# Scattering Amplitudes and Asymptotic Symmetries

*By*

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## List of Publications arising from the thesis

**Journals**

1. ‘‘*MHV gluon scattering in the massive scalar background and celestial OPE*’’, Shamik Banerjee, **Raju Mandal**, Akavoor Manu and Partha Paul, JHEP 10 (2023) 007, [arXiv: 2302.10245 [hep-th]].

2. ‘‘*All OPEs invariant under the infinite symmetry algebra for gluons on the celestial sphere*’’, Shamik Banerjee, **Raju Mandal**, Sagnik Misra, Sudhakar Panda and Partha Paul, Phys.Rev.D 110 (2024) 2, 026020, [arXiv: 2311.16796 [hep-th]].

3. ‘‘*Singularity structure of the four point celestial leaf amplitudes*’’, **Raju Mandal**, Sagnik Misra, Partha Paul and Baishali Roy, JHEP 02 (2025) 152, [arXiv: 2410.13969 [hep-th]].

4. ‘‘*Holographic symmetry algebra for the MHV sector revisited*’’, Shamik Banerjee, Mousumi Maitra, **Raju Mandal** and Milan Patra, JHEP 12 (2025) 175, [arXiv:2508.02098 [hep-th]].

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## DEDICATIONS

***Dedicated to My Parents...***

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# CONTENTS

# ABSTRACT

The success of the AdS/CFT correspondence has motivated efforts to extend the holographic principle to more realistic spacetimes, particularly those that are asymptotically flat. Celestial holography has emerged as a leading framework in this pursuit. It aims to recast the S-matrix elements of gravity and gauge theories in $(3+1)$-dimensional asymptotically flat spacetimes as correlation functions of a two-dimensional celestial conformal field theory (CCFT) living on the celestial sphere at null infinity.

Most progress in celestial holography has focused on the scattering of massless particles, where celestial amplitudes are obtained by Mellin transforming momentum-space S-matrix elements. These amplitudes transform covariantly under global conformal transformations, yet they differ in key ways from conventional 2D CFT correlators. Notably, CCFTs feature infinite-dimensional current algebra symmetries, which are directly connected to the soft factorization properties of the S-matrix in gauge theory and gravity. These symmetries have no counterpart in standard 2D CFTs and impose strong constraints on celestial amplitudes.

A striking manifestation of these constraints is the emergence of null decoupling equations for celestial MHV (Maximally Helicity Violating) amplitudes in trivial backgrounds. While these equations have been solved in some specific cases, this thesis demonstrates that they do not have unique solutions. In particular, we show that pure Yang-Mills theory chirally coupled to a massive scalar background also satisfies the same set of null decoupling equations.

Recent advances have revealed that gluon scattering amplitudes are governed by the $S$ algebra, while graviton amplitudes are controlled by the wedge subalgebra of $w_{1+\infty}$, both generated by positive helicity soft particles. This thesis extends the study of the $S$ algebra by deriving the general structure of $S$-invariant operator product expansions (OPEs) up to $\mathcal{O}(1)$ and identifying Knizhnik-Zamolodchikov (KZ)-type null states associated with these symmetries. Our results point to the existence of infinitely many unexplored sectors of pure Yang-Mills theory in flat space, with MHV and self-dual Yang-Mills (SDYM) being the only currently known examples.

Furthermore, this thesis addresses a gap in the literature by identifying KZ-type null states for negative helicity gluons and gravitons in celestial MHV amplitudes.

In a recent development, it was proposed that celestial amplitudes in Klein space can be decomposed into more fundamental building blocks known as celestial leaf amplitudes, which are smooth functions on the celestial sphere. The full distributional nature of celestial amplitudes is then recovered by summing over timelike and spacelike leaf amplitudes. We extend this line of inquiry by investigating the singularity structure of four-point celestial leaf amplitudes for both MHV gluons and massless scalars.

# LIST OF FIGURES

# Chapter 1

# Introduction

Einstein's general theory of relativity, formulated as a classical field theory, provides an outstanding description of gravity in terms of spacetime geometry. However, developing a consistent theory that incorporates gravity within a quantum framework remains a long-standing challenge in theoretical physics. Various approaches have been explored, with string theory being a leading candidate for achieving a unified framework that encompasses quantum gravity along with the other fundamental forces. An alternative and conceptually rich approach is provided by the holographic principle, originally proposed by Gerard 't Hooft [1] and further developed by Leonard Susskind [2]. This principle was motivated by the observation that Bekenstein-Hawking entropy of black holes scales as area of the black hole horizon rather than volume. The holographic principle proposes an equivalence between a quantum theory of gravity in a higher-dimensional bulk spacetime and a lower-dimensional, non-gravitational quantum field theory living on the boundary of the spacetime. The precise nature of the boundary theory depends on asymptotic structure of the spacetime under consideration. For example, if the spacetime is the anti-de Sitter (AdS) then the boundary is timelike, whereas if the spacetime is asymptotically flat then the boundary is a null hypersurface. The best-established realization of this principle is the $AdS/CFT$ correspondence [3], which establishes a duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang–Mills (SYM) theory on the four dimensional boundary. The remarkable success of $AdS/CFT$ has motivated efforts to explore whether holographic principles can be extended beyond anti-de Sitter space, particularly to asymptotically flat spacetimes (AFS), which are more relevant to our universe. Asymptotically flat spacetime provides a very good model for a wide range of phenomena, from the collider physics (via the S-matrix) to large-scale astrophysical phenomena well below cosmological scales. AFS are particularly interesting due to their rich asymptotic symmetry structure first uncovered by Bondi, Burg, Metzner, and Sachs [4, 5]. These symmetries are related to observable effects such as the gravitational memory effect [6]. There has been renewed interest in the asymptotic symmetries of AFS following the developments of [7–10], which conjectured that in a quantum theory of gravity in four-dimensional asymptotically flat spacetime, the global Lorentz symmetry may be enhanced to an infinite-dimensional Virasoro algebra. This insight has led to two independent holographic approaches to flat space quantum gravity: Celestial Holography and Carrollian Holography.

Celestial holography [11–29] proposes a duality between a quantum theory of gravity in a $(d+2)$-dimensional asymptotically flat spacetime and a celestial conformal field theory (CCFT) living on the $d$-dimensional celestial sphere at null infinity. Unlike the $AdS/CFT$ correspondence, the boundary theory in Celestial holography lives on a co-dimension two surface. This is supported by the fact that the Lorentz group $SO(1, d+1)$ acts as the group of global conformal transformations on the celestial sphere [30]. In this framework, celestial amplitudes, which are obtained by expressing momentum space S matrix elements in a boost eigenbasis, serve as the fundamental observables. The celestial bases for massless and massive particles were constructed in [11, 12]. This choice of basis naturally highlights the conformal structure of scattering amplitudes [11, 12, 20], while obscuring the bulk spacetime translation symmetry, which no longer appears explicitly. In the standard construction, the scaling dimensions of celestial primary operators take continuous values, $\Delta = 1 + i\mathbb{R}$, corresponding to the principal series; this follows from the delta-function normalization of boost eigenstates. More recently, an alternative complete basis for celestial CFT with discrete integer scaling dimensions was constructed in [29], providing a complementary perspective on the operator spectrum.

In this thesis, we investigate the scattering of massless particles in (3+1)-dimensional flat spacetime and its holographic description in terms of dual celestial conformal field theories (CCFTs) defined on the two-dimensional celestial sphere at null infinity.

Although celestial correlation functions transform covariantly under conformal transformations, they exhibit several distinctive features. In particular, low-point correlators possess distributional supports on the celestial sphere, arising from the momentum-conserving delta functions of the bulk amplitudes [13]. The analytic structure of celestial amplitudes has been explored in [31]. Symmetries play a central role in celestial holography. This framework has proven very convenient for analyzing infrared structures of the flat-space S-matrix by recasting soft factorization theorems in gauge theory and gravity as the Ward identities of asymptotic symmetries.

In the boost eigenbasis, the conventional notion of an energetically soft limit is obscured. Nevertheless, soft theorems can be recovered by analytically continuing the conformal scaling dimension $\Delta$ away from the principal series, a procedure known as the conformally soft limit [32–38] in the celestial dictionary. Conformally soft theorems reformulate momentum-space soft theorems as Ward identities of global symmetries in celestial CFT, leading to infinite-dimensional current algebras that govern the scattering amplitudes. These symmetries have no direct analogue in conventional two-dimensional CFTs.

For example, the symmetry algebra implied by the leading and subleading soft graviton theorems is the semidirect product of supertranslations and $\widehat{sl}_2$ current algebra [21]. This structure was later shown to extend to the wedge subalgebra of $w_{1+\infty}$ [23, 24], associated with the infinite tower of tree-level soft factorization theorems [39]. A candidate stress tensor [40] in celestial CFT was constructed using the shadow transform [41] of the subleading soft graviton operator [42, 43]. Obstructions to constructing the $TT$ ope from double soft limit, and possible modifications, were discussed in [44, 45].

Another key factorization property of the flat-space S-matrix is collinear factorization: when two bulk momenta become parallel, the amplitude factorizes into a lower-point amplitude multiplied by a universal splitting function [46]. On the celestial sphere, this corresponds to the operator product expansion (OPE) of two celestial primary operators. The celestial OPE can be derived directly from the collinear limit of celestial amplitudes [34, 47], and its structure can also be fixed using asymptotic symmetries [47]. In particular, the singular terms in the OPE of conformally soft operators encode the underlying soft symmetry algebra. The symmetries generated by these soft operators are genuine symmetries of the flat-space S-matrix, as they are associated with nontrivial conserved charges.

At tree level, graviton scattering amplitudes are governed by the wedge subalgebra of $w_{1+\infty}$, while gluon amplitudes are controlled by an infinite-dimensional $S$ algebra [23, 24]. However, these algebras are generated solely by positive-helicity soft modes. Generic graviton and gluon amplitudes are therefore expected to exhibit a richer symmetry structure. Determining the complete nontrivial symmetry algebra of generic tree-level amplitudes remains an open problem, and the symmetry principles governing loop-level amplitudes are even less well understood.

The representation of these symmetry algebras discussed above contains the null states which play an important role in the study of celestial CFT. These null states have proven very useful in classifying celestial CFTs [25, 48] and in the computation of specific class of scattering amplitudes using asymptotic symmetries alone [21, 22, 49, 50]. With this, we conclude the introductory discussion of celestial holography and now turn to an alternative approach to flat-space holography.

**Carrollian Holography:** An alternative approach to flat-space holography in asymptotically flat spacetimes is provided by Carrollian holography. In this framework, gravity in asymptotically flat spacetimes (AFS), endowed with BMS symmetry, is reformulated in terms of a Carrollian conformal field theory ($\mathfrak{C}$CFT) living on the null boundary of spacetime. Unlike celestial CFTs, where the theory resides on the celestial sphere, the boundary theory in Carrollian holography is defined on a codimension-one null hypersurface. This proposal is rooted in the BMS symmetries [4, 5] of the bulk asymptotically flat spacetimes (AFS). The BMS algebra, $\mathfrak{bms}_{d+1}$ of a $(d+1)$-dimensional AFS, has been shown to be isomorphic to the $d$-dimensional Carrollian conformal algebra ($\mathfrak{CCarr}_d$) [51, 52], which forms the algebraic backbone of this holographic correspondence. The appearance of Carrollian symmetries from the Inönü-Wigner contraction of Poincaré algebra was first demonstrated in [53, 54]. Boundary Carrollian conformal field theories can be obtained via a Carrollian contraction of relativistic CFTs in the ultra-relativistic limit, where the speed of light tends to zero, $c \to 0$. This framework has led to several notable results, particularly in the context of three- dimensional bulk spacetimes with two-dimensional boundaries, including computations of entanglement entropy [55–57], stress tensors [58], and entropy matching [59–61]. Beyond holography, Carrollian symmetries have found applications across diverse areas of physics, ranging from condensed matter systems such as fractons [62] and flat-band systems [63] to ultra-relativistic hydrodynamics relevant for the quark-gluon plasma [64, 65]. A comprehensive review of recent developments in carrollian holography can be found in [66]. Moreover, a well-established connection between the carrollian and celestial approaches has been elucidated in [67].

This thesis presents a comprehensive analysis of the soft symmetry structures governing scattering amplitudes in flat-space gravity and gauge theories within the framework of celestial holography, with particular emphasis on the role of null states in the associated symmetry algebras.

The thesis is organized as follows:

Section 1.1 reviews the essential background material. In Section 1.1.1, we begin with the Penrose diagram of $(3+1)$-dimensional Minkowski spacetime and introduce the notions of null infinity and the celestial sphere. Section 1.1.2 constructs the celestial basis for scattering processes involving both massless and massive particles. We then review maximally helicity-violating (MHV) amplitudes in momentum space and their celestial counterparts in Section 1.1.3, which play a central role throughout this work. In Section 1.1.4, we discuss soft theorems in both the momentum and boost eigenbases, highlighting their reformulation in terms of asymptotic symmetries in the boost basis. Section 1.1.5 provides a brief discussion of collinear factorization and the celestial operator product expansion (OPE). A concise introduction to the soft symmetry algebras governing gluon and graviton amplitudes is presented in Section 1.1.6. In Section 1.1.7, we review an example of a two-dimensional CFT endowed with a non-abelian Kac-Moody symmetry and explain how the Knizhnik-Zamolodchikov (KZ) equations arise in such theories. Finally, Section 1.1.8 focuses on null states and their crucial role in the computation of celestial amplitudes, thereby concluding the introductory part of the thesis.

Chapter 2 investigates MHV gluon scattering in the presence of a massive scalar background and analyzes the corresponding celestial OPE structure. In Chapter 3, we construct all $S$-invariant OPEs on the celestial sphere. Chapter 4 reviews the construction of leaf amplitudes and studies the singularity structure of four-point scattering amplitudes for massless scalars and gluons. Finally, in Chapter 5, we revisit the holographic symmetry algebra in the MHV sector for both gluons and gravitons, clarifying its structure and implications.

## 1.1 Background material

### 1.1.1 Penrose diagram of Minkowski spacetime

The Penrose-Carter diagram is a useful tool for analyzing the asymptotic structure and causal properties of spacetimes. In this section, we present the Penrose diagram of $(3{+}1)D$ Minkowski spacetime $(M, g)$, this representation allows us to bring infinity to a finite distance via a conformal compactification. In this construction, we introduce important concepts such as null infinity and the celestial sphere, which play a crucial role in the study of scattering amplitudes of massless particles in the context of flat space holography. We begin with the metric on $(3+1)$D Minkowski spacetime in spherical polar coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \;; \qquad r \geq 0 \tag{1.1.1}$$

where

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{1.1.2}$$

Introduce the null coordinates $u$ and $v$,

$$\begin{aligned} u &= t - r \\ v &= t + r \end{aligned} \tag{1.1.3}$$

where

$$(u, v) \in (-\infty, \infty) \;\text{ and }\; u \leq v. \tag{1.1.4}$$

The metric in $(u, v, \theta, \phi)$ coordinates becomes,

$$ds^2 = -dudv + \left(\frac{u-v}{2}\right)^2 d\Omega_2^2 \tag{1.1.5}$$

To make the coordinate range finite we do the following coordinate transformations

$$\begin{aligned} u &= \tan U \\ v &= \tan V \end{aligned} \tag{1.1.6}$$

where

$$(U, V) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \;\text{ and }\; U \leq V. \tag{1.1.7}$$

In this coordinate system, the metric reads

$$\begin{aligned} ds^2 &= \frac{1}{4\cos^2 U \cos^2 V}\bigg( -4dUdV + \sin^2(V-U)\bigg) d\Omega_2^2 \\ &= \Lambda^{-2}(U,V)\bigg( -4dUdV + \sin^2(V-U)\bigg) d\Omega_2^2 \end{aligned} \tag{1.1.8}$$

where $\Lambda^2 = 4\cos^2 U \cos^2 V$. The above metric diverges as we go towards the boundary of Minkowski spacetime i.e in the limit $U, V \to \pm\frac{\pi}{2}$. Now we define a conformally rescaled metric to bring these points to finite affine parameters

$$d\tilde{s}^2 = \Lambda^2 ds^2. \tag{1.1.9}$$

which remains finite at the boundaries and preserves the causal structure of Minkowski spacetime, though not its distances. This process is known as *conformal compactification*. We can now extend the spacetime by including boundary points at infinity $(U, V) = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$, subject to $U \leq V$. See the figure 1.2.

To see this, it is convenient to introduce new timelike and radial coordinates, $\tau$ and $\chi$ defined by

$$\begin{aligned} \tau &= U + V \quad \in (-\pi, \pi) \\ \chi &= V - U \quad \in (0, \pi) \end{aligned} \tag{1.1.10}$$

where $\chi \geq 0$ since $V \geq U$.

The Minkowski metric in these coordinates becomes

$$ds^2 = \frac{1}{4\cos^2(\frac{\chi-\tau}{2})\cos^2(\frac{\chi+\tau}{2})}\Big( -d\tau^2 + d\chi^2 + \sin^2\chi d\Omega_2^2\Big) = \Omega^{-2} d\tilde{s}^2 \tag{1.1.11}$$

where $\Omega = (\cos\tau + \cos\chi)$.

We define a rescaled metric $d\tilde{s}^2$ by dropping out the conformal factor

$$d\tilde{s}^2 = \Big( - d\tau^2 + d\chi^2 + \sin^2\chi d\Omega_2^2 \Big). \tag{1.1.12}$$

The Minkowski metric is conformal to a patch of Einstein static universe, $\mathbb{R}_T \times S^3$ as shown in figure 1.1.

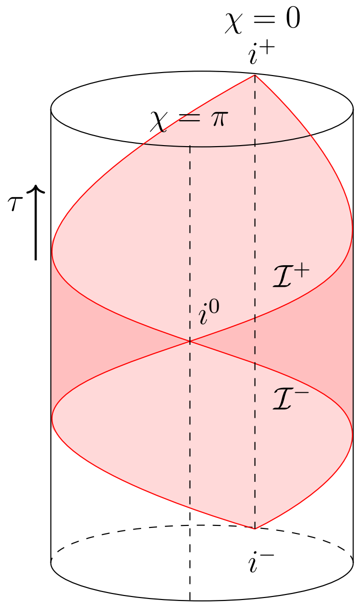


Figure 1.1: Penrose diagram of Minkowski spacetime as a patch of Einstein static universe.

The boundaries of the Minkowski spacetime $(M, g)$ are at $\Omega|_{\partial M} = 0$. We discuss these asymptotic regions below:

**Timelike infinity** $(i^{\pm})$: We denote the *past timelike infinity* by $i^-$ which is located at $(\chi, \tau) = (0, -\pi)$ and the *future timelike infinity* is denoted by $i^+$ located at $(\chi, \tau) = (0, \pi)$. Massive particels with timelike geodesics start at *past timelike infinity* $(i^-)$ and end at *future timelike infinity* $(i^+)$.

**Null infinity** $(\mathcal{I}^{\pm})$: We denote the *past null infinity* by $\mathcal{I}^-$ and it is parameterized by $U = -\frac{\pi}{2}$ and $V \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The *future null infinity* is denoted by $\mathcal{I}^+$ parameterized by $V = \frac{\pi}{2}$ and $U \in (-\frac{\pi}{2}, \frac{\pi}{2})$. In terms of $(\tau, \chi)$ coordinates $\mathcal{I}^-$ is given by $R \in (0, \pi)$ and with $\tau = -\pi + R$ and $\mathcal{I}^+$ is parameterized by $\tau = \pi - R$. Massless particles with null geodesics start at *past null infinity* ($\mathcal{I}^-$) and end up at *future null infinity* ($\mathcal{I}^+$).

**Spacelike infinity**($i^0$): All the spacelike geodesics begin and end at *spacelike infinity*($i^0$), parameterized by $(\chi, \tau) = (\pi, 0)$.

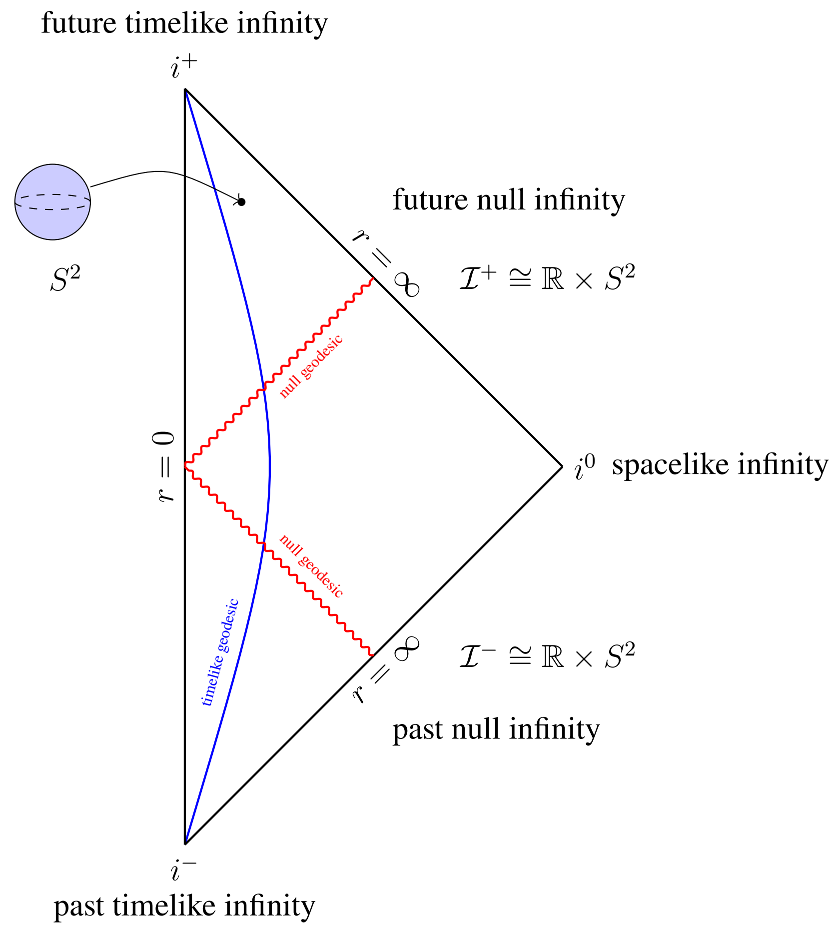


Figure 1.2: Penrose diagram of Minkowski spacetime. Timelike geodesics begin at $i^-$ and end at $i^+$ and null geodesics start at $\mathcal{I}^-$ and end at $\mathcal{I}^+$. Each point on this diagram represents a 2-sphere ($S^2$).

The surfaces $\mathcal{I}^{\pm}$ are null hypersurfaces with topology $\mathbb{R} \times S^2$. The 2-sphere at null infinity is known as the celestial sphere $(CS^2)$, which serves as the stage for celestial conformal field theories (CCFTs).

### 1.1.2 Celestial amplitudes

#### 1.1.2.1 Massless particles

In this section, we construct the clestial basis for massless particles and define the celestial amplitudes for massless scattering processes.

We parameterize the null momentum in $(-,+,+,+)$ signature as

$$p_k^\mu = \epsilon_k \omega_k (1 + z_k \bar{z}_k, z_k + \bar{z}_k, -i(z_k - \bar{z}_k), 1 - z_k \bar{z}_k), \;\; p_k^2 = 0 \tag{1.1.13}$$

where $\epsilon_k$ is $+1$ for outgoing particles and $-1$ for incoming particles. In momentum space, one particle states are represented by $|\omega, \sigma, z, \bar{z}\rangle$, where $\sigma$ denotes the helicty of the massless particle and a generic $n$-point S matrix element is given by

$$\langle \text{out}|S|\text{in}\rangle = \big\langle \{q_i^\mu, \alpha_i\}_{i=m+1,\cdots,n} |S| \{p_j^\mu, \alpha_j\}_{j=1,\cdots,m} \big\rangle \tag{1.1.14}$$

where $p_j^\mu$s are the momenta of incoming particles and $q_j^\mu$s are the momenta of outgoing particles and, $\{\alpha_j\}$ collectively denotes other internal quantum numbers.

For massless particles, one particle states in boost eigenbasis [11–13, 20] can be constructed via a Mellin trasform of the momentum space state

$$|\Delta, \sigma, z, \bar{z}\rangle = \int_0^\infty d\omega \omega^{\Delta - 1} \, |\omega, \sigma, z, \bar{z}\rangle \tag{1.1.15}$$

where $\Delta = 1 + i\lambda$ with $\lambda \in \mathbb{R}$, which ensures the following normalization

$$\langle \lambda_1, \sigma_1, z_1, \bar{z}_1 | \lambda_2, \sigma_2, z_2, \bar{z}_2 \rangle = \delta(\lambda_1 - \lambda_2)\delta^2(z_1 - z_2)\delta_{\sigma_1,\sigma_2} \tag{1.1.16}$$

and $\sigma$ is the helicity of the massless particle.

Celestial amplitude for $n$-gluon scattering amplitude can be obtained by performing the Mellin transform of $n$-particle momentum space S-matrix elements [11, 12],

$$\mathcal{A}_n(\{z_i, \bar{z}_i, h_i, \bar{h}_i, a_i\}) = \langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i, \bar{z}_i) \rangle = \prod_{i=1}^{n} \int_0^\infty d\omega_i \, \omega_i^{\Delta_i - 1} \mathcal{S}_n(\{\omega_i, z_i, \bar{z}_i, \sigma_i, a_i\}) \tag{1.1.17}$$

Where $a_i$ is the Lie algebra index of the $i$-th particle and scaling dimensions $(h_i, \bar{h}_i)$ are defined as

$$h_i = \frac{\Delta_i + \sigma_i}{2}, \quad \bar{h}_i = \frac{\Delta_i - \sigma_i}{2}. \tag{1.1.18}$$

The $(3 + 1)$D Lorentz group $SO(3, 1)$ acts as global conformal group $SL(2, \mathbb{C})$ on the celestial sphere and the celestial amplitudes $\mathcal{A}_n$ obtained in (1.1.17) transform as the correlation functions of primary operators of conformal weights $(h_i, \bar{h}_i)$ of a 2D CFT [11, 12, 20, 28, 29],

$$\mathcal{A}_n(\{z_i, \bar{z}_i, h_i, \bar{h}_i, a_i\}) = \prod_{i=1}^{n} \frac{1}{(cz_i + d)^{2h_i}} \frac{1}{(\bar{c}\bar{z}_i + \bar{d})^{2\bar{h}_i}} \mathcal{A}_n\left(\left\{\frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}, h_i, \bar{h}_i, a_i\right\}\right). \tag{1.1.19}$$

To make the conformal nature of celestial amplitudes manifest, it is useful to write them as correlation functions of conformal primary operators. We define the gluon conformal primary operator [22] on the celestial sphere as the Mellin transform of a momentum space annihilation or creation operator of a gluon with helicity $\sigma$ and Lie algebra index $a$

$$\mathcal{O}^{a,\epsilon}_{h,\bar{h}}(z, \bar{z}) = \int_0^\infty d\omega \omega^{\Delta - 1} A^a(\epsilon\omega, z, \bar{z}, \sigma) \tag{1.1.20}$$

where $\epsilon = \pm$. Under global conformal transformations, the operator defined by (1.1.20) transforms as the primary operator of a 2D CFT with conformal weights $(h, \bar{h})$

$$\mathcal{O}'^{a,\epsilon}_{h,\bar{h}}(z,\bar{z}) = (cz+d)^{-2h}(\bar{c}\bar{z}+\bar{d})^{-2\bar{h}}\mathcal{O}^{a,\epsilon}_{h,\bar{h}}\left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right). \tag{1.1.21}$$

We can now write the celestial amplitude (1.1.17) as the correlation function of the primary operators defined in (1.1.20) as

$$\mathcal{A}_n(\{z_i, \bar{z}_i, h_i, \bar{h}_i, a_i\}) = \left\langle \prod_{i=1}^{n} \mathcal{O}^{a,\epsilon}_{h,\bar{h}}(z_i, \bar{z}_i) \right\rangle. \tag{1.1.22}$$

Although celestial correlators exhibit global conformal symmetry, they differ from traditional 2D CFT correlators. For instance, lower point correlation functions typically vanish on the celestial sphere except at specific distributional supports [13], a direct consequence of bulk spacetime translation symmetry of the S-matrix elements. Below we present schematically how this distributional nature appears in the lower point celestial amplitudes. For example, let's consider Mellin transform of the color-stripped 4-point tree-level gluon amplitude given by the Parke-Taylor formula (boxed term in (1.1.23) below),

$$\begin{aligned}
&\tilde{\mathcal{A}}(1^-2^-3^+4^+) \\
&= \prod_{j=1}^{4}\int_0^\infty d\omega_j\ \omega_j^{\Delta_j-1}\boxed{\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\delta^{(4)}\left(\sum_{i=1}^{4} p_i^\mu\right)} \\
&= \prod_{j=1}^{4}\int_0^\infty d\omega_j\ \omega_j^{\Delta_j-1}\mathcal{F}(\{\omega_i, z_i, \bar{z}_i\})\delta(\omega_1-\omega_1^\star)\delta(\omega_2-\omega_2^\star)\delta(\omega_3-\omega_3^\star)\delta(z_{12}z_{34}\bar{z}_{13}\bar{z}_{24}-z_{13}z_{24}\bar{z}_{12}\bar{z}_{34})
\end{aligned} \tag{1.1.23}$$

where $\mathcal{F}$ is a smooth function of $(\{\omega_i, z_i, \bar{z}_i\})$. The momentum conserving delta function gives four constraints allowing us to solve for maximum three $\omega$'s for a four-point amplitude. After performing Mellin transforms, $\delta(z_{12}z_{34}\bar{z}_{13}\bar{z}_{24} - z_{13}z_{24}\bar{z}_{12}\bar{z}_{34})$ remains which make the amplitude distributional on the celestial sphere. Similarly, the two-point and three-point celestial correlation functions are also distributional. Beyond four point this does not occur because the number external states exceeds the number of constraints. Moreover, CCFTs are enriched with the various infinite-dimensional current algebra symmetries [21–24], which are associated with the soft factorization theorems of the bulk S-matrix. These additional symmetries do not have any analogs in conventional 2D CFT.

#### 1.1.2.2 Massive particles

In the previous section, we discussed celestial amplitudes for massless scattering processes. We now briefly review the construction of celestial amplitudes for massive scattering processes.

On-shell massive momenta are parameterized on the hyperbolic slice $H_3$ of Minkowski space. For an outgoing massive particle, one may choose the following embedding map $\hat{p}^\mu : H_3 \to \mathbb{R}^{1,3}$ describing the upper hyperboloid:

$$\hat{p}^\mu = \frac{1}{2y}\left(1 + y^2 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - y^2 - z\bar{z}\right) \tag{1.1.24}$$

which satisfies $\hat{p}^2 = -1$, Here $(y, z, \bar{z})$ are Poincaré coordinates on $H_3$, equipped with the metric,

$$ds^2_{H_3} = \frac{dy^2 + dzd\bar{z}}{y^2}, \qquad y^2 > 0. \tag{1.1.25}$$

The action of $SL(2,\mathbb{C})$ on these coordinates is given by,

$$\begin{aligned} z \to z' &= \frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}y^2}{|cz+d|^2 + |c|^2y^2}, \\ \bar{z} \to \bar{z}' &= \frac{(\bar{a}\bar{z}+\bar{b})(cz+d) + \bar{a}cy^2}{|cz+d|^2 + |c|^2y^2}, \\ y \to y' &= \frac{y}{|cz+d|^2 + |c|^2y^2} \end{aligned} \tag{1.1.26}$$

where, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$.

The conformal boundary of $H_3$ space is located at $y = 0$, where the hyperbolic slice asymptotes to celestial sphere.

Given a momentum-space $n$-point amplitude $\mathcal{M}_n(m_i\hat{p}_i)$ involving massive scalars with momenta $p_i^\mu = \pm m\hat{p}^\mu(y_i, z_i, \bar{z}_i)$, the corresponding celestial amplitude is defined by the transform [11]

$$\begin{aligned} &\widetilde{\mathcal{M}}_n(\Delta_i; w_i, \bar{w}_i) \\ &= \prod_{k=1}^{n} \int_{H_3} \frac{d^3\hat{p}_k}{\hat{p}_k^0} G_{\Delta_k}(\hat{p}_k^\mu(y_k, z_k, \bar{z}_k); q_k(w_k, \bar{w}_k))\mathcal{M}_n(m_i\hat{p}_i) \end{aligned} \tag{1.1.27}$$

where $G_\Delta(\hat{p}_k^\mu(y,z,\bar{z}); q(w,\bar{w}))$ is the scalar bulk-to-boundary propagator given by

$$G_\Delta(\hat{p}_k^\mu(y,z,\bar{z}); q(w,\bar{w})) = \left(\frac{y}{y^2 + |z-w|^2}\right)^\Delta . \tag{1.1.28}$$

The extension of this construction to massive spinning particles has been discussed in [11, 68–70].

### 1.1.3 Maximally Helicity Violating (MHV) amplitude and the dual celestial correlation function

Scattering amplitudes provide the natural observables of gauge theory and gravity and form the backbone of flat space holography. However, their computation using traditional Feynman-diagram techniques is highly inefficient. The number of diagrams grows rapidly with the number of external particles, and individual diagrams are not gauge invariant, even though the final amplitude must be. Consequently, the simplicity and hidden structures of the full result are obscured in a diagrammatic approach.

A dramatic simplification occurs in four spacetime dimensions, where massless particles are labeled by definite helicity, positive or negative. At tree level, Yang-Mills amplitudes display a striking helicity selection structure: many seemingly allowed processes vanish identically. For example, in the all-outgoing convention,

$$A_n\left(1^+2^+3^+4^+\cdots n^+\right) = 0. \tag{1.1.29}$$

By crossing symmetry, this process can be written as

$$1^-2^- \to 3^+4^+\cdots n^+ \tag{1.1.30}$$

where particles $(1,2)$ are treated as incoming and the rest as outgoing.
Let's consider the amplitude $A_n\left(1^+2^+3^-4^+\cdots n^+\right)$ in the all-outgoing convention. Crossing symmetry relates it to

$$1^-2^- \to 3^-4^+\cdots n^+ \tag{1.1.31}$$

Despite being less helicity-violating than the previous configuration, this amplitude also vanishes at tree level in pure Yang-Mills theory.
A qualitatively different situation arises when two gluons carry negative helicity and the remaining ones positive helicity. In the all-outgoing convention, the relevant amplitude is $A_n\left(1^+2^+3^-4^-\cdots n^+\right)$, which, by crossing symmetry, corresponds to

$$1^-2^- \to 3^-4^-\cdots n^+. \tag{1.1.32}$$

Although this configuration maximally violates helicity conservation among nonvanishing tree amplitudes, it yields a remarkably simple and nontrivial result. Such amplitudes are known as *maximally helicity violating* (MHV) amplitudes.

The tree-level color-stripped $n$-point MHV gluon amplitude is given by the celebrated

Parke-Taylor formula [71],

$$A_n\left(1^+2^+3^-4^-\cdots n^+\right) = \frac{\langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n1\rangle} \tag{1.1.33}$$

Here the gluon momenta are expressed in spinor-helicity variables (see, for example, [72] for conventions and a detailed review). The amplitudes with two positive-helicity gluons and all others negative, $A_n\left(1^-2^-3^+4^+\cdots n^-\right)$ are known as anti-MHV amplitudes and are obtained from (1.1.33) by replacing angle brackets with square brackets.

The extraordinary simplicity of the Parke-Taylor formula has inspired powerful on-shell methods for computing amplitudes. In particular, MHV gluon amplitudes can be derived recursively from three-point amplitudes using the BCFW (Britto-Cachazo-Feng-Witten) recursion relations [73], which rely solely on on-shell data and bypass the use of a Lagrangian description.

In gravity, the tree-level MHV graviton amplitude admits an equally elegant representation, given by Hodges' formula [74],

$$\mathbb{A}_n(1^-,2^-,3^+,\cdots,n^+) = \langle 12\rangle^8 \frac{\det(M^{ijk}_{pqr})}{\langle ij\rangle\langle ik\rangle\langle jk\rangle\langle pq\rangle\langle pr\rangle\langle qr\rangle} \tag{1.1.34}$$

where $M^{ijk}_{pqr}$ is a $(n-3)\times(n-3)$ matrix obtained by removing the rows $(i,j,k)$ and columns $(p,q,r)$ from a $n\times n$ matrix $M$, the elements of which are given by,

$$M_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle} & \text{if } i\neq j, \\ -\sum_{k\neq i}\frac{[ik]\langle xk\rangle\langle yk\rangle}{\langle ik\rangle\langle xi\rangle\langle yi\rangle} & \text{if } i=j \end{cases} \tag{1.1.35}$$

with $x$ and $y$ denoting reference spinors.

Tree-level MHV graviton amplitudes can alternatively be obtained by "squaring" the corresponding gauge-theory amplitudes through the double copy construction, also known as BCJ duality [75].

In this thesis, we study the celestial amplitudes associated with MHV gluon and graviton scattering processes. For example, the three-point MHV gluon amplitude in the celestial basis takes the form [22],

$$\tilde{\mathcal{A}}(1^{-a_1}, 2^{+a_2}, 4^{-x}) \sim f^{a_1 a_2 x} \frac{z_{14}^3}{z_{12}^3 z_{24}} \delta(\bar{z}_{14}) \delta(\bar{z}_{24}) \prod_{i=1}^{3} \Theta(\epsilon_i \sigma_{i,1}) \tag{1.1.36}$$

where $\sigma_{i,1}$ are the functions of coordinates $\{z_i, \bar{z}_i\}$ on the celestial sphere.

MHV amplitudes provide a powerful testing ground for exploring the structure of celestial conformal field theory (Celestial CFT), and they play a central role in the developments presented in this thesis.

### 1.1.4 Tree level soft theorems and asymptotic symmetries

Soft theorems establish universal relations between a scattering amplitude with an additional soft gauge boson and the corresponding amplitude without the soft particle. The associated factorization is universal in the sense that it is independent of the detailed dynamics of the hard particles. Instead, the soft factor depends only on the momenta and charges of the external states, as well as on the species of the soft particle. This universality strongly suggests the presence of an underlying symmetry principle governing soft behavior that has indeed been realized.

Recent developments [19, 76–78] have uncovered a remarkable connection between soft theorems and the asymptotic symmetries of the S-matrix. In particular, soft theorems can be reinterpreted as Ward identities associated with these symmetries. In this section, we briefly review tree-level soft theorems in both the momentum and boost bases, and then discuss their relation to asymptotic symmetries in the boost basis.

#### 1.1.4.1 Momentum space soft theorems

In momentum space formulation of quantum field theory, scattering amplitudes of gauge theory and gravity admit a universal factorization, where amplitudes with an additional gauge boson factorize into lower-point amplitudes in the limit of soft momentum, i.e., when the momentum of that gauge boson approaches zero. These are known as energetically soft theorems [42, 79–83], illustrated in figure 1.3.

At tree level, soft factorization theorems in momentum space can be expressed as

$$\begin{aligned} S_{N+1}\left(\{p_a,\sigma_a\},\{\epsilon^h,\omega\hat{q}\}\right) &\overset{\omega\to 0}{=} \frac{1}{\omega}\mathbb{S}_{-1}\left(\{p_a,\sigma_a\},\{\epsilon^h,\hat{q}\}\right)S_N\left(\{p_a,\sigma_a\}\right) \\ &+\mathbb{S}_0\left(\{p_a,\sigma_a\},\{\epsilon^h,\hat{q}\}\right)S_N\left(\{p_a,\sigma_a\}\right)+\mathcal{O}(q) \end{aligned} \tag{1.1.37}$$

where $S_{N+1}$ is an $(N+1)$-point amplitude with a soft gauge boson of momenta $q^\mu \sim \omega\hat{q}^\mu$ and polarization $\epsilon^h$, where $h$ denotes the helicity of soft the particle. Here $p_a$ denote the momenta of the hard particles and $\sigma_a$ collectively represent their remaining quantum numbers and the coefficient $\mathbb{S}_n$ is the *soft factor* at $\mathcal{O}(\omega^n)$. At tree level, they depend only on the polarization of the soft particle and on the momenta and charges of the hard particles; they are insensitive to the detailed interactions among the hard states.

The leading and subleading soft factors can be extracted using the following projection operators:

$$\lim_{\omega\to 0}\omega S_{N+1}\left(\{p_a,\sigma_a\};\{\epsilon^h,\omega\hat{q}\}\right)=\mathbb{S}_{-1}\left(\{p_a,\sigma_a\};\{\epsilon^h,\hat{q}\}\right)S_N\left(\{p_a,\sigma_a\}\right) \tag{1.1.38}$$

and

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)S_{N+1}\left(\{p_a,\sigma_a\};\{\epsilon^h,\omega\hat{q}\}\right)=\mathbb{S}_{0}\left(\{p_a,\sigma_a\};\{\epsilon^h,\hat{q}\}\right)S_N\left(\{p_a,\sigma_a\}\right). \tag{1.1.39}$$

We now present two explicit examples of soft theorems in momentum space: the soft gluon theorem and the soft graviton theorem at tree level.

We begin with the tree-level soft gluon theorems [84, 85]. A gluon scattering amplitude $S^a_{n+1}(p_1, \cdots, p_n; q)$ with an additional soft gluon ($q \to 0$) factorizes into lower-point amplitudes $S_n(p_1, \cdots, p_n)$ as,

$$S^a_{n+1}(p_1, \cdots, p_n; q) = \left( \sum_{k=1}^{n} \frac{\epsilon_\mu p_k^\mu}{q \cdot p_k} T^a_k + \sum_{k=1}^{n} \frac{i\epsilon_\mu q_\nu \mathcal{J}_k^{\mu\nu}}{q \cdot p_k} T^a \right) S_n(p_1, \cdots, p_n) + \mathcal{O}(q). \tag{1.1.40}$$

where $\mathcal{J}_k^{\mu\nu}$ is the total angular momentum of the $k$'th particle and $T^a_k$s are generators of the non-abelian gauge group.

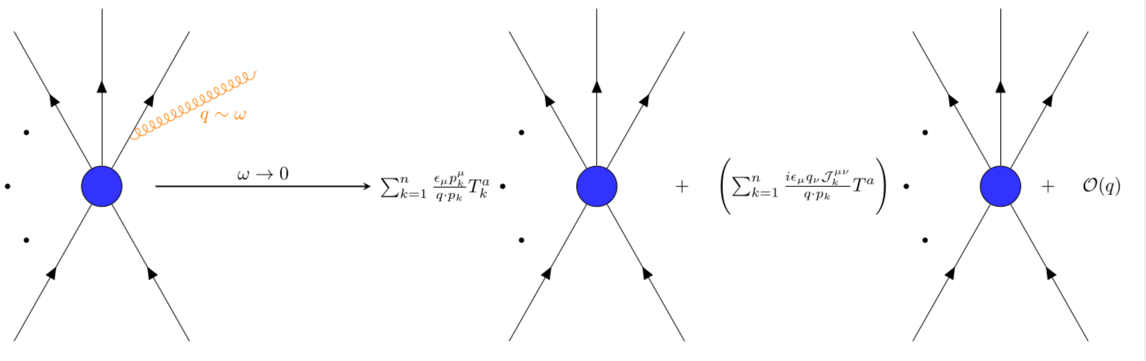


Figure 1.3: Tree level soft factorization of scattering amplitude at leading and subleading order with an additional soft gluon

Similarly tree level graviton scattering amplitude with an additional soft graviton ($q \to 0$) factorizes into lower point amplitudes as

$$S_{n+1}(p_1, \cdots, p_n; q) = \left[ \frac{1}{\omega} S^{(0)} + S^{(1)} + \omega S^{(2)} \right] S_n(p_1, \cdots, p_n) + \mathcal{O}(\omega^2) \tag{1.1.41}$$

where $q^\mu = \omega \hat{q}^\mu$ is the momentum of the soft graviton and the leading [79], subleading[42, 86–88] and sub-subleading[42] soft graviton factors are given by

$$S^{(0)} = \frac{\kappa}{2} \sum_{j=1}^{n} \frac{\epsilon^{\mu\nu}(q) p_{j\mu} p_{j\nu}}{p_j \cdot \hat{q}}, \tag{1.1.42}$$

$$S^{(1)} = -\frac{i\kappa}{2} \sum_{j=1}^{n} \frac{\epsilon_{\mu\nu}(q) p_j^\mu q_\lambda \mathcal{J}_j^{\lambda\nu}}{p_j \cdot \hat{q}}, \tag{1.1.43}$$

and

$$S^{(2)} = -\frac{\kappa}{4} \sum_{j=1}^{n} \frac{\epsilon_{\mu\nu}(q) q_\rho \mathcal{J}_j^{\rho\mu} q_\sigma \mathcal{J}_j^{\sigma\nu}}{p_j \cdot \hat{q}} \tag{1.1.44}$$

respectively. Where $\epsilon^{\mu\nu}(q)$ is the polarization tensor of the soft graviton and $\mathcal{J}_j^{\mu\nu}$ is the total angular momentum of the $j$-th particle.

An infinite set of soft theorems for tree level soft photon, gluon and graviton amplitudes was derived in [39]. At loop level, the soft theorems (1.1.37) get replaced by logarithmic soft theorems [89]. We do not discuss these results in detail in this thesis. Instead, we briefly comment on recent progress in the symmetry interpretation of logarithmic soft theorems in Chapter 6.

#### 1.1.4.2 Asymptotic symmetries

Symmetries are central to theoretical physics, and a complete classification of all the true symmetries of the S-matrix remains an open question. By true symmetries, we refer to those that act non-trivially on the Hilbert space of physical states. According to Noether's theorem, continuous global symmetries give rise to conserved currents and associated charges. Consider, a continuous symmetries generated by a set of parameters $\{\omega_a\}$. Following the Noether's procedure, one can identify the conserved currents, $j_a^\mu(x)$ from the variation of action $\mathcal{A}$,

$$\delta\mathcal{A} = -\int d^4x \, j_a^\mu \partial_\mu \omega_a(x) \tag{1.1.45}$$

and they are conserved classically, on shell:

$$\partial_\mu j_a^\mu(x)|_{\text{on-shell}} = 0. \tag{1.1.46}$$

The conserved charges are obtained by the following integral

$$Q_a = \int j_a^0(x) d^3x. \tag{1.1.47}$$

We are interested in the conserved charges associated with the gauge invariance [1]. In gauge theories, these charges often reduce to boundary terms. For example, in Maxwell theory, the gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \alpha(x) \tag{1.1.48}$$

lead to conserved charges of the form:

$$Q_\alpha = \int_{s^2_\infty} \alpha(x) \vec{E} \cdot d\vec{S} \tag{1.1.49}$$

where $S^2_\infty$ denotes the 2-sphere at infinity. So there are infinite number of conserved charges corresponding to the functions $\alpha(x)$.

From 1.1.49, it can be seen that if $\alpha(x)$ has compact support in spacetime (i.e., vanishes at infinity), the corresponding charge $Q_\alpha$ vanishes. These transformations are called *small gauge transformations*. In contrast, *large gauge transformations* are characterized by parameters that do not vanish at the boundary and therefore lead to non-zero charges.

*Asymptotic symmetries are defined as the symmetries generated by the large gauge transformations.*

In quantum theory, the charges associated with small gauge transformations act trivially on physical states:

$$\hat{Q}_\alpha^{\text{small}} = 0 \implies \hat{Q}_\alpha^{\text{small}} |\text{Phys}\rangle = 0 \tag{1.1.50}$$

[1] see Shamik Banerjee's talk on 'Asymptotic Symmetries in Gauge theory and Gravity' at IISER Kolkata

which expresses the redundancy of gauge invariance. Small gauge transformations do not generate physical symmetries. On the other hand, charges associated with large gauge transformations act non-trivially:

$$\hat{Q}_\alpha^{\text{large}} \neq 0 \implies \hat{Q}_\alpha^{\text{large}} |\text{Phys}\rangle \neq 0. \tag{1.1.51}$$

and thus correspond to genuine symmetries of the quantum theory.

The quantum statement of charge conservation is captured by the Ward identity:

$$\langle f|\hat{Q}_\alpha S - S\hat{Q}_\alpha|i\rangle = 0. \tag{1.1.52}$$

This identity imposes an infinite set of constraints on the S-matrix, corresponding to each choice of large gauge parameter $\alpha(x)$.

#### 1.1.4.3 Conformally soft factorization theorems and Ward identities of asymptotic symmetries

In this section we discuss the soft factorization theorems for gluons and gravitons in boost (Mellin) basis. In this basis, external massless particles are labeled not by energy, but by their boost weight $\Delta$ and helicity $\sigma$ or equivalently by their $SL(2,\mathbb{C})$ conformal weights $(h,\bar{h})$. In this framework, the usual momentum-space *energetically soft theorems* are replaced by *conformally soft theorems* [24, 32–38]. We begin by introducing soft operators in the boost eigenbasis and then derive the corresponding soft theorems in this representation.

**1.1.4.3.1 Conformally soft operators in boost eigenbasis:** In 1.1.2, we defined celestial primary operator $\mathcal{O}_\Delta^+(z,\bar{z})$ for a massless particle as the Mellin transform of the momentum space creation/annihilation operator $a(\pm\omega, z\bar{z})$ w.r.t the energy $\omega$ of the particle which is given by,

$$\mathcal{O}^{+}_{\Delta}(z,\bar{z}) = \int_0^\infty d\omega \omega^{\Delta-1} a(\omega,z,\bar{z}) \tag{1.1.53}$$

To extract the soft behavior, consider the regulated Mellin integral

$$\lim_{\Delta\to k}(\Delta-k)\int_0^{\omega_\star} d\omega\omega^{\Delta-1}a(\omega,z,\bar{z}) \tag{1.1.54}$$

Assuming the insertion of $a(\omega,z,\bar{z})$ inside the scattering amplitude decay sufficiently rapidly as $\omega\to\infty$, we can expand the operator around $\omega=0$ in the region $\omega\leq\omega_\star$,

$$a(\omega,z,\bar{z}) = \sum_n \omega^n H^{-n}(z,\bar{z}) \tag{1.1.55}$$

The soft factorization theorem (1.1.37) implies that, when inserted into tree-level amplitudes, the sum runs over $n=-1,0,1,\cdots,\infty$

Substituting (1.1.55) into (1.1.54), we obtain

$$\begin{aligned}
&\lim_{\Delta\to k}(\Delta-k)\int_0^{\omega_\star} d\omega\omega^{\Delta-1}a(\omega,z,\bar{z})\\
&=\lim_{\Delta\to k}\int_0^{\omega_\star} d\omega(\Delta-k)\sum_{n=-1}^{\infty}\omega^{\Delta+n-1}H^{-n}(z,\bar{z})\\
&=\lim_{\Delta\to k}\sum_{n=-1}^{\infty}(\Delta-k)\frac{\omega_\star^{\Delta+n}}{\Delta+n}H^{-n}(z,\bar{z})\\
&=H^k(z,\bar{z})
\end{aligned} \tag{1.1.56}$$

with $k=-n=1,0,-1,-2,\cdots$ These are known as *conformally soft limits* [32–38] of the primary operator in Celestial CFT. As it is seen from (1.1.56), the limit $\Delta\to k$ picks out the coefficient of $\omega^k$ in the expansion (1.1.55). The operators $H^k(z,\bar{z})$ are called **conformally soft operators**, defined as

$$H^k(z,\bar{z}) = \lim_{\Delta\to k}(\Delta-k)\mathcal{O}^{+}_{\Delta}(z,\bar{z}), \quad k=1,0,-1,-2,\cdots \tag{1.1.57}$$

These are equivalent to the soft operators in momentum basis. Inserting a conformally soft operator $H^k(z,\bar{z})$ inside the celestial amplitude will reproduce $\mathcal{O}(\omega^k)$ soft theorem in Mellin basis. We now show explicitly that these limits are equivalent to the standard momentum-space soft theorems [21]. Let us consider the following limit in the Mellin transformation of an $(N{+}1)$-point momentum space amplitude $S_{N+1}\left(\omega, z, \bar{z}, , \sigma=2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right)$ with a graviton operator parameterized by the momentum $(\omega, z, \bar{z})$ and helicity, $\sigma = 2$,

$$
\begin{aligned}
&\lim_{\Delta\to 1}(\Delta-1)\int_0^\infty d\omega\ \ \omega^{\Delta-1}S_{N+1}\left(\omega, z, \bar{z}, , \sigma=2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right)\\
&=\lim_{\Delta\to 1}(\Delta-1)\int_{-\infty}^\infty d\omega\ \ \theta(\omega)\omega^{\Delta-2}\,\omega S_{N+1}\left(\omega, z, \bar{z}, , \sigma=2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right)\\
&=\lim_{\Delta\to 1}(\Delta-1)\int_{-\infty}^\infty d\omega\ \ \left(\frac{\delta(\omega)}{\Delta-1}-\frac{\delta'(\omega)}{\Delta}+\frac{1}{2}\frac{\delta''(\omega)}{\Delta+1}+\cdots\right)\omega S_{N+1}\left(\omega, z, \bar{z}, , \sigma=2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right)
\end{aligned}
\tag{1.1.58}
$$

where we have used the distributional identity,

$$
\omega^{\Delta-2}\theta(\omega)\sim\frac{\delta(\omega)}{\Delta-1}-\frac{\delta'(\omega)}{\Delta}+\frac{1}{2}\frac{\delta''(\omega)}{\Delta+1}+\cdots \tag{1.1.59}
$$

Since the integrand is supported on $\delta(\omega)$, we can expand the amplitude $S_{N+1}$ around $\omega = 0$ in (1.1.58) using (1.1.41),

$$
\begin{aligned}
&\lim_{\Delta\to 1}(\Delta-1)\int_0^\infty d\omega\ \ \omega^{\Delta-1}S_{N+1}\left(\omega, z, \bar{z}, , \sigma=2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right)\\
&=\lim_{\Delta\to 1}(\Delta-1)\int_{-\infty}^\infty d\omega\ \ \left(\frac{\delta(\omega)}{\Delta-1}-\frac{\delta'(\omega)}{\Delta}+\frac{1}{2}\frac{\delta''(\omega)}{\Delta+1}+\cdots\right)\\
&\quad\times\omega\left(\frac{1}{\omega}\mathbb{S}_{-1}+\mathbb{S}_0+\omega\mathbb{S}_1+\cdots\right)S_N(\{\omega_i, z_i, \bar{z}_i, \sigma_i\})\\
&=\mathbb{S}_{-1}S_N(\{\omega_i, z_i, \bar{z}_i, \sigma_i\})\\
&=\lim_{\omega\to 0}\omega S_{N+1}\left(\omega, z, \bar{z}, , \sigma=2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right)
\end{aligned}
\tag{1.1.60}
$$

Hence, the limit $\Delta \to 1$ reproduces the leading soft graviton theorem in momentum space.

Similarly, if we consider the limit $\Delta \to 0$ we obtain,

$$\begin{aligned}
&\lim_{\Delta\to 0} \Delta \int_0^\infty d\omega \;\; \omega^{\Delta-1} S_{N+1}\left(\omega, z, \bar{z}, , \sigma = 2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right) \\
&= \lim_{\Delta\to 0} \Delta \int_{-\infty}^\infty d\omega \;\; \left(\frac{\delta(\omega)}{\Delta - 1} - \frac{\delta'(\omega)}{\Delta} + \frac{1}{2}\frac{\delta''(\omega)}{\Delta+1} + \cdots\right) \\
&\quad \times \omega \left(\frac{1}{\omega}\mathbb{S}_{-1} + \mathbb{S}_0 + \omega \mathbb{S}_1 + \cdots\right) S_N(\{\omega_i, z_i, \bar{z}_i, \sigma_i\}) \\
&= \mathbb{S}_0 S_N(\{\omega_i, z_i, \bar{z}_i, \sigma_i\}) \\
&= \lim_{\omega\to 0}(1 + \omega\partial_\omega) S_{N+1}\left(\omega, z, \bar{z}, , \sigma = 2; \{\omega_i, z_i, \bar{z}_i, \sigma_i\}\right).
\end{aligned} \tag{1.1.61}$$

which is equivalent to the subleading soft graviton theorems in momentum space.

We therefore conclude that conformally soft limits in the Mellin basis are precisely equivalent to the standard momentum-space soft theorems. In the following, we discuss the tree-level soft theorems for gluons and gravitons in detail, together with the associated asymptotic symmetries and their Ward identities.

**1.1.4.3.2 Conformally soft gluon theorems:** In celestial holography, the conformally soft [32–38] positive helicity gluon operators [23, 24] are defined as

$$R^{k,a}(z, \bar{z}) := \lim_{\Delta\to k}(\Delta - k)\mathcal{O}^a_{\Delta,+}(z, \bar{z}), \quad k = 1, 0, -1, -2, -3, \cdots \tag{1.1.62}$$

where $\mathcal{O}^a_{\Delta,+}(z, \bar{z})$ denotes a positive helicity gluon conformal primary operator of conformal dimension $\Delta$ on the celestial sphere at the point $(z, \bar{z})$ and $a$ is the color index.

In this setup, soft theorems are recast as Ward identities of the asymptotic symmetries of celestial amplitudes[21, 22].

The tree-level leading conformally soft gluon theorem for positive helicity gluons takes the form:

$$\left\langle R^{1,a}(z)\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = -\sum_{k=1}^{n}\frac{T^a_k}{z-z_k}\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle \tag{1.1.63}$$

where the leading conformally soft positive helicity gluon operator is defined as,

$$R^{1,a}(z) = \lim_{\Delta\to 1}(\Delta-1)\mathcal{O}^a_{\Delta,+}(z,\bar{z}) \tag{1.1.64}$$

and $T^a$s are generators of the non-abelian gauge group, acting on the gluon primary operators as

$$T^a_k\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = if^{aa_ix}\mathcal{O}^{x}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\delta_{ik}. \tag{1.1.65}$$

The modes $R^{1,a}_n$ of the above current satisfy the following level zero Kac-Moody algebra

$$[R^{1,a}_m, R^{1,b}_n] = -if^{abc}R^{1,c}_{m+n}. \tag{1.1.66}$$

Similarly, a subleading conformally soft positive helicity gluon operator is defined as

$$R^{0,a}(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}^a_{\Delta,+}(z,\bar{z}) \tag{1.1.67}$$

and the subleading conformally soft gluon factorization theorem takes the following form in the celestial basis

$$\begin{aligned}&\left\langle R^{0,a}(z,\bar{z})\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle\\ &= -\sum_{k=1}^{n}\frac{\epsilon_k}{z-z_k}(-2\bar{h}_k+1+(\bar{z}-\bar{z}_k)\bar{\partial}_k)T^a_kP^{-1}_k\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle\end{aligned} \tag{1.1.68}$$

where the operator $P^{-1}_k$ acts on the gluon as,

$$P^{-1}_k\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = \mathcal{O}^{a_i}_{h_i-\frac{1}{2},\bar{h}_i-\frac{1}{2}}(z_i,\bar{z}_i)\delta_{ki}. \tag{1.1.69}$$

Now we expand $R^{0,a}(z,\bar{z})$ in powers of $\bar{z}$ and define the two currents $R^{0,a}_{\frac{1}{2}}(z)$ and $R^{0,a}_{-\frac{1}{2}}(z)$ as

$$R^{0,a}(z,\bar{z}) = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \frac{R^{0,a}_m(z)}{\bar{z}^{m-1/2}} = \bar{z}R^{0,a}_{-\frac{1}{2}}(z) + R^{0,a}_{\frac{1}{2}}(z) \tag{1.1.70}$$

Substituting this into (1.1.68), we obtain the Ward identities for these currents

$$\left\langle R^{0,a}_{\frac{1}{2}}(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = -\sum_{k=1}^{n} \frac{\epsilon_k}{z-z_k}(-2\bar{h}_k+1-\bar{z}_k\bar{\partial}_k)T^a_k P^{-1}_k \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle \tag{1.1.71}$$

and

$$\left\langle R^{0,a}_{-\frac{1}{2}}(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = -\sum_{k=1}^{n} \frac{\epsilon_k}{z-z_k}\bar{\partial}_k T^a_k P^{-1}_k \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle . \tag{1.1.72}$$

We can further mode expand these currents as

$$R^{0,a}_{\frac{1}{2}}(z) = \sum_{\alpha\in\mathbb{Z}-1/2} \frac{R^{0,a}_{\alpha,\frac{1}{2}}}{z^{\alpha+1/2}}, \quad R^{0,a}_{-\frac{1}{2}}(z) = \sum_{\beta\in\mathbb{Z}-1/2} \frac{R^{0,a}_{\beta,-\frac{1}{2}}}{z^{\beta+1/2}} \tag{1.1.73}$$

But the modes $\{R^{0,a}_{\alpha,\frac{1}{2}}, R^{0,a}_{\beta,-\frac{1}{2}}\}$ do not form a closed algebra.

**1.1.4.3.3 Conformally soft graviton theorems:** In this section, we discuss tree level soft graviton theorems and associated underlying symmetry structures in Mellin basis [21]. The tree level positive helicity leading soft graviton theorem [79] in Mellin basis takes the following form

$$\langle H^1(z,\bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \sum_{k=1}^{n} \left(\frac{\bar{z}-\bar{z}_k}{z-z_k}\right) H^1_{-\frac{1}{2},-\frac{1}{2}}(k)\langle \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \tag{1.1.74}$$

where the positive helicity leading conformally soft graviton $H^1(z,\bar{z})$ is defined as

$$H^1(z,\bar{z}) = \lim_{\Delta\to 1}(\Delta-1)G^+_\Delta(z,\bar{z}) \tag{1.1.75}$$

and the operator $H^1_{-\frac{1}{2},-\frac{1}{2}}(k)$ acts on a primary operator as

$$H_{-\frac{1}{2},-\frac{1}{2}}(k)\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = -\delta_{ik}\phi_{h_i+\frac{1}{2},\bar{h}_i+\frac{1}{2}}(z_i,\bar{z}_i) \tag{1.1.76}$$

The $\bar{z}$ expansion of RHS of (1.1.74) leads us to define the following two currents

$$H^1(z,\bar{z}) = H^1_{\frac{1}{2}}(z) + \bar{z}H^1_{-\frac{1}{2}}(z). \tag{1.1.77}$$

Substituting (1.1.77) into (1.1.74), we can write the Ward identities for currents $H^1_{\frac{1}{2}}(z)$ and $H^1_{-\frac{1}{2}}(z)$

$$\langle H^1_{\frac{1}{2}}(z)\prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = -\sum_{k=1}^n\left(\frac{\bar{z}_k}{z-z_k}\right)H^1_{-\frac{1}{2},-\frac{1}{2}}(k)\langle\prod_{i=1}^n\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \tag{1.1.78}$$

and

$$\langle H^1_{-\frac{1}{2}}(z)\prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \sum_{k=1}^n\left(\frac{1}{z-z_k}\right)H^1_{-\frac{1}{2},-\frac{1}{2}}(k)\langle\prod_{i=1}^n\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \tag{1.1.79}$$

We can further mode expand these currents as

$$H^1_{\pm\frac{1}{2}}(z) = \sum_{m\in\mathbb{Z}-\frac{3}{2}}\frac{H^1_{m,\pm\frac{1}{2}}}{z^{m+\frac{3}{2}}}. \tag{1.1.80}$$

The modes $H^1_{m,\pm\frac{1}{2}}$ generate supertranslations and commute among themselves

$$[H^1_{\alpha,m}, H^1_{\beta,n}] = 0; \quad m, n = \pm\frac{1}{2}. \tag{1.1.81}$$

We will now discuss the subleading soft graviton theorem [42] and the associated current algebra symmetries. The tree level subleading soft graviton theorem in Mellin basis takes the following form

$$\langle H^0(z,\bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \sum_{k=1}^{n} \frac{(\bar{z}-\bar{z}_k)^2}{z-z_k} \left[\frac{2\bar{h}_k}{\bar{z}-\bar{z}_k} - \bar{\partial}_k\right] \langle \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \tag{1.1.82}$$

where the positive helicity conformally subleading soft graviton operator $H^0(z,\bar{z})$ is defined as

$$H^0(z,\bar{z}) = \lim_{\Delta\to 0} \Delta G^{+}_{\Delta}(z,\bar{z}) \tag{1.1.83}$$

From the RHS of (1.1.82) we can mode expand the soft graviton operator in $\bar{z}$ and define the following currents

$$H^0(z,\bar{z}) = \sum_{n=-1}^{1} \frac{H^0_n(z)}{\bar{z}^{n-1}} \tag{1.1.84}$$

Substituting this into (1.1.82) and comparing the powers of $\bar{z}$ we can obtain the Ward identities of the currents $H^0_{\pm 1}(z)$ and $H^0_0(z)$

$$\langle H^0_1(z) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = -\sum_{k=1}^{n} \frac{\bar{L}_1(k)}{z-z_k} \langle \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \tag{1.1.85}$$

$$\langle H^0_0(z)\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \sum_{k=1}^{n}\frac{2\bar{L}_0(k)}{z-z_k}\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \tag{1.1.86}$$

and

$$\langle H^0_{-1}(z)\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = -\sum_{k=1}^{n}\frac{\bar{L}_{-1}(k)}{z-z_k}\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \tag{1.1.87}$$

where the operators, $\bar{L}_1(k)$, $\bar{L}_0(k)$ and $\bar{L}_{-1}(k)$ are given by

$$\bar{L}_1(k) = \bar{z}_k^2\bar{\partial}_k + 2\bar{h}_k\bar{z}_k, \tag{1.1.88}$$

$$\bar{L}_0(k) = \bar{z}_k\bar{\partial}_k + \bar{h}_k, \tag{1.1.89}$$

$$\bar{L}_{-1}(k) = \frac{\partial}{\partial\bar{z}_k}. \tag{1.1.90}$$

which generate the anti-holomorphic Lorentz transformations or $sl(2,R)_R$ transformations.

The currents defined in (1.1.84) admit the usual Laurent expansion in $z$ variable

$$H^0_m(z) = \sum_{\alpha\in\mathbb{Z}-1}\frac{H^0_{\alpha,m}}{z^{\alpha+1}} \tag{1.1.91}$$

with $m = +1, 0, -1$ The modes $H^0_{\alpha,m}$ generate the $\widehat{sl}_2(R)$ current algebra and with the following identifications

$$J^a_1 = -H^0_{a,1}, \quad J^a_0 = \frac{1}{2}H^0_{a,0}, \quad J^a_{-1} = -H^0_{a,-1} \tag{1.1.92}$$

they satisfy the commutation relations

$$[J^a_m, J^b_n] = (m-n)J^{a+b}_{m+n}. \tag{1.1.93}$$

where $m, n = 0, \pm 1$ and $a, b \in \mathbb{Z}$.

The complete symmetry algebra implied by leading and subleading soft graviton theorems is given by the semidirect of (1.1.81) and (1.1.93), which is *supertranslations* $\ltimes \widehat{sl}_2(R)$. The global modes of $\widehat{sl}_2(R)$ currents (1.1.92) generate antiholomorphic Lorentz transformations. It is important to note that, in Celestial CFT, supertranslations are generated only by the currents $H^1_{\frac{1}{2}}(z)$ and $H^1_{-\frac{1}{2}}(z)$.

These symmetry algebras ((1.1.81) and (1.1.93)) have been used to compute the scattering amplitudes in a more efficient way. For example, in [21, 22] it was shown that these symmetries can completely determine the tree level MHV gluon and graviton scattering amplitudes using the null states of these algebras. In the next section, we introduce the celestial operator product expansion (OPE), which will be a key tool in finding these null states. We discuss the role of null states in computing celestial amplitudes in Section 1.1.8. In celestial CFT, the symmetry algebras extracted from infinite tower of tree-level soft factorization theorems have been shown to close into the wedge subalgebra of $w_{1+\infty}$, we discuss this in detail in Section 1.1.6.

### 1.1.5 Collinear factorization and celestial OPE

The operator product expansion (OPE) plays an important role in conformal field theory, encapsulating the short-distance behavior of local operator insertions. It expresses the product of two local operators, in the limit where their insertion points approach one another, as a sum over local operators evaluated at a single point. In the celestial formulation of scattering amplitudes, an important analogue of this structure arises through collinear factorization. Specifically, the collinear limit—where two massless particles in the bulk become parallel—maps onto the OPE limit on the celestial sphere, in which the insertion points of two celestial primary operators approach one another. In this limit, celestial amplitudes factorize into lower point amplitudes. This correspondence allows one to extract celestial OPE between two hard primary operators on the celestial sphere directly from the collinear limits of scattering amplitudes, as shown in [34, 47]. As such, collinear factorization provides a powerful bridge between bulk kinematics and the operator algebra of the celestial CFT.

For example, the four point gluon scattering amplitude factorizes in the following way [22] when the 3rd and 4th gluon primary operators become collinear to each other i.e when $p_3 \cdot p_4 = 0$

$$\mathcal{A}_4(1^{a_1}_{\Delta_1,-}, 2^{a_2}_{\Delta_2,+}, 3^{a_3}_{\Delta_3,+}, 4^{a_4}_{\Delta_4,-}) \xrightarrow[p_3 \cdot p_4 = 0]{z_{34} \to 0} -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1) \mathcal{A}_3(1^{a_1}_{\Delta_1,-}, 2^{a_2}_{\Delta_2,+} 4^{x}_{\Delta_3+\Delta_4-1,-}) + \text{subleading in } z_{34} + ... \tag{1.1.94}$$

From the above factorization, we extract the leading OPE structure between a positive and a negative helicity gluon primary operators

$$\mathcal{O}^{a_3}_{\Delta_3,+}(z_3, \bar{z}_3) \mathcal{O}^{a_4}_{\Delta_4,-}(z_4, \bar{z}_4) \sim -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1) \mathcal{O}^{x}_{\Delta_3+\Delta_4-1,-}(z_4, \bar{z}_4) \tag{1.1.95}$$

The OPE structure at higher orders can be extracted similarly from the above expansion (1.1.94).

The OPE between two graviton primaries can be obtained similarly. For example, the leading OPE structure between two outgoing positive helicity graviton operators has the following form

$$G^+_{\Delta_1}(z_1,\bar{z}_1)G^+_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2-1)G^+_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)+\cdots \tag{1.1.96}$$

Moreover, these OPEs are not merely kinematic consequences of collinear limits—they can also be systematically derived from the underlying symmetry algebra of celestial CFT. In particular, the infinite-dimensional current algebra symmetries associated with soft theorems impose strong constraints on the form of the OPEs [21–23, 47]. This dual perspective—deriving OPEs both from collinear factorization and from symmetry principles—highlights the rich algebraic structure of celestial amplitudes and their deep connection to the infrared behavior of quantum field theories. For more discussions on celestial OPE please see [15, 16, 34, 47, 90–93].

### 1.1.6 Review of holographic symmetry algebras for gluons and gravitons

In this section, we review soft symmetry algebras for tree level gluon and graviton scattering amplitudes generated by the infinite tower of conformally soft positive helicity gluons and gravitons.

#### 1.1.6.1 Gluons:

Let us start with the leading OPE between two positive helicity outgoing gluon conformal primary operators

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{if^{ab}{}_c}{z_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \tag{1.1.97}$$

where $z_{12} = (z_1 - z_2)$ and $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler beta function and $O^{a,+}_{\Delta}(z,\bar{z})$ denotes a positive helicity outgoing gluon conformal primary operator of dimension $\Delta$ and adjoint group index $a$ at the point $(z,\bar{z})$ on the celestial sphere. This was computed from the collinear singularity in [34] and using asymptotic symmetries in [33]. The OPE coefficient has poles at the conformal weights $\Delta = 1, 0, -1, -2, -3, \cdots$ With this observations, we define infinite tower of conformally soft positive helicity gluons as

$$R^{k,a}(z,\bar{z}) = \lim_{\Delta \to k} (\Delta - k) O^{a,+}_{\Delta}(z,\bar{z}), \qquad k = 1, 0, -1, ... \tag{1.1.98}$$

The singular terms in the OPE between two positive helicity outgoing gluons operators including the contributions from $SL(2,\mathbb{R})_R$ descendants were derived in [23],

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{if^{ab}{}_{c}}{z_{12}} \sum_{n=0}^{\infty} B(\Delta_1 + n - 1, \Delta_2 - 1) \frac{\bar{z}^n_{12}}{n!} \bar{\partial}^n_2 \mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2). \tag{1.1.99}$$

The structure of the OPE (1.1.99) allows us to do the following truncated mode expansion for conformally soft gluons $R^{k,a}(z,\bar{z})$ in $\bar{z}$ variable

$$R^{k,a}(z,\bar{z}) = \sum_{n=\frac{k-1}{2}}^{\frac{1-k}{2}} \frac{R^{k,a}_n(z)}{\bar{z}^{n+\frac{k-1}{2}}} \tag{1.1.100}$$

The holomorphic currents $R^{k,a}_n(z)$ has usual Laurent expansion in $z$

$$R^{k,a}_n(z) = \sum_{\alpha \in \mathbb{Z} - \frac{k+1}{2}} \frac{R^{k,a}_{\alpha,n}}{z^{\alpha+\frac{k+1}{2}}} \tag{1.1.101}$$

The OPE between two conformally soft positive helicity gluons is given by

$$R^{k,a}(z_1,\bar{z}_1)R^{l,b}(z_2,\bar{z}_2) \sim -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{1-k}\binom{2-k-l-n}{1-l}\frac{\bar{z}_{12}^n}{n!}\bar{\partial}^n R^{k+l-1,c}(z_2,\bar{z}_2) \quad (1.1.102)$$

The modes of $R^{k,a}(z,\bar{z})$ satisfy the holographic symmetry algebra [23]

$$[R^{k,a}_{\alpha,m}, R^{l,b}_{\beta,n}] = -if^{ab}{}_c\frac{(\frac{1-k}{2}-m+\frac{1-l}{2}-n)!}{(\frac{1-k}{2}-m)!(\frac{1-l}{2}-n)!}\frac{(\frac{1-k}{2}+m+\frac{1-l}{2}+n)!}{(\frac{1-k}{2}+m)!(\frac{1-l}{2}+n)!}R^{k+l-1,c}_{\alpha+\beta,m+n} \quad (1.1.103)$$

Now we define the positive helicity light transformed soft gluon operators as [24]

$$S^{q,a}_{\alpha,m} = (q-m-1)!(q+m-1)!R^{3-2q,a}_{\alpha,m} \quad (1.1.104)$$

where $p,q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ with the following restriction on $m$

$$1-p \le m \le p-1 \quad (1.1.105)$$

The algebra of the light transformed positive helicity gluon operators is the the $S$ algebra

$$[S^{p,a}_{\alpha,m}, S^{q,b}_{\beta,n}] = -if^{ab}{}_c S^{p+q-1,c}_{\alpha+\beta,m+n}. \quad (1.1.106)$$

#### 1.1.6.2 Gravitons:

Now we will briefly review the holographic symmetry algebra for gravitons. We denote a positive helicity outgoing graviton conformal primary operator of dimension $\Delta$ at $(z,\bar{z})$ on the celestial sphere by $G^+_\Delta(z,\bar{z})$. The singular terms in the celestial OPE between two positive helicity outgoing gravitons are given by [23]

$$G^+_{\Delta_1}(z_1,\bar{z}_1)G^+_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{n=0}^{\infty}B(\Delta_1-1+n,\Delta_2-1)\frac{\bar{z}_{12}^{n+1}}{n!}\bar{\partial}^n G^+_{\Delta_1+\Delta_2}(z_2,\bar{z}_2) \quad (1.1.107)$$

The infinite tower of conformally soft positive helicity graviton operators is defined as

$$H^k\left(z,\bar{z}\right)=\lim_{\Delta\to k}\left(\Delta-k\right)G^{+}_{\Delta}\left(z,\bar{z}\right),\quad k=2,1,0,-1,\cdots \tag{1.1.108}$$

with weights

$$(h,\bar{h})=\left(\frac{k+2}{2},\frac{k-2}{2}\right). \tag{1.1.109}$$

and we defined the following truncated mode expansion

$$H^k(z,\bar{z})=\sum_{n=\frac{k-2}{2}}^{\frac{2-k}{2}}\frac{H^k_n(z)}{\bar{z}^{n+\frac{k-2}{2}}} \tag{1.1.110}$$

We further expand the holomorphic currents $H^k(z)$ in following Laurent expansion

$$H^k_n(z)=\sum_{\alpha\in\mathbb{Z}-\frac{k+2}{2}}\frac{H^k_{\alpha,n}}{z^{\alpha+\frac{k+2}{2}}}. \tag{1.1.111}$$

The OPE between conformally soft positive helicity gravitons (1.1.108) is given by

$$H^k(z_1,\bar{z}_1)H^l(z_2,\bar{z}_2)\sim-\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{n=0}^{1-k}\binom{2-k-l-n}{1-l}\frac{\bar{z}_{12}^{n+1}}{n!}\bar{\partial}^n H^{k+l}(z_2,\bar{z}_2) \tag{1.1.112}$$

The modes of the conformally soft positive helicity gravitons $H^k(z,\bar{z})$ satisfy the holographic symmetry algebra [23]

$$\begin{aligned}\left[H^k_{\alpha,m},H^l_{\beta,n}\right]=&-\frac{\kappa}{2}\Big[n(2-k)-m(2-l))\Big]\\&\times\frac{\left(\frac{2-k}{2}-m+\frac{2-l}{2}-n-1\right)!}{\left(\frac{2-k}{2}-m\right)!\left(\frac{2-k_2}{2}-m_2\right)!}\frac{\left(\frac{2-k}{2}+m+\frac{2-l}{2}+n-1\right)!}{\left(\frac{2-k}{2}+m\right)!\left(\frac{2-l}{2}+n\right)!}H^{k+l}_{\alpha+\beta,m+n}\end{aligned} \tag{1.1.113}$$

The light transformed positive helicity graviton operators are defined as

$$w^p_m = \frac{1}{\kappa}(p-1-m)!(p-1+m)!H_m^{-2p+4} \tag{1.1.114}$$

where $p, q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ and the restriction on $m$ becomes

$$1 - p \leq m \leq p - 1 \tag{1.1.115}$$

In terms of light transformed generators $w^p_m$, the algebra (1.1.113) becomes

$$[w^p_m, w^q_n] = [m(q-1) - n(p-1)]w^{p+q-2}_{m+n} \tag{1.1.116}$$

which is the wedge subalgebra of $w_{1+\infty}$. The representation of these infinite dimensional symmetry algebras admit the presence of a class of null states which contain the descendants of $L_{-1}$. Decoupling of these states from the physical Hilbert space gives rise to a set of differential equations which are similar to Knizhnik-Zamolodchikov (KZ) type null states. In Celestial CFT literature, these are known as Banerjee-Ghosh (BG) equations. Before discussing the null states in Celestial CFT in section 1.1.8, we will first briefly introduce the KZ-type null states in the next section.

### 1.1.7 2D conformal field theories with non-abelian Kac-Moody symmetries and Knizhnik-Zamolodchikov (KZ) equations

As discussed earlier, celestial CFTs possess not only global conformal symmetry but also various infinite-dimensional current algebra symmetries. These additional symmetries originate from the soft factorization of the four-dimensional scattering amplitude. To better understand the structure and consequences of such current algebras, it is useful to review a well-known two-dimensional quantum field theory in which these features are realized explicitly: the Wess-Zumino-Witten (WZW) model. In particular, we will focus on its underlying Virasoro and non-abelian Kac-Moody symmetries and the resulting

Knizhnik-Zamolodchikov (KZ) equations.

The WZW model can be viewed as a two-dimensional non-linear sigma model with an additional Wess-Zumino term. Its action is given by

$$\mathcal{A}_{\lambda,k}(g) = \frac{1}{4\lambda^2}\int Tr\left(\partial_\mu g^{-1}\partial_\mu g\right) d^2\xi + k\Gamma(g) \tag{1.1.117}$$

where the field $g(\xi)$ takes values in a semisimple Lie group $G$, and $\xi^\mu = (\xi^1, \xi^2)$ are coordinates on the two-dimensional spacetime. The parameters $\lambda^2$ and $k$ are dimensionless couplings.

The Wess-Zumino term $\Gamma(g)$ is defined by the following integral over three-dimensional ball with coordinates $Y^\mu$, the boundary of which is identified with the 2-dimensional space parameterized by $\xi^\mu$,

$$\Gamma(g) = \frac{1}{24\pi}\int d^3Y e^{\mu\nu\rho} Tr\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g\right). \tag{1.1.118}$$

While the classical theory is well defined for arbitrary $k$, quantum consistency requires the path integral weight $e^{-\mathcal{A}_{\lambda,k}(g)}$ to be single-valued. This condition forces $k$ to be an integer.

For $k = 0$, the theory reduces to the ordinary sigma model, which is classically conformal but generally develops a nonvanishing beta function at the quantum level. In contrast, for integer $k > 0$, the theory flows to an infrared fixed point at

$$\lambda^2 = \frac{4\pi}{k}. \tag{1.1.119}$$

At this fixed point, the action takes the form

$$\mathcal{A}_k(g) = k\left(\frac{1}{16\pi}\int Tr\left(\partial_\mu g^{-1}\partial_\mu g\right) d^2\xi + \Gamma(g)\right) \tag{1.1.120}$$

which we will refer to as the Wess-Zumino-Witten model.

At the conformal fixed point, the theory exhibits a rich symmetry structure. Introducing complex coordinates

$$\begin{aligned} z &= \xi^1 + i\xi^2 \\ \bar{z} &= \xi^1 - i\xi^2, \end{aligned} \tag{1.1.121}$$

it is easy to see that the action (1.1.120) has $G \times G$ symmetry with the following transformations,

$$g(z,\bar{z}) \to \Omega_L(z) g(z,\bar{z}) \Omega_R^{-1}(\bar{z}). \tag{1.1.122}$$

These symmetries give rise to conserved holomorphic and anti-holomorphic currents,

$$\begin{aligned} J = J^a t^a &= -\frac{1}{2} k \partial_z g g^{-1}, \\ \bar{J} = \bar{J}^a t^a &= -\frac{1}{2} k g^{-1} \partial_{\bar{z}} g \end{aligned} \tag{1.1.123}$$

where $\partial_z = \frac{\partial}{\partial z}$ and $t^a$ are the generators of the Lie group satisfying

$$[t^a, t^b] = f^{abc} t^c \tag{1.1.124}$$

with $f^{abc}$ being the structure constants of the Lie algebra. The conservation equations,

$$\partial_{\bar{z}} J = 0, \;\; \partial_z \bar{J} = 0, \tag{1.1.125}$$

imply that the currents are purely chiral,

$$J^a = J^a(z), \;\; \bar{J}^a = \bar{J}^a(\bar{z}). \tag{1.1.126}$$

The action (1.1.120) are also invariant under infinite dimensional conformal transformations as the conformal anomaly vanishes at fixed point. The conformal symmetries are generated by the holomorphic and anti-holomorphic components of the stress tensor, $T(z)$ and $\bar{T}(\bar{z})$, respectively.

The operator product expansions (OPEs) in the holomorphic sector of a Wess-Zumino-

Witten model are given by

$$\begin{aligned} T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \text{regular terms}, \\ T(z)J^a(w) &= \frac{1}{(z-w)^2}J^a(w) + \frac{1}{z-w}\partial J^a(w) + \text{regular terms}, \\ J^a(z)J^b(w) &= \frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{abc}}{z-w}J^c(w) + \text{regular terms}, \end{aligned} \tag{1.1.127}$$

with the asymptotic conditions on the currents $T(z)$ and $J^a(z)$ are given by,

$$T(z) \sim \frac{1}{z^4}, \quad J^a(z) \sim \frac{1}{z^2} \qquad \text{as } z \to \infty. \tag{1.1.128}$$

For a primary field $\phi_l(z,\bar{z})$ of the theory with conformal weights $(\Delta_l, \bar{\Delta}_l)$, the singular terms in the OPE with currents are given by the following expressions,

$$\begin{aligned} T(z)\phi_l(w,\bar{w}) &= \frac{\Delta_l}{(z-w)^2}\phi_l(w,\bar{w}) + \frac{1}{z-w}\frac{\partial}{\partial z}\phi_l(w,\bar{w}) + \cdots \\ J^a(z)\phi_l(w,\bar{w}) &= \frac{t_l^a}{z-w}\phi_l(w,\bar{w}) + \cdots \end{aligned} \tag{1.1.129}$$

where $t_l^a$ is the left representation of generators of the Lie group $G$ for the field $\phi_l$.

The OPEs (1.1.129) imply following standard Ward identities,

$$\begin{aligned} \langle T(z)\phi_1(z_1,\bar{z}_1)\cdots\phi_N(z_N,\bar{z}_N)\rangle &= \sum_{j=1}^{N}\left(\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right)\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_N(z_N,\bar{z}_N)\rangle \\ \langle J^a(z)\phi_1(z_1,\bar{z}_1)\cdots\phi_N(z_N,\bar{z}_N)\rangle &= \sum_{j=1}^{N}\frac{t_j^a}{z-z_j}\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_N(z_N,\bar{z}_N)\rangle. \end{aligned} \tag{1.1.130}$$

Using asymptotic conditions (1.1.128) in (1.1.130) one obtains the global Ward identities,

$$\sum_{j=1}^{N}\left(z_j^{n+1}\frac{\partial}{\partial z_j} + (n+1)\Delta_j z_j^n\right)\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_N(z_N,\bar{z}_N)\rangle = 0 \tag{1.1.131}$$

for $n = -1, 0, 1$ which are the Ward identities corresponding to the global conformal symmetries and

$$\sum_{j=1}^{N} t_j^a \langle \phi_1(z_1,\bar{z}_1)\cdots\phi_N(z_N,\bar{z}_N)\rangle = 0 \tag{1.1.132}$$

is the Ward identity corresponding to the global gauge symmetry.

The stress tensor $T(z)$ and the left chiral current $J^a(z)$ admit the following Laurent expansion in $z$,

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \\ J^a(z) &= \sum_{n\in\mathbb{Z}} \frac{J^a_n}{z^{n+1}}. \end{aligned} \tag{1.1.133}$$

Primary fields of WZW models satisfy the following conditions,

$$\begin{aligned} L_n\phi_l &= J^a_n\phi_l = 0 \quad \text{for } n \geq 0 \\ L_0\phi_l &= \Delta_l\phi_l, \quad J^a_0\phi_l = t^a_l\phi_l. \end{aligned} \tag{1.1.134}$$

The complete set of local fields $\{A_j\}$ in a WZW theory consists of primary fields $\phi_l$ and their descendants of the form

$$L_{-n_1}\cdots L_{-n_N}\bar{L}_{-\bar{n}_1}\cdots\bar{L}_{-\bar{n}_N}J^{a_1}_{-m_1}\cdots J^{a_M}_{-m_M}\bar{J}^{b_1}_{-\bar{m}_1}\cdots\bar{J}^{b_M}_{-\bar{m}_M}\phi_l \tag{1.1.135}$$

with positive integers $n, \bar{n}, m, \bar{m}$.

From the singular terms in the OPEs (1.1.127), one obtains the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{1.1.136}$$

$$[L_m, J^a_n] = -nJ^a_{m+n} \tag{1.1.137}$$

$$[J^a_m, J^b_n] = f^{abc}J^c_{m+n} + \frac{1}{2}kn\delta^{ab}\delta_{m+n,0}. \tag{1.1.138}$$

Equation (1.1.136) defines the *Virasoro algebra*, while (1.1.138) is the *affine Kac-Moody algebra*. The full symmetry algebra is their semidirect product. The anti-holomorphic modes satisfy analogous relations.

A key assumption of Knizhnik and Zamolodchikov is that the spectrum contains a primary field $g(z,\bar{z})$ with weights $\Delta_g = \bar{\Delta}_g = \Delta$ satisfying the quantum relations

$$\begin{aligned} \kappa\partial_z g(z,\bar{z}) &=: J^a(z) t^a g(z,\bar{z}) :, \\ \kappa\partial_{\bar{z}} g(z,\bar{z}) &=: \bar{J}^a(\bar{z}) g(z,\bar{z}) t^a : \end{aligned} \tag{1.1.139}$$

where $\kappa$ is a numerical factor which will be determined later. The OPE $J^a(w)t^a g(z,\bar{z})$ has the following general structure,

$$J^a(w)t^a g(z,\bar{z}) = \frac{c_g}{w-z} g(z,\bar{z}) + \sum_{n=1}^{\infty} (w-z)^{n-1} t^a J^a_{-n} g(z,\bar{z}) \tag{1.1.140}$$

with

$$t^a t^a = c_g I. \tag{1.1.141}$$

The normal ordered OPE $: J^a(z)t^a g(z,\bar{z}) :$ is defined as,

$$: J^a(z)t^a g(z,\bar{z}) := \lim_{w\to z} \left( J^a(w) - \frac{t^a}{w-z} \right) t^a g(z,\bar{z}) \tag{1.1.142}$$

Now using (1.1.139), we see that the $\mathcal{O}(1)$ term of (1.1.140) must coincide with $\kappa\partial_z g$, so we can write

$$J^a(w)t^a g(z,\bar{z}) = \frac{c_g}{w-z} g(z,\bar{z}) + \kappa\partial_z g(z,\bar{z}) + \mathcal{O}(w-z). \tag{1.1.143}$$

Now comparing the $\mathcal{O}(1)$ term from (1.1.140) and (1.1.143), one obtains

$$\xi \equiv \left( J^a_{-1} t^a - \kappa L_{-1} \right) g = 0 \tag{1.1.144}$$

where we have used $L_{-1}\phi(z,\bar{z}) = \partial_z \phi(z,\bar{z})$. The relation (1.1.144) can also be obtained by using the *Sugawara* form of the stree-tensor, discussed in [94].

The existence of (1.1.144) implies the representation is *degenerate* and $\xi$ is a *null vector*. The consistency then requires that the field $\xi$ will satisfy the conditions of being primary operator,

$$L_0\xi = (\Delta + 1)\xi, \quad J_0^a\xi = t^a\xi \tag{1.1.145}$$

$$L_n\xi = J_n^a\xi = 0 \quad \text{for } n > 0. \tag{1.1.146}$$

One can show that (1.1.146) is satisfied provided

$$\begin{aligned} c_g + 2\Delta\kappa &= 0, \\ c_V + k + 2\kappa &= 0, \end{aligned} \tag{1.1.147}$$

with

$$f^{acd}f^{bcd} = c_V\delta^{ab}. \tag{1.1.148}$$

From (1.1.147), one can find the dimension $\Delta$ of the field $g$ and the parameter $\kappa$,

$$\begin{aligned} \Delta &= \frac{c_g}{c_V + k}, \\ \kappa &= -\frac{1}{2}(c_V + k). \end{aligned} \tag{1.1.149}$$

In general conformal field theory, correlation functions of degenerate fields satisfy linear differential equations known as the BPZ equations, after Belavin, Polyakov, and Zamolodchikov [95]. Now let's look for the correlation function of degenerate fields in WZW theory and consider the following correlation function,

$$t_i^a\langle J^a(z)g(z_1,\bar{z}_1),\cdots g(z_N,\bar{z}_N)\rangle = \left(\frac{c_g}{z-z_i} + \sum_{j\neq i}^{N}\frac{t_i^at_j^a}{z-z_j}\right)\langle g(z_1,\bar{z}_1)\cdots g(z_N,\bar{z}_N)\rangle \tag{1.1.150}$$

Using the OPE (1.1.143) in the l.h.s. of (1.1.150), one obtains the following set of partial differential equations,

$$\left(\kappa\frac{\partial}{\partial z_i} - \sum_{j\neq i}^{N}\frac{t_i^at_j^a}{z-z_j}\right)\langle g(z_1,\bar{z}_1)\cdots g(z_N,\bar{z}_N)\rangle = 0 \tag{1.1.151}$$

where $i = 1, 2, \cdots, N$, together with analogous anti-holomorphic equations.

These are the *Knizhnik-Zamolodchikov* (KZ) equations, first derived by Knizhnik and Zamolodchikov in [94]. In two-dimensional celestial CFT, analogous relations arise, known as the *Banerjee-Ghosh* (BG) equations [22], which will be discussed in the next section.

### 1.1.8 Null states in Celestial CFT and null decoupling equations

In this section, we briefly review the role of null states in celestial conformal field theories (CFTs) and their importance in constraining scattering amplitudes within the framework of flat space holography. As discussed previously, celestial CFTs are enriched by various infinite-dimensional current algebra symmetries. Null states arise as descendants of these algebras that also satisfy the conditions of being primary operators. For instance, the criteria for primaries under the $S$ algebra are outlined in Appendix B.1, and hence null states associated with this algebra must also satisfy those conditions.

#### 1.1.8.1 Null states of the MHV gluon sector

Here we discuss null states in the MHV-sector for gluons. These null states can be obtained as follows [22, 25]. We start with the OPE between two outgoing positive helicity gluon operators

$$\begin{aligned}\mathcal{O}^a_{\Delta,+}(z,\bar{z})\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1) = -iB(\Delta-1,\Delta_1-1)\Bigg[\frac{f^{aa_1x}}{z-z_1} + \frac{\Delta-1}{\Delta+\Delta_1-2}f^{aa_1x}L_{-1}\\ +i\left(\frac{\Delta-1}{\Delta+\Delta_1-2}\delta^{ax}\delta^{a_1y} + \frac{\Delta_1-1}{\Delta+\Delta_1-2}\delta^{ay}\delta^{a_1x}\right)R^{1,y}_{-1,0}\Bigg]\mathcal{O}^x_{\Delta+\Delta_1-1,+}(z_1,\bar{z}_1) + \ldots\end{aligned} \tag{1.1.152}$$

After taking the conformally soft limit $\Delta \to k$ in (1.1.152) and comparing $\mathcal{O}(z^0\bar{z}^0)$ term from both sides, we obtain the following expression

$$R^{k,a}_{-\frac{(1+k)}{2},\frac{(1-k)}{2}}\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1) = -i\frac{(-1)^{1-k}}{(1-k)!}\frac{\Gamma(\Delta_1-1)}{\Gamma(\Delta_1+k-2)}\Bigg[\frac{k-1}{\Delta_1+k-2}f^{aa_1x}L_{-1}$$

$$+\,i\left(\frac{k-1}{\Delta_1+k-2}\delta^{ax}\delta^{a_1y}+\frac{\Delta_1-1}{\Delta_1+k-2}\delta^{ay}\delta^{a_1x}\right)R^{1,y}_{-1,0}\Bigg]\mathcal{O}^{x}_{\Delta_1+k-1,+} \tag{1.1.153}$$

where $k = 1, 0, -1, -2, ...$

Now we use following expression of the null state found in [22] at $\mathcal{O}(1)$

$$if^{aa_1x}L_{-1}\mathcal{O}^{x}_{\Delta_1-1,+}(z_1,\bar{z}_1) - R^{1,a_1}_{-1,0}\mathcal{O}^{a}_{\Delta_1-1,+}(z_1,\bar{z}_1) + R^{0,a}_{-1/2,1/2}\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1)$$
$$+\,(\Delta_1-1)R^{1,a}_{-1,0}\mathcal{O}^{a_1}_{\Delta_1-1,+}(z_1,\bar{z}_1) = 0 \tag{1.1.154}$$

to replace $L_{-1}\mathcal{O}^{x_1}_{\Delta_1+k-1}$ in (1.1.153).

Now setting $k = -j$ and shifting $\Delta_1$ to $\Delta_1 + j$ we can get the following set of null states,

$$R^{-j,a}_{\frac{j-1}{2},\frac{j+1}{2}}\mathcal{O}^{a_1}_{\Delta_1+j,+}(z_1,\bar{z}_1) - \frac{(-1)^j j}{(1+j)!}\frac{\Gamma(\Delta_1+j-1)}{\Gamma(\Delta_1-2)}R^{1,a}_{-1,0}\mathcal{O}^{a_1}_{\Delta_1-1,+}(z_1,\bar{z}_1)$$
$$-\,\frac{(-1)^j(j+1)}{(1+j)!}\frac{\Gamma(\Delta_1+j-1)}{\Gamma(\Delta_1-1)}R^{0,a}_{-1/2,1/2}\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1) = 0 \tag{1.1.155}$$

with $j = 1, 2, 3, \cdots$

These null null states imposes powerful constraints on celestial amplitudes. In particular, Knizhnik–Zamolodchikov (KZ)-type null states, which involve descendants of the holomorphic translation generator $L_{-1}$, play a central role. These null states are derived using the OPE commutativity and consistency with the OPE between a soft and hard primary operators on the celestial sphere. In [22], the authors have found these null states of the MHV gluon sector.

For example, the null state at $\mathcal{O}(1)$ in the celestial MHV gluon sector obtained in [22], is

$$\xi^a(\Delta) = C_A L_{-1}\mathcal{O}^{a,+}_{\Delta} - (\Delta+1)R^{1,b}_{-1,0}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta} - R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta+1}. \tag{1.1.156}$$

The existence of KZ-type null states leads to a set of differential equations for celestial amplitudes, known as the Banerjee-Ghosh (BG) equations [21, 22, 25]

$$\left[\frac{C_A}{2}\frac{\partial}{\partial z_i} - h_i\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{T_i^a T_j^a}{z_i - z_j} - \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\left(2\bar{h}_j - 1 - (\bar{z}_i - \bar{z}_j)\frac{\partial}{\partial \bar{z}_j}\right)}{z_i - z_j}T_j^a P_j^{-1} T_i^a H^1_{-\frac{1}{2},-\frac{1}{2}}(i)\right] \times \left\langle \prod_{k=1}^{n}\mathcal{O}^{a_k}_{h_k,\bar{h}_k}(z_k,\bar{z}_k)\right\rangle_{MHV} = 0 \tag{1.1.157}$$

where $i$ runs over positive helicity gluons only and $T^a$ is the Lie algebra generator in adjoint representation and $C_A$ is the quadratic Casimir. And actions of the operators $P_j^{-1}$ and $H^1_{-\frac{1}{2},-\frac{1}{2}}(i)$ on gluon primary operator are given by,

$$P_j^{-1}\mathcal{O}^{a_k}_{h_k,\bar{h}_k}(z_k,\bar{z}_k) = \mathcal{O}^{a_k}_{h_k-\frac{1}{2},\bar{h}_k+\frac{1}{2}}(z_i,\bar{z}_i)\delta_{jk} \tag{1.1.158}$$

and

$$H^1_{-\frac{1}{2},-\frac{1}{2}}(i)\mathcal{O}^{a_k}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = -\epsilon_i\mathcal{O}^{a_i}_{h_i+\frac{1}{2},\bar{h}_i+\frac{1}{2}}(z_i,\bar{z}_i). \tag{1.1.159}$$

These null decoupling equations (1.1.157) have been successfully solved in a number of cases to determine celestial amplitudes [21, 22, 49, 50]. In the next section, we will briefly discuss the MHV null states for gravitons.

#### 1.1.8.2 Null states of the MHV graviton sector

The null states of the MHV graviton sector have been discussed in detail in [21, 96]. For example, the null states at $\mathcal{O}(z_{12}^0\bar{z}_{12}^0)$ in the MHV graviton sector are given by

$$\Phi_k(\Delta) = \left[H^{1-k}_{\frac{k-3}{2},\frac{k+1}{2}}\left(-H^1_{-\frac{1}{2},-\frac{1}{2}}\right)^k - \frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-2)}H^1_{-\frac{3}{2},\frac{1}{2}}\right]G^+_{\Delta-1} \tag{1.1.160}$$

and MHV null states that appear at $\mathcal{O}(z_{12}^0\bar{z}_{12}^1)$ are given by

$$\begin{aligned}\Psi_k(\Delta) = \Bigg[H^{-k}_{\frac{k-2}{2},\frac{k}{2}}\left(-H^1_{-\frac{1}{2},-\frac{1}{2}}\right)^{k+1} &- \frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-2)}H^1_{-1,0}\left(-H^1_{-\frac{1}{2},-\frac{1}{2}}\right)\\ &- \frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-3)}H^1_{-\frac{3}{2},-\frac{1}{2}}\Bigg]G^+_{\Delta-2}.\end{aligned} \tag{1.1.161}$$

These null states have been very useful to explore different sectors of $w_{1+\infty}$ invariant theories on the celestial sphere [26, 48].

The MHV graviton sector also admit the presence of Knizhnik–Zamolodchikov (KZ)-type null states. These null states are obtained from the graviton-graviton OPE and using the following consistency condition

$$G^+_{\Delta_1}(z_1,\bar{z}_1)G^+_{\Delta_2}(z_2,\bar{z}_2) = G^+_{\Delta_2}(z_2,\bar{z}_2)G^+_{\Delta_1}(z_1,\bar{z}_1). \tag{1.1.162}$$

Now, using the expression of OPE and taking different soft limits one can find the KZ-type null states that appear at a particular order. In [21], the authors have found KZ-type null states in the MHV graviton sector.

The KZ-type null state at $\mathcal{O}(z^0\bar{z}^1)$ obtained from plus-plus OPE is given by

$$\boxed{L_{-1}G^+_\Delta + H^0_{0,-1}H^1_{-\frac{3}{2},\frac{1}{2}}G^+_{\Delta-1} + H^0_{-1,0}G^+_\Delta + (\Delta-1)H^1_{-\frac{3}{2},-\frac{1}{2}}G^+_{\Delta-1} = 0.} \tag{1.1.163}$$

Inserting these null states inside the correlation function lead to a set of differential equations for the celestial MHV amplitudes [21].

However, this analysis led to an apparent conceptual puzzle: in the Wess–Zumino–Witten (WZW) model, an $n$-point correlation function is subject to $n$ KZ equations. In contrast, in the celestial MHV sector, only $(n-2)$ such BG equations were identified. The two missing constraints could not be recovered solely from the current algebras associated with the leading and subleading soft gluon (graviton) theorems.

Subsequent work [23, 24] extended the known celestial symmetries: for instance, the symmetry algebra generated by conformally soft positive helicity gluons was enhanced to the $S$ algebra (1.1.103),(1.1.104), while that for positive helicity gravitons was extended to the infinite-dimensional $w_{1+\infty}$ algebra (1.1.113),(1.1.116). Despite this progress, the missing differential constraints remained unresolved. A resolution to this puzzle is presented in Chapter 5.

While celestial holography has produced numerous interesting results about scattering amplitudes in asymptotically flat spacetimes, the field still remains in its early stages. A fundamental challenge that persists is the intrinsic construction of the Celestial CFT.

# Chapter 2

# MHV Gluon Scattering in the Massive Scalar Background and Celestial OPE

This chapter is based on [50].

## 2.1 Introduction

As shown in [13], the Mellin transform of tree level three and four point gluon scattering amplitudes are distributional in nature due to the momentum conserving delta function and the conformal invariance of the tree level Yang-Mills theory. The momentum conservation constraint can be removed by supplying background momentum to the YM theory which breaks translational invariance explicitly [49]. In [49], this was achieved by coupling the YM theory to background massless dilaton field. But this does not break the conformal invariance of the YM theory. Translation invariance breaking solution was also considered in [97].

A somewhat different set up was considered in [98]. They chirally coupled the Yang-Mills theory to a massive dilaton background. As a result the space-time translation as well as the scale invariance of the tree-level Yang-Mills theory was explicitly broken but the (Lorentz) 2D conformal invariance was preserved. The interesting fact about this coupled theory is that, when written in the celestial basis, the 3-point amplitudes take the usual 2D CFT form. It was shown [98] that the leading soft gluon theorem and the leading OPE structure remain unchanged for this chirally coupled theory. In this chapter we show that the subleading soft gluon theorem and subleading OPE structures also remain unchanged, giving rise to the same null state relation as obtained in [22]. We expect this to be true as the leading and subleading soft gluon theorems don't require space-time translation or scale invariance.

This chapter is organized as follows. In section 2.3, following [98] we briefly review the Yang-Mills amplitude chirally coupled to a massive dilaton background and Mellin transform it to get the celestial amplitude. OPE factorization of the 4-point celestial amplitude is given in section 2.4. In section 2.5 we show that the subleading soft gluon theorem remains the same for this coupled theory. By demanding the consistency of the OPE at order 1 with the soft theorem we get the same null state relation under the soft current algebra as obtained in [22]. We also show that the 3-point amplitude satisfies the BG equation in section A.2. Finally we end with discussion and future directions in section 2.6.

## 2.2 Notations and conventions

We will work in (-,+,-,+) signature in four spacetime dimensions. This is also called Klein space. The geometry of this space has been discussed in great detail in [99]. The scattering amplitudes, written in the boost eigenstates, behave as a correlation functions in a Lorentzian CFT on the celestial torus [99, 100]. We review some of the elementary equations of the celestial amplitude for this chapter to be self-contained.

In Klein space the null momentum ($p_i$) of $i$-th hard massless particle is parametrized as:

$$p_i = \epsilon_i \omega_i q(z_i, \bar{z}_i) \tag{2.2.1}$$

where $\omega_i$ is a real positive number, $\epsilon_i = +1(-1)$ corresponds to an outgoing (incoming) particle and

$$q(z_i, \bar{z}_i) = \{1 + z_i\bar{z}_i, z_i + \bar{z}_i, z_i - \bar{z}_i, 1 - z_i\bar{z}_i\} \tag{2.2.2}$$

$z_i, \bar{z}_i$ are two real independent variables. The map from a creation/annihilation operator of a massless particle in the bulk to an operator on a celestial torus is given by the Mellin transformation

$$\phi^{a,\epsilon}_{h,\bar{h}}(z, \bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1} A^a(\epsilon\omega, \sigma, z, \bar{z}) \tag{2.2.3}$$

where $A^a(\epsilon\omega, \sigma, z, \bar{z})$ is an annihilation ($\epsilon = +1$)/creation ($\epsilon = -1$) operator in the adjoint representation of an $SU(N)$ gauge theory, $\sigma$ is the helicity of the corresponding massless particle and

$$h = \frac{\Delta + \sigma}{2}, \; \bar{h} = \frac{\Delta - \sigma}{2} \tag{2.2.4}$$

The momentum space amplitudes for $n$ number of external massless states written in the celestial conformal primary basis (2.2.3) then takes the form of a 2D conformal correlator

$$\begin{aligned} \mathcal{M}_n(\{a_i, \epsilon_i, z_i, \bar{z}_i, h_i, \bar{h}_i\}) &= \left\langle \prod_{j=1}^n \phi^{a_j,\epsilon_j}_{h_j,\bar{h}_j}(z_j, \bar{z}_j) \right\rangle \\ &= \left( \prod_{j=1}^n \int d\omega_j\, \omega_j^{\Delta_j - 1} \right) \mathcal{A}_n(a_i, \epsilon_i\omega_i, z_i, \bar{z}_i, \sigma_i) \end{aligned} \tag{2.2.5}$$

For an on-shell massive scalar particle, the conformal primary wavefunction is obtained through the bulk-to-boundary propagator [11, 12]. For concreteness we take the outgoing particle with unit mass. In $(2, 2)$ signature Klein space, the momentum $Q^\mu$ of a massive scalar of unit mass satisfying the on-shell condition $Q^2 = -1$ can be parameterized using the coordinates of $\mathrm{AdS}_3/\mathbb{Z}$ as [1]

$$Q^\mu = \frac{1}{2y}\{1 + y^2 + z\bar{z}, z + \bar{z}, (z - \bar{z}), 1 - y^2 - z\bar{z}\} \tag{2.2.7}$$

The mapping of a massive scalar particle to a conformal primary wavefunction is given through the bulk-to-boundary propagator

$$\Phi^\epsilon_\Delta(X, z, \bar{z}) = \int_0^\infty \frac{dy}{y^3} \int dz d\bar{z}\, G_\Delta(y, z, \bar{z}; w, \bar{w})\, e^{i\epsilon Q^\mu \cdot X_\mu} \tag{2.2.8}$$

where the bulk-to-boundary propagator is given by

$$G_\Delta(y, z, \bar{z}; w, \bar{w}) = \left(\frac{y}{y^2 + (z + \bar{w})(z - \bar{w})}\right)^\Delta \tag{2.2.9}$$

## 2.3 Celestial MHV amplitudes in the massive scalar background

An $n$-point MHV gluon amplitude in a massive complex scalar background was computed in [98]. The theory considered there was a Yang-Mills theory chirally coupled to a massive complex scalar. The massive scalar was coupled to the anti-self dual curvature tensor. An $(n+1)$-point scalar-gluon (one scalar and $n$ gluons) single trace, color-ordered tree-level amplitude in this set up is given by

$$\mathcal{A}_{n+1}(\phi, 1^{\epsilon_1}_-, 2^{\epsilon_2}_-, 3^{\epsilon_3}_+, \cdots, n^{\epsilon_n}_+) = \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 23\rangle \cdots \langle n1\rangle} \delta^4\left(\sum_{i=1}^n p_i + Q\right) \tag{2.3.1}$$

[1]The metric on the $\mathrm{AdS}_3/\mathbb{Z}$ is given by

$$ds^2_{H_3} = \frac{dy^2 + dz d\bar{z}}{y^2} \tag{2.2.6}$$

The above amplitude is exactly the same as $n$-point MHV amplitude except that the scalar momentum now appears in the momentum conserving delta function. The scalar field can be treated as a background and the amplitude (2.3.1) is coupled to this background by integrating over the scalar phase space [98]

$$\mathcal{A}_n^{\phi}(1_-^{\epsilon_1}, 2_-^{\epsilon_2}, 3_+^{\epsilon_3}, \cdots, n_+^{\epsilon_n}) = \int \widetilde{d^3Q}\, g(Q) \mathcal{A}_{n+1}(\phi, 1_-^{\epsilon_1}, 2_-^{\epsilon_2}, 3_+^{\epsilon_3}, \cdots, n_+^{\epsilon_n}) \tag{2.3.2}$$

where $g(Q)$ is the Fourier coefficient of the scalar field $\phi(X)$ and $\widetilde{d^3Q}$ is the invariant measure. This amplitude is called the $n$-point MHV amplitude in the massive scalar background and hence is denoted by $\mathcal{A}_n^{\phi}(1_-^{a_1,\epsilon_1}, 2_-^{a_2,\epsilon_2}, 3_+^{a_3,\epsilon_3}, \cdots, n_+^{a_n,\epsilon_n})$. We are interested in the OPE factorization of this amplitude on the celestial torus. Hence we Mellin transform the amplitude (2.3.2) and get $n$-point correlators on the celestial torus. To simplify calculations we will work with the 5-point scalar gluon amplitude, i.e., the 4-point MHV amplitude in the massive scalar background.

## 2.4 OPE factorization from the 4-point celestial amplitude

In this section we factorize the $4$-point amplitude into 3-point amplitude and determine the leading and subleading terms in the OPE between two positive helicity outgoing ($\epsilon_3 = \epsilon_4 = +1$) gluons. We show that the OPE remains the same as the MHV case [22] . Let us start with the full 5-point scalar-gluon amplitude, given by

$$\mathcal{A}_5(\phi, 1_-^{a_1,\epsilon_1}, 2_-^{a_2,\epsilon_2}, 3_+^{a_3,\epsilon_3}, 4_+^{a_4,\epsilon_4}) = \left\{\mathcal{A}_4[1_-^{\epsilon_1} 2_-^{\epsilon_2} 3_+^{\epsilon_3} 4_+^{\epsilon_4}]\, tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{perm (234)}\right\} \times \delta^{(4)}\left(\sum_{i=1}^{4} p_i + Q\right) \tag{2.4.1}$$

where $\mathcal{A}_4[i^{\epsilon_i}_{\sigma_i} j^{\epsilon_j}_{\sigma_j} k^{\epsilon_k}_{\sigma_k} l^{\epsilon_l}_{\sigma_l}]$ are color ordered partial MHV amplitudes given by

$$\mathcal{A}_4[i^{\epsilon_i}_{-} j^{\epsilon_j}_{+} k^{\epsilon_k}_{-} l^{\epsilon_l}_{+}] = \frac{\langle ik\rangle^4}{\langle ij\rangle \langle jk\rangle \langle kl\rangle \langle li\rangle} \tag{2.4.2}$$

After substituting the explicit form of the color ordered amplitude (2.4.2) in (2.4.1) and using $[T^a, T^b] = if^{abc}T^c,\ tr(T^aT^b) = \delta^{ab}$ we get

$$\begin{aligned} &\mathcal{A}_5(\phi, 1^{a_1,\epsilon_1}_{-}, 2^{a_2,\epsilon_2}_{-}, 3^{a_3,\epsilon_3}_{+}, 4^{a_4,\epsilon_4}_{+}) \\ &= \frac{\langle 12\rangle^3}{\langle 23\rangle \langle 34\rangle \langle 41\rangle} \left( f^{a_1a_2x} f^{xa_3a_4} - \frac{z_{14}z_{23}}{z_{13}z_{24}} f^{a_1a_3x} f^{xa_2a_4} \right) \delta^{(4)} \left( \sum_{i=1}^{4} p_i + Q \right) \end{aligned}$$

Substituting this expression in (2.3.2) for the $n = 4$ and Mellin integrating over the energies we finally get the 4-point celestial MHV amplitude in the massive scalar background, given by [98]

$$\begin{aligned} &\widetilde{\mathcal{M}}_4^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-}, 2^{a_2,\epsilon_2}_{\Delta_2,-}, 3^{a_3,\epsilon_3}_{\Delta_3,+}, 4^{a_4,\epsilon_4}_{\Delta_4,+}\right) \\ &= \frac{\mathcal{N}_4}{(2\pi)^4} \frac{z_{12}^3}{z_{23}z_{34}z_{41}} \left( f^{a_1a_2x} f^{xa_3a_4} - \frac{z_{12}z_{34}}{z_{13}z_{24}} f^{a_1a_3x} f^{xa_2a_4} \right) \Gamma(\Delta_1 + 1) \\ &\times \Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\Gamma(\Delta_4-1) \int \widetilde{d^3\hat{x}} (-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1} (-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1} \\ &\times (-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1} (-q(z_4,\bar{z}_4)\cdot\hat{x})^{-\Delta_4+1} \int_0^{i\infty} d\tau\, \tau^{-1-\beta_4} \phi_B(\tau) \left(e^{2\pi i\beta_4} - 1\right) \end{aligned}$$

where $\beta_4 = \sum_{j=1}^{4} (\Delta_j - 1)$, $\mathcal{N}_4 = \prod_{j=1}^{4} (-i\epsilon_j)^{\Delta_j - \sigma_j}$. To clarify the notations let us note that a bulk point $X^\mu$ is parameterized as $\tau\hat{x}^\mu$ with $\hat{x}^2 = -1$. $\widetilde{d^3\hat{x}}$ is a measure on the $\hat{x}^2 = -1$ slice [2]. One can also compute the 3-point function in the same way and it is

[2] For more details please see [98].

given by

$$\widetilde{\mathcal{M}}_3^{\Phi}\left(1_{\Delta_1,-}^{a_1,\epsilon_1},2_{\Delta_2,-}^{a_2,\epsilon_2},4_{\Delta_4,+}^{a_4,\epsilon_4}\right)=\frac{\widetilde{\mathcal{N}}_3}{(2\pi)^4}\frac{2z_{12}^3}{z_{24}z_{41}}\left(if^{a_1a_2a_4}\right)\Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_4-1)$$
$$\times\int\widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}(-q(z_4,\bar{z}_4)\cdot\hat{x})^{-\Delta_4+1}$$
$$\times\int_0^{i\infty}d\tau\,\tau^{-1-\tilde{\beta}_3}\phi_B(\tau)\left(e^{2\pi i\tilde{\beta}_3}-1\right)\tag{2.4.3}$$

where

$$\tilde{\beta}_3=\sum_{j=1,j\neq 3}^{4}(\Delta_j-1)\,,\,\widetilde{\mathcal{N}}_3=\prod_{j=1,j\neq 3}^{4}(-i\epsilon_j)^{\Delta_j-\sigma_j}\tag{2.4.4}$$

.

We now take the OPE limit $z_3\to z_4,\ \bar{z}_3\to\bar{z}_4$ in (2.4.3). To do that let us first note that in the OPE limit we have

$$(-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1}$$
$$=(-q(z_4,\bar{z}_4)\cdot\hat{x})^{-\Delta_3+1}\left[1+\frac{\Delta_3-1}{y^2+|z-z_4|^2}((\bar{z}-\bar{z}_4)z_{34}+(z-z_4)\bar{z}_{34}+z_{34}\bar{z}_{34})\right]+\cdots\tag{2.4.5}$$

This will be useful in the next two subsections to extract the OPE from (2.4.3).

### 2.4.1 Leading order

The leading order OPE between two positive helicity gluons was computed in [98] and it was shown that the leading term doesn't get any correction in the massive scalar background. Here for the sake of completeness we reproduce their results and then we move to the subleading terms in the next subsection. The leading order term of (2.4.3) in the OPE expansion is

$$
\begin{aligned}
&\widetilde{\mathcal{M}}_4^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-}, 2^{a_2,\epsilon_2}_{\Delta_2,-}, 3^{a_3,\epsilon_3}_{\Delta_3,+}, 4^{a_4,\epsilon_4}_{\Delta_4,+}\right) \\
&= \frac{\mathcal{N}_4}{(2\pi)^4}\frac{z_{12}^3}{z_{24}z_{41}}\frac{1}{z_{34}} f^{a_1a_2x}f^{xa_3a_4}\Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\Gamma(\Delta_4-1) \\
&\times \int \widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}(-q(z_4,\bar{z}_4)\cdot\hat{x})^{-\Delta_3-\Delta_4+2} \\
&\times \int_0^{i\infty} d\tau\, \tau^{-1-\beta_4}\phi_B(\tau)\left(e^{2\pi i\beta_4}-1\right)
\end{aligned}
$$

Replacing $\Delta_4 \to \Delta_3+\Delta_4-1$ in the 3-point amplitude (2.4.3), we get

$$
\begin{aligned}
&\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-}, 2^{a_2,\epsilon_2}_{\Delta_2,-}, 4^{a_4,\epsilon_4}_{\Delta_3+\Delta_4-1,+}\right) \\
&= \frac{\mathcal{N}_4}{(2\pi)^4}\frac{2z_{12}^3}{z_{24}z_{41}}\left(if^{a_1a_2a_4}\right)\Gamma(\Delta_1+1)\Gamma(\Delta_2+1) \\
&\times \Gamma(\Delta_3+\Delta_4-2)\int \widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}(-q(z_4,\bar{z}_4)\cdot\hat{x})^{-\Delta_3+\Delta_4+2} \\
&\times \int_0^{i\infty} d\tau\, \tau^{-1-\beta_4}\phi_B(\tau)\left(e^{2\pi i\beta_4}-1\right)
\end{aligned} \tag{2.4.6}
$$

Hence at leading order we can write

$$
\begin{aligned}
&\widetilde{\mathcal{M}}_4^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-}, 2^{a_2,\epsilon_2}_{\Delta_2,-}, 3^{a_3,\epsilon_3}_{\Delta_3,+}, 4^{a_4,\epsilon_4}_{\Delta_4,+}\right) = -\frac{1}{2z_{34}}B(\Delta_3-1,\Delta_4-1)if^{xa_3a_4}\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1}, 2^{a_2,\epsilon_2}_{\Delta_2}, 4^{x,\epsilon_4}_{\Delta_3+\Delta_4-1}\right) \\
&\Rightarrow \left\langle \mathcal{O}^{a_1,\epsilon_1}_{\Delta_1,-}(z_1,\bar{z}_1)\mathcal{O}^{a_2,\epsilon_2}_{\Delta_2,-}(z_2,\bar{z}_2)\mathcal{O}^{a_3,\epsilon_3}_{\Delta_3,+}(z_3,\bar{z}_3)\mathcal{O}^{a_4,\epsilon_4}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle \\
&= -\frac{1}{2z_{34}}B(\Delta_3-1,\Delta_4-1)if^{xa_3a_4}\left\langle \mathcal{O}^{a_1,\epsilon_1}_{\Delta_1,-}(z_1,\bar{z}_1)\mathcal{O}^{a_2,\epsilon_2}_{\Delta_2,-}(z_2,\bar{z}_2)\mathcal{O}^{a_4,\epsilon_4}_{\Delta_3+\Delta_4-1,+}(z_4,\bar{z}_4)\right\rangle
\end{aligned}
$$

At the level of OPE the above equation reads

$$
\mathcal{O}^{a_3,+1}_{\Delta_3,+}(z_3,\bar{z}_3)\mathcal{O}^{a_4,+1}_{\Delta_4,+}(z_4,\bar{z}_4) \sim -\frac{1}{2z_{34}}B(\Delta_3-1,\Delta_4-1)if^{a_3a_4x}\mathcal{O}^{x,+1}_{\Delta_3+\Delta_4-1,+}(z_4,\bar{z}_4) \tag{2.4.7}
$$

### 2.4.2 Subleading terms: $\mathcal{O}(1)$

In this section we show that the subleading ($\mathcal{O}(1)$) term in the OPE expansion remains same as the MHV case. As we know from the study of MHV gluon amplitudes [22], the descendants of the leading soft gluon symmetry algebra appears at $\mathcal{O}(1)$ in the OPE expansion. Here we only write down the action of the relevant operators on the celestial gluon amplitude. The leading conformally soft gluon operator [23, 24, 32–36] for positive helicity is defined as

$$R_0^{1,a}(z) = \lim_{\Delta\to 1}(\Delta - 1)\mathcal{O}^a_{\Delta,+}(z,\bar{z}) \tag{2.4.8}$$

where $\mathcal{O}^a_{\Delta,+}(z,\bar{z})$ is a positive helicity primary gluon operator with scaling dimension $\Delta$. The soft current $R_0^{1,a}(z)$ is a Kac-Moody current [19, 101–106]. The modes of the current $R_0^{1,a}(z)$ are denoted by $R^{1,a}_{p,0}$. For our purpose we only mention the correlation function of the descendants $R^{1,a}_{-p,0}\,\mathcal{O}^b_{\Delta,\sigma}(z,\bar{z}),\ p \geq 1$ with a collection of gluon primaries. These are given by[3] [22]

$$\left\langle R^{1,a}_{-p,0}\,\mathcal{O}^b_{\Delta,\sigma}(z,\bar{z})\prod_{i=1}^{n}\mathcal{O}^{a_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle = \mathcal{R}^{1,a}_{-p,0}(z)\left\langle \mathcal{O}^b_{\Delta,\sigma}(z,\bar{z})\prod_{i=1}^{n}\mathcal{O}^{a_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle \tag{2.4.9}$$

where the operator $\mathcal{R}^{1,a}_{-p,0}(z)$ is defined as

$$\mathcal{R}^{1,a}_{-p,0}(z)\left\langle \mathcal{O}^b_{\Delta,\sigma}(z,\bar{z})\prod_{i=1}^{n}\mathcal{O}^{a_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle = \sum_{k=1}^{n}\frac{T^a_k}{(z_k - z)^p}\left\langle \mathcal{O}^b_{\Delta,\sigma}(z,\bar{z})\prod_{i=1}^{n}\mathcal{O}^{a_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle \tag{2.4.10}$$

[3] Here we are using the notation of [24].

Let us now consider the $\mathcal{O}(1)$ term in the OPE expansion of (2.4.3), given by

$$\begin{aligned}
&\widetilde{\mathcal{M}}_4^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-},2^{a_2,\epsilon_2}_{\Delta_2,-},3^{a_3,\epsilon_3}_{\Delta_3,+},4^{a_4,\epsilon_4}_{\Delta_4,+}\right)\big|_{\mathcal{O}(1)}\\
&=\frac{\mathcal{N}_4}{(2\pi)^4}\frac{z_{12}^3}{z_{24}z_{41}}\Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\Gamma(\Delta_4-1)\\
&\times\int\widetilde{d^3\hat{x}}\left[\frac{f^{a_1a_2x}f^{xa_3a_4}}{z_{14}}+\frac{f^{a_2a_3x}f^{xa_4a_1}}{z_{24}}+(\Delta_3-1)\frac{(\bar{z}-\bar{z}_4)}{y^2+|z-z_4|^2}f^{a_1a_2x}f^{xa_3a_4}\right](-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}\\
&\times(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}(-q(z_4,\bar{z}_4)\cdot\hat{x})^{-\Delta_3-\Delta_4+2}\int_0^{i\infty}d\tau\,\tau^{-1-\beta_4}\phi_B(\tau)\left(e^{2\pi i\beta_4}-1\right)
\end{aligned}$$

With the help of the operator (2.4.10), RHS of the above equation can be written as

$$\begin{aligned}
&\widetilde{\mathcal{M}}_4^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-},2^{a_2,\epsilon_2}_{\Delta_2,-},3^{a_3,\epsilon_3}_{\Delta_3,+},4^{a_4,\epsilon_4}_{\Delta_4,+}\right)\big|_{\mathcal{O}(1)}=\frac{1}{2}B(\Delta_3-1,\Delta_4-1)\left[-\frac{(\Delta_3-1)}{(\Delta_3+\Delta_4-2)}if^{xa_3a_4}\mathcal{L}_{-1}(4)\right.\\
&\times\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1},2^{a_2,\epsilon_2}_{\Delta_2},4^{x,\epsilon_4}_{\Delta_3+\Delta_4-1}\right)+\frac{(\Delta_4-1)}{(\Delta_3+\Delta_4-2)}\mathcal{R}^{1,a_3}_{-1,0}(4)\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1},2^{a_2,\epsilon_2}_{\Delta_2},4^{a_4,\epsilon_4}_{\Delta_3+\Delta_4-1}\right)\\
&\left.+\frac{(\Delta_3-1)}{(\Delta_3+\Delta_4-2)}\mathcal{R}^{1,a_4}_{-1,0}(4)\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1},2^{a_2,\epsilon_2}_{\Delta_2},4^{a_3,\epsilon_4}_{\Delta_3+\Delta_4-1}\right)\right]
\end{aligned} \tag{2.4.11}$$

where the argument $(4)$ in the operators $\mathcal{L}_{-1},\mathcal{R}^{1,a}_{-1,0}$ implies that these modes are acting on the last particle of the 3-point amplitude. At the level of the OPE we have [4]

$$\begin{aligned}
&\mathcal{O}^{a_3,+1}_{\Delta_3,+}(z_3,\bar{z}_3)\mathcal{O}^{a_4,+1}_{\Delta_4,+}(z_4,\bar{z}_4)\big|_{\mathcal{O}(1)}\sim\frac{1}{2}B(\Delta_3-1,\Delta_4-1)\left[-\frac{(\Delta_3-1)}{(\Delta_3+\Delta_4-2)}if^{xa_3a_4}L_{-1}\right.\\
&\left.+\left(\frac{(\Delta_4-1)}{(\Delta_3+\Delta_4-2)}\delta^{a_3y}\delta^{a_4x}+\frac{(\Delta_3-1)}{(\Delta_3+\Delta_4-2)}\delta^{a_4y}\delta^{a_3x}\right)R^{1,y}_{-1,0}\right]\mathcal{O}^{x,+1}_{\Delta_3+\Delta_4-1,+}(z_4,\bar{z}_4)
\end{aligned} \tag{2.4.12}$$

One can recognize that this is the $\mathcal{O}(1)$ term in the OPE obtained by [22, 107] in the MHV case. Thus we can see that $\mathcal{O}(1)$ OPE also doesn't change in the presence of the massive scalar background.

[4] We would like to emphasize that (3.4.2) does *not* hold beyond MHV sector. In the $N^kMHV$ sector the soft symmetry algebra changes because of the existence of the negative helicity soft gluons and as a result the $O(z^0\bar{z}^0)$ term in the OPE has to change. This is also the case for pure YM theory and has nothing to do with the existence of the massive scalar background.

## 2.5 Subleading soft gluon theorem in massive scalar background

In this section we show that the subleading conformal soft gluon theorem remains same in a massive scalar background. More precisely, we show that the subleading conformal soft limit, $\Delta_4 \to 0$, of (2.4.3) is equivalent to the action of the subleading soft operator [32–36] of the 4-th particle, on the 3-pt correlation function (2.4.3).

Hence we start with the 4-point amplitude (2.4.3) and take the conformal soft limit $\Delta_4 \to 0$ to get

$$\begin{aligned}
&\lim_{\Delta_4\to 0} \Delta_4 \widetilde{\mathcal{M}}_4^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-}, 2^{a_2,\epsilon_2}_{\Delta_2,-}, 3^{a_3,\epsilon_3}_{\Delta_3,+}, 4^{a_4,\epsilon_4}_{\Delta_4,+}\right)\\
&= -i\frac{\mathcal{N}_3}{(2\pi)^4}\frac{z_{12}^3}{z_{23}z_{34}z_{41}}\left(f^{a_1a_2x}f^{xa_3a_4} - \frac{z_{12}z_{34}}{z_{13}z_{24}}f^{a_1a_3x}f^{xa_2a_4}\right)\\
&\times \Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\int \widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}\\
&\times(-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1}(-q(z_4,\bar{z}_4)\cdot\hat{x})\int_0^{i\infty} d\tau\, \tau^{-\beta_3}\phi_B(\tau)\left(e^{2\pi i\beta_3}-1\right)
\end{aligned}$$

The above result can be written as

$$\begin{aligned}
&\lim_{\Delta_4\to 0} \Delta_4 \widetilde{\mathcal{M}}_4^{\Phi}(1^{a_1,\epsilon_1}_{\Delta_1,-}, 2^{a_2,\epsilon_2}_{\Delta_2,-}, 3^{a_3,\epsilon_3}_{\Delta_3,+}, 4^{a_4,\epsilon_4}_{\Delta_4,+}) = \left(\frac{c_1}{z_{14}}+\frac{c_2}{z_{24}}+\frac{c_3}{z_{34}}\right) i\frac{\mathcal{N}_3}{(2\pi)^4}\frac{z_{12}^3}{z_{23}z_{31}}\\
&\times \Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\int \widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}\\
&\times(-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1}(-q(z_4,\bar{z}_4)\cdot\hat{x})\int_0^{i\infty} d\tau\, \tau^{-\beta_3}\phi_B(\tau)\left(e^{2\pi i\beta_3}-1\right)
\end{aligned} \tag{2.5.1}$$

where $c_1 = f^{xa_2a_3}f^{a_4a_1x}$, $c_2 = f^{a_1xa_3}f^{a_4a_2x}$ and $c_3 = f^{a_1a_2x}f^{a_4a_3x}$.

We shall now show that the above expression for the conformal soft limit of the celestial correlation function is the same as the action of the subleading soft operator on the 3-pt correlation function. The subleading soft gluon theorem in Mellin space is given by [47],

$$\left\langle R^{0,a_4}(z_4,\bar{z}_4)\prod_{i=1}^{3}\mathcal{O}^{a_i,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle = -\sum_{k=1}^{3}\frac{\epsilon_k}{z_{4k}}(-2\bar{h}_k+1+\bar{z}_{4k}\bar{\partial}_k)\,T^{a_4}_k\mathcal{P}^{-1}_k\left\langle\prod_{i=1}^{3}\mathcal{O}^{a_i,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle \tag{2.5.2}$$

where $R^{0,a}(z,\bar{z})$ is the subleading conformally soft gluon operator defined by

$$R^{0,a}(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}^a_{\Delta,+}(z,\bar{z}) \tag{2.5.3}$$

$T^a_k$ is the lie algebra generator in the adjoint representation of the gauge group and $\mathcal{P}^{-1}_k$ is a dimension lowering operator acting on the $k$-th primary field. The action of both is

$$T^a_k\mathcal{O}^{a_i,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i) = if^{aa_ix}\,\mathcal{O}^{x,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\,\delta_{k,i},\;\; \mathcal{P}^{-1}_k\mathcal{O}^{a_i,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i) = \mathcal{O}^{a_i,\epsilon_i}_{\Delta_i-1,\sigma_i}(z_i,\bar{z}_i)\delta_{ki} \tag{2.5.4}$$

By repeated use of the following equation,

$$(-q_i\cdot\hat{x}) = \frac{y^2+(z-z_i)(\bar{z}-\bar{z}_i)}{y} \tag{2.5.5}$$

and the explicit expression for 3-point function given in (2.4.3) on the RHS of (2.5.2), one can show that (appendix A.1)

$$\begin{aligned}-\sum_{k=1}^{3}\frac{\epsilon_k}{z_{4k}}(-2\bar{h}_k+1+\bar{z}_{4k}\bar{\partial}_k)\,T^{a_4}_k\mathcal{P}^{-1}_k\left\langle\prod_{i=1}^{3}\mathcal{O}^{a_i,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle &= \left(\frac{c_1}{z_{14}}+\frac{c_2}{z_{24}}+\frac{c_3}{z_{34}}\right)i\frac{\mathcal{N}_3}{(2\pi)^4}\,\frac{2z_{12}^3}{z_{23}z_{31}}\\ \times\Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\int\widetilde{d^3\hat{x}}&(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}\\ \times(-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1}&(-q(z_4,\bar{z}_4)\cdot\hat{x})\int_0^{i\infty}d\tau\,\tau^{-\beta_3}\phi_B(\tau)\left(e^{2\pi i\beta_3}-1\right)\end{aligned} \tag{2.5.6}$$

Comparing (2.5.1), (2.5.2) and (2.5.6) we get

$$\begin{aligned}\lim_{\Delta_4\to 0}\ \Delta_4\widetilde{\mathcal{M}}_4^{\Phi}(1^{a_1,\epsilon_1}_{\Delta_1,-},2^{a_2,\epsilon_2}_{\Delta_2,-},3^{a_3,\epsilon_3}_{\Delta_3,+},4^{a_4,\epsilon_4}_{\Delta_4,+}) &= \left\langle R^{0,a_4}(z_4,\bar{z}_4)\prod_{i=1}^{3}\mathcal{O}^{a_i,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle \\ &= -\frac{1}{2}\sum_{k=1}^{3}\frac{\epsilon_k}{z_{4k}}(-2\bar{h}_k+1+\bar{z}_{4k}\bar{\partial}_k)\,T^{a_4}_k\mathcal{P}^{-1}_k\left\langle\prod_{i=1}^{3}\mathcal{O}^{a_i,\epsilon_i}_{\Delta_i,\sigma_i}(z_i,\bar{z}_i)\right\rangle\end{aligned} \tag{2.5.7}$$

Thus we see that subleading soft gluon theorem for a positive helicity soft gluon does not change if we couple the Yang-Mills with the massive complex scalar in a chiral way mentioned in section 2.3.

### 2.5.1 BG equations in massive scalar background

A set of differential equations for the gluon MHV amplitudes were obtained in [22] by demanding the consistency between the subleading soft gluon theorem and OPE factorization at $\mathcal{O}(1)$ [5]. Here we have shown that, even in the case of MHV amplitudes in a massive scalar background the subleading soft gluon theorem and the OPE factorization at $\mathcal{O}(1)$ do not change. Hence we conclude that the celestial MHV amplitudes coupled to a massive scalar background also satisfy BG equations. This is not surprising because the existence of the null state, which gives rise to the differential equations, is guaranteed by the leading and the subleading soft gluon theorems and $SL(2,C)$ invariance. The presence of a massive dilaton background breaks the scaling as well as translational invariance but the soft gluon theorems remain unchanged. So the BG equations should also not change.

[5] Although these equations look very similar to the KZ equation which appears in WZW models, they are qualitatively different. In particular, they cannot be derived by any Sugawara construction on the celestial sphere. One way to see this is the following. Sugawara construction leads to the quantization of dimensions of primary operators which we know is not the case in celestial holography.

## 2.6 Discussions

In this chapter we extracted the OPE between two positive helicity outgoing gluons from the Mellin amplitude of the Yang-Mills theory chirally coupled to massive scalar background. The leading order term was already computed in [98]. Here we have computed a subleading term. One of our motivation behind this work was to check if the scattering amplitudes in this theory is also a solution of the BG equations. We have shown in this chapter that this is indeed the case[6]. Though the scaling and translation symmetry were explicitly broken for the theory we considered, the leading and subleading soft theorems remain unchanged. The OPE factorization at $\mathcal{O}(1)$ is completely determined in terms of the descendants of the $SL(2, C)$ and the leading soft symmetry algebra in the same way as the MHV amplitudes. On the other hand we have also shown that the subleading soft gluon theorem does not change in the massive background. Thus by comparing the $\mathcal{O}(1)$ OPE with the subleading soft gluon theorem we get the same BG equations as the MHV amplitudes. More generally, we can say that the scattering amplitudes of all the theories which respect the symmetries coming from the leading and subleading soft gluon theorems should satisfy the BG equations.

However, if one considers the graviton scattering amplitude and breaks some of the symmetries considered here, the situation will change completely. This is because of the fact that the leading soft graviton theorem is a consequence of supertranslation invariance. So if in a gravitational theory the translation symmetry is broken, the leading soft graviton theorem would no longer holds. Then one can ask the question that what happens to the decoupling equations obtained for the MHV graviton scattering amplitudes in [21].

It would also be interesting to check how the analysis of our work would change in the

[6]This seems to raise a puzzle because we are saying that the MHV amplitudes in pure YM and the YM coupled to massive dilaton both satisfy the BG equation. The resolution of this is the following: The MHV amplitudes in pure YM are not only the solutions of BG equations but they also have to satisfy the Ward identities coming from the space time translation invariance and the scale invariance of the YM theory. But when coupled to the massive dilaton background both these Ward identities are no longer valid and so we get a different kind of MHV amplitude by solving the BG equation.

context of deformed soft algebras for gauge theories recently considered in [108]. We hope to answer some of these questions in the near future.

# Chapter 3

# All OPEs Invariant under the Infinite Symmetry Algebra for Gluons on the Celestial Sphere

This chapter is based on the work [25].

## 3.1 Introduction

Celestial amplitudes are the primary observables in celestial conformal field theory (CCFT). For massless particles, these amplitudes are obtained by performing Mellin transforms of momentum-space scattering amplitudes, as discussed in Section 1.1.2. Celestial CFTs differ significantly from conventional two-dimensional CFTs, most notably in their symmetry structure. In particular, CCFTs exhibit an extended set of infinite-dimensional current algebra symmetries, which have no direct analog in standard 2D CFTs. These enhanced asymptotic symmetries [19, 21–24, 40, 43, 45, 76–78, 90, 96, 97, 108–123] are intimately connected to soft factorization theorems in gauge theories and gravity.

Some of these novel features of CCFT were introduced in Chapter 1. In Section 1.1.6.1, we explored the so-called $S$ algebra—an infinite-dimensional soft symmetry algebra generated by conformally soft positive-helicity gluon operators. Additionally, Section 1.1.5 includes a brief discussion on how celestial operator product expansions (OPEs) can be derived from collinear factorization and asymptotic symmetry considerations. Operator Product Expansion (OPE) in CCFT correspond to the collinear limit in the bulk and it plays a very important role in the study of the dual theory [21, 23, 26, 34, 47–50, 92, 96, 100, 107, 115, 117, 124–130]. In a previous work [48], the authors have studied the $w_{1+\infty}$ invariant OPEs in theories of gravity. They showed that there are an infinite number of theories on the celestial sphere which are $w_{1+\infty}$ invariant. By deriving the OPE from graviton scattering amplitudes they have also shown explicitly in [26] that the self dual gravity is one example of this infinite family.

In this chapter we perform a similar analysis for gluons. In the case of gluons the infinite symmetry algebra is know as the $S$ algebra [23, 24]. We write down all possible $S$ invariant OPE structures between two positive helicity outgoing gluons. We find that there is a (discrete) infinite number of such structures and presumably, each one of them corresponds to a $S$ invariant theory of gluons in the bulk. However, a more explicit Lagrangian description of these theories are not known to us.

There is an important difference between the analyses of $w_{1+\infty}$ and $S$-invariant theories, which we want to point out. $S$ algebra does not contain the Poincare generators. Therefore the consistent OPEs need not be Poincare invariant. However in this chapter we make sure that all the OPEs are (conformal) Lorentz invariant and this plays an important role. This is along the line of [49, 50, 98, 127].

We reviewed the soft gluon symmetry algebra known as the $S$ algebra briefly in section 1.1.6.1. In section 3.2, the general structure of the OPE between two positive helicity outgoing gluons on the celestial sphere has been discussed. We have argued, how the null states of the MHV-sector can be used to write down the general OPE. In section 3.3, we have written down the null states that appear at $\mathcal{O}(z^0\bar{z}^0)$ of the gluon-gluon OPE in the MHV-sector. These are not the complete set of null states that the MHV sector has at $\mathcal{O}(z^0\bar{z}^0)$. There are more of them. We talk about them later in section 3.7, where we have discussed the Knizhnik-Zamolodchikov (KZ)-type null states. Section 3.4 explicitly shows how to organise the OPE at every order. For simplicity we focus on the $\mathcal{O}(z^0\bar{z}^0)$ terms in the OPE. We have also discussed the transformation properties of MHV-null states under the $S$ algebra in this section, which are required to organise the OPE. Section 3.5 shows the invariance of the $\mathcal{O}(z^0\bar{z}^0)$ OPE under $S$ algebra. In section 3.6, we have argued, how an infinite number theories can exist on the celestial sphere. We conclude the chapter with the discussion of the results found in this chapter and some future directions in section 3.9.

## 3.2 General structure of the OPE between two positive helicity outgoing gluons

We can write the general structure of the OPE between two positive helicity gluons invariant under the $S$ algebra as

$$\begin{aligned} O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2) &= \frac{-if^{abc}}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1-1+n,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}^n O^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \\ &+ \sum_{p,q=0}^{\infty} z_{12}^p\bar{z}_{12}^q\sum_{k=1}^{\tilde{n}_{p,q}}\tilde{C}^k_{p,q}(\Delta_1,\Delta_2)\tilde{O}^{ab}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar{z}_2) \end{aligned} \tag{3.2.1}$$

where in the second line we have now added the $S$ algebra descendants of a positive helicity gluon. The sum over $k$ could be finite or infinite depending on the theory. Our goal is to determine the descendants $\tilde{O}^{ab}_{k,p,q}$ and the OPE coefficients $\tilde{C}^k_{p,q}$ in a general $S$-invariant theory.

In the gravity case [48], it was found that any $w$-invariant OPE can be written in terms of the MHV OPE and the MHV null states. We have also checked by detailed calculation that this structure holds in the self-dual gravity theory [26] which is $w$ invariant. The same reasoning also goes through for the $S$ algebra and gluons. We summarize the argument below.

Since the $S$ algebra is universal, i.e the *same* [1] algebra holds in any $S$ invariant theory, it is reasonable to assume that there is a Master OPE which holds in *all* $S$ invariant theories. Let us now consider the gluon-gluon OPE in the (tree-level) MHV sector of the pure YM theory. Since the MHV sector is $S$ invariant the Master OPE, when inserted in a MHV gluon scattering amplitude, should reproduce the known MHV sector OPE. Therefore one can write

$$\text{Master OPE} = \text{MHV-sector OPE} + R \tag{3.2.2}$$

where $R$ should *vanish* inside an MHV scattering amplitude. This is possible only if $R$ is a linear combination of MHV *null states*. Now since the MHV-sector OPE already contains the universal singular terms (1.1.99) of the gluon-gluon OPE, $R$ consists only of non-singular terms. So we can write,

$$\begin{aligned} O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{Any Theory}} &= O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{MHV}} \\ &+ \sum_{p,q=0}^{\infty} z_{12}^{p}\bar{z}_{12}^{q}\sum_{k=1}^{\tilde{n}_{p,q}}\tilde{C}^{k}_{p,q}(\Delta_1,\Delta_2)M^{ab}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar{z}_2) \end{aligned} \tag{3.2.3}$$

where $M^{ab}_{k,p,q}$ are the MHV null states. So when "Any Theory" is taken to be the MHV sector, $M^{ab}_{k,p,q}$ vanishes and we get back the MHV sector OPE by construction.

We now describe the MHV null states which are of interest to us. In this chapter we apply this general procedure to write down the OPE at $\mathcal{O}(z_{12}^{0}\bar{z}_{12}^{0})$.

[1] For example, this is not true in the conventional $2-D$ CFTs because different CFTs have different Virasoro central charges and so different conformal symmetry algebras.

## 3.3 Null states in the MHV sector

The general null state at order $z_{12}^0 \bar{z}_{12}^0$ is given by [2]

$$\Psi_j^{ab}(\Delta) = R^{-j,a}_{\frac{j-1}{2},\frac{j+1}{2}} \mathcal{O}^{b,+}_{\Delta+j} - \frac{(-1)^j j}{\Gamma(j+2)} \frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-2)} R^{1,a}_{-1,0} \mathcal{O}^{b,+}_{\Delta-1} - \frac{(-1)^j}{\Gamma(j+1)} \frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)} R^{0,a}_{-\frac{1}{2},\frac{1}{2}} \mathcal{O}^{b,+}_{\Delta} \tag{3.3.1}$$

Here we have ignored the $(p,q)$ index and have simply written $M_k^{ab}$ instead of $M_{k,0,0}^{ab}$ for the order $z_{12}^0 \bar{z}_{12}^0$ MHV null states.

Now it turns out that the following basis of null states

$$M_k^{ab}(\Delta) = \sum_{i=1}^{k} \frac{1}{\Gamma(k-i+1)} \frac{\Gamma(\Delta+k-1)}{\Gamma(\Delta+i-1)} \Psi_i^{ab}(\Delta) \tag{3.3.2}$$

is more convenient because they transform nicely under the $S$-algebra. We will discuss their transformation law in the next section.

We conclude this section by defining the antisymmetric part of the null states $M_k^{ab}(\Delta)$ as

$$M_k^a(\Delta) = f^{abc} M_k^{bc}, \tag{3.3.3}$$

## 3.4 Organizing the OPE at every order

Since the MHV sector is $S$ invariant the MHV null states must form a representation of the $S$ algebra. In other words every generator of the $S$ algebra must map any MHV null state to another MHV null state. Our analysis shows that this representation is reducible and different $S$ invariant theories correspond to different irreducible components of this representation. So our first job is to study the action of the $S$ algebra generators on the

[2] These null states can be obtained by taking soft limits of the gluon-gluon MHV OPE [22]. The relevant terms in the gluon-gluon OPE in the MHV sector which gives rise to these null states are given in (3.5.2).

MHV null states. This is facilitated by the following observation.

We have discussed in Section 1.1.6.1 that the $S$ algebra is generated by an infinite number of holomorphic soft currents $\{R^{k,a}_p(z)\}$ [3] where $k = 1, 0, -1, -2, \cdots$ . is the dimension $\Delta$ of the soft operator and $\frac{k-1}{2} \leq p \leq \frac{1-k}{2}$. For a fixed $k$, the soft currents $R^{k,a}_{\frac{1-k}{2}}, \cdots, R^{k,a}_{\frac{k-1}{2}}$ transform in a $(2-k)$ dimensional representation of the $\overline{sl_2(R)}$ [4]. This can be seen from the following commutation relations:

$$\begin{gathered}\left[H^0_{0,-1}, R^{k,a}_{m,p}\right] = \frac{1}{2}(2p+k-3)R^{k,a}_{m,p-1} \text{ for } p > \frac{k-1}{2}, \qquad \left[H^0_{0,-1}, R^{k,a}_{m,\frac{k-1}{2}}\right] = 0 \\ \left[H^0_{0,0}, R^{k,a}_{m,p}\right] = -2p\, R^{k,a}_{m,p} \\ \left[H^0_{0,1}, R^{k,a}_{m,p}\right] = \frac{1}{2}(2p-k+3)R^{k,a}_{m,p+1} \text{ for } p < -\frac{k-1}{2}, \qquad \left[H^0_{0,1}, R^{k,a}_{m,-\frac{k-1}{2}}\right] = 0\end{gathered} \tag{3.4.1}$$

Now let us consider the currents $R^{1,a}_0, R^{0,a}_{\frac{1}{2}}, R^{-1,a}_1, \cdots$ with the lowest $\overline{sl_2(R)}$ weights. Starting from $R^{1,a}_0$ all the currents in this family can be obtained by applying the global subleading soft gluon operator $R^{0,b}_{\frac{1}{2},\frac{1}{2}}$. This can be seen from the the following commutation relations

$$\left[R^{0,a}_{\frac{1}{2},\frac{1}{2}}, R^{k,b}_{m,\frac{1-k}{2}}\right] = -if^{abc}(2-k)R^{k-1,c}_{m+\frac{1}{2},\frac{2-k}{2}} \tag{3.4.2}$$

Equations (3.4.1) and (3.4.2) show that we can write any generator of the $S$ algebra as a sum of products of the generators $\left(R^{1,a}_{n,0}, R^{0,a}_{\frac{1}{2},\frac{1}{2}}, H^0_{0,0}, H^0_{0,\pm1}\right)$. Therefore in order to study the action of the $S$ algebra generators on the MHV null states we just need to focus on these finite number of generators.

[3] Here $-p$ is the antiholomorphic weight of the current.

[4] Note that we are assuming the theory to be (conformal) Lorentz invariant.

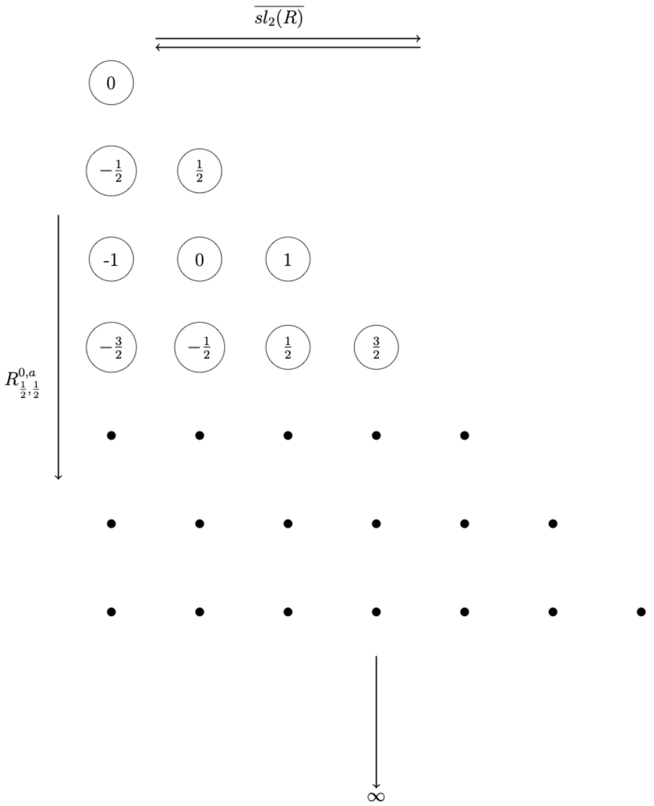


Figure 3.1: The figure shows the soft gluon currents arranged in representations of $\overline{sl_2(R)}$. The $\overline{sl_2(R)}$ generators move the currents horizontally in both directions whereas the global subleading soft gluon symmetry generator $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ moves the currents vertically downward.

### 3.4.1 Transformation properties of the null states under $\overline{sl_2(R)}$ algebra

Using the action of different generators of $\overline{sl_2(R)}$ algebra on the gluon primary operators and the commutation relations (3.4.1) it is easy to show that

$$H^0_{0,1}\Psi^{ab}_k(\Delta) = 0. \tag{3.4.3}$$

Thus (3.3.2) implies that

$$H^0_{0,1}M^{ab}_k(\Delta) = 0. \tag{3.4.4}$$

Therefore the null states $M_k^{ab}$ are $\overline{sl_2(R)}$ primaries.

### 3.4.2 Transformation properties of the null states under the leading soft gluon current algebra

One can easily check that under the leading soft gluon current algebra, the null states (3.3.2) transform as

$$\begin{gathered} R_{0,0}^{1,a} M_k^{bc}(\Delta) = -i f^{abd} M_k^{dc}(\Delta) - i f^{acd} M_k^{bd}(\Delta) \\ R_{n,0}^{1,a} M_k^{bc}(\Delta) = 0,\ n > 0 \end{gathered} \tag{3.4.5}$$

Therefore the null states $M_k^{ab}$ are the leading soft gluon current algebra primaries.

### 3.4.3 Transformation properties of the null states under subleading soft gluon operator $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$

This is perhaps the most important transformation property because it mixes the null states $M_k^{ab}$ with different $k$ values. The action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ on $M_k^{bc}(\Delta)$ is given by,

$$R^{0,a}_{\frac{1}{2},\frac{1}{2}} M_k^{bc}(\Delta) = -(k+2) i f^{abx} M_{k+1}^{xc}(\Delta-1) + (\Delta+k-2)\left[i f^{acx} M_k^{bx}(\Delta-1) + i f^{abx} M_k^{xc}(\Delta-1)\right] \tag{3.4.6}$$

We have used (B.2.1) to derive the above equation.

Now let us consider the set of null states:

$$M_k^{bc}(\Delta),\ k = 1, 2, \cdots, n. \tag{3.4.7}$$

From (3.4.6) we can see that if we set

$$M_{k+1}^{ab}(\Delta) = 0, k \geq n \geq 0. \tag{3.4.8}$$

then the set (3.4.7) is closed under the action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$. Moreover it follows from (3.4.6) that the infinite set of equations (3.4.8) is also invariant under the action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ because the index $k$ mixes only with $k' \geq k$. Therefore the truncation (3.4.8) is $S$ algebra invariant and we can get an $S$ invariant OPE if we keep only the finite set (3.4.7). Let us emphasize that the integer $n$ is in no way restricted by the $S$ invariance.

## 3.5 $\mathcal{O}(z_{12}^0 \bar{z}_{12}^0)$ OPE and its invariance under the $S$ algebra

Let us now consider the $\mathcal{O}(1)$ terms in the OPE when we keep only the finite set of MHV null states (3.4.7). In particular, we show that the $\mathcal{O}(1)$ terms in the OPE with the following coefficients are $S$-invariant:

$$\boxed{\begin{aligned} \mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\mathcal{O}(1)} &= \mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\text{MHV OPE at }\mathcal{O}(1)} \\ &+\sum_{k=1}^{n} B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2) \end{aligned}} \tag{3.5.1}$$

where $\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\text{MHV OPE at }\mathcal{O}(1)}$ is given by [22, 107]

$$\begin{aligned} \mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\text{MHV OPE at }\mathcal{O}(1)} = B(\Delta_1-1,\Delta_2-1)\Big[\Delta_1\, R^{1,a}_{-1,0}\mathcal{O}^{b,+}_{\Delta_1+\Delta_2-1}(0,0) \\ +\frac{\Delta_1-1}{\Delta_1+\Delta_2-2}R^{0,a}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{b,+}_{\Delta_1+\Delta_2}(0,0)\Big] \end{aligned} \tag{3.5.2}$$

Let us first apply $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ on the OPE (3.5.1). After some straightforward algebra we get,

$$\begin{aligned} R^{0,x}_{\frac{1}{2},\frac{1}{2}}\Big(\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\mathcal{O}(1)}\Big) - R^{0,x}_{\frac{1}{2},\frac{1}{2}}\Big[\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\text{MHV OPE at }\mathcal{O}(1)} \\ +\sum_{k=1}^{n} B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2)\Big] = if^{xay}(n+2)B(\Delta_1+n,\Delta_2-1)M^{yb}_{n+1}(\Delta_1+\Delta_2-1) \end{aligned} \tag{3.5.3}$$

Now, we have argued in the previous section that if the $\mathcal{O}(1)$ OPE of a $S$ invariant theory truncates at $k = n$, then $M^{ab}_{n+1}(\Delta)$ will be a null state of that theory. Thus, we can set the RHS of (3.5.3) to 0 and hence (3.5.1) is invariant under the action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$. Using (3.4.5), one can also verify that, (3.5.1) is invariant under the actions of $R^{1,a}_{n,0}$.

In [22], it was shown that the OPE in the MHV-sector is invariant under the action of $H^0_{0,1}$. We can also see from (3.4.3) that the null states $M^{ab}_k(\Delta)$ are annihilated by $H^0_{0,1}$. Thus, the OPE (3.5.1) is also invariant under $H^0_{0,1}$. Hence we conclude that the truncated OPE (3.5.1) is invariant under the $S$ algebra.

## 3.6 Infinite family of $S$ invariant theories

In section (3.4.3) we have shown that the following set of equations

$$M^{ab}_{k+1} = 0, k \geq n \geq 0. \tag{3.6.1}$$

are $S$ invariant. Thus at $\mathcal{O}(z^0\bar{z}^0)$ we can truncate the OPE (3.2.3) at an arbitrary $n$ in an $S$-invariant way. That is to say $S$-invariance does not fix the value of the integer $n$. Hence we can get a discrete infinite family of $S$-invariant OPEs for different choices of the integer $n$. Each of these consistent OPEs correspond to a $S$ invariant theory of gluons. But, at present we do not know the Lagrangian description of these theories except perhaps the self-dual Yang-Mills theory.

## 3.7 Knizhnik-Zamolodchikov type null states

Knizhnik-Zamolodchikov (KZ) type null states contain descendants of the holomorphic translation generator $L_{-1}$ on the celestial sphere. They can be obtained algebraically by determining the relevant primary descendant but in our case we can bypass this tedious procedure if we use the OPE commutativity

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) = \mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1) \tag{3.7.1}$$

The reason behind this is that the $\mathcal{O}(z_{12}^0\bar{z}_{12}^0)$ terms of the OPE, as written in (3.5.1), are not manifestly symmetric under the exchange (3.7.1). Therefore OPE commutativity imposes non-trivial constraints on the OPE coefficients and one such constraint is essentially the KZ equation. The process can be further simplified if make the operator $\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)$ leading soft by taking the limit $\Delta_2 \to 1$. Now a straightforward calculation gives the KZ type null state

$$\boxed{K^a(\Delta) = \xi^a(\Delta) - i\sum_{k=1}^{n} M^a_k(\Delta+1) = 0.} \tag{3.7.2}$$

where

$$\xi^a(\Delta) = C_A L_{-1}\mathcal{O}^{a,+}_{\Delta} - (\Delta+1)R^{1,b}_{-1,0}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta} - R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta+1} \tag{3.7.3}$$

is the KZ type null state in the MHV-sector [22] and $M^a_k(\Delta)$ is the antisymmetric part of the null state $M^{ab}_k(\Delta)$ defined in (3.3.3). We have also used the identity $f^{abx}f^{aby} = C_A\delta^{xy}$ in deriving the KZ type null state equation (3.7.2).

Another null state equation involving the descendant $L_{-1}\mathcal{O}^{a,+}_{\Delta}$ can be obtained from (3.7.1) in a similar way by taking the subleading conformal soft limit $\Delta_2 \to 0$. It is given by

$$(\Delta-1)\xi^a(\Delta) - \sum_{k=1}^{n}(\Delta+k)M^a_k(\Delta+1) = 0. \tag{3.7.4}$$

Now multiplying equation (3.7.2) by $(\Delta-1)$ and then subtracting it from (3.7.4) we get the following (current algebra) null state

$$\chi_n^{1,a}(\Delta) = \sum_{k=1}^{n}(k+1)M_k^a(\Delta) = 0 \tag{3.7.5}$$

One can continue this procedure and get other current algebra null states by taking conformal soft limits $\Delta_2 \to k,\ k \leq -1$. We can denote them by $\{\chi_n^1(\Delta), \chi_n^2(\Delta), \cdots\}$. However, it can be shown that after a finite iteration this procedure stops due to the truncation (3.4.8).

### 3.7.1 S invariance of the KZ type null state

In this section we show that the KZ type null state (3.7.2) is $S$-invariant.

First of all, the states $M_k^{ab}(\Delta)$, and as a result $M_k^a(\Delta) = f^{abc}M_k^{bc}(\Delta)$, are annihilated by $H^0_{0,1}$. Therefore the state $K^a(\Delta)$ is a primary of $\overline{sl_2(R)}$ because the KZ type null state (3.7.3) in the MHV-sector is annihilated [22] by $H^0_{0,1}$.

Similarly, one can show after some algebra that the following relation holds

$$\begin{aligned} R^{0,c}_{\frac{1}{2},\frac{1}{2}}\, K^a(\Delta) &= (\Delta-2)if^{cax}K^x(\Delta-1) - (n+2)f^{cbx}f^{aby}M^{xy}_{n+1}(\Delta) + f^{cax}\chi_n^{1,x}(\Delta) \\ &+ f^{cby}f^{bax}\left[\sum_{k=1}^{n}(k+1)M_k^{yx}(\Delta) + 2E^{yx}(\Delta)\right] + f^{cab}f^{byx}E^{yx}(\Delta) \end{aligned} \tag{3.7.6}$$

where

$$E^{yx}(\Delta) = (\Delta-2)R^{1,y}_{-1,0}\mathcal{O}^{x,+}_{\Delta-1} + R^{0,y}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{x,+}_{\Delta} \tag{3.7.7}$$

Now we know that, in a theory in which the $\mathcal{O}(z^0_{12}\bar{z}^0_{12})$ OPE truncates at $k = n$, i.e, (3.4.8) holds, both $M^{bc}_{n+1}(\Delta)$ and $\chi_n^{1,a}(\Delta)$ are null states. Thus we can set them to 0 and get,

$$\begin{aligned} &R^{0,c}_{\frac{1}{2},\frac{1}{2}}\, K^a(\Delta) \\ &= (\Delta-2)if^{cax}K^x(\Delta-1) + f^{cby}f^{bax}\left[\sum_{k=1}^{n}(k+1)M_k^{yx}(\Delta) + 2E^{yx}(\Delta)\right] + f^{cab}f^{byx}E^{yx}(\Delta) \end{aligned} \tag{3.7.8}$$

We show in Appendix (B.3) that the second and the third terms on the RHS of (3.7.8) are actually zero. Taking this into account we get

$$R^{0,c}_{\frac{1}{2},\frac{1}{2}}\, K^a(\Delta) = (\Delta - 2) i f^{cax} K^x(\Delta - 1) \tag{3.7.9}$$

Thus we see that $R^{0,c}_{\frac{1}{2},\frac{1}{2}}$ maps the KZ type null state $K^a(\Delta)$ to linear combination of other null states in the theory. Hence, the null state equation

$$K^a(\Delta) = 0 \tag{3.7.10}$$

is $S$ invariant.

## 3.8 Example: celestial OPE in self dual Yang Mills

We now consider the example of self-dual Yang Mills (SDYM) theory which is known to be $S$ invariant. In particular, we write the $\mathcal{O}(1)$ terms explicitly and show that it can be written completely in terms of MHV OPE and MHV null states. The colour-dressed self dual Yang-Mills (SDYM) amplitude is given by [131, 132],

$$\mathcal{A}^{(1)}_{n\,\text{SDYM}} = \sum_{\sigma \in S_{n-1}/R} c^{a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}} A^{(1)}_{n\,\text{SDYM}} \left(\sigma(1)\sigma(2)\cdots\sigma(n)\right) \tag{3.8.1}$$

where the sum is over non-cyclic permutations, modulo reflection of the list $\sigma$. The cyclic $n$-gluon colour factors is given by,

$$c^{a_1 a_2 \cdots a_n} = f^{b_1 a_1 b_2} f^{b_2 a_2 b_3} \cdots f^{b_n a_n b_1} \tag{3.8.2}$$

and $A^{(1)}_{\text{SDYM}}\left(\sigma(1)\sigma(2)\cdots\sigma(n)\right)$ is the colour ordered amplitudes, given by [131, 132],

$$A^{(1)}_{n\,\text{SDYM}}\left(123\cdots n\right) = M_n \sum_{1\le i_1<i_2<i_3<i_4\le n} \frac{\langle i_1 i_2\rangle\, [i_2 i_3]\, \langle i_3 i_4\rangle\, [i_4 i_1]}{\langle 12\rangle\, \langle 23\rangle \cdots \langle n1\rangle} \tag{3.8.3}$$

where $M_n$ is a numerical normalisation constant. Here we are working in the split signature $(-,+,-,+)$ and in this signature the null momentum of a massless particle is parametrized as,

$$p_i = \epsilon_i \omega_i \left(1 + z_i \bar{z}_i, z_i + \bar{z}_i, z_i - \bar{z}_i, 1 - z_i \bar{z}_i\right) \tag{3.8.4}$$

with $\epsilon_i = +1(-1)$ for outgoing (incoming) particles. $(z_i, \bar{z}_i)$ are the coordinates on the celestial torus. In our notation the angle and square brackets are given by,

$$\langle ij \rangle = -2\epsilon_i \epsilon_j \sqrt{\omega_i \omega_j} z_{ij}, \ [ij] = 2\sqrt{\omega_i \omega_j} \bar{z}_{ij} \tag{3.8.5}$$

Here, we are interested in the 5-point color dressed amplitudes only. For $n = 5$, the sum in (3.8.3) will give 20 terms. Let's start with the following term [133],

$$A^{(1)}_{5\,\mathrm{SDYM}}\,(12345) = \frac{s_{12}s_{23} + s_{45}s_{51} + s_{25}s_{45} + s_{25}s_{14} + \langle 24 \rangle [14] \langle 15 \rangle [52]}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \tag{3.8.6}$$

where $s_{ij} = (p_i + p_j)^2$. The Mellin transformation for 5-point colour-ordered amplitudes is given by,

$$\mathcal{M}_{5\,\mathrm{SDYM}}(1^+_{\Delta_1}, 2^+_{\Delta_2}, \cdots 5^+_{\Delta_5}) = \left(\prod_{k=1}^{5} \int_0^\infty d\omega_k \, \omega_k^{\Delta_k - 1}\right) A^{(1)}_{5\,\mathrm{SDYM}}\,(12345)\, \delta^{(4)}(\sum_{k=1}^{5} p_k) \tag{3.8.7}$$

We are interested in the OPE limit $4 \to 5$. For our purpose, the following parametrization of the 5-point momentum conserving delta function will be convenient [21],

$$\delta^{(4)}\left(\sum_{i=1}^{5} \epsilon_i \omega_i q_i\right)$$
$$= \frac{1}{4\omega_p} \delta(\omega_1 - \omega_1^*) \delta(\omega_2 - \omega_2^*) \delta(\omega_3 - \omega_3^*)$$
$$\times \delta\left(x - \bar{x} - t z_{45}\left(\frac{x}{z_{35}} - \frac{\bar{x}}{z_{25}}\right) - t \bar{z}_{45}\left(\frac{x}{\bar{z}_{25}} - \frac{\bar{x}}{\bar{z}_{35}}\right) + t z_{45} \bar{z}_{45}\left(\frac{x}{z_{35}\bar{z}_{25}} - \frac{\bar{x}}{z_{25}\bar{z}_{35}}\right)\right)$$

where we have used the parametrisation,

$$\omega_4 = t\omega_p,\ \omega_5 = (1-t)\omega_p \tag{3.8.8}$$

and for $i = \{1, 2, 3\}$ we have

$$\omega_i^* \;=\; \epsilon_i\,\omega_p\,(\sigma_{i,1} + tz_{45}\sigma_{i,2} + t\bar{z}_{45}\sigma_{i,3} + tz_{45}\bar{z}_{45}\sigma_{i,4}) \tag{3.8.9}$$

The $\sigma_{i,1},\ x,\ \bar{x}$ are given by

$$\sigma_{1,1} \;=\; -\frac{z_{25}\bar{z}_{35}}{z_{12}\bar{z}_{13}} \tag{3.8.10}$$

$$\sigma_{2,1} \;=\; \frac{z_{15}\bar{z}_{35}}{z_{12}\bar{z}_{23}} \tag{3.8.11}$$

$$\sigma_{3,1} \;=\; -\frac{z_{25}\bar{z}_{15}}{z_{23}\bar{z}_{13}} \tag{3.8.12}$$

$$x \;=\; z_{12}z_{35}\bar{z}_{13}\bar{z}_{25},\ \bar{x} = z_{13}z_{25}\bar{z}_{12}\bar{z}_{35} \tag{3.8.13}$$

Since, we are interested in the $\mathcal{O}(1)$ term only we will not require the other $\sigma_{i,j}$'s. However, one can find their expressions in [26].

Now, we will perform the OPE decomposition of the 5-point amplitude (3.8.7). We apply the strategy of [26]. First we write $A^{(1)}_{5\,\text{SDYM}}\,(12345)$, given by (3.8.6), in terms of $\{\omega_i, z_i, \bar{z}_i\}$ and substitute it in (3.8.7). Next, using (3.8.8), we can easily perform the $\omega_i,\ \{i = 1, 2, 3\}$ integrals. After expanding around $z_{45} = \bar{z}_{45} = 0$, one can then perform the $\omega_p$ and $t$ integrals. The leading term is already known in the literature. Here we concentrate on the $\mathcal{O}(1)$ terms. The $\mathcal{O}(1)$ terms are given by,

$$\mathcal{M}_{5\,\text{SDYM}}(1^+_{\Delta_1}, 2^+_{\Delta_2}, \cdots, 5^+_{\Delta_5})\big|_{\mathcal{O}(1)} = \delta\Big(\sum_{i=1}^{5}\Delta_i - 5\Big)\sum_{k=0}^{2} B(\Delta_4 - 1 + k, \Delta_5 - 1) \times\mathcal{F}_k(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\}) \tag{3.8.14}$$

where the explicit form of the functions $\mathcal{F}_k(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\})$ are not required for our purpose. For the other 19 terms one can check that the structure of the $B$-function is same. Hence including all those terms we write our final colour dressed 5-point celestial amplitude as,

$$\mathcal{M}_{5\,\text{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},\cdots 5^{a_5,+}_{\Delta_5})\big|_{\mathcal{O}(1)} = \delta\Big(\sum_{i=1}^{5}\Delta_i - 5\Big)\sum_{k=0}^{2} B(\Delta_4 - 1 + k, \Delta_5 - 1) \times \mathcal{G}_k^{(\{a_i\})}(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\}) \tag{3.8.15}$$

Now, to factorise the above 5-point SDYM amplitude into 4-point amplitude we will adopt here the same strategy as was done in the case of the self dual gravity in [26]. We do this by taking different conformally soft limits of the fourth gluon primary and replace the functions $\mathcal{G}_k^{(\{a_i\})}(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\})$ by correlation functions of the $S$ algebra descendants of the fifth gluon[5] and the first, second and the third gluons. For example, we can make the fourth gluon leading conformally soft ($\Delta_4 \to 1$) and use the leading soft factorization theorem to replace the function $\mathcal{G}_0^{(\{a_i\})}(\{z_i, \bar{z}_i, \Delta_i\})$ and by doing that we get

$$\begin{aligned}&\mathcal{M}_{5\,\text{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},\cdots 5^{a_5,+}_{\Delta_5})\big|_{\mathcal{O}(1)}\\ &= B(\Delta_4 - 1, \Delta_5 - 1) R^{1,a_4}_{-1,0}\mathcal{M}_{4\,\text{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},3^{a_3,+}_{\Delta_3},5^{a_5,+}_{\Delta_4+\Delta_5-1})\\ &\qquad + \delta\Big(\sum_{i=1}^{5}\Delta_i - 5\Big)\sum_{k=1}^{2} B(\Delta_4 - 1 + k, \Delta_5 - 1)\mathcal{G}_k^{(\{a_i\})}(\{z_i, \bar{z}_i, \Delta_i\}).\end{aligned} \tag{3.8.16}$$

We can repeat this procedure to find other two functions $\mathcal{G}_1$ and $\mathcal{G}_2$ by taking subleading ($\Delta_4 \to 0$) and sub-subleading ($\Delta_4 \to -1$) conformally soft limits of the fourth gluon primary respectively in (3.8.16). Finally we can write the 5-point SDYM amplitude in the

[5]Note that we are taking the $4 \to 5$ OPE.

following factorized form

$$
\begin{aligned}
&\mathcal{M}_{5\,\text{SDYM}}(1^{a_1,+}_{\Delta_1}, 2^{a_2,+}_{\Delta_2}, \cdots 5^{a_5,+}_{\Delta_5})\big|_{\mathcal{O}(1)} \\
&= \frac{1}{2}\frac{\Gamma(\Delta_4+2)}{\Gamma(\Delta_4)}B(\Delta_4-1,\Delta_5-1)R^{1,a_4}_{-1,0}\mathcal{M}_{4\,\text{SDYM}}(1^{a_1,+}_{\Delta_1}, 2^{a_2,+}_{\Delta_2}, 3^{a_3,+}_{\Delta_3}, 5^{a_5,+}_{\Delta_4+\Delta_5-1}) \\
&\quad + \frac{\Gamma(\Delta_4+2)}{\Gamma(\Delta_4+1)}B(\Delta_4,\Delta_5-1)R^{0,a_4}_{-\frac{1}{2},\frac{1}{2}}\mathcal{M}_{4\,\text{SDYM}}(1^{a_1,+}_{\Delta_1}, 2^{a_2,+}_{\Delta_2}, 3^{a_3,+}_{\Delta_3}, 5^{a_5,+}_{\Delta_4+\Delta_5}) \\
&\qquad + \frac{\Gamma(\Delta_4+2)}{\Gamma(\Delta_4+2)}B(\Delta_4+1,\Delta_5-1)R^{-1,a_4}_{0,1}\mathcal{M}_{4\,\text{SDYM}}(1^{a_1,+}_{\Delta_1}, 2^{a_2,+}_{\Delta_2}, 3^{a_3,+}_{\Delta_3}, 5^{a_5,+}_{\Delta_4+\Delta_5+1}).
\end{aligned}
\tag{3.8.17}
$$

At the OPE level, it can be written as,

$$
\begin{aligned}
&\mathcal{O}^{a_4,+}_{\Delta_4}(z_4,\bar{z}_4)\mathcal{O}^{a_5,+}_{\Delta_5}(z_5,\bar{z}_5)\big|^{\text{SDYM}}_{\mathcal{O}(1)} \\
&= B(\Delta_4-1,\Delta_5-1)\left[\Delta_4 R^{1,a_4}_{-1,0}\mathcal{O}^{a_5,+}_{\Delta_4+\Delta_5-1} + \frac{\Delta_4-1}{\Delta_4+\Delta_5-2}R^{0,a_4}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{a_5,+}_{\Delta_4+\Delta_5}\right](z_5,\bar{z}_5) \\
&\quad + B(\Delta_4+1,\Delta_5-1)M^{a_4a_5}_1(\Delta_4+\Delta_5)(z_5,\bar{z}_5) \\
&= \mathcal{O}^{a_4,+}_{\Delta_4}(z_4,\bar{z}_4)\mathcal{O}^{a_5,+}_{\Delta_5}(z_5,\bar{z}_5)\big|^{\text{MHV}}_{\mathcal{O}(1)} + B(\Delta_4+1,\Delta_5-1)M^{a_4a_5}_1(\Delta_4+\Delta_5)(z_5,\bar{z}_5)
\end{aligned}
\tag{3.8.18}
$$

where $M^{ab}_1$ is the MHV null state defined in (3.3.2). Hence, we see that the OPE of two positive helicity gluon primaries at $\mathcal{O}(1)$ in SDYM theory can be written as the MHV OPE at $\mathcal{O}(1)$ plus a MHV null state.

This precisely matches with our result (3.5.1) when it is truncated at $n = 1$.

## 3.9 Discussion

In celestial CFT the sector with no negative helicity soft gluon is governed by the infinite dimensional $S$ algebra. In this chapter we found the most general form of the $S$ invariant

OPE of two positive helicity gluons at $\mathcal{O}(z_{12}^0 \bar{z}_{12}^0)$. It is given by

$$\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\mathcal{O}(1)} = \mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\big|_{\text{MHV OPE at }\mathcal{O}(1)} + \sum_{k=1}^{n} B(\Delta_1 + k, \Delta_2 - 1) M^{ab}_k(\Delta_1 + \Delta_2) \tag{3.9.1}$$

In this equation $M^{ab}_k(\Delta)$ is a MHV null state which transforms in a simple manner under the $S$ algebra. We have also shown that for $n = 1$ (3.9.1) gives the correct $\mathcal{O}(z_{12}^0 \bar{z}_{12}^0)$ term in the gluon-gluon OPE in the self-dual Yang-Mills theory. This is an important consistency check for the OPE formula.

Although the OPE coefficients and the descendants $M^{ab}_k(\Delta)$ which can appear at $\mathcal{O}(z_{12}^0 \bar{z}_{12}^0)$ is fixed by the $S$ invariance, the integer $n$ is not fixed by the symmetry. We saw that for $n = 1$ we get the OPE of the self -dual Yang-Mills theory but, the underlying theories for $n = 2$ and higher are not known. Presumably they are exactly solvable theories of gluon which generalize the self-dual Yang-Mills theory. An interesting question for future research is to find out these theories. We have also found out the KZ type null states for these (unknown) theories. They may be of some help in the search for these theories.

In an interesting recent work a celestial dual for the MHV gluon scattering amplitudes has been found in [134]. The theories which underlie the OPE (3.9.1) can be thought of as deformations of [134] which preserve the $S$ invariance. Our results suggest that there is a *discrete* infinite number of such deformations. It will be interesting to see if this picture is correct.

Another point we would like to emphasize is that for every ''theory'' there are only a finite number of descendants which contribute to the subleading OPE. This is somewhat unexpected because the spectrum of operator dimensions in celestial CFT is not bounded from below [6]. This might point to a reformulation of celestial CFT where operator dimensions are discrete and bounded from below. Proposals along this line have been made in [28,

[6] See for example the recent work [135].

29]. It will be interesting to find out the relation between [28, 29] and our observation in this chapter.

Let us now discuss some interesting questions whose study we leave to the future.

If there are S-invariant field theories on the celestial sphere that lack bulk-Lorentz invariance, can we provide a physical interpretation for these theories? Additionally, can we find reasons, such as mathematical consistency, to rule out S-invariant field theories on the celestial sphere that lack bulk-Lorentz invariance? In this chapter we have assumed bulk Lorentz invariance from the beginning. However, space-time translational invariance has not been assumed.

Could S-invariant non-Lorentz-invariant theories on the celestial sphere emerge from a spontaneous breakdown of Lorentz-invariant theories? Could the Goldstone modes associated with the breakdown of Lorentz invariance be analogous to the soft modes responsible for the breakdown of BMS symmetry, as suggested by Hawking-Perry-Strominger [136] in the context of the black hole information paradox?

Are there constraints on the Lagrangian formulation of these S-invariant theories? Alternatively, from the perspective of axiomatic conformal field theory, is it conceivable that there is no Lagrangian formulation of a CFT, and instead, the focus should be on verifying whether either $w_{1+\infty}$ or $S$ invariant celestial CFTs adhere to axioms such as the Osterwalder-Schrader axioms? The null states found in this chapter and in [48] place tight constraints on the Lagrangian formulation of the celestial dual theories. One way to see this is that in celestial CFT the spectrum of operator dimensions is the *same* for every theory, at least in its current formulation. Therefore, different theories are *not* distinguished by their operator spectrum but by their null states. So any Lagrangian formulation has to produce all the correct null states and this may be useful in constraining the form of the Lagrangian. We leave these very interesting questions for future study.

# Chapter 4

# Singularity Structure of the Four Point Celestial Leaf Amplitudes

## 4.1 Introduction

In this chapter we discuss the singularity structure of the four-point celestial leaf amplitudes for massless scalars and gluons. This chapter is based on [137]. It is well known that celestial amplitudes for massless scattering processes are obtained from the momentum space scattering amplitudes by Mellin integrating the energies of the external massless particles [12, 20]. These amplitudes transform as 2D conformal correlators under global (Lorentz) conformal transformations. Thus we can apply the powerful 2D CFT techniques to constrain the quantum gravity scattering amplitudes. However, one of the caveats about this approach is that, lower point celestial amplitudes are heavily constrained due to the spacetime translation symmetries. They take distributional forms which are unfamiliar from the usual 2D CFT perspective. Several efforts have been made to construct smooth conformally invariant celestial amplitudes by breaking translational symmetries [49, 50, 97, 98, 120, 121, 134, 138–143].

Recently, authors in [144], have considered three-point MHV gluon amplitudes in $(2, 2)$ signature Klein spacetime and showed that the translationally invariant celestial amplitudes can be written as sums of generically smooth amplitudes given by $\mathrm{AdS}_3/\mathbb{Z}$-Witten diagrams. These amplitudes are called *leaf amplitudes*. Klein space can be divided into timelike ($X^2 < 0$) and spacelike ($X^2 > 0$) wedges, each of which is foliated by hyperbolic slices. These slices are geometrically $\mathrm{AdS}_3/\mathbb{Z}$ whose boundaries are Lorentzian torus. Using the Fourier representation of the momentum-conserving delta function, celestial amplitudes can be written directly in position space as the weighted integrals of Witten diagrams on these slices/leaves. These leaf amplitudes enjoy Lorentz/conformal symmetry but they are not translationally invariant. Thus they take the familiar 2D CFT form on the Lorentzian torus. For a discussion on 2D CFT correlators on Lorentzian torus see [145].

Here we study the four-point leaf amplitudes of massless scalar and MHV gluon scattering [1]. In general these four-point leaf amplitudes have branch point singularities as a function of two real independent cross ratios $z$ and $\bar{z}$. The total conformal weights of the massless scalars or gluons are constrained since the bulk theories under consideration are scale invariant. On the support of these constraints, we compute the discontinuities around the branch point singularities and show that the four-point leaf amplitudes are non vanishing everywhere on the cross-ratio space defined by $z, \bar{z}$ and develop a *simple pole singularity at* $z = \bar{z}$. The distributional nature of the four point celestial amplitudes will be recovered by adding the leaf amplitudes in both timelike and spacelike regions.

Interestingly, similar singularity structure of the four-point boundary correlators have appeared in other studies also [146–149]. Recently, in [146], an interesting approach has been discussed to study the flat space holography [2]. More specifically, by considering the Euclidean path integral of a quantum field theory (QFT) the authors of [146] have defined boundary correlation functions with Dirichlet boundary conditions. The $S$-matrix

[1] Since two- and three-point functions are completely fixed by symmetries, they don't provide any information about the bulk dynamics.

[2] See [150] for a connection between their approach and celestial holography.

of the QFT can be obtained by smearing these boundary correlators with the free particle wave functions. In the case of massless scalar scattering, after analytically continuing to Minkowski spacetime they have found that their four-point boundary correlators develop a simple pole singularity at $z = \bar{z}$, where $z, \bar{z}$ are the conformal cross ratios on the celestial sphere. In the AdS/CFT context, on the other hand, it was shown that there exists certain singularity structure for the four-point Lorentzian boundary correlators, known as the bulk point singularity and this is directly related to the bulk locality in [148, 149]. The residues of these singularities give the flat space $S$-matrix elements. However, one needs to consider certain limits to get the $S$-matrix elements from the boundary correlators.

In our analysis, we have considered two examples: massless scalar and MHV gluon scattering and, have shown that the four point leaf amplitudes correspond to simple pole singularity on the support of the delta function constraint involving total conformal weights. It will be interesting to see if our conclusion holds without this constraint as well. Another interesting question is to check the singularity structure of the four point leaf amplitudes corresponding other bulk scattering. We hope to answer some these questions in future.

In this chapter, we also show that the leaf amplitudes satisfy the Banerjee-Ghosh (BG) equations first derived in [22] for MHV gluon celestial amplitudes. By considering the ward identities of leading [19, 32–36, 101–106] and subleading [22–24] positive helicity soft gluon theorems and the OPE between two outgoing positive helicity gluons, the authors in [22], found some null state relation under the leading and subleading soft gluon algebra. Decoupling of these null states from the MHV gluon celestial amplitudes leads to differential equations for the latter. Later, this was generalized to the whole $S$-algebra and it was found that there exists an (discrete) infinite number of theories invariant under $S$-algebra, each having a null state relation [25]. Like celestial MHV gluon amplitudes, leaf amplitudes are also governed by the same infinite dimensional soft gluon algebra [151], known as $S$-algebra. Thus, it is expected that they will satisfy the BG equations. By computing the subleading terms in the OPE between two outgoing positive helicity hard gluon primary operators and comparing it to the OPE between a soft and a hard gluon primary operator, we derive the BG equations for leaf amplitudes.

The rest of the chapter is organized as follows. We summarize our main results in the next section. In section 4.3, we give a brief review of the geometry of Klein space and the construction of the leaf amplitudes. Section 4.4 discusses the singular behavior of the four-point leaf amplitudes in the cross ratio space. We consider two examples: scalar contact diagram and MHV gluon amplitudes. In section 4.5, by extracting the subleading order ($\mathcal{O}(1)$) OPE from the four-point MHV gluon leaf amplitudes we show that they satisfy the BG equations. Appendix C.1 computes the integral appeared in the four-point scalar leaf amplitude in detail. In Appendix C.2, we discuss some of the properties of $H$-function used in the main section. Finally in Appendix C.3 we have computed the tree level celestial four-point scalar amplitude to match with the results obtained in subsection 4.4.1.1. Let us now start with the summary of the main results.

## 4.2 Summary of the main results

We denote the conformally invariant part of the $4$-point celestial leaf amplitudes for tree level massless scalar scattering by $\mathcal{S}_4(z, \bar{z})$, where $z$ and $\bar{z}$ are two real conformally invariant cross ratios. This is the leaf amplitude in the timelike region of the Klein space. Similarly, for spacelike region we have $\overline{\mathcal{S}_4}(z, \bar{z})$. The full leaf amplitudes will be obtained by multiplying the appropriate conformally covariant (non-unique) pre-factor. On the support of the constraints on the imaginary part $\beta$, of the total conformal weights coming from the bulk scale invariance, these leaf amplitudes develop a simple pole singularity at $z = \bar{z}$. We know that the conformal dimension $\Delta_i$ of the $i$-th primary operator on the celestial torus lies on the principle continuous series, i.e., $\Delta_i = 1 + i\lambda_i$ [12]. For simplicity we take, $\lambda_i = \lambda, \forall i$. Then $\beta$ becomes $\beta = 4\lambda$. Then our main results for $4$-point scalar leaf amplitudes are given by,

$$\begin{aligned} \delta(\lambda)\mathcal{S}_4(z,\bar{z})\big|_{z\to\bar{z}} &= \delta(\lambda)\frac{i\pi}{2}\frac{4\pi^2}{\bar{z}-z+i\epsilon},\ z>1 \\ \delta(\lambda)\overline{\mathcal{S}_4}(z,\bar{z})\big|_{z\to\bar{z}} &= -\delta(\lambda)\frac{i\pi}{2}\frac{4\pi^2}{\bar{z}-z-i\epsilon},\ z>1. \end{aligned} \tag{4.2.1}$$

The celestial amplitudes will be obtained by adding the two leaf amplitudes on the support of $\delta(\lambda)$ and we will recover the distributional nature of the celestial amplitudes as explained in detail in section 4.4.1.

Similarly, for MHV gluon scattering we define $\mathcal{G}_4(z,\bar{z})$ and $\overline{\mathcal{G}_4}(z,\bar{z})$ the two conformally invariant part of the $4$-point leaf amplitudes in timelike and spacelike regions respectively. We then have,

$$\begin{aligned} \delta(\lambda)\mathcal{G}_4(z,\bar{z})\big|_{z\to\bar{z}} &= \delta(\lambda)\frac{i\pi}{2}\frac{z}{z-1}\frac{4\pi^2}{\bar{z}-z+i\epsilon},\ z>1 \\ \delta(\lambda)\overline{\mathcal{G}_4}(z,\bar{z})\big|_{z\to\bar{z}} &= -\delta(\lambda)\frac{i\pi}{2}\frac{z}{z-1}\frac{4\pi^2}{\bar{z}-z-i\epsilon},\ z>1. \end{aligned} \tag{4.2.2}$$

Once again the addition of the above two leaf amplitudes will give the distributional nature of the conformally invariant part of the $4$-point celestial MHV gluon amplitude. For details see 4.4.2.

## 4.3 Klein space and Celestial leaf amplitudes

In this section, we will first briefly discuss the geometry of Klein space. A detailed discussion on Klein space can be found in [99, 152]. Then we construct celestial leaf amplitudes in Klein space following the work [144].

### 4.3.1 The Geometry of Klein Space

Klein space ($\mathbb{K}^{2,2}$) is a flat space with signature $(2,2)$ and the metric in Cartesian coordinates is given by

$$ds^2 = -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \tag{4.3.1}$$

We want to study the conformal compactification of $\mathbb{K}^{2,2}$ and the conformal geometry of null infinity, denoted as $\mathcal{I}$. We can defne the following polar coordinates,

$$X^0 + iX^1 = qe^{i\psi},\ X^2 + iX^3 = re^{i\phi} \tag{4.3.2}$$

the metric now reads

$$ds^2 = -dq^2 - q^2 d\psi^2 + dr^2 + r^2 d\phi^2 \tag{4.3.3}$$

Let's define a new set of coordinates $(U,V)$ which are useful to study the null infinity $\mathcal{I}$,

$$q - r = \tan U,\ q + r = \tan V \tag{4.3.4}$$

Since $q$ and $r$ both are radial coordinates, we have $0 < q, r < \infty$, so the coordinate ranges for $(U,V)$ become $-\frac{\pi}{2} < U < \frac{\pi}{2},\ |U| < V < \frac{\pi}{2}$. In $(U,V,\psi,\phi)$ coordinates, the metric (4.3.3) becomes,

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left(-dUdV - \frac{1}{4}\sin^2(V+U)d\psi^2 + \frac{1}{4}\sin^2(V-U)d\phi^2\right) \tag{4.3.5}$$

The timelike infinity $(i')$ can be reached at the limit $q \to \infty$ where $\left(U = \frac{\pi}{2}, V = \frac{\pi}{2}\right)$ and spacelike infinity is reached by taking $r \to \infty$, where $\left(U = -\frac{\pi}{2}, V = \frac{\pi}{2}\right)$. Null infinity is at $V = \frac{\pi}{2}$ and it is parametrized by the null coordinate $-\frac{\pi}{2} < U < \frac{\pi}{2}$ and the periodic coordinates $(\psi,\phi)$. Taking the limit $V \to \frac{\pi}{2}$ and rescaling the metric (4.3.5) by $\cos^2 V$, we get the conformal metric on $\mathcal{I}$ as given by,

$$ds^2 = -d\psi^2 + d\phi^2, \qquad \psi \sim \psi + 2\pi,\ \phi \sim \phi + 2\pi. \tag{4.3.6}$$

So in Klein space the geometry of the null infinity is a Lorentzian torus ($\mathcal{LT}^2 = S^1 \times S^1$) times a null interval. The Penrose diagram of the Klein space is presented in figure 4.1. We see that it has only one connected component and we can not define incoming and outgoing particles both, hence we can't define an $S$-matrix like in Minkowski spacetime ($\mathbb{M}^{1,3}$), rather we have an $S$-vector [153]. The authors in [153] have shown that one can define $S$-matrix in $\mathbb{M}^{1,3}$ from $S$-vector in $\mathbb{K}^{2,2}$ using a suitable analytic continuation.

The lightcone $X^2 = 0$ divides the Klein space into two regions: timelike $X^2 < 0$ and spacelike $X^2 > 0$ (See figure 4.1). The parameterize the coordinates in timelike and spacelike wedges as:

$$\begin{aligned} \text{Timelike} : X^\mu = \tau \hat{x}^\mu_+,\ \hat{x}^2_+ = -1 \\ \text{Spacelike} : X^\mu = \tau \hat{x}^\mu_-,\ \hat{x}^2_- = +1. \end{aligned} \tag{4.3.7}$$

with $\tau \in (0, \infty)$. Constant $\tau$ surfaces give the hyperbolic foliations of $\mathbb{K}^{2,2}$, similar to the hyperoblic foliation of Minkowski space discussed in [154]. To study the geometry of the constant $\tau$ slices or leaves, we will use following parameterization. For timelike leaves $\hat{x}^2_+ = -1$ we use the global coordinates

$$\hat{x}^0_+ + i\hat{x}^1_+ = e^{i\psi} \cosh\rho,\ \hat{x}^2_+ + i\hat{x}^3_+ = e^{i\phi} \sinh\rho \tag{4.3.8}$$

with $0 < \rho < \infty$ along each of the constant $\tau$-slices. With these parameterization the metric (4.3.1) becomes,

$$ds^2 = -d\tau^2 + \tau^2 \left(-\cosh^2\rho\, d\psi^2 + \sinh^2\rho\, d\phi^2 + d\rho^2\right) \tag{4.3.9}$$

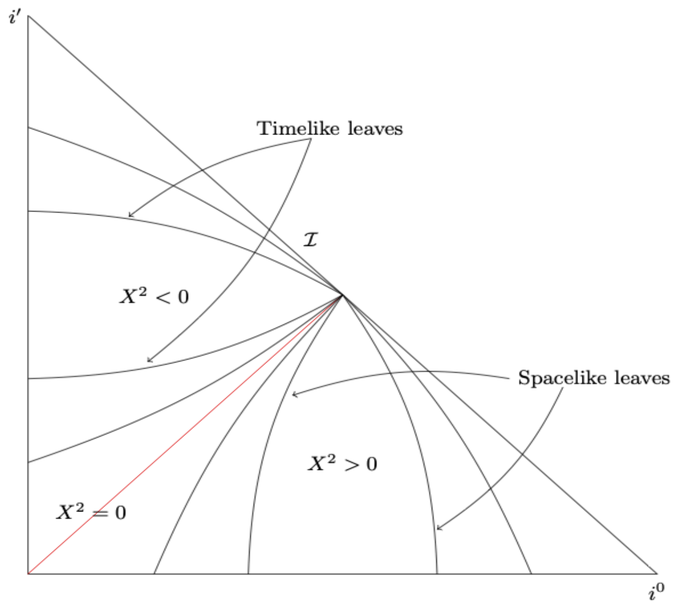


Figure 4.1: Penrose diagram of Klein space $\mathbb{K}^{2,2}$ where the hyperbolic foliation in timelike and spacelike region is shown. Each point on this diagram denotes a Lorentzian torus.

From (4.3.9), we see that each constant $\tau$ leaf is given geometrically by $\mathrm{AdS}_3/\mathbb{Z}$ with a periodically identified time coordinate $\psi$. For the constant $\tau$-slices in spacelike region we use the following parameterization

$$\hat{x}^0_- + i\hat{x}^1_- = e^{i\psi}\sinh\rho, \ \hat{x}^2_- + i\hat{x}^3_- = e^{i\phi}\cosh\rho \tag{4.3.10}$$

and the metric in the spacelike region is

$$ds^2 = d\tau^2 - \tau^2\left(-\cosh^2\rho\, d\phi^2 + \sinh^2\rho\, d\psi^2 + d\rho^2\right) \tag{4.3.11}$$

One can reach to the conformal boundary of each $\mathrm{AdS}_3/\mathbb{Z}$ leaf (4.3.9) or (4.3.11) by sending $\rho \to \infty$ and it is is given by the Lorentzian torus (4.3.6). This Lorentzian torus on $\mathcal{I}$ is referred as celestial torus $\mathcal{CT}^2$.

Let's introduce the null coordinates (also called the global coordinates)

$$\sigma = \frac{\psi + \phi}{2}, \ \bar{\sigma} = \frac{\psi - \phi}{2} \tag{4.3.12}$$

on $\mathcal{CT}^2$. We first compute the leaf amplitudes in global coordinates and then we can express the amplitudes in planar coordinates which are defined as

$$z = \tan\sigma,\ \bar{z} = \tan\bar{\sigma}, \qquad z, \bar{z} \in \mathbb{R} \tag{4.3.13}$$

The action of the Lorentz group $(\mathrm{SL}(2,\mathbb{R}) \times \overline{\mathrm{SL}}(2,\mathbb{R}))$ on the planar coordinates is as follows

$$\begin{aligned} z &\to \frac{az+b}{cz+d}, \qquad (a,b,c,d) \in \mathbb{R}, \qquad ad-bc=1. \\ \bar{z} &\to \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \qquad (\bar{a},\bar{b},\bar{c},\bar{d}) \in \mathbb{R}, \qquad \bar{a}\bar{d}-\bar{b}\bar{c}=1. \end{aligned} \tag{4.3.14}$$

From (4.3.13) one can see that the planar coordinates $(z, \bar{z})$ can cover only half of the celestial torus. Rather, the coordinates $(z, \bar{z})$ can be understood as the local coordinates of a two dimensional diamond in $(1, 1)$ signature and the whole celestial torus except the point $z\bar{z} = 0$ can be covered by two such diamonds. However, it should be noted that the distinction between two diamonds is just a choice of the coordinate patches, and in the global coordinates $(\sigma, \bar{\sigma})$ we don't have any such distinction.

### 4.3.2 Leaf amplitudes

We parameterize a null momentum $p_k^\mu$ of $k$-th massless particle, satisfying $p_k^2 = 0$, in $(-,-,+,+)$ signature in planar coordinates as:

$$p_k^\mu = \epsilon_k \omega_k q_k^\mu,\ q_k^\mu = (1 - z_k\bar{z}_k, z_k + \bar{z}_k, 1 + z_k\bar{z}_k, z_k - \bar{z}_k)\,. \tag{4.3.15}$$

where $\omega_k \in [0, \infty)$ represents the magnitude of the frequency of the $k$-th massless particle and $\epsilon_k = \pm$ denotes the sign of the frequency. In global coordinates, we parameterize momentum (4.3.15) through the relation (4.3.13) and (4.3.12),

$$p_k^\mu = \frac{\omega_k \hat{p}_k^\mu}{|\cos\sigma_k \cos\sigma_k|} \tag{4.3.16}$$

where $\hat{p}_k^\mu$ is a null vector parametrized by the points $(\psi_k, \phi_k)$ on $\mathcal{CT}^2$ as,

$$\hat{p}_k^0 + i\hat{p}_k^1 = e^{i\psi_k},\ \hat{p}_k^2 + i\hat{p}_k^3 = e^{i\phi_k} \tag{4.3.17}$$

and we have identified $\epsilon_k = \frac{p_k^0 + p_k^2}{2\omega_k} = \text{sgn}(\cos\sigma_k \cos\bar{\sigma}_k)$.

With the above parameterization, let's start with an $n$-point tree level momentum space amplitude with all the external state particles as massless,

$$A_n(1^{\vartheta_1}, 2^{\vartheta_2}, \ldots, n^{\vartheta_n}) = A_n(\{\epsilon_i\omega_i q_i, \vartheta_i\})\,\delta\left(\sum_{k=1}^{n} \epsilon_k \omega_k q_k^\mu\right) \tag{4.3.18}$$

where $(\vartheta_1, \vartheta_2, \cdots, \vartheta_n)$ are the helicities of the massless particles. The celestial amplitude is obtained by doing Mellin transform of (4.3.18) with respect to energies of the external states,

$$\mathcal{M}_n\left(1^{(h_1,\bar{h}_1)}, 2^{(h_2,\bar{h}_2)}, \ldots, n^{(h_n,\bar{h}_n)}\right) = \prod_{j=1}^{n} \int_0^\infty d\omega\, \omega^{\Delta_j - 1} e^{-\epsilon\omega_j} A_n(1^{\vartheta_1}, 2^{\vartheta_2}, \ldots, n^{\vartheta_n}) \tag{4.3.19}$$

with $\epsilon$ being the UV regulator and $h_j, \bar{h}_j$ are the conformal weights given by,

$$h_j = \frac{\Delta_j + \vartheta_j}{2},\ \bar{h}_j = \frac{\Delta_j - \vartheta_j}{2}. \tag{4.3.20}$$

We will use the following representation of the delta function in (4.3.19) before doing the Mellin transform

$$\delta\left(\sum_{k=1}^{n} \epsilon_k \omega_k q_k^\mu\right) = \int \frac{d^4X}{(2\pi)^4} e^{i\sum_{k=1}^{n} \epsilon_k \omega_k q_k \cdot X} \tag{4.3.21}$$

Thus we can write the amplitude (4.3.19) as,

$$\mathcal{M}_n\left(1^{(h_1,\bar{h}_1)},2^{(h_2,\bar{h}_2)},\ldots,n^{(h_n,\bar{h}_n)}\right)=\prod_{j=1}^{n}\int_0^\infty d\omega_j\,\omega_j^{\Delta_j-1}e^{-\epsilon\omega_j}$$
$$\times A_n(\{\epsilon_i\omega_i q_i,\vartheta_i\})\int\frac{d^4X}{(2\pi)^4}e^{i\sum_{k=1}^{n}\epsilon_k\omega_k q_k\cdot X} \tag{4.3.22}$$

Now, we first perform the Mellin integral of the plane wave $e^{i\epsilon_k\omega_k q_k\cdot X}$ which gives the scalar conformal primary wavefunction $\Phi_{\Delta_k,\epsilon_k}(X,q_k)$ given by (4.5.5). Then we perform the integral over Klein space. Dividing the $X^\mu$ integral in (4.3.22) into two parts according to (4.3.7) and replacing $\omega_i\to\frac{\omega_i}{\tau}$ we get,

$$\mathcal{M}_n\left(1^{(h_1,\bar{h}_1)},2^{(h_2,\bar{h}_2)},\ldots,n^{(h_n,\bar{h}_n)}\right)=\frac{1}{(2\pi)^4}\int_0^\infty\tau^{3-n-\beta}d\tau\left[\int_{\hat{x}_+^2=-1}d^3\hat{x}_+\prod_{j=1}^{n}\int_0^\infty d\omega_j\,\omega_j^{\Delta_j-1}\right.$$
$$\times A_n\left(\left\{\frac{\epsilon_i\omega_i q_i}{\tau},\vartheta_i\right\}\right)e^{i\sum_{k=1}^{n}\epsilon_k\omega_k q_k\cdot\hat{x}_+-\epsilon\omega_k}+\int_{\hat{x}_+^2=1}d^3\hat{x}_-\prod_{j=1}^{n}\int_0^\infty d\omega_j\,\omega_j^{\Delta_j-1}A_n\left(\left\{\frac{\epsilon_i\omega_i q_i}{\tau},\vartheta_i\right\}\right)$$
$$\left.\times e^{i\sum_{k=1}^{n}\epsilon_k\omega_k q_k\cdot\hat{x}_--\epsilon\omega_k}\right] \tag{4.3.23}$$

where $\beta=\sum_{k=1}^{n}(\Delta_k-1)=\sum_{k=1}^{n}(2\bar{h}_k+\vartheta_k-1)$. An $n$-point stripped scattering amplitude in any scale invariant theory behaves as the following under scaling

$$A_n\left(\left\{\frac{\epsilon_i\omega_i q_i}{\tau},\vartheta_i\right\}\right)\to\tau^{n-4}A_n\left(\{\epsilon_i\omega_i q_i,\vartheta_i\}\right) \tag{4.3.24}$$

which follows from the mass dimension of the $n$-point amplitude. The mass dimension of an $n$-point scalar or gluon amplitudes is $[A_n]\sim(\text{mass})^{4-n}$ and hence the scaling behavior is given by (4.3.24) [155]. The $\tau$ integral in (4.3.23) produce $\delta(\beta)$ and the resulting expression can be written as

$$\mathcal{M}_n\left(1^{(h_1,\bar{h}_1)},2^{(h_2,\bar{h}_2)},\ldots,n^{(h_n,\bar{h}_n)}\right)=\frac{\delta(\beta)}{(2\pi)^3}\left[\mathcal{L}_n(\sigma_i,\bar{\sigma}_i)+\overline{\mathcal{L}}_n(\sigma_i,\bar{\sigma}_i)\right] \tag{4.3.25}$$

where $\mathcal{L}_n(\sigma_i, \bar{\sigma}_i)$ and $\overline{\mathcal{L}}_n(\sigma_i, \bar{\sigma}_i)$ are given by

$$\mathcal{L}_n(\sigma_i, \bar{\sigma}_i) = \int_{\hat{x}_+^2=-1} d^3\hat{x}_+ \prod_{j=1}^{n} \int_0^\infty d\omega_j\, \omega_j^{\Delta_j - 1} A_n\left(\{\epsilon_i \omega_i q_i, \vartheta_i\}\right) e^{i\sum_{k=1}^{n} \epsilon_k \omega_k q_k \cdot \hat{x}_+ - \epsilon\omega_k}$$
$$\overline{\mathcal{L}}_n(\sigma_i, \bar{\sigma}_i) = \int_{\hat{x}_+^2=1} d^3\hat{x}_- \prod_{j=1}^{n} \int_0^\infty d\omega_j\, \omega_j^{\Delta_j - 1} A_n\left(\{\epsilon_i \omega_i q_i, \vartheta_i\}\right) e^{i\sum_{k=1}^{n} \epsilon_k \omega_k q_k \cdot \hat{x}_- - \epsilon\omega_k} \tag{4.3.26}$$

The amplitudes $\mathcal{L}_n$ and $\overline{\mathcal{L}}_n$ are called $n$-point celestial leaf amplitudes. We have expressed them in global coordinates. One can show that the second integral in (4.3.26) can be obtained from the first one by sending $\bar{\sigma}_i \to -\bar{\sigma}_i$ i.e.,

$$\overline{\mathcal{L}}_n(\sigma_i, \bar{\sigma}_i) = \mathcal{L}_n(\sigma_i, -\bar{\sigma}_i) \tag{4.3.27}$$

In the next section, we will study the 4-point celestial leaf amplitudes for scalars and gluons and we investigate the singularity structures of these leaf amplitudes on the support of $\delta(\beta)$.

## 4.4 Singularity structure of the four-point leaf amplitudes on the support of $\delta(\beta)$

In the previous section, we have discussed how one can construct the leaf amplitudes in global coordinates from a momentum space amplitude. Now we construct the four-point leaf amplitudes in planar coordinates for tree level scalar and MHV gluon scattering and show that each celestial leaf amplitude shows a simple pole type singularity at $z = \bar{z}$ on the support of $\delta(\beta)$.

### 4.4.1 Scalar contact diagram

In this section, we focus on the tree level scattering amplitude of massless $\phi^4$ theory which is given by the following contact diagram

$$A_4(p_1, p_2, p_3, p_4) = -i(2\pi)^4\tilde{\lambda}\,\delta^{(4)}(p_1 + p_2 + p_3 + p_4) \tag{4.4.1}$$

with the coupling constant $\tilde{\lambda}$. From (4.3.18), we can write the stripped amplitude

$$A_4\left(\{\epsilon_i\omega_i q_i, \vartheta_i\}\right) = -i(2\pi)^4\tilde{\lambda} \tag{4.4.2}$$

Hence, from (4.3.26) the 4-point leaf amplitudes in this case are given by

$$\begin{aligned} \mathcal{L}_4^s(\sigma_i, \bar{\sigma}_i) &= -i(2\pi)^4\tilde{\lambda}\,\mathcal{C}_4(\sigma_i, \bar{\sigma}_i), \\ \overline{\mathcal{L}^s}_4(\sigma_i, \bar{\sigma}_i) &= \mathcal{L}_4^s(\sigma_i, -\bar{\sigma}_i) = -i(2\pi)^4\tilde{\lambda}\,\mathcal{C}_4(\sigma_i, -\bar{\sigma}_i) \end{aligned} \tag{4.4.3}$$

where $\mathcal{C}_4(\sigma_i, \bar{\sigma}_i)$ is given by

$$\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = \int_{\hat{x}_+^2=-1} d^3\hat{x}_+ \prod_{j=1}^{4}\int_0^\infty d\omega_j\,\omega_j^{2\bar{h}_j-1} e^{i\sum_{k=1}^4 \epsilon_k\omega_k q_k\cdot\hat{x}_+ - \epsilon\omega_k} \tag{4.4.4}$$

The superscript $s$ on $\mathcal{L}, \overline{\mathcal{L}}$ denote scalars. Note that for scalars we have $\vartheta_j = 0,\ \forall j$, hence the conformal weights $h_j, \bar{h}_j$ are given by $h_j = \frac{\Delta_j}{2} = \frac{1+i\lambda_j}{2} = \bar{h}_j$. However, we will evaluate the integrals without using any specific values for $\bar{h}_j$'s as the integral (4.4.4) will appear for MHV gluon scattering also. Let's introduce the abbreviation for the torus separation

$$s_{ij} := \sin(\sigma_{ij}),\ \bar{s}_{ij} := \sin(\bar{\sigma}_{ij}). \tag{4.4.5}$$

where $\sigma_{ij} = \sigma_i - \sigma_j,\ \bar{\sigma}_{ij} = \bar{\sigma}_i - \bar{\sigma}_j$.

The detailed computation of the integral (4.4.4) is in appendix C.1. The final result in terms of $S_{ij} = s_{ij}\bar{s}_{ij}$ is

$$\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = \frac{i\pi}{2}\Gamma\left(\bar{h}-1\right)(S_{13}+i\epsilon)^{d_1}(S_{34}+i\epsilon)^{d_2}(S_{24}+i\epsilon)^{d_3}(S_{23}+i\epsilon)^{d_4}H\left(u_+, v_+\right)$$
$$-\frac{i\pi}{2}\Gamma\left(\bar{h}-1\right)(S_{13}-i\epsilon)^{d_1}(S_{34}-i\epsilon)^{d_2}(S_{24}-i\epsilon)^{d_3}(S_{23}-i\epsilon)^{d_4}H\left(u_-, v_-\right) \tag{4.4.6}$$

where

$$d_1 = -2\bar{h}_1,\ d_2 = \bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4,\ d_3 = -\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4,\ d_4 = \bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4. \tag{4.4.7}$$

and $u_\pm, v_\pm$ are given by

$$u_\pm = \frac{(s_{12}\bar{s}_{12})(s_{34}\bar{s}_{34} \pm i\epsilon)}{(s_{13}\bar{s}_{13} \pm i\epsilon)(s_{24}\bar{s}_{24} \pm i\epsilon)},\ v_\pm = \frac{(s_{14}\bar{s}_{14})\left(s_{23}\bar{s}_{23} \pm i\epsilon\right)}{(s_{13}\bar{s}_{13} \pm i\epsilon)(s_{24}\bar{s}_{24} \pm i\epsilon)} \tag{4.4.8}$$

The $H$-functions in (4.4.6) has been discussed in [156]. We have also reviewed it in appendix C.2. We have suppressed their first four arguments for notational convenience. However, we will write them explicitly whenever it will be required. We want to express the leaf amplitudes in planar coordinates. The planar coordinates are related to the global coordinates by the following transformations

$$s_{ij} = z_{ij}\cos\sigma_i\cos\sigma_j,\ \bar{s}_{ij} = \bar{z}_{ij}\cos\bar{\sigma}_i\cos\bar{\sigma}_j, \epsilon_i = \text{sgn}(\cos\sigma_i\cos\bar{\sigma}_i) \tag{4.4.9}$$

After doing the coordinate transformations and taking care of the Jacobian factor $\prod_{i=1}^{4}|\cos\sigma_i|^{2h_i}|\cos\bar{\sigma}_i|^{2\bar{h}_i}$ we obtain the leaf amplitudes in the planar coordinates $(z_i, \bar{z}_i)$. Now in planar coordinates $\mathcal{C}_4(z_i, \bar{z}_i)$ reads

$$\mathcal{C}_4(z_i, \bar{z}_i) = \frac{i\pi}{2}\Gamma\left(\bar{h}-1\right)(\epsilon_1\epsilon_3 z_{13}\bar{z}_{13}+i\epsilon)^{d_1}(\epsilon_3\epsilon_4 z_{34}\bar{z}_{34}+i\epsilon)^{d_2}(\epsilon_2\epsilon_4 z_{24}\bar{z}_{24}+i\epsilon)^{d_3}$$
$$\times(\epsilon_2\epsilon_3 z_{23}\bar{z}_{23}+i\epsilon)^{d_4}H\left(u_+, v_+\right) - \frac{i\pi}{2}\Gamma\left(\bar{h}-1\right)(\epsilon_1\epsilon_3 z_{13}\bar{z}_{13}-i\epsilon)^{d_1}$$
$$\times(\epsilon_3\epsilon_4 z_{34}\bar{z}_{34}-i\epsilon)^{d_2}(\epsilon_2\epsilon_4 z_{24}\bar{z}_{24}-i\epsilon)^{d_3}(\epsilon_2\epsilon_3 z_{23}\bar{z}_{23}-i\epsilon)^{d_4}H\left(u_-, v_-\right) \tag{4.4.10}$$

where in planar coordinates, $u_\pm$ and $v_\pm$ are given by,

$$u_\pm = \frac{(\epsilon_1\epsilon_2 z_{12}\bar{z}_{12})(\epsilon_3\epsilon_4 z_{34}\bar{z}_{34} \pm i\epsilon)}{(\epsilon_1\epsilon_3 z_{13}\bar{z}_{13} \pm i\epsilon)(\epsilon_2\epsilon_4 z_{24}\bar{z}_{24} \pm i\epsilon)}, v_\pm = \frac{(\epsilon_1\epsilon_4 z_{14}\bar{z}_{14})\,(\epsilon_2\epsilon_3 z_{23}\bar{z}_{23} \pm i\epsilon)}{(\epsilon_1\epsilon_3 z_{13}\bar{z}_{13} \pm i\epsilon)(\epsilon_2\epsilon_4 z_{24}\bar{z}_{24} \pm i\epsilon)} \tag{4.4.11}$$

In planar coordinates the signs of the frequencies $\epsilon_i$ reappear because the celestial torus $\mathcal{CT}^2$ gets divided into two diamonds and the signs $\epsilon_i$ determine the choice of the diamonds. This is equivalent to the identification of in and out states in Minkowski signature. In global coordinates there is no distinction between the diamonds that we had in planar coordinates, they were removed by the relation $\epsilon_i = \mathrm{sgn}(\cos\sigma_i \cos\bar{\sigma}_i)$ .

Now, in planar coordinates, we analyze the singularity structure of the scalar leaf amplitude (4.4.3) in the cross ratio space given by,

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}},\ \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}. \tag{4.4.12}$$

We will first consider the timelike leaf amplitude. Since $\mathcal{L}_4^s(z_i, \bar{z}_i)$ and $\mathcal{C}_4(z_i, \bar{z}_i)$ are related by a constant multiplicative factor, we will concentrate on $\mathcal{C}_4(z_i, \bar{z}_i)$ only. Let's now focus on the scalar case, for which $\bar{h}_j = \frac{1+i\lambda_j}{2}$. For the sake of simplicity, we consider the case where the imaginary parts of the conformal dimensions of all the four external particles are same, i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$. With this choice, let us write $H$-function with all the arguments (see C.2.1),

$$H(1 + i\lambda, 1 + i\lambda, 1 + 2i\lambda, 2 + 2i\lambda; u_\pm, v_\pm) \tag{4.4.13}$$

For all $\lambda_j = \lambda$ the constraint on the total conformal weights becomes $\delta(\beta) = \frac{1}{4}\delta(\lambda)$. On the support of this constraint $H$-function simplifies to $H(1, 1, 1, 2; u, v)$. It has been shown [156] that $H$-function with these specific arguments can be expressed in terms of log and $\mathrm{Li}_2$ functions (See appendix C.2 for details). With the proper care of the monodromies of these functions, it was shown [146–148] that $H(1, 1, 1, 2; u, v)$ exhibit simple pole singularity in the limit of $z$ goes to $\bar{z}$, where $z$ and $\bar{z}$ are two independent real cross-ratios. We now present this in detail. For concreteness we choose the following sign assignment for the frequencies

$$\epsilon_1 = \epsilon_2 = -1, \ \epsilon_3 = \epsilon_4 = +1. \tag{4.4.14}$$

Now using conformal symmetry we can send send three points to $0, 1$ and $\infty$

$$z_1, \bar{z}_1 \to \infty, z_2 = \bar{z}_2 = 1, \ z_3 = z, \bar{z}_3 = \bar{z}, z_4 = \bar{z}_4 = 0. \tag{4.4.15}$$

We define the following function,

$$\begin{aligned}
\mathcal{S}_4(z, \bar{z}) &= \lim_{z_1, \bar{z}_1 \to \infty} z_1^{2h_1} \bar{z}_1^{2\bar{h}_1} \mathcal{C}_4(z_i, \bar{z}_i), \ \ h_j = \bar{h}_j = \frac{1 + i\lambda}{2}, \forall j \\
&= \frac{i\pi}{2} \Gamma\left(1 + 2i\lambda\right) e^{2\pi\lambda} H(1 + i\lambda, 1 + i\lambda, 1 + 2i\lambda, 2 + 2i\lambda; u_+, v_+) \\
&- \frac{i\pi}{2} \Gamma\left(1 + 2i\lambda\right) e^{-2\pi\lambda} H(1 + i\lambda, 1 + i\lambda, 1 + 2i\lambda, 2 + 2i\lambda; u_-, v_-)
\end{aligned} \tag{4.4.16}$$

and, $u_\pm$, $v_\pm$ become,

$$u_\pm = z\bar{z} \pm (1 + z\bar{z}) i\epsilon, \ v_\pm = (1 - z)(1 - \bar{z}) \pm \{(1 - z)(1 - \bar{z}) - 1\} i\epsilon. \tag{4.4.17}$$

We have obtain the above expressions of $u_\pm$ and $v_\pm$ using (4.4.14), (4.4.15) in (4.4.11), then expanding the latter around $\epsilon = 0$ and keeping terms only upto $\mathcal{O}(\epsilon)$. To analyze the singularity structure of $\mathcal{C}_4(z_i, \bar{z}_i)$ as $z \to \bar{z}$, we will work with equation (4.4.16). Since, $\mathcal{C}_4(z_i, \bar{z}_i)$ is obtained from $\mathcal{S}_4(z, \bar{z})$ by multiplying a conformally covariant prefactor which does not show any singular behavior as $z \to \bar{z}$, so working with $\mathcal{S}_4(z, \bar{z})$ is as good as $\mathcal{C}_4(z_i, \bar{z}_i)$.

We start by writing explicitly the expression of the $H$-function,

$$\begin{aligned}
H(1 + i\lambda, 1 + i\lambda, 1 + 2i\lambda, 2 + 2i\lambda; u_\pm, v_\pm) &= \frac{1}{1 - x_\pm - y_\pm} \left[ \log\{x_\pm(1 - y_\pm)\} \log\left(\frac{y_\pm}{1 - x_\pm}\right) \right. \\
&\left. + 2\{Li_2(1 - y_\pm) - Li_2(x_\pm)\} \right] + \mathcal{O}(\lambda)
\end{aligned} \tag{4.4.18}$$

and we define $u_\pm = x_\pm(1-y_\pm),\ v_\pm = y_\pm(1-x_\pm)$. The derivation of (4.4.18) is given in appendix C.2 for the sake of completeness. As explained in C.2, we can identify $x_\pm = z \mp i\epsilon$ and $1-y_\pm = \bar{z} \pm i\epsilon$ provided that $\bar{z} > z > 1$. Hence, we assume $\bar{z} > z > 1$ and write $u_\pm$ and $v_\pm$ as

$$u_\pm = (z \mp i\epsilon)(\bar{z} \pm i\epsilon),\ v_\pm = (1 - z \pm i\epsilon)(1 - \bar{z} \mp i\epsilon). \tag{4.4.19}$$

Then on the support of $\delta(\lambda)$ our $H$-functions in (4.4.16) become

$$\begin{aligned} &\delta(\lambda)H(1+i\lambda, 1+i\lambda, 1+2i\lambda, 2+2i\lambda; u_\pm, v_\pm) \\ = &\frac{\delta(\lambda)}{\bar{z} - z \pm i\epsilon}\left[\{\log(z \mp i\epsilon) + \log(\bar{z} \pm i\epsilon)\}\{\log(1-\bar{z} \mp i\epsilon) - \log(1 - z \pm i\epsilon)\}\right. \\ &\left. -2\mathrm{Li}_2(z \mp i\epsilon) + 2\mathrm{Li}_2(\bar{z} \pm i\epsilon)\right] \end{aligned} \tag{4.4.20}$$

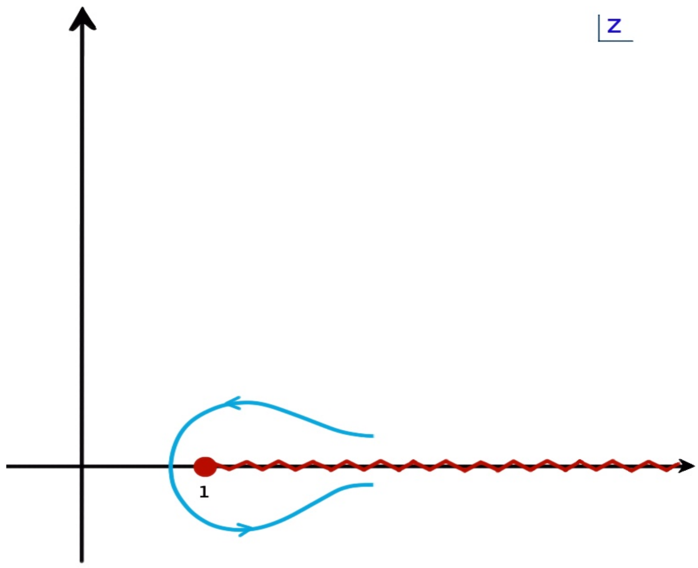


Figure 4.2: Path traversed by $z$ for timelike leaf amplitude.

Note, $\log(1-x), \mathrm{Li}_2(x)$ and $\log x$ have nontrivial monodromies at branch point singularities $1$ and $\infty$. We use the branch cut prescription in the complex $z, \bar{z}$ plane inspired by [148]. We place a branch cut from $1$ to $\infty$ on the positive real axis in the $z$ plane and assume that $z$ moves around the branch cut counterclockwise (see Fig 4.2), whereas there is no branch cut in the $\bar{z}$ plane. With this prescription we can take $\epsilon \to 0$ limit in all the log and $\mathrm{Li}_2$ functions involving $\bar{z}$ without encountering any discontinuities. On the other hand for $z$ we have,

$$\begin{gathered}\log(z+i\epsilon)-\log(z-i\epsilon)=-2\pi i,\ \log(1-z+i\epsilon)-\log(1-z-i\epsilon)=2\pi i,\\ \mathrm{Li}_2(z+i\epsilon)-\mathrm{Li}_2(z-i\epsilon)=2\pi i\ \log(z)\end{gathered} \tag{4.4.21}$$

Using the above discontinuities in the first $H$-function of (4.4.16) we obtain

$$\delta(\lambda)\mathcal{S}_4(z,\bar{z})=\frac{i\pi}{2}\frac{\delta(\lambda)}{\bar{z}-z+i\epsilon}\left[4\pi^2+2\pi i\ln\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right],\ z,\bar{z}>1. \tag{4.4.22}$$

Thus we show that on the support of the delta function arising from bulk scale invariance, timelike leaf amplitude shows a simple pole singularity as $z \to \bar{z}$. Spacelike leaf amplitude also exhibit the simple pole type singularity that we will show in the next subsection. We will also discuss how to get the celestial amplitude by adding the timelike and spacelike leaf amplitudes.

#### 4.4.1.1 Recovering celestial amplitude from leaf amplitudes

We have computed the timelike leaf amplitude in the previous section. After calculating the monodromies carefully, we have shown that it exhibits a simple pole singularity at $z = \bar{z}$. Similarly one can calculate the singularity structure of leaf amplitude in the spacelike wedge. The calculation for the spacelike leaf amplitudes with arbitrary conformal weights in global coordinates is in appendix C.1 (see (C.1.27)). The procedure is similar to what we discussed in the previous subsection. We first write the spacelike leaf amplitude in planar coordinates $\overline{\mathcal{C}_4}(z_i,\bar{z}_i)$ from the spacelike leaf amplitude in global coordinates $\mathcal{C}_4(\sigma_i,-\bar{\sigma}_i)$.

We define $\overline{\mathcal{S}_4}(z,\bar{z})$ using conformal symmetry. The final expression for $\overline{\mathcal{S}_4}(z,\bar{z})$ is

$$\begin{aligned}\overline{\mathcal{S}_4}(z,\bar{z}) = &-\frac{i\pi}{2}\Gamma(1+2i\lambda)H(1+i\lambda,1+i\lambda,1+2i\lambda,2+2i\lambda;u_+,v_+)\\ &+\frac{i\pi}{2}\Gamma(1+2i\lambda)H(1+i\lambda,1+i\lambda,1+2i\lambda,2+2i\lambda;u_-,v_-)\end{aligned} \tag{4.4.23}$$

For spacelike case, the choice of paths traversed by $z,\bar{z}$ are opposite to the case of timelike leaf amplitude, i.e., $\bar{z}$ moves clockwise around the branch cut placed from $1$ to $\infty$ in the complex $\bar{z}$ plane and there is no branch cut in the $z$ plane. Then calculating the monodromies of the log and $\text{Li}_2$ functions involving $\bar{z}$, in the second $H$-function of (4.4.23), and expanding around $\lambda = 0$ we obtain the following result,

$$\delta(\lambda)\overline{\mathcal{S}_4}(z,\bar{z}) = \frac{i\pi}{2}\frac{\delta(\lambda)}{\bar{z}-z-i\epsilon}\left[-4\pi^2 - 2\pi i\ln\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right],\ z,\bar{z}>1. \tag{4.4.24}$$

According to (4.3.25), celestial amplitude is obtained by adding the two leaf amplitudes and multiplying it by $\frac{\delta(\beta)}{(2\pi)^3}$. Since we are working with the conformally invariant functions $\mathcal{S}_4(z,\bar{z})$ and $\overline{\mathcal{S}_4}(z,\bar{z})$, by adding them we only obtain the conformally invariant part of the celestial amplitude. Multiplying this with the conformally covariant prefactor which is fixed by conformal symmetries gives us the full celestial amplitude. Hence, from (4.4.22) and (4.4.24), the conformally invariant part of the celestial amplitude is given by,

$$\begin{aligned}\widetilde{\mathcal{M}}_4(z,\bar{z}) &= (-(2\pi)^4\tilde{\lambda})\frac{\delta(\lambda)}{16}\Theta(z-1)\left[\frac{i}{\bar{z}-z+i\epsilon} - \frac{i}{\bar{z}-z-i\epsilon}\right]\\ &= (-(2\pi)^4\tilde{\lambda})\delta(\lambda)\frac{\pi}{8}\Theta(z-1)\delta(z-\bar{z})\end{aligned} \tag{4.4.25}$$

where

$$\begin{aligned}\Theta(z-1) &= 1,\ z>1\\ &= 0,\ \text{otherwise}\end{aligned} \tag{4.4.26}$$

Our result (4.4.25) exactly matches with the conformally invariant part of the 4-point tree level scalar celestial amplitude derived in the appendix C.3 (see equation (C.3.8)). One can restore the conformally covariant prefactor using the conformal symmetry. We want to emphasize that there are other choices to calculate the monodromies of the functions appearing in leaf amplitudes. But not all the choices will lead to the required delta function of the cross ratios.

### 4.4.2 MHV gluon scattering

In this section, we investigate the singularity structure of the 4 point clestial leaf amplitudes for MHV gluon scattering processes. We first consider the MHV gluon leaf amplitudes in the timelike wedge as the two cross ratios $z$ and $\bar{z}$ approach each other. The color ordered 4-point MHV gluon amplitude is given by,

$$A_4\left(1^-,2^-,3^+,4^+\right)=\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\delta^{(4)}(p_1+p_2+p_3+p_4) \tag{4.4.27}$$

We first write the timelike leaf amplitude in the global coordinates. In global coordinates the spinor brackets are given by,

$$\langle ij\rangle=\sqrt{\omega_i\omega_j}\sin\sigma_{ij}=\sqrt{\omega_i\omega_j}s_{ij} \tag{4.4.28}$$

From equation (4.3.18) we get,

$$A_4\left(\{\epsilon_i\omega_i q_i,\vartheta_i\}\right)=\frac{\omega_1\omega_2}{\omega_3\omega_4}\frac{s_{12}^3}{s_{23}s_{34}s_{41}} \tag{4.4.29}$$

Substituting (4.4.29) in the first equation of (4.3.26) one can obtain the timelike 4-point leaf amplitude for MHV gluon scattering as,

$$\mathcal{L}_4^g(\sigma_i,\bar{\sigma}_i) = \frac{s_{12}^3}{s_{23}s_{34}s_{41}}\mathcal{C}_4(\sigma_i,\bar{\sigma}_i) \tag{4.4.30}$$

where $\mathcal{C}_4(\sigma_i,\bar{\sigma}_i)$ is given by (4.4.6) with

$$\bar{h}_1 = 1+\frac{i\lambda_1}{2},\ \bar{h}_2 = 1+\frac{i\lambda_2}{2},\ \bar{h}_3 = \frac{i\lambda_3}{2},\ \bar{h}_4 = \frac{i\lambda_4}{2}. \tag{4.4.31}$$

We now follow the same procedure as discussed in the scalar case in previous subsections. We first express the amplitude (4.4.30) in the planar coordinates using (4.4.9) and multiplying by the appropriate Jacobian factor. For the sake of simplicity, we take $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$ and use (4.4.14). We similarly define the following function from the leaf amplitude in planar coordinates $\mathcal{L}_4^g(z_i,\bar{z}_i)$ in the configuration (4.4.15):

$$\begin{aligned}
\mathcal{G}_4(z,\bar{z}) &= \lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\mathcal{L}_4^g(z_i,\bar{z}_i)\\
&= \frac{i\pi}{2}\Gamma\left(1+2i\lambda\right)e^{2\pi\lambda}\frac{z\bar{z}^2}{z-1}H(2+i\lambda,2+i\lambda,3+2i\lambda,4+2i\lambda;u_+,v_+)\\
&-\frac{i\pi}{2}\Gamma\left(1+2i\lambda\right)e^{-2\pi\lambda}\frac{z\bar{z}^2}{z-1}H(2+i\lambda,2+i\lambda,3+2i\lambda,4+2i\lambda;u_-,v_-)
\end{aligned} \tag{4.4.32}$$

where $u_\pm,\ v_\pm$ are given by (4.4.19). $H$-functions appeared in (4.4.32) satisfy the following two identities [156, 157],

$$H(\alpha,\beta,\gamma,\delta;u,v) = H(\alpha,\beta,\alpha+\beta-\delta+1,\alpha+\beta-\gamma+1;v,u) \tag{4.4.33}$$

$$(\delta-\alpha-\beta)\,H(\alpha,\beta,\gamma,\delta;u,v) = H(\alpha,\beta,\gamma,\delta+1;u,v) - v\,H(\alpha+1,\beta+1,\gamma,\delta+1;u,v) \tag{4.4.34}$$

First using the identity (4.4.33), then (4.4.34) and finally (4.4.33) again, we can write

$$H\left(2+i\lambda,2+i\lambda,3+2i\lambda,4+2i\lambda;u,v\right)=\frac{1}{u}\left[H(1+i\lambda,1+i\lambda,1+2i\lambda,2+2i\lambda;u,v)\right.$$
$$\left.+(1+2i\lambda)H(1+i\lambda,1+i\lambda,2+2i\lambda,2+2i\lambda;u,v)\right] \tag{4.4.35}$$

Using (4.4.35), equation (4.4.32) can be rewritten as

$$\mathcal{G}_4(z,\bar{z})=\mathcal{G}_4^{\text{sing}}(z,\bar{z})+\mathcal{G}_4^{\text{reg}}(z,\bar{z}) \tag{4.4.36}$$

where

$$\begin{aligned}\mathcal{G}_4^{\text{sing}}(z,\bar{z})&=\frac{i\pi}{2}\Gamma\left(1+2i\lambda\right)e^{2\pi\lambda}\frac{\bar{z}}{z-1}H(1+i\lambda,1+i\lambda,1+2i\lambda,2+2i\lambda;u_+,v_+)\\&-\frac{i\pi}{2}\Gamma\left(1+2i\lambda\right)e^{-2\pi\lambda}\frac{\bar{z}}{z-1}H(1+i\lambda,1+i\lambda,1+2i\lambda,2+2i\lambda;u_-,v_-)\\\mathcal{G}_4^{\text{reg}}(z,\bar{z})&=\frac{i\pi}{2}\Gamma\left(2+2i\lambda\right)e^{2\pi\lambda}\frac{\bar{z}}{z-1}H(1+i\lambda,1+i\lambda,2+2i\lambda,2+2i\lambda;u_+,v_+)\\&-\frac{i\pi}{2}\Gamma\left(2+2i\lambda\right)e^{-2\pi\lambda}\frac{\bar{z}}{z-1}H(1+i\lambda,1+i\lambda,2+2i\lambda,2+2i\lambda;u_-,v_-)\end{aligned} \tag{4.4.37}$$

The $H$-function in the first equation of (4.4.37) is the one that appeared in the scalar case and we have showed that they contain a simple pole singularity at $z=\bar{z}$. As discussed in [156] (equation C.11), the $H$-function in the second equation of (4.4.37) does not have any singularity at $z=\bar{z}$. Thus using the same prescription for the paths of $z,\bar{z}$ as described in the paragraph above (4.4.21), we can show that for $\bar{z}>z>1$,

$$\delta(\lambda)\mathcal{G}_4(z,\bar{z})=\frac{i\pi}{2}\frac{\bar{z}}{z-1}\frac{\delta(\lambda)}{\bar{z}-z+i\epsilon}\left[4\pi^2+2\pi i\log\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right]+\delta(\lambda)\text{Reg}^t. \tag{4.4.38}$$

where $\text{Reg}^t$ is the regular terms as $z\to\bar{z}$. Similarly, for $\bar{z}>z>1$ the spacelike leaf amplitude is given by

$$\delta(\lambda)\overline{\mathcal{G}_4}(z,\bar{z}) = \frac{i\pi}{2}\frac{\bar{z}}{z-1}\frac{\delta(\lambda)}{\bar{z}-z+i\epsilon}\left[-4\pi^2 - 2\pi i \log\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right] + \delta(\lambda)\text{Reg}^s. \quad (4.4.39)$$

By adding (4.4.38) and (4.4.39) and multiplying $\frac{1}{(2\pi)^3}\frac{1}{4}$, we get

$$\delta(\lambda)\frac{\pi}{8}\Theta(z-1)\frac{z}{z-1}\delta(z-\bar{z}) + \delta(\lambda)\,(\text{Regular terms as } z \to \bar{z}) \quad (4.4.40)$$

The distributional part in the above equation exactly matches with conformally invariant part of the $4$-point celestial MHV gluon amplitudes. This can be seen by applying our configuration (4.4.14),(4.4.15) to 4-point MHV gluon celestial amplitude computed in [13] and extracting the conformally invariant part which only depends on the cross ratios $z$ and $\bar{z}$. We expect that the regular terms will vanish from the final celestial amplitude, i.e. form (4.4.40). We hope to address this in future.

## 4.5 BG equations for MHV gluon leaf amplitudes

MHV gluon leaf amplitudes can be shown to satisfy BG equations first derived in [22] [3] by considering the subleading ($\mathcal{O}(1)$) terms in the OPE between two positive helicity outgoing gluon operators and soft gluon theorems. In this section we derive the BG equations for the timelike leaf amplitudes in the same way. We will work with the $4$-point amplitude with $\epsilon_1 = \epsilon_2 = -1, \epsilon_3 = \epsilon_4 = +1$. We first write the $4$-point leaf amplitude in the planar coordinates in a form that is appropriate for OPE decomposition. We also restore the color indices. So we start with the full $4$-point tree level MHV gluon amplitude in momentum

[3] See [126] for momentum space origin of these differential equations.

space as given by:

$$A_4(1^{-,a_1},2^{-,a_2},3^{+,a_3},4^{+,a_4}) = g_{YM}^2\left\{A_4[1^-2^-3^+4^+]\text{tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{perm}(234)\right\} \times\delta^{(4)}\left(\sum_{i=1}^4 p_i\right) \tag{4.5.1}$$

where $g_{YM}$ is the coupling constant and $A_4[i^{\vartheta_i}j^{\vartheta_j}k^{\vartheta_k}l^{\vartheta_l}]$ are color ordered partial MHV amplitudes given by,

$$A_4[i^-j^+k^-l^+] = \frac{\langle ik\rangle^4}{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle} \tag{4.5.2}$$

Substituting (4.5.2) in (4.5.1) we get [4],

$$A_4(1^{-,a_1},2^{-,a_2},3^{+,a_3},4^{+,a_4}) = -g_{YM}^2\frac{\omega_1\omega_2}{\omega_3\omega_4}\frac{z_{12}^3}{z_{23}z_{34}z_{41}}\left[f^{a_1a_2x}f^{xa_3a_4} - \frac{z_{12}z_{34}}{z_{13}z_{24}}f^{a_1a_3x}f^{xa_2a_4}\right]\delta^{(4)}\left(\sum_{i=1}^4 p_i\right) \tag{4.5.3}$$

In planar coordinates the 4-point leaf amplitude in the timelike region corresponding to the above amplitude is given by,

$$\mathcal{L}_4^g(1^{-,a_1}_{\Delta_1},2^{-,a_2}_{\Delta_2},3^{+,a_3}_{\Delta_3},4^{+,a_4}_{\Delta_4}) = -g_{YM}^2\frac{z_{12}^3}{z_{23}z_{34}z_{41}}\left[f^{a_1a_2x}f^{xa_3a_4} - \frac{z_{12}z_{34}}{z_{13}z_{24}}f^{a_1a_3x}f^{xa_2a_4}\right] \times\int_{\hat{x}_+^2=-1} d^3\hat{x}_+\left(\prod_{j=1}^2\int_0^\infty d\omega_j\,\omega_j^{\Delta_j}\right)\left(\prod_{j=3}^4\int_0^\infty d\omega_j\,\omega_j^{\Delta_j-2}\right)e^{\sum_{k=1}^4(i\epsilon_k\omega_kq_k\cdot\hat{x}_+-\epsilon\omega_k)} \tag{4.5.4}$$

where $q_k^\mu$ in terms of $(z_k,\bar{z}_k)$ is given by (4.3.15) and we have used a different notation for the leaf amplitudes which is convenient for OPE factorization. $i^{\pm}_{\Delta_i}$ inside $\mathcal{L}_4^g$ means $i$-th gluon with helicity $\vartheta_i = \pm 1$ and conformal dimension $\Delta_i$. In terms of the scalar conformal primary wavefunction

$$\Phi_{\Delta_i,\epsilon_i}(\hat{x},q_i) = \int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}e^{i\epsilon_i\omega_iq_i\cdot\hat{x}-\epsilon\omega_i} \tag{4.5.5}$$

[4]Throughout this section we will use planar coordinates and follow the same parametrization of the spinor brackets as given in [144].

we can write the leaf amplitude (4.5.4) as,

$$\mathcal{L}_4^g(1_{\Delta_1}^{-,a_1}, 2_{\Delta_2}^{-,a_2}, 3_{\Delta_3}^{+,a_3}, 4_{\Delta_4}^{+,a_4}) = -g_{YM}^2 \frac{z_{12}^3}{z_{23}z_{34}z_{41}} \left[ f^{a_1a_2x} f^{xa_3a_4} - \frac{z_{12}z_{34}}{z_{13}z_{24}} f^{a_1a_3x} f^{xa_2a_4} \right] \times \int_{\hat{x}_+^2=-1} d^3\hat{x}_+ \left( \prod_{j=1}^{2} \Phi_{\Delta_j+1,-}(\hat{x}_+, q_j) \right) \left( \prod_{j=3}^{4} \Phi_{\Delta_j-1,+}(\hat{x}_+, q_j) \right) \tag{4.5.6}$$

We want to compute the OPE between the positive helicity outgoing [5] conformal gluon operators inserted at points $(z_3, \bar{z}_3)$ and $(z_4, \bar{z}_4)$ on the celestial torus. Different modes of the leading and subleading soft gluon symmetry algebra will appear in the subleading order ($\mathcal{O}(1)$) of the OPE. Hence, following [23, 24, 32–36], we define the leading and subleading conformally soft gluon operators as,

$$R^{k,a}(z, \bar{z}) := \lim_{\Delta \to k} (\Delta - k) \mathcal{O}_{\Delta}^{+,a}(z, \bar{z}), \ k = 1, 0. \tag{4.5.7}$$

where $\mathcal{O}_{\Delta}^{+,a}(z, \bar{z})$ denote a positive helicity outgoing gluon conformal primary operator of dimension $\Delta$ at the point $(z, \bar{z})$ on the celestial torus. In fact one can define a tower of conformally soft gluon operators and the corresponding conserved currents follow a symmetry algebra known as $S$-algebra [24]. It was shown in [151] that MHV gluon leaf amplitudes respect this symmetry algebra. However, for our purpose we will restrict to leading and subleading soft gluon symmetry. The soft current $R_0^{1,a}(z)$ is a Kac-Moody current [19, 101–106].

The explicit expressions of the actions of the modes $R_{\alpha,n}^{k,a}$ of the operators $R^{k,a}(z, \bar{z})$ ($k = 1, 0$) on the amplitudes [22] can be found out from the known soft theorems. In particular, for our purpose the leading soft gluon mode that will play an important role, is given by

[5] Here we abuse the notation and call the conformal gluon operator corresponding to positive frequency solutions in the bulk as outgoing operator.

$R^{1,a}_{-1,0}$. Its action on the 3-point leaf amplitudes

$$\mathcal{L}^g_3\left(1^{-,a_1}_{\Delta_1}, 2^{-,a_2}_{\Delta_2}, 4^{+,a_4}_{\Delta_3+\Delta_4-1}\right) = -2ig_{YM}f^{a_1a_2a_4}\frac{z_{12}^3}{z_{24}z_{41}}\int d^3\hat{x}_+\left(\prod_{j=1}^{2}\Phi_{\Delta_j+1,-}(\hat{x}_+,q_j)\right) \times\Phi_{\Delta_3+\Delta_4-2,+}\left(\hat{x}_+,q_4\right) \tag{4.5.8}$$

can be determined from the leading positive helicity soft gluon theorem and it is given by [6],

$$\begin{aligned}\mathcal{R}^{1,a_3}_{-1,0}\mathcal{L}^g_3\left(1^{-,a_1}_{\Delta_1}, 2^{-,a_2}_{\Delta_2}, 4^{+,a_4}_{\Delta_3+\Delta_4-1}\right) &= -2g_{YM}\left(\frac{f^{a_1a_3x}f^{xa_2a_4}}{z_{14}}+\frac{f^{xa_2a_3}f^{xa_4a_1}}{z_{24}}\right)\\ &\times\frac{z_{12}^3}{z_{24}z_{41}}\int d^3\hat{x}_+\left(\prod_{j=1}^{2}\Phi_{\Delta_j+1,\epsilon_j}(\hat{x}_+,q_j)\right)\Phi_{\Delta_3+\Delta_4-2,+}\left(\hat{x}_+,q_4\right)\end{aligned} \tag{4.5.9}$$

Let us now move to the OPE factorization.

### 4.5.1 OPE factorization

We will extract the OPE by factorizing the four point gluon leaf amplitudes. Since we are interested in computing the OPE between 3rd and 4th gluon primaries, we will concentrate on the product of the following two scalar conformal primaries,

$$\Phi_{\Delta_3-1,+}\left(\hat{x}_+,q_3\right)\Phi_{\Delta_4-1,+}\left(\hat{x}_+,q_4\right) = \int_0^\infty d\omega_3\,\omega_3^{\Delta_3-2}\int_0^\infty d\omega_4\,\omega_4^{\Delta_4-2}e^{i(\omega_3q_3+\omega_4q_4)\cdot\hat{x}_+} \tag{4.5.10}$$

Using the following parametrization:

$$\omega_3 = t\omega,\ \omega_4 = (1-t)\omega \tag{4.5.11}$$

[6]Equation (4.5.9) can be obtained by applying equation (3.9) of [22] to the 3-point leaf amplitudes and by identifying $\mathcal{J}^a_{-1} = \mathcal{R}^{1,a}_{-1,0}$.

and expanding around $z_{34}=0, \bar{z}_{34}=0$ we get from (4.5.10),

$$\begin{aligned}\Phi_{\Delta_3-1,+}\left(\hat{x}_+,q_3\right)\Phi_{\Delta_4-1,+}\left(\hat{x}_+,q_4\right) &= B(\Delta_3-1,\Delta_4-1)\Phi_{\Delta_3+\Delta_4-2,+}\left(\hat{x}_+,q_4\right)\\ &+z_{34}\,B(\Delta_3,\Delta_4-1)\frac{\partial}{\partial z_4}\Phi_{\Delta_3+\Delta_4-2,+}\left(\hat{x}_+,q_4\right)+\cdots\end{aligned} \tag{4.5.12}$$

We use the above equation in (4.5.6) and expanding around $z_{34}=0, \bar{z}_{34}=0$ we get,

$$\begin{aligned}\mathcal{L}_4^g\left(1_{\Delta_1}^{-,a_1},2_{\Delta_2}^{-,a_2},3_{\Delta_3}^{+,a_3},4_{\Delta_4}^{+,a_4}\right) &= -g_{YM}^2\frac{1}{z_{34}}B(\Delta_3-1,\Delta_4-1)f^{a_1a_2x}f^{xa_3a_4}\\ &\times\frac{z_{12}^3}{z_{24}z_{41}}\int d^3\hat{x}_+\left(\prod_{j=1}^{2}\Phi_{\Delta_j+1,\epsilon_j}(\hat{x}_+,q_j)\right)\Phi_{\Delta_3+\Delta_4-2,+}\left(\hat{x}_+,q_4\right)\\ &-g_{YM}^2B(\Delta_3-1,\Delta_4-1)\left(\frac{f^{a_1a_3x}f^{xa_2a_4}}{z_{14}}+\frac{f^{xa_2a_3}f^{xa_4a_1}}{z_{24}}\right)\\ &\times\frac{z_{12}^3}{z_{24}z_{41}}\int d^3\hat{x}_+\left(\prod_{j=1}^{2}\Phi_{\Delta_j+1,\epsilon_j}(\hat{x}_+,q_j)\right)\Phi_{\Delta_3+\Delta_4-2,+}\left(\hat{x}_+,q_4\right)\\ &-g_{YM}^2f^{a_1a_2x}f^{xa_3a_4}\,B(\Delta_3,\Delta_4-1)\frac{z_{12}^3}{z_{24}z_{41}}\\ &\times\frac{\partial}{\partial z_4}\int d^3\hat{x}_+\left(\prod_{j=1}^{2}\Phi_{\Delta_j+1,\epsilon_j}(\hat{x}_+,q_j)\right)\Phi_{\Delta_3+\Delta_4-2,+}\left(\hat{x}_+,q_4\right)+\cdots\end{aligned} \tag{4.5.13}$$

We are now ready to write down the OPE. The leading order term was calculated in [151] and is given by,

$$\mathcal{O}_{\Delta_3}^{+,a_3}(z_3,\bar{z}_3)\mathcal{O}_{\Delta_4}^{+,a_4}(z_4,\bar{z}_4)=-\frac{g_{YM}}{2}\frac{1}{z_{34}}B(\Delta_3-1,\Delta_4-1)if^{a_3a_4x}\mathcal{O}_{\Delta_3+\Delta_4-1}^{+,x}(z_4,\bar{z}_4) \tag{4.5.14}$$

This is the same as the leading order OPE between two positive helicity gluons obtained from celestial MHV amplitude [47]. We now discuss the $\mathcal{O}(1)$ term.

#### 4.5.1.1 Subleading terms: $\mathcal{O}(1)$

Using (4.5.9) in (4.5.13) at $\mathcal{O}(1)$ we have,

$$\mathcal{L}_4^g\left(1^{-,a_1}_{\Delta_1},2^{-,a_2}_{\Delta_2},3^{+,a_3}_{\Delta_3},4^{+,a_4}_{\Delta_4}\right)\big|_{\mathcal{O}(1)}=\frac{g_{YM}}{2}\Big[B(\Delta_3,\Delta_4-1)(-if^{xa_3a_4})\mathcal{L}_{-1}(4)\mathcal{L}_3^g\left(1^{-,a_1}_{\Delta_1},2^{-,a_2}_{\Delta_2},4^{+,x}_{\Delta_3+\Delta_4-1}\right)$$
$$+B(\Delta_3-1,\Delta_4-1)\mathcal{R}^{1,a_3}_{-1,0}\mathcal{L}_3^g\left(1^{-,a_1}_{\Delta_1},2^{-,a_2}_{\Delta_2},4^{+,a_4}_{\Delta_3+\Delta_4-1}\right)-B(\Delta_3,\Delta_4-1)\mathcal{R}^{1,a_3}_{-1,0}\mathcal{L}_3^g\left(1^{-,a_1}_{\Delta_1},2^{-,a_2}_{\Delta_2},\right.$$
$$\left.4^{+,a_4}_{\Delta_3+\Delta_4-1}\right)+B(\Delta_3,\Delta_4-1)\mathcal{R}^{1,a_4}_{-1,0}\mathcal{M}_3^{+}\left(1^{-,a_1}_{\Delta_1},2^{-,a_2}_{\Delta_2},4^{+,a_3}_{\Delta_3+\Delta_4-1}\right)\Big] \tag{4.5.15}$$

After some simplification the above equation can be written in terms of the correlators as follows,

$$\left\langle\mathcal{O}^{-,a_1}_{\Delta_1,-}(z_1,\bar z_1)\mathcal{O}^{-,a_2}_{\Delta_2,-}(z_2,\bar z_2)\mathcal{O}^{+,a_3}_{\Delta_3,+}(z_3,\bar z_3)\mathcal{O}^{+,a_4}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle\big|_{\mathcal{O}(1)}$$
$$=\frac{g_{YM}}{2}\Big[B(\Delta_3,\Delta_4-1)(-if^{xa_3a_4})\left\langle\mathcal{O}^{-,a_1}_{\Delta_1,-}(z_1,\bar z_1)\mathcal{O}^{-,a_2}_{\Delta_2,-}(z_2,\bar z_2)L_{-1}\mathcal{O}^{+,x}_{\Delta_3+\Delta_4-1,+}(z_4,\bar z_4)\right\rangle$$
$$+B(\Delta_3-1,\Delta_4-1)\left[\frac{\Delta_4-1}{\Delta_3+\Delta_4-2}\left\langle\mathcal{O}^{-,a_1}_{\Delta_1,-}(z_1,\bar z_1)\mathcal{O}^{-,a_2}_{\Delta_2,-}(z_2,\bar z_2)R^{1,a_3}_{-1,0}\mathcal{O}^{+,a_4}_{\Delta_3+\Delta_4-1,+}(z_4,\bar z_4)\right\rangle\right.$$
$$\left.+\frac{\Delta_3-1}{\Delta_3+\Delta_4-2}\left\langle\mathcal{O}^{-,a_1}_{\Delta_1,-}(z_1,\bar z_1)\mathcal{O}^{-,a_2}_{\Delta_2,-}(z_2,\bar z_2)R^{1,a_4}_{-1,0}\mathcal{O}^{+,a_3}_{\Delta_3+\Delta_4-1,+}(z_4,\bar z_4)\right\rangle\right]\Big] \tag{4.5.16}$$

Hence, at the level of OPE we have,

$$\mathcal{O}^{+,a_3}_{\Delta_3}(z_3,\bar z_3)\mathcal{O}^{+,a_4}_{\Delta_4}(z_4,\bar z_4)\big|_{\mathcal{O}(1)}=\frac{g_{YM}}{2}\Big[B(\Delta_3,\Delta_4-1)(-if^{xa_3a_4})L_{-1}\mathcal{O}^{+,x}_{\Delta_3+\Delta_4-1}(z_4,\bar z_4)$$
$$+B(\Delta_3-1,\Delta_4-1)\left[\frac{\Delta_4-1}{\Delta_3+\Delta_4-2}R^{1,a_3}_{-1,0}\mathcal{O}^{+,a_4}_{\Delta_3+\Delta_4-1}(z_4,\bar z_4)\right.$$
$$\left.+\frac{\Delta_3-1}{\Delta_3+\Delta_4-2}R^{1,a_4}_{-1,0}\mathcal{O}^{+,a_3}_{\Delta_3+\Delta_4-1}(z_4,\bar z_4)\right]\Big] \tag{4.5.17}$$

This OPE exactly matches with the OPE obtained from celestial MHV gluon amplitudes in [22, 107].

### 4.5.2 BG equations

Now, we can take the subleading soft limit in (4.5.17) and get the following null equation [22],

$$\begin{aligned} & i f^{abc} L_{-1} \mathcal{O}^{+,c}_{\Delta-1,+}(z,\bar{z}) + R^{0,a}_{-\frac{1}{2},\frac{1}{2}} \mathcal{O}^{+,b}_{\Delta,+}(z,\bar{z}) - R^{1,b}_{-1,0} \mathcal{O}^{+,a}_{\Delta-1,+}(z,\bar{z}) \\ & \quad + (\Delta-1) R^{1,a}_{-1,0} \mathcal{O}^{+,b}_{\Delta-1,+}(z,\bar{z}) = 0 \end{aligned} \tag{4.5.18}$$

decoupling of which from the leaf amplitude will lead to BG equations [22] for the latter. In a theory which respects leading and subleading positive helicity soft gluon theorems is guaranteed to have the above null state. Since, $S$-algebra does not change for leaf amplitudes, it was expected that the leaf amplitudes will satisfy the BG equations, which are the differential equations corresponding to the null state (4.5.18).

# Chapter 5

# Revisiting Holographic Symmetry Algebra for the MHV Sector

## 5.1 Introduction

In this chapter, we revisit the holographic symmetry algebra [23, 24] of celestial MHV graviton and gluon scattering amplitudes. At tree level, it was shown that MHV graviton sector is governed by the wedge subalgebra of $w_{1+\infty}$, while the MHV gluon sector is controlled by the $S$ algebra described in 1.1.6. In both cases, the algebra is generated by positive-helicity gravitons or gluons, respectively. The role of negative-helicity states was not addressed in earlier studies of celestial MHV amplitudes, since the MHV condition prevents taking a negative-helicity particle to be energetically soft. In this work, we use the notion of *conformal softness* [32–37] and show that negative-helicity particles generate an additional abelian algebra in both the MHV graviton and gluon sectors. For gravitons, this extends the soft symmetry algebra to a semidirect product of the wedge subalgebra of $w_{1+\infty}$ with the newly identified abelian algebra. An analogous extension arises in the gluon case. We further demonstrate that this extended symmetry structure resolves a puzzle in the celestial MHV literature.

To set the stage, let us briefly review known results for celestial MHV graviton amplitudes. As shown in [21], these amplitudes are completely fixed by the symmetry algebra implied by the tree-level leading and subleading soft graviton theorems for positive-helicity gravitons. The leading soft theorem generates supertranslations, while the subleading soft theorem gives rise to a $\widehat{sl}_2$ current algebra. Together, they form a semidirect product algebra.

The representation of this algebra is degenerate and contains a class of null states, analogous to the Knizhnik-Zamolodchikov (KZ) null states in WZW models. Imposing the decoupling of these null states yields $(n-2)$ differential equations for an $n$-point MHV graviton amplitude. Importantly, these null states involve only descendants of positive-helicity gravitons. By contrast, in a WZW model one finds $n$ KZ equations for an $n$-point correlator. The soft symmetry algebra derived from the leading and subleading soft theorems does not account for the KZ-type null states built from descendants of negative-helicity gravitons, leaving two missing constraints. This discrepancy has remained unresolved.

Although the soft symmetry algebra in the MHV graviton sector was later extended to the wedge subalgebra of $w_{1+\infty}$, this infinite-dimensional enhancement still did not resolve the puzzle. In the present work, we show that the extended algebra given by the semidirect product of the wedge subalgebra of $w_{1+\infty}$ with the abelian algebra generated by negative-helicity gravitons naturally produces the two missing KZ-type null states. These involve the $L_{-1}$ descendants of negative-helicity gravitons and complete the expected set of constraints on the MHV amplitudes.

A similar story exists for the MHV gluon scattering amplitudes which we discuss in detail in this chapter. This chapter is based on [27].

The chapter is organized as follows: We start with the mixed helicity celestial OPE between two graviton primary operators in the MHV-sector and write down the corresponding symmetry algebra in section 5.2. We also review the $w_{1+\infty}$ algebra that is obtained from the OPE between two positive-helicity graviton primaries. In section 5.4, we discuss the symmetry algebra for MHV gluon scattering amplitudes. In section 5.3, we talk about new KZ-type null states for negative-helicity gravitons and we find the same for gluons in section 5.5. Appendix D.1 and appendix D.2 contain the derivations of the new symmetry algebra from the mixed helicity OPE of the MHV-sector for gravitons and gluons respectively. The derivation of the KZ-type null states for negative-helicity gluons is in appendix D.3. We write down the conditions on the graviton primary operators for the symmetry algebra under consideration in appendix D.4.

## 5.2 Symmetry algebra from negative helicity graviton in the MHV-sector

It is well known that holographic (soft) symmetry algebra governing celestial MHV grvaiton amplitudes is given by the wedge subalgebra of $w_{1+\infty}$, generated solely by conformally soft positive-helicity grvaitons. This algebra arises from the singular terms in the OPE between two positive helicity graviton operators. In this section, we derive additional symmetry generators associated with conformally soft negative-helicity gravitons and determine the complete symmetry algebra of the gravitational MHV sector. We denote the positive/negative helicity outgoing graviton conformal primary operators of dimension $\Delta$ at $(z, \bar{z})$ on the celestial sphere by $G^{\pm}_{\Delta}(z, \bar{z})$. The singular terms in the mixed helicity celestial OPE between two graviton primaries in the MHV sector are given by,

$$G^{-}_{\Delta_1}(z_1, \bar{z}_1)G^{+}_{\Delta_2}(z_2, \bar{z}_2) \sim -\frac{\bar{z}_{12}}{z_{12}} \sum_{n=0}^{\infty} B(\Delta_1 + 3 + n, \Delta_2 - 1)\frac{(\bar{z}_{12})^n}{n!}\bar{\partial}_2^n G^{-}_{\Delta_1+\Delta_2}(z_2, \bar{z}_2) \tag{5.2.1}$$

Let's define the infinite tower of conformally soft negative helicity graviotons $H^k(z, \bar{z})$ as

$$\bar{H}^k(z, \bar{z}) = \lim_{\Delta \to k}(\Delta - k)G^{-}_{\Delta}(z, \bar{z}), \quad k = -2, -3, -4, -5, \cdots \tag{5.2.2}$$

with weights

$$(h,\bar{h}) = \left(\frac{k-2}{2}, \frac{k+2}{2}\right). \tag{5.2.3}$$

It is important to note that the limit (5.2.2) is different from the ‘‘energetic soft limit” because in the MHV sector energetic soft limit of negative helicity gravitons is identically zero.

From the structure of the OPE coefficients in (5.2.1) one can see that the OPE truncates in $\bar{z}$ when $k = -2, -3, -4\cdots$. So we can do a truncated mode expansion of $\bar{H}^k(z,\bar{z})$ as follows

$$\bar{H}^k(z,\bar{z}) = \sum_{m=\frac{k+2}{2}}^{-\frac{k+2}{2}} \frac{\bar{H}^k_m(z)}{\bar{z}^{m+\frac{k+2}{2}}} \tag{5.2.4}$$

We can further expand the currents $\bar{H}^k_m(z)$ in Laurent expansion

$$\bar{H}^k_m(z) = \sum_{\alpha\in\mathbb{Z}-\frac{k-2}{2}} \frac{\bar{H}^k_{\alpha,m}}{z^{\alpha+\frac{k-2}{2}}}. \tag{5.2.5}$$

Similarly, the infinite tower of conformally soft positive-helicity graviton operators are defined as

$$H^k\left(z,\bar{z}\right) = \lim_{\Delta\to k}\left(\Delta-k\right)G^+_\Delta\left(z,\bar{z}\right), \quad k = 2,1,0,-1,\cdots \tag{5.2.6}$$

with conformal weights

$$(h,\bar{h}) = \left(\frac{k+2}{2}, \frac{k-2}{2}\right). \tag{5.2.7}$$

The modes of the operators $H^k(z,\bar{z})$ satisfy the holographic symmetry algebra [23]

$$\begin{aligned}\left[H^{k_1}_{\alpha_1,m_1}, H^{k_2}_{\alpha_2,m_2}\right] = &-\left[m_2(2-k_1)-m_1(2-k_2))\right] \\ &\times \frac{\left(\frac{2-k_1}{2}-m_1+\frac{2-k_2}{2}-m_2-1\right)!}{\left(\frac{2-k_1}{2}-m_1\right)!\left(\frac{2-k_2}{2}-m_2\right)!}\frac{\left(\frac{2-k_1}{2}+m_1+\frac{2-k_2}{2}+m_2-1\right)!}{\left(\frac{2-k_1}{2}+m_1\right)!\left(\frac{2-k_2}{2}+m_2\right)!}H^{k_1+k_2}_{\alpha_1+\alpha_2,m_1+m_2}\end{aligned} \tag{5.2.8}$$

Similarly, the OPE (5.2.1) give rise to the commutator algebra between the modes of $H^k(z,\bar{z})$ and $\bar{H}^k(z,\bar{z})$ (the derivation is in Appendix D.1)

$$\begin{aligned}\left[H^{k_1}_{\alpha_1,m_1},\bar{H}^{l_1}_{\beta_1,n_1}\right] = &-\Big[n_1(2-k_1)+m_1(2+l_1)\Big]\\ &\times \frac{\left(\frac{2-k_1}{2}-m_1-\frac{2+l_1}{2}-n_1-1\right)!}{\left(\frac{2-k_1}{2}-m_1\right)!\left(-\frac{2+l_1}{2}-n_1\right)!}\frac{\left(\frac{2-k_1}{2}+m_1-\frac{2+l_1}{2}+n_1-1\right)!}{\left(\frac{2-k_1}{2}+m_1\right)!\left(-\frac{2+l_1}{2}+n_1\right)!}\bar{H}^{k_1+l_1}_{\alpha_1+\beta_1,m_1+n_1}\end{aligned} \tag{5.2.9}$$

Since the OPE between two negative helicity gravitons does not contain a pole term in the MHV sector, so we get

$$\left[\bar{H}^{l_1}_{\beta_1,n_1},\bar{H}^{l_2}_{\beta_2,n_2}\right]=0. \tag{5.2.10}$$

One can now define the light transformed operators for positive-helicity and negative-helicity conformally soft gravitons as [24]

$$w^p_m = \frac{1}{\kappa}(p-1-m)!(p-1+m)!H^{-2p+4}_m \tag{5.2.11}$$

$$\bar{w}^q_n = \frac{1}{\kappa}(q-1-n)!(q-1+n)!\bar{H}^{-2q}_n \tag{5.2.12}$$

where $p,q=1,\frac{3}{2},2,\frac{5}{2},...$ and $m$ and $n$ are restricted to take the values

$$1-p\le m\le p-1;\quad 1-q\le n\le q-1 \tag{5.2.13}$$

The algebra of these light transformed operators is given by

$$\begin{aligned}[w^p_m,w^q_n] &= [m(q-1)-n(p-1)]w^{p+q-2}_{m+n}\\ [w^p_m,\bar{w}^q_n] &= [m(q-1)-n(p-1)]\bar{w}^{p+q-2}_{m+n}\\ [\bar{w}^p_m,\bar{w}^q_n] &= 0\end{aligned} \tag{5.2.14}$$

This is the complete symmetry algebra which govern the MHV graviton scattering amplitudes, which is a semidirect product of the $w_{1+\infty}$ and the Abelian algebra. The conformally soft positive helicity gravitons give rise to $w_{1+\infty}$ and the negative helicity conformally soft gravitons give rise to an infinite Abelian algebra.

## 5.3 Knizhnik-Zamolodchikov type null states for negative helicity gravitons

In the previous section we discussed the complete symmetry algebra (5.2.14) that governs the celestial MHV graviton scattering amplitudes. The representation of this symmetry algebra admits null states. In this section we look for the Knizhnik-Zamolodchikov (KZ) type null states which contain the $L_{-1}$ descendant of the (hard) graviton operator on the celestial sphere. Such KZ type null states involving the positive helicity soft particles in the MHV sector were found in [21].

However, using the symmetry algebra generated by positive helicity (conformally soft) gravitons we get only $(n-2)$ KZ type equations for an $n$-point MHV amplitude. These null states contain the $L_{-1}$ descendant of the *positive* helicity hard graviton only. So it was not possible to obtain a complete list of KZ type null states in [21]. In this section obtain the two missing null states using the infinite Abelian symmetry algebra generated by the conformally soft negative helicity gravitons in the MHV sector.

Let's start with the mixed helicity OPE between two graviton conformal-primary operators in the MHV-sector up to $\mathcal{O}(\bar{z})$ [21]

$$\begin{aligned}G^{+}_{\Delta_1}(z_1,\bar{z}_1)\,G^{-}_{\Delta_2}(z_2,\bar{z}_2) =&\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+3)\,H^{1}_{-\frac{1}{2},-\frac{1}{2}}G^{-}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\\ &+B(\Delta_1-1,\Delta_2+3)\,H^{1}_{-\frac{3}{2},\frac{1}{2}}G^{-}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\\ &+\bar{z}_{12}B(\Delta_1-1,\Delta_2+3)\left[\frac{\Delta_1-1}{\Delta_1+\Delta_2+2}H^{0}_{-1,0}G^{-}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)\right.\\ &\left.+\Delta_1\,H^{1}_{-\frac{3}{2},-\frac{1}{2}}G^{-}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\right]+\cdots\end{aligned} \tag{5.3.1}$$

Taking $\Delta_2 \to -3$ soft limit in (5.3.1) we get

$$\begin{aligned}G^{+}_{\Delta_1}(z_1,\bar{z}_1)\bar{H}^{-3}(z_2,\bar{z}_2) =&-\frac{\bar{z}_{12}}{z_{12}}G^{-}_{\Delta_1-3}(z_2,\bar{z}_2)+H^{1}_{-\frac{3}{2},\frac{1}{2}}G^{-}_{\Delta_1-4}(z_2,\bar{z}_2)\\ &+\bar{z}_{12}\left[H^{0}_{-1,0}G^{-}_{\Delta_1-3}(z_2,\bar{z}_2)+\Delta_1\,H^{1}_{-\frac{3}{2},-\frac{1}{2}}G^{-}_{\Delta_1-4}(z_2,\bar{z}_2)\right]+\cdots\end{aligned} \tag{5.3.2}$$

Expanding the RHS of (5.3.2) around $(z_1,\bar{z}_1)$

$$z_2 \to z_1 - z_{12}, \quad \bar{z}_2 \to \bar{z}_1 - \bar{z}_{12}\ ,$$

and demanding consistency with the OPE between $\bar{H}^{-3}(z_2,\bar{z}_2)$ and $G^{+}_{\Delta}(z_1,\bar{z}_1)$ leads to the following equation

$$\begin{aligned}-\bar{z}_{12}\bar{H}^{-3}_{\frac{5}{2},-\frac{1}{2}}G^{+}_{\Delta_1}(z_1,\bar{z}_1) =&\bar{z}_{12}\Big[L_{-1}G^{-}_{\Delta_1-3}(z_1,\bar{z}_1)+H^{0}_{0,-1}H^{1}_{-\frac{3}{2},\frac{1}{2}}G^{-}_{\Delta_1-4}(z_1,\bar{z}_1)\\ &+H^{0}_{-1,0}G^{-}_{\Delta_1-3}(z_1,\bar{z}_1)+\Delta_1\,H^{1}_{-\frac{3}{2},-\frac{1}{2}}G^{-}_{\Delta_1-4}(z_1,\bar{z}_1)\Big]\end{aligned} \tag{5.3.3}$$

which is equivalent to the following null state involving the $L_{-1}$ descendant of the negative helicity graviton

$$\boxed{L_{-1}G_{\Delta}^{-} + H^{0}_{-1,0}G_{\Delta}^{-} + (\Delta+3)\, H^{1}_{-\frac{3}{2},-\frac{1}{2}}G_{\Delta-1}^{-} + H^{0}_{0,-1}H^{1}_{-\frac{3}{2},\frac{1}{2}}G_{\Delta-1}^{-} + \bar{H}^{-3}_{\frac{5}{2},-\frac{1}{2}}G_{\Delta+3}^{+} = 0} \tag{5.3.4}$$

Whereas the KZ type null state involving the $L_{-1}$ descendant of the positive helicity graviton, as obtained in [21] using plus-plus OPE, is

$$\boxed{L_{-1}G_{\Delta}^{+} + H^{0}_{0,-1}H^{1}_{-\frac{3}{2},\frac{1}{2}}G_{\Delta-1}^{+} + H^{0}_{-1,0}G_{\Delta}^{+} + (\Delta-1)H^{1}_{-\frac{3}{2},-\frac{1}{2}}G_{\Delta-1}^{+} = 0.} \tag{5.3.5}$$

In this section, we have derived two new KZ-type null states, (5.3.4), associated with the two negative-helicity gravitons in the MHV sector. The KZ-type null states for the remaining $(n{-}2)$ positive-helicity gravitons are given in (5.3.5). Altogether, this yields a complete set of $n$ KZ equations for an $n$-point MHV graviton amplitude, placing the structure of these amplitudes on the same conceptual footing as correlators in WZW models.

## 5.4 Symmetry algebra from negative helicity gluon in the MHV-sector

In this section we discuss the additional symmetry generators arising out of conformally soft negative helicity gluons in the MHV-sector. The analysis is same as that of the gravity case discussed earlier. We begin with the singular terms of the OPE between a negative and a positive-helicity gluon conformal primary operators in the MHV-sector which are given by [23]

$$O^{a,-}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{if^{ab}_{\ \ c}}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+1+n,\Delta_2-1)\frac{(\bar{z}_{12})^n}{n!}\bar{\partial}_2^n O^{c,-}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2). \tag{5.4.1}$$

where we denote $O_{\Delta}^{a,\pm}(z,\bar{z})$ as a positive/negative helicity outgoing gluon conformal primary operator of dimension $\Delta$ and adjoint group index $a$ at the point $(z,\bar{z})$ on the celestial torus. Let's define the ''conformally soft'' negative-helicity gluon operators $\bar{R}^{k,a}(z,\bar{z})$ as

$$\bar{R}^{k,a}(z,\bar{z}) := \lim_{\Delta\to k}(\Delta-k)O_{\Delta}^{a,-}(z,\bar{z}), \quad k=-1,-2,-3,\cdots \tag{5.4.2}$$

with conformal weights

$$(h,\bar{h}) = \left(\frac{k-1}{2},\frac{k+1}{2}\right) \ .$$

From the structure of OPE (5.4.1) one can do the following truncated mode expansion of $\bar{R}^{k,a}(z,\bar{z})$ in $\bar{z}$-variable

$$\bar{R}^{k,a}(z,\bar{z}) = \sum_{m=\frac{k+1}{2}}^{-\frac{k+1}{2}} \frac{\bar{R}_m^{k,a}(z)}{\bar{z}^{m+\frac{k+1}{2}}} \tag{5.4.3}$$

The holomorphic currents $\bar{R}_m^{k,a}(z)$ can be further mode expanded in Laurent expansion in $z$-variable

$$\bar{R}_m^{k,a}(z) = \sum_{\alpha\in\mathbb{Z}-\frac{k-1}{2}} \frac{\bar{R}_{\alpha,m}^{k,a}}{z^{\alpha+\frac{k-1}{2}}} \tag{5.4.4}$$

Similarly, an infinite tower of conformally soft positive-helicity gluon operators are defined as

$$R^{k,a}\left(z,\bar{z}\right) = \lim_{\Delta\to k}\left(\Delta-k\right)O_{\Delta}^{a,+}\left(z,\bar{z}\right), \quad k=1,0,-1,\cdots \tag{5.4.5}$$

with weights

$$(h,\bar{h}) = \left(\frac{k+1}{2},\frac{k-1}{2}\right) \ .$$

The modes of $R^{k,a}(z,\bar{z})$ satisfy the following holographic symmetry algebra [23]

$$\left[R_{\alpha,n}^{k,a},R_{\alpha',n'}^{l,b}\right] = -if^{ab}_{\ \ c}\,\frac{\left(\frac{1-k}{2}-n+\frac{1-l}{2}-n'\right)!\left(\frac{1-k}{2}+n+\frac{1-l}{2}+n'\right)!}{\left(\frac{1-k}{2}-n\right)!\left(\frac{1-l}{2}-n'\right)!\left(\frac{1-k}{2}+n\right)!\left(\frac{1-l}{2}+n'\right)!}R_{\alpha+\alpha',n+n'}^{k+l-1,c} \tag{5.4.6}$$

Similarly, one can obtain the following commutator algebra between the modes of $R^{k,a}(z,\bar{z})$ and $\bar{R}^{l,b}(z,\bar{z})$ from the mixed helicity OPE (5.4.1) (the derivation is in Appendix D.2)

$$\left[R^{k,a}_{\alpha,n}, \bar{R}^{l,b}_{\alpha',n'}\right] = -if^{ab}_{\ \ c}\, \frac{\left(\frac{1-k}{2} - n - \frac{l+1}{2} - n'\right)!\left(\frac{1-k}{2} + n - \frac{l+1}{2} + n'\right)!}{\left(\frac{1-k}{2} - n\right)!\left(-\frac{l+1}{2} - n'\right)!\left(\frac{1-k}{2} + n\right)!\left(-\frac{l+1}{2} + n'\right)!} \bar{R}^{k+l-1,c}_{\alpha+\alpha',n+n'} \tag{5.4.7}$$

Since the OPE between two negative helicity gluons does not a have pole term, we get

$$\left[\bar{R}^{k,a}_{\alpha,n}, \bar{R}^{l,b}_{\alpha',n'}\right] = 0. \tag{5.4.8}$$

Let's define the conformally soft light transformed positive and negative helicity gluon operators as [24]

$$S^{p,a}_m = (p-1-m)!(p-1+m)! R^{3-2p,a}_m \tag{5.4.9}$$

$$\bar{S}^{q,a}_n = (q-1-n)!(q-1+n)! \bar{R}^{1-2q,a}_m \tag{5.4.10}$$

where $p, q = 1, \frac{3}{2}, 2, \frac{5}{2}, ...$ and the restrictions on $m$ and $n$ become

$$1-p \leq m \leq p-1; \quad 1-q \leq n \leq q-1. \tag{5.4.11}$$

The light transformed soft gluon operators satisfy the following algebra

$$[S^{p,a}_m, S^{q,b}_n] = -if^{ab}_{\ \ c} S^{p+q-1,c}_{m+n} \tag{5.4.12}$$

$$[S^{p,a}_m, \bar{S}^{q,b}_n] = -if^{ab}_{\ \ c} \bar{S}^{p+q-1,c}_{m+n} \tag{5.4.13}$$

$$[\bar{S}^{p,a}_m, \bar{S}^{q,b}_n] = 0 \tag{5.4.14}$$

Maximally helicity violating (MHV) gluon scattering amplitudes in Minkowski spacetime are governed by the symmetry algebra described above. Conformally soft positive helicity gluons generate the S-algebra (5.4.12), while conformally soft negative helicity gluons generate an infinite-dimensional Abelian algebra (5.4.14). The full symmetry algebra underlying MHV gluon scattering is given by the semi-direct product of the S-algebra with this Abelian algebra.

## 5.5 Knizhnik-Zamolodchikov type null states for *negative* helicity gluons

In this section we obtain the KZ type null states involving the negative helicity gluons using the additional symmetries found in section 5.4. The procedure is same as that of the gravity case. The OPE between between a positive and negative helicity outgoing gluon primary operators up to $\mathcal{O}(1)$ is given by [22]

$$\begin{aligned} O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,-}_{\Delta_2}(z_2,\bar{z}_2) =& B\left(\Delta_1-1,\Delta_2+1\right)\Bigg[-\frac{if^{ab}_{\ \ c}}{z_{12}}+\Delta_1\delta^{bc}R^{1,a}_{-1,0} \\ &+\frac{\Delta_1-1}{\Delta_1+\Delta_2}\delta^{bc}R^{0,a}_{-\frac{1}{2},\frac{1}{2}}\left(-H^{1}_{-\frac{1}{2},-\frac{1}{2}}\right)\Bigg]O^{c,-}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \end{aligned} \tag{5.5.1}$$

Now we take the $\Delta_2\to -1$ soft limit in (5.5.1) and demand the consistency with the OPE between $\bar{R}^{-1,b}$ and $O^{a,+}_{\Delta_1}$. Comparing the $\mathcal{O}(1)$ terms from both sides, we obtain the following null state relations involving negative helicity gluon (the details of the computation is given in appendix D.3)

$$\boxed{C_A L_{-1}O^{a,-}_{\Delta}-(\Delta+2)R^{1,b}_{-1,0}R^{1,b}_{0,0}O^{a,-}_{\Delta}-R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}O^{a,-}_{\Delta+1}-\bar{R}^{-1,b}_{1,0}R^{1,b}_{0,0}O^{a,+}_{\Delta+2}=0.} \tag{5.5.2}$$

with $C_A$ being the quadratic Casimir of the adjoint representation.

On the other hand, the KZ type null state involving $L_{-1}$ descendant of the positive helicity gluon, as derived in [22] using plus-plus OPE, is

$$\boxed{C_A L_{-1}\mathcal{O}^{a,+}_{\Delta} - (\Delta+1)R^{1,b}_{-1,0}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta} - R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta+1} = 0.} \tag{5.5.3}$$

In this section, we have identified the two previously missing KZ-type null states, (5.5.2), associated with the two negative-helicity gluons in the MHV sector. Together with the $(n-2)$ KZ-type null states (5.5.3) corresponding to the $(n-2)$ positive-helicity gluons, these results furnish a complete set of $n$ KZ equations for an $n$-point MHV amplitude. This resolves the puzzle in the MHV gluon sector.

# Chapter 6

# Conclusion and Future Directions

In this thesis, we have investigated the soft symmetries of celestial amplitudes that emerge from tree-level soft factorization theorems and explored their implications. In particular, we have studied the constraints these symmetries impose on celestial amplitudes in both gauge theory and gravity. We conclude by summarizing the main results of this work and outlining several promising directions for future research.

In Chapter 2, we studied the tree-level leading and subleading soft gluon theorems in pure Yang-Mills theory chirally coupled to a massive scalar background field. The presence of a massive background explicitly breaks bulk spacetime translation and scaling symmetries of the pure Yang-Mills theory. This naturally raises the question: do scattering amplitudes in such a background continue to satisfy the same null decoupling equations that constrain pure Yang-Mills amplitudes in a trivial background?

We found that they do. We first demonstrated that both the leading and subleading soft gluon theorems remain unmodified in the presence of the massive scalar background. We then analyzed the leading and subleading operator product expansion (OPE) between two positive-helicity gluons and showed that its structure is likewise unchanged. These results strongly suggest that gluon amplitudes in the massive background obey the same null decoupling equations as in pure Yang-Mills theory in a trivial background. We verified this explicitly for the three-point Yang-Mills amplitude coupled to the massive scalar background. An interesting direction for future work would be to investigate graviton scattering amplitudes in this setup, since the leading soft graviton theorem is expected to be modified in the presence of a massive background.

In Chapter 3, we constructed an infinite class of $S$-invariant OPEs between two outgoing positive-helicity gluons on the celestial sphere. These OPEs correspond to distinct $S$-invariant celestial CFTs, characterized by the number of MHV null states appearing in the expansion at subleading order. We also derived KZ-type null states for all such theories. This work illustrates how the $S$ algebra and MHV null states can be used in celestial CFT to probe different sectors of pure Yang-Mills theory in the bulk. We showed that the OPEs derived from MHV gluon amplitudes and from self-dual Yang-Mills amplitudes fit precisely into our classification scheme. It would be highly interesting to determine which additional sectors of pure Yang-Mills theory are captured by the other $S$-invariant OPEs constructed in this work.

A notable feature of our construction is that, although the scaling dimension in celestial CFT is unbounded from below, only a finite number of null states appear in the OPE at subleading order. This observation hints at a possible reformulation of celestial CFT in terms of discrete and bounded scaling dimensions. Discrete bases for celestial CFT have been constructed in the literature, and it would be valuable to relate those constructions to the finite truncation of null states observed here.

In Chapter 4, we analyzed the singularity structure of four-point celestial leaf amplitudes for massless scalars and MHV gluons and demonstrated that they satisfy the same null decoupling equations as those obeyed by the full celestial amplitudes. These leaf amplitudes serve as fundamental building blocks for celestial amplitudes in Klein space. They are particularly useful because they are non-distributional on the celestial torus and admit a natural realization within the framework of the $AdS_3/CFT_2$ correspondence.

Our analysis shows that four-point celestial leaf amplitudes exhibit simple pole-type singularities in cross-ratio space. Although we established this result using constraints derived from bulk scaling invariance, it would be worthwhile to investigate whether this behavior can be derived more generally and can be extended to more generic scattering processes in future work.

In Chapter 5, we revisited the symmetry algebra governing MHV sectors of graviton and gluon scattering amplitudes. Prior to our work, MHV graviton amplitudes were known to be controlled by the wedge subalgebra of $w_{1+\infty}$, generated by positive-helicity gravitons. We showed that the full symmetry algebra is instead a semidirect product of $w_{1+\infty}$ with an infinite abelian algebra generated by conformally soft negative-helicity gravitons. Representations of this extended algebra contain additional KZ-type null states associated with negative-helicity gravitons, which are not found within the $w_{1+\infty}$ wedge algebra alone. While the latter contains $(n-2)$ KZ-type null states for an $n$-point amplitude, our analysis reveals a total of $n$ KZ-type equations in the celestial MHV sector closely paralleling the structure of KZ equations in the WZW model.

Throughout this thesis, we have focused exclusively on tree-level soft theorems and their associated symmetries. We demonstrated how these symmetries can be exploited to constrain and compute tree-level scattering amplitudes in gauge theory and gravity. A natural and important question is how this symmetry structure extends beyond tree level. At loop level, soft theorems for photons and gravitons receive corrections beginning at subleading order. In momentum space, tree-level soft theorems are replaced by logarithmic soft theorems at loop level [89]. However, the full symmetry interpretation of loop-corrected soft theorems remains incomplete.

Recent progress [158] has attempted to provide a symmetry interpretation of loop-level soft photon theorems in the celestial basis. Extending this understanding to loop-level soft graviton theorems would be a particularly significant development, with profound implications for flat-space holography. Although much of the progress in celestial holography has relied on bulk scattering amplitudes, constructing a formulation of celestial CFT that is defined intrinsically on the celestial sphere would mark a major step toward establishing a robust conceptual foundation for flat-space holography.

# Appendix A

# Appendices for Chapter 2

## A.1 Subleading soft gluon theorem

In this appendix we derive (2.5.6). Let us start with the $3$-point correlation function given by (2.4.3),

$$\begin{aligned}\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-},2^{a_2,\epsilon_2}_{\Delta_2,-},3^{a_3,\epsilon_3}_{\Delta_3,+}\right) &= \frac{\mathcal{N}_3}{(2\pi)^4}\frac{2z_{12}^3}{z_{23}z_{31}}\left(if^{a_1a_2a_3}\right)\Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\\ &\times\int\widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}(-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1}\\ &\times\int_0^{i\infty}d\tau\,\tau^{-1-\beta_3}\phi_B(\tau)\left(e^{2\pi i\beta_3}-1\right)\end{aligned} \tag{A.1.1}$$

where

$$\beta_3=\sum_{j=1}^{3}\left(\Delta_j-1\right),\ \mathcal{N}_3=\prod_{j=1}^{3}(-i\epsilon_j)^{\Delta_j-\sigma_j} \tag{A.1.2}$$

.

So, the action of the subleading soft operator on the 3-point correlation function is

$$
\begin{aligned}
&-\sum_{k=1}^{3} \frac{\epsilon_k}{z_{4k}}(-2\bar{h}_k + 1 + \bar{z}_{4k}\bar{\partial}_k)\, T_k^{a_4}\mathcal{P}_k^{-1}\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-}, 2^{a_2,\epsilon_2}_{\Delta_2,-}, 3^{a_3,\epsilon_3}_{\Delta_3,+}\right) \\
&= \epsilon_1 \frac{c_1}{z_{41}}(-\Delta_1 + \bar{z}_{41}\bar{\partial}_1)\, \widetilde{\mathcal{M}}_3^{\Phi}(1_{\Delta_1-1,-}, 2_{\Delta_2,-}, 3_{\Delta_3,+}) + \epsilon_2 \frac{c_2}{z_{42}}(-\Delta_2 + \bar{z}_{42}\bar{\partial}_2)\, \widetilde{\mathcal{M}}_3^{\Phi}(1_{\Delta_1,-}, 2_{\Delta_2-1,-}, 3_{\Delta_3,+}) \\
&\qquad + \epsilon_3 \frac{c_3}{z_{43}}(-\Delta_3 + 2 + \bar{z}_{43}\bar{\partial}_3)\, \widetilde{\mathcal{M}}_3^{\Phi}(1_{\Delta_1,-}, 2_{\Delta_2,-}, 3_{\Delta_3-1,+})
\end{aligned}
\tag{A.1.3}
$$

We have pulled out the colour factors $(if)$ out of the amplitude and $\widetilde{\mathcal{M}}_3^{\Phi}(1_{\Delta_1-1,-}, 2_{\Delta_2,-}, 3_{\Delta_3,+})$ etc. are the colour stripped amplitude in eq.(A.1.3). The right hand side of eq.(A.1.3) can be written as

$$
\begin{aligned}
&\left(\frac{c_1}{z_{14}} + \frac{c_2}{z_{24}} + \frac{c_3}{z_{34}} + (c_1 + c_2 + c_3)\partial_4\right) i\frac{\mathcal{N}_3}{(2\pi)^4}\frac{2z_{12}^3}{z_{23}z_{31}}\Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1) \\
&\times \int \widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}(-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1}(-q(z_4,\bar{z}_4)\cdot\hat{x}) \\
&\qquad \times \int_0^{i\infty} d\tau\, \tau^{-1-\beta_3}\phi_B(\tau)\left(e^{2\pi i\beta_3} - 1\right)
\end{aligned}
\tag{A.1.4}
$$

where we have used the following equation

$$
\bar{z}_{4k}\bar{\partial}_k(-q_k\cdot\hat{x}) = -(-q_k\cdot\hat{x}) + (1 + z_{k4}\partial_4)(-q_4\cdot\hat{x}) \tag{A.1.5}
$$

Now using the Jacobi identity

$$
c_1 + c_2 + c_3 = 0 \tag{A.1.6}
$$

we finally get

$$-\sum_{k=1}^{3}\frac{\epsilon_k}{z_{4k}}(-2\bar{h}_k+1+\bar{z}_{4k}\bar{\partial}_k)\,T_k^{a_4}\mathcal{P}_k^{-1}\widetilde{\mathcal{M}}_3^{\Phi}\left(1^{a_1,\epsilon_1}_{\Delta_1,-},2^{a_2,\epsilon_2}_{\Delta_2,-},3^{a_3,\epsilon_3}_{\Delta_3,+}\right)=\left(\frac{c_1}{z_{14}}+\frac{c_2}{z_{24}}+\frac{c_3}{z_{34}}\right)i\frac{\mathcal{N}_3}{(2\pi)^4}\frac{2z_{12}^3}{z_{23}z_{31}}$$
$$\times\Gamma(\Delta_1+1)\Gamma(\Delta_2+1)\Gamma(\Delta_3-1)\int\widetilde{d^3\hat{x}}(-q(z_1,\bar{z}_1)\cdot\hat{x})^{-\Delta_1-1}(-q(z_2,\bar{z}_2)\cdot\hat{x})^{-\Delta_2-1}\,(-q(z_3,\bar{z}_3)\cdot\hat{x})^{-\Delta_3+1}$$
$$\times(-q(z_4,\bar{z}_4)\cdot\hat{x})\int_0^{i\infty}d\tau\,\tau^{-\beta_3}\phi_B(\tau)\left(e^{2\pi i\beta_3}-1\right)\tag{A.1.7}$$

which is same as (2.5.6).

## A.2 Solution of the BG equations

In subsection 2.5.1, we have argued that the BG equations remain same if we put the MHV amplitudes in a massive scalar background. Here we show explicitly that the three point MHV amplitude in the massive background satisfies the BG equations. We first write down the most general form of the 3-point amplitude using the $SL(2,C)$ symmetry and then derive the constraints for the 3-point coefficient imposed by the BG equations.

Let us start with the color ordered $SL(2,C)$-covariant 3-point amplitude given by,

$$\widetilde{\mathcal{M}}_3(1^-_{\Delta_1}2^-_{\Delta_2}3^+_{\Delta_3})=C(\Delta_1,\Delta_2,\Delta_3)z_{12}^{h_3-h_1-h_2}z_{13}^{h_2-h_1-h_3}z_{23}^{h_1-h_2-h_3}\bar{z}_{12}^{\bar{h}_3-\bar{h}_1-\bar{h}_2}\bar{z}_{13}^{\bar{h}_2-\bar{h}_1-\bar{h}_3}\bar{z}_{23}^{\bar{h}_1-\bar{h}_2-\bar{h}_3}\tag{A.2.1}$$

There are two sets of decoupling equations for the color ordered amplitudes [22, 126].

They are

$$\left(\partial_3 - \frac{\Delta_3}{z_{13}} - \frac{1}{z_{23}}\right) \widetilde{\mathcal{M}}_3(1^-_{\Delta_1} 2^-_{\Delta_2} 3^+_{\Delta_3}) + \epsilon_1\epsilon_3 \frac{\Delta_1 - \sigma_1 - 1 + \bar{z}_{13}\bar{\partial}_1}{z_{13}} \widetilde{\mathcal{M}}_3(1^-_{\Delta_1 - 1} 2^-_{\Delta_2} 3^+_{\Delta_3+1}) = 0$$
$$\left(\partial_3 - \frac{\Delta_3}{z_{23}} - \frac{1}{z_{13}}\right) \widetilde{\mathcal{M}}_3(1^-_{\Delta_1} 2^-_{\Delta_2} 3^+_{\Delta_3}) + \epsilon_2\epsilon_3 \frac{\Delta_2 - \sigma_2 - 1 + \bar{z}_{23}\bar{\partial}_2}{z_{23}} \widetilde{\mathcal{M}}_3(1^-_{\Delta_1} 2^-_{\Delta_2 - 1} 3^+_{\Delta_3+1}) = 0 \tag{A.2.2}$$

Using (A.2.1) in (A.2.2) we get the following constraints on the 3-point coefficient

$$C(\Delta_1 - 1, \Delta_2, \Delta_3 + 1) = \epsilon_1\epsilon_3 \frac{(\Delta_1 - \Delta_2 - \Delta_3 + 1)}{(\Delta_3 - \Delta_1 - \Delta_2 - 1)} C(\Delta_1, \Delta_2, \Delta_3) \tag{A.2.3}$$
$$C(\Delta_1, \Delta_2 - 1, \Delta_3 + 1) = \epsilon_2\epsilon_3 \frac{(\Delta_2 - \Delta_1 - \Delta_3 + 1)}{(\Delta_3 - \Delta_1 - \Delta_2 - 1)} C(\Delta_1, \Delta_2, \Delta_3) \tag{A.2.4}$$

Now, one can check that the 3-point coefficient given by [98]

$$C(\Delta_1, \Delta_2, \Delta_3) = \mathcal{N}_3\, \Gamma\left(\frac{\Delta_1 + \Delta_2 - \Delta_3 + 3}{2}\right) \Gamma\left(\frac{\Delta_1 - \Delta_2 + \Delta_3 - 1}{2}\right) \Gamma\left(\frac{-\Delta_1 + \Delta_2 + \Delta_3 - 1}{2}\right) f(\beta) \tag{A.2.5}$$

satisfies (A.2.3), (A.2.4), where $f(\beta)$ is any function with $\beta = \sum_{i=1}^{3} \Delta_i$ and $\mathcal{N}_3$ is given by (A.1.2).

# Appendix B

# Appendices for Chapter 3

## B.1 $S$ algebra primaries

In this Appendix, we write down the conditions on the primary operators that follow from the OPE between two positive helicity outgoing gluon primaries (1.1.99). They are obtained by taking different soft limits in (1.1.99) and comparing both the sides of the OPE:

$$\begin{aligned} R^{k,a}_{p-\frac{k+1}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}_{\Delta}(0,0) &= 0, p \geq 2 \\ R^{k,a}_{\frac{1-k}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}_{\Delta}(0,0) &= -if^{abc}\frac{(-1)^{k+q+1}}{\Gamma(-k-q+2)}\frac{\Gamma(\Delta-1)}{\Gamma(\Delta+q+k-2)}\frac{\bar{\partial}^q}{q!}\mathcal{O}^{c,+}_{\Delta+k-1}(0,0) \end{aligned} \tag{B.1.1}$$

where $0 \leq q \leq 1-k,\ k = 1, 0, -1, \cdots$. These conditions have been used in writing down the transformation properties of the MHV null states. For more details of how to obtain these conditions one can check Appendix F of [26]. The analyses there was done for $w_{1+\infty}$ primaries, but the methodology is same for $S$ algebra also.

## B.2 Transformation properties of the $\Psi_j^{bc}$-null states under the leading soft gluon operator $R_{0,0}^{1,a}$ and the subleading soft gluon operator $R_{\frac{1}{2},\frac{1}{2}}^{0,a}$

Using (3.3.1), (1.1.103) and (B.1.1), one can show that,

$$\begin{aligned} R_{0,0}^{1,a}\Psi_j^{bc}(\Delta) &= -if^{abx}\Psi_j^{xc}(\Delta) - if^{acx}\Psi_j^{bx}(\Delta) \\ R_{\frac{1}{2},\frac{1}{2}}^{0,a}\Psi_j^{bc}(\Delta) &= -(j+2)\,if^{abx}\Psi_{j+1}^{xc}(\Delta-1) + (\Delta+j-2)\,if^{acx}\Psi_j^{bx}(\Delta-1) \\ &\quad + 2\frac{(-1)^j}{\Gamma(j+1)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)}if^{abx}\Psi_1^{xc}(\Delta-1) \end{aligned} \tag{B.2.1}$$

These equations have been used in sections 3.4.2 and 3.4.3.

## B.3 Proof that the KZ-type null states are closed under the action of $R_{\frac{1}{2},\frac{1}{2}}^{0,a}$

We write the second and third term in (3.7.8) as,

$$\Sigma^{ca}(\Delta) = f^{cby}f^{bax}\left[\sum_{k=1}^{n}(k+1)M_k^{yx}(\Delta) + 2E^{yx}(\Delta)\right] + f^{cab}f^{byx}E^{yx}(\Delta) \tag{B.3.1}$$

The above equation can be decomposed into symmetric and antisymmetric part in the following way,

$$\Sigma^{ca}(\Delta) = \Sigma_A^{ca}(\Delta) + \Sigma_S^{ca}(\Delta) \tag{B.3.2}$$

where

$$\Sigma_A^{ca}(\Delta) = \frac{1}{2}\left[\Sigma^{ca}(\Delta) - \Sigma^{ac}(\Delta)\right]$$
$$\Sigma_S^{ca}(\Delta) = \frac{1}{2}\left[\Sigma^{ca}(\Delta) + \Sigma^{ac}(\Delta)\right] \tag{B.3.3}$$

Now, using the Jacobi identity

$$f^{acb}f^{bxy} + f^{xab}f^{bcy} + f^{xcb}f^{aby} = 0 \tag{B.3.4}$$

one can show that

$$\Sigma_A^{ca}(\Delta) = -\frac{1}{2}f^{cab}\chi_n^{1,b}(\Delta) = 0 \tag{B.3.5}$$

We now simplify the symmetric part (B.3.3) and get,

$$\Sigma_S^{ca}(\Delta) = \frac{1}{2}f^{cby}f^{bax}\left[\sum_{k=1}^{n}(k+1)\left(M_k^{xy}(\Delta) + M_k^{yx}(\Delta)\right) + 2\left(E^{xy}(\Delta) + E^{yx}(\Delta)\right)\right] \tag{B.3.6}$$

The leading and subleading soft limits of (3.7.1) and some straightforward algebra then gives,

$$\sum_{k=1}^{n}(k+1)\left(M_k^{xy}(\Delta) + M_k^{yx}(\Delta)\right) + 2\left(E^{xy}(\Delta) + E^{yx}(\Delta)\right) = 0. \tag{B.3.7}$$

Hence we conclude that,

$$\Sigma^{ca}(\Delta) = 0. \tag{B.3.8}$$

# Appendix C

# Appendices for chapter 4

## C.1 Detailed calculation: 4-point scalar leaf amplitude

In this section we perform the integral (4.4.4) in detail and calculate the 4-point scalar leaf amplitudes both in time-like and space-like regions. We will use the techniques described in [144] and [159]. The detailed notations are given in section 4.3. In global coordinates the measure is given by,

$$d^3\hat{x}_+ = \sinh\rho \cosh\rho \, d\rho d\psi d\phi \tag{C.1.1}$$

Using (4.3.8) and (4.3.17), one can compute the following,

$$\hat{p}_k \cdot \hat{x}_+ = \cos(\phi - \phi_k) \sinh\rho - \cos(\psi - \psi_k) \cosh\rho \tag{C.1.2}$$

Then the integral (4.4.4) becomes,

$$\begin{aligned}\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = \int_0^\infty d\rho \, \sinh\rho \, \cosh\rho \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \left( \prod_{k=1}^{4} \int_0^\infty d\omega_k \omega_k^{2\bar{h}_k - 1} e^{-\epsilon\omega_k} \right) \\ \times e^{ix\cos\phi + iy\sin\phi} e^{i\bar{x}\cos\psi + i\bar{y}\sin\psi}\end{aligned} \tag{C.1.3}$$

where

$$
\begin{aligned}
x &= \sinh\rho \sum_{k=1}^{4} \omega_k \cos\phi_k,\ y = \sinh\rho \sum_{k=1}^{4} \omega_k \sin\phi_k \\
\bar{x} &= -\cosh\rho \sum_{k=1}^{4} \omega_k \cos\psi_k,\ \bar{y} = -\cosh\rho \sum_{k=1}^{4} \omega_k \sin\psi_k
\end{aligned}
\tag{C.1.4}
$$

Remember that $\sigma_i, \bar{\sigma}_i$ are the global coordinates on celestial torus and they are related to $\psi_i, \phi_i$ via (4.3.12). We can perform the $\phi$ and $\psi$ integrals in (C.1.3) to get,

$$
\begin{aligned}
\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = 4\pi^2 \int_0^\infty d\rho\, \sinh\rho\, \cosh\rho \left( \prod_{k=1}^{4} \int_0^\infty d\omega_k \omega_k^{2\bar{h}_k - 1} e^{-\epsilon\omega_k} \right) \\
\times J_0\left(\sqrt{x^2+y^2}\right) J_0\left(\sqrt{\bar{x}^2+\bar{y}^2}\right)
\end{aligned}
\tag{C.1.5}
$$

One can check using (C.1.4) that,

$$
\begin{aligned}
\sqrt{x^2+y^2} &= \sinh\rho \sqrt{\sum_{k=1}^{4} \omega_k^2 + 2\sum_{j<k}^{4} \omega_j\omega_k \cos\phi_{jk}} = \sinh\rho\, \Phi \\
\sqrt{\bar{x}^2+\bar{y}^2} &= \cosh\rho \sqrt{\sum_{k=1}^{4} \omega_k^2 + 2\sum_{j<k}^{4} \omega_j\omega_k \cos\psi_{jk}} = \cosh\rho\, \Psi
\end{aligned}
\tag{C.1.6}
$$

where $\phi_{ij} = \phi_i - \phi_j,\ \psi_{ij} = \psi_i - \psi_j$. Next we make a change of variables $y = \sinh\rho$ in (C.1.5) to get,

$$
\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = 4\pi^2 \left( \prod_{k=1}^{4} \int_0^\infty d\omega_k \omega_k^{2\bar{h}_k - 1} e^{-\epsilon\omega_k} \right) \int_0^\infty dy y J_0\left(y\Phi\right) J_0\left(\sqrt{1+y^2}\Psi\right) \tag{C.1.7}
$$

Using the following representation of the Bessel function

$$
J_0\left(\sqrt{1+y^2}\Psi\right) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{d\xi}{\xi} e^{\xi - \frac{(1+y^2)\Psi^2}{4\xi}} \tag{C.1.8}
$$

and rescaling $\xi \to \frac{\xi\Psi^2}{4}$ in (C.1.7) gives,

$$\begin{aligned}\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = -2\pi i \left( \prod_{k=1}^{4} \int_0^\infty d\omega_k \omega_k^{2\bar{h}_k - 1} e^{-\epsilon\omega_k} \right) \int_0^\infty dy y J_0\left(y\Phi\right) \\ \times \int_{\delta - i\infty}^{\delta + i\infty} \frac{d\xi}{\xi} e^{\frac{\xi\Psi^2}{4} - \frac{(1+y^2)}{\xi}}\end{aligned} \tag{C.1.9}$$

Then one can easily perform the $y$-integral in the above equation and get,

$$\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = -\pi i \left( \prod_{k=1}^{4} \int_0^\infty d\omega_k \omega_k^{2\bar{h}_k - 1} e^{-\epsilon\omega_k} \right) \int_{\delta - i\infty}^{\delta + i\infty} d\xi \, e^{-\frac{1}{\xi} + \frac{\xi}{4}(\Psi^2 - \Phi^2)} \tag{C.1.10}$$

with

$$\Psi^2 - \Phi^2 = -4 \sum_{j<k}^{4} \omega_j \omega_k s_{jk} \bar{s}_{jk} \tag{C.1.11}$$

Hence, the scalar leaf amplitude can be written as,

$$\mathcal{C}_4(\sigma_i, \bar{\sigma}_i) = -\pi i \int_{\delta - i\infty}^{\delta + i\infty} d\xi \, e^{-\frac{1}{\xi}} I_4 \tag{C.1.12}$$

where

$$I_4 = \left( \prod_{k=1}^{4} \int_0^\infty d\omega_k \omega_k^{2\bar{h}_k - 1} e^{-\epsilon\omega_k} \right) e^{-\xi \sum_{j<k}^{4} \omega_j \omega_k s_{jk} \bar{s}_{jk}} \tag{C.1.13}$$

We now compute the above integral. There are six terms in the summation, namely,

$$\omega_1\omega_2 s_{12}\bar{s}_{12} + \omega_1\omega_3 s_{13}\bar{s}_{13} + \omega_1\omega_4 s_{14}\bar{s}_{14} + \omega_2\omega_3 s_{23}\bar{s}_{23} + \omega_2\omega_4 s_{24}\bar{s}_{24} + \omega_3\omega_4 s_{34}\bar{s}_{34} \tag{C.1.14}$$

We use the Mellin-Barnes representation for the following two exponentials,

$$\begin{aligned}e^{-\xi\omega_1\omega_2 s_{12}\bar{s}_{12}} e^{-\xi\omega_1\omega_4 s_{14}\bar{s}_{14}} = \int_{c - i\infty}^{c + i\infty} \frac{ds}{2\pi i} \Gamma(s) \left(\xi\omega_1\omega_4 s_{14}\bar{s}_{14}\right)^{-s} \\ \times \int_{c' - i\infty}^{c' + i\infty} \frac{dr}{2\pi i} \Gamma(r) (\xi\omega_1\omega_2 s_{12}\bar{s}_{12})^{-r}\end{aligned} \tag{C.1.15}$$

Let's first write $I_4$ as,

$$I_4 = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\Gamma(s)\left(\xi s_{14}\bar{s}_{14}\right)^{-s} \int_{c'-i\infty}^{c'+i\infty} \frac{dr}{2\pi i}\Gamma(r)(\xi s_{12}\bar{s}_{12})^{-r} \left(\prod_{k=2}^{4}\int_0^{\infty} d\omega_k \omega_k^{2\bar{h}_k-1} e^{-\epsilon\omega_k}\right)$$
$$\times \omega_4^{-s}\omega_2^{-r} e^{-\xi(\omega_2\omega_3 s_{23}\bar{s}_{23}+\omega_2\omega_4 s_{24}\bar{s}_{24}+\omega_3\omega_4 s_{34}\bar{s}_{34})} \int_0^{\infty} d\omega_1 \omega_1^{2\bar{h}_1-r-s-1} e^{-\omega_1(\xi\omega_3 s_{13}\bar{s}_{13}+\epsilon)} \tag{C.1.16}$$

Then performing the $\omega_1$ integral we get,

$$I_4 = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\Gamma(s)\left(\xi s_{14}\bar{s}_{14}\right)^{-s} \int_{c'-i\infty}^{c'+i\infty} \frac{dr}{2\pi i}\Gamma(r)(\xi s_{12}\bar{s}_{12})^{-r} \frac{\Gamma(2\bar{h}_1-r-s)}{(\xi s_{13}\bar{s}_{13}+\epsilon)^{2\bar{h}_1-r-s}}$$
$$\times \left(\prod_{k=2}^{4}\int_0^{\infty} d\omega_k \omega_k^{2\bar{h}_k-1} e^{-\epsilon\omega_k}\right) \omega_4^{-s}\omega_2^{-r}\omega_3^{r+s-2\bar{h}_1} e^{-\xi(\omega_2\omega_3 s_{23}\bar{s}_{23}+\omega_2\omega_4 s_{24}\bar{s}_{24}+\omega_3\omega_4 s_{34}\bar{s}_{34})} \tag{C.1.17}$$

Let us now make the change of variable $\omega_k = \frac{\sqrt{t_2t_3t_4}}{t_k}$ and perform the $t$ integrals. Thus $I_4$ becomes,

$$I_4 = \frac{1}{2}(\xi s_{13}\bar{s}_{13}+\epsilon)^{-2\bar{h}_1}(\xi s_{34}\bar{s}_{34}+\epsilon)^{\bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4}(\xi s_{24}\bar{s}_{24}+\epsilon)^{-\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4}$$
$$\times(\xi s_{23}\bar{s}_{23}+\epsilon)^{\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4}\mathcal{F}_4(\tilde{u},\tilde{v}) \tag{C.1.18}$$

where

$$\mathcal{F}_4(\tilde{u},\tilde{v}) = \int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{dr}{2\pi i}\Gamma(s)\Gamma(r)\Gamma(s-\bar{h}_1+\bar{h}_2+\bar{h}_3-\bar{h}_4)$$
$$\times\Gamma(r-\bar{h}_1-\bar{h}_2+\bar{h}_3+\bar{h}_4)\Gamma(2\bar{h}_1-r-s)\Gamma(-r-s+\bar{h}_1+\bar{h}_2-\bar{h}_3+\bar{h}_4)\tilde{u}^{-r}\tilde{v}^{-s} \tag{C.1.19}$$

and we have defined,

$$\tilde{u} = \frac{(\xi s_{12}\bar{s}_{12})(\xi s_{34}\bar{s}_{34}+\epsilon)}{(\xi s_{13}\bar{s}_{13}+\epsilon)(\xi s_{24}\bar{s}_{24}+\epsilon)}, \tilde{v} = \frac{(\xi s_{14}\bar{s}_{14})\,(\xi s_{23}\bar{s}_{23}+\epsilon)}{(\xi s_{13}\bar{s}_{13}+\epsilon)(\xi s_{24}\bar{s}_{24}+\epsilon)} \tag{C.1.20}$$

We now change $r \to -r,\ s \to -s$ in the integral (C.1.19) and get,

$$\mathcal{F}_4(\tilde{u},\tilde{v}) = \int_{-c-i\infty}^{-c+i\infty} \frac{ds}{2\pi i} \int_{-c'-i\infty}^{-c'+i\infty} \frac{dr}{2\pi i} \Gamma(-s)\Gamma(-r)\Gamma(-s-\bar{h}_1+\bar{h}_2+\bar{h}_3-\bar{h}_4)$$
$$\times\Gamma(-r-\bar{h}_1-\bar{h}_2+\bar{h}_3+\bar{h}_4)\Gamma(2\bar{h}_1+r+s)\Gamma(r+s+\bar{h}_1+\bar{h}_2-\bar{h}_3+\bar{h}_4)\tilde{u}^r\tilde{v}^s \tag{C.1.21}$$

$c, c' > 0$. The above integral is given by (B.9) of [156] and can be written in terms of the $H$-function defined in that paper. More specifically, we can write,

$$\mathcal{F}_4(\tilde{u},\tilde{v}) = H\left(2\bar{h}_1, \bar{h}_1+\bar{h}_2-\bar{h}_3+\bar{h}_4, 2\bar{h}_1+2\bar{h}_2-1, 2\bar{h}_1+2\bar{h}_2; \tilde{u},\tilde{v}\right) \tag{C.1.22}$$

We will ignore writing the first four arguments of the $H$-function unless their explicit expressions are required. Since the $\xi$-dependence in $H$-function is only through $\tilde{u},\tilde{v}$ (and hence becomes $\xi$-independent by giving proper $i\epsilon$ factors), we can easily perform the $\xi$ integral. Let us write the scalar leaf amplitudes $\mathcal{C}_4(\sigma_i,\bar{\sigma}_i)$ in terms of the $H$-function,

$$\mathcal{C}_4(\sigma_i,\bar{\sigma}_i) = -\frac{i\pi}{2}\int_{\delta-i\infty}^{\delta+i\infty} d\xi\, e^{-\frac{1}{\xi}}(\xi s_{13}\bar{s}_{13}+\epsilon)^{-2\bar{h}_1}(\xi s_{34}\bar{s}_{34}+\epsilon)^{\bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4}$$
$$\times(\xi s_{24}\bar{s}_{24}+\epsilon)^{-\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4}(\xi s_{23}\bar{s}_{23}+\epsilon)^{\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4}H\left(\tilde{u},\tilde{v}\right) \tag{C.1.23}$$

Following [144] we now make a change of variable $\xi = \delta + iy$ to get,

$$\mathcal{C}_4(\sigma_i,\bar{\sigma}_i) = \frac{\pi}{2}\int_{-\infty}^{\infty} dy\, e^{\frac{i}{y-i\delta}}(iys_{13}\bar{s}_{13}+\epsilon)^{-2\bar{h}_1}(iys_{34}\bar{s}_{34}+\epsilon)^{\bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4}$$
$$\times(iys_{24}\bar{s}_{24}+\epsilon)^{-\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4}(iys_{23}\bar{s}_{23}+\epsilon)^{\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4} \tag{C.1.24}$$
$$\times H\left(\frac{(iys_{12}\bar{s}_{12})(iys_{34}\bar{s}_{34}+\epsilon)}{(iys_{13}\bar{s}_{13}+\epsilon)(iys_{24}\bar{s}_{24}+\epsilon)}, \frac{(iys_{14}\bar{s}_{14})\,(iys_{23}\bar{s}_{23}+\epsilon)}{(iys_{13}\bar{s}_{13}+\epsilon)(iys_{24}\bar{s}_{24}+\epsilon)}\right)$$

where we have used $(\xi s_{ij}\bar{s}_{ij}) = (iys_{ij}\bar{s}_{ij}+\epsilon)$ in the $\delta \to 0^+$ limit. Now we break the integrals into two parts depending on $y > 0$ and $y < 0$. This gives

$$\begin{aligned}\mathcal{C}_4(\sigma_i,\bar{\sigma}_i) = \frac{\pi}{2}\int_0^\infty & dy\, e^{\frac{i}{y}} y^{-\bar{h}} e^{-i\frac{\pi}{2}\bar{h}} (s_{13}\bar{s}_{13}-i\epsilon)^{-2\bar{h}_1}(s_{34}\bar{s}_{34}-i\epsilon)^{\bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4} \\ & \times (s_{24}\bar{s}_{24}-i\epsilon)^{-\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4}(s_{23}\bar{s}_{23}-i\epsilon)^{\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4} \\ & \times H\left(\frac{(s_{12}\bar{s}_{12})(s_{34}\bar{s}_{34}-i\epsilon)}{(s_{13}\bar{s}_{13}-i\epsilon)(s_{24}\bar{s}_{24}-i\epsilon)}, \frac{(s_{14}\bar{s}_{14})\,(s_{23}\bar{s}_{23}-i\epsilon)}{(s_{13}\bar{s}_{13}-i\epsilon)(s_{24}\bar{s}_{24}-i\epsilon)}\right) \\ + \frac{\pi}{2}\int_0^\infty dy\, e^{-\frac{i}{y}} y^{-\bar{h}} e^{i\frac{\pi}{2}\bar{h}} & (s_{13}\bar{s}_{13}+i\epsilon)^{-2\bar{h}_1}(s_{34}\bar{s}_{34}+i\epsilon)^{\bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4}(s_{24}\bar{s}_{24}+i\epsilon)^{-\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4} \\ \times (s_{23}\bar{s}_{23}+i\epsilon)^{\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4} & H\left(\frac{(s_{12}\bar{s}_{12})(s_{34}\bar{s}_{34}+i\epsilon)}{(s_{13}\bar{s}_{13}+i\epsilon)(s_{24}\bar{s}_{24}+i\epsilon)}, \frac{(s_{14}\bar{s}_{14})\,(s_{23}\bar{s}_{23}+i\epsilon)}{(s_{13}\bar{s}_{13}+i\epsilon)(s_{24}\bar{s}_{24}+i\epsilon)}\right)\end{aligned} \tag{C.1.25}$$

where we have defined $\bar{h} = \sum_{k=1}^{4}\bar{h}_k$. By substituting $y \to \frac{1}{y}$ and performing the $y$ integral we will get (4.4.6).

To obtain the other scalar leaf amplitude we have to send $\bar{\sigma}_i \to -\bar{\sigma}_i$ and use,

$$(-x \pm i\epsilon)^{\Delta} = e^{\pm i\pi\Delta}(x \mp i\epsilon)^{\Delta} \tag{C.1.26}$$

Then we have,

$$\begin{aligned}\mathcal{C}_4(\sigma_i,-\bar{\sigma}_i) = \frac{i\pi}{2}\Gamma\left(\bar{h}-1\right) & e^{-i\pi\bar{h}} (s_{13}\bar{s}_{13}-i\epsilon)^{-2\bar{h}_1}(s_{34}\bar{s}_{34}-i\epsilon)^{\bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4} \\ & \times (s_{24}\bar{s}_{24}-i\epsilon)^{-\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4}(s_{23}\bar{s}_{23}-i\epsilon)^{\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4} \\ & \times H\left(\frac{(s_{12}\bar{s}_{12})(s_{34}\bar{s}_{34}-i\epsilon)}{(s_{13}\bar{s}_{13}-i\epsilon)(s_{24}\bar{s}_{24}-i\epsilon)}, \frac{(s_{14}\bar{s}_{14})\,(s_{23}\bar{s}_{23}-i\epsilon)}{(s_{13}\bar{s}_{13}-i\epsilon)(s_{24}\bar{s}_{24}-i\epsilon)}\right) \\ -\frac{i\pi}{2}\Gamma\left(\bar{h}-1\right) e^{i\pi\bar{h}} & (s_{13}\bar{s}_{13}+i\epsilon)^{-2\bar{h}_1}(s_{34}\bar{s}_{34}+i\epsilon)^{\bar{h}_1+\bar{h}_2-\bar{h}_3-\bar{h}_4}(s_{24}\bar{s}_{24}+i\epsilon)^{-\bar{h}_1-\bar{h}_2+\bar{h}_3-\bar{h}_4} \\ \times (s_{23}\bar{s}_{23}+i\epsilon)^{\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4} & H\left(\frac{(s_{12}\bar{s}_{12})(s_{34}\bar{s}_{34}+i\epsilon)}{(s_{13}\bar{s}_{13}+i\epsilon)(s_{24}\bar{s}_{24}+i\epsilon)}, \frac{(s_{14}\bar{s}_{14})\,(s_{23}\bar{s}_{23}+i\epsilon)}{(s_{13}\bar{s}_{13}+i\epsilon)(s_{24}\bar{s}_{24}+i\epsilon)}\right)\end{aligned} \tag{C.1.27}$$

## C.2 H-function

$H$-function and its properties can be found in [156]. It is defined in terms of $G$-functions by the following equation ( see (5.9) of [156]),

$$H(\alpha,\beta,\gamma,\delta;u,v) = \frac{\Gamma(1-\gamma)}{\Gamma(\delta)}\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha)\Gamma(\delta-\beta)G(\alpha,\beta,\gamma,\delta;u,,1-v)$$
$$+\frac{\Gamma(\gamma-1)}{\Gamma(\delta-2\gamma+2)}\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)\Gamma(\delta-\gamma-\alpha+1)\Gamma(\delta-\gamma-\beta+1)u^{1-\gamma}$$
$$\times G(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\delta-2\gamma+2;u,1-v) \tag{C.2.1}$$

From Appendix C.1, we know that,

$$\alpha = 2\bar{h}_1,\ \beta = \bar{h}_1+\bar{h}_2-\bar{h}_3+\bar{h}_4,\ \gamma = 2\bar{h}_1+2\bar{h}_2-1,\ \delta = 2\bar{h}_1+2\bar{h}_2 \tag{C.2.2}$$

i.e., $\delta = \gamma+1$. For this special relation between $\gamma$ and $\delta$, $G$-functions can be written in terms of Hypergeometric functions using

$$G(\alpha,\beta,\gamma,\gamma+1;u,1-v) = \frac{1}{1-x-y}((1-y)\,{}_2F_1(\alpha-1,\beta-1;\gamma;x)\,{}_2F_1(\alpha,\beta;\gamma+1;1-y)$$
$$-x\,{}_2F_1(\alpha,\beta;\gamma+1;x)\,{}_2F_1(\alpha-1,\beta-1;\gamma;1-y)) \tag{C.2.3}$$

where, $u = x(1-y)$ and $v = y(1-x)$.

### C.2.1 Scalar case

In scalar case we have,

$$\alpha = 1+i\lambda_1$$
$$\beta = 1+\frac{1}{2}(i\lambda_1+i\lambda_2-i\lambda_3+i\lambda_4)$$
$$\gamma = 1+i\lambda_1+i\lambda_2$$
$$\delta = 2+i\lambda_1+i\lambda_2$$

As mentioned in the main section we are interested in the configuration where $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$. Then $H$-function is given by,

$$\begin{aligned} & H(1+i\lambda, 1+i\lambda, 1+2i\lambda, 2+2i\lambda; u, v) \\ = & -\frac{\pi}{\sin[2\pi i\lambda]}\left[\frac{\Gamma(1+i\lambda)^4}{\Gamma(1+2i\lambda)\Gamma(2+2i\lambda)}G(1+i\lambda, 1+i\lambda, 1+2i\lambda, 2+2i\lambda; u, 1-v)\right. \\ & \left.-\frac{\Gamma(1-i\lambda)^4}{\Gamma(1-2i\lambda)\Gamma(2-2i\lambda)}u^{-2i\lambda}G(1-i\lambda, 1-i\lambda, 1-2i\lambda, 2-2i\lambda; u, 1-v)\right] \end{aligned} \tag{C.2.4}$$

On the support of $\delta(\lambda)$ we can expand this $H$-function around $\lambda = 0$ and keep upto $\mathcal{O}(\lambda^0)$ terms only. Using (C.2.3) and the following expansion of Hypergeometric functions,

$$\begin{aligned} {}_2F_1(i\lambda, i\lambda; 1+2i\lambda; x) &= 1 + \mathcal{O}(\lambda^2) \\ x\, {}_2F_1(1+i\lambda, 1+i\lambda; 2+2i\lambda; x) &= -(1+2i\lambda)\log(1-x) - 2i\lambda\, \mathrm{Li}_2(x) + \mathcal{O}(\lambda^2) \end{aligned} \tag{C.2.5}$$

one can show that,

$$\begin{aligned} G(1+i\lambda, 1+i\lambda, 1+2i\lambda, 2+2i\lambda; u, 1-v) = \frac{1}{1-x-y}&\left[-(1+2i\lambda)\log\left(\frac{y}{1-x}\right)\right. \\ & \left.-2i\lambda\{\mathrm{Li}_2(1-y) - \mathrm{Li}_2(x)\}\right] + \mathcal{O}(\lambda^2) \end{aligned} \tag{C.2.6}$$

Using the above equation in (C.2.4) and, expanding around $\lambda = 0$, we can get (4.4.18).

We know from (4.4.17), that in our case, $u, v$ are given by, $u_\pm = z\bar{z} \pm (1 + z\bar{z})i\epsilon$ and $v_\pm = (1-z)(1-\bar{z}) \pm \{(1-z)(1-\bar{z}) - 1\}i\epsilon$. Hence, we can write,

$$x_\pm(1-y_\pm) = z\bar{z} \pm (1+z\bar{z})i\epsilon,\ y_\pm(1-x_\pm) = (1-z)(1-\bar{z}) \pm \{(1-z)(1-\bar{z})-1\}i\epsilon \tag{C.2.7}$$

The above equations can be solved for $x_\pm, y_\pm$. The solutions are given by,

$$x_\pm = z \mp \frac{z^2+z-1}{\bar{z}-z}i\epsilon,\ 1-y_\pm = \bar{z} \pm \frac{\bar{z}^2+\bar{z}-1}{\bar{z}-z}i\epsilon \tag{C.2.8}$$

Let $\zeta = \text{sgn}\left(\frac{z^2+z-1}{\bar{z}-z}\right)$, $\bar{\zeta} = \text{sgn}\left(\frac{\bar{z}^2+\bar{z}-1}{\bar{z}-z}\right)$. Thus (4.4.18) becomes,

$$\begin{aligned}
&\delta(\lambda)H(1+i\lambda, 1+i\lambda, 1+2i\lambda, 2+2i\lambda; (z \mp \zeta i\epsilon)(\bar{z} \pm \bar{\zeta} i\epsilon), (1-z \pm \zeta i\epsilon)(1-\bar{z} \mp \bar{\zeta} i\epsilon)) \\
&= \frac{1}{\bar{z}-z \pm (\zeta+\bar{\zeta})i\epsilon}\left[\{\log(z \mp \zeta i\epsilon) + \log(\bar{z} \pm \bar{\zeta} i\epsilon)\}\{\log(1-\bar{z} \mp \bar{\zeta} i\epsilon) - \log(1-z \pm \zeta i\epsilon)\}\right. \\
&\qquad\qquad \left. -2\text{Li}_2(z \mp \zeta i\epsilon) + 2\text{Li}_2(\bar{z} \pm \bar{\zeta} i\epsilon)\right]
\end{aligned} \tag{C.2.9}$$

We take $\zeta = \bar{\zeta} = +1$. This can be achieved by letting $z > \frac{\sqrt{5}-1}{2},\ \bar{z} > z$. However, in the region $\frac{\sqrt{5}-1}{2} < z < 1$ we don't have any singularities for log or $\text{Li}_2$ functions. Thus, in the limit $z \to \bar{z}$, $H$-function does not develop any simple pole. Hence, we take $z > 1, \bar{z} > z$, i.e. $z$ can approach $\bar{z}$ from below. In this regime of $z, \bar{z}$ we obtain (4.4.20).

### C.2.2 Gluon case

For MHV gluon scattering from (4.4.31) and (C.2.2), we have

$$\alpha = 2 + i\lambda_1,\ \beta = 2 + \frac{1}{2}(i\lambda_1 + i\lambda_2 - i\lambda_3 + i\lambda_4),\ \gamma = 3 + i\lambda_1 + i\lambda_2,\ \delta = 4 + i\lambda_1 + i\lambda_2 \tag{C.2.10}$$

With $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$ one will get the $H$-functions appeared in (4.4.32).

## C.3 Tree level 4-point celestial amplitudes for massless scalars

The tree level momentum space 4-point amplitude is given by (4.4.1). In this section of the appendix, we compute the 4-point celestial amplitude in planar coordinates. As mentioned in the main section we take $\epsilon_1 = \epsilon_2 = -1,\ \epsilon_3 = \epsilon_4 = +1$. The momentum conserving delta function can be parametrized as,

$$\delta^{(4)}\left(\sum_{k=1}^{4} p_k^{\mu}\right) = \frac{1}{4\omega_4}\delta\left(\omega_1-\omega_1^*\right)\delta\left(\omega_2-\omega_2^*\right)\delta\left(\omega_3-\omega_3^*\right)\delta(r-\bar{r}) \tag{C.3.1}$$

where

$$\begin{aligned}&\omega_1^* = \omega_4\frac{z_{24}\bar{z}_{34}}{z_{12}\bar{z}_{13}},\ \omega_2^* = \omega_4\frac{z_{41}\bar{z}_{34}}{z_{12}\bar{z}_{23}},\ \omega_3^* = \omega_4\frac{z_{24}\bar{z}_{41}}{z_{23}\bar{z}_{13}},\\ &r = z_{12}z_{34}\bar{z}_{13}\bar{z}_{24}, \bar{r} = \bar{z}_{12}\bar{z}_{34}z_{13}z_{24}\end{aligned} \tag{C.3.2}$$

The 4-point celestial amplitude is given by,

$$\mathcal{M}_4(\{z_i,\bar{z}_i,\bar{h}_i\}) = \prod_{k=1}^{4}\int d\omega_k\,\omega_k^{2\bar{h}_k-1}A_4(p_1,p_2,p_3,p_4) \tag{C.3.3}$$

where $2\bar{h}_k = \Delta_k = 1+i\lambda_k$. Now, substituting $A_4$ from (4.4.1) and using (C.3.1) we get,

$$\begin{aligned}\mathcal{M}_4(\{z_i,\bar{z}_i,\bar{h}_i\}) = &-\frac{i\pi(2\pi)^4\tilde{\lambda}}{2z_{13}z_{24}\bar{z}_{13}\bar{z}_{24}}\Theta\left(\frac{z_{24}\bar{z}_{34}}{z_{12}\bar{z}_{13}}\right)\Theta\left(\frac{z_{41}\bar{z}_{34}}{z_{12}\bar{z}_{23}}\right)\Theta\left(\frac{z_{24}\bar{z}_{41}}{z_{23}\bar{z}_{13}}\right)\delta(z-\bar{z})\\ &\times\left(\frac{z_{24}\bar{z}_{34}}{z_{12}\bar{z}_{13}}\right)^{2\bar{h}_1-1}\left(\frac{z_{41}\bar{z}_{34}}{z_{12}\bar{z}_{23}}\right)^{2\bar{h}_2-1}\left(\frac{z_{24}\bar{z}_{41}}{z_{23}\bar{z}_{13}}\right)^{2\bar{h}_3-1}\delta(\beta)\end{aligned} \tag{C.3.4}$$

where we have defined $\beta = \sum_{k=1}^{4}\lambda_k$. The cross ratios are given by,

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}},\ \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} \tag{C.3.5}$$

Using the conformal symmetry we take three points to $0, 1$ and $\infty$. More precisely we define an amplitude in the following way,

$$\begin{aligned}\widetilde{\mathcal{M}_4}(z,\bar{z},\{\bar{h}_i\}) = \lim_{z_1,\bar{z}_1\to\infty}(z_1\bar{z}_1)^{2\bar{h}_1}\mathcal{M}_4(z_1,\bar{z}_1\to\infty, z_2=\bar{z}_2=1,\\ z_3=z,\bar{z}_3=\bar{z},z_4=\bar{z}_4=0,\{\bar{h}_i\})\end{aligned} \tag{C.3.6}$$

Remember that for scalars we have $h_i = \bar{h}_i$. We obtain,

$$\widetilde{\mathcal{M}_4}(z,\bar{z},\{\bar{h}_i\}) = \Big(-i(2\pi)^4\tilde{\lambda}\Big)\,\frac{\pi}{2}\Theta\,(z-1)\,\delta(z-\bar{z})\,z^{2\bar{h}_1+2\bar{h}_2-2}\,(z-1)^{2-2\bar{h}_2-2\bar{h}_3}\,\delta(\beta) \tag{C.3.7}$$

If the imaginary part of the complex dimensions of all the scalars are same, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$, then we have,

$$\widetilde{\mathcal{M}_4}(z,\bar{z},\lambda) = \Big(-i(2\pi)^4\tilde{\lambda}\Big)\,\frac{\pi}{8}\Theta\,(z-1)\,\delta(z-\bar{z})\,\delta(\lambda) \tag{C.3.8}$$

# Appendix D

# Appendices for chapter 5

## D.1 Mixed helicity OPE and the soft symmetry algebra for gravitons

The mixed helicity OPE between two graviton operators in the MHV sector is given by

$$G^{+}_{\Delta_1}(z_1,\bar{z}_1)G^{-}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\bar{z}_{12}}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1-1+n,\Delta_2+3)\frac{(\bar{z}_{12})^n}{n!}\bar{\partial}_2^n G^{-}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2). \tag{D.1.1}$$

We define the ''conformally soft'' negative-helicity graviton operator $\bar{H}^k(z,\bar{z})$ as

$$\bar{H}^k(z,\bar{z}) = \lim_{\Delta\to k}(\Delta-k)G^{-}_{\Delta}(z,\bar{z}), \quad k=-3,-4,-5,... \tag{D.1.2}$$

The operators $\bar{H}^k(z,\bar{z})$ admit the following truncated anti-holomorphic mode expansion

$$\bar{H}^k(z,\bar{z}) = \sum_{m=\frac{k+2}{2}}^{-\frac{k+2}{2}} \frac{\bar{H}^k_m(z)}{\bar{z}^{m+\frac{k+2}{2}}} \tag{D.1.3}$$

Similarly, we define the conformally soft positive-helicity graviton as

$$H^k(z,\bar{z}) = \lim_{\Delta\to k}(\Delta - k)\, G^+_\Delta(z,\bar{z}), \quad k = 1, 0, -1, \cdots \tag{D.1.4}$$

with weights $\left(\frac{k+2}{2}, \frac{k-2}{2}\right)$.

$H^k(z,\bar{z})$ also admits the following truncated mode expansion

$$H^k(z,\bar{z}) = \sum_{m=\frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_m(z)}{\bar{z}^{m+\frac{k-2}{2}}} \tag{D.1.5}$$

The OPE between two conformally soft mixed helicity gravitons is given by

$$H^k(z_1,\bar{z}_1)\bar{H}^l(z_2,\bar{z}_2) \sim -\frac{\bar{z}_{12}}{z_{12}} \sum_{n=0}^{1-k} \frac{(-k-l-n-2)!}{(-l-3)!\,(1-k-n)!} \frac{(\bar{z}_{12})^n}{n!} \bar{\partial}_2^n \bar{H}^{k+l}(z_2,\bar{z}_2) \tag{D.1.6}$$

The currents $H^k_n(z)$ and $\bar{H}^l_n(z)$ are given by

$$H^k_n(z) = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+\frac{k-4}{2}} H^k(z,\bar{z}) \tag{D.1.7}$$

and

$$\bar{H}^l_{n'}(z) = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n'+\frac{l}{2}} \bar{H}^l(z,\bar{z}). \tag{D.1.8}$$

Using OPE (D.1.6) we can write the follwoing commutator as

$$\left[H^k_n, \bar{H}^l_{n'}\right](z_2) = \oint_{|\bar{z}_1|<\epsilon} \frac{d\bar{z}_1}{2\pi i} \bar{z}_1^{n+\frac{k-4}{2}} \oint_{|\bar{z}_2|<\epsilon} \frac{d\bar{z}_2}{2\pi i} \bar{z}_2^{n'+\frac{l}{2}} \oint_{|z_{12}|<\epsilon} \frac{dz_1}{2\pi i} H^k(z_1,\bar{z}_1)\bar{H}^l(z_2,\bar{z}_2) \tag{D.1.9}$$

We Perform the $\bar{z}_1$ integral first and use the following results

$$\oint_{|\bar{z}_1|<\epsilon} \frac{d\bar{z}_1}{2\pi i} \bar{z}_1^{n+\frac{k-4}{2}} \left(\bar{z}_{12}\right)^{m+1} = 0, \qquad\qquad -1 \leq m < -n - \frac{k}{2}$$
$$= \frac{1}{\left(1-\frac{k}{2}-n\right)!} \frac{(m+1)!}{\left(m+n+\frac{k}{2}\right)!} \left(-\bar{z}_2\right)^{m+n+\frac{k}{2}}, \qquad -n-\frac{k}{2} \leq m \leq 1-k \tag{D.1.10}$$

Then the commutator (D.1.9) with the above result becomes

$$\left[H^k_n, \bar{H}^l_{n'}\right](z_2) = -\sum_{m=-n-\frac{k}{2}}^{1-k} \frac{(-k-l-m-2)!}{(-l-3)!\,(1-k-m)!} \frac{(m+1)\,(-1)^{m+n+\frac{k}{2}}}{\left(1-\frac{k}{2}-n\right)!\left(m+n+\frac{k}{2}\right)!}$$
$$\oint_{|\bar{z}_2|<\epsilon} \frac{d\bar{z}_2}{2\pi i} \left(\bar{z}_2\right)^{m+n+\frac{k}{2}+n'+\frac{l}{2}} \partial^m_{\bar{z}_2} \bar{H}^{k+l}\left(z_2, \bar{z}_2\right) \tag{D.1.11}$$

Then performing $\bar{z}_2$ integral and substituting the mode expansionn for $\bar{H}^{k+l}$ we get

$$\oint_{|\bar{z}_2|<\epsilon} \frac{d\bar{z}_2}{2\pi i} \left(\bar{z}_2\right)^{m+n+\frac{k}{2}+n'+\frac{l}{2}} \partial^m_{\bar{z}_2} \bar{H}^{k+l}\left(z_2, \bar{z}_2\right) = \oint_{|\bar{z}_2|<\epsilon} \frac{d\bar{z}_2}{2\pi i} \left(\bar{z}_2\right)^{m+n+\frac{k}{2}+n'+\frac{l}{2}} \partial^m_{\bar{z}_2} \sum_{m'=\frac{k+l+2}{2}}^{-\frac{k+l+2}{2}} \frac{\bar{H}^{k+l}_{m'}(z_2)}{\bar{z}_2^{m'+\frac{k+l+2}{2}}}$$
$$= \frac{\left(-n-n'-\frac{k+l+2}{2}\right)!}{\left(-n-n'-\frac{k+l+2}{2}-m\right)!} \bar{H}^{k+l}_{n+n'}(z_2)$$

Substituting the result in (D.1.11) and doing the sum we have

$$\left[H^k_n, \bar{H}^l_{n'}\right](z_2) = -\left[n'(2-k)+n(2+l)\right]$$
$$\times \frac{\left(\frac{2-k}{2}-n-\frac{2+l}{2}-n'-1\right)!}{\left(\frac{2-k}{2}-n\right)!\left(-\frac{2+l}{2}-n'\right)!} \frac{\left(\frac{2-k}{2}+n-\frac{2+l}{2}+n'-1\right)!}{\left(\frac{2-k}{2}+n\right)!\left(-\frac{2+l}{2}+n'\right)!} \bar{H}^{k+l}_{n+n'}(z_2) \tag{D.1.12}$$

This is the conformal soft symmetry algebra for gravitons arising from mixed helicity OPE.

## D.2 Mixed helicity OPE and the soft symmetry algebra for gluons

The mixed helicity OPE between two gluon conformal primary operators is given by

$$O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,-}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{if^{abc}}{z_{12}}\sum_{m=0}^{\infty} B(\Delta_1 - 1 + m, \Delta_2 + 1)\frac{(\bar{z}_{12})^m}{m!}\bar{\partial}_2^m O^{c,-}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \tag{D.2.1}$$

Let's define the ''conformally soft'' negative helicity gluon operator $\bar{R}^{k,a}(z,\bar{z})$ as

$$\bar{R}^{k,a}(z,\bar{z}) := \lim_{\Delta\to k}(\Delta - k)O^{a,-}_{\Delta}(z,\bar{z}) \tag{D.2.2}$$

The OPE (D.2.1) allows us to do the following truncated mode expansion of $\bar{R}^{k,a}(z,\bar{z})$

$$\bar{R}^{k,a}(z,\bar{z}) = \sum_{m=\frac{k+1}{2}}^{-\frac{k+1}{2}} \frac{\bar{R}^{k,a}_m(z)}{\bar{z}^{m+\frac{k+1}{2}}} \tag{D.2.3}$$

For positive helicity gravitons we similarly define

$$R^{k,a}(z,\bar{z}) = \lim_{\Delta\to k}(\Delta - k)\,O^{a,+}_{\Delta}(z,\bar{z}), \quad k = 1, 0, -1, \cdots \tag{D.2.4}$$

Similarly, we have the following truncated mode expansion

$$R^{k,a}(z,\bar{z}) = \sum_{m=\frac{k-1}{2}}^{\frac{1-k}{2}} \frac{R^{k,a}_m(z)}{\bar{z}^{m+\frac{k-1}{2}}} \tag{D.2.5}$$

The OPE between $R^{k,a}(z_1,\bar{z}_1)$ and $\bar{R}^{l,b}(z_2,\bar{z}_2)$ is given by

$$R^{k,a}(z_1,\bar{z}_1)\bar{R}^{l,b}(z_2,\bar{z}_2) \sim -\frac{if^{ab}_{\ \ c}}{z_{12}}\sum_{m=0}^{1-k}\frac{(-k-l-m)!}{(-l-1)!\,(1-k-m)!}\frac{(\bar{z}_{12})^m}{m!}\partial^m_{\bar{z}_2}\bar{R}^{k+l-1,c}(z_2,\bar{z}_2). \tag{D.2.6}$$

The commutator between $R^{k,a}_n$ and $\bar{R}^{l,b}_{n'}$ can be found using the above OPE

$$\left[R^{k,a}_n,\bar{R}^{l,b}_{n'}\right](z_2) = \oint_{|\bar{z}_1|<\epsilon}\frac{d\bar{z}_1}{2\pi i}\bar{z}_1^{n+\frac{k-3}{2}}\oint_{|\bar{z}_2|<\epsilon}\frac{d\bar{z}_2}{2\pi i}\bar{z}_2^{n'+\frac{l-1}{2}}\oint_{|z_{12}|<\epsilon}\frac{dz_1}{2\pi i}R^{k,a}(z_1,\bar{z}_1)\bar{R}^{l,b}(z_2,\bar{z}_2) \tag{D.2.7}$$

Now we substitute (D.2.6) in the r.h.s. Then we first perform the $z_1$ integral and use the following

$$\begin{aligned}\oint_{|\bar{z}_1|<\epsilon}\frac{d\bar{z}_1}{2\pi i}\bar{z}_1^{n+\frac{k-3}{2}}\,(\bar{z}_{12})^m &= 0, && 0\le m<\frac{1-k}{2}-n\\ &= \frac{m!}{\left(\frac{1-k}{2}-n\right)!\left(m+n+\frac{k-1}{2}\right)!}\,(-\bar{z}_2)^{m+n+\frac{k-1}{2}}, && \frac{1-k}{2}-n\le m\le 1-k\end{aligned} \tag{D.2.8}$$

We obtain the following commutator

$$\begin{aligned}\left[R^{k,a}_n,\bar{R}^{l,b}_{n'}\right](z_2) =& -if^{ab}_{\ \ c}\sum_{m=\frac{1-k}{2}-n}^{1-k}\frac{(-k-l-m)!}{(-l-1)!\,(1-k-m)!}\frac{(-1)^{m+n+\frac{k-1}{2}}}{\left(\frac{1-k}{2}-n\right)!\left(m+n+\frac{k-1}{2}\right)!}\\ &\times\oint_{|\bar{z}_2|<\epsilon}\frac{d\bar{z}_2}{2\pi i}\bar{z}_2^{m+n+\frac{k-1}{2}+n'+\frac{l-1}{2}}\partial^m_{\bar{z}_2}\bar{R}^{k+l-1,c}(z_2,\bar{z}_2)\end{aligned} \tag{D.2.9}$$

Next we perform the remaining integral by substituting the modes of $\bar{R}^{k+l-1,c}$

$$\oint_{|\bar{z}_2|<\epsilon} \frac{d\bar{z}_2}{2\pi i} \bar{z}_2^{m+n+\frac{k-1}{2}+n'+\frac{l-1}{2}} \partial_{\bar{z}_2}^m \bar{R}^{k+l-1,c}(z_2,\bar{z}_2) = \oint_{|\bar{z}_2|<\epsilon} \frac{d\bar{z}_2}{2\pi i} \bar{z}_2^{m+n+\frac{k-1}{2}+n'+\frac{l-1}{2}} \partial_{\bar{z}_2}^m \sum_{m'=\frac{k+l}{2}}^{-\frac{k+l}{2}} \frac{\bar{R}_{m'}^{k+l-1,c}(z_2)}{\bar{z}_2^{m'+\frac{k+l}{2}}}$$
$$= \frac{\left(-n-n'-\frac{k+l}{2}\right)!}{\left(-n-n'-m-\frac{k+l}{2}\right)!} \bar{R}_{n+n'}^{k+l-1,c}(z_2) \tag{D.2.10}$$

Substituting the result in (D.2.9) and performing the sum we get the holographic symmetry algbra for gluons

$$\left[R_n^{k,a}, \bar{R}_{n'}^{l,b}\right](z_2) = -if^{ab}_{\ \ c} \frac{\left(\frac{1-k}{2}-n-\frac{l+1}{2}-n'\right)!\left(\frac{1-k}{2}+n-\frac{l+1}{2}+n'\right)!}{\left(\frac{1-k}{2}-n\right)!\left(-\frac{l+1}{2}-n'\right)!\left(\frac{1-k}{2}+n\right)!\left(-\frac{l+1}{2}+n'\right)!} \bar{R}_{n+n'}^{k+l-1,c}(z_2). \tag{D.2.11}$$

## D.3 The $KZ$ type null states for negative helicity gluons

In this appendix, we derive the Knizhnik-Zamolodchikov (KZ) type null states involving negaive-helicity gluon operators. These null states emerge from analyzing the soft limits in the operator product expansion (OPE) and requiring consistency with the OPE between $\bar{R}$ and the positive-helicity operator $O_{\Delta}^{a,+}$. The OPE between a positive and a negative-helicity gluon primary operators up to $\mathcal{O}(1)$ takes the form given by [22]

$$O_{\Delta_1}^{a,+}(z_1,\bar{z}_1)O_{\Delta_2}^{b,-}(z_2,\bar{z}_2) = B(\Delta_1-1,\Delta_2+1)\left[-\frac{if^{ab}_{\ \ c}}{z_{12}} + \Delta_1\delta^{bc}R_{-1,0}^{1,a} + \frac{\Delta_1-1}{\Delta_1+\Delta_2}\delta^{bc}R_{-\frac{1}{2},\frac{1}{2}}^{0,a}\left(-H_{-\frac{1}{2},-\frac{1}{2}}^{1}\right)\right]O_{\Delta_1+\Delta_2-1}^{c,-}(z_2,\bar{z}_2) \tag{D.3.1}$$

Taking the soft limit $\Delta_2 \to -1$ on both side of (D.3.1) we get

$$O_{\Delta_1}^{a,+}(z_1,\bar{z}_1)\bar{R}^{-1,b}(z_2,\bar{z}_2) = \left[-\frac{if^{ab}_{\ \ c}}{z_{12}} + \Delta_1\delta^{bc}R_{-1,0}^{1,a} + \delta^{bc}R_{-\frac{1}{2},\frac{1}{2}}^{0,a}\left(-H_{-\frac{1}{2},-\frac{1}{2}}^{1}\right)\right]O_{\Delta_1-2}^{c,-}(z_2,\bar{z}_2) \tag{D.3.2}$$

Now we demand the consistency with the OPE between $\bar{R}^{-1,b}(z_2,\bar{z}_2)$ and $O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)$ and expand $z_1$ in the above expression by

$$z_2 \to z_1 - z_{12}, \quad \bar{z}_2 \to \bar{z}_1 - \bar{z}_{12} \,.$$

Comparing the $\mathcal{O}(1)$ terms, we have

$$\bar{R}^{-1,b}_{1,0} O^{a,+}_{\Delta_1}(z_1,\bar{z}_1) = i f^{ab}_{\ \ c} L_{-1} O^{c,-}_{\Delta_1-2}(z_1,\bar{z}_1) + \Delta_1 R^{1,a}_{-1,0} O^{b,-}_{\Delta_1-2}(z_1,\bar{z}_1) + R^{0,a}_{-\frac{1}{2},\frac{1}{2}} O^{b,-}_{\Delta_1-1}(z_1,\bar{z}_1) \tag{D.3.3}$$

Hence, the KZ type null states involving the $L_{-1}$ descendant of negative helicity gluons in the MHV-sector is

$$i f^{ab}_{\ \ c} L_{-1} O^{c,-}_{\Delta} + (\Delta+2) R^{1,a}_{-1,0} O^{b,-1}_{\Delta} + R^{0,a}_{-\frac{1}{2},\frac{1}{2}} O^{b,-}_{\Delta+1} - \bar{R}^{-1,b}_{1,0} O^{a,+}_{\Delta+2} = 0. \tag{D.3.4}$$

Multiplying (D.3.4) with $-if^{abd}$ and using $f^{aa_1b} f^{aa_1c} = C_A \delta^{bc}$, we obtain the following relations

$$C_A L_{-1} O^{a,-}_{\Delta} - (\Delta+2) R^{1,b}_{-1,0} R^{1,b}_{0,0} O^{a,-}_{\Delta} - R^{0,b}_{-\frac{1}{2},\frac{1}{2}} R^{1,b}_{0,0} O^{a,-}_{\Delta+1} - \bar{R}^{-1,b}_{1,0} R^{1,b}_{0,0} O^{a,+}_{\Delta+2} = 0. \tag{D.3.5}$$

where $C_A$ is the quadratic Casimir of the adjoint representation.

## D.4 Graviton primaries of the new symmetry algebra

We derive the conditions on the primary operators that follow from the OPE between a positive and negative helicity graviton conformal-primary operators in this chapter. We follow the same procedure to derive these conditions as was done in Appendix F of [26]. By taking the $\Delta_1 \to k$ soft limit and doing the mode expansion we get the following conditions

$$H^{k}_{-\frac{k+2}{2}+m,\frac{2-k}{2}-n-1}G^{-}_{\Delta}\left(0,0\right)=-\frac{(-1)^{1-k-n}}{(1-k-n)!}\frac{\Gamma\left(\Delta+3\right)}{\Gamma\left(k+\Delta+n+2\right)}\frac{1}{n!}\bar{\partial}^{n}G^{-}_{\Delta+k}(0,0) \tag{D.4.1}$$

for $m=1$ and $0\leq n\leq 1-k$ and $k=1,0,-1,\cdots$ and

$$H^{k}_{-\frac{k+2}{2}+m,\frac{2-k}{2}-n-1}G^{-}_{\Delta}\left(0,0\right)=0 \tag{D.4.2}$$

for $m>1$ and $0\leq n\leq 1-k$ and $k=1,0,-1,\cdots$

We get another condition

$$H^{k}_{-\frac{k+2}{2}+m,\frac{2-k}{2}}G^{-}_{\Delta}\left(0,0\right)=0 \tag{D.4.3}$$

for $m\geq 1$ and $k=1,0,-1,\cdots$ .

Similarly, from the OPE between a negative and positive helicity gravitons we get the following conditions

$$\bar{H}^{k}_{m-\frac{k-2}{2},-\frac{k+2}{2}-n-1}G^{+}_{\Delta}(0,0)=-\frac{(-1)^{-k-n-3}}{(-k-n-3)!}\frac{\Gamma\left(\Delta-1\right)}{\Gamma\left(k+\Delta+n+2\right)}\frac{1}{n!}\bar{\partial}^{n}G^{-}_{\Delta+k}(0,0) \tag{D.4.4}$$

for $m=1$ and $0\leq n\leq -k-3$ . and

$$\bar{H}^{k}_{-\frac{k-2}{2}+m,-\frac{k+2}{2}-n-1}G^{+}_{\Delta}(0,0)=0 \tag{D.4.5}$$

for $m>1$ and $0\leq n\leq -k-3$ .

And for $m\geq 1$ we get

$$\bar{H}^{k}_{-\frac{k-2}{2}+m,-\frac{k+2}{2}}G^{+}_{\Delta}\left(0,0\right)=0. \tag{D.4.6}$$

The OPE between two positive-helicity outgoing gravitons gives rise to the following conditions

$$H^k_{-\frac{k+2}{2}+m,\frac{2-k}{2}-n-1} G^+_\Delta(0,0) = -\frac{(-1)^{1-k-n}}{(1-k-n)!} \frac{\Gamma(\Delta-1)}{\Gamma(k+\Delta+n-2)} \frac{1}{n!} \bar{\partial}^n G^+_{\Delta+k}(0,0) \tag{D.4.7}$$

for $m = 1$ and $0 \leq n \leq 1-k$ and $k = 1, 0, -1, \cdots$ and

$$H^k_{-\frac{k+2}{2}+m,\frac{2-k}{2}-n-1} G^+_\Delta(0,0) = 0 \tag{D.4.8}$$

for $m > 1$ and $0 \leq n \leq 1-k$ and $k = 1, 0, -1, \cdots$

And for $m \geq 1$ and $k = 1, 0, -1, \cdots$ we have

$$H^k_{-\frac{k+2}{2}+m,\frac{2-k}{2}} G^+_\Delta(0,0) = 0. \tag{D.4.9}$$

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