$ such that $d(f(x), 0) \leq kd(x, 0)$ for all $x \in M$. The set of all extensively bounded mappings from M to N is denoted as $E(M, N)$.

Theorem 6.2.2. *Let M and N be metric spaces with 0 as distinguished points.*

1. *The map $d_e : E(M, N) \times E(M, N) \to [0, \infty)$ given by*

$$d_e(f, g) := \sup_{x \neq 0} \frac{d(f(x), g(x))}{d(x, 0)},$$

   for all $f, g \in E(M, N)$ determines a metric on $E(M, N)$.

2. *$(E(M, N), d_e)$ is a complete metric space whenever N is so.*

3. *If N replaced by a (real) normed linear space Y, then $E(M, Y)$ becomes a vector space and d_e induces a norm $\| \cdot \|_e$ on $E(M, Y)$. Consequently, $(E(M, Y), \| \cdot \|_e)$ is a Banach space whenever Y is so.*

Proof. Let $f, g \in E(M, N)$. Then there exist $k > 0$ and $l > 0$ such that $d(f(x), 0) \leq kd(x, 0)$ and $d(g(x), 0) \leq ld(x, 0)$ for all $x \in M$. Now using the triangle inequality, we get

$$d(f(x), g(x)) \leq d(f(x), 0) + d(g(x), 0) \leq (k + l)d(x, 0)$$

for any $x \in M$. Thus for any $x \in M \setminus \{0\}$, we have

$$\frac{d(f(x), g(x))}{d(x, 0)} \leq \frac{d(f(x), 0)}{d(x, 0)} + \frac{d(g(x), 0)}{d(x, 0)} \leq k + l.$$

Therefore, d_e is well-defined. It follows from the definition that $d_e(f, g) = d_e(g, f)$ for any $f, g \in E(M, N)$.

Assume that $d_e(f, g) = 0$ for some $f, g \in E(M, N)$. Then for all $x \in M \setminus \{0\}$, we have $d(f(x), g(x)) = 0$. Also $f(0) = 0 = g(0)$ so that $f = g$. Thus d_e is a metric on $E(M, N)$.

Let (f_n) be a Cauchy sequence in $(E(M,N), d_e)$. Then given $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $d_e(f_m, f_n) \leq \epsilon$ for all $m, n \geq n_0$. In other words, $d(f_m(x), f_n(x)) \leq \epsilon d(x, 0)$ for all $x \in M$ if $m, n \geq n_0$. In particular, for each $x \in M$, $(f_n(x))$ is Cauchy in N. Since N is complete, $(f(x_n))$ converges to some $y \in N$. As y is determined by $x \in M$ uniquely, we get a map $f : M \to N$ such that f_n converges to f point-wise. Also then

$$d(f(x), f_n(x)) = \lim_{m \to \infty} d(f_m(x), f_{n_0}(x)) \leq \epsilon d(x, 0)$$

for all $x \in M$, if $n \geq n_0$. Therefore,

$$d(f(x), 0) \leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), 0) \leq (\epsilon + d(f_{n_0}, 0)) d(x, 0)$$

for all $x \in M$. Thus $f \in E(M, N)$. Also then

$$d_e(f, f_n) = \sup_{x \neq 0, x \in M} \frac{d(f(x), f_n(x))}{d(x, 0)} \leq \epsilon$$

for all $n \geq n_0$. Thus $(E(M,N), d_e)$ is complete.

Now consider Y to be a (real) normed linear space. It is routine to check that $E(M, Y)$ is a (real) vector space under pointwise addition and scalar multiplication. Again, for $f, g \in E(M, Y)$,

$$d_e(f, g) = \sup_{x \neq 0} \frac{\|f(x) - g(x)\|}{d(x, 0)} = d_e(f - g, 0).$$

Further, for $\alpha \in \mathbb{K}$,

$$d_e(\alpha f, \alpha g) = \sup_{x \neq 0} \frac{\|\alpha f(x) - \alpha g(x)\|}{d(x, 0)} = |\alpha| \sup_{x \neq 0} \frac{\|f(x) - g(x)\|}{d(x, 0)} = |\alpha| d_e(f, g).$$

Thus the metric d_e induces a norm $\| \cdot \|_e$ on $E(M, Y)$ given by

$$\|f\|_e = d_e(f, 0)$$

for all $f \in E(M, Y)$ whenever Y is a (real) normed linear space.

In particular, if Y is a Banach space, then $(E(M, Y), \| \cdot \|_e)$ is also a Banach space. $\square$

Remark 6.2.3. In general, there is no specific metric defined on $Lip_0(M, N)$. However, being a subspace of $E(M, N)$, we can provide the restriction of d_e on it. But we do not know whether, in general, $Lip_0(M, N)$ is complete with this metric or not, even assuming the completeness of N.

Example 6.2.4. *In general, $E(M, N)$ and $C(M, N)$ are distinct spaces. For example, let $M = N = \mathbb{R}$ and consider the functions*

$$g(x) = \begin{cases} 2 & \textit{if } x = 1 \\ x & \textit{if } x \neq 1 \end{cases}$$

and $h(x) = x^2$. Then $g \in E(\mathbb{R}, \mathbb{R}) \setminus C(\mathbb{R}, \mathbb{R})$ and $h \in C(\mathbb{R}, \mathbb{R}) \setminus E(\mathbb{R}, \mathbb{R})$.

Now we consider the space $E_c(M, Y) = C(M, Y) \cap E(M, Y)$ as a normed subspace of $E(M, Y)$.

Proposition 6.2.5. *Consider $E_c(M, N) := C(M, N) \cap E(M, N)$ where M and N are metric spaces with distinguished points as 0. Then $E_c(M, N)$ is a closed subspace of $E(M, N)$ and hence is complete whenever N is so.*

Proof. Let (f_n) be a sequence in $E_c(M, N)$ such that it converges to $f \in E(M, N)$. We show that f is continuous.

Step I. Continuity of f at $0 \in M$.

Since $f_n \to f$, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_e(f_n, f) < \epsilon$ for all $n \geq n_0$. Thus $d(f_n(x), f(x)) \leq \epsilon d(x, 0)$ for all $n \geq n_0$ and $x \in M$. Now f_{n_0} is continuous at 0 so there exists $\delta > 0$ such that $d(f_{n_0}(x), 0) < \frac{\epsilon}{2}$ whenever $d(x, 0) < \delta$. (In fact, $f_{n_0}(0) = 0$.) Consider $\delta' = min\{\delta, \frac{1}{2}\} > 0$, then for $d(x, 0) < \delta'$, we have

$$\begin{aligned} d(f(x), f(0)) = d(f(x), 0) &\leq d(f_{n_0}(x), f(x)) + d(f_{n_0}(x), 0) \\ &< \epsilon\, d(x, 0) + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus f is continuous at 0.

Step II. Continuity of f at $x_0 \in M \setminus \{0\}$.

Since $f_n \to f$, given $\epsilon > 0$, we can find $n_1 \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) \leq \frac{\epsilon}{4\, d(x_0, 0)} d(x, 0)$$

for all $n \geq n_1$ and $x \in M$. Also, as f_{n_0} is continuous at x_0, there exists $\delta > 0$ such that $d(f_{n_1}(x), f_{n_1}(x_0)) < \frac{\epsilon}{4}$, whenever $d(x, x_0) < \delta$. Choose $\delta' = min\{d(x_0, 0), \delta\} > 0$. Then, for $x \in M$ with $d(x, x_0) < \delta'$, we have

$$\begin{aligned}
d(f(x), f(x_0)) &\leq d(f(x), f_{n_1}(x)) + d(f_{n_1}(x), f_{n_1}(x_0)) \\
&+ d(f_{n_1}(x_0), f(x_0)) \\
&< \frac{\epsilon}{4\, d(x_0, 0)} d(x, 0) + \frac{\epsilon}{4} + \frac{\epsilon}{4\, d(x_0, 0)} d(x_0, 0) \\
&\leq \frac{\epsilon\, (d(x, x_0) + d(x_0, 0))}{4\, d(x_0, 0)} + \frac{\epsilon}{2} \\
&< \frac{3\, \epsilon}{4} + \frac{\epsilon}{4\, d(x_0, 0)} \delta' \\
&\leq \frac{3\, \epsilon}{4} + \frac{\epsilon}{4\, d(x_0, 0)} d(x_0, 0) = \epsilon.
\end{aligned}$$

Thus f is continuous in M, that is, $f \in E_c(M, N)$ and consequently, we conclude that $E_c(M, N)$ is a closed subspace of $E(M, N)$. □

Example 6.2.6. *1. Consider $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = xsin(x)$ then $f \in E_c(\mathbb{R}, \mathbb{R})$ with $\|f\|_e = 1$, but f is not uniformly continuous as $f(2n\pi + \frac{1}{n}) - f(2n\pi)$ converges to 2π; hence f is not Lipschitz. Thus $f \in E_c(\mathbb{R}, \mathbb{R}) \setminus Lip_0(\mathbb{R}, \mathbb{R})$.*

2. From the above example, we have that $E_c(\mathbb{R}, \mathbb{R})$ is not contained in the set of all uniformly continuous maps from $\mathbb{R}$ to $\mathbb{R}$. Also, the map $f(x) = \frac{1}{1+x^2}$ $\forall x \in \mathbb{R}$ is uniformly continuous but not in $E_c(\mathbb{R}, \mathbb{R})$.

Remark 6.2.7. 1. Let M be a metric space with the distinguished point 0 and Y be a normed linear space, then $Lip_0(M,Y) \subset E_c(M,Y)$ with $\|f\|_e \leq Lip(f)$ for any $f \in Lip_0(M,Y)$. In fact, if $f \in Lip_0(M,Y)$, then $\|f(x)\| \leq Lip(f)d(x,0)$ for all $x \in M$. Thus $\|f\|_e \leq Lip(f)$.

2. For any normed linear spaces X and Y, we have

$$L(X,Y) \subset Lip_0(X,Y) \subset E_c(X,Y)$$

with $\|T\| = Lip(T) = \|T\|_e$, whenever $T \in L(X,Y)$, where $L(X,Y)$ is the space of all bounded linear operators from X to Y. To see this, let $T \in L(X,Y)$. Then

$$\|T\|_e = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} = \|T\|$$

and

$$Lip(T) = \sup_{x \neq y} \frac{\|T(x) - T(y)\|}{\|x - y\|} = \|T\|.$$

Remark 6.2.8. 1. In general, $Lip(\cdot)$ and $\|\cdot\|_e$ are not equivalent in $Lip_0(M,\mathbb{R})$. To show this, we consider $M = [0,1]$ and the sequence of functions $f_n(x) = x^n$. Then $f_n \in Lip_0([0,1],\mathbb{R})$ with $\|f_n\|_e = 1$ but $Lip(f_n) = n$ for all $n \in \mathbb{N}$.

2. By Remark 6.2.7(1), we have $\overline{Lip_0(M,\mathbb{R})}^{Lip(\cdot)} \subset \overline{Lip_0(M,\mathbb{R})}^{\|\cdot\|_e}$. Also, by above remark (1), the inclusion is proper.

Example 6.2.9. *Consider the metric space $M = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with 0 as the distinguished point. Then*

$$E(M,\mathbb{R}) = \{(x_n) \in c_0 : |x_n| \leq k.\frac{1}{n} \textit{ for all } n \in \mathbb{N}\}.$$

Proof. Let $f \in E(M,\mathbb{R})$. Then $|f(\frac{1}{n})| \leq \frac{1}{n}\|f\|_e$ and $f(0) = 0$. Put $x_n = f(\frac{1}{n})$. Then $(x_n) \in c_0$ and $|x_n| \leq \|f\|_e.\frac{1}{n}$ for all $n \in \mathbb{N}$. Conversely, consider $(x_n) \in c_0$ with $|x_n| \leq k.\frac{1}{n}$ for some $k > 0$ and all $n \in \mathbb{N}$. Define $f(\frac{1}{n}) = x_n$ for $n \in \mathbb{N}$ and $f(0) = 0$ then $f \in E(M,\mathbb{R})$. □

6.2.1 Linearization of extensively bounded map

Before we proceed further, we shall fix some notations. We denote the space $E(M,\mathbb{R})$ by M^e and therefore, $E(E(M,\mathbb{R}),\mathbb{R})$ by M^{ee} and so on. In this section, we shall prove that an extensively bounded mapping between metric spaces can be dilated to a Lipschitz mapping between bigger metric spaces and an extensively bounded continuous mapping from a metric space into a normed linear space can be dilated to a bounded linear mapping between (bigger) Banach spaces.

Proposition 6.2.10. *Let M be a metric space with 0 as a distinguished point. For $x \in M$ we define $\delta_x^e : M^e \to \mathbb{R}$ given by $\delta_x^e(f) = f(x)$ for all $f \in M^e$. Then δ_x^e is a bounded linear functional on M^e with $\|\delta_x^e\| = d(x,0)$. In general, $\|\delta_x^e - \delta_y^e\| \geq d(x,y)$ for $x, y \in M$. Next, consider the mapping $\delta_M^e : M \to (M^e)^*$ given by $\delta_M^e(x) = \delta_x^e$ for all $x \in M$. Then δ_M^e is extensively bounded.*

Proof. Clearly, δ_x^e is linear. Also for any $f \in E(M,\mathbb{R})$ with $\|f\|_e \leq 1$, we have

$$|\delta_x^e(f)| = |f(x)| \leq d(x,0)$$

so that δ_x^e is bounded with $\|\delta_x^e\| \leq d(x,0)$. Consider $f_1 : M \to \mathbb{R}$ given by $f_1(z) = d(z,0)$ for all $z \in M$. Then $f_1 \in M^e$ with $\|f_1\|_e = 1$. Thus $\|\delta_x^e\| \geq \delta_x^e(f_1) = f_1(x) = d(x,0)$ so that $\|\delta_x^e\| = d(x,0)$ for all $x \in M$. Hence, the map δ_M^e is extensively bounded.

Let $x, y \in M$ and define $g : M \to \mathbb{R}$ given by $g(z) = d(y,z) - d(y,0)$ for all $z \in M$. Then $g \in M^e$ with $\|g\|_e \leq 1$. Therefore,

$$\|\delta_x^e - \delta_y^e\| = \sup_{f \in M^e,\ \|f\|_e \leq 1} |f(x) - f(y)| \geq |g(x) - g(y)| = d(x,y).$$

□

Remark 6.2.11. Since δ_x^e is a linear map, it follows from Remark 6.2.7 (2) that $\|\delta_x^e\| = Lip(\delta_x^e) = \|\delta_x^e\|_e$.

When M is replaced by a normed linear space X, $x \mapsto \delta^e_x$ is a (non-linear) norm preserving mapping from X into $(X^e)^*$. We consider the closed subspace $F_e(M) = \overline{span\,\{\delta^e_x : x \in M\}}$ of $(M^e)^*$. Then $\delta^e_M : M \to F_e(M)$ given by $\delta^e_M(x) = \delta^e_x$ is a norm preserving (and therefore, extensively bounded) non-linear map such that $\overline{span\,\{\delta^e_M(M)\}} = F_e(M)$. Let us recall that when M^e is replaced by $Lip_0(M, \mathbb{R})$, a space similar to $F_e(M)$ is called the Lipschitz-free space of M. As is in the case of Lipschitz functions, we show that M^e is also a dual space. Actually, we prove more. Before proceeding further, we establish additional properties of $F_e(M)$.

Proposition 6.2.12. $F_e(\mathbb{N} \cup \{0\}) = F_e(\mathbb{N})$ *is isometrically isomorphic to* $\ell_1(\mathbb{N})$.

Proof. $F_e(\mathbb{N} \cup \{0\}) = F_e(\mathbb{N})$ follows directly as $\delta^e_0 = 0$.

Let us define $\Lambda : \ell_1(\mathbb{N}) \to F_e(\mathbb{N})$ given by

$$\Lambda((x_n)) = \sum_{n=1}^{\infty} \frac{x_n}{n} \delta^e_n.$$

Since $\sum\limits_{n=1}^{\infty} \|\frac{x_n}{n}\delta^e_n\| \leq \sum\limits_{n=1}^{\infty} \|x_n\|$, and $F_e(\mathbb{N})$ is a Banach space Λ is well-defined. Also, we can immediately verify that Λ is linear. In fact the well-definedness part provides that Λ is a contraction.

Now we show that Λ is injective. For all $i \in \mathbb{N}$ consider the maps

$$f_i(n) = \begin{cases} n & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$

Directly it follows that $f_i \in B_{(\mathbb{N}\cup\{0\})^e}$ for all $i \in \mathbb{N}$. Now $\Lambda((x_n)) = 0$ gives $\sum\limits_{n=1}^{\infty} \frac{x_n}{n}\delta^e_n = 0$. Therefore $\left(\sum\limits_{n=1}^{\infty} \frac{x_n}{n}\delta^e_n\right)(f_i) = 0$ for all $i \in \mathbb{N}$. This implies $x_i = 0$ for all $i \in \mathbb{N}$. Hence Λ is injective.

Again for surjectivity of Λ; suppose $\gamma \in F_e(\mathbb{N})$. Then $\gamma = \sum\limits_{n=1}^{\infty} \alpha_n \delta^e_n$, for some $(\alpha_n) \subset \mathbb{R}$.

Put $x_n = n\alpha_n$, for all $n \in \mathbb{N}$. Also consider the map $f_0 : \mathbb{N} \to \mathbb{R}$ given by

$$f_0(n) = sgn(\alpha_n)n.$$

Then $f_0 \in B_{(\mathbb{N}\cup\{0\})^e}$. Now $\gamma = \sum_{n=1}^{\infty} \alpha_n \delta_n^e \in F_e(\mathbb{N})$ implies $\left(\sum_{n=1}^{\infty} \alpha_n \delta_n^e\right)(f_0) \in \mathbb{R}$. That is

$$\sum_{n=1}^{\infty} \alpha_n f_0(n) = \sum_{n=1}^{\infty} \alpha_n sgn(\alpha_n)n = \sum_{n=1}^{\infty} n|\alpha_n| \in \mathbb{R}.$$

Thus we have $(x_n) \in \ell_1(\mathbb{N})$, and it is immediate that $\Lambda((x_n)) = \gamma$. Therefore from the open mapping theorem Λ^{-1} exists as a bounded linear map. Next we show that Λ^{-1} is also a contraction. We prove this by showing that $\Lambda\left(B_{\ell_1(\mathbb{N})}\right) = B_{F_e(\mathbb{N})}$. Since Λ is a contraction $\Lambda\left(B_{\ell_1(\mathbb{N})}\right) \subset B_{F_e(\mathbb{N})}$. Again from the surjectivity part it follows that for any $\gamma \in B_{F_e(\mathbb{N})}$, $\Lambda^{-1}(\gamma) \in B_{\ell_1(\mathbb{N})}$. , Λ is a contraction and its inverse is also a contraction. Hence $\ell_1(\mathbb{N})$ is isometrically isomorphic to $F_e(\mathbb{N})$. $\square$

Proposition 6.2.13. *Let M and N be two metric spaces with distinguished point as 0. $T : M \to N$ be surjective metric isometry. Then $F_e(M)$ and $F_e(N)$ are linearly isometric isomorphism.*

Proof. We already have seen that the map $T^e : N^e \to M^e$ given by $T^e(g) = g \circ T$ is bounded linear. Now, we show that it is a surjective isometry.

$$\|T^e(g)\|_e = \sup_{x\in M\setminus\{0\}} \frac{|g(Tx)|}{d(x,o)} = \sup_{x\in M\setminus\{0\}} \frac{|g(Tx)|}{d(Tx,0)} = \sup_{y\in N\setminus\{0\}} \frac{|g(y)|}{d(y,0)} = \|g\|_e.$$

Let $f \in M^e$, then $g = f \circ T^{-1}$ be the preimage of f under T^e in N^e. This shows that M^e and N^e are linearly isometric isomorphic.

Now $(T^e)^* : (M^e)^* \to (N^e)^*$ is also linearly isometric isomorphism. Since $(T^e)^*(F_e(M)) = F_e(N)$, the proof follows. $\square$

Corollary 6.2.14. *Let X and Y be two linear isometrically isomorphic Banach spaces and T be the linear isometry. Then $F_e(X)$ is isometrically isomorphic to $F_e(Y)$.*

Proof. Let us define $\Lambda : span\left\{\delta_x^e : x \in X\right\} \to span\left\{\delta_y^e : y \in Y\right\}$ given by $\Lambda\left(\sum\limits_{n=1}^{m} \alpha_n \delta_{x_n}^e\right) = \sum\limits_{n=1}^{m} \alpha_n \delta_{Tx_n}^e$. Now

$$\begin{aligned}
\left\|\sum_{n=1}^{m} \alpha_n \delta_{Tx_n}^e\right\| &= \sup_{g \in B_{Y^e}} \left|\sum_{n=1}^{m} \alpha_n g \circ T(x_n)\right| \\
&\leq \sup_{f \in B_{X^e}} \left|\sum_{n=1}^{m} \alpha_n f(x_n)\right| \\
&= \left\|\sum_{n=1}^{m} \alpha_n \delta_{x_n}^e\right\|
\end{aligned}$$

shows that Λ is a well-defined linear contraction.

Similarly the map $\Theta : span\left\{\delta_y^e : y \in Y\right\} \to span\left\{\delta_x^e : x \in X\right\}$ given by

$$\Theta\left(\sum_{n=1}^{m} \alpha_n \delta_{y_n}^e\right) = \sum_{n=1}^{m} \alpha_n \delta_{T^{-1}y_n}^e,$$

is a well-defined linear contraction. In fact Θ is the inverse of Λ. Now extending these maps Λ and Θ to the closure of their respective domains, we complete the proof of the proposition. $\square$

Theorem 6.2.15. *Let M be a metric space and Y a Banach space and consider $T \in E(M, Y)$. Then there exists a unique bounded linear map $\hat{T} : F_e(M) \to Y$ such that $\hat{T} \circ \delta_M^e = T$ with $\|\hat{T}\| = \|T\|_e$. Further, the mapping $\Phi_M^Y : E(M, Y) \to L(F_e(M), Y)$, given by $\Phi_M^Y(T) = \hat{T}$ for $T \in E(M, Y)$, is a surjective linear isometry. We shall write simply Φ for Φ_M^Y, if it is clear from the context.*

Proof. Define $\hat{T} : span\{\delta_x^e : x \in M\} \to Y$ by $\hat{T}\left(\sum\limits_{i=1}^{n} \alpha_i \delta_{x_i}^e\right) = \sum\limits_{i=1}^{n} \alpha_i T(x_i)$. Then $\hat{T}$ is linear with $\hat{T} \circ \delta_X^e = T$. Also

$$\begin{aligned}
\left\|\sum_{i=1}^{n} \alpha_i T(x_i)\right\| &= \sup_{y^* \in B_{Y^*}} \left\|\sum_{i=1}^{n} \alpha_i y^*(T(x_i))\right\| \\
&= \|T\|_e \sup_{y^* \in B_{Y^*}} \left|\sum_{i=1}^{n} \alpha_i y^*(\frac{T}{\|T\|_e}(x_i))\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|T\|_e \sup\left\{\left|\sum_{i=1}^{n}\alpha_i f(x_i)\right| : f \in X^e, \|f\|_e \leq 1\right\}\\
&= \|T\|_e \sup\left\{\left|\left(\sum_{i=1}^{n}\alpha_i \delta^e_{x_i}\right)(f)\right| : f \in X^e, \|f\|_e \leq 1\right\}\\
&= \|T\|_e \left\|\sum_{i=1}^{n}\alpha_i \delta^e_{x_i}\right\|.
\end{aligned}$$

Thus $\hat{T}$ is bounded and linear in $span\,\{\delta^e_x : x \in M\}$ with $\|\hat{T}\| \leq \|T\|_e$. Hence it can be extended uniquely to $F_e(M)$ as a bounded linear mapping, denoted again as $\hat{T}$, with $\|\hat{T}\| \leq \|T\|_e$.

Further,

$$\|T\|_e = \sup_{x\neq 0}\frac{\|Tx\|}{d(x,0)} = \sup_{x\neq 0}\frac{\|\hat{T}(\delta^e_x)\|}{\|\delta^e_x\|} \leq \|\hat{T}\|.$$

Thus Φ is a linear isometry. We prove that Φ is surjective. In fact, if $\xi \in L(F_e(M), Y)$, then $\xi \circ \delta^e_M \in E(M, Y)$ and $\Phi(\xi \circ \delta^e_M) = \xi$. This completes the proof. $\square$

Remark 6.2.16. Taking $Y = \mathbb{R}$, we note that M^e is dual of $F_e(M)$.

Definition 6.2.17. For $T \in E(M, Y)$ we define $T^e : Y^e \to M^e$ by

$$T^e(f) = f \circ T.$$

Then T^e is a bounded linear map with $\|T^e\| = \|T\|_e$. Now consider $T^{ee}(= (T^e)^e) : M^{ee} \to Y^{ee}$. Then $T^{ee}(\phi) = \phi \circ T^e$. Thus for $g \in Y^e$,

$$\left(T^{ee}(\delta^e_x)\right)(g) = (\delta^e_x \circ T^e)(g) = \delta^e_x(g \circ T) = g(T(x)) = \delta^e_{(T(x))}(g),$$

so that $T^{ee}(\delta^e_x) = \delta^e_{(T(x))}$. Moreover, $\|T^{ee}\| = \|T^e\| = \|T\|_e$.

Lemma 6.2.18. *Let X be a Banach space. Then there exists a linear contraction $\beta^e_X : F_e(X) \to X$.*

Proof. Define $\beta_X^e : span\{\delta_x^e : x \in X\} \to X$ given by

$$\beta_X^e\Big(\sum_{i=1}^n \alpha_i \delta_{x_i}^e\Big) = \sum_{i=1}^n \alpha_i x_i.$$

Then β_X^e is a well-defined linear map. Also,

$$\begin{aligned}
\|\sum_{i=1}^n \alpha_i x_i\| &= \sup\Big\{\big|\sum_{i=1}^n \alpha_i x^*(x_i)\big| : x^* \in X^*, \|x^*\| \leq 1\Big\} \\
&\leq \sup\Big\{\big|\sum_{i=1}^n \alpha_i f(x_i)\big| : f \in X^e, \|f\|_e \leq 1\Big\} \\
&= \sup\Big\{\Big|\Big(\sum_{i=1}^n \alpha_i \delta_{x_i}^e\Big)(f)\Big| : f \in X^e, \|f\|_e \leq 1\Big\} \\
&= \|\sum_{i=1}^n \alpha_i \delta_{x_i}^e\|.
\end{aligned}$$

Thus β_X^e is a bounded linear contraction with $\beta_X^e \circ \delta_X^e = Id_X$ and hence can be extended uniquely to $F_e(X)$. In fact it is a left inverse of δ_X^e. □

6.2.2 The (non-linear) operator algebra of extensively bounded mappings

Let X, Y, Z be normed linear spaces. Then for $f, f_1, f_2 \in E(X,Y)$ and $g, g_1, g_2 \in E(Y,Z)$, we have $g \circ f \in E(X,Z)$, $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$. But, in general, $g \circ (f_1 + f_2) \neq g \circ f_1 + g \circ f_2$. However, when $g \in L(Y,Z) \subset E(Y,Z)$, we have $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$. Further

$$\|g \circ f\|_e = \sup_{x \neq 0} \frac{\|g(f(x))\|}{\|x\|} \leq \|g\|_e \sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|} = \|g\|_e \|f\|_e.$$

These properties exhibit a 'normed operator ideal' like structure on the space of extensively bounded maps of Banach spaces with a pinch of non-linearity. This situation is observed in the case of Lipschitz maps as well. In this sense, $E(X,X)$ is a (non-linear) normed algebra with unity and therefore, it is a (non-linear) unital Banach algebra whenever X is complete.

Let us continue with the above discussion. We observe that since $g \circ f \in E(X, Z)$, $\Phi_X^Z(g \circ f)$ makes sense. However, as $\Phi_Y^Z(g)$ can not composed with $\Phi_X^Y(f)$, Φ is not 'multiplicative'. We dilate Theorem 6.2.15 further to get a 'multiplicative' linearization.

Theorem 6.2.19. *Let X and Y be normed linear spaces, and $T \in E(X, Y)$. Then there exists a unique bounded linear map $\widetilde{T} : F_e(X) \to F_e(Y)$ such that $\widetilde{T} \circ \delta_X^e = \delta_Y^e \circ T$ with $\|\widetilde{T}\| = \|T\|_e$.*

Proof. Let us first define the map

$$\widetilde{T} : span\,\{\delta_x^e : x \in X\} \to span\,\{\delta_y^e : y \in Y\}$$

given by

$$\widetilde{T}\left(\sum_{i=1}^{n} \alpha_i \delta_{x_i}^e\right) = \sum_{i=1}^{n} \alpha_i \delta_{Tx_i}^e.$$

Therefore by definition $\widetilde{T}$ is a linear map and $\widetilde{T} \circ \delta_X^e = \delta_Y^e \circ T$. Now for $x_1, ..., x_n \in X$ and the scalars $\alpha_i, i = 1, 2, ..., n$

$$\begin{aligned}
\left\|\sum_{i=1}^{n} \alpha_i \delta_{x_i}^e\right\| &= \sup\left\{\left|\sum_{i=1}^{n} \alpha_i f(x_i)\right| : f \in X^e, \|f\|_e \leq 1\right\} \\
&\geq \frac{1}{\|T\|_e} \sup\left\{\left|\sum_{i=1}^{n} \alpha_i (g \circ T)(x_i)\right| : g \in Y^e, \|g\|_e \leq 1\right\} \\
&= \frac{1}{\|T\|_e} \sup\left\{\left|\sum_{i=1}^{n} \alpha_i \delta_{(T(x_i))}^e(g)\right| : g \in Y^e, \|g\|_e \leq 1\right\} \\
&= \frac{1}{\|T\|_e} \left\|\sum_{i=1}^{n} \alpha_i \delta_{Tx_i}^e\right\|.
\end{aligned}$$

Thus

$$\left\|\widetilde{T}\left(\sum_{i=1}^{n} \alpha_i \delta_{x_i}^e\right)\right\| = \left\|\sum_{i=1}^{n} \alpha_i \delta_{Tx_i}^e\right\| \leq \|T\|_e \left\|\sum_{i=1}^{n} \alpha_i \delta_{x_i}^e\right\|$$

so that $\widetilde{T}$ is bounded. We extend $\widetilde{T}$ uniquely to $F_e(X)$ and retain the same symbol. Thus $\widetilde{T} \in L(F_e(M), F_e(Y))$ with $\|\widetilde{T}\| \leq \|T\|_e$. Again,

$$\|T\|_e = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \neq 0} \frac{\|\delta_{Tx}^e\|}{\|\delta_x^e\|} = \sup_{x \neq 0} \frac{\|\widetilde{T}(\delta_x^e)\|}{\|\delta_x^e\|} \leq \|\widetilde{T}\|$$

so that $\|\widetilde{T}\| = \|T\|_e$.

Let $S \in L(F_e(X), F_e(Y))$ be such that $S \circ \delta_X^e = \delta_Y^e \circ T$. Then $\widetilde{T} \circ \delta_X^e = S \circ \delta_X^e$ so that $\widetilde{T}(\delta_x^e) = S(\delta_x^e)$ for all $x \in X$. Since $\widetilde{T}$ and S are linear and bounded, we get $\widetilde{T}(\gamma) = S(\gamma)$ for all $\gamma \in F_e(X)$. Thus $S = \widetilde{T}$. $\square$

Lemma 6.2.20. *Let W, X, Y, Z be normed linear spaces and $U \in E(W, X)$, $T \in E(X, Y)$, $S \in E(Y, Z)$. Then $(S \circ T)\hat{} = \hat{S} \circ \widetilde{T}$ and $\widetilde{S \circ T} = \widetilde{S} \circ \widetilde{T}$.*

Proof. Applying Theorems 6.2.15 and 6.2.19, we get unique linear maps $\hat{S} \in L(F_e(Y), Z)$, $\widetilde{S} \in L(F_e(Y), F_e(Z))$ and $\widetilde{T} \in L(F_e(X), F_e(Y))$ such that $\hat{S} \circ \delta_Y^e = S$, $\widetilde{S} \circ \delta_Y^e = \delta_Z^e \circ S$ and $\widetilde{T} \circ \delta_X^e = \delta_Y^e \circ T$. Further

$$\hat{S} \circ \widetilde{T} \circ \delta_X^e = \hat{S} \circ \delta_Y^e \circ T = S \circ T = (S \circ T)\hat{} \circ \delta_X^e$$

and

$$(\widetilde{S} \circ \widetilde{T}) \circ \delta_X^e = (\widetilde{S} \circ \delta_Y^e) \circ T = \delta_Z^e \circ (S \circ T) = \widetilde{S \circ T} \circ \delta_X^e.$$

Thus $(S \circ T)\hat{} = \hat{S} \circ \widetilde{T}$ and $\widetilde{S \circ T} = \widetilde{S} \circ \widetilde{T}$. $\square$

Theorem 6.2.21. *Let X be a Banach space. Then $E(X, X)$ is a Banach algebra and the map $\Psi_X^X : E(X, X) \to L(F_e(X), F_e(X))$ defined by $\Psi_X^X(T) = \widetilde{T}$, is an injective, unital, norm preserving and multiplicative extensively bounded mapping. We simply write Ψ for Ψ_X^X when it is clear from the context.*

Proof. From the above discussion, we know that $E(X, X)$ is a non-linear Banach algebra such that $\|.\|_e$ is submultiplicative. By Theorem 6.2.19 and the Lemma 6.2.20, we observe that Ψ is a norm preserving, multiplication preserving, extensively bounded map. Also

$$\delta_x^e = \delta_X^e \circ Id_X(x) = \Psi(Id_X) \circ \delta_X^e(x) = \Psi(Id_X)(\delta_x^e)$$

for all $x \in X$. Thus $\Psi(Id_X) = Id_{F_e(X)}$. and to prove the injectivity of Ψ, consider $T, S \in E(X, X)$ such that $\widetilde{T} = \widetilde{S}$. Therefore composing β_X^e (defined in Lemma 6.2.18) and δ_X^e from the left and right side respectively we get $\beta_X^e \circ \widetilde{T} \circ \delta_X^e = \beta_X^e \circ \widetilde{S} \circ \delta_X^e$. *i.e.* $T = S$. This completes the proof. □

Remark 6.2.22. Note that, though $L(F_e(X), F_e(X))$ is a unital Banach algebra in the usual sense, $\Psi(E(X,X))$ is not, as Ψ is not linear. In fact, $\Psi(E(X,X))$ is not even a (non-linear) Banach algebra like $E(X,X)$.

Proposition 6.2.23. *Let X, Y, Z and W be Banach spaces. Assume $U \in E(X,Y)$, $T \in E(Y,Z)$ and $S \in E(Z,W)$. Then*

$$\Phi_X^W(S \circ T \circ U) = \Phi_Z^W(S) \circ \Psi_Y^Z(T) \circ \Psi_X^Y(U).$$

In particular, if $S \in L(Z,W)$, then

$$\Phi_X^W(S \circ T \circ U) = S \circ \Phi_Y^Z(T) \circ \Psi_X^Y(U).$$

Proof. Applying Theorem 6.2.19, we get unique $\Psi_X^Y(U) \in L(F_e(X), F_e(Y))$ and $\Psi_Y^Z(T) \in L(F_e(Y), F_e(Z))$ such that $\Psi_X^Y(U) \circ \delta_X^e = \delta_Y^e \circ U$ and $\Psi_Y^Z(T) \circ \delta_Y^e = \delta_Z^e \circ T$. Similarly, by Theorem 6.2.15, there exist unique $\Phi_Y^Z(T) \in L(F_e(Y), Z)$ and $\Phi_Z^W(S) \in L(F_e(Z), W)$ such that $\Phi_Y^Z(T) \circ \delta_Y^e = T$ and $\Phi_Z^W(S) \circ \delta_Z^e = S$. Therefore,

$$\begin{aligned} \Phi_Z^W(S) \circ \Psi_Y^Z(T) \circ \Psi_X^Y(U) \circ \delta_X^e &= \Phi_Z^W(S) \circ \Psi_Y^Z(T) \circ \delta_Y^e \circ U \\ &= \Phi_Z^W(S) \circ \delta_Z^e \circ T \circ U \\ &= S \circ T \circ U \\ &= \Phi_X^W(S \circ T \circ U) \circ \delta_X^e \end{aligned}$$

and

$$\begin{aligned} S \circ \Phi_Y^Z(T) \circ \Psi_X^Y(U) \circ \delta_X^e &= S \circ \Phi_Y^Z(T) \circ \delta_Y^e \circ U \\ &= S \circ T \circ U \end{aligned}$$

$$= \Phi_X^W(S \circ T \circ U) \circ \delta_X^e.$$

Thus by Theorem 6.2.15, we get the result. □

Remark 6.2.24. The notions introduced in 6.2.17 can be rephrased in the following way using the maps Φ and Ψ as defined in 6.2.15 and 6.2.21, respectively. Let $T \in E(X, Y)$. Then $T^e \in L(Y^e, X^e)$ so for $f \in Y^e$, we have

$$\begin{aligned}
\left(\Phi_X^{\mathbb{R}}\right)^{-1} \circ \Psi(T)^* \circ \Phi_Y^{\mathbb{R}}(f) &= \left(\Phi_X^{\mathbb{R}}\right)^{-1}\left(\Psi(T)^*\left(\Phi_Y^{\mathbb{R}}(f)\right)\right) \\
&= \Psi(T)^*\left(\Phi_Y^{\mathbb{R}}(f)\right) \circ \delta_X^e \\
&= \Phi_Y^{\mathbb{R}}(f) \circ \Psi(T) \circ \delta_X^e \\
&= \Phi_Y^{\mathbb{R}}(f) \circ \delta_Y^e \circ T \\
&= f \circ T = T^e(f),
\end{aligned}$$

Thus $T^e = \left(\Phi_X^{\mathbb{R}}\right)^{-1} \circ \Psi(T)^* \circ \Phi_Y^{\mathbb{R}}$. In a similar manner, we can show that $T^e\big|_{Y^*} = \left(\Phi_X^{\mathbb{R}}\right)^{-1} \circ \Phi(T)^*$.

Chapter 7

Operator ideals of extensively bounded maps

As an application of the theory developed in the previous chapter, we propose to establish a theory of *extensively bounded operator ideals* in this chapter. We introduce the notion of an extensively bounded operator ideal (or, E-operator ideal, in short), replicating the notion of operator ideal of linear mappings as presented in [20] and [49]. In addition to this, we describe two basic examples. We introduce the notion of (non-linear) finite rank extensively bounded mappings and that of extensively compact mappings. We also discuss the corresponding extensively bounded operator ideals. We begin with the following notion.

Definition 7.0.1. For Banach spaces X and Y, we say that $T \in E(X, Y)$ is of *finite rank* whenever $span(T(X))$ is a finite-dimensional subspace of Y. The set of all finite rank extensively bounded maps from X to Y is denoted by $F_E(X, Y)$.

Definition 7.0.2. Let E denote the class of extensively bounded mappings and let L denote the class of bounded linear mappings between an arbitrary pair of Banach spaces. An *extensively bounded operator ideal* (or, *E-operator ideal*, in short) $\mathcal{I}_E$ is a subclass of E satisfying the following conditions:

1. $\mathcal{I}_E(X, Y)$ is a subspace of $E(X, Y)$.

2. $F_E(X, Y) \subset \mathcal{I}_E(X, Y)$.

3. For any $S \in E(Z, X), T \in \mathcal{I}_E(X, Y)$ and $U \in L(Y, W)$, we have $UTS \in \mathcal{I}_E(Z, W)$.

whenever X, Y, Z and W are Banach spaces. If, in addition, for each pair of Banach spaces (X, Y) there is a norm, say α, on $\mathcal{I}_E(X, Y)$ satisfying:

1. $(\mathcal{I}_E(X,Y),\alpha)$ is a normed linear space;

2. $\alpha(f\otimes y)=\|f\|_e\|y\|$ for each $f\in X^e$ and $y\in Y$; and

3. $\alpha(UTS)\leq\|U\|\alpha(T)\|S\|_e$ whenever X,Y,Z,W are Banach spaces, $S\in E(Z,X)$, $T\in\mathcal{I}_E(X,Y)$ and $U\in L(Y,W)$,

then $(\mathcal{I}_E,\alpha)$ is called a *normed E-operator ideal*. It is called a *Banach E-operator ideal* when $(\mathcal{I}_E(X,Y),\alpha)$ is a Banach space for all Banach spaces X and Y.

We remark that composing with only bounded linear maps on the left instead of arbitrary extensively bounded maps is a compromise to retain the linear structure in $\mathcal{I}_E$. A similar practice is followed in the case of Lipschitz operator ideals.

Next, we describe our first (and the smallest) E-operator ideal.

Lemma 7.0.3. *Let X and Y be Banach spaces and $T\in E(X,Y)$. Then $T\in F_E(X,Y)$ if and only if $\Phi(T)\in F(F_e(X),Y)$.*

Proof. Let $T\in E(X,Y)$ so that $\Phi(T)\in L(F_e(X),Y)$. Since $F_e(X)=\overline{span\,\{\delta^e_x : x\in X\}}$, we get

$$\Phi(T)(F_e(X))=\overline{span\,\{\Phi(T)(\delta^e_x) : x\in X\}}=\overline{span\,\{T(x) : x\in X\}}.$$

Thus for $T\in F_E(X,Y)$, $\Phi(T)(F_e(X))=span(T(X))$ is a finite dimensional subspace of Y, that is, $\Phi(T)\in F(F_e(X),Y)$. The proof of the converse can be traced back in a similar way. $\square$

Proposition 7.0.4. *F_E is the smallest operator ideal.*

Proof. Let $S,T\in F_E(X,Y)$ and $\alpha\in\mathbb{R}$, then

$$span((\alpha S+T)(X))\subset span(S(X))+span(T(X))$$

and thus $\alpha S + T \in F_E(X, Y)$. Again, if $S \in E(Z, X)$, $T \in F_E(X, Y)$ and $U \in L(Y, W)$, then $span(UTS(X)) = U(span(T(S(X))))$. This gives the ideal property of $F_E(X, Y)$. Further we can easily check that $\|UTS\|_e \leq \|U\|\|T\|_e\|S\|_e$. Now, by definition $F_E(X, Y) \subset \mathcal{I}_E(X, Y)$ whenever $\mathcal{I}_E$ is a normed E-operator ideal. $\square$

Now we discuss our next E-operator ideal.

7.1 Extensively compact mapping

Definition 7.1.1. Let X and Y be Banach spaces and $T \in E(X, Y)$. We say that T is called an extensively compact mapping (in short, E-compact mapping) if the set $\left\{\frac{T(x)}{\|x\|} : x \neq 0\right\}$ is pre-compact subset of Y and the set of all E-compact mappings is denoted by $K_E(X, Y)$.

Lemma 7.1.2. *Let X be a Banach space. Then the closed unit ball of $F_e(X)$ is the closed absolute convex hull of $\left\{\frac{\delta_x^e}{\|x\|} : x \in X \setminus \{0\}\right\}$.*

Proof. Put $S = \left\{\frac{\delta_x^e}{\|x\|} : x \in X \setminus \{0\}\right\} \subset F_e(X)$. Then the polar of S is

$$\begin{aligned}
S^\circ &= \left\{\xi \in F_e(X)^* : \left|\xi\left(\frac{\delta_x^e}{\|x\|}\right)\right| \leq 1 : x \in X \setminus \{0\}\right\} \\
&= \left\{\Phi_X^{\mathbb{R}}(T) \in F_e(X)^* : \left|\Phi_X^{\mathbb{R}}(T)\left(\frac{\delta_x^e}{\|x\|}\right)\right| \leq 1 : T \in X^e,\ x \in X \setminus \{0\}\right\} \\
&= \left\{\Phi_X^{\mathbb{R}}(T) \in F_e(X)^* : \left|\frac{\delta_x^e(T)}{\|x\|}\right| \leq 1 : T \in X^e,\ x \in X \setminus \{0\}\right\} \\
&= \left\{\Phi_X^{\mathbb{R}}(T) \in F_e(X)^* : \left|\frac{Tx}{\|x\|}\right| \leq 1 : T \in X^e,\ x \in X \setminus \{0\}\right\} \\
&= \left\{\Phi_X^{\mathbb{R}}(T) \in F_e(X)^* : \|T\|_e \leq 1 : T \in X^e\right\} \\
&= \Phi_X^{\mathbb{R}}\left(B_{X^e}\right).
\end{aligned}$$

Again the pre-polar of $\Phi_X^{\mathbb{R}}\left(B_{X^e}\right)$ is

$$\begin{aligned}
{}^\circ\big(\Phi_X^{\mathbb{R}}\left(B_{X^e}\right)\big) &= \left\{\gamma \in F_e(X) : \left|\Phi_X^{\mathbb{R}}(T)(\gamma)\right| \leq 1 \text{ for all } T \in B_{X^e}\right\} \\
&= \left\{\gamma \in F_e(X) : |\gamma(T)| \leq 1 \text{ for all } T \in B_{X^e}\right\}
\end{aligned}$$

$$\begin{aligned} &= \{\gamma \in F_e(X) : \|\gamma\| \leq 1\} \\ &= B_{F_e(X)}. \end{aligned}$$

Moreover as $B_{F_e(X)}$ is absolutely convex and closed, the bipolar theorem yeilds

$$B_{F_e(X)} = {}^{\circ}(S^{\circ}) = \overline{aco}\left\{\frac{\delta_x^e}{\|x\|} : x \in X \setminus \{0\}\right\}.$$

$\square$

Lemma 7.1.3. *Let X and Y be Banach spaces and $T \in E(X,Y)$. The following are equivalent:*

1. $T \in K_E(X,Y)$.
2. $\Phi(T) \in K(F_e(X),Y)$.
3. $T^e\big|_{Y^*}$ *is a linear compact operator from* Y^* *to* X^e.

Proof. By Lemma 7.1.2, we get

$$\begin{aligned} \Phi(T)\left(B_{F_e(X)}\right) &= \Phi(T)\left(\overline{aco}\left\{\frac{\delta_x^e}{\|x\|} : x \in X \setminus \{0\}\right\}\right) \\ &\subset \overline{aco}\left\{\frac{Tx}{\|x\|} : 0 \neq x \in X\right\} \end{aligned}$$

for all $T \in E(X,Y)$. Let $T \in K_E(X,Y)$. Then $\left\{\frac{Tx}{\|x\|} : 0 \neq x \in X\right\}$ is precompact. Thus $aco\left\{\frac{Tx}{\|x\|} : 0 \neq x \in X\right\}$ is also precompact so that $\Phi(T) \in K_E(F_e(X),Y)$.

Since $\left\{\frac{Tx}{\|x\|} : 0 \neq x \in X\right\} \subset \Phi(T)\left(B_{F_e(X)}\right)$, we get $T \in K_E(X,Y)$ whenever $\Phi(T) \in K_E(F_e(X),Y)$. This proves the equivalence of (1) and (2).

Finally, since $T^e\big|_{Y^*} = \Phi^{-1}\circ(\Phi(T))^*$ for any $T \in E(X,Y)$, (2) is equivalent to (3). $\square$

Proposition 7.1.4. *Let X and Y be Banach spaces, and $T \in K_E(X,Y)$. Then $T = C \circ g$ for some $C \in E(g(X),Y)$ and $g \in K_E(X,c_0)$.*

Proof. Let $T \in E_K(X,Y)$. Then by Lemma 7.1.3, $T^e\big|_{Y^*} \in K(Y^*, X^e)$. Thus $T^e\big|_{Y^*}(B_{Y^*}) \subset \overline{co\{u_n\}}$ for some $(u_n) \in c_0(X^e)$. Consider the map $g : X \to c_0$, defined as $g(x) = (u_n(x))$ for all $x \in X$. Since $\left\{\frac{g(x)}{\|x\|} : x \neq 0\right\} \subset co\{(\|u_n\|) : n \in \mathbb{N}\}$, we have $g \in K_E(X, c_0)$.

Let $x_1, x_2 \in X$ be such that $u_n(x_1) = u_n(x_2)$ for all $n \in \mathbb{N}$. We show that $T(x_1) = T(x_2)$. Let $y^* \in B_{Y^*}$. Then there exists a sequence $(\alpha_n) \in B_{\ell_1}$ such that $(T^e)^*(y^*) = \sum\limits_{n=1}^{\infty} \alpha_n u_n$. Thus

$$\begin{aligned} y^*(T(x_1)) &= (T^e)^*(y^*)(x_1) \\ &= \sum_{n=1}^{\infty} \alpha_n u_n(x_1) \\ &= \sum_{n=1}^{\infty} \alpha_n u_n(x_2) \\ &= (T^e)^*(y^*)(x_2) \\ &= y^*(T(x_2)). \end{aligned}$$

Since $y^* \in B_{Y^*}$ is arbitrary, we get $T(x_1) = T(x_2)$. Therefore, the map $C : g(X) \to Y$ given by $C(g(x)) = T(x)$ is a well-defined. We show that C is extensively bounded. Fix $x \in X$. If $y^* \in B_{Y^*}$, then as above $y^*(T(x)) = (T^e)^*(x) = \sum\limits_{n=1}^{\infty} \alpha_n u_n(x)$. Thus

$$|y^*(T(x))| = |\sum_{n=1}^{\infty} \alpha_n u_n(x)| \leq \sup\{\|u_n(x)\| : n \in \mathbb{N}\} = \|g(x)\|.$$

Hence

$$\|C(g(x))\| = \|T(x)\| = \sup_{y^* \in B_{Y^*}} |y^*(T(x))| \leq \|g(x)\|.$$

so that C is extensively bounded. $\square$

Theorem 7.1.5. *$(K_E, \|\cdot\|_e)$ is a Banach E-operator ideal.*

Proof. Fix Banach spaces X and Y. Let $S, T \in K_E(X,Y)$ and (x_n) be any sequence in $X \setminus \{0\}$. As $S \in K_E(X,Y)$, $\left(\frac{S(x_{n_k})}{\|x_{n_k}\|}\right)$ is Cauchy in Y for some subsequence (x_{n_k}) of (x_n). Again, as $T \in K_E(X,Y)$, we further get that $\left(\frac{T(x_{n_{k_l}})}{\|x_{n_{k_l}}\|}\right)$ is Cauchy in Y for some subsequence $(x_{n_{k_l}})$ of (x_{n_k}). Therefore, $\left(\frac{(S+T)(x_{n_{k_l}})}{\|x_{n_{k_l}}\|}\right)$ is a Cauchy subsequence of $\left(\frac{(S+T)(x_n)}{\|x_n\|}\right)$ so that $S+T \in K_E(X,Y)$. Similarly, we can show that for $\alpha \in \mathbb{R}$, we have $\alpha S \in K_E(X,Y)$. Thus $K_E(X,Y)$ is a subspace of $E(X,Y)$.

Next, we show that $(K_E(X,Y), \|\cdot\|_e)$ is a Banach space.

Let (T_i) be a sequence in $K_E(X,Y)$ such that $\|T_i - T\|_e \to 0$ as $i \to \infty$ for some $T \in E(X,Y)$. We show that $T \in K_E(X,Y)$. Let (x_n) be a sequence in $X \setminus \{0\}$. Since T_1 is compact there exists a subsequence $(x_{n_{1,k}})_{k\in\mathbb{N}}$ such that $\left(\frac{T_1(x_{n_{1,k}})}{\|x_{n_{1,k}}\|}\right)_{k\in\mathbb{N}}$ is a Cauchy subsequence. Similarly T_2 is compact, therefore there exists $(x_{n_{2,k}})_{k\in\mathbb{N}}$ such that $\left(\frac{T_2(x_{n_{2,k}})}{\|x_{n_{2,k}}\|}\right)_{k\in\mathbb{N}}$ is also cauchy. Continuing inductively, we have for each $i \in \mathbb{N}$, $\left(\frac{T_i(x_{n_{i,k}})}{\|x_{n_{i,k}}\|}\right)_{k\in\mathbb{N}}$ converge in Y. Consider the diagonal subsequence $x_{n_k} := x_{n_{k,k}}$ for $k \in \mathbb{N}$ and fix $\epsilon > 0$. We prove that $\left(\frac{T(x_{n_k})}{\|x_{n_k}\|}\right)_{k\in\mathbb{N}}$ converges in Y as well. Since (T_i) converges to T, there exists $i_0 \in \mathbb{N}$ such that $\|T - T_i\|_e < \frac{\epsilon}{3}$ for all $i \geq i_0$. Again $\left(\frac{T_i(x_{n_k})}{\|x_{n_k}\|}\right)_{k\in\mathbb{N}}$ converges implies there exists a natural number k_0 such that $\left\|\frac{T_i(x_{n_k})}{\|x_{n_k}\|} - \frac{T_i(x_{n_l})}{\|x_{n_l}\|}\right\| < \frac{\epsilon}{3}$ for all $k, l \geq k_0$. Now for all $k, l \geq \max\{k_0, i_0\}$

$$\begin{aligned}
\left\|\frac{T(x_{n_k})}{\|x_{n_k}\|} - \frac{T(x_{n_l})}{\|x_{n_l}\|}\right\| &\leq \left\|\frac{T(x_{n_k})}{\|x_{n_k}\|} - \frac{T_{i_0}(x_{n_k})}{\|x_{n_k}\|}\right\| + \left\|\frac{T_{i_0}(x_{n_k})}{\|x_{n_k}\|} - \frac{T_{i_0}(x_{n_l})}{\|x_{n_l}\|}\right\| \\
&+ \left\|\frac{T_{i_0}(x_{n_l})}{\|x_{n_l}\|} - \frac{T(x_{n_l})}{\|x_{n_l}\|}\right\| \\
&\leq \|T - T_{i_0}\|_e + \left\|\frac{T_{i_0}(x_{n_k})}{\|x_{n_k}\|} - \frac{T_{i_0}(x_{n_l})}{\|x_{n_l}\|}\right\| + \|T_{i_0} - T\|_e \\
&< \epsilon.
\end{aligned}$$

Hence $K_E(X,Y)$ is a Banach space.

Now, we prove the E-operator ideal conditions. It follows from Lemmas 7.0.3 and 7.1.3 that $\Phi(F_E(X,Y)) = F(F_e(X),Y)$ and $\Phi(K_E(X,Y)) = K(F_e(X),Y)$. Since $F(F_e(X),Y) \subset K(F_e(X),Y)$, we deduce that $F_E(X,Y) \subset K_E(X,Y)$.

Finally, let W, X, Y and Z be Banach spaces and assume that $U \in E(Z,X)$, $T \in K_E(X,Y)$ and $S \in L(Y,W)$. Then by Theorem 6.2.19, there exists a unique $\tilde{U} \in L(F_e(Z), F_e(X))$

such that $\tilde{U} \circ \delta_Z^e = \delta_X^e \circ U$. Also by Lemma 7.1.3, there exists a unique $\hat{T} \in K(F_e(X), Y)$ such that $\hat{T} \circ \delta_Y^e = T$. We write $\tilde{U} = \Psi_Z^X(U)$ and $\hat{T} = \Phi_X^Y(T)$. Thus, from the Proposition 6.2.23 we have $\Phi_Z^W(S \circ T \circ U) = S \circ \Phi_X^Y(T) \circ \Psi_Z^X(U)$. Since K is a Banach operator ideal, we have

$$\Phi_Z^W(S \circ T \circ U) = S \circ \Phi_X^Y(T) \circ \Psi_Z^X(U) \in K(F_e(Z), W).$$

Hence by Lemma 7.1.3, $S \circ T \circ U \in K_E(Z, W)$. □

7.2 Extensively p-summing mapping

7.2.1 Summability with respect to X^e

Let X be a normed linear space and $1 \leq p < \infty$. Recall that a sequence (x_n) in X is said to be *weakly p-summable* if $\sup_{f \in B_{X^*}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}}$ is finite. However, when we replace B_{X^*} by B_{X^e}, we find $\|(x_n)\|_p^s = \sup_{f \in B_{X^e}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}}$.

Lemma 7.2.1. *Let X be a normed linear space and (x_n) be a sequence in X. Then $(x_n) \in \ell_p^s(X)$ if and only if $\sup_{f \in B_{X^e}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}}$ is finite. Moreover, in this case $\|(x_n)\|_p^s = \sup_{f \in B_{X^e}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}}$.*

Proof. Let (x_n) be a sequence in the normed linear space X such that $\sup_{f \in B_{X^e}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}}$ is finite.

We consider the map $g(z) = \|z\|$ for all $z \in X$. Then $g \in B_{X^e}$ and

$$\sup_{f \in B_{X^e}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}} \geq \left(\sum_{n=1}^{\infty} |g(x_n)|^p \right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} = \|(x_n)\|_p^s.$$

Therefore, $(x_n) \in \ell_p^s$, as $\sup_{f \in B_{X^e}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}}$ is finite. Conversely, let $(x_n) \in \ell_p^s(X)$. Now we show that $\sup_{f \in B_{X^e}} \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}} = \|(x_n)\|_p^s$. Let $f \in B_{X^e}$. Then $|f(x_n)| \leq$

$\|f\|_e\|x_n\|$. Thus, $|f(x_n)| \leq \|x_n\|$ for all $f \in B_{X^e}$ and $n \in \mathbb{N}$. This implies that $\sum\limits_{n=1}^{\infty} |f(x_n)|^p \leq \sum\limits_{n=1}^{\infty} \|x_n\|^p$ for all $f \in B_{X^e}$. Hence $\sup\limits_{f\in B_{X^e}} \left(\sum\limits_{n=1}^{\infty} |f(x_n)|^p\right)^{\frac{1}{p}} \leq \|(x_n)\|_p^s$ so that $\sup\limits_{f\in B_{X^e}} \left(\sum\limits_{n=1}^{\infty} |f(x_n)|^p\right)^{\frac{1}{p}} = \|(x_n)\|_p^s$. This completes the proof. $\square$

Now, let us first recall the definition of a linear p-summing operator. A linear operator $T : X \to Y$ is linear p-summing if there exists a constant $c \geq 0$ such that regardless of the natural number m and regardless of the choice of $x_1, x_2, ..., x_m$ in X we have

$$\left(\sum_{n=1}^{m} \|T(x_n)\|^p\right)^{\frac{1}{p}} \leq c \cdot \sup_{f\in B_{X^*}} \left(\sum_{n=1}^{m} |f(x_n)|^p\right)^{\frac{1}{p}}.$$

The smallest such c (depends on T) will be denoted by $\pi_p(T)$.

Observation 7.2.2. *1. Let $m \in \mathbb{N}$, $x_1, x_2, ..., x_m \in X$ and $T \in E(X,Y)$. Then by the Lemma 7.2.1*

$$\left(\sum_{n=1}^{m} \|T(x_n)\|^p\right)^{\frac{1}{p}} \leq \|T\|_e \left(\sum_{n=1}^{m} \|x_n\|^p\right)^{\frac{1}{p}} = \|T\|_e \sup_{f\in B_{X^e}} \left(\sum_{n=1}^{m} |f(x_n)|^p\right)^{\frac{1}{p}}.$$

Thus, any extensively bounded map appears to be extensively p-summing if we directly adapt the definition of linear p-summing operators, replacing X^* by X^e. Therefore, we look at an alternative.

2. Let $T : X \to Y$ be linear p-summing. Suppose $m, r \in \mathbb{N}$ and consider the finite subsets $\left\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\}$ of X and $\left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\}$ of $\mathbb{R}$. Then

$$\begin{aligned}\left(\sum_{i=1}^{m} \left\|\sum_{n=1}^{r} \alpha_n^{(i)} T(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} &\leq \pi_p(T) \sup_{f\in B_{X^*}} \left(\sum_{i=1}^{m} \left\|f\left(\sum_{n=1}^{r} \alpha_n^{(i)} x_n^{(i)}\right)\right\|^p\right)^{\frac{1}{p}} \\ &= \pi_p(T) \sup_{f\in B_{X^*}} \left(\sum_{i=1}^{m} \left\|\sum_{n=1}^{r} \alpha_n^{(i)} f\left(x_n^{(i)}\right)\right\|^p\right)^{\frac{1}{p}}\end{aligned}$$

Further suppose $T \in L(X,Y)$ satisfies

$$\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}T(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} \leq c\cdot \sup_{f\in B_{X^*}}\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}f\left(x_n^{(i)}\right)\right\|^p\right)^{\frac{1}{p}}, \qquad (7.2.1)$$

for some $c \geq 0$ (depends only on T), and all $\alpha_n^{(i)} \in \mathbb{R}$, $x_n^{(i)} \in X$, $n = 1, ..., r$ and $i = 1, .., m$, where $r, m \in \mathbb{N}$. Now, if we consider $r = 1$, $\alpha_n^{(i)} = 1$ for all $i = 1, ..., m$, the equation 7.2.1 implies T is linear p-summing. Therefore, $T \in L(X,Y)$ is linear p-summing if and only if it satisfies the equation 7.2.1.

We take a cue from the above observation to extend the notion of p-summing map in the context of extensively bounded mappings.

Definition 7.2.3. Let $1 \leq p < \infty$. $T : X \to Y$ is said to be an extensively p-summing (E-p-summing, in short) operator, if there exists $0 \leq c$(depends only on T) such that

$$\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}T(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} \leq c \sup_{f\in B_{X^e}}\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}f(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}},$$

for any $m, r \in \mathbb{N}$ and any finite subsets $\left\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\}$ of X and $\left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\}$ of $\mathbb{R}$.

Remark 7.2.4.

1. Infimum over all such $c \geq 0$ will be denoted by $\pi_p^e(T)$. Also, the set of all extensively p-summing mappings will be denoted by $E_{\pi_p}(X,Y)$. Note that if $T \in E_{\pi_p}(X,Y)$, then it is an extensively bounded. At the end of the paper, we provide an example (Example 7.3.4) to justify that every extensively bounded mapping is not E-p-summing. Moreover, if both T and f are linear, then the Definition 7.2.3 coincides with the notion of classical linear p-summing operator.

2. We notice that a similar-looking notion of factorable p-dominated operator has been discussed in [48]. A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is said to be a *factorable p-dominated* if there exists a constant $C \geq 0$ such that for every

choice of vectors $x^i_{k,j} \in X_k$ and scalars $\lambda^i_j \in \mathbb{R}$ (for $1 \le j \le m_1, 1 \le i \le m_2$), and all positive integers m_1, m_2 we have

$$\left(\sum_{j=1}^{m_1}\left\|\sum_{i=1}^{m_2}\lambda^i_j T\left(x^i_{1,j},\ldots,x^i_{n,j}\right)\right\|^p\right)^{1/p} \le C \sup_{\substack{\varphi_k\in B_{X^*_k}\\ 1\le k\le n}}\left(\sum_{j=1}^{m_1}\left|\sum_{i=1}^{m_2}\lambda^i_j\varphi_1\left(x^i_{1,j}\right)...\varphi_n\left(x^i_{n,j}\right)\right|^p\right)^{1/p}.$$

If we consider T to be one linear, then the condition matches with the observation 7.2.2[2]. Similarly in [1, Section 3], if we consider the definition of Factorable strongly p-summing for a polynomial $P \in \mathcal{P}(\,^1X, Y)$ (for details see the mentioned article), it matches with this Observation 7.2.2[2].

The next result will be used to show that elements of $F_E(X, Y)$ are extensively p-summing.

Lemma 7.2.5. *Let $T \in F_E(X, Y)$. Then T has the representation as $T = \sum_{i=1}^{m} y_i f_i$, for some $f_1, ..., f_m \in X^e$ and $y_1, ..., y_m \in Y$.*

Proof. Let $T \in F_E(X, Y)$. Then by Lemma 7.0.3 $\Phi(T) \in F(F_e(X), Y)$. Therefore $\Phi(T) = \sum_{i=1}^{m} \Phi^{\mathbb{R}}_X(f_i) \otimes y_i$, for some $f_i \in X^e$ and $y_i \in Y$ $(i = 1, ..., m)$. Let $x \in X$. Thus $T = \Phi(T) \circ \delta^e_X$ provides

$$T(x) = \left(\sum_{i=1}^{m} \Phi^{\mathbb{R}}_X(f_i) \otimes y_i\right) \circ \delta^e_X(x).$$

This implies $T(x) = \sum_{i=1}^{m} \left(\Phi^{\mathbb{R}}_X(f_i)(\delta^e_x)\right) y_i$. Hence $T(x) = \sum_{i=1}^{m} f_i(x) y_i$, that is $T = \sum_{i=1}^{m} y_i f_i$. $\square$

Proposition 7.2.6. *Let X and Y be Banach spaces. Then (E_{π_p}, π^e_p) is a Banach E-operator ideal.*

Proof. Let X and Y be Banach spaces. First we show that $E_{\pi_p}(X, Y)$ is a subspace of $E(X, Y)$.

Let $T_1, T_2 \in E_{\pi_p}(X,Y)$. Fix $m, r \in \mathbb{N}$ and finite subsets

$$\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\} \subset X$$

and

$$\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\} \subset \mathbb{R}$$

Put $z_i = \sum\limits_{n=1}^{r} \alpha_n^{(i)} T_1(x_n^{(i)})$ and $w_i = \sum\limits_{n=1}^{r} \alpha_n^{(i)} T_2(x_n^{(i)})$. Then using Minkowski's inequality we have

$$\begin{aligned}
&\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}(T_1+T_2)(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}T_1(x_n^{(i)})+\sum_{n=1}^{r}\alpha_n^{(i)}T_2(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{m}\|z_i+w_i\|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{m}\|z_i\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m}\|w_i\|^p\right)^{\frac{1}{p}} \\
&\leq \left(\pi_p^e(T_1)+\pi_p^e(T_2)\right) \sup_{f\in B_{X^e}} \left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}f(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, $T_1 + T_2 \in E_{\pi_p}(X,Y)$ with $\pi_p^e(T_1+T_2) \leq \pi_p^e(T_1) + \pi_p^e(T_2)$. It is easy to check $\alpha T \in E_{\pi_p}(X,Y)$ whenever $T \in E_{\pi_p}(X,Y)$, for any $\alpha \in \mathbb{R}$.

Let $x \in X\backslash\{0\}$. Then $\|T(x)\| \leq \pi_p^e(T) \sup\limits_{f\in B_{X^e}} |f(x)|$ so that $\frac{\|T(x)\|}{\|x\|} \leq \pi_p^e(T) \sup\limits_{f\in B_{X^e}} \frac{|f(x)|}{\|x\|}$. Thus $\|T\|_e \leq \pi_p^e(T)$ and hence π_p^e is a norm on $E_{\pi_p}(X,Y)$.

Now we investigate the E-operator ideal properties. Put $T = f.y \in F_E(X,Y)$ where $f \in X^e$ and non-zero $y \in Y$. Then we have $T \in E_{\pi_p}(X,Y)$ with $\pi_p^e(T) \leq \|y\|.\|f\|_e$. Thus by Lemma 7.2.5, $E_{\pi_p}(X,Y)$ contains all finite rank extensively bounded maps.

Next let $S \in E(Z,X),\ T \in E_{\pi_p}(X,Y),\ U \in L(Y,W)$, where X, Y, Z, W are Banach spaces. We show that $UTS \in E_{\pi_p}(Z,W)$ with $\|UTS\|_e \leq \|U\|.\pi_p^e(T).\|S\|_e$. Fix $m, r \in \mathbb{N}$

and finite subsets

$$\left\{z_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset Z$$

and

$$\left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset \mathbb{R}.$$

Then

$$\begin{aligned}
\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}UTS(z_n^{(i)})\right\|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{m}\left\|U\left(\sum_{n=1}^{r}\alpha_n^{(i)}T\left(S(z_n^{(i)})\right)\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \|U\|\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}T\left(S(z_n^{(i)})\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \|U\|.\pi_p^e(T)\sup_{f\in B_{X^e}}\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}f\left(S(z_n^{(i)})\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \|U\|.\pi_p^e(T)\|S\|_e\sup_{f\in B_{X^e}}\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\frac{\alpha_n^{(i)}}{\|S\|_e}f\left(S(z_n^{(i)})\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \|U\|.\pi_p^e(T)\|S\|_e\sup_{g\in B_{Z^e}}\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}g(z_n^{(i)})\right\|^p\right)^{\frac{1}{p}}
\end{aligned}$$

Thus (E_{π_p}, π_p^e) is normed E-operator ideal.

The proof of the completeness of $(E_{\pi_p}(X,Y), \pi_p^e(\cdot))$ will be deduced from Corollary 7.2.10. □

Proposition 7.2.7. (Injectivity) *Let X, Y and Y_0 be Banach , $T \in E(X,Y)$ and $\mathfrak{J} : Y \to Y_0$ be a linear isometry. Then $T \in E_{\pi_p}(X,Y)$ if and only if $\mathfrak{J}T \in E_{\pi_p}(X,Y_0)$. In this case, $\pi_p^e(T) = \pi_p^e(\mathfrak{J}T)$.*

Proof. Let $T \in E(X,Y)$ and $\mathfrak{J} : Y \to Y_0$ be a linear isometry such that $\mathfrak{J}T \in E_{\pi_p}(X,Y_0)$. We show that $T \in E_{\pi_p}(X,Y)$.

Suppose $m, r \in \mathbb{N}$. Consider the finite sets

$$\left\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset X$$

and

$$\left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset \mathbb{R}.$$

Then

$$\begin{aligned}
\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}T(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{m}\left\|\mathfrak{J}\left(\sum_{n=1}^{r}\alpha_n^{(i)}T(x_n^{(i)})\right)\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{m}\left\|\left(\sum_{n=1}^{r}\alpha_n^{(i)}\mathfrak{J}T(x_n^{(i)})\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \pi_p^e(\mathfrak{J}T). \sup_{f\in B_{X^e}}\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}f(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus $T \in E_{\pi_p}(X, Y_0)$. We can prove the converse using the ideal property of E_{π_p}. Moreover, as $\mathcal{I}$ is an isometry, we can further deduce that $\pi_p^e(T) = \pi_p^e(\mathfrak{J}T)$. $\square$

Remark 7.2.8. Since X^* is a closed subspace of X^e, it follows that $\pi_p(X, Y) \subset E_{\pi_p}(X, Y)$, and $\pi_p(X, Y)$ is a proper subspace of $E_{\pi_p}(X, Y)$.

Proposition 7.2.9. *Let X and Y be Banach spaces and $T \in E(X, Y)$. Then T is E-p-summing if and only if $\Phi_X^Y(T) \in L(F_e(X), Y)$ is linear p-summing with $\pi_p^e(T) = \pi_p(\Phi(T))$.*

Proof. Let $T \in E(X, Y)$ be such that $\Phi_X^Y(T) \in L(F_e(X), Y)$ is linear p-summing. We show that T is E-p-summing.

Let $r, m \in \mathbb{N}$ and consider $\left\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset X$ and $\left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset \mathbb{R}$. Then

$$\begin{aligned}
\left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}T(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{m}\left\|\sum_{n=1}^{r}\alpha_n^{(i)}\left(\Phi_X^Y(T)(\delta^e_{x_n^{(i)}})\right)\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{m}\left\|\Phi_X^Y(T)\left(\sum_{n=1}^{r}\alpha_n^{(i)}\delta^e_{x_n^{(i)}}\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \pi_p(\Phi_X^Y(T)) \sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^{m}\left\|\xi\left(\sum_{n=1}^{r}\alpha_n^{(i)}\delta^e_{x_n^{(i)}}\right)\right\|^p\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \pi_p(\Phi_X^Y(T)) \sup_{\xi \in B_{F_e(X)^*}} \left(\sum_{i=1}^m \left\| \sum_{n=1}^r \alpha_n^{(i)} \xi\left(\delta^e_{x_n^{(i)}}\right) \right\|^p \right)^{\frac{1}{p}} \\
&= \pi_p(\Phi_X^Y(T)) \sup_{f \in B_{X^e}} \left(\sum_{i=1}^m \left\| \sum_{n=1}^r \alpha_n^{(i)} \Phi_X^{\mathbb{R}}(f)\left(\delta^e_{x_n^{(i)}}\right) \right\|^p \right)^{\frac{1}{p}} \\
&= \pi_p(\Phi_X^Y(T)) \sup_{f \in B_{X^e}} \left(\sum_{i=1}^m \left\| \sum_{n=1}^r \alpha_n^{(i)} f(x_n^{(i)}) \right\|^p \right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus T is E-p-summing with $\pi_p^e(T) \leq \pi_p(\Phi_X^Y(T))$.

Conversely let $T \in E_{\pi_p(X,Y)}$. We prove that $\Phi_X^Y(T)$ is linear p-summing. Suppose $\gamma_1, .., \gamma_m \in F_e(X)$ and $\epsilon > 0$. Then for each $i = 1, .., m$ there exist $\left\{\alpha_n^{(i)} : n = 1, ..., r\right\} \subset \mathbb{R}$ and $\left\{x_n^{(i)} : n = 1, ..., r\right\} \subset X$ such that

$$
\left\| \gamma_i - \sum_{n=1}^r \alpha_n^{(i)} \delta^e_{x_n^{(i)}} \right\| < \frac{\epsilon}{m^{\frac{1}{p}}}.
$$

Now

$$
\begin{aligned}
\left(\sum_{i=1}^m \left\| \Phi_X^Y(T)(\gamma_i) \right\|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^m \left\| \Phi_X^Y(T)\left(\gamma_i - \sum_{n=1}^r \alpha_n^{(i)} \delta^e_{x_n^{(i)}} \right) \right\|^p \right)^{\frac{1}{p}} \\
&+ \left(\sum_{i=1}^m \left\| \Phi_X^Y(T)\left(\sum_{n=1}^r \alpha_n^{(i)} \delta^e_{x_n^{(i)}} \right) \right\|^p \right)^{\frac{1}{p}} \\
&\leq \|\Phi_X^Y(T)\| \left(\sum_{i=1}^m \left\| \gamma_i - \sum_{n=1}^r \alpha_n^{(i)} \delta^e_{x_n^{(i)}} \right\|^p \right)^{\frac{1}{p}} \\
&+ \left(\sum_{i=1}^m \left\| \left(\sum_{n=1}^r \alpha_n^{(i)} \Phi_X^Y(T)\left(\delta^e_{x_n^{(i)}}\right) \right) \right\|^p \right)^{\frac{1}{p}} \\
&= \|\Phi_X^Y(T)\| \left(\sum_{i=1}^m \left\| \gamma_i - \sum_{n=1}^r \alpha_n^{(i)} \delta^e_{x_n^{(i)}} \right\|^p \right)^{\frac{1}{p}} \\
&+ \left(\sum_{i=1}^m \left\| \sum_{n=1}^r \alpha_n^{(i)} T(x_n^{(i)}) \right\|^p \right)^{\frac{1}{p}} \\
&\leq \|\Phi_X^Y(T)\|\epsilon + \pi_p^e(T) \sup_{f \in B_{X^e}} \left(\sum_{i=1}^m \left\| \sum_{n=1}^r \alpha_n^{(i)} f(x_n^{(i)}) \right\|^p \right)^{\frac{1}{p}}.
\end{aligned}
$$

Also

$$\begin{aligned}
\sup_{f\in B_{X^e}}\left(\sum_{i=1}^m\left\|\sum_{n=1}^r\alpha_n^{(i)}f(x_n^{(i)})\right\|^p\right)^{\frac{1}{p}} &= \sup_{f\in B_{X^e}}\left(\sum_{i=1}^m\left\|\sum_{n=1}^r\alpha_n^{(i)}\Phi_X^{\mathbb{R}}(f)\left(\delta^e_{x_n^{(i)}}\right)\right\|^p\right)^{\frac{1}{p}}\\
&= \sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\left\|\sum_{n=1}^r\alpha_n^{(i)}\xi\left(\delta^e_{x_n^{(i)}}\right)\right\|^p\right)^{\frac{1}{p}}\\
&= \sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\left\|\xi\left(\sum_{n=1}^r\alpha_n^{(i)}\delta^e_{x_n^{(i)}}\right)\right\|^p\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\left\|\xi\left(\sum_{n=1}^r\alpha_n^{(i)}\delta^e_{x_n^{(i)}}-\gamma_i\right)\right\|^p\right)^{\frac{1}{p}}\\
&+ \sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\|\xi(\gamma_i)\|^p\right)^{\frac{1}{p}}\\
&\leq \left(\sum_{i=1}^m\left\|\sum_{n=1}^r\alpha_n^{(i)}\delta^e_{x_n^{(i)}}-\gamma_i\right\|^p\right)^{\frac{1}{p}}\\
&+ \sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\|\xi(\gamma_i)\|^p\right)^{\frac{1}{p}}\\
&\leq \epsilon+\sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\|\xi(\gamma_i)\|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, from the above two estimations we get

$$\left(\sum_{i=1}^m\left\|\Phi_X^Y(T)(\gamma_i)\right\|^p\right)^{\frac{1}{p}}\leq\|\Phi_X^Y(T)\|\epsilon\ +\ \pi_p^e(T)\epsilon\ +\ \pi_p^e(T)\sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\|\xi(\gamma_i)\|^p\right)^{\frac{1}{p}}.$$

Since $\epsilon>0$ is arbitrary, we get

$$\left(\sum_{i=1}^m\left\|\Phi_X^Y(T)(\gamma_i)\right\|^p\right)^{\frac{1}{p}}\leq\pi_p^e(T)\sup_{\xi\in B_{F_e(X)^*}}\left(\sum_{i=1}^m\|\xi(\gamma_i)\|^p\right)^{\frac{1}{p}}.$$

Thus $\Phi_X^Y(T)$ is linear p-summing with $\pi_p(\Phi_X^Y(T))\leq\pi_p^e(T)$ and this completes the proof.

□

Corollary 7.2.10. *For Banach spaces X and Y, $(E_{\pi_p}(X,Y),\pi_p^e(\cdot))$ is (linearly) isometrically isomorphic to $(\Pi_p(F_e(X),Y),\pi_p(\cdot))$. In particular, $(E_{\pi_p}(X,Y),\pi_p^e(\cdot))$ is a Banach space.*

Proof. From the above proposition it follows that

$$\Phi : (E_{\pi_p}(X,Y), \pi_p^e(\cdot)) \to (\Pi_p(F_e(X),Y), \pi_p(\cdot))$$

is a linear isometric isomorphism. Therefore $(E_{\pi_p}(X,Y), \pi_p^e(\cdot))$ is a Banach space. $\square$

7.3 Domination and factorization Theorems

Theorem 7.3.1. *Let X and Y be two Banach spaces and $T \in E(X,Y)$. Then the following are equivalent:*

1. *T is an E-p-summing operator.*

2. *There exists a probability measure μ on $(B_{X^e}, weak^*)$ such that*

$$\sum_{i=1}^{m} \left\| \sum_{n=1}^{r} \alpha_n^{(i)} T(x_n^{(i)}) \right\|^p \leq \left(\pi_p^e(T)\right)^p \sum_{i=1}^{m} \int_{B_{X^e}} \left| \sum_{n=1}^{r} \alpha_n^{(i)} f(x_n^{(i)}) \right|^p d\mu(f),$$

for any finite sets

$$\left\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset X$$

and

$$\left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset \mathbb{R},$$

where $r, m \in \mathbb{N}$.

Proof. Let $T \in E_{\pi_p}(X,Y)$. Consider

$$\mathcal{A} = \{F \in C(B_{X^e}, \mathbb{R}) : F(f) > 0, \ \forall f \in B_{X^e}\}.$$

Then $\mathcal{A}$ is an open convex set. For any finite subsets M of X and M' of $\mathbb{R}$, we define $g_{M,M'} : B_{X^e} \to \mathbb{R}$ given by

$$g_{M,M'}(f) = \sum_{x \in M} \left(\left\| \sum_{\alpha \in M'} \alpha T x \right\|^p - \left(\pi_p^e(T)\right)^p \left| \sum_{\alpha \in M'} \alpha f(x) \right|^p \right),$$

for all $f \in B_{X^e}$. As X^e is a dual space, we consider $weak^*$ topology on B_{X^e}. Therefore $g_{M,M'}$ is continuous for all finite $M \subset X$ and $M' \subset \mathbb{R}$. Again let $\mathcal{B}$ be the cone generated by

$$\left\{g_{M,M'} : M \subset X, M' \subset \mathbb{R},\ M, M' \text{are finite}\right\}.$$

Then $\mathcal{B}$ is also a convex subset of $C(B_{X^e}, \mathbb{R})$. Further as $T \in E_{\pi_p}(X,Y)$, $\mathcal{A} \cap \mathcal{B} = \phi$. So the geometric version of Hahn-Banach theorem and Riesz-Representation theorem provides us with a $\mu \in C(B_{X^e}, \mathbb{R})^*$ such that, for some $c \in \mathbb{R}$,

$$\int_{B_{X^e}} G d\mu(f)\ \leq c\ < \int_{B_{X^e}} F d\mu(f),$$

for all $G \in \mathcal{B}$ and $F \in \mathcal{A}$. Since $0 = g_{\{0\},\{0\}} \in \mathcal{B}$, we have $c \geq 0$. Also as every positive constant function belongs to $\mathcal{A}$, we get $c \leq 0$. Thus $c = 0$. Further, since $\int_{B_{X^e}} d\mu(f)$ is positive on the positive cone $\mathcal{A}$ of $C(B_{X^e}, \mathbb{R})$, we can assume that μ is a positive regular Borel measure on B_{X^e}. Therefore, the above inequality becomes

$$\int_{B_{X^e}} G d\mu(f)\ \leq 0\ < \int_{B_{X^e}} F d\mu(f),$$

for all $G \in \mathcal{B}$ and $F \in \mathcal{A}$. Further scaling by a positive real number, we can assume that μ is a probability measure.

Now fix $r, m \in \mathbb{N}$ and put

$$M = \left\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset X$$

and

$$M' = \left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset \mathbb{R}.$$

Then $\int_{B_{X^e}} g_{M,M'}\, d\mu(f)\ \leq 0$ so that

$$\sum_{i=1}^{m} \left\| \sum_{n=1}^{r} \alpha_n^{(i)} T(x_n^{(i)}) \right\|^p \leq \left(\pi_p^e(T)\right)^p \sum_{i=1}^{m} \int_{B_{X^e}} \left| \sum_{n=1}^{r} \alpha_n^{(i)} f(x_n^{(i)}) \right|^p d\mu(f).$$

Conversely, we assume that (2) holds. Let $r, m \in \mathbb{N}$ and consider

$$\left\{x_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset X$$

and

$$\left\{\alpha_n^{(i)} : n = 1, ..., r; i = 1, ..., m\right\} \subset \mathbb{R}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{m} \left\| \sum_{n=1}^{r} \alpha_n^{(i)} T(x_n^{(i)}) \right\|^p &\leq \left(\pi_p^e(T)\right)^p \sum_{i=1}^{m} \int_{B_{X^e}} \left| \sum_{n=1}^{r} \alpha_n^{(i)} f(x_n^{(i)}) \right|^p d\mu(f) \\
&\leq \left(\pi_p^e(T)\right)^p \sup_{f \in B_{X^e}} \left(\sum_{i=1}^{m} \left\| \sum_{n=1}^{r} \alpha_n^{(i)} f(x_n^{(i)}) \right\|^p \right) \int_{B_{X^e}} d\mu(f) \\
&= \left(\pi_p^e(T)\right)^p \sup_{f \in B_{X^e}} \left(\sum_{i=1}^{m} \left\| \sum_{n=1}^{r} \alpha_n^{(i)} f(x_n^{(i)}) \right\|^p \right).
\end{aligned}$$

This completes the proof. □

Let X be a Banach space. For each $x \in X$, we define $\chi_x : B_{X^e} \to \mathbb{R}$ given by $\chi_x(f) = f(x)$ for all $f \in B_{X^e}$. Then χ_x is continuous with respect to the $weak^*$ topology on B_{X^e}. Now we define $\chi : X \to C(B_{X^e})$ given by $\chi(x) = \chi_x$. Then

$$\|\chi(x)\| = \sup_{f \in B_{X^e}} |f(x)| = \|x\|,$$

so that $\chi \in E(X, C(B_{X^e}))$. Thus $\chi : X \to C(B_{X^e})$ is a norm-preserving extensively bounded map.

Theorem 7.3.2. *Let $1 \leq p < \infty$, X and Y be Banach spaces. For every $T \in E(X, Y)$, the following statements are equivalent:*

1. *T is E-p-summing.*

2. *There exist a regular borel probability measure μ on $(B_{X^e}, weak^*)$, a closed subspace X_p of $L^p(\mu)$ and a bounded linear operator $A : X_p \to Y$ such that $T = Aj_p\chi$; where*

$j_p : C(B_{X^e}) \to L^p(\mu)$ is the canonical mapping (formal identity) and $\chi : X \to C(B_{X^e})$ is a norm-preserving extensively bounded map.

Proof. Let $T \in E_{\pi_p}(X, Y)$. By the Theorem 7.3.1, there exists a regular borel probability measure μ on B_{X^e}. Put $\hat{X} := \overline{span\ \chi(X)} \subset C(B_{X^e})$. Consider the canonical mapping $j_p : C(B_{X^e}) \to L^p(\mu)$. Then j_p is linear p-summing with $\pi_p(j_p) = 1$(see Example 2.9(b) in [22]). Furthermore, we denote the range of $j_p(\hat{X})$ by S and define $A : S \to Y$ given by

$$A\left(\sum_{i=1}^{n} \alpha_i j_p \chi(x_i)\right) = \sum_{i=1}^{n} \alpha_i T(x_i).$$

Let $\sum\limits_{i=1}^{n} \alpha_i j_p \chi(x_i) = 0$. Then $\sum\limits_{i=1}^{n} \alpha_i \chi(x_i) = 0$ so that $\sum\limits_{i=1}^{n} \alpha_i f(x_i) = 0$ for all $f \in B_{X^e}$. Now

$$\begin{aligned}
\left\|\sum_{i=1}^{n} \alpha_i T(x_i)\right\| &= \sup_{y^* \in B_{Y^*}} \left| y^* \left(\sum_{i=1}^{n} \alpha_i T(x_i)\right)\right| \\
&= \|T\|_e \sup_{y^* \in B_{Y^*}} \left|\left(\sum_{i=1}^{n} \alpha_i \frac{y^* \circ T}{\|T\|_e}(x_i)\right)\right| \\
&\leq \|T\|_e \sup_{f \in B_{X^e}} \left|\sum_{i=1}^{n} \alpha_i f(x_i)\right| = 0.
\end{aligned}$$

This shows that A is a well-defined linear map from $j_p\,(span\ \chi(X))$ to Y with $Aj_p\chi(x) = Tx$ for all $x \in X$. Further

$$\begin{aligned}
\left\|A\left(\sum_{i=1}^{n} \alpha_i j_p \chi(x_i)\right)\right\| &= \left\|\sum_{i=1}^{n} \alpha_i T(x_i)\right\| \\
&\leq \|T\|_e \sup_{f \in B_{X^e}} \left|\sum_{i=1}^{n} \alpha_i f(x_i)\right| \\
&= \|T\|_e \sup_{f \in B_{X^e}} \left|\sum_{i=1}^{n} \alpha_i \chi(x_i)(f)\right| \\
&= \|T\|_e \left\|\sum_{i=1}^{n} \alpha_i \chi(x_i)\right\|_\infty \\
&= \|T\|_e \left\|\sum_{i=1}^{n} \alpha_i j_p \chi(x_i)\right\|_\infty .
\end{aligned}$$

Therefore A is bounded linear and hence it can be extended uniquely to $X_p := \overline{S} \supset A(\hat{X})$, and we still denote it by A. This proves $(1) \implies (2)$. The other implication directly follows from the ideal property and the fact that j_p is linear p-summing; with $\pi_p^e(T) \leq \|A\|$. $\square$

Let us consider the natural embedding $i_Y : Y \to \ell^\infty(B_{Y^*})$. Since $\ell^\infty(B_{Y^*})$ is injective, in the proof of Theorem 7.3.2, $i_Y \circ A : X_p \to \ell^\infty(B_{Y^*})$ has an extension $\tilde{A} : L^p(\mu) \to \ell^\infty(B_{Y^*})$ with $\|\tilde{A}\| = \|i_Y A\|$.

Similarly, we can factor j_p using canonical mappings

$$j_p : C(B_{X^e}) \xrightarrow{j_\infty} L^\infty(\mu) \xrightarrow{i_p} L^p(\mu),$$

where $i_p : L^\infty(\mu) \to L^p(\mu)$ be the formal inclusion map. If we put $U = j_\infty \chi$, we can deduce the following result.

Corollary 7.3.3. *Let X and Y be Banach spaces and $T \in E(X,Y)$. Then the following are equivalent:*

1. *$T \in E_{\pi_p}(X,Y)$.*
2. *There exist a regular Borel probability measure μ on B_{X^e} and an operator $\tilde{A}$: $L^p(\mu) \to \ell^\infty(B_{Y^*})$ such that $i_Y T = \tilde{A} j_p \chi$.*
3. *There exist a probability space (Ω, Σ, μ) a linear operator $\tilde{A} : L^p(\mu) \to \ell^\infty(B_{Y^*})$, and an extensively bounded map $U : X \to L^\infty(\mu)$ such that $i_Y T = \tilde{A} i_p U$.*

Example 7.3.4. *We show $F_e(\mathbb{R})$ is an infinite dimensional vector space and then deduce that $\delta_{\mathbb{R}}^e \in E(\mathbb{R}, F_e(\mathbb{R})) \setminus E_{\pi_p}(\mathbb{R}, F_e(\mathbb{R}))$.*

Let $r, s \in \mathbb{R}$ such that $0 < |r| < |s|$. We prove that δ_r^e and δ_s^e are linearly independent. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha\delta_r^e + \beta\delta_s^e = 0$. Consider

$$g(x) = \begin{cases} x & \text{if } x \in B_{|r|}(0) \\ \frac{rx}{\|x\|} & \text{if } x \notin B_{|r|}(0). \end{cases}$$

Then $g \in E(\mathbb{R},\mathbb{R})$. Now $(\alpha\delta_r^e + \beta\delta_s^e)(Id_{\mathbb{R}}) = 0$ and $(\alpha\delta_r^e + \beta\delta_s^e)(g) = 0$ yield $\alpha r + \beta s = 0$ and $\alpha r + \beta\frac{rs}{|s|} = 0$, or equivalently, $\alpha|s| + \beta s = 0$. Hence $\alpha = 0 = \beta$. Since there are infinitely many such kind of choices of real numbers r and s, $F_e(\mathbb{R})$ contains infinitely many linear independent vectors. Now the linearization map of $\delta_{\mathbb{R}}^e$ is $\Phi(\delta_{\mathbb{R}}^e) = Id_{F_e(\mathbb{R})} \notin \pi_p(F_e(\mathbb{R}), F_e(\mathbb{R}))$, as $F_e(\mathbb{R})$ is infinite dimensional.

7.4 p-compact operators in $E(X,Y)$

Definition 7.4.1. Let X and Y be normed linear spaces. $T \in E(X,Y)$ is said to be an extensively p-compact mapping (in short, E-p-compact map) if $\left\{\frac{Tx}{\|x\|} : x \neq 0\right\}$ is relatively p-compact subset of Y.

We denote the set of all E-p-compact mapping by $E_{K_p}(X,Y)$, also we set

$$\begin{aligned} k_p^e(T) &:= m_p\left(\left\{\frac{Tx}{\|x\|} : x \neq 0\right\}\right) \\ &= \inf\left\{\|(y_n)\|_p : \left\{\frac{Tx}{\|x\|} : x \neq 0\right\} \subset p - conv\{(y_n)\}\right\}. \end{aligned}$$

Remark 7.4.2. The definition of E-p-compact mapping generalizes the definition of the classical p-compact operator. i.e. $K_p(X,Y) \subset E_{K_p}(X,Y)$. For classical linear p- compact operators and its ideal we refer [55], [25], [50].

Lemma 7.4.3. *For any $T \in E_{K_p}(X,Y)$, $\Phi(T) \in K_p(F_e(X),Y)$ with $k_p^e(T) = k_p(\Phi(T))$ and further $\left(E_{K_p}(X,Y), k_p^e(\cdot)\right)$ is a Banach space whenever Y is so.*

Proof. We have already proved $B_{F_e(X)} = \overline{aco}\left\{\frac{\delta_x^e}{\|x\|} : x \neq 0\right\}$. Since $T \in E_{K_p}(X,Y)$, $\left\{\frac{Tx}{\|x\|} : x \neq 0\right\}$ is relatively p-compact, from [40, p. 1205] we have

$$\begin{aligned} m_p\left(\left\{\frac{Tx}{\|x\|} : x \neq 0\right\}\right) &= m_p\left(\Phi(T)\left(\overline{aco}\left\{\frac{\delta_x^e}{\|x\|} : x \neq 0\right\}\right)\right) \\ &= m_p(\Phi(T)(B_{F_e(X)})). \end{aligned}$$

Again for $T, S \in E_{K_p}(X,Y)$ and $\alpha \in \mathbb{K}$, $\Phi(\alpha T + S) \in K_p(F_e(X), Y)$. Therefore by uniqueness of the map Φ we have

$$\Phi(\alpha T + S) = \alpha\Phi(T) + \Phi(S).$$

This proves that $\alpha T + S = (\alpha\Phi(T) + \Phi(S)) \circ \delta_X^e \in E_{K_p}(X,Y)$, and hence $E_{K_p}(X,Y)$ is a subspace of $E(X,Y)$, $k_p^e(\cdot)$ is a norm on $E_{K_p}(X,Y)$ and $\left(E_{K_p}(X,Y), k_p^e(\cdot)\right)$ is isometrically isomorphic to $(K_p(F_e(X),Y), k_p(\cdot))$. Moreover $\left(E_{K_p}(X,Y), k_p^e(\cdot)\right)$ is a Banach space whenever Y is a Banach space as it is isometrically isomorphic to $(K_p(F_e(X),Y), k_p(\cdot))$. □

Proposition 7.4.4. *$\left(E_{K_p}(X,Y), k_p^e(\cdot)\right)$ is a Banach operator ideal.*

Proof. We have already proved that $\Phi\left(F_E(X,Y)\right) = F(F_e(X),Y)$ and $\Phi\left(E_{K_p}(X,Y)\right) = K_p(F_e(X),Y)$. Since $K_p(F_e(X),Y)$ is an operator ideal

$$\Phi\left(F_E(X,Y)\right) = F(F_e(X),Y) \subset K_p(F_e(X),Y) = \Phi\left(E_{K_p}(X,Y)\right).$$

Therefore $F_E(X,Y) \subset E_{K_p}(X,Y)$.

Now for the ideal property, suppose $U \in E(X,Z), T \in E_{K_p}(Z,W), S \in L(W,Y)$, for some normed linear spaces X, Z, W and the Banach space Y. Now

$$\Phi(STU)(\delta_x^e) = S \circ \Phi(T) \circ \Phi(U)(\delta_x^e)$$

ensures that $\Phi(STU) = S \circ \Phi(T) \circ \Phi(U)$. Then as $K_p(F_e(X),Y)$ is an operator ideal $\Phi(STU) \in K_p(F_e(X),Y)$. Therefore $STU \in E_{K_p}(X,Y)$. Hence, using the above lemma, the proof of the proposition follows. □

Proposition 7.4.5. *Let X and Y be two normed linear spaces. Then the following are equivalent:*

1. *$T \in E_{K_p}(X,Y)$.*

2. $\|T^e(y^*)\| \leq \left(\sum\limits_{n=1}^{\infty} |y^*(y_n)|^p\right)^{\frac{1}{p}}$*, for all* $y^* \in Y^*$*, where* $(y_n) \in \ell_p^s(Y)$ *be such that* $\left\{\frac{T(x)}{\|x\|} : x \neq 0\right\} \subset p - conv\{(y_n)\}$.

Proof. Let $\epsilon > 0$ and $\left\{\frac{T(x)}{\|x\|} : x \neq 0\right\} \subset \overline{p - conv\{(y_n)\}}$ for some $(y_n) \in \ell_p^s(Y)$. Then for $y^* \in Y^*$ there exist $x \in X \setminus \{0\}$ such that

$$\|T^e(y^*)\|_e \quad < \quad \left\|y^*\left(\frac{Tx}{\|x\|}\right)\right\| + \frac{\epsilon}{2}.$$

Now for $x \in X \setminus \{0\}$ there exists $(\alpha_n) \in B_{\ell^{p^*}}$, where $p^* \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{p^*} = 1$ such that

$$\left\|\frac{T(x)}{\|x\|} - \sum_{n=1}^{\infty} \alpha_n y_n\right\| < \frac{\epsilon}{2}.$$

Therefore

$$\begin{aligned}
\|T^e(y^*)\|_e \quad &< \quad \left\|y^*\left(\frac{Tx}{\|x\|}\right)\right\| + \frac{\epsilon}{2} \\
&\leq \quad \left|y^*\left(\frac{T(x)}{\|x\|} - \sum_{n=1}^{\infty} \alpha_n y_n\right)\right| + \left|\sum_{n=1}^{\infty} \alpha_n y^*(y_n)\right| + \frac{\epsilon}{2} \\
&\leq \quad \|y^*\|\frac{\epsilon}{2} + \|(\alpha_n)\|_{p^*}\left(\sum_{n=1}^{\infty} |y^*(y_n)|^p\right)^{\frac{1}{p}} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we have $\|T^e(y^*)\| \leq \left(\sum\limits_{n=1}^{\infty} |y^*(y_n)|^p\right)^{\frac{1}{p}}$, for all $y^* \in Y^*$.

For the other way around i.e. to prove $(2) \implies (1)$, suppose $\left\{\frac{T(x)}{\|x\|} : x \neq 0\right\} \nsubseteq \overline{p - conv\{(y_n)\}}$ for some $(y_n) \in \ell_p^s(Y)$. Then there exists $x_0 \in X \setminus \{0\}$ such that $\frac{Tx_0}{\|x_0\|} \notin \overline{p - conv\{(y_n)\}}$. Now $\{\frac{Tx_0}{\|x_0\|}\}$ is compact and $\overline{p - conv\{(y_n)\}}$ is closed. Therefore by geometric version of Hahn-Banach theorem there exist $\alpha > 0$ and $y^* \in Y^*$ such that $\left|y^*(\frac{Tx_0}{\|x_0\|})\right| \geq \alpha$ and $|y^*(y)| \leq \alpha$ for all $y \in \overline{p - conv\{(y_n)\}}$. Further

$$\begin{aligned}
\alpha < \left|y^*(\frac{Tx_0}{\|x_0\|})\right| \quad &\leq \quad \|T^e(y^*)\|_e \\
&\leq \quad \left(\sum_{n=1}^{\infty} |y^*(y_n)|^p\right)^{\frac{1}{p}}
\end{aligned}$$

$$= \sup_{(\alpha_n)\in B_{\ell_{p^*}}} \left| \sum_{n=1}^{\infty} y^*(\alpha_n y_n) \right|$$
$$= \sup_{(\alpha_n)\in B_{\ell_{p^*}}} \left| y^*\left(\sum_{n=1}^{\infty} \alpha_n y_n \right) \right| \leq \alpha,$$

which is a contradiction and thus $(2) \implies (1)$. □

7.4.1 Duality of E-p- compact operator

Definition 7.4.6. Let X and Y be normed linear spaces. For $T \in E(X, Y)$ we say $T \in E_{K_p^d}(X, Y)$ if $T^e(B_{Y^*})$ is relatively p - compact subset of X^e and we set $k_p^{ed}(T) = k_p\left(T^e|_{Y^*}\right)$.

Lemma 7.4.7. *Let X, Y, and Z be normed linear spaces, and $I \in L(Y, Z)$ be an isometric isomorphism, and $T \in K_p(X, Y)$, then $k_p(IT) = k_p(T)$.*

Proof. As $T \in K_p(X, Y)$, there exists $y = (y_n) \in \ell_p^s(Y)$ such that $T(B_E) \subset p - conv\{(y_n)\}$, which is equivalent to $IS(B_X) \subset p - conv\{(I(y_n))\}$. Hence, the proof follows. □

Proposition 7.4.8. *$T \in E_{K_p^d}(X, Y)$ if and only if $\Phi(T) \in K_p^d(X, Y)$ with $k_p^d(\Phi(T)) = k_p^{ed}(T)$.*

Proof. For any $T \in E(X, Y)$ from [36, Remark 4.14] we have $T^e|_{Y^*} = \left(\Phi_X^{\mathbb{R}}\right)^{-1} \circ \Phi(T)^*$. This gives that $T \in E_{K_p^d}(X, Y)$ if and only if $\Phi(T) \in K_p^d(X, Y)$, as $K_p^d(X, Y)$ is an operator ideal.

Again, $k_p^{ed}(T) = k_p\left(T^e|_{Y^*}\right) = k_p\left(\left(\Phi_X^{\mathbb{R}}\right)^{-1} \circ \Phi(T)^*\right)$. Then using the Lemma 7.4.7 we have $k_p^{ed}(T) = k_p(\Phi(T)^*) = k_p^d(\Phi(T))$. □

The proof of the next corollary follows using the above lemma, proposition, and similar technique as we have used in Proposition 7.4.4.

Corollary 7.4.9. *$k_p^{ed}(\cdot)$ forms a norm on $E_{K_p^d}(X,Y)$ and being isometric isomorphic to $\left(K_p^d(F_e(X),Y),k_p^d(\cdot)\right)$, $\left(E_{K_p^d}(X,Y),k_p^{ed}(\cdot)\right)$ is a Banach operator ideal, whenever Y is a Banach space.*

Corollary 7.4.10. *Let X and Y be two normed linear spaces. Then $K_p^d(X,Y)\subset E_{K_p^d}(X,Y)$.*

Proof. Suppose $T\in K_p^d(X,Y)$, then $T^e\big|_{Y^*}=T^*\in K_p(Y^*,X^*)\subset E_{K_p}(Y^*,X^e)$. Hence, the proposition follows. □

7.4.2 Factorization of $E_{K_p^d}$

Proposition 7.4.11. *Let X and Y be two normed linear spaces, $1\le p<\infty$, then $T\in E_{K_p^d}(X,Y)$ if and only if there exists a Banach space Z, $R\in E_K(X,Z)$, $U\in\pi_p(Z,Y)$ such that $T=UR$.*

Proof. Suppose $T\in E_{K_p}(X,Y)$. This is the same as $\Phi(T)\in K_p^d(F(X),Y)$. Therefore, by [55, Theorem 3.1] there exists a Banach space Z, $V\in K(F(X),Z)$ and $U\in\pi_p(Z,Y)$ such that $\Phi(T)=UV \implies T=\Phi(T)\circ\delta_X^e=UV\delta_X^e=UR$ where $R=V\delta_X^e\in E_K(X,Z)$ and $U\in\pi_p(Z,Y)\subset E_{\pi_p}(Z,Y)$.

Conversely, let Z be a Banach space, $T\in E(X,Y)$, and there exists $R\in E_K(X,Z)$, $U\in\pi_p(Z,Y)$ such that $T=UR$. Now it is easy to verify that $\Phi(T)=U\Phi(R)\in\mathcal{K}_p^d(F(X),Y)$, which implies that $T\in E_{K_p^d}(X,Y)$. □

Chapter 8

Extensively bounded p-integral and p-nuclear operator ideal

In the literature, Lipschitz operator ideals have been studied in the context of Lipschitz functions between Banach spaces that map 0 to 0. These ideals are considered a natural extension of the classical operator ideal, a specific class of continuous linear operators between Banach spaces. Such operator ideals on Lipschitz space have been studied in recent years in [2], [34], [3], [56]. This chapter seeks to expand the scope and establish a related theory within a more general framework.

One of our goals is to investigate further classes of E-operator ideals, including E-p-integral, E-p-nuclear, and E-p-compact operators. These generalize the classical concepts of linear p-integral, p-nuclear, and p-compact operators.

In Section 8.1, we introduce extensively p-integral operators (E-p-integral, in short) as a generalization of linear p-integral operators. We demonstrate that the class of E-p-integral operators constitute a Banach E-operator ideal under a specific norm. Additionally, we explore the relationship between this class of operators and E- p-summing operators.

We also introduce extensively (r, p, q)-nuclear operators in Section 8.2, inspired by their classical counterparts, effectively generalizing the traditional linear version. In our exploration of the E-(r, p, q)-nuclear operator ideal, we discover that specific spaces of extensively bounded mappings are, in fact, isometrically isomorphic to specific vector-valued sequence spaces, particularly in Corollary 8.2.4, Lemma 8.2.5 in Section 8.3.

8.1 p-integral operators in $E(X, Y)$

Definition 8.1.1. Let $1 \leq p < \infty$, we say $T : X \to Y$ is an extensively p-integral operator (in short, E-p-integral operator) if there are a probability measure space (Ω, Σ, μ), $A \in L(L^p(\mu), Y^{**})$, $B \in E(X, L^\infty(\mu))$ such that $\mathfrak{J}_Y T = A\mathfrak{i}_p^\mu B$, where $\mathfrak{J}_Y : Y \to Y^{**}$ is the canonical embedding and $\mathfrak{i}_p^\mu : L^\infty(\mu) \to L^p(\mu)$ is the formal identity map.

The collection of all such operators will be denoted as $E_I^p(X, Y)$ and let us put $i_p^e(T) = \inf \|A\|\|B\|_e$, where infimum is taken over all such kind of factorization of T.

We define extensively p-integral operators in the above manner so that the operator ideal property in the E-sense is preserved. Moreover, this definition ensures that every extensively p-integral operator is extensively p-summing.

Proposition 8.1.2. *$E_I^p(X, Y)$ is a subspace of $E(X, Y)$.*

Proof. Let $\alpha \in \mathbb{K}$ and $T_i \in E_I^p(X, Y)$ for $i = 1, 2$. Then it is easy to show that $\alpha T_i \in E_I^p(X, Y)$. Since $T_i \in E_I^p(X, Y)$ there exists probability measure spaces $(\Omega_i, \Sigma_i, \mu_i)$, with $A_i \in L(L^p(\mu_i), Y^{**})$, $B_i \in E(X, L^\infty(\mu_i))$ $i = 1, 2$ such that $\mathfrak{J}_Y T_i = A\mathfrak{i}_p^{\mu_i} B$.

We may assume that $\Omega_1 \cap \Omega_2 = \phi$ and put $\Omega := \Omega_1 \cup \Omega_2$, and $\Sigma := \{S \subset \Omega : S \cap \Omega_i \in \Sigma_i \text{ for } i = 1, 2\}$. Then Σ is a sigma-algebra on Ω with the probability measure

$$\mu(S) = \frac{\|A_1\|\mu_1(S \cap \Omega_1) + \|A_2\|\mu_2(S \cap \Omega_2)}{\|A_1\| + \|A_2\|}.$$

Now define $A : L^p(\mu) \to Y^{**}$ and $B : X \to L^\infty(\mu)$ given by $A(f) = A_1(f\big|_{\Omega_1}) + A_2(f\big|_{\Omega_2})$ and $B(x) = B_1(x).\chi_{\Omega_1} + B_2(x).\chi_{\Omega_2}$, respectively. Then we can easily prove that $A \in L(L^p(\mu), Y^{**})$, $B \in E(X, L^\infty(\mu))$ with $\|A\| \leq \|A_1\| + \|A_2\|$ and $\|B\|_e \leq \|B_1\|_e + \|B_2\|_e$. Also, for $x \in X$

$$\begin{aligned} A\mathfrak{i}_p^\mu B(x) &= A\mathfrak{i}_p^\mu \left(B_1(x).\chi_{\Omega_1} + B_2(x).\chi_{\Omega_2}\right) \\ &= A_1\mathfrak{i}_p^{\mu_1} B_1(x) + A_2\mathfrak{i}_p^{\mu_2} B_2(x) \end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{J}_Y T_1(x) + \mathfrak{J}_Y T_2(x) \\
&= \mathfrak{J}_Y (T_1 + T_2)(x).
\end{aligned}
$$

This completes the proof. □

Proposition 8.1.3. *$F_E(X, Y)$ is a subspace of $E_I^p(X, Y)$*

Proof. We show that $E_I^p(X, Y)$ contains rank one extensively bounded operator, and hence it will follow. Suppose $T = f.y \in F_E(X, Y)$ with $f \in X^e$ and non-zero $y \in Y$, and $x_0 \in X$. Consider $\Omega = \{x_0\}$, $\Sigma = \{\phi, \Omega\}$, $\mu : \Sigma \to \mathbb{R}$ such that $\mu(\phi) = 0$, $\mu(\Omega) = 1$. then $L^p(\mu)$ and $L^\infty(\mu)$ consist of constant functions only. Define $A : L^p(\mu) \to Y^{**}$ and $B : X \to L^\infty(\mu))$ given by $A(\alpha.\mathbb{1}) = \alpha \mathfrak{J}_Y(y)$ and $B(x) = f(x)\mathbb{1}$, respectively, where $\mathbb{1}$ is the constant one function. Then $A \in L(L^p(\mu), Y^{**})$ and $B \in E(X, L^\infty(\mu))$ with $\|A\| = \|y\|$ and $\|B\|_e = \|f\|_e$. Now

$$
\begin{aligned}
\mathfrak{J}_Y T(x) = \mathfrak{J}_Y(f.y)(x) &= \mathfrak{J}_Y(f(x)y) = f(x)\mathfrak{J}_Y(y) = f(x)A(\mathbb{1}) \\
&= A(f(x)\mathbb{1}) = A\mathrm{i}_p^\mu(f(x)\mathbb{1}) = A\mathrm{i}_p^\mu B(x).
\end{aligned}
$$

Hence, the proof follows. □

Proposition 8.1.4. *$i_p^e : E_I^p(X, Y) \to [0, \infty)$ given by $i_p^e(T) = \inf \|A\|\|B\|_e$, where infimum is taken over all such kind of factorization of $T \in E_I^p(X, Y)$ as defined, forms a norm and $\|T\|_e \leq i_p^e(T)$.*

Proof. Let $T \in E_I^p(X, Y)$ and $\mathfrak{J}_Y T = A\mathrm{i}_p^\mu B$, where T has such factorization as in 8.1. Easily we can show that

$$
\|T\|_e = \|\mathfrak{J}_Y T\|_e = \|A\mathrm{i}_p^\mu B\|_e \leq \|A\|\|B\|_e
$$

and hence $\|T\|_e \leq i_p^e(T)$. This gives us that $i_p^e(T) \geq 0$ and the technique of proving $i_p^e(T) = 0$ if and only if $T = 0$ and $i_p^e(\alpha T) = |\alpha|\|T\|$ are same as we proved in the Proposition 7.2.6 in case of π_p^e.

Now for triangle inequality suppose $\epsilon > 0$ and $T_i \in E_I^p(X, Y)$ with $\mathfrak{J}_Y T_i = A_i \mathfrak{i}_p^{\mu_i} B_i$ for $i = 1, 2$; where T_i have the factorization as in the Proposition 8.1.2, with $\|B_i\|_e = \frac{1}{2}$ and $\|A_i\| < i_p^e(T_i) + \frac{\epsilon}{2}$. Then $\|B\|_e \leq 1$ and $\mathfrak{J}_Y(T_1 + T_2) = A\mathfrak{i}_p^{\mu_i} B$. This implies

$$i_p^e(T_1 + T_2) \leq \|A\|\|B\|_e \leq \|A\| \leq i_p^e(T_1) + i_p^e(T_2) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary the triangle inequality part follows. □

Proposition 8.1.5. *$E_I^p(X, Y)$ is an E-operator ideal.*

Proof. Let X, Y, Z, W be normed linear spaces and $S \in E(Z, X)$, $T \in E_I^p(X, Y)$, $U \in L(Y, W)$. We show that $UTS \in E_I^p(Z, W)$ with $i_p^e(UTS) \leq \|U\|.i_p^e(T).\|S\|_e$. We first prove the following lemma, and using that, we complete the proof.

Lemma 8.1.6. $U^{**} \circ \mathfrak{J}_Y = \mathfrak{J}_W \circ U$.

Proof. Let $y \in Y$, $m^* \in W^*$. Then

$$(\mathfrak{J}_W \circ U(y))(m^*) = (\mathfrak{J}_W(U(y)))(m^*) = m^*(U(y)).$$

Again,

$$\begin{aligned} ((U^{**} \circ \mathfrak{J}_Y)(y))(m^*) &= (U^{**}(\mathfrak{J}_Y(y))(m^*) \\ &= (\mathfrak{J}_Y(y) \circ U^*)(m^*) \\ &= \mathfrak{J}_Y(y)(U^*(m^*)) \\ &= (U^*(m^*))(y) \\ &= (m^* \circ U)(y) \\ &= m^*(U(y)). \end{aligned}$$

This completes the proof of the lemma. □

Proof of the proposition: Since $T \in E_I^p(X,Y)$, consider the factorization as defined with $\mathfrak{J}_Y T = A\mathrm{i}_p^\mu B$, where (Ω,Σ,μ) is the probability measure space, $A \in E(L^p(\mu),Y^{**}),\ B \in E(X,L^\infty(\mu))$. Now

$$U^{**}A\mathrm{i}_p^\mu B = U^{**}\mathfrak{J}_Y T = \mathfrak{J}_W UT(\text{ by the lemma}).$$

Hence, $\mathfrak{J}_W UTS = U^{**}A\mathrm{i}_p^\mu BS$. Therefore the proof of the proposition follows and $i_p^e(UTS) \leq \|U\|.i_p^e(T).\|S\|_e$ part can be verified easily. $\square$

Proposition 8.1.7. *$\left(E_I^p(X,Y), i_p^e(\cdot)\right)$ is a Banach space.*

Proof. Let (T_n) be a sequence in $E_I^p(X,Y)$ such that $\sum\limits_{i=1}^{\infty} i_p^e(T_n) < \infty$. Therefore, $\sum\limits_{n=1}^{\infty} \|T_n\|_e < \infty$. Since $(E(X,Y), \|\cdot\|_e)$ is a Banach space $\sum\limits_{i=1}^{\infty} T_n = T \in E(X,Y)$ with respect to $\|\cdot\|_e$.

Claim: $\sum\limits_{n=1}^{\infty} T_n = T$ in $i_p^e(\cdot)$ norm.

Let $\epsilon > 0$. Now for all $n \in \mathbb{N}$ there exist probability measure spaces $(\Omega_n,\Sigma_n,\mu_n)$, and operators $A_n \in L(L^p(\mu_n),Y^{**}),\ B_n \in E(X,L^\infty(\mu_n))$ with $\|B_n\|_e = \frac{1}{2^n}$ and $\|A_n\| \leq i_p^e(T_n) + \frac{\epsilon}{2^n}$ such that $\mathfrak{J}_Y T_n = A_n\mathrm{i}_p^{\mu_n} B_n$.

We may assume Ω_n are pair wise disjoint and consider

$$\Omega = \dot{\bigcup_{n\in\mathbb{N}}} \Omega_n,\ \Sigma = \{S \subset \Omega : S \cap \Omega_n \in \Sigma_n \text{ for all } n \in \mathbb{N}\}.$$

Then Σ is an sigma-algebra on Ω and $\mu(S) = \frac{\sum\limits_{n=1}^{\infty} \mu_n(S\cap\Omega_n)\|A_n\|}{\sum\limits_{n=1}^{\infty} \|A_n\|_e}$ is a probability measure on Ω.

Let us now define $A : L^p(\mu) \to Y^{**}$ and $B : X \to L^\infty(\mu))$ given by

$$A(f) = \sum_{n=1}^{\infty} A_n(f\big|_{\Omega_n}) \text{ and } B(x) = \sum_{n=1}^{\infty} B_n(x).\chi_{\Omega_n}.$$

Then $A \in L(L^p(\mu),Y^{**}),$ and $B \in E(X,L^\infty(\mu))$ with

$$\|A\| \leq \sum_{n=1}^{\infty} \|A_n\| \leq \sum_{n=1}^{\infty} i_p^e(T_n) + \epsilon \text{ and } \|B\|_e \leq 1.$$

Now

$$\begin{aligned}
A\mathfrak{i}_p^\mu B(x) &= A\left(\sum_{n=1}^{\infty}\mathfrak{i}_p^\mu.B_n(x).\chi_{\Omega_n}\right)\\
&= \sum_{m=1}^{\infty}A_m\left(\left(\sum_{n=1}^{\infty}\mathfrak{i}_p^\mu.B_n(x).\chi_{\Omega_n}\right)|_{\Omega_m}\right)\\
&= \sum_{m=1}^{\infty}A_m.\mathfrak{i}_p^\mu.B_m(x)\\
&= \sum_{m=1}^{\infty}\mathfrak{J}_Y T_m(x)\\
&= \mathfrak{J}_Y T(x)
\end{aligned}$$

Also, $i_p^e(T) \le \|A\|\|B\|_e \le \|A\| \le \sum_{n=1}^{\infty} i_p^e(T_n)+\epsilon$. This shows that $T \in E_I^p(X,Y)$.

Again define $\tau_n : L^p(\mu) \to Y^{**}$ given by $\tau_n(f) = \sum_{k=n+1}^{\infty} A_k(f\big|_{\Omega_k})$.Thus $\tau_n \in E(L^p(\mu), Y^{**})$ with $\|\tau_n\|_e \le \sum_{k=n+1}^{\infty}\|A_k\|$. Hence $\lim_{n\to\infty}\|\tau_n\|_e = 0$ and

$$\begin{aligned}
\tau_n\mathfrak{i}_p^\mu B(x) &= \tau_n\mathfrak{i}_p^\mu\left(\sum_{m=1}^{\infty}B_m(x).\chi_{\Omega_m}\right)\\
&= \tau_n\left(\sum_{m=1}^{\infty}\mathfrak{B}_m(x).\chi_{\Omega_m}\right)\\
&= \sum_{k=n+1}^{\infty}A_k\left(\left(\sum_{m=1}^{\infty}B_m(x).\chi_{\Omega_m}\right)|_{\Omega_k}\right)\\
&= \left(\sum_{k=n+1}^{\infty}T_k\right)(x)
\end{aligned}$$

This implies $T - \sum_{k=1}^{n} T_k = \tau_n\mathfrak{i}_p^\mu B$ with $\lim_{n\to\infty} i_p^e(T - \sum_{k=1}^{n} T_k) = 0$. □

Proposition 8.1.8. *Let $T \in E(X,Y)$ be such that $\Phi(T) \in I_p(F_e(X),Y)$. Then T is an E-p-integral operator.*

Proof. Since $\Phi(T) \in I_p(F_e(X),Y)$ there are a probability measure space (Ω,Σ,μ), $A \in L(L^p(\mu),Y^{**})$, $B \in L(X,L^\infty(\mu))$ such that $\mathfrak{J}_Y\Phi(T) = A\mathfrak{i}_p^\mu B$. Therefore $\mathfrak{J}_Y \circ T = \mathfrak{J}_Y \circ \Phi(T) \circ \delta_X^e = A \circ \mathfrak{i}_p^\mu \circ B \circ \delta_X^e$ and it immediately proves the proposition. □

Now, we only state the next proposition, whose proof follows similarly.

Proposition 8.1.9. *Let X, Y, Z be normed linear spaces. $T \in I_p(Z, Y)$ and $S \in E(X, Z)$ then $T \circ S \in E_I^p(X, Y)$. That is $I_p \circ E \subset E_I^p$.*

8.1.1 Relation between E-p-summing and E-p-integral operators

Let $T \in E_I^p(X, Y)$. Therefore $\mathfrak{J}_Y T = A \mathfrak{i}_p^\mu B$ for some suitable factorization, where $\mathfrak{i}_p^\mu \in \pi_p(X, Y) \subset E_{\pi_p}(X, Y)$. Now the ideal property of E_{π_p} gives that $\mathfrak{J}_Y T = A \mathfrak{i}_p^\mu B \in E_{\pi_p}(X, Y^{**})$ and by Proposition 7.2.7 we have $T \in E_{\pi_p}(X, Y)$ with $\pi_p^e(T) \leq i_p^e(T)$.

8.2 Some particular classes of extensively bounded maps

In our study of the factorization of extensively (r, p, q)-operators in coming sections, we encounter certain spaces of extensively bounded mappings that are ultimately linearly isometrically isomorphic to vector-valued sequence spaces. This isometric relationship serves as a key tool for establishing the factorization of extensively (r, p, q)-nuclear operators.

Proposition 8.2.1. *Let (M, d) be a metric space with the distinguished point 0 and Y as a normed linear space. Then $E(M, Y)$ is isometrically embedded into $L(Y^*, M^e)$ linearly.*

Proof. Given $T \in E(M, Y)$, we define $T^e : Y^e \to M^e$ by $T^e(f) = f \circ T$ for all $f \in Y^e$. Then T^e is linear with $\|T^e(f)\|_e \leq \|f\|_e d_e(T, 0)$. Thus $T^e \in L(Y^e, M^e)$ with

$$\|T^e\| = \|T^e\|_e \leq d_e(T, 0).$$

Also

$$\|T^e\| = \sup_{f \neq 0} \frac{\|f \circ T\|_e}{\|f\|_e} = \sup_{f \neq 0} \sup_{x \neq 0} \frac{|f(T(x))|}{\|f\|_e d(x, 0)} \geq \sup_{x \neq 0} \frac{\|Tx\|}{d(x, 0)} = d_e(T, 0).$$

Thus $\|T^e\| = \|T^e\|_e = d_e(T, 0)$.

Now define the map $\xi : E(M,Y) \to L(Y^*, M^e)$ given by $\xi(T) = T^e\big|_{Y^*}$, for all $T \in E(M,Y)$. Then ξ is linear and

$$\begin{aligned}\|T^e\big|_{Y^*}\| &= \sup_{f\neq 0, f\in Y^*} \frac{\|f\circ T\|_e}{\|f\|} \\ &= \sup_{f\neq 0, f\in Y^*} \sup_{x\neq 0} \frac{\|f(T(x))\|}{\|f\|\|x\|} \\ &= \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \|T\|_e.\end{aligned}$$

Hence, ξ is an isometry. □

Remark 8.2.2. In particular, if $Y = \ell_p$, for $1 \leq p < \infty$, the map ξ is surjective also.

Corollary 8.2.3. *Let X be a normed linear space and $1 < p < \infty$. Then $(E(X,\ell_p), \|\cdot\|_e)$ is isometrically isomorphic to $L(\ell_{p^*}, X^e)$ and $(E(X,\ell_1), \|.\|_e)$ is isometrically isomorphic to $L(c_0, X^e)$, where p^* be the conjugate exponent of p.*

Proof. If $1 < p < \infty$:

We have already proved

$$\xi : E(X,\ell_p) \to L((\ell_p)^*, X^e)$$

given by $\xi(T) = T^e\big|_{(\ell_{p^*})}$, for all $T \in E(X,\ell_p)$ is linear isometry. Now we show that ξ is surjective. Suppose $S \in L(\ell_{p^*}, X^e)$. Put $S(e_n) = f_n \in X^e$ and define $T(x) = (f_n(x))$ for all $x \in X$. Since $S \in L(\ell_{p^*}, X^e) \left(\cong \ell_p^w(X^e)\right)$, $\sum\limits_{n=1}^{\infty} \alpha_n f_n(x)$ is real number for all $(\alpha_n) \in \ell_{p^*}$. Therefore, $(f_n(x)) \in \ell_p$ for all $x \in X$. This shows that T is well-defined. Again,

$$\begin{aligned}\|Tx\|_p &= \|\left(f_n(x)\right)\|_p \\ &= \sup_{(\alpha_n)\in B_{\ell_{p^*}}} |\langle(\alpha_n), (f_n(x))\rangle| \\ &= \sup_{(\alpha_n)\in B_{\ell_{p^*}}} |S\left((\alpha_n)\right)(x)|\end{aligned}$$

$$\leq \|S\|\|x\|.$$

This implies that $T \in E(X, \ell_p)$. Also,

$$T^e\left((e_n)\right)(x) = f_n(x) = S(e_n)(x)$$

gives that $\xi(T) = T^e\big|_{\ell_{p^*}} = S$. This completes the case $1 < p < \infty$ proof.

For $p = 1$:

Consider the map $\xi : E(X, \ell_1) \to L\left(c_0, X^e\right)$ given by $\xi(T) = T^e\big|_{c_0}$, for all $T \in E(X, \ell_1)$. Then, similarly to the above, we get the linear isometric isomorphism. $\square$

Corollary 8.2.4. *For any normed linear space X and $1 \leq p < \infty$, $(E(X, \ell_p), \|\cdot\|_e)$ is linearly isometrically isomorphic to $\left(\ell_p^w(X^e), \|\cdot\|_p^w\right)$.*

Lemma 8.2.5. *For any normed linear space X, $(E(X, c_0), \|\cdot\|_e)$ is isometrically isomorphic to $(c_0^w(X^e), \|\cdot\|_\infty^w)$.*

Proof. Consider the map $\xi : E(X, c_0) \to c_0^w(X^e)$ given by

$$\xi(T) = \left(e_n \circ T\right),$$

for all $T \in E(X, c_0)$. First, we prove that the map ξ is well-defined. From [36, Remark 4.4] we have $X^e \cong F_e(X)^*$. Therefore, $(X^e)^* \cong F_e(X)^{**}$. Let $T \in E(X, c_0)$. Then $Tx \in c_0$ for all $x \in X$. Now $Tx = (\delta_x^e(e_n \circ T))$. As c_0 is a Banach space $(\gamma(e_n \circ T)) \in c_0$ for all $\gamma \in F_e(X)$. Therefore $(X^e)^* \cong F_e(X)^{**}$ provides that $(\lambda(e_n \circ T)) \in c_0$ for all $\lambda \in (X^e)^*$. This implies $(e_n \circ T) \in c_0^w(X^e)$. It is easy to check that ξ is linear. Further

$$\begin{aligned}
\|\xi(T)\|_\infty^s &= \sup_{n \in \mathbb{N}} \|e_n \circ T\|_e \\
&= \sup_{n \in \mathbb{N}} \sup_{x \neq 0} \frac{|e_n(Tx)|}{\|x\|} \\
&= \sup_{x \neq 0} \frac{\|Tx\|_\infty}{\|x\|}
\end{aligned}$$

$$= \|T\|_e,$$

and using the fact $\|\cdot\|_\infty^s = \|\cdot\|_\infty^w$ we have $\|\xi(T)\|_\infty^s = \|\xi(T)\|_\infty^w = \|T\|_e$. This implies that the map ξ is linear isometry. Now, we show that ξ is also surjective. Suppose $(f_n) \in c_0^w(X^e)$. Put $S(x) = (f_n(x))$, for all $x \in X$. Again, [36, Remark 4.4 and Lemma 5.6] give

$$\begin{aligned}
\|(f_n)\|_\infty^w &= \sup_{\gamma \in B_{F_e(X)}} \|(\gamma(f_n))\|_\infty \\
&\geq \sup_{x \neq 0} \left\| \left(\frac{\delta_x^e(f_n)}{\|x\|} \right) \right\|_\infty \\
&= \sup_{x \neq 0} \frac{\|(f_n(x))\|_\infty}{\|x\|}.
\end{aligned}$$

Therefore, $\|(f_n(x))\|_\infty \leq \|(f_n)\|_\infty^w \|x\|$ for all $x \in X$. This shows that $S \in E(X, c_0)$. Further $e_n \circ S(x) = e_n(S(x)) = f_n(x)$ for all $x \in X$. Hence, $\xi(S) = (f_n)$ and ξ is surjective, also. This completes the proof. □

8.3 Extensively (r, p, q)-nuclear operator

Definition 8.3.1. Let $0 < r \leq \infty; 1 \leq p, q < \infty$, and $1 + \frac{1}{r} \geq \frac{1}{p} + \frac{1}{q}$. A map $T : X \to Y$, between two normed linear spaces X and Y; is called extensively (r, p, q)-nuclear if

$$T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n,$$

for some $(\lambda_n) \in \ell_r$, $(f_n) \in \ell_{q^*}^w(X^e)$, and $(y_n) \in \ell_{p^*}^w(Y)$.

The set of all extensively (r, p, q)-nuclear operator is denoted by $E_N^{(r,p,q)}(X, Y)$.

Proposition 8.3.2. *Let $0 < r \leq \infty; 1 \leq p, q < \infty$, and $1 + \frac{1}{r} \geq \frac{1}{p} + \frac{1}{q}$. Then for $T \in E(X, Y)$ the following are equivalent:*

1. *T is an extensively (r, p, q)-nuclear operator.*

2. *$T = AM_\lambda B$, for some $M_\lambda \in L(\ell_{q^*}, \ell_p)$; $\lambda \in \ell_r$, $A \in L(\ell_p, Y)$, and $B \in E(X, \ell_{q^*})$.*

Proof. Let $T \in E_N^{(r,p,q)}(X,Y)$. Then

$$T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n,$$

for some $(\lambda_n) \in \ell_r$, $(f_n) \in \ell_{q^*}^w(X^e)$, and $(y_n) \in \ell_{p^*}^w(Y)$.

Put $A((\alpha_n)) = \sum\limits_{n=1}^{\infty} \alpha_n y_n$, for all $(\alpha_n) \in \ell_p$. Then from [22, Proposition 2.2] we have $A \in L(\ell_p, Y)$. Again consider $B(x) = (f_n(x))$, for all $x \in X$. Then Corollary 8.2.4 provides that $B \in E(X, \ell_{q^*})$. Further $M_\lambda : \ell_{q^*} \to \ell_p$ be the multiplication map $(a_n) \mapsto (\lambda_n a_n)$.

Suppose $x \in X$. Thus using the fact $\frac{1}{r} + \frac{1}{q^*} \geq \frac{1}{p}$ we get

$$T(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n = A(\lambda_n f_n(x)) = AM_\lambda(f_n(x)) = AM_\lambda B(x).$$

This shows the part (1) implies (2).

$\underline{(2) \implies (1)}$ Suppose T has such a factorization as in (2). Consider $y_n = A(e_n)$, $f_n(x) = \pi_n \circ B(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Therefore, repeated use of Corollary 8.2.4 gives $(f_n) \in \ell_{q^*}^w(X^e)$, and $(y_n) \in \ell_{p^*}^w(Y)$. Moreover, for any $x \in X$

$$\begin{aligned} Tx &= AM_\lambda B(x) \\ &= A\left(\sum_{n=1}^{\infty} \lambda_n \pi_n \circ B(x)\right) \\ &= \sum_{n=1}^{\infty} \lambda_n f_n(x) A(e_n) \\ &= \left(\sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n\right)(x). \end{aligned}$$

Hence, the proof follows. $\square$

Remark 8.3.3. In the classical case, whenever $r = p$ and $q = 1$, a linear (r,p,q)-nuclear operator is simply a p-nuclear operator. Here, we also see a similar phenomenon in the Extensively bounded map setting.

8.3.1 Extensively p-nuclear operator

Definition 8.3.4. Let $1 \leq p < \infty$. Then $T \in E(X, Y)$ is said to be an extensively p-nuclear operator (in short, E-p-nuclear) if there exist $(f_n) \in \ell_p^s(X^e)$, $(y_n) \in \ell_{p^*}^w(Y)$ such that

$$T = \sum_{n=1}^{\infty} f_n \otimes y_n \text{ with } \sum_{n=1}^{\infty} \|f_n\|_e \|y_n\| < \infty.$$

Let us consider $\nu_p^e(T) = \inf \|(f_n)\|_p^s \|(y_n)\|_{p^*}^w$, where infimum is taken over all such representation of T, and denote the space of all such operators by $E_N^p(X, Y)$.

Proposition 8.3.5. *$E_N^p(X, Y)$ is a subspace and $\nu_p^e(\cdot)$ is a norm on $E_N^p(X, Y)$.*

Proof. by definition $\nu_p^e(T) \geq 0$ for all $T \in E_N^p(X, Y)$. Let $T \in E_N^p(X, Y)$ have one such representation. Now $T(x) = \sum_{n=1}^{\infty} f_n(x) y_n$. Thus

$$\begin{aligned} \|T(x)\| &= \|\sum_{n=1}^{\infty} f_n(x) y_n\| \\ &= \sup_{\|y^*\| \leq 1} \left| y^* \left(\sum_{n=1}^{\infty} f_n(x) y_n \right) \right| \\ &\leq \sup_{\|y^*\| \leq 1} \sum_{n=1}^{\infty} |f_n(x)| |y^*(y_n)| \\ &\leq \|x\| \|(f_n)\|_p^s \|(y_n)\|_{p^*}^w. \end{aligned}$$

This shows that $\|T\|_e \leq \nu_p^e(T)$, and using this we get $\nu_p^e(T) = 0$ if and only if $T = 0$. Also, it is easy to verify that $\nu_p^e(\alpha T) = |\alpha| \nu_p^e(T)$.

For the triangle inequality, suppose $T, S \in E_N^p(X, Y)$, and $\epsilon > 0$. Choose representations of T and S such that $T = \sum_{n=1}^{\infty} f_{2n-1} \otimes y_{2n-1}$, and $S = \sum_{n=1}^{\infty} f_{2n} \otimes y_{2n}$ with

$$\|(f_{2n-1})\|_p^s < \nu_p^e(T) + \epsilon, \ \|(f_{2n})\|_p^s < \nu_p^e(S) + \epsilon, \ \|(y_{2n-1})\|_{p^*}^w = \|(y_{2n})\|_{p^*}^w = 1.$$

Fix $l, k > 0$ and put $\tilde{f}_{2n-1} := l.f_{2n-1}$ and $\tilde{f}_{2n} := k.f_{2n}$ for all $n \in \mathbb{N}$. Then $(\tilde{f}_n) \in \ell_p^s(X^e)$ with $\|(\tilde{f}_n)\|_p^p \leq l^p(\nu_p^e(T) + \epsilon)^p + k^p(\nu_p^e(S) + \epsilon)^p$. Again considering $\tilde{y}_{2n-1} := l^{-1}.y_{2n-1}$ and $\tilde{y}_{2n} := k^{-1}.y_{2n}$ for all $n \in \mathbb{N}$ we get $(\tilde{y}_n) \in \ell_{p^*}^w(Y)$ with $\left(\|(\tilde{y}_n)\|_{p^*}^w\right)^{p^*} \leq l^{-p^*} + k^{-p^*}$.

Now choosing $l = (\nu_p^e(T)+\epsilon)^{-\frac{1}{p^*}}$ and $k = (\nu_p^e(S)+\epsilon)^{-\frac{1}{p^*}}$ we get

$$\nu_p^e(T+S) \le \nu_p^e(T) + \nu_p^e(S) + 2\epsilon,$$

where $T+S = \sum\limits_{n=1}^{\infty} \tilde{f}_n \otimes \tilde{y}_n$. Since $\epsilon > 0$ is arbitrary, the triangle inequality part for the norm follows. $\square$

Proposition 8.3.6. *$\left(E_N^p(X,Y), \nu_p^e(\cdot)\right)$ is a Banach operator ideal.*

Proof. The proof follows a similar pattern to the classical case. $\square$

8.3.2 Factorization of E-p-nuclear operators

Proposition 8.3.7. *Let $T \in E_N^p(X,Y)$, then the followings hold:*

1. *T can be written as $T = \sum\limits_{n=1}^{\infty} \lambda_n g_n \otimes y_n$, with $\sum\limits_{n=1}^{\infty} |\lambda_n| \|y_n\| < \infty$, where $(\lambda_n) \in \ell_p^s$, $(g_n) \subset B_{X^e}$, $(y_n) \in \ell_{p^*}^w(Y)$.*

2. *T will have a factorization as $T = AM_sB$, where $A \in L(\ell_p, Y)$, $M_s \in L(\ell_\infty, \ell_p)$ (the standard linear multiplication operator), and $B \in E(X, c_0)$.*

Proof. As $T \in E_N^p(X,Y)$, there exist $(f_n) \in \ell_p^s(X^e)$, $(y_n) \in \ell_{p^*}^w(Y)$ such that $T = \sum\limits_{n=1}^{\infty} f_n \otimes y_n$, with $\sum\limits_{n=1}^{\infty} \|f_n\|_e \|y_n\| < \infty$. Now consider $\lambda_n = \|f_n\|_e$, and

$$g_n = \begin{cases} \frac{f_n}{\|f_n\|_e} & \text{if } f_n \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then, (1) immediately follows.

To prove (2), we use the first part. Let $T = \sum\limits_{n=1}^{\infty} \lambda_n g_n \otimes y_n$, as in (1). Define $A : \ell_p \to Y$ as $A((\alpha_n)) = \sum\limits_{n=1}^{\infty} \alpha_n y_n$. Then from [22, Proposition 2.2] it follows that $A \in L(\ell_p, Y)$ with $\|A\| = \|(y_n)\|_{p^*}^w$.

Since $(\lambda_n) \in \ell_p^s$, it has the factorization as $\lambda_n = r_n.s_n$ for all $n \in \mathbb{N}$, where $(r_n) \in c_0$, $(s_n) \in \ell_p^s$. Using this factorization we define $B : X \to c_0$ as $B(x) = (r_n g_n(x))$ and

$M_s : \ell_\infty \to \ell_p$ as $M_s((t_n)) = (s_n t_n)$. Easily we can verify that $M_s \in L(\ell_\infty, \ell_p)$ with $\|M_s\| = \|(s_n)\|_p$.

Now we show that $B \in E(X, c_0)$. Put $B_n(x) = r_n g_n(x)$. Since $(r_n g_n) \in c_0^s(X^e) \subset c_0^w(X^e)$, by Lemma 8.2.5 $B \in E(X, c_0)$. This completes the proof of the proposition. □

Proposition 8.3.8. *Let $A \in L(\ell_p, Y), M_s \in L(\ell_\infty, \ell_p)$ (the standard linear multiplication operator with $s \in \ell_p$), and $B \in E(X, c_0)$. Then AM_sB is an E-p-nuclear operator.*

Proof. Define $f_n : X \to \mathbb{R}$ given by $f_n(x) = s_n \pi_n B(x)$. Then each f_n is an extensively bounded map. In fact $|f_n(x)| \leq |s_n| \|B(x)\|_\infty$ implies that $f_n \in X^e$ with $\|f_n\|_e \leq |s_n| \|B\|_e$. Again

$$\left(\sum_{n=1}^{\infty} \|f_n\|_e^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |s_n|^p \|B\|_e^p\right)^{\frac{1}{p}} \leq \|B\|_e \|(s_n)\|_p^s$$

proves that $(f_n) \in \ell_p^s(X^e)$.

Further define $y_n = A(e_n)$. Therefore by [22, Proposition 2.2] $(y_n) \in \ell_{p^*}^w(Y)$ as $A \in L(\ell_p, Y)$.

Now $AM_sB(x) = A\left(\sum\limits_{n=1}^{\infty} f_n(x) e_n\right) = \sum\limits_{n=1}^{\infty} f_n(x) y_n = \left(\sum\limits_{n=1}^{\infty} f_n \otimes y_n\right)(x)$. Hence, the proof follows. □

Remark 8.3.9. Since $M_s \in \pi_p(\ell_\infty, \ell_p)$, AM_sB is E-p-summing. Hence by 8.1.1 we get $E_N^p(X, Y) \subset E_I^p(X, Y)$.

Remark 8.3.10. For $1 \leq p < \infty$, Extensively $(p, p, 1)$-nuclear operator is Extensively p-nuclear operator. The proof of this fact directly follows from the factorization of E-p-nuclear operators.

Proposition 8.3.11. *Let $T \in E(X, Y)$ be such that $\Phi(T) \in N_p(F_e(X), Y)$. Then T is an E-p-nuclear operator.*

Proof. Since $\Phi(T) \in N_p(F_e(X), Y)$ there exist $A \in L(\ell^p, Y),\ B \in L(X, \ell^\infty)$ and $\lambda \in \ell^p$ such that $\phi(T) = AM_\lambda B$, where M_λ is the multiplication operator in $L(\ell^\infty, \ell^p)$. Therefore, $T = \phi(T) \circ \delta_X^e = A \circ M_\lambda \circ B \circ \delta_X^e$ and it immediately provide the proof of the proposition using the factorization result above. $\square$

Now, we state the next proposition, whose proof follows similarly.

Proposition 8.3.12. *Let X, Y, Z be normed linear spaces. $T \in N_p(Z, Y)$ and $f \in E(X, Z)$ then $T \circ f \in E_N^p(X, Y)$. That is $N_p \circ E \subset E_N^p$.*

References

[1] D. Achour, A. Alouani, P. Rueda, and K. Saadi, *Factorable strongly p-nuclear m-homogeneous polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 2, 969–986.

[2] D. Achour, P. Rueda, E. A. Sánchez-Pérez, and R. Yahi, *Lipschitz operator ideals and the approximation property*, J. Math. Anal. Appl. **436** (2016), no. 1, 217–236.

[3] D. Achour, E. Dahia, and P. Turco, *Lipschitz p-compact mappings*, Monatsh. Math. **189** (2019), no. 4, 595–609.

[4] N. Albarracín and P. Turco, *On the Lipschitz operator ideal* $\mathrm{Lip}_0 \circ \mathcal{A} \circ \mathrm{Lip}_0$, Studia Math. **277** (2024), no. 3, 243–269.

[5] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, *Lipschitz free spaces isomorphic to their infinite sums and geometric applications*, Trans. Amer. Math. Soc. **374** (2021), no. 10, 7281–7312.

[6] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, *Lipschitz algebras and Lipschitz-free spaces over unbounded metric spaces*, Int. Math. Res. Not. IMRN (2022), no. 20, 16327–16362.

[7] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, *Structure of the Lipschitz free p-spaces $\mathcal{F}_p(\mathbb{Z}^d)$ and $\mathcal{F}_p(\mathbb{R}^d)$ for $0 < p \leq 1$*, Collect. Math. **73** (2022), no. 3, 337–357.

[8] R. J. Aliaga and E. Pernecká, *Integral representation and supports of functionals on Lipschitz spaces*, Int. Math. Res. Not. IMRN (2023), no. 4, 3004–3072.

[9] R. F. Arens and J. Eells, Jr., *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397–403.

[10] Y. Benyamini and Y. Sternfeld, *Spheres in infinite-dimensional normed spaces are Lipschitz contractible*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 439–445.

[11] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[12] H. Bustos Domecq, *Vector-valued invariant means revisited*, J. Math. Anal. Appl. **275** (2002), no. 2, 512–520.

[13] L. Candido and P. L. Kaufmann, *On Lipschitz-free spaces over spheres of Banach spaces*, J. Math. Anal. Appl. **500** (2021), no. 1, Paper No. 125093, 4.

[14] c. Cobzaş, R. Miculescu, and A. Nicolae, *Lipschitz functions*, Lecture Notes in Mathematics, vol. 2241, Springer, Cham, 2019.

[15] M. Cúth and M. Doucha, *Lipschitz-free spaces over ultrametric spaces*, Mediterr. J. Math. **13** (2016), no. 4, 1893–1906.

[16] M. Cúth, M. Doucha, and P. a. Wojtaszczyk, *On the structure of Lipschitz-free spaces*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 3833–3846.

[17] M. Cúth, O. F. K. Kalenda, and P. Kaplický, *Isometric representation of Lipschitz-free spaces over convex domains in finite-dimensional spaces*, Mathematika **63** (2017), no. 2, 538–552.

[18] A. Dalet, *Free spaces over countable compact metric spaces*, Proc. Amer. Math. Soc. **143** (2015), no. 8, 3537–3546.

[19] K. de Leeuw, *Banach spaces of Lipschitz functions*, Studia Math. **21** (1961/62), 55–66.

[20] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co., Amsterdam, 1993.

[21] J. Diestel, S. A. Morris, and S. A. Saxon, *Varieties of linear topological spaces*, Trans. Amer. Math. Soc. **172** (1972), 207–230.

[22] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.

[23] M. Dubei, E. D. Tymchatyn, and A. Zagorodnyuk, *Free Banach spaces and extension of Lipschitz maps*, Topology **48** (2009), no. 2-4, 203–212.

[24] Y. Dutrieux and V. Ferenczi, *The Lipschitz free Banach spaces of $C(K)$-spaces*, Proc. Amer. Math. Soc. **134** (2006), no. 4, 1039–1044.

[25] J. Fourie and J. Swart, *Banach ideals of p-compact operators*, Manuscripta Math. **26** (1978/79), no. 4, 349–362.

[26] D. H. Fremlin, *Measure theory. Vol. 2*, Torres Fremlin, Colchester, 2003, Broad foundations, Corrected second printing of the 2001 original.

[27] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, vol. 159, 2003, Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday, pp. 121–141.

[28] G. Godefroy, *A survey on Lipschitz-free Banach spaces*, Comment. Math. **55** (2015), no. 2, 89–118.

[29] A. P. Goucher and T. Kania, *Invariant means on Abelian groups capture complementability of Banach spaces in their second duals*, Studia Math. **260** (2021), no. 1, 91–101.

[30] P. Hájek, G. Lancien, and E. Pernecká, *Approximation and Schur properties for Lipschitz free spaces over compact metric spaces*, Bull. Belg. Math. Soc. Simon Stevin **23** (2016), no. 1, 63–72.

[31] P. Hájek and M. e. Novotný, *Some remarks on the structure of Lipschitz-free spaces*, Bull. Belg. Math. Soc. Simon Stevin **24** (2017), no. 2, 283–304.

[32] K. Hamidi, E. Dahia, D. Achour, and A. Tallab, *Two-Lipschitz operator ideals*, J. Math. Anal. Appl. **491** (2020), no. 2, 124346, 21.

[33] T. M. Jenkins, *BANACH SPACES OF LIPSCHITZ FUNCTIONS OF AN ABSTRACT METRIC SPACE*, ProQuest LLC, Ann Arbor, MI, 1968, Thesis (Ph.D.)–Yale University.

[34] A. Jiménez-Vargas, J. M. Sepulcre, and M. Villegas-Vallecillos, *Lipschitz compact operators*, J. Math. Anal. Appl. **415** (2014), no. 2, 889–901.

[35] T. Kania, *Vector-valued invariant means revisited once again*, J. Math. Anal. Appl. **445** (2017), no. 1, 797–802.

[36] A. K. Karn and A. Mandal, *A generalization of lipschitz mappings*, 2025.

[37] A. K. Karn and A. Mandal, *Position of $L(X,Y)$ in $Lip_0(X,Y)$*, 2025.

[38] T. Kochanek and E. Pernecká, *Lipschitz-free spaces over compact subsets of superreflexive spaces are weakly sequentially complete*, Bull. Lond. Math. Soc. **50** (2018), no. 4, 680–696.

[39] C. S. Kubrusly, , Universitext, Springer, Cham, [2023] ©2023.

[40] S. Lassalle and P. Turco, *On p-compact mappings and the p-approximation property*, J. Math. Anal. Appl. **389** (2012), no. 2, 1204–1221.

[41] H. Lebesgue, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz*, Bulletin de la Société Mathématique de France **38** (1910), 184–210 (fr).

[42] J. Lindenstrauss, *On nonlinear projections in Banach spaces*, Michigan Math. J. **11** (1964), 263–287.

[43] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), no. 12, 837–842.

[44] M. Messerschmidt, *A pointwise Lipschitz selection theorem*, Set-Valued Var. Anal. **27** (2019), no. 1, 223–240.

[45] E. Michael, *A short proof of the Arens-Eells embedding theorem*, Proc. Amer. Math. Soc. **15** (1964), 415–416.

[46] H. Mirkil, *Continuous translation of Hölder and Lipschitz functions*, Canadian J. Math. **12** (1960), 674–685.

[47] K. F. Ng, *On a theorem of Dixmier*, Math. Scand. **29** (1971), 279–280.

[48] D. Pellegrino, P. Rueda, and E. A. Sánchez-Pérez, *Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **110** (2016), no. 1, 285–302.

[49] A. Pietsch, *Operator ideals*, North-Holland Mathematical Library, vol. 20, North-Holland Publishing Co., Amsterdam-New York, 1980, Translated from German by the author.

[50] A. Pietsch, *The ideal of p-compact operators and its maximal hull*, Proc. Amer. Math. Soc. **142** (2014), no. 2, 519–530.

[51] W. Rudin, *Functional analysis*, , International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[52] R. A. Ryan, , Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2002.

[53] K. Saadi, *On the composition ideals of Lipschitz mappings*, Banach J. Math. Anal. **11** (2017), no. 4, 825–840.

[54] E. A. Sánchez Pérez, *Product spaces generated by bilinear maps and duality*, Czechoslovak Math. J. **65(140)** (2015), no. 3, 801–817.

[55] D. P. Sinha and A. K. Karn, *Compact operators which factor through subspaces of* l_p, Math. Nachr. **281** (2008), no. 3, 412–423.

[56] N. Weaver, *Lipschitz algebras*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018, Second edition of [MR1832645].

[57] N. Weaver, *Lipschitz algebras*, second ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

[58] N. Weaver, *On the unique predual problem for Lipschitz spaces*, Math. Proc. Cambridge Philos. Soc. **165** (2018), no. 3, 467–473.