Black Holes and Relativistic Fluids from the perspective of Near-Equilibrium Dynamics

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3. S. Bhattacharyya, S. Mitra, **S. Roy**, R. Singh, *"Field redefinition and its impact in relativistic hydrodynamics"*, [arXiv:2409.15387]

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Presentations at conferences and workshops

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DEDICATION

This thesis is dedicated to
Mamai
Maa
Baba
Dadamoni
Shyamashree Madam

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Summary

In this thesis, we look into the near-equilibrium dynamics of two very familiar systems with well-known thermodynamic properties: black holes and fluids. Part I explores the effect of reparametrizations of the horizon's null generators on the entropy production on the horizon of a black hole in Einstein-Gauss-Bonnet theory. Part II attempts to understand the relationship between stability and causality in two well-known stable-causal models of relativistic hydrodynamics: the Müller-Israel-Stewart (MIS) model and the Bemfica-Disconzi-Noronha-Kovtun (BDNK) model, first by Lorentz transforming to ultra-high boosted frames, and then by field redefinitions of the thermodynamic variables. In all the cases, analysis was performed up to the linearized order in amplitude dynamics.

Recent advances in the second law of black hole thermodynamics for higher-derivative gravity theories have shown that there exists an entropy density and an entropy current on the dynamical horizons of black holes of these theories, which, by construction, have a total non-negative divergence for linearized amplitude perturbations about a stationary solution. However, the formulation of this entropy density and current depends on the spatial slicing of the horizon along its affinely-parametrized null generators.

In the first work 3 of Part 1, we study the non-trivial changes in entropy density and current under a local reparametrization of the affinely-parametrized null-generators to another family of affinely-parametrized null-generators. We find that the entropy density and entropy current change such that their divergence, and hence the net entropy production on the horizon, remain invariant.

In the second work 4, we dualize this entropy density and entropy current to an entropy current for a fluid residing on the boundary of an asymptotically AdS Einstein-Gauss-Bonnet black-brane solution. The boundary coordinates used to describe the fluid's entropy current correspond to a non-affine parametrization of the null generator on the horizon. Although the Gauss-Bonnet

coupling doesn't lead to any corrections to the fluid entropy current in the first order in boundary-derivative expansion, there are non-trivial corrections in the second order dependent on the horizon-to-boundary mapping functions, which aren't necessarily expressible solely in terms of fluid variables. Hence, we conclude that for generic situations, the boundary entropy current thus obtained doesn't admit a derivative expansion.

One of the most difficult challenges in relativistic hydrodynamics has been to formulate hydrodynamic theories that admit perturbations about local equilibrium that are causal (i.e., do not exit the light cone) and stable (i.e., decay down with time). The decades-old MIS and the recently developed BDNK are two such formalisms with some regions in their parameter space where the theories are stable and causal.

The first work 5 of Part 2 investigates the connection between stability and causality properties using these two theories as case studies. Here, we utilize linearized stability analysis to obtain the causality criteria for these two theories unambiguously. We find that the regions of the parameter spaces of both these theories which are stable at an ultra-high boost (i.e., boost velocity = speed of light), are stable at all other boost velocities and, hence, causal. The causality criteria thus obtained from a low-wavenumber analysis match the asymptotic causality criteria performed at a high-wavenumber of the theories.

In the second work of this part 6, we rewrite the conformal BDNK stress tensor in the "Landau frame" by redefining the temperature and velocity fields. We show that to maintain stability and causality in the "Landau frame", one either has to have an infinite number of derivative corrections or has to include new 'non-fluid' variables in the formalism. Moreover, we find that this incorporation of 'non-fluid' variables is a non-unique procedure.

Prelude: Introduction, Background

Chapter 1

Introduction

To understand the thermodynamic properties of any system, its equilibrium is often a safe harbor from which to start venturing from. Akin to a sailor starting his adventures from close to the coastline, the proverbial theoretical physicist often limits their analysis to the linearized regime of dynamics only, where extraction and interpretation of analytic results are less complicated.

From a historical perspective, the field of thermodynamics has emerged to understand the relationship between the different phenomenological quantities that label the state of a system in and around its thermodynamic equilibrium. These physical quantities are called macroscopic state variables and comprise different quantities like the system's pressure, volume, energy density, temperature, the number density of constituent particles etc., and thermodynamic equilibrium refers to the situation where there is no flow of energy or particles between two systems in contact. The four laws of thermodynamics were developed out of experimental observations in the eighteenth and nineteenth centuries, and works by the likes of Carnot, Gibbs, Thomson, Clausius, and Boltzmann in these directions further went on to shape the course of science as well as the history of humankind in the form of the industrial revolution.

Classically, the laws of thermodynamics establish relationships between the macroscopic variables without getting into the microscopic details of the system. Besides, they also function as no-go theorems, restricting the possibility of unphysical phenomena from occurring. However, the pursuit of understanding the underlying microscopic structures and constructing models consistent with the observed macroscopic behavior of the systems has led to the formulation of kinetic theory. With the advent of the atomic picture and, thence, the quantum theory as a possible microscopic framework, the field of statistical mechanics rose to prominence as a way of connecting these underlying microscopic degrees of freedom to the macroscopic variables like pressure, temperature etc. Statistical mechanics has since then been applied to a wide range of problems in different areas, and the essence of any problem then boils down to identifying the microscopic degrees of freedom of a system and constructing consistent solutions for them. One would then expect the macroscopic variables corresponding to these solutions to follow the laws of thermodynamics.

In this thesis, we'll focus on black holes and fluids, two of the most ideal and very potent instruments in a theorist's toolkit to probe the laws of thermodynamics in a variety of setups in nature. The dynamics of fluids, alias "Hydrodynamics", has been an instrumental theory for understanding a wide variety of phenomena, ranging from everyday steady flows of water or air to those in violent astrophysical plasma or heavy-ion collisions. Einstein's theory of general relativity, on the other hand, has enjoyed more than a century's success in explaining gravitational phenomena, from black holes to gravitational waves on the extremities, with those in our solar systems somewhere in between. Though seemingly very different, the dynamics of these two derivative expandable theories are strikingly similar, and in some particular cases, exact correspondences can be drawn between them. The Fluid/Gravity correspondence, since its development in the 2000s, has been instrumental in shedding light on this deep connection between these two theories, often leading to predictions in one theory from analyses performed in its dual theory. In the following sections of this chapter, we'll briefly discuss some of the interesting questions and developments in all these fields and try to establish the works presented in this thesis in the context of these, with the bigger picture in the background.

1.1 Black Holes in General Relativity and Beyond

1.1.1 Gravitational theories as derivative expansions

Throughout the centuries, scientists and philosophers have been baffled by the observation that all massive objects fall towards the surface of the earth. The idea of gravity as an attractive force had been speculated for centuries by stalwarts like Aristotle and Galileo, among others, until it reached its culmination in the form of Newton's law of gravitation. Newton's law of gravitation can be considered the earliest attempt at unification, as it connected the laws governing the motion of celestial bodies with those followed by everyday objects around us on Earth. Another important breakthrough was achieved by Einstein in the form of the General theory of relativity, which modified the idea of gravity as a force with the notion of gravity as a curvature in spacetime due to the presence of matter. Einstein's equation (actually a set of nonlinear partial differential equations) of general relativity is a relation between the curvature tensors of spacetime and the stress tensor of the external matter that causes the curvature. The curvature tensors themselves are functions of spacetime derivatives of the metric tensor, which defines the line element on a particular spacetime. In a theory of pure gravity, the metric is the dynamic degree of freedom, and Einstein's equations are the corresponding equations of motion. These equations can also be derived from an action principle, where the corresponding Lagrangian, being a scalar, can contain only an even number of derivatives. The least non-trivial action, the Einstein-Hilbert action, contains two derivatives and gives rise to the renowned equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.1}$$

where $R_{\mu\nu}, R, T_{\mu\nu}$ are the Ricci curvature tensor, Ricci scalar and the stress tensor of the external matter field, respectively. G is Newton's gravitational constant and c is the speed of light.

Another parallel development has been in understanding the nature of the fundamental constituents of matter, where the advent of quantum mechanics has led to a significant paradigm shift. The attempt to reconcile the laws of the minuscule and the laws of the gigantic in the form of a consistent quantum theory of gravity has been one of the biggest puzzles in the last century, and a conclusive answer still continues to elude us. The problem lies in the fact that, while trying to take loop corrections into account in a two-derivative theory of gravity, one encounters non-renormalizable divergences in different physical quantities that cannot be absorbed by redefinitions of the fields or the coupling constants [4]. Hence, a two-derivative theory of gravity cannot be a UV complete quantum theory of gravity (a theory valid in all energy scales). On the other hand, an attempt to construct an effective field theory of quantum gravity would inevitably lead to a Lagrangian containing an infinite number of higher-derivative correction terms, as in an EFT, one must take into account all terms consistent with the symmetries of the theory (in this case diffeomorphism-invariance). One needs a UV-complete theory to fix the expansion coefficients of the various higher-derivative terms, as it is not possible to do so using only an EFT. String theory, one of the leading candidates of a theory of quantum gravity, also leads to such higher-derivative corrections fixing the expansion coefficients. [1] This leads to the expectation that any viable UV complete theory of gravity when expanded in the low-energy limit, would give rise to a series of higher-derivative corrections on the two-derivative Einstein-Hilbert term. Thus, it is of interest to study higher-derivative theories with arbitrary coefficients to explore the properties of UV complete gravity theories in more generality while staying in the low-energy regime itself.

Of special interest among these higher-derivative terms is a combination called the Lovelock Theory. The Lagrangian in the Lovelock theory is given by an appropriate linear combination of contractions of the Riemann tensor that results in a second-order equation of motion. Ignoring the proportionality constants, the Lagrangian can be expressed as [7]

$$\begin{aligned} S &= \int d^d x \sqrt{-g} \left(R + \sum_{m=2}^{\infty} \alpha_m l_s^{2m-2} \mathcal{L}_m \right) \\ \mathcal{L}_m &= \delta^{\mu_1 \nu_1 \cdots \mu_m \nu_m}_{\rho_1 \sigma_1 \cdots \rho_m \sigma_m} R^{\rho_1 \ \ \sigma_1}_{\ \ \mu_1 \ \ \nu_1} \cdots R^{\rho_m \ \ \sigma_m}_{\ \ \mu_m \ \ \nu_m} \end{aligned} \tag{1.2}$$

where l_s is the length scale at which the higher-derivative terms begin to appear, and α_m is the coefficient for each of the contributions. $\sqrt{-g}$ is the metric determinant.

The leading term in (1.2)

$$S = \int d^d x \sqrt{-g} R \tag{1.3}$$

corresponds to the Einstein-Hilbert Lagrangian. The first non-trivial term in the series of Lovelock theory is called the Gauss-Bonnet term and has the form

$$\mathcal{L}_2 = \mathcal{L}_{GB} = R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \tag{1.4}$$

In four spacetime dimensions, the Gauss-Bonnet term is topological and equates to the Euler characteristic of the spacetime. Hence, contributions from the Gauss-Bonnet term become meaningful

[1] One can try to construct renormalizable quantum gravity theories with higher-derivative Lagrangians as in [5,6]

only at $d > 4$. In the first part of the thesis I, we'll specialize our analyses to the Gauss-Bonnet theory.

Also worth noticing is the fact that the presence of solutions in the form of derivative expansion is very reminiscent of derivative expansions in hydrodynamics, and as it was discovered later, one can write exact mappings between the two systems. Some parallels were already being drawn in [8] where, using the membrane paradigm, it was shown that a generic black hole horizon can behave like a fluid with its own electrical conductivity, shear and bulk viscosity.

1.1.2 Black holes and their entropy

Black holes are a rich class of solutions to Einstein's equations of general relativity with interesting thermodynamic properties. Recent advances in astronomy like LIGO [9,10] and EHT [11–13] have elevated them from purely theoretical constructs to tangible physical entities with considerable observational signatures. Classically, black holes can be visualized as ideal absorbers of radiation, thus indicating the presence of some possible thermodynamic behavior. Following the seminal works of Hawking, Bardeen, Carter and later Bekenstein [14–17] it was established that connections indeed exist between the geometric parameters characterizing a black hole and its thermodynamic behavior [18–20] [2]. In particular, for Einstein's theory with a two-derivative Lagrangian, the temperature of the black hole is given by its surface gravity, and the area of the event horizon corresponds to its entropy. Using the Raychaudhuri equation [24], it was shown that the area of a black hole never decreases.

Black-hole solutions can be shown to exist in higher-derivative gravity theories as well, and probing into the thermodynamics of such solutions provides deeper insights into possible quantum gravity theories. However, the identification between geometrical quantities with thermodynamic variables there often becomes progressively non-trivial and in some cases, deriving the laws

[2] The third law is more of a conjecture than an actual inviolable law because there are several indications for its violations in thermodynamic systems as well as in black hole mechanics. For example, spin ice systems found in experiments in condensed matter [21] have non-vanishing entropy at ground state due to the degeneracy of the ground state, and Kerr-Newman black holes [22, 23] in general relativity have non-zero entropy at vanishing surface gravity, which is not a universal constant but depends on its mass and angular momentum

Law	Thermodynamics	Black Hole
Zeroth Law	At thermal equilibrium , temperature is constant throughout a body.	Surface gravity κ is constant on the horizon of a stationary black hole.
First Law	Energy is conserved between two thermodynamic states by $\delta U = T\delta S + \delta W$	Perturbations in mass, area, angular velocity, and charge around stationary black holes are related by $\delta M = \frac{\kappa}{8\pi}\delta A + J\,\delta\Omega + \Phi\,\delta Q$
Second Law	The entropy of an isolated system never decreases.	The area of a black hole's horizon never decreases. $\delta A \geq 0$
Third Law	Entropy of a system must go to zero (or a universal constant) at zero temperature.	Entropy of a black hole should go to zero (or a universal constant) at $\kappa = 0$.

Table 1.1: The four laws of thermodynamics and their counterparts in black hole mechanics.

of thermodynamics for general higher-derivative theories can be very difficult. Wald's formalism [25, 26] provides one such possible identification for the laws of thermodynamics in general diffeomorphism-invariant theories using Noether charges as in classical mechanics. The entropy thus derived is called the "Wald Entropy" and was found to satisfy the first law of thermodynamics by construction [26]. It was later found that although Wald entropy gives the entropy at equilibrium but out of equilibrium, the definition of entropy is riddled with ambiguities [27–29].

Entropy of Black holes: Why is it important?

At this juncture, it would be a good point to pause and ponder the following question: Why does the entropy of a black hole hold a position of high importance? The answer lies in the fact that a black hole's entropy can be considered a sort of bridge between the classical understanding of a black hole and its underlying quantum nature. Classically, a black hole doesn't radiate any energy. Also, following the "No hair theorem", a generic classical black hole solution is characterized only by its mass, angular momentum and charge.

Historically, the four laws as derived in [14] were treated only as a correspondence since black holes don't radiate classically and hence, area can't actually be interpreted as its entropy. However, Bekenstein later showed that to maintain the second law of thermodynamics in the rest of the universe, the black hole should also have some entropy associated with it and that it should be proportional to the horizon area [15]. Using information theory arguments, the proportionality constant was found to be related to the Planck length, thus indicating a connection with quantum mechanics. Finally, Hawking's semi-classical calculation based on a quantum field near the horizon of a classical black hole background shows that a black hole does radiate with a thermal spectrum at a temperature called the Hawking temperature and an entropy proportional to its area [16]. Now, instead of using a two-derivative theory where the entropy is given by area, one can use an arbitrary diffeomorphism-invariant theory where entropy is given by Wald entropy, and the proportionality constant would then be fixed from quantum field theoretic analysis.

All of this hints that the entropy of a black hole is a possible window to peer into its quantum nature. Also, the existence of entropy in any system has a foundation in statistical mechanics, where a counting of some underlying some underlying microscopic degrees of freedom (or microstates) gives us entropy. Hence, from a statistical perspective, the existence of entropy in black holes provokes one to think about some underlying quantum microstates of the black holes, counting which one can calculate its entropy. Thus, the problem of black hole entropy can be translated into a counting problem. For specific extremal and near-extremal solutions at a large-charge limit, progress has been made in this direction in [30,31] (see [32] for a review), but it remains to be seen whether this can be achieved for any generic black hole solution.

Another avenue towards which black hole entropy guides is the holographic nature of information in gravitational theories. The area dependence of entropy instead of volume, despite its extensive nature, is a piece of strong evidence of the same. Motivated by t'Hooft's observation [33] that a reconciliation of quantum mechanics and gravity indicates a possibility of encoding gravitational degrees of freedom in a lower spacetime dimension and thus constraining the set of plausible quantum theories of gravity, Susskind proposed the holographic principle in [34] and explored how it might be realizable. As we will see in 1.3, there are indeed theories of holography like AdS/CFT [35] where the bulk dynamics in some special spacetimes can be captured in the dynamics of some theory residing on the boundary.

Recent developments on black hole thermodynamics

We shall conclude this section by reporting some recent progress made on the thermodynamics of black holes in higher-derivative theories. Following Wald's procedure, [36] developed a framework for studying the second law for black holes near equilibrium under a linearized approximation of amplitude dynamics. [7] then attempted a non-perturbative construction of entropy for Lovelock-theory, which satisfies the second law. In [37], the second law was formulated up to the linearized order in amplitude dynamics as the combined divergence of an entropy density and an entropy current on the horizon with non-negative divergence by construction and then in [38], this was generalized for arbitrary higher-derivative theories of gravity. For an orientation towards the first part of the thesis, we'll review the coordinate systems and some necessary details from [37] and [38] briefly in 2. [3] considers the effect of the event horizon's null generator's reparametrizations on the local entropy production for the Gauss-Bonnet theory and [2] generalizes it to arbitrary higher-derivative theories with an explicit expression for the transformation of entropy density and entropy current under such reparametrizations. [39] extends the proof of the second law to nonlinear order in amplitude dynamics in an effective field theory framework and [40, 41] study it further in the non-perturbative regime. Parallelly, in [42–44], the authors extend the proof for arbitrary diffeomorphism invariant gravity theories non-minimally coupled to matter fields. [2] also studies the impact of constructional ambiguities (called Iyer-Wald ambiguities) in the entropy current on its transformation under reparametrizations. Recent works have also tried to address the issue of possible violations in the Bousso bound [45] of entropy for higher-derivative theories using Wald entropy [46].

1.2 Relativistic Hydrodynamics: A Brief Introduction

The field of fluid dynamics has been an active area of research for many centuries for theoretical as well as experimental reasons. Theoretically, they are one of the most commonly used models to understand the physical properties of continuum matter, and experimentally, they are one of the most frequently encountered systems around us. Hydrodynamics is essentially a low-energy effective theory that uses gradient expansions of conserved quantities at equilibrium to describe systems near equilibrium. Being a low-energy effective theory, it has a cutoff energy (or length scale) below (or above) to which the derivative expansion is applicable. The derivatives of fluid variables must be small compared to this cutoff scale, allowing us to treat these terms perturbatively [3]. Relativistic hydrodynamics deals with systems at very high energies where the underlying symmetry is Lorentz symmetry [49]. One of the most celebrated real-world successes of relativistic hydrodynamics has been in explaining the physics behind heavy-ion collision experiments [50, 51]. The recent discovery of collective flow in mini-jets [52, 53] also has shown that hydrodynamic behavior can be observed in systems of very small sizes and densities.

1.2.1 Stability and Causality in Relativistic Hydrodynamics

Entering the domain of derivative expansions in relativistic hydrodynamics, one naturally encounters the question of whether the added derivative corrections to the equilibrium theory are physically plausible. In principle, Lorentz symmetry allows one to add derivative corrections to the equilibrium theory with arbitrary undetermined coefficients. But then, such solutions can often lead to conflicts with physical principles. Thus, demanding that the solutions conform to physical observations like entropy production, stability of solutions at equilibrium, or causal signal propagation lays constraints on possible structures that can be added to an equilibrium theory, and also on the transport coefficients associated with these structures. For example, it has been demonstrated

[3]One such ratio is the Knudsen number, which is a ratio between the mean free path in the fluid and the system size. [47, 48]

in [54] that one can use the second law of thermodynamics (i.e., local entropy production, quantified by the non-negative divergence of a covariant entropy current) as one such physical guiding principle to constrain an entropy current for a relativistic fluid. However, these constraints are clearly not enough, as can be illustrated by the presence of unphysical solutions in the relativistic Navier-Stokes equations at the first order in derivative expansion in the "Landau frame". The equations of motion for viscous flows, the well-known Navier-Stokes equations, are found to lead to pathological solutions for relativistic fluids in the "Landau fluid frame" [55] (a particular choice of off-equilibrium definitions of hydrodynamic variables). Specifically, the theory leads to acausal solutions, and on Lorentz boosting, new modes pop up which may or may not be stable [56, 57]. People have tried to remedy these issues by formulating the theory in different ways, of which the two most well-known ones are the Müller-Israel-Stewart (MIS) [58–60] and the Bemfica-Disconzi-Noronha-Kovtun (BDNK) [61–64] theories. In the MIS formulation, the viscous corrections to the conserved currents at equilibrium are promoted to new degrees of freedom with their own relaxation-equation-like equations of motion. In the BDNK formulation, the conserved currents are written in some generalized fluid frame, away from the Landau fluid frame. To derive these theories from some microscopic degrees of freedom, a plethora of models and formulations have come up with different underlying principles ranging from AdS/CFT to kinetic theory [65–68].

1.3 Correspondences: AdS/CFT, and thence, Fluid/Gravity

1.3.1 AdS/CFT Correspondence

The fact that the entropy of black holes, which is an extensive property, is given by the area of black holes instead of its volume was a strong hint towards the holographic nature of information in gravitational theories. This idea reached its pinnacle with the AdS/CFT correspondence, which connects the dynamics of certain strongly coupled conformal field theories to the dynamics of quantum gravity theories within the bulk of an AdS spacetime (a maximally symmetric spacetime with a negative cosmological constant) of one higher dimension [35, 69–71]. Since its advent, it has been studied and applied in a variety of contexts ranging from black holes, strongly coupled plasmas, heavy-ion collisions, holographic superconductors and superfluids [72]. Some of the interesting results of relevance to this proposal have been in the form of obtaining expressions of shear viscosity and other linear transport coefficients for a holographic fluid [73, 74]. It has also been derived from the correspondence that the ratio between this shear viscosity coefficient and the entropy density can be significantly smaller than what was previously suggested by perturbative calculations [75]. Some other investigations have also been conducted to explore connections between the quasinormal modes of a black-brane lying in the bulk of the AdS spacetime and certain correlators of the boundary dual CFT [76–80]. All of these developments already set the stage for a deeper dive into deriving the correspondence between holographic fluids and gravitational systems in an AdS spacetime.

1.3.2 Fluid/Gravity Correspondence

The culmination of all these ideas led to the development of the Fluid/Gravity Correspondence [1, 81, 82], where it was shown that the dynamics of a fluid residing on the AdS boundary can be derived from the dynamics of a black brane metric inside the AdS bulk spacetime in the long wavelength limit. Of special importance is the fact that this correspondence can be extended to nonlinear order in amplitude perturbations, though it is perturbatively expanded in terms of boundary coordinate derivatives. The velocity and temperature of the black brane are promoted to the temperature and velocity of the boundary fluid, and solving Einstein equations for the bulk metric leads to the Navier-Stokes equation for the boundary fluid. Thus, one can start with a stationary metric corresponding to the boundary fluid at equilibrium and generate higher-order derivative corrected metric solutions using this technique. Using the correspondence, one can also calculate various quantities for the boundary fluid, like its stress tensor [81, 83], or a number of other transport coefficients in various systems [65, 84]. One can also incorporate the effects of higher curvature corrections to the gravitational system and study its effects on the boundary fluid [78, 79, 83]. Another very important application of the correspondence has been in using the area increase theorem for a black brane horizon in the bulk to derive an entropy current for the boundary fluid with a non-negative divergence [85]. Thus, the second law of black hole thermodynamics in a two-derivative theory has been dualized to an ultra-local second law for a relativistic fluid on the boundary. For higher-derivative corrected gravity theories, the entropy is given by Wald entropy. For relativistic fluids, it was already known that entropy increase can be quantified as an entropy current with a non-negative divergence and that the second law can be utilized to constrain the transport coefficients and possible structures appearing in a fluid entropy current. The aforementioned duality between the second law statements for a black brane in the bulk and a fluid on the boundary then automatically makes one expect some kind of an ultra-local statement of the second law for black hole horizons in line with an existent general ultra-local second law in fluids. This ultra-local form of the second law in black holes was later worked out in the linear order in amplitude fluctuations in [38]. As in [65, 67, 68], one can try to construct stable-causal hydrodynamic theories with the bulk gravitational theory dictating the underlying microstructure of the fluid. Also, similar to the stability and causality issues that plague relativistic hydrodynamic theories, one can find instabilities and causality issues in the context of gravity. Besides Gregory-Laflamme instabilities in higher dimensional black strings and branes [86,87], or extremal horizons becoming unstable under perturbations [88], one can find stability and causality issues in spacetimes upon including higher derivative corrections to the Lagrangian [89, 90]. Works like [91–93] also provide a window to peer into such issues in depth on the gravity side and fluid/gravity correspondence can be a potent tool to draw connections between stability and causality criteria on the two sides.

1.4 Motivation and Outline of the Thesis

Building on the background presented in the preceding sections, the works presented in this thesis start off as a follow-up on the analysis in [38]. As it will be seen in 2, the class of coordinate systems used in [38] to prove the second law uses an affine parameter along the null generators

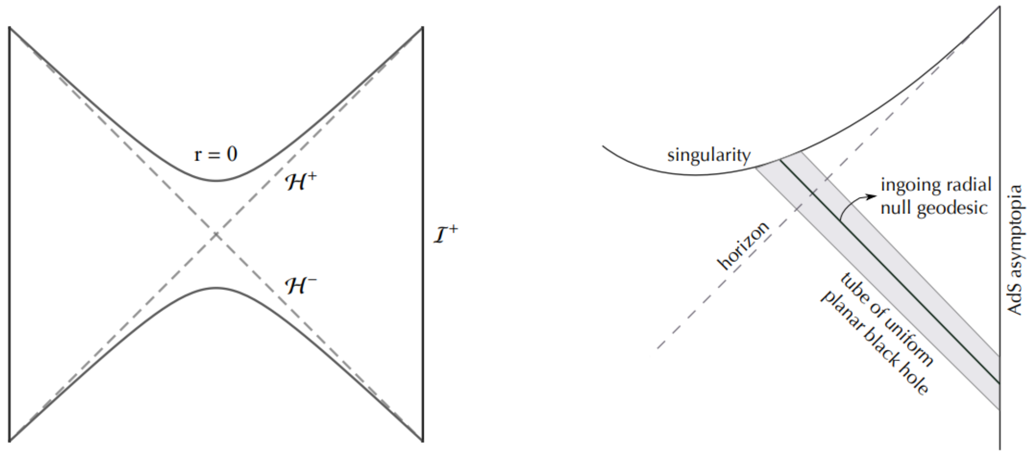


Figure 1.1: Penrose diagram of a uniform black hole (left) and a representation of the projection of the horizon on the boundary (right). The shaded tube represents an area of the spacetime over which the metric solution can locally be well-approximated to that of a corresponding uniform black hole [1].

of the horizon as one of the coordinates (called the v coordinate); another null coordinate (the r coordinate) that takes an observer away from the horizon is also affinely parametrized everywhere on the spacetime. It was found that the form of the metric remains invariant under a rescaling of these two coordinates of the form

$$r \to \frac{r}{\lambda}, \qquad v \to \lambda v \tag{1.5}$$

where λ is a constant number.

A natural question that arises then is this: what happens if instead of constant λs, we consider a λ which depends on the spatial coordinates on the horizon? Because if we rescale the v coordinate in a v independent way, we'd still get an affine parameter along the null generators, thus keeping the metric in the same class of coordinates as we started with; as a result, all of the analysis on the second law via entropy densities and spatial entropy currents follows through. But non-triviality is now introduced into the setup due to the fact that choosing a local reparametrization of the null coordinate leads to the tangent vectors mixing up on a spatial slice of the horizon, resulting in a modified slicing of the space-like slices of the horizon. Since the entropy density and spatial entropy current depend on the slicing, their form should also now be modified. But, since these entropy density and spatial entropy current structures are not covariant objects to begin with, hence, their transformations under these reparametrizations can be highly non-trivial. As described in Chapter 3 based on [3], here we try to find out the transformation of this combination of entropy density and spatial entropy current for the case of the Gauss-Bonnet theory.

Now, the existence of the entropy current as in [38], or the further proof of the second law for nonlinear dynamics of amplitude in an effective field theory sense as in [39], use the amplitude of perturbations about a stationary black hole background solution as a perturbation parameter. Another possible expansion can be in terms of boundary derivatives on the boundary of an asymptotically AdS spacetime. A derivative expansion scheme like this can often allow one to probe into the non-perturbative amplitude dynamics regime while staying in the vicinity of long-wavelength perturbations about equilibrium. Furthermore, for an entropy current for a relativistic fluid valid in the linear regime of amplitude dynamics, there exist algorithms to extend it to entropy currents that work in the nonlinear regime of amplitude dynamics [94]. Given all this existing framework, one can aspire to explore the statement of the second law in a non-perturbative regime by constructing a fluid entropy current from the entropy density and the spatial entropy current on the horizon using fluid-gravity duality. The first step towards this long and ambitious goal would then be to construct a fluid entropy current from the horizon entropy current for some higher-derivative theory of gravity. Based on the algorithm presented in [85], we present this analysis in Chapter 4 of the thesis based on [95].

Now, for a relativistic hydrodynamic theory, there already exists a conceptual tension in analyzing causality criteria. Specifically, while hydrodynamics is a low-energy effective theory, traditional causality analysis is mostly performed in the high-energy limit, also called asymptotic causality analysis. Motivated by the principle that the stability property of the causal parameter space of a theory should remain invariant in all reference frames connected by Lorentz boosts, Chapter 5 based on [96] is an attempt to utilize this for causality analysis via stability analysis in the MIS and BDNK theories.

Again, as illustrated in [97], the stable-causal MIS theory can actually be visualized as an all-order corrected theory in the Landau frame. The equation of motion of the viscous correction to the stress tensor can be rewritten in the form of an infinite summation of derivative corrections of fluid variables with a particular form of the associated transform coefficients. This shows that in a stable-causal theory written in the Landau frame, one can trade off the extra degrees of freedom (introduced to maintain stability and causality of the solutions) for an infinite order derivative corrections to the equilibrium theory. In the same spirit, Chapter 6 based on [98] is a rewriting of the BDNK stress tensor from its generalized hydrodynamic frame into the Landau frame, utilizing infinite-order field redefinitions and then using new degrees of freedom.

Finally, Chapter 7 concludes the thesis with a brief summary of all the works presented in the thesis and the future directions where the results presented here can be useful.

Chapter 2

Technical Background

In this chapter, we'll briefly discuss some of the formulations and techniques that form the basis of the works in this thesis. In the first section 2.1, we'll mostly review the framework developed in [7, 36, 38] to derive the second law on the horizon for arbitrary higher-derivative theories of gravity using entropy currents. We'll briefly discuss the "boost symmetry" of the metric and the algorithm to get to the second law using this symmetry to constrain possible structures along with appropriate boundary conditions. In the second section 2.2, we'll review the basics of stability and causality analysis in relativistic hydrodynamics and the MIS and BDNK theories, where we perform the rest of the analyses in II.

2.1 Near-Horizon Coordinates and the Second Law for Higher-Derivative Theories

This section will be based on Appendix A of [7], Appendix A of [37], [38], and Appendix A of [2].

2.1.1 Coordinates and the metric adapted to the horizon

To begin with, let us consider a $(d+1)$-dimensional spacetime, with a d-dimensional null-hypersurface foliated by $(d-1)$-dimensional slices. [1] Let the coordinates on the $d-1$-dimensional slice on the null hypersurface be denoted by x^i with their associated tangent vectors ∂_i. Let ∂_τ be the null generator along the d-dimensional null hypersurface. Let another set of null vectors ∂_ρ shoot off this d-dimensional null-hypersurface into the full $d+1$-dimensional spacetime, and the corresponding coordinate ρ measures the distance away from this d-dimensional surface. This null hypersurface is actually the event horizon of a dynamic black hole in spacetime.

[1] Throughout this section, the Greek indices μ, ν etc. will refer to the full $d+1$-dimensional coordinates and the Latin indices i, j etc. will refer to the $d-1$-dimensional coordinates.

On the horizon, let us choose the coordinates such that the tangent vectors have the following inner products among them

$$(\partial_\tau, \partial_i) = 0, \qquad (\partial_\tau, \partial_\tau) = 0 \tag{2.1}$$

The second equality just follows from the fact that ∂_τ are the null generators of the horizon. Note that these relations are valid only on the horizon.

Next, we choose the ρ coordinate such that the horizon is located at $\rho = 0$, the ∂_ρ vectors are null everywhere, and orthogonal to the ∂_i vectors everywhere. With the ∂_τ vectors, they have an inner product of 1 everywhere. These conditions translate to the following equations valid all over the $d+1$-dimensional spacetime

$$(\partial_\rho, \partial_\rho) = (\partial_\rho, \partial_i) = 0, \qquad (\partial_\rho, \partial_\tau) = 1 \tag{2.2}$$

With these conditions, the metric written in (ρ, τ, x^i) coordinates can be expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 2d\tau\, d\rho - (\rho C(\tau, x^i) + \rho^2 X(\rho, \tau, x^i))d\tau^2 + 2\rho\omega(\rho, \tau, x^i)d\tau\, dx^i + h_{ij}(\rho, \tau, x^i)dx^i\, dx^j \tag{2.3}$$

Now, let us consider a stationary black hole with a Killing horizon where τ is the Killing coordinate. This tells us that all the metric components are now independent of τ and (2.3) now becomes

$$ds^2 = 2d\tau\, d\rho - (\rho C(x^i) + \rho^2 X(\rho, x^i))d\tau^2 + 2\rho\omega(\rho, x^i)d\tau\, dx^i + h_{ij}(\rho, x^i)dx^i\, dx^j \tag{2.4}$$

At this point, we can refer to the Zeroth law of black hole mechanics, which states that for a stationary black hole, the temperature of the black hole is related to $(\partial_\rho g_{\tau\tau})_{\rho=0}$ and that it is a constant all over the horizon. This tells us that $C(\tau, x^i)$ is actually independent of τ as well as the x^i coordinates. Hence, (2.3) becomes

$$ds^2 = 2d\tau\, d\rho - (\rho C + \rho^2 X(\rho, x^i))d\tau^2 + 2\rho\omega(\rho, x^i)d\tau\, dx^i + h_{ij}(\rho, x^i)dx^i\, dx^j \tag{2.5}$$

Another important point worth mentioning here is that the ∂_τ vectors are non-affinely parametrized null generators on the horizon $\rho = 0$.

Now if we want to transform the metric to a different set of coordinates (r, v, x^i) where r satisfies all of the properties as ρ, but v is now an affine parameter along the null generators, this leads us to an additional constraint on the horizon as

$$(\partial_r G_{vv})_{r=0} = 0 \tag{2.6}$$

Following [37], one can always transform to such a system of coordinates for a stationary black hole metric using the transformations

$$\rho = \frac{C}{2} r v, \qquad \tau = \frac{2}{C} \log\left(\frac{Cv}{2}\right) \tag{2.7}$$

and the metric (2.5) can now be expressed in (r, v, x^i) coordinates as

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = 2dv\, dr - r^2 X\left(\frac{Crv}{2}, x^i\right) dv^2 + 2r\omega_i \left(\frac{Crv}{2}, x^i\right) dv dx^i + h_{ij}\left(\frac{Crv}{2}, x^i\right) dx^i dx^j \tag{2.8}$$

The Killing vector ∂_τ also transforms to

$$\partial_\tau = \frac{C}{2}(v\partial_v - r\partial_r) \tag{2.9}$$

Even for the generic case of an event horizon of a dynamical black hole without a Killing vector, one can still transform to a set of coordinates where one of the coordinates (v in this case) is an affine parameter along the null generators on the horizon. The transformation isn't simply (2.7), but receives corrections due to the τ and x^i dependence of the metric components. In the (r, v, x^i) coordinates, the metric near the horizon of a dynamical black hole can then be expressed as

$$ds^2 = 2dv\, dr - r^2 X\left(r, v, x^i\right) dv^2 + 2r\omega_i \left(r, v, x^i\right) dv dx^i + h_{ij}\left(r, v, x^i\right) dx^i dx^j \tag{2.10}$$

One more noteworthy point at this juncture is that these horizon-adapted coordinates may not be a good set of global coordinates, but locally, sufficiently near the horizon, one can always find such a form. Explicit checks in [44] show that while the Schwarzschild metric can be globally described by these coordinates, the Kerr metric can only be described in these coordinates very close to the horizon.

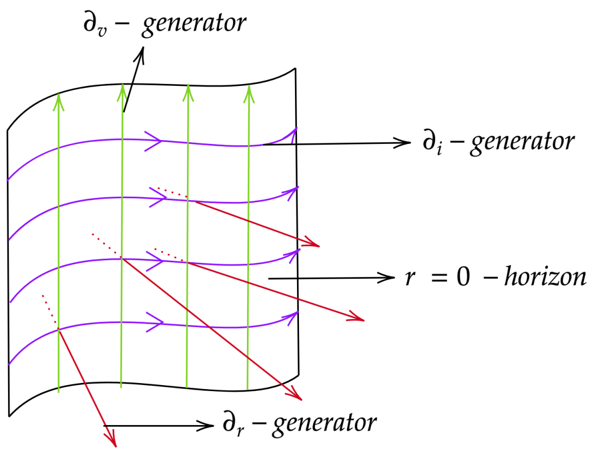


Figure 2.1: A schematic representation of the horizon adapted coordinates in (2.10) from [2]

2.1.2 "Boost symmetry" of the metric

The coordinate choice in (2.10) doesn't fix the metric completely. We are still left with some room for further coordinate reparametrizations on the constant r or v slices without going out of the gauge of the metric

1. A coordinate transformation of the form

$$v \to v' = f_0(x^i) + f_1(x^i)v \tag{2.11}$$

   on the horizon still keeps the null generators in the new coordinates affinely parametrized. Away from the horizon, we'd need to appropriately redefine all the coordinates to stay in the gauge. This form of transformation essentially is a redefinition in choosing the constant v-slicing of the horizon.

2. Another coordinate transformation that only mixes the x^i coordinates on the constant v-slice of the horizon as

$$x^i \to y^i = f_2(x^i) \tag{2.12}$$

   can also be done without departing from our choice of gauge. This essentially captures our freedom to choose the spatial coordinates on every constant v-slice of the horizon. This freedom also allows us to convert the partial derivatives ∂_i to ∇_i.

A special class of (2.11) corresponding to $f_0(x^i) = 0, f_1(x^i) = \lambda$ with λ a constant number is called the "boost" transformation [36]. The full transformation of the coordinates given by

$$v' = \lambda v, \qquad r' = \frac{r}{\lambda} \tag{2.13}$$

keeps the form of the metric (2.10) preserved to

$$ds^2 = 2dv'\,dr' - r'^2 X\left(\lambda r', \frac{v'}{\lambda}, x^i\right) dv'^2 + 2r'\omega_i\left(\lambda r', \frac{v'}{\lambda}, x^i\right) dv' dx^i + h_{ij}\left(\lambda r', \frac{v'}{\lambda}, x^i\right) dx^i dx^j \tag{2.14}$$

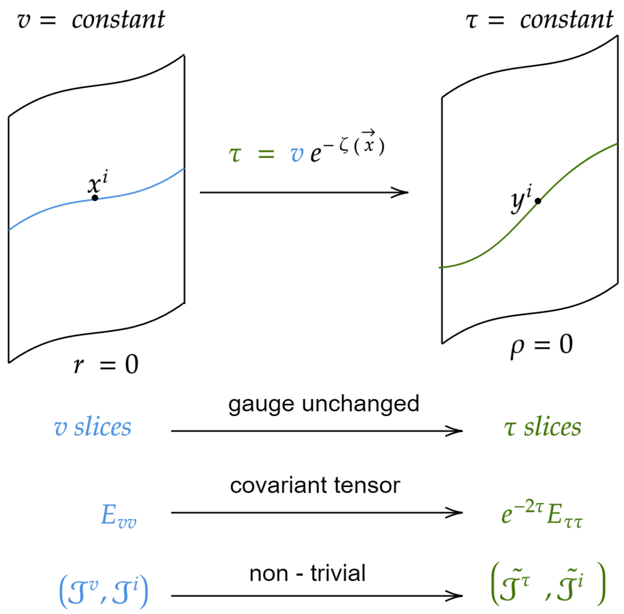


Figure 2.2: Another special case of (2.11) where the v and x^i coordinates map to τ and y^i where τ is another affine parameter. Comparing with (2.11), this corresponds to a case of $(f_0 = 0, f_1 = e^{-\zeta(x^i)})$. This reparametrization has been used in Chapter 3 [2,3].

Since for a stationary black hole solution, all the r and v dependence of the metric components occurs as products of rv, and therefore, under the (2.13) transformation, such a metric remains totally invariant. Infinitesimal boost transformation is generated by the vector

$$\xi^\mu \partial_\mu = v\partial_v - r\partial_r \tag{2.15}$$

which is proportional to the Killing vector of stationarity with a factor of $\frac{C}{2}$. Since the stationary part of the metric remains invariant under (2.13), hence, if we decompose the full metric into a stationary part and fluctuations around it, then only the latter transforms under this transformation. If we write the full metric as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon \delta g_{\mu\nu} \tag{2.16}$$

where $g^{(0)}_{\mu\nu}$ is the stationary background metric, ϵ is a parameter that quantifies the departure from equilibrium, and $\delta g_{\mu\nu}$ is the fluctuation in the metric about equilibrium, then under an infinitesimal transformation generated by ξ^μ, the change in the metric is given by

$$\mathcal{L}_\xi g_{\mu\nu} = \epsilon \mathcal{L}_\xi \delta g_{\mu\nu} \tag{2.17}$$

For $\epsilon << 1$, we can treat $\delta g_{\mu\nu}$ as a very small fluctuation about equilibrium. In the further sections, we'll work in a linearized approximation, where the equations will be considered only up to $O(\epsilon^1)$.

Any covariant tensor in these coordinates will be constructed out of the following building blocks:

1. **Metric coefficients**: A scalar $X(r, v, x^i)$, a vector $\omega_i(r, v, x^i)$ and a tensor $h_{ij}(r, v, x^i)$ with respect to the symmetry transformations among the x^i coordinates.

2. **Derivative operators**: Two scalar operators ∂_v and ∂_r, and a vector operator ∇_i with respect to the aforementioned transformations.

A generic covariant tensor can then be expressed as

$$\mathcal{T} \sim (\partial_r)^{m_r} (\partial_v)^{m_v} \mathcal{Q} \tag{2.18}$$

where $\mathcal{Q}$ consists of X, ω_i, h_{ij} and only actions of ∇_i on them. For such structures, $\mathcal{T}$ will be zero at equilibrium on the horizon $(r = 0)$ if $m_v > m_r$ due to extra factors of r coming out from the operation. Also, out of equilibrium, for such structures with $m_v > m_r$, contributions on the horizon will always come out at $O(\epsilon)$ owing to the fact that the equilibrium contribution goes to 0.

For ease in calculations, we define the notion of boost weight w of a covariant tensor $\mathcal{T}$ as follows: under a boost transformation of r and v coordinates as in (2.13), a tensor with boost weight $\mathcal{T}$ transforms as

$$\mathcal{T} \to \lambda^w \mathcal{T} \tag{2.19}$$

The previously discussed constraints related to m_v and m_r can be interpreted in terms of boost weight as follows. Any positive boost weight structure vanishes on the horizon at equilibrium, and out of equilibrium contributes at $O(\epsilon)$ or higher.

At this point, it should be noted that this boost symmetry is a special form of the reparametrization of the null generators where the affine parameter along them is scaled by a constant number. In 3, we will generalize this to a class of reparametrizations where this scaling would be dependent on the spatial coordinates on the codimension-2 slice of the horizon. Further, the scenario we consider in 4 is a more general case where the reparametrized null generators are non-affine. This happens because when the null generators on the horizon are expressed in terms of the coordinates used to describe the boundary hydrodynamic theory, they correspond to non-affinely parametrized vectors.

2.1.3 An off-shell identity and the second law

Based on the principles outlined in Section 2.1.2, one can constrain the possible covariant structures that can exist on the horizon up to the linearized fluctuation regime up to $O(\epsilon)$ [2] Using these, one can further lay constraints on the equation of motion of a diffeomorphism-invariant arbitrary higher-derivative theory of gravity.

Considering a Lagrangian where the only dynamical degrees of freedom are the metric $g_{\mu\nu}$, the Riemann tensor $R_{\mu\nu\alpha\beta}$, and symmetrized covariant derivatives of the Riemann tensor of the form $D_{(\alpha_1} \cdots D_{\alpha_n)} R_{\mu\nu\alpha\beta}$

$$S = \int d^d x \sqrt{-g}\, L(g_{\mu\nu}, R_{\mu\nu\alpha\beta}, D_{(\alpha_1} \cdots D_{\alpha_n)} R_{\mu\nu\alpha\beta}) \tag{2.20}$$

[2]For more details on such constraints, one can refer to [38].

Varying the Lagrangian with respect to the metric, one can obtain

$$\delta[\sqrt{-g}L] = \sqrt{-g}[E_{\mu\nu}(g)\delta g^{\mu\nu} + D_\mu \Theta^\mu(\delta g)] \tag{2.21}$$

where $E_{\mu\nu}$ is the equation of motion and Θ^μ is a boundary term depending on both $g_{\mu\nu}$ and $\delta g_{\mu\nu}$ for some arbitrary variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$.

It can be shown that up to $O(\epsilon)$, the $(\mu, \nu = v, v)$ component of $E_{\mu\nu}$ in the (r, v, x^i) coordinates and the metric of (2.10) can be arranged on the horizon into the form

$$(E_{vv})_{r=0} = \partial_v \left[\frac{1}{\sqrt{h}} \partial_v(\sqrt{h} J^v) + \nabla_i J^i \right] + O(\epsilon^2) \tag{2.22}$$

where, J^v and J^i have boost weights 0 and 1 respectively. Since J^i has a positive boost weight, it vanishes out of equilibrium. Although J^v is a boost weight 0 quantity, it gets contributions at equilibrium as well as out of equilibrium. The equilibrium contribution of J^v corresponds to Wald entropy as defined in [26] and the out-of-equilibrium contributions contribute to JKM ambiguities [28], which vanish at equilibrium. For these reasons, J^v and J^i are called the entropy density and the spatial entropy current respectively.

Now, if we consider some matter field to be there in the Lagrangian, the full equation of motion would be given by

$$\mathcal{E}_{\mu\nu} = E_{\mu\nu} + T_{\mu\nu} \tag{2.23}$$

where $T_{\mu\nu}$ corresponds to the matter sector's stress tensor and $E_{\mu\nu}$ is the contribution from the purely gravitational part of the Lagrangian. Now, if the matter stress tensor satisfies the Null Energy Condition [3], then on the horizon, it satisfies

$$T_{vv} = 0 \tag{2.24}$$

[3] The null energy condition states that for any future-directed null vector field k^μ, the stress tensor satisfies $T_{\mu\nu}k^\mu k^\nu \geq 0$. [99]

because ∂_v are null vectors on the horizon. If we consider the theory to be on-shell, then

$$\begin{aligned}
&\mathcal{E}_{\mu\nu} = 0 \Rightarrow E_{\mu\nu} + T_{\mu\nu} = 0 \\
&\Rightarrow (E_{vv} + T_{vv})_{r=0} = 0 \\
&\Rightarrow \partial_v \left[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_i J^i \right] + T_{vv} + O(\epsilon^2) = 0 \\
&\Rightarrow \partial_v \left[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_i J^i \right] + O(\epsilon^2) \leq 0
\end{aligned} \tag{2.25}$$

If we now impose a physical condition that the black hole settles down to equilibrium at future infinity in time, i.e., $v \to \infty$, then

$$v \to \infty \Rightarrow \partial_v \left[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_i J^i \right] \to 0 \tag{2.26}$$

Since J^v and J^i have boost weights 0 and 1 respectively, hence at equilibrium,

$$\left[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_i J^i \right] \to 0 \tag{2.27}$$

From this, we can conclude that

$$\partial_v \left[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_i J^i \right] + O(\epsilon^2) \geq 0 \tag{2.28}$$

for all finite v. This shows us that J^v and J^i together satisfy an ultra-local form of the second law on the black hole horizon. J^v denotes the change of entropy density in time, whereas J^i captures the spatial flux of entropy on a constant v-slice of the horizon. As has been shown in [2, 38], an integral of the combined divergence of the entropy density and the spatial entropy current gives us the entropy production between two nearby equilibrium states of the black hole. This entropy production can be captured as

$$\delta S = \int dv \int d^{d-1}x \left(\partial_v (\sqrt{h} J^v) + \partial_i J^i \right) + O(\epsilon^2) \tag{2.29}$$

This form of the change in entropy actually corresponds to the "Physical process version of the first law" [29].

As will be seen in 3, (2.22) and (2.29) act as crucial inputs in understanding the reparametrization symmetry of local entropy production on the horizon.

2.2 Relativistic Hydrodynamics: A Primer

In this section, we'll briefly review the tools and techniques one can use to analyze stability and causality in hydrodynamic theories. We'll also review some commonly used choices of hydrodynamic frames and the formulations of MIS and BDNK theories.

2.2.1 Linearized stability analysis in a theory

Consider any theory with a set of fields $\{\Phi_i(x,t)\}$ as its degrees of freedom. Let the equations of motion of the system be given by a set of nonlinear coupled partial differential equations of the form $E(\{\Phi_i(x,t)\}) = 0$. Let $\{\Phi_i^e\}$ be a set of exact solutions of $E(\{\Phi_i(x,t)\}) = 0$ conforming to its symmetries. $E(\{\Phi_i(x,t)\})$ is a differential operator consisting of the field variables and their spacetime derivatives [4]. In hydrodynamics, $\{\Phi_i(x,t)\}$ would be given by the fluid's temperature, velocity, chemical potential etc. $E(\{\Phi_i(x,t)\})$ would be given by the conservation equations of the stress tensor $T^{\mu\nu}$ or conserved charge currents J^μ as

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu J^\mu = 0 \tag{2.30}$$

Let $\{\Phi_i^e\}$ be given by the values of the fields at global equilibrium. To find the dispersion relation, we'd first consider a linear expansion of $\{\Phi_i(x,t)\}$ about $\{\Phi_i^e\}$ of the form

$$\{\Phi_i^s(x,t)\} = \{\Phi_i^e\} + \int d\omega d^3\vec{k}\; \epsilon\; e^{i(\vec{k}\cdot\vec{x}-\omega t)}\{\delta\Phi_i(\omega,\vec{k})\} \tag{2.31}$$

where $k = \sqrt{\vec{k}\cdot\vec{k}}$ and $\epsilon << 1$. The deviations from global equilibrium are Fourier expanded and quantified by the set of fluctuations $\{\delta\Phi_i(\omega,\vec{k})\}$. Assuming that $\{\Phi_i^s(x,t)\}$ solves $E(\{\Phi_i(x,t)\}) = 0$ up to linear order in ϵ

$$E(\{\Phi_i(x,t)\} = \{\Phi_i^s(x,t)\}) + O(\epsilon^2) = 0 \tag{2.32}$$

for arbitrary values of $\{\delta\Phi_i(\omega,\vec{k})\}$. Since the equations are all linear in $\{\delta\Phi_i(\omega,\vec{k})\}$, hence, we can

[4] This kind of linearized perturbation analysis isn't just limited to hydrodynamics but is used in many other fields like gravity. Hence, we keep the fields and equations here in full generality.

express the full set of equations as a matrix equation of the form

$$E(\{\Phi_i(x,t)\}) = \{\Phi_i^s\} + O(\epsilon^2) = 0 \Rightarrow \sum_b M_{ab}(\{\Phi_i\},\omega,k)\ \ \delta\Phi_b(\omega,\vec{k}) = 0 \tag{2.33}$$

Since this must be satisfied for all values of $\{\delta\Phi_i(\omega,\vec{k})\}$, this can be possible only if

$$det(M(\{\Phi_i\},\omega,k)) = 0 \tag{2.34}$$

The resultant equation gives us a set of polynomials of ω and k whose roots give us the spectrum of ω. For a rotationally invariant spacetime, the polynomials are only functions of k^2 and ω and are of the form

$$f(\omega,k^2) = 0 \tag{2.35}$$

A fluctuation around equilibrium is called linearly stable if it decays down to 0 with increasing time. In the rest frame of the fluid, this is given by

$$Im(\omega) \leq 0 \tag{2.36}$$

Since hydrodynamics is a low-energy effective theory, hence in general, one is interested in finding solutions of ω as infinite series of k near $k \to 0$ as

$$\omega = \sum_{n=0}^{\infty} c_n k^n \tag{2.37}$$

Modes that have $\omega = 0$ at the limit $k \to 0$ are defined as 'hydrodynamic' or 'massless' modes, and those with $\omega \neq 0$ at $k \to 0$ are called 'non-hydrodynamic' or massive modes. The non-hydro modes are named thus for the following reasons. The global equilibrium in a hydrodynamic system is characterized by conserved charges that can take any constant values, and dynamics in time are generated only when there is some spatial variation in their value. In this perspective, each fluid variable is associated with some conserved charge, and hydro modes are the ones whose frequency (variation in time) vanishes as soon as there is no spatial variation (i.e., at $k \to 0$). Hence, one can think of hydrodynamics as the collective dynamics of the massless modes of a system slightly away from global equilibrium. From this point of view, the non-hydro modes can be interpreted as those that do not vanish at zero momentum and, hence, cannot be associated with any conserved charges of the system at equilibrium. This $\omega_{(k=0)} \neq 0$ nature also resembles the rest mass energy of massive particles in relativity where $E = \sqrt{p^2 + m^2}$ and $E_{(p=0)} \neq 0$, hence, the name massive modes.

Recent developments suggest that the inclusion of non-hydro modes is necessary to maintain the stability and causality of the solution [100, 101]. Different schemes to introduce them in a hydrodynamic theory include the MIS formulation and the BDNK formulation, which will be discussed in Section 2.2.3.

Routh-Hurwitz stability analysis

In Chapter 5, our analysis focuses on the non-hydro modes of two stable-causal hydrodynamic theories at the $k \to 0$ limit, i.e. where $\omega(k = 0) = c_0$. The stability criteria $Im(\omega) < 0$ in this case then becomes $Im(c_0) < 0$. The rest of the discussion in this section on stability analysis using Routh-Hurwitz criteria [102] will focus on this special case.

The imaginary part of a_0 can be extracted as

$$C_0 \equiv Im(c_0) = -i\, c_0 \tag{2.38}$$

The Routh-Hurwitz stability analysis is a way to analyze the stability of the roots of a polynomial without explicitly calculating their roots. It involves constructing an array from the coefficients of the polynomial called the 'Routh array' as follows. For a polynomial of the form

$$\sum_{n=0}^{N} a_n x^n = a_0 + a_1 x^1 + \cdots + a_N x^N = 0 \tag{2.39}$$

the Routh array is defined as

$$\begin{bmatrix} a_N & a_{N-2} & \cdots \\ a_{N-1} & a_{N-3} \cdots & \\ b_{N-1} & b_{N-3} \cdots & \\ c_{N-1} & c_{N-3} \cdots & \\ \vdots & \vdots & \end{bmatrix} \tag{2.40}$$

where b_is and c_is are defined as

$$\begin{aligned} b_{N-1} &= -\frac{1}{a_{N-1}} \begin{vmatrix} a_N & a_{N-2} \\ a_{N-1} & a_{N-3} \end{vmatrix} \\ b_{N-3} &= -\frac{1}{a_{N-1}} \begin{vmatrix} a_N & a_{N-4} \\ a_{N-1} & a_{N-5} \end{vmatrix} \\ c_{N-1} &= -\frac{1}{b_{N-1}} \begin{vmatrix} a_{N-1} & a_{N-3} \\ b_{N-1} & b_{N-3} \end{vmatrix} \\ c_{N-3} &= -\frac{1}{b_{N-1}} \begin{vmatrix} a_{N-3} & a_{N-5} \\ b_{N-3} & b_{N-5} \end{vmatrix} \end{aligned} \tag{2.41}$$

In this way, for an $O(x^N)$ polynomial, one can get $N+1$ expressions in the first column of the Routh array. Now, as one counts down this first column of the array, the no. of sign changes between consecutive elements indicates the number of roots of the polynomial $\sum_{n=0}^{N} a_n x^n = 0$ lying in the right half of the complex plane, i.e., unstable roots. Hence, for all roots of the polynomial to be stable, all the elements of the first column of the Routh array must have the same sign, whether positive or negative. In our case in Chapter 5, we'll need to perform this Routh-Hurwitz analysis on the C_0s as defined above.

2.2.2 Causality analysis in linearized regime: Asymptotic causality

One of the most fundamental principles in relativity is that of causality, which ensures that the time ordering of causally connected spacetime events remains preserved, i.e., cause always precedes its effect. From a perturbation theory perspective, perturbations around the equilibrium of a system should never exit the light cone for the theory to be deemed causal. From a physical perspective, these perturbations can be assumed to originate from some source localized in spacetime, and causality analysis then deals with the question/fact/scenario of whether they time evolve to exit the light cone of the source. Since a perturbation localized in spacetime has a spread in the energy-momentum (or frequency-wavenumber) space, hence, causality analysis of such a localized perturbation should consider the contributions from all of these frequencies and wavenumbers.

Imposing the principle of causality in a relativistic hydrodynamic theory with a finite number of transport coefficients puts constraints on the phase space of these parameters, thus constraining the values that these coefficients can take. In this way, it serves as a good benchmark criterion for different microscopic models to be possible physically realizable theories.

One popular choice for analyzing a theory's causality in the linearized fluctuation regime is calculating the perturbation's group velocity at an infinite wavenumber. A group velocity lesser than the speed of light implies the perturbation being restricted to within the light cone, thus retaining the causality property of the system [103]. [5]

$$\begin{aligned} \text{Group velocity: } v_g &\equiv \frac{d\omega}{dk} \\ \text{System is causal } &\Rightarrow\ lim_{k\to\infty}|v_g| < 1 \end{aligned} \tag{2.42}$$

Since the analysis is performed at the $k \to \infty$ limit, hence, it is also called the "asymptotic causality" analysis. Satisfying asymptotic causality is a necessary criterion for a well-behaved theory but not sufficient; a hydrodynamic theory that violates asymptotic causality necessarily leads to acausal modes [104]. Also, as discussed in Chapter 1, asymptotic causality analysis lies outside the hydrodynamic regime due to its being a high-wavenumber analysis.

In another chain of recent works, the authors have attempted to understand the principle of causality by staying within the low-k regime only [100, 105]. From a relativistic quantum field theory analysis [106], one can derive necessary constraints on $Im(\omega)$ and $Im(k)$ as

$$Im(\omega(k)) \leq |Im(k)|. \tag{2.43}$$

Using these conditions, further bounds can be imposed on the expansion coefficients c_n of the modes (2.37) [100, 105, 107]. Chapter 5 is a pursuit following these principles to derive causality criteria from a low-k stability analysis in conformal, uncharged MIS and BDNK theories.

Recent works like [108, 109] have explored the causality properties of a theory in the nonlinear regime by evaluating the characteristic velocities of the partial differential equations of motion (see Appendix A of [101] for a discussion). In these cases, the propagation velocities are calculated by calculating the normals to the characteristic curves, and the subluminality of these velocities (i.e.,

[5] As shown in [104], as long as the asymptotic causality condition is fulfilled, causality is not violated even if the perturbation's group velocity exceeds the speed of light at some intermediate wavenumber.

being less than the speed of light) makes the system causal. Satisfying nonlinear causality criteria is sufficient for a theory to be causal. Since the analyses performed in Part II are linear, we'll restrict our attention to linearized causality analysis only in further discussion.

Schur stability of polynomials

Checking for the asymptotic causality of a dispersion polynomial often involves finding the roots of a polynomial and imposing other conditions on it (e.g. that the roots be real, lie between -1 to $+1$ etc). In most of the cases, the polynomials are higher-order than quadratic and analytic computations of causality criteria by root extraction methods can be very cumbersome. In Chapter 5, we use a novel method to extract causality criteria for a dispersion polynomial: checking for the "Schur stability" of polynomials. Schur stability analysis checks for the existence of roots of the polynomial within a unit disc on the complex plane without directly extracting its roots. For this reason, it can be a much easier method to extract causality criteria analytically. We'll briefly review the method in this section following [110].

Consider a polynomial in complex x of the form

$$P(x) = \sum_{n=0}^{N} c_n x^n \tag{2.44}$$

For this polynomial, Schur stability analysis checks for the existence of roots in the unit disc on the complex-x plane. This is analyzed by performing the following Möbius transformation on the complex-x plane that maps the unit disc onto the entire left half-plane

$$w = \frac{x+1}{x-1} \;\; (-1 \neq x \in \mathbb{C}), \qquad x = \frac{w+1}{w-1} \;\; (1 \neq w \in \mathbb{C}) \tag{2.45}$$

The polynomial in x is then converted into a polynomial in w as

$$\psi(w) = (w-1)^N P\left(\frac{w+1}{w-1}\right) \tag{2.46}$$

where the rescaling with $(w-1)^N$ is performed to cancel the $(w-1)$ factors in the denominators. $\psi(w)$ is thus an $O(w^N)$ polynomial with its roots lying in the left half plane due to the applied

Möbius transformation. Now, we can apply the Routh-Hurwitz stability check on $\psi(w)$ as detailed in the previous section, to check for the existence of stable w roots. Since the unit disc in the x plane was mapped to the left half plane in the w plane, hence, RH stable w roots of $\psi(w)$ would indicate Schur stable x roots of $P(x)$.

However, this is not the end of the story as, although Schur stability analysis allows us to check for the existence of roots within the unit disc, it doesn't tell us whether the roots are real or complex. The group velocities that we require are real quantities. Hence, one needs to impose further constraints on the polynomial, like the positivity of the discriminant, to ensure that the roots lying within the unit disc are real. The polynomials in Chapter 5 were at most quadratic in v_g^2, thus providing a less difficult setup to apply the positive discriminant criteria. Although imposing the positive discriminant criteria in higher-order polynomials can be more non-trivial, the total method of asymptotic causality analysis by Schur stability analysis should be less cumbersome to execute than the usual root extraction methods.

2.2.3 Stress tensors and hydrodynamic frames

In the rest of this section, we'll discuss some particular choices of hydrodynamic frames and the form of stress tensors in those frames. A choice of hydrodynamic frame essentially refers to a choice of out-of-equilibrium definitions of hydrodynamic fields like temperature, velocity, chemical potential etc. Since we'll deal only with conformal uncharged fluids in this thesis, the discussion in the rest of this section will only consider conformal fluids without any charge. Also, throughout the section, we'll consider the background spacetime to be flat and four-dimensional with the metric $\eta^{\mu\nu} = \text{diagonal}(-1, 1, 1, 1)$.

One more terminology that is used in linearized stability analysis is 'shear' and 'sound channel'. These are nothing but the relations between the directions of $\vec{k}$ of the perturbation and the velocity fluctuation $\delta\vec{u}$. 'Shear channel' refers to the case where $\vec{k} \perp \delta\vec{u}$ and 'Sound channel' refers to the case where $\vec{k} \parallel \delta\vec{u}$

Stress tensor of an ideal fluid

A fluid without any dissipative effects is called an ideal fluid. The stress tensor for a relativistic fluid is the conserved current corresponding to its translation symmetry. For a generic, uncharged fluid, the equilibrium stress tensor has the form

$$T^{\mu\nu}_{(0)} = \mathcal{E} u^{\mu} \nu + \mathcal{P} \Delta^{\mu\nu} \tag{2.47}$$

where $\mathcal{E}, \mathcal{P}, u^{\mu}$ and $\Delta^{\mu\nu}$ are the fluid's energy density, pressure, four-velocity and the projection tensor orthogonal to the velocity, respectively. The projection tensor is defined as

$$\Delta^{\mu\nu} = \eta^{\mu\nu} + u^{\mu} u^{\nu} \tag{2.48}$$

The stress tensor being conserved supplies us with the equations of motion of the fluid's degrees of freedom (viz. temperature, velocity etc.)

$$\partial_{\mu} T^{\mu\nu} = 0 \tag{2.49}$$

For a conformal fluid in four dimensions, energy and pressure are related by the equation of state

$$\mathcal{E} = 3\mathcal{P} \tag{2.50}$$

and in four dimension, the relation between $\mathcal{E}$ and temperature $\mathcal{T}$ is given by

$$\mathcal{E} = \kappa \mathcal{T}^4 \tag{2.51}$$

where κ is a constant related to the Stefan-Boltzmann constant. Thus, the stress tensor of an ideal conformal uncharged fluid is given by

$$T^{\mu\nu} = \kappa \mathcal{T}^4 \left(u^{\mu} u^{\nu} + \frac{1}{3} \Delta^{\mu\nu} \right) \tag{2.52}$$

Dissipative corrections and Landau frame

Hydrodynamic fields like velocity and temperature are well-defined for a fluid at global equilibrium, but when going out of equilibrium, they lose their meaning. To take into account this slightly out-of-equilibrium scenario, dissipative corrections are added to the equilibrium theory in the form of derivative corrections of the equilibrium hydrodynamic fields. One then has to make a choice of the out-of-equilibrium definitions of these fields, termed a “choice of hydrodynamic frame” or, more colloquially, a ‘frame choice’. This frame choice then further regulates the possible structures that can be added in these dissipative corrections.

For a general hydrodynamic theory, dissipative corrections can be added to the equilibrium theory as follows (for simplicity, we’ll consider a conformal uncharged fluid here):

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{(0)} + T^{\mu\nu}_{vis} \\ T^{\mu\nu}_{vis} &= S_1 u^\mu u^\nu + S_2 \Delta^{\mu\nu} + 2u^{(\mu}V^{\nu)} + \mathcal{T}^{\mu\nu} \\ \partial_\mu T^{\mu\nu} &= 0 \end{aligned} \tag{2.53}$$

where $S_1, S_2, V^\mu, \mathcal{T}^{\mu\nu}$ contribute $O(\partial^1)$ onwards. Examples of possible structures in these corrections include $\partial_\mu \mathcal{T}, \partial_\mu u_\nu$ and their higher-order derivatives. In addition, V^μ and $\mathcal{T}^{\mu\nu}$ satisfy $V \cdot u = \mathcal{T}^{\mu\nu} u_\mu = 0$. Moreover, the added corrections should respect all the symmetries of the equilibrium theory, and the equation of motion would still be given by the conservation of the full stress tensor (conformal invariance in this case). Frame choices, as we shall see, are essentially choices of these $S_1, S_2, V^\mu, \mathcal{T}^{\mu\nu}$ structures.

The traditional strategy of frame choice involves first defining the fluid variables in terms of some microscopic quantities (field theory operators) and then exploring the structure of the equations and their consequences. The “Landau frame” condition is one such example of a frame choice where the equilibrium values of the hydrodynamic fields are chosen to be maintained even out of equilibrium. This puts severe restrictions on the possible corrections as follows. In the Landau frame, the fluid velocity is chosen to be a unit normalized timelike eigenvector of the stress tensor with the negative energy density as the corresponding eigenvalue.

$$\begin{gathered} T^{\mu\nu} u_\mu = -\mathcal{E} u^\nu \quad (u_\mu u^\mu = -1) \\ \Rightarrow T^{\mu\nu}_{visc} u_\mu = 0 \Rightarrow T^{\mu\nu}_{visc} = S_2 \Delta^{\mu\nu} + \mathcal{T}^{\mu\nu} \end{gathered} \tag{2.54}$$

For a conformal fluid, a non-zero S_2 violates conformal symmetry, therefore, $S_2 = 0$. Physically, the Landau frame choice has a nice interpretation: the fluid velocity is chosen to be the velocity of the energy flow. Hence, temperature and velocity at every point are well-defined as in local equilibrium, ignoring the effects of dissipation. Dissipative effects are accounted for in the viscous correction terms only, which are entered as corrections in pressure or in the traceless tensor sector.

Another commonly used frame choice is the Eckart frame, where the fluid velocity is chosen along the velocity of the particle flow. Here, $T^{\mu\nu}_{visc}$ can have a non-zero V^{μ}.

Pathologies in the Landau frame

Ideally, a systematic attempt to add dissipative effects to the equilibrium theory should involve adding derivative corrections order by order, starting from the first order. Since an $O(\partial^n)$ quantity is considered to be much smaller than another $O(\partial^{n-1})$ quantity, it is a natural expectation that such derivative corrections can be added without causing any major upsets in the physical properties of the equilibrium solutions. Higher-order derivative corrections should not impact the stability or causality properties of the lower-order solution.

For a conformal fluid in the Landau frame, the only possible dissipative correction at $O(\partial)$ is of the form

$$\mathcal{T}^{\mu\nu} = -2\eta\sigma^{\mu\nu} \tag{2.55}$$

where η is called the 'coefficient of shear viscosity' and $\sigma^{\mu\nu}$ is the shear tensor defined as

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_{(\alpha}u_{\beta)} - \frac{1}{3}\eta_{\alpha\beta}\,\partial\cdot u\right) \tag{2.56}$$

Using the second law of thermodynamics, η can be constrained to be a positive number.

The total stress tensor of the first-order derivative corrected conformal fluid in the Landau frame, therefore, comes out to be

$$T^{\mu\nu} = \kappa\mathcal{T}^4\left(u^{\mu}u^{\nu} + \frac{1}{3}\Delta^{\mu\nu}\right) - 2\eta\sigma^{\mu\nu} \tag{2.57}$$

It can be seen that upon Lorentz boosting the dispersion polynomial of the above stress tensor by some finite non-zero boost and solving for ω in terms of k, new modes pop up which are non-hydrodynamic in nature. Moreover, these new non-hydro modes have their leading terms (the $k = 0$

expansion coefficient) to be inversely proportional to the boost velocity. Since boost velocity can take values arbitrarily close to 0 and even be equal to 0 in the rest frame of the fluid, these modes can possibly diverge and indicate some pathologies of the theory [56, 111]. One can also find that Green's function corresponding to these dispersion polynomials has finite support outside the light-cone [50], thus indicating possible propagation of these perturbations outside the light-cone, thus violating causality. Thus, we can conclude from here that the relativistic first-order Navier-Stokes equation in the Landau frame is acausal and unstable.

The problems here actually have to do with the following facts. Since without a specific microscopic theory, we are agnostic to the microscopic interactions and dynamics in the underlying quantum field theory, hydrodynamics, to us, is just an effective theory. Hence, one should, in principle, have to add an infinite number of derivative corrections to the equilibrium theory. Now, numerical computations with infinite-order corrected theories are difficult, and for practical purposes, one has to truncate these infinite series at some finite order. As has been shown in [97], it is impossible to restore causality or stability to a viscous stress-tensor in the Landau frame adding correction terms only up to a finite order in derivative expansion. These problems also can be understood from the perspective of non-hydro modes: these dispersion polynomials lack any non-hydro modes, and hence, a scale that determines up to what energy scale it is valid to take the hydrodynamic approximation (i.e., long-wavelength approximation). As explored in [97] and in Chapter 6 based on [98], incorporating the infinite series of derivative corrections can be a possible way to cure these pathologies.

In the rest of the section, we'll review two well-established formalisms to take into account these non-hydro modes, thereby restoring stability and causality in the theories.

Müller-Israel-Stewart (MIS) formalism

The MIS formalism is a method to remedy the stability and causality issues staying in the Landau frame by promoting viscous contributions to conserved currents as new degrees of freedom. These new degrees of freedom come with their own equations of motion, which are like relaxation equations with their associated relaxation timescales. Due to the presence of time derivatives in the relaxation equations, the full set of dispersion relations can now account for the non-hydro modes, thus restoring stability and causality. Since the viscous degrees of freedom are not the result of any microscopic quantum field theoretic operators, hence, they are not associated with any conserved quantities. Thus, they are defined only out of equilibrium and lack any equilibrium counterparts. Therefore, these are termed as 'non-fluid' degrees of freedom.

For an uncharged conformal fluid, the $T^{\mu\nu}_{visc}$ in MIS formalism is expressed as

$$\begin{aligned} T^{\mu\nu}_{visc} &= \Pi^{\mu\nu} \\ \Pi^{\mu\nu}_{(0)} &= 0 \\ \Pi^{\mu\nu} + \tau_{\Pi} u \cdot \partial \Pi^{\mu\nu} &= -2\eta\sigma^{\mu\nu} \end{aligned} \tag{2.58}$$

where $\Pi^{\mu\nu}$ is the shear viscous flux and τ_{Π} is the corresponding relaxation timescale.

The MIS formalism has successfully studied numerical simulations and phenomenological models in colliding systems, both large and small [53]. Since it gives finite truncated equations of motion and works with the equilibrium definitions of velocity and temperature, it is a very suitable framework for practical purposes. However, the conceptual hurdle lies in finding physical motivations for the new degrees of freedom that are currently being investigated [112–114].

As explored in [97] and [98], incorporating new non-fluid degrees of freedom is just another way of packaging the infinite number of derivative corrections required to render the theory causal.

Bemfica-Disconzi-Noronha-Kovtun (BDNK) formalism

The BDNK formalism addresses the stability-causality issue of relativistic hydrodynamics in a different way; it accounts for the non-hydro modes using the hydrodynamic degrees of freedom solely by trading off the Landau frame condition. Essentially, it uses field redefinitions of the hydrodynamic variables to adopt a more general out-of-equilibrium definition of the fields instead of the Landau frame condition. In this 'generalized hydrodynamic frame,' one can write a truncated stable-causal stress tensor without introducing any new degrees of freedom. All components of the stress tensor and its viscous corrections can be written as derivatives of temperature, velocity etc. But in the process, the well-defined out-of-equilibrium notions of fluid variables are lost, and one is left with ambiguities in their definitions out of equilibrium. Consequently, in terms of microscopic operators, one doesn't know what temperature or velocity means out of equilibrium anymore. Rather, they act more like auxiliary variables to define the stress tensor in terms of the quantities comprising the constitutive relations (like energy density). Moreover, one now has to include viscous corrections in the constitutive relations like energy density and pressure. To account for the non-hydro modes, these corrections must include temporal derivatives.

Historically, the stability-causality issues of the first-order relativistic Navier-Stokes equation had been known for a long time, and it was a general impression that it is impossible to write a stable-causal theory with only first-order corrections to the stress-tensor. With the advent of the BDNK formalism, it became clear that by compromising the Landau frame choice using field redefinitions, it is possible to write a first-order stable-causal theory.

The BDNK stress tensor for a conformal uncharged fluid is formulated as follows

$$\begin{aligned} T^{\mu\nu} &= (\mathcal{E}+\mathcal{A})\left[u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3}\right] + [u^\mu Q^\nu + u^\nu Q^\mu] - 2\eta\sigma^{\mu\nu} \\ \mathcal{A} &= \chi\left(3\frac{u\cdot\partial T}{T} + \partial\cdot u\right) \\ Q^\mu &= \theta\left(\frac{\Delta^{\mu\nu}\partial_\nu T}{T} + u\cdot\partial u^\mu\right) \end{aligned} \tag{2.59}$$

The temporal derivatives $(u\cdot\partial)$ are the ones that account for the non-hydro modes and restore causality in the equations.

A naive attempt to write this stress tensor from the generalized hydro frame to the Landau frame would be by setting χ and θ to zero.

However, stability analysis reveals that the non-hydro modes come out with χ and θ in their denominators, making it impossible to write the stress tensor in the Landau frame without compromising stability and causality yet again. As has been shown in Chapter 6, it is possible to rewrite the BDNK stress tensor in the Landau frame using an infinite tower of derivative corrections in field redefinitions. This again emphasizes the fact that the BDNK formulation is a finite-truncated

UV complete theory of relativistic hydrodynamics.

Part I

Local Entropy Current on a Black Hole Horizon and its Reparametrizations

Chapter 3

Reparametrization symmetry of local entropy production on a dynamical horizon

This chapter is based on [3].

The construction of entropy density and the current in [38] relies on a very specific choice of the coordinate system 2.1 where the affine parameter along the null generator of the horizon is one of the coordinates. Now, it is possible to reparameterize the null generators of the horizon in a nontrivial way without affecting the affine nature of the parameter (2.11). The expressions for both the entropy density and the spatial current change under this reparametrization. But we expect the net entropy production, given by the ‘time’ derivative of the entropy density plus the divergence of the spatial current, should be something physical and, therefore, independent of our choice of affine parameters.

In this chapter, our goal is to verify the above expectation for the special case of Gauss-Bonnet theory where both the entropy density and the current have been explicitly computed in [37].

We have found that under this transformation, the ’time’ derivative of the entropy density, as well as the divergence of the spatial entropy current, change individually in a very nontrivial way; however, they precisely cancel each other. Apart from being a consistency check for the results described in [37], it also says why a spatial entropy current is necessary to make the laws of entropy production independent of our choices of coordinates.

Though, at the moment, all the calculations are linear in the amplitude of the dynamics, we eventually would like to have some construction of entropy density and the entropy current that satisfy the first and the second law at all nonlinear orders and, if possible, without using any perturbation. Now, in any such construction, a full knowledge of the underlying symmetries might turn out to be very useful. The requirement that the entropy current and the density must transform in such a way so that the net entropy production has some particular symmetry could be constraining for all the nonlinear terms. [1] In other words, it would be very interesting if, instead of verifying the symmetry in a particular theory, we could use it to predict some relation between the structure of entropy density and the spatial entropy current in a theory-independent manner. We expect that our explicit computation in the simple case of Gauss-Bonnet theory would help us to gain experience for further progress in this direction.

The contents of this chapter are organized as follows. In section 3.1, we have described the setup of our problem. In section 3.2, we have described the reparametrization symmetry. In section 3.3, we have explicitly verified that the entropy density and the entropy current do maintain this symmetry in the particular case of Gauss-Bonnet theory. Finally, in section 3.4, we have concluded. The details of the calculation are explained in the appendix A.

3.1 Set up

In this section, we shall briefly review the coordinate system used in the analysis of [37] for the sake of continuity and the expression for entropy current and entropy density for the Gauss-Bonnet theory.

3.1.1 Coordinate system

As mentioned before, the geometry we are considering is of the black-hole type containing a codimension one null surface as the horizon. The coordinate system is constructed with the horizon being the base, i.e., we first choose $(D-1)$ coordinates on the horizon. Let ∂_v be the generator of the horizon, which is a null geodesic with v being the affine parameter and x^a, $\{a = 1, \cdots, D-2\}$ are the spatial coordinates along the constant v slices of the horizon. So $\{v, x^a\}$ together constitute a coordinate system on the horizon.

Once the coordinates on the horizon are fixed, we shoot off affinely parametrized null rays

[1] In [39], which came up shortly after our work, the authors have included an elaborate discussion on this issue.

∂_r, making specific angles with horizon coordinates. The affine parameter r along these rays is a measure of the distance away from the horizon. The angles are chosen so that the inner product between ∂_r and ∂_v on the horizon is 1 and the same between ∂_r and ∂_as are zero. After imposing all these conditions, the metric takes the following form (see [37] for more details)

$$\begin{aligned} ds^2 &= 2\,dv\,dr - r^2\,X(r,v,x^a)\,dv^2 \\ &\quad + 2\,r\,\omega_a(r,v,x^b)\,dv\,dx^a + h_{ab}(r,v,x^a)\,dx^a\,dx^b \end{aligned} \tag{3.1}$$

3.1.2 Gauss-Bonnet theory

We are considering a theory of pure gravity with a maximum of four derivatives. We shall be even more specific in choosing the theory; we'll work with the Gauss-Bonnet theory of gravity with the following Action.

$$S = \int d^D x \sqrt{-G}\left[R + \alpha^2\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right)\right] \tag{3.2}$$

Here R, $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are the Ricci scalar, Ricci tensor, and Riemann tensor[2] of the full spacetime respectively. All raising and lowering of indices have been done using the bulk metric $g_{\mu\nu}$.

The entropy density (J^v) and the entropy current (J^a) on the horizon have the following structure

$$\begin{aligned} J^v &= \left(1 + 2\alpha^2\mathcal{R}\right) \\ J^a &= \alpha^2\left[-4\nabla_b K^{ab} + 4\nabla^a K\right] \end{aligned} \tag{3.3}$$

Here $\mathcal{R}$ is the intrinsic Ricci scalar of the constant v slices of the horizon (i.e., the Ricci scalar computed using the metric h_{ab}). K_{ab} is the extrinsic curvature of the null horizon, and ∇_a is the covariant derivative with respect to h_{ab}

$$K_{ab} \equiv \frac{1}{2}\partial_v h_{ab}, \quad K \equiv h^{ab}K_{ab} \tag{3.4}$$

[2] According to our convention,

$$\begin{aligned} &R \equiv g^{\mu\nu}R_{\mu\nu}, \quad R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} \\ &R^{\mu}{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\rho\nu} \end{aligned}$$

The sole reason for choosing this theory is its simplicity. Despite being a four-derivative theory, the equation of motion remains two derivatives, and both the entropy density and the current could be constructed entirely from h_{ab} and its v and x^a derivatives evaluated on the horizon, which simplifies our task to a large extent. However, we must emphasize that the symmetry that we are going to describe in the next section is expected to hold in any higher derivative theory of gravity.

3.2 Symmetry

In section 3.1, we have chosen a coordinate system adapted to the horizon so that the metric takes the form as described in equation (3.1). However, this form does not fix the coordinates completely; some residual gauge freedom is still left, and both the entropy density and entropy current do change non-trivially under this unfixed coordinate freedom.

On the other hand, as we have explained in the introduction, the expression

$$\left[\frac{1}{\sqrt{h}}\partial_v(\sqrt{h}J^v) + \nabla_i J^i\right]$$

(where J^v and J^i are the entropy density and the spatial entropy current, respectively) is related to the local entropy production along every point of the dynamical horizon and therefore, we expect it to be invariant under the reparametrization of the null generators.

In this section, we shall first describe this residual freedom of coordinate transformation that is not fixed by our choice of gauge. Next, we shall use the details of this transformation to make our intuition about 'invariance' more precise.

3.2.1 Reparametrization of the null generator

The starting points in setting up our bulk coordinate system are the affinely parametrized null generators of the horizon and the coordinates along its spatial slices. Once we fix the horizon coordinates, our gauge conditions uniquely fix the coordinates along the bulk. It follows that the residual symmetry that we are going to discuss here must involve a transformation of the horizon coordinates, maintaining the affineness of the null generators. For convenience, let us use a bar on all the coordinates of the horizon to distinguish them from the bulk coordinates. For example, $\{\bar{v}, \bar{x}^a\}$ will denote the affine parameter along the null generator and spatial coordinates along the constant $\bar{v}$ slices of the horizon only.

Now, an affine parameter will remain an affine parameter if we scale it in a $\bar{v}$ independent manner. So, we shall consider the following transformation on the horizon ($r = 0$ hypersurface).

$$\bar{v} \to \bar{\tau} = \bar{v}\, e^{-\zeta(\bar{x}^a)}, \quad \bar{x}^a \to \bar{y}^a = \bar{x}^a \tag{3.5}$$

As mentioned before, both $\bar{v}$ and $\bar{\tau}$ are affine parameters along the null generators of the horizon. However, constant $\bar{v}$ slices are not the same as the constant $\bar{\tau}$ slices. In other words, the tangent vectors along the constant $\bar{v}$ slices given as $\bar{\partial}_a^{(x)}$ are different from the tangent vectors $\bar{\partial}_a^{(y)}$ along the constant $\bar{\tau}$ slices. They are related as follows

$$\bar{\partial}_a^{(x)} = \bar{\partial}_a^{(y)} - \left(\frac{\partial \zeta}{\partial \bar{y}^a}\right) \bar{\tau} \partial_{\bar{\tau}} \tag{3.6}$$

Since the tangent vectors on the horizon change under this transformation, we need to transform the r coordinate also so that the tangents along the constant $\{\tau, y^a\}$ lines (or the coordinate vectors pointing away from the horizon) maintain the same angle with the coordinate vectors along the horizon. This will firstly lead to a redefinition of the r coordinate, and also, it will correct the coordinate transformation (3.6) as one goes away from the horizon.

$$\begin{aligned} v &= e^{\zeta(y)}\tau \left[1 + \sum_{n=1} (\rho\, \tau)^n\, V_{(n)}(\tau, \vec{y})\right] \\ r &= e^{-\zeta(y)}\rho \left[1 + \sum_{n=1} (\rho\, \tau)^n\, R_{(n)}(\tau, \vec{y})\right] \\ x^a &= y^a + \sum_{n=1} (\rho\, \tau)^n\, Z^a_{(n)}(\tau, \vec{y}) \end{aligned} \tag{3.7}$$

Let us briefly motivate the choice of the above ansatz .

As mentioned before, the coordinate transformation is generated due to the scaling function $\zeta(\bar{y})$

defined only on the horizon, and once this horizon function is given, the rest of the coordinates throughout the bulk are uniquely determined by our gauge condition. Clearly, it is impossible to solve these gauge conditions exactly for a generic spacetime. However, the problem is very well-suited for a near-horizon expansion since, geometrically, our choice of gauge is a two-step process where we first choose coordinates on the horizon and then shoot out null geodesics with precise angles to extend them away from the horizon.

As is often true with perturbative expansions, our ansatz also involves a few conventions and assumptions. First note that each of the expansion coefficients ($V_{(n)}, R_{(n)}$ and $Z^a_{(n)}$), including the function $e^{\pm\zeta}$ strictly speaking should depend only on the horizon coordinates $\{\bar{\tau}, \bar{y}^a\}$. Whenever we are writing them as functions of bulk coordinates $\{\tau, y^a\}$, it involves an extension of these functions to the bulk, which is rather arbitrary. It is always possible to redefine the expansion coefficients at any given order by adding functions that vanish on the horizon without affecting the lower-order coefficients. Similarly, ζ itself might admit a power series expansion in a distance from the horizon (in fact, if we choose to write $\zeta(y^a)$ in terms of $\{x^a\}$ coordinates, this will happen). However, such redefinition, geometrically, does not mean that we are choosing new curves for coordinate axes since we know all coordinates are uniquely determined by our gauge choice once we fix the coordinates on the horizon. This is simply a rearrangement redundancy that is built into our perturbative technique of solving the gauge choices. However, here we have chosen the most naive bulk extension of all these horizon quantities by simply replacing all the $\{\bar{\tau}, \bar{y}^a\}$ dependence with bulk coordinates $\{\tau, y^a\}$ (which need not be the simplest choice in terms of the final form of the expansion coefficients).

Next, we shall come to the second unusual choice we made in our ansatz. A near horizon expansion in our coordinates simply means an expansion in powers of ρ (and not in the powers of the product $(\rho\tau)$ as we have done here). However, note that there is no loss of generality in expanding the powers of the product $(\rho\tau)$ if we keep the τ dependence in the expansion coefficients completely free. The reason behind this choice of expansion parameter is related to equilibrium (stationary) horizons. We know that in stationary black holes, the radial dependence of the metric components is always through the boost-invariant product $(\rho\tau)$ or (rv) [37]. This will be true provided the coordinate transformation has the structure as described above with coefficient functions independent of τ coordinates. In other words, in our $(\rho\tau)$ expansion, the expansion coefficients will depend on τ only when the horizon is evolving with time, thus enabling us to clearly distill out the effect of dynamics from that of the stationary case.

Fortunately, all these subtle issues about the form of the coordinate transformation turn out to be completely irrelevant to the present analysis of Einstein Gauss-Bonnet gravity. For this theory, both the entropy density and entropy current are entirely constructed out of the induced spatial metric of the horizon (denoted as h_{ab}) and its derivative along the tangents of the horizon (i.e., ∂_a and ∂_v only and no ∂_r). Here we do not need to know the metric components away from the horizon and therefore there is no need to determine the coordinate transformation for nonzero ρ.[3] The induced metric on the horizon remains invariant under the reparametrization as

$$\tilde{h}_{ij} = h_{ij} + \mathcal{O}(r) \tag{3.8}$$

3.2.2 Why we expect this transformation to be a symmetry

Here, we shall present a heuristic argument of why we expect such a symmetry to be there in the first place. The argument is very similar to what one uses to prove 'the physical process version of the first law.'

Following the setup in [29], consider a stationary black hole. The horizon is a Killing horizon in the absence of any perturbation. At some Killing time t_0, matter fields are perturbed. If we treat the amplitude of the field perturbation as of $(\mathcal{O}(\delta))$, then typically, the fluctuation in the matter stress tensor would be of order $\mathcal{O}(\delta^2)$ and the same would be the order of the metric fluctuation (which, at later sections, has been denoted as $\epsilon \sim \delta^2$). It follows that the local entropy production

[3] Higher order corrections to the metric coefficients are going to be computed in an upcoming work.

$S_p \equiv \left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,J^v\right) + \nabla_i J^i\right]$, which is constructed solely out of metric fluctuation, is also of order $\mathcal{O}\left(\delta^2\right)$. Note that the Killing equation will remain true up to order $\mathcal{O}(\delta)$ and therefore to compute the leading order ($\mathcal{O}(\delta^2)$) expression for the entropy production, it makes sense to integrate S_p between two constant 'Killing time' slices of the horizon, namely initial equilibrium (at 'Killing time $t = -\infty$) to final equilibrium (at Killing time $t = \infty$). Now we could relate the 'Killing time' to the affine parameter of the null generators where $t = -\infty$ will correspond to $v = 0$, and $t = \infty$ will correspond to $v = \infty$ (see [29] for the details). So, the net entropy production could be expressed as [29, 37, 38, 115–119] [4]

$$\begin{aligned}\Delta S &= \int_0^\infty dv \int_{\Sigma_v} d^n\vec{x}\,\sqrt{h}\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,J^v\right) + \nabla_i J^i\right] \\ &= S_{Equilibrium_2} - S_{Equilibrium_1}\end{aligned} \tag{3.9}$$

where Σ_v is the constant v slices of the horizon and $n = D - 2$.

But the total entropy in equilibrium or for a stationary black hole is unambiguously defined through Wald entropy, which is independent of how we parametrize the null generators of the horizon, and the same must be true of their difference. Now under the reparametrization that we are discussing, the measure of the above integration changes as

$$\sqrt{h}\,dv\,d^n\vec{x} = e^{\zeta(y)}\sqrt{h}\,d\tau\,d^n\vec{y}$$

If we want ΔS to be invariant under the reparametrization of the null generators, then the expression $\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,J^v\right) + \nabla_i J^i\right]$, once written in terms of quantities defined in $\{\tau, \vec{y}\}$ coordinates, must have an overall factor of $e^{-\zeta}$.

$$\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\,J^v\right) + \nabla_a J^a\right] = e^{-\zeta}\left[\frac{1}{\sqrt{\tilde{h}}}\partial_\tau(\sqrt{\tilde{h}}\tilde{J}^\tau) + \tilde{\nabla}_a\tilde{J}^a\right] \tag{3.10}$$

Here the LHS is expressed in $\{v, \vec{x}\}$ coordinates and RHS is in $\{\tau, \vec{y}\}$ coordinates.

[4] We thank the referee for clarifying this point to us.

Now we shall come to an algebraic reason why the expression for net entropy production should transform exactly as predicted in equation (3.10). We shall restrict this discussion to the theories of pure gravity.

The key equation that leads to the entropy current on the horizon is the following

$$E_{vv}|_{r=0} = \partial_v \Big[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_a J^a \Big], \tag{3.11}$$

Here E_{vv} is the (vv) component of the equation of motion. This is a component of a covariant tensor, and therefore, we know how it transforms under the above coordinate transformation for every possible gravity action. On the horizon (i.e., at $\rho = 0$ hypersurface) the transformation becomes particularly simple.

$$E_{vv}|_{r=0} = e^{-2\zeta} E_{\tau\tau}|_{r=0} \tag{3.12}$$

Now, in $\{\rho, \tau, y^a\}$ coordinates, the metric has the same form as in equation (3.1). Therefore, $E_{\tau\tau}$ could also be expressed as in equation (3.11) for some $\tilde{J}^\tau$ and $\tilde{J}^a$.

$$E_{\tau\tau}|_{r=0} = \partial_\tau \Big[\frac{1}{\sqrt{\tilde{h}}} \partial_\tau (\sqrt{\tilde{h}} \tilde{J}^\tau) + \tilde{\nabla}_a \tilde{J}^a \Big]$$

Note $\tilde{J}^\tau$ and $\tilde{J}^a$ are not components of covariant tensors on bulk space, and therefore, they do not transform in any well-defined way. But combining the above equation with equations (3.12) and (3.11) we get the following prediction.

$$\begin{aligned} E_{vv}|_{r=0} &= \partial_v \Big[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_a J^a \Big] \\ &= e^{-\zeta}\, \partial_\tau \Big[\frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_a J^a \Big] \\ &= e^{-2\zeta} E_{\tau\tau} \\ &= e^{-2\zeta}\, \partial_\tau \Big[\frac{1}{\sqrt{\tilde{h}}} \partial_\tau (\sqrt{\tilde{h}} \tilde{J}^\tau) + \tilde{\nabla}_a \tilde{J}^a \Big] \\ \Rightarrow \frac{1}{\sqrt{h}} \partial_v (\sqrt{h} J^v) + \nabla_a J^a &= e^{-\zeta} \Big[\frac{1}{\sqrt{\tilde{h}}} \partial_\tau (\sqrt{\tilde{h}} \tilde{J}^\tau) + \tilde{\nabla}_a \tilde{J}^a \Big] \end{aligned} \tag{3.13}$$

In the last line, both LHS and RHS (up to the factor of $e^{-\zeta}$) are related to the net entropy production in the two coordinate systems discussed here. It follows that though the entropy density and the entropy current might change in a very nontrivial way with several terms dependent on derivatives of ζ, in the final expression of entropy production, they must cancel, leaving just an overall $e^{-\zeta}$ factor. Further, the equation (3.13) also says that this nontrivial cancellation must be true in all higher derivative theories of gravity. In the next section, we shall verify this claim in the simplest case of Gauss-Bonnet theory. [5]

3.3 Verification for Gauss-Bonnet Theory

In this section, for the special case of Gauss-Bonnet theory, we would like to explicitly verify whether the local entropy production on the horizon transforms the way we have predicted in the previous sections. We know

$$E_{vv}|_{r=0} = \partial_v \Big[\frac{1}{\sqrt{h}}\partial_v(\sqrt{h}J^v) + \nabla_a J^a\Big], \tag{3.14}$$

where

$$J^v = 1 + 2\alpha^2 \mathcal{R}, \tag{3.15}$$

$$J^a = \alpha^2 \left[-4\nabla_b K^{ab} + 4\nabla^a K\right] \tag{3.16}$$

On the horizon, the reparametrization we are considering is the following

$$v = \tau\, e^{\zeta(y)}, \tag{3.17}$$

$$x^a = y^a. \tag{3.18}$$

[5] It might seem that the heuristic justification provided at the very beginning of this subsection is not very different from the algebraic one involving E_{vv}. Indeed, if we follow the argument presented in [29], we see that at linearized order, the net entropy production has been first related to the integration of the $\{vv\}$ component of the matter stress tensor and then by the equation of motion is related to the integration of E_{vv}. So, the covariance of the integrand in (eqn 9) is effectively the same as the covariance of E_{vv} at least in this order. However, the covariance of the integrand has a scope for further generalization if we want to extend this construction to higher orders in amplitude expansion. Following [39], we could see that as we go in higher order, this local entropy current can no longer be derived just from E_{vv}, but the other components of $E_{\mu\nu}$ also contribute, and it becomes quite complicated to figure out the net transformation property of this combination of equations. However, if we expect the ultra-local form of entropy production to be valid at higher orders, then there must be an integration formula for ΔS, and the integrand must transform in a covariant manner once the corrections to Killing equations have been appropriately taken care of.

Clearly, the $\mathcal{O}(\alpha^0)$ piece (contribution from Einstein gravity) in J^v does not transform. So now we have to determine how the order $\mathcal{O}(\alpha^2)$ pieces of J^v and J^a transform. Both of them will receive non-trivial shifts generated by derivatives of the function $\zeta(\vec{y})$. But these shifts will be such that eventually in the expression of $\left[\frac{1}{\sqrt{h}}\partial_v(\sqrt{h}J^v) + \nabla_a J^a\right]$ they will precisely cancel up to a factor of overall $e^{-\zeta}$.

Now we shall first describe how all the relevant quantities individually transform under this reparametrization.

The derivatives transform as

$$\partial_v = e^{-\zeta(y)}\partial_\tau, \tag{3.19}$$

$$\partial_a = \tilde{\partial}_a - (\tilde{\partial}_a\zeta)\tau\partial_\tau. \tag{3.20}$$

The Christoffel connection transforms as

$$\begin{aligned}\Gamma_{a,bc} &= \frac{1}{2}(\partial_b h_{ac} + \partial_c h_{ab} - \partial_a h_{bc}),\\ &= \tilde{\Gamma}_{a,bc} - \tau(\xi_b\tilde{K}_{ac} + \xi_c\tilde{K}_{ab} - \xi_a\tilde{K}_{bc}),\end{aligned} \tag{3.21}$$

where,

$$\xi_a = \partial_a\zeta = \tilde{\partial}_a\zeta, \tag{3.22}$$

$$\tilde{K}_{ab} = \frac{1}{2}\partial_\tau h_{ab}. \tag{3.23}$$

The Ricci scalar is given as

$$\tilde{\mathcal{R}} = (h^{ad}h^{bc} - h^{ac}h^{bd})(\tilde{\partial}_d\tilde{\Gamma}_{a,bc} - h^{pq}\tilde{\Gamma}_{p,ad}\tilde{\Gamma}_{q,bc}). \tag{3.24}$$

Under the change of coordinates, the Ricci Scalar transforms as

$$\begin{aligned}\mathcal{R} = \tilde{\mathcal{R}} + 2(h^{ad}h^{bc} - h^{ac}h^{bd})\Big[&(-\tau)\{\xi_{bd} + (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b) - \xi_b\xi_d\}\tilde{K}_{ac}\\ &+ \tau\tilde{\Gamma}^p_{ad}(\xi_b\tilde{K}_{pc} + \xi_c\tilde{K}_{pb} - \xi_p\tilde{K}_{bc}) + \tau^2\xi_b\xi_d\partial_\tau\tilde{K}_{ac}\Big].\end{aligned} \tag{3.25}$$

This implies that the order $\mathcal{O}(\alpha^2)$ piece of the entropy density transforms as

$$\begin{aligned} J^v &= 2\mathcal{R} = 2\tilde{\mathcal{R}} \\ &+ 4(h^{ad}h^{bc} - h^{ac}h^{bd})[(-\tau)\{\xi_{bd} + (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b) - \xi_b\xi_d\}\tilde{K}_{ac} \\ &+ \tau\tilde{\Gamma}^p_{ad}(\xi_b\tilde{K}_{pc} + \xi_c\tilde{K}_{pb} - \xi_p\tilde{K}_{bc}) + \tau^2\xi_b\xi_d\partial_\tau\tilde{K}_{ac}]. \end{aligned} \tag{3.26}$$

Now we know that $J^\tau|_{\mathcal{O}(\alpha^2)} \equiv 2\tilde{R}$, then

$$\begin{aligned} &\frac{1}{\sqrt{h}}\partial_v(\sqrt{h}J^v)|_{\mathcal{O}(\alpha^2)} = e^{-\zeta}\frac{1}{\sqrt{h}}\partial_\tau(\sqrt{h}J^\tau)|_{\mathcal{O}(\alpha^2)} \\ &\quad + 4e^{-\zeta}(h^{ad}h^{bc} - h^{ac}h^{bd})\Big[-(\xi_{bd}\tilde{K}_{ac}) - (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\tilde{K}_{ac} \\ &\quad + \tilde{\Gamma}^p_{ad}(\xi_b\tilde{K}_{pc} + \xi_c\tilde{K}_{pb} - \xi_p\tilde{K}_{bc}) - \tau\{\xi_{bd} + (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\}(\partial_\tau\tilde{K}_{ac}) \\ &\quad + \tau\tilde{\Gamma}^p_{ad}(\xi_b\partial_\tau\tilde{K}_{pc} + \xi_c\partial_\tau\tilde{K}_{pb} - \xi_p\partial_\tau\tilde{K}_{bc}) + \xi_b\xi_d\tilde{K}_{ac} + 3\tau\xi_b\xi_d\partial_\tau\tilde{K}_{ac} + \xi_b\xi_d\tau^2\partial_\tau^2\tilde{K}_{ac}\Big] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3.27}$$

The entropy current is given as

$$J^a = -4(h^{ad}h^{bc} - h^{cd}h^{ab})\nabla_b K_{cd}, \tag{3.28}$$

hence

$$\nabla_a J^a = -4(h^{ad}h^{bc} - h^{ac}h^{bd})\nabla_b\nabla_d K_{ac}. \tag{3.29}$$

The extrinsic curvature in the two coordinate systems are related as

$$K_{ac} = e^{-\zeta}\tilde{K}_{ac}. \tag{3.30}$$

This implies

$$\nabla_d K_{ac} = e^{-\zeta}[\tilde{\nabla}_d\tilde{K}_{ac} - \xi_d(\tilde{K}_{ac} + \tau\partial_\tau\tilde{K}_{ac})] \tag{3.31}$$

$$\begin{aligned} \nabla_b\nabla_d K_{ac} = e^{-\zeta}\Big[&\tilde{\nabla}_b\tilde{\nabla}_d\tilde{K}_{ac} - (\xi_{bd}\tilde{K}_{ac}) - (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\tilde{K}_{ac} + \tilde{\Gamma}^p_{ad}(\xi_b\tilde{K}_{pc} + \xi_c\tilde{K}_{pb} - \xi_p\tilde{K}_{bc}) \\ &- \tau\{\xi_{bd} + (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\}(\partial_\tau\tilde{K}_{ac}) + \tau\tilde{\Gamma}^p_{ad}(\xi_b\partial_\tau\tilde{K}_{pc} + \xi_c\partial_\tau\tilde{K}_{pb} - \xi_p\partial_\tau\tilde{K}_{bc}) \\ &+ \xi_b\xi_d\tilde{K}_{ac} + 3\tau\xi_b\xi_d\partial_\tau\tilde{K}_{ac} + \xi_b\xi_d\tau^2\partial_\tau^2\tilde{K}_{ac}\Big] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3.32}$$

Hence, the divergence of the Entropy current transforms as

$$\begin{aligned}
&\nabla_a J^a = e^{-\zeta}\tilde{\nabla}_a\tilde{J}^a \\
&- 4e^{-\zeta}(h^{ad}h^{bc} - h^{ac}h^{bd})\Big[- (\xi_{bd}\tilde{K}_{ac}) - (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\tilde{K}_{ac} + \tilde{\Gamma}^p_{ad}(\xi_b\tilde{K}_{pc} + \xi_c\tilde{K}_{pb} - \xi_p\tilde{K}_{bc}) \\
&- \tau\{\xi_{bd} + (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\}(\partial_\tau\tilde{K}_{ac}) + \tau\tilde{\Gamma}^p_{ad}(\xi_b\partial_\tau\tilde{K}_{pc} + \xi_c\partial_\tau\tilde{K}_{pb} - \xi_p\partial_\tau\tilde{K}_{bc}) \\
&+ \xi_b\xi_d\tilde{K}_{ac} + 3\tau\xi_b\xi_d\partial_\tau\tilde{K}_{ac} + \xi_b\xi_d\tau^2\partial_\tau^2\tilde{K}_{ac}\Big] + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{3.33}$$

From equations (3.27) and (3.33), we find that terms linear in $\tilde{K}_{ab}$, i.e., $\mathcal{O}(\epsilon)$ terms cancel exactly leaving an overall factor of $e^{-\zeta}$ in the zeroth order term. Hence, we have

$$\frac{1}{\sqrt{h}}\partial_v(\sqrt{h}J^v) + \nabla_a J^a = e^{-\zeta}\left[\frac{1}{\sqrt{h}}\partial_\tau(\sqrt{h}J^\tau) + \tilde{\nabla}_a\tilde{J}^a\right] + \mathcal{O}(\epsilon^2). \tag{3.34}$$

3.4 Conclusion

In this chapter, we have verified the general expectation that net entropy production in a dynamical gravity should not depend on how we choose coordinates along the horizon. First, in section 3.2, we have outlined a general proof of why the entropy production should transform in the way we physically expect (see equation (3.13) and the discussion around). Then, in the next section, we verified the claim for the particular case of Gauss-Bonnet theory by explicit computation. This provides a consistency check on the construction of the entropy current in Einstein-Gauss-Bonnet theory.

It might seem that apart from the consistency check mentioned above, our computation is not of much use since we already have a general proof that this symmetry must work. However, as we have already mentioned in the introduction, our final goal is to have some construction of entropy current and entropy density that works without any perturbation. In this context, it would be interesting to analyze this symmetry in a more systematic manner so that we could use it to constrain the structure of the entropy density and the entropy current in a theory-independent manner. Note that the existence of entropy density and the spatial entropy current has been predicted using the special case of the transformation considered here, namely boost symmetry generated by a constant ζ [38, 120]. It is natural to expect more constraints in the whole structure if we use a larger symmetry where ζ is a function of all spatial coordinates. This work is a small step towards this goal, which would give us more experience in dealing with the symmetries of null surfaces and the corresponding transformation of the relevant physical quantities.

One very natural extension of this work might be to perform similar calculations for other four-derivative theories where the cancellations can be slightly non-trivial due to the presence of off-the-horizon terms in the entropy current and entropy density.

Another interesting future direction to take can be to explore the existence of any possible relations between this reparametrization symmetry and the BMS or Carrollian symmetries. Recently in [121–124], the authors have shown the presence of extended BMS-like symmetries on the black hole horizon called Carrollian symmetries. Any possible connections of this symmetry with supertranslations or superrotations of the others can be useful in our understanding of the rich symmetric structure of the horizon.

Chapter 4

Entropy current and fluid-gravity duality in Gauss-Bonnet theory

This chapter is based on [95].

4.1 Introduction

In general, it is a challenging task to find dynamical black hole solutions even in Einstein gravity. One either has to use some perturbation or numerics. The perturbation in terms of the amplitude of the dynamics around a stationary solution is one such analytic technique to generate dynamical black hole solutions and as mentioned above, this is the one that has been used for the construction of the entropy density and the current on the horizon. In this chapter, we would like to extend this construction of horizon entropy current to another class of dynamical black hole/brane solutions generated using derivative expansion [1, 81, 82, 125].

Derivative expansion is a technique that could be applied to slowly varying dynamics (not necessarily of small amplitude). In [81], this technique has been used to generate solutions to Einstein equations in the presence of a negative cosmological constant and in [83], it has been further extended to Einstein-Gauss-Bonnet theory. These solutions are asymptotically AdS and are dual to conformal hydrodynamics with a very specific value of shear viscosity that gets corrected once the Gauss-Bonnet terms are added to the gravity action. The dual theory of hydrodynamics lives on the boundary of the AdS space, a co-dimension one hypersurface with flat metric. Such a theory of hydrodynamics always admits an entropy current - a covariant vector under the boundary Lorentz transformation, which has non-negative divergence on every solution of the fluid equations. It is natural to expect that the entropy along the dynamical horizon could be recast into one candidate for the boundary entropy current in any higher derivative theory of gravity as long as the black hole solution admits a fluid dual (see [126, 127] for such constructions).

In the case of Einstein gravity, where the horizon area plays the role of entropy density in the black hole, one could lift the horizon entropy to the boundary by using some (non-unique) horizon-boundary map. This map finally results in an entropy current in the fluid theory, expressed entirely in terms of fluid variables and with non-negative divergence, guaranteed by the 'horizon area increase' theorem [85]. In other words, in two derivative theories of gravity with a negative cosmological constant, the entropy production at every point on the dynamical horizon (with a degenerate metric) could be neatly mapped to the similar ultra-local (point by point) entropy production in the dual fluid dynamics, living on the boundary (with simple flat metric).

Clearly, this whole algorithm of lifting the horizon entropy density to the fluid entropy current crucially depends on how we map the points on the horizon to the points on the boundary. From the perspective of the boundary fluid, the mapping functions, which relate every point on the null horizon to a point on the time-like boundary, are some external variables. One of the key outcomes of the analysis in [85] is that for dynamical black holes/branes in Einstein gravity, it is possible to choose these mapping functions in a way so that the local entropy density on the horizon is a local function of the fluid variables only.

The reason that allows one to make such a choice is as follows.
For black holes in two derivative theories, the second law of thermodynamics is a consequence of the 'horizon area increase' theorem. The proof of this theorem does not need any form of perturbation or approximation on the horizon dynamics [15, 17, 99]. Also, the candidate for the entropy density - the area of the spatial sections of the horizon, is entirely independent of how we choose to parametrize the null generators of the horizon. This is why in two derivative theories, one is free to choose the mapping functions that are compatible with the description of the boundary fluid.

In fact, the choice of mapping used in [85] explicitly breaks the Lorentz covariance of the boundary coordinates, and the applicability of derivative expansion is implicitly assumed at all intermediate steps. It was the final answer for entropy current that was independently checked for Lorentz covariance and then covariantized entirely in terms of fluid variables and their boundary derivatives.

Now, the construction of entropy current in [37, 38] in higher derivative theories depends very much on how we choose the spatial sections of the dynamical horizon. So, a priori, it is not clear whether in such higher derivative theories also

1. we could lift the horizon entropy current to the boundary and rewrite the entropy production as a divergence of a current covariant with respect to the boundary metric;

2. the covariant boundary entropy current, thus constructed, is a legitimate entropy current in the dual theory of hydrodynamics, expressible entirely in terms of fluid variables.

In this chapter, we shall see that the answer to the first question is positive. We have been able to construct a manifestly covariant formula for boundary entropy current by rearranging the expressions for the entropy current and entropy density on the horizon with the mapping functions. These mapping functions are left arbitrary in our construction. They appear in the final formula of the boundary entropy current as new variables, much like the fluid variables. However, these new variables need not admit any derivative expansion.

In the case of two derivative theories of gravity, dependence on these mapping functions cancels out in the final formula as a consequence of the 'reparametrization invariance' of horizon area. This provides another justification of why the procedure used in [85], despite explicitly breaking the Lorentz invariance and translation invariance at every intermediate step, has worked so beautifully.

But in higher derivative theories, the construction of the entropy density and the entropy current need a very specific choice of coordinates on the horizon, where the null generators are affinely parametrized. Therefore, unlike the two derivative theories, the mapping functions here are not completely free; they have to be compatible with the horizon-adapted coordinates used in [37, 38] to parametrize the rate of entropy production along the null generator. Further, to generate a legitimate fluid entropy current on the boundary fluid, the mapping functions should not violate the applicability of derivative expansion in terms of the boundary coordinates. It turns out that these two conditions are not easy to satisfy simultaneously. We applied our construction to the horizon entropy current in Einstein-Gauss-Bonnet theory, whose fluid dual has already been constructed in [83]. But even before using the details of the of the bulk metric here, we could see that the covariant entropy current in the boundary theory, constructed by dualizing the horizon entropy, will have non-trivial dependence on the mapping functions, which do not get cancelled and also most likely will not admit any derivative expansion.

To summarise, the answer to the second question posed above is generically negative.

However, this is probably not a complete ‘no go’ theorem about the possibility of dualizing the ‘horizon entropy current’ to a legitimate fluid entropy current. It is still possible that for some special higher derivative theory, these dependencies on the mapping functions do cancel among themselves. Also, we have one construction of the boundary entropy current, but we do not have any proof that this is a unique construction. For example, any expression of the current could be modified by adding terms that are identically conserved without affecting its divergence. Similarly, the entropy current and entropy density on the horizon also have a number of ambiguities [3,27–29]. It is worth exploring whether all the terms that are not compatible with derivative expansion or fluid dynamics could be removed by fixing these ambiguities in a certain way. We leave these for future work.

This chapter is organized as follows. In the next subsection, we give a summary of the main results. Then in section 4.2 we have described how we could construct the horizon to boundary map. Next, in section 4.3, we have used this map to translate the horizon current to a covariant boundary current. In section 4.4, we have applied this construction to the dynamical black holes of Einstein-Gauss-Bonnet theory in the presence of a negative cosmological constant. In section 4.5, we explore some future directions. Finally, in section 4.6, we conclude.

4.1.1 The Result

As mentioned before, the main result in this chapter is a formula for the boundary entropy current whose divergence is equal to the rate of local entropy production on the dynamical horizon.

In [37, 38], it has been shown that in higher derivative theories of gravity, one could always construct an entropy density (denoted as j^v) and a spatial entropy current (denoted as j^i) on every black hole solution with a dynamical horizon such that

$$\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}j^v\right)+\nabla_i j^i\right]\geq 0 \tag{4.1}$$

provided the amplitude of the dynamics remains small throughout the evolution of the black hole till it settles to equilibrium.

Here v is the affine parameter along the null generators of the horizon; the sub/superscript 'i' denotes the spatial coordinates along the constant v slices of the horizon and ∇_i is the covariant derivative with respect to the induced metric along the constant v slices.

In this chapter, using a set of mapping functions from the horizon to the boundary (a map between the horizon coordinates $\{v, \alpha^i\}$ and boundary coordinates $\{x^\mu\}$) we have constructed an expression for entropy current J^μ on the boundary such that

$$D_\mu J^\mu = \frac{1}{\sqrt{h}}\partial_v(\sqrt{h}\, j^v)+\nabla_i j^i \tag{4.2}$$

where D_μ denotes the covariant derivative with respect to the boundary metric.

The expression for J^μ turns out to be

$$\begin{aligned}
J^\mu &= \frac{1}{\sqrt{g^{(b)}}}\frac{\sqrt{H}}{\sqrt{t^\alpha t^\beta g^{(b)}_{\alpha\beta}}}\left(j^v t^\mu + j^i l^\mu_i\right)\\
H &\equiv \frac{\hat{n}_\mu \hat{n}_\nu \epsilon^{\mu\mu_1\mu_2\cdots\mu_n}\epsilon^{\nu\nu_1\nu_2\cdots\nu_n}\chi_{\mu_1\nu_1}\cdots\chi_{\mu_n\nu_n}}{n!}\\
\hat{n}^\mu &= \frac{t^\mu}{\sqrt{t^\alpha t^\beta g^{(b)}_{\alpha\beta}}}
\end{aligned} \tag{4.3}$$

where t^μ and l_i^μ are vectors related to the map of $\{v, \alpha^i\}$ coordinates on the horizon to $\{x^\mu\}$ coordinates on the boundary and are defined as follows

$$t^\mu \equiv \frac{\partial x^\mu}{\partial v}, \quad l_i^\mu \equiv \frac{\partial x^\mu}{\partial \alpha^i}$$

And $\chi_{\mu\nu}$ is the degenerate induced metric on the horizon expressed in terms of the boundary coordinates or, more precisely if the bulk metric dual to the boundary fluid is denoted as $G_{AB}(r, x^\mu)$ with $r = 0$ being the horizon, then

$$\chi_{\mu\nu} = G_{\mu\nu}|_{r\to 0}$$

The symbol $\epsilon^{\mu\mu_1\cdots\mu_n}$ denotes the completely antisymmetric $(n+1)$ indexed tensor with each component equal to either 0 or ± 1. Note that in our convention, this epsilon tensor does not have any factor like the determinant of the metric.

We have explicitly constructed the boundary entropy current for the case of Einstein-Gauss-Bonnet theory, for which the horizon current is already determined in [37].

$$J^\mu = \frac{1}{\sqrt{g^{(b)}}}\frac{\sqrt{H}}{\sqrt{t^\alpha t^\beta g^{(b)}_{\alpha\beta}}}\left[\left(1+\alpha^2\mathcal{R}\right)t^\mu - 4\alpha^2\left(\bar{\chi}^{\gamma\alpha}\bar{\chi}^{\mu\beta} - \bar{\chi}^{\gamma\mu}\bar{\chi}^{\alpha\beta}\right)\left(\mathcal{D}_\gamma \mathcal{K}_{\alpha\beta}\right)\right]$$

with the following notation

$$\mathcal{R} \equiv \left(\bar{\chi}^{\mu_1\nu_1}\bar{\chi}^{\mu_2\nu_2} - \bar{\chi}^{\mu_1\nu_2}\bar{\chi}^{\mu_2\nu_1}\right)\left[\partial_{\mu_1}\Gamma_{\nu_1,\mu_2\nu_2} - \bar{\chi}^{\alpha_1\alpha_2}\Gamma_{\alpha_1,\mu_1\nu_1}\Gamma_{\alpha_2,\mu_2\nu_2} - 2t^\alpha\Gamma_{\alpha,\mu_1\nu_1}\left(\partial_{\mu_2}\tilde{t}_{\nu_2}\right)\right]_{r=0}$$

$$\mathcal{K}_{\alpha\beta} \equiv -t^\mu\Gamma_{\mu,\alpha\beta}, \quad \bar{\chi}^{\mu\nu} \equiv \left(\delta^\mu_\alpha - t^\mu\tilde{t}_\alpha\right)\left(\delta^\nu_\beta - t^\nu\tilde{t}_\beta\right)G^{\alpha\beta}(r=0)$$

and

$$\mathcal{D}_\alpha\mathcal{K}_{\mu\nu} \equiv \partial_\alpha\mathcal{K}_{\mu\nu} - \tilde{\Gamma}^\beta_{\ \alpha\mu}\mathcal{K}_{\beta\nu} - \tilde{\Gamma}^\beta_{\ \alpha\nu}\mathcal{K}_{\mu\beta} \tag{4.4}$$

where

$$\tilde{t}_\mu \equiv \frac{\partial v}{\partial x^\mu} \text{ such that } t^\mu\tilde{t}_\mu = 1, \ l_i^\mu\tilde{t}_\mu = 0$$
$$\Gamma_{\alpha,\mu\nu} = \frac{1}{2}\left(\partial_\mu\chi_{\nu\alpha} + \partial_\nu\chi_{\mu\alpha} - \partial_\alpha\chi_{\mu\nu}\right), \quad \tilde{\Gamma}^\alpha_{\mu\nu} \equiv \bar{\chi}^{\alpha\beta}\Gamma_{\beta,\mu\nu} + t^\alpha\partial_\nu\tilde{t}_\mu \tag{4.5}$$

Note that for a generic case, these mapping functions will enter the expression of the boundary entropy current through the two vectors t^μ and $\tilde{t}_\mu$. And as we have mentioned before, these two vectors need not admit a derivative expansion. The reason is as follows.

t^μ, being the tangent vector to the affinely parametrized null generators of the horizon (located at $r = 0$), must be proportional to the normal of the $r = 0$ hypersurface. This normal is given by $n^\mu \equiv G^{\mu r}(r = 0)$, which according to fluid-gravity correspondence, must admit a derivative expansion in terms of fluid variables. Let us denote the proportionality factor as $e^{\phi(x)}$.

$$t^\mu = e^{\phi(x)} n^\mu = e^{\phi(x)} \, G^{\mu r}|_{r=0}$$

The affine parameter v could be related to $\phi(x)$ as (see section 4.3.3)

$$v \equiv e^{-\phi} L = e^{-\phi} \sum_{k=0}^{\infty} L_{(k)}, \text{ where } \frac{L_{(k)}}{L_{(0)}} = -\left[(n \cdot \partial) L_{(k-1)}\right], \quad L_{(0)} = -\left[\left(\frac{n^\mu n^\nu}{2}\right) [\partial_r \chi_{\mu\nu}]_{r=0}\right]^{-1}$$

Therefore $\tilde{t}_\mu = \left(\frac{\partial v}{\partial x^\mu}\right)$ must have a term proportional to $\partial_\mu \phi$. Now $\partial_\mu \phi$ must be a zeroth order vector since its component along the direction of n^μ is of zeroth order in derivative expansion. It satisfies the equation (follows from the fact that t^μ is an affinely parametrized geodesic, see section 4.2.2)

$$(n \cdot \partial)\phi = \left(\frac{n^\mu n^\nu}{2}\right) [\partial_r \chi_{\mu\nu}]_{r=0}$$

However at zeroth order in derivative expansion, only vector that could be expressed entirely in terms of fluid variables is the fluid velocity u^μ itself. So $\partial_\mu \phi$ has to be proportional to u_μ with proportionality factor being some function of temperature. But any gradient vector field like $\partial_\mu \phi$ or $\partial_\mu v$ could not be proportional to fluid velocity whenever the velocity has nonzero vorticity. This shows that any generic situation $\partial_\mu \phi$ are the 'non-fluid' terms, that will remain there in the boundary entropy current constructed dualizing the horizon current.

Finally we have evaluated the boundary current (4.16) on slowly varying black holes in Einstein-Gauss-Bonnet theory up to correction of order $\mathcal{O}(\partial^2)$. Up to this order in derivative expansion, the 'non-fluid' mapping functions (functions that do not admit a derivative expansion in terms of fluid variables) do not contribute. In fact just like the fluid dual to Einstein gravity, the $\mathcal{O}(\partial)$ contribution to the entropy current vanishes which is also what is expected for an uncharged fluid. The expression of J^μ turns out to be the following

$$J^\mu = r_H^3\, \hat{n}^\mu + \mathcal{O}\left(\alpha^4, \partial^2\right) \tag{4.6}$$

where r_H is the length scale associated with the temperature of the Black hole or the dual fluid as defined in (4.8).

4.2 The map between horizon and boundary

As described before, the dynamical black brane solution that we are considering here, is always perturbative. Two different types of perturbations are used to describe the solution. For the entropy density and the current constructed on the horizon as in [37–39, 120], the perturbation parameter is the amplitude of the dynamics whereas in [81] it is the derivatives of the boundary fluid data (velocity and temperature) that play the role of the small parameter. In both cases, the starting point is a stationary black hole/brane metric. In both cases, we could choose a gauge where the horizon is at the origin of the radial coordinate (the coordinate that measures the distance away from the horizon). In amplitude expansion, the black hole metric is parametrized by its components evaluated at the horizon, whereas in the case of derivative expansion, it is parametrized by the metric components evaluated at the AdS boundary expressed in terms of the variables of the dual fluid description.

The key part of this chapter is about a map between the points on the horizon and the points on the boundary. To define any such map we first need to set up coordinate systems on both horizon and the boundary. In this section first we shall briefly describe the two coordinate systems that are used to describe the entropy current on the horizon [37, 38] and the fluid dynamics living on the boundary [81, 125]. We shall refer to them as 'horizon adapted coordinates' and 'boundary coordinates' respectively.

Then in the final subsection we shall relate this two coordinates to get a point by point map from the horizon to the boundary.

4.2.1 Horizon adapted coordinate system

The entropy density and current, defined on the horizon are expressed in a very special choice of coordinates, tuned to the structure of null hypersurface. We shall denote this coordinate system as 'horizon adapted coordinate system'. In these coordinates the metric takes the following form

$$ds^2 = 2d\rho\, dv - \rho^2 X(\rho, v, \vec{\alpha})\, dv^2 + 2\rho\, \omega_i(\rho, v, \vec{\alpha})\, dv\, d\alpha^i + h_{ij}(\rho, v, \vec{\alpha})\, d\alpha^i d\alpha^j \tag{4.7}$$

where X, ω_i and h_{ij} are arbitrary nonzero functions of ρ, v and $\vec{\alpha} = \{\alpha^i\}$. In this metric, the horizon is located at the $\rho = 0$ hypersurface. At $\rho = 0$, the vector ∂_v is affinely patrametrized null generator of the horizon, with v being the affine parameter. ∂_i s are the spatial vectors on the constant v slices of the horizon. The entropy current is defined on the horizon and therefore could depend only on the metric functions X, ω_i and h_{ij} and their ∂_i and ∂_v derivatives.

In a stationary solution, the ρ and v dependence of the metric would be constrained. The functions X, ω_i and h_{ij} will only depend on the product of ρ and v. The stationary metric will be completely invariant under the transformation

$$v \to \lambda v, \quad \rho \to \frac{\rho}{\lambda}$$

While constructing the horizon entropy current, a departure from this invariance has been treated as the small parameter, characterizing the amplitude of the dynamics.

4.2.2 Boundary coordinates

In hydrodynamics, the local velocity of the field denoted as $u^\mu(x)$ is a special vector. While writing the dual metric, the most convenient choice of gauge turns out to be related to this velocity field. In this choice of gauge (with coordinates denoted as $\{r, y^\mu\}$), the metric takes the following general structure

$$
\begin{split}
&ds^2 = -2u_\mu\, dy^\mu\, dr + \chi_{\mu\nu}\, dy^\mu dy^\nu\\
&\chi_{\mu\nu} \text{ could be further decomposed as}\\
&\chi_{\mu\nu} \equiv S_1 u_\mu u_\nu + S_2 P_{\mu\nu} + (V_\mu u_\nu + V_\nu u_\mu) + \mathcal{T}_{\mu\nu}\\
\text{such that }\ & u^\mu V_\mu = u^\mu \mathcal{T}_{\mu\nu} = 0, \quad P_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu
\end{split}
\tag{4.8}
$$

Here $r \to \infty$ is the boundary, and the metric takes the form of Poincare patch AdS as we approach the boundary. Here also, we shall choose the origin of the r coordinate at the horizon. Therefore $r = 0$ is a null hypersurface by construction, which further implies

$$
G^{rr}(r=0) = 0 \ \text{ and } \ n^\mu \partial_\mu = G^{r\mu}(r=0)\partial_\mu \ \text{ is a null vector at the horizon}
$$

The vector $n^\mu = G^{r\mu}|_{r=0}$ must be identified with the null generator of the horizon (though not affinely parametrized).

Using the fact that the null generator of the horizon is just the dual vector of the one form dr or, in other words, $n^A G_{AB} = \delta^r_B$, we get the following identities for the n^μ vector, which would turn out to be useful at a later point.

$$
\begin{split}
&\delta^r_B = n^A G_{AB}|_{r=0} = n^\mu G_{\mu B}\\
\Rightarrow\ & n^\mu G_{\mu r} = -u_\mu n^\mu = 1\\
\Rightarrow\ & n^\mu G_{\mu\nu}|_{r=0} = n^\mu \chi_{\mu\nu}|_{r=0} = 0
\end{split}
\tag{4.9}
$$

S_1, S_2, V_μ and $\mathcal{T}_{\mu\nu}$ all are functions of r and y^μ , but the y^μ dependence is known only perturbatively where the perturbation parameters are the derivatives of the fluid variables. In fact the derivative expansion would be valid only when the fluid variables are slowly varying with respect to some scale, in this case, the temperature of the fluid. The more the number of derivatives, the more suppressed the terms are. [1]

[1]Note in [81, 83] the choice of gauge was quite different from the one we are using here. In case of fluid gravity correspondence, it makes sense to parametrize the metric in terms of fluid variables defined with respect to the boundary stress tensor. The horizon in the initial papers of fluid-gravity correspondence is not located at $r = 0$ but given by

4.2.3 The horizon to boundary map

The metric described in section 4.2.1 is in a completely different gauge than that of hydrodynamic metric in section 4.2.2. However, the construction of the horizon entropy current is very much tied to the choice of coordinates as given in 4.2.1. It is obvious that, to translate the horizon entropy current in terms of the fluid variables the first step would be to establish a dictionary between these two coordinate systems.

We shall transform the fluid metric (as given in eq:(4.8)) to the gauge described in section (4.2.1). This will allow us to describe metric functions (X, ω_i and h_{ij}) as they appeared in equation (4.7) in terms of the fluid variables (velocity and temperatures) and their appropriate derivatives.

In other words, we shall express r and x^μ as functions of $\{\rho, v, \vec{\alpha}\}$ such that the following gauge conditions are satisfied.

$$\begin{aligned}
G_{\rho\rho} = 0 \Rightarrow & -2u_\mu \left(\frac{\partial x^\mu}{\partial\rho}\right)\left(\frac{\partial r}{\partial\rho}\right) + \chi_{\mu\nu}\left(\frac{\partial x^\mu}{\partial\rho}\right)\left(\frac{\partial x^\nu}{\partial\rho}\right) = 0\\
G_{\rho v} = 1 \Rightarrow & -u_\mu\left[\left(\frac{\partial r}{\partial\rho}\right)\left(\frac{\partial x^\mu}{\partial v}\right) + \left(\frac{\partial r}{\partial v}\right)\left(\frac{\partial x^\mu}{\partial\rho}\right)\right] + \chi_{\mu\nu}\left[\left(\frac{\partial x^\nu}{\partial\rho}\right)\left(\frac{\partial x^\mu}{\partial v}\right)\right] = 1\\
G_{\rho\alpha_i} = 0 \Rightarrow & -u_\mu\left[\left(\frac{\partial r}{\partial\rho}\right)\left(\frac{\partial x^\mu}{\partial\alpha_i}\right) + \left(\frac{\partial r}{\partial\alpha_i}\right)\left(\frac{\partial x^\mu}{\partial\rho}\right)\right] + \chi_{\mu\nu}\left[\left(\frac{\partial x^\nu}{\partial\rho}\right)\left(\frac{\partial x^\mu}{\partial\alpha_i}\right)\right] = 0
\end{aligned} \tag{4.11}$$

Now it is difficult to solve these equations exactly, even in just the radial coordinate. However, for our entropy current, it is enough to have the near horizon structure of the metric. So we shall be solving the gauge conditions (4.11) in an expansion in ρ.

$r = r_H(y^\mu)$ whose value is related to the local temperature of the dynamical black brane being considered. We can translate between these two gauges by a simple shift of r coordinate

$$r \to r + r_H(y^\mu)$$

This step adds a little modification to the fluid metric without affecting its general structure. The net result of this shift is just a shift in $\chi_{\mu\nu}$ as follows

$$\chi_{\mu\nu} \to \chi_{\mu\nu} - (u_\mu\partial_\nu + u_\nu\partial_\mu)r_H \tag{4.10}$$

In our solution r_H will simply be length scale, with respect to which the slow variation or the derivative expansion is defined.

We shall take the following ansatz for the coordinate transformations:

$$
\begin{aligned}
r &= \rho\, r_{(1)}(v,\alpha_i) + \rho^2\, r_{(2)}(v,\alpha_i) + \cdots \\
x^\mu &= x^\mu_{(0)}(v,\alpha_i) + \rho\, x^\mu_{(1)}(v,\alpha_i) + \rho^2\, x^\mu_{(2)}(v,\alpha_i) + \cdots
\end{aligned}
\tag{4.12}
$$

In the above coordinate transformation the functions $x^\mu_{(0)}(v,\alpha^i)$ will be effectively taken as input functions. All the rest, namely $\{x^\mu_{(n)}(v,\alpha^i)\}$ and $r_{(n)}(v,\alpha^i)$ will be determined in terms of the functions $x^\mu_{(0)}(v,\alpha_i)$. In Appendix B.2 we have determined the first few coefficients of the above transformation equations (equation (4.12)).

Note that the input functions $x^\mu_{(0)}(v,\alpha_i)$ are not entirely free. The vector $t^\mu \equiv \left(\frac{\partial x^\mu_{(0)}}{\partial v}\right)$ must be an affinely parametrized null geodesic with respect to the full metric.

Let us define the following set of vectors that are tangent to the horizon

$$
t^\mu \equiv \left(\frac{\partial x^\mu_{(0)}}{\partial v}\right), \quad l^\mu_i \equiv \left(\frac{\partial x^\mu_{(0)}}{\partial \alpha_i}\right)
$$

t^μ, being the null generator of the horizon, is also a normal to the horizon.

Hence it follows that t^μ must be proportional to n^μ of the fluid metric we defined in the previous step. In other words

$$
t^\mu = e^\phi n^\mu = e^\phi G^{r\mu}(r=0)
$$

where ϕ is a scalar function of $\{x^\mu\}$ so that t^μ becomes a affinely parametrized null geodesic. Processing this condition we get the following equation for the field $\phi(x)$

$$
(n\cdot\partial)\phi = \left(\frac{n^\mu n^\nu}{2}\right)\left[\partial_r \chi_{\mu\nu}\right]_{r=0}
\tag{4.13}
$$

Note that the RHS of equation (4.13) is nonzero even at zeroth order in derivative. Therefore, it is not ϕ but its derivative along the direction of n^μ that satisfies the derivative expansion. At this stage we are free to choose the dependence of ϕ along the directions perpendicular to n^μ.

Now ϕ is an external scalar field from the perspective of boundary fluid dynamics and generically the fluid entropy current would depend on the choice of ϕ. We should be able to choose ϕ in a way so that the final fluid entropy current is entirely expressible in terms of the fluid variables like velocity and temperature only.

4.3 Translating the horizon current to the boundary current

In this section, we shall find out an abstract expression for entropy current J^μ in the boundary such that

$$D_\mu J^\mu(j^v, j^i) = \frac{1}{\sqrt{h}}\partial_v(\sqrt{h}\, j^v) + \nabla_i j^i \tag{4.14}$$

where j^v and j^i are defined in equation (4.1). Here the RHS of the above equation is written in the horizon adapted coordinates whereas the LHS is in terms of the boundary coordinates. D_μ denotes the covariant derivative with respect to the boundary metric. In the first subsection, we shall describe how to determine J^μ, given j^v and j^i. The final expression for J^μ turns out to be the following

$$J^\mu = \frac{1}{\sqrt{g^{(b)}}}\sqrt{H}\left(j^v t^\mu + j^i l_i^\mu\right) \text{ where } H \equiv \frac{\tilde{t}_\mu \tilde{t}_\nu \epsilon^{\mu\mu_1\mu_2\cdots\mu_n}\epsilon^{\nu\nu_1\nu_2\cdots\nu_n}\chi_{\mu_1\nu_1}\cdots\chi_{\mu_n\nu_n}}{n!} \tag{4.15}$$

Here j^v and j^i have to be read off from the expression of the horizon current. t^μ, $\tilde{t}_\mu$ and l_i^μ are vectors related to the map.

$$t^\mu \equiv \frac{\partial x^\mu}{\partial v}, \quad \tilde{t}_\mu \equiv \frac{\partial v}{\partial x^\mu}, \quad l_i^\mu \equiv \frac{\partial x^\mu}{\partial \alpha^i}, \quad l_\mu^i \equiv \frac{\partial \alpha^i}{\partial x^\mu}$$

Using the fact that $t^\mu = e^\phi n^\mu$ and $t^\mu \tilde{t}_\mu = 1$ the expression for H and current could be simplified further

$$\begin{aligned}
J^\mu &= \frac{1}{\sqrt{g^{(b)}}}\frac{\sqrt{H}}{\sqrt{t^\alpha t^\beta g^{(b)}_{\alpha\beta}}}\left(j^v t^\mu + j^i l_i^\mu\right) \\
H &\equiv \frac{\hat{n}_\mu \hat{n}_\nu \epsilon^{\mu\mu_1\mu_2\cdots\mu_n}\epsilon^{\nu\nu_1\nu_2\cdots\nu_n}\chi_{\mu_1\nu_1}\cdots\chi_{\mu_n\nu_n}}{n!} \\
\hat{n}^\mu &= \frac{n^\mu}{\sqrt{n^\alpha n^\beta g^{(b)}_{\alpha\beta}}} = \frac{t^\mu}{\sqrt{t^\alpha t^\beta g^{(b)}_{\alpha\beta}}}
\end{aligned} \tag{4.16}$$

4.3.1 Constructing J^μ

In this subsection, we shall determine an algorithm to determine J^μ out of j^v and j^i. The key issue here is to re-express the entropy production formula on the horizon (i.e., the expression in the RHS of equation (4.14)) as a divergence of a current covariant with respect to the boundary coordinates. It turns out that if we could rewrite the equation (4.14) in a 'metric independent' language using n and $(n-1)$ forms, it helps to identify the J^μ.

Let us first define the following two n-forms.

$$
\begin{split}
J_{temp} &\equiv \sqrt{h}\; j^v \; \frac{\epsilon_{i_1 i_2 \cdots i_n} d\alpha^{i_1} \wedge d\alpha^{i_2} \cdots \wedge d\alpha^{i_n}}{n!} \\
J_{space} &\equiv -\sqrt{h}\; j^k \; \frac{\epsilon_{k i_2 i_3 \cdots i_n} dv \wedge d\alpha^{i_2} \wedge d\alpha^{i_3} \cdots \wedge d\alpha^{i_n}}{(n-1)!}
\end{split}
\tag{4.17}
$$

Here, $\epsilon_{i_1 i_2 \cdots i_n}$ is the completely antisymmetric n indexed tensor with each component equal to 0 or ± 1.

One could show that the exterior derivative of $(J_{temp} + J_{space})$ is proportional to the top form on the horizon, where the proportionality constant is the RHS of equation (4.14).

$$
d(J_{temp} + J_{space}) = \left[\partial_v \left(\sqrt{h}\; j^v\right) + \sqrt{h}\; \nabla_i j^i\right] \left[\frac{\epsilon_{i_1 i_2 \cdots i_n} dv \wedge d\alpha^{i_1} \wedge d\alpha^{i_2} \cdots \wedge d\alpha^{i_n}}{n!}\right] \tag{4.18}
$$

Here d denotes the exterior derivative.

Now we shall rewrite J_{temp} and J_{space} in terms of the boundary coordinates using the fact that

$$
dv = \tilde{t}_\mu dx^\mu, \quad d\alpha^i = l^i_\mu dx^\mu
$$

We need to use the following identities.

$$
\begin{split}
& l^{i_1}_{\mu_1} \cdots l^{i_n}_{\mu_n} \epsilon_{i_1 i_2 \cdots i_n} = \Delta\; t^\mu \; \epsilon_{\mu \mu_1 \cdots \mu_n} \\
& l^{\mu_1}_{i_1} \cdots l^{\mu_n}_{i_n} \epsilon^{i_1 i_2 \cdots i_n} = \left(\frac{1}{\Delta}\right) \; \tilde{t}_\mu \; \epsilon^{\mu \mu_1 \cdots \mu_n} \\
& \epsilon_{\mu \mu_1 \mu_2 \cdots \mu_n} \left(\frac{dx^\nu \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_n}}{n!}\right) = \delta^\nu_\mu \epsilon_{\mu_1 \cdots \mu_{n+1}} \left(\frac{dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n+1}}}{(n+1)!}\right) \\
& \epsilon_{\mu \alpha \mu_2 \cdots \mu_n} \left(\frac{dx^\nu \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_n}}{(n-1)!}\right) = \left(\delta^\nu_\alpha \epsilon_{\mu \mu_1 \cdots \mu_n} - \delta^\nu_\mu \epsilon_{\alpha \mu_1 \cdots \mu_n}\right) \left(\frac{dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}}{n!}\right)
\end{split}
\tag{4.19}
$$

Here Δ is the Jacobian of the coordinate transformation

$$
\Delta = det\left[\frac{\partial\{v, \alpha^i\}}{\partial\{x^\mu\}}\right], \quad \frac{1}{\Delta} = det\left[\frac{\partial\{x^\mu\}}{\partial\{v, \alpha^i\}}\right],
$$

First, we shall write an expression for $\sqrt{h}$ in terms of the boundary coordinates.

$$
\begin{aligned}
h = det[h_{ij}] &= \frac{\epsilon^{i_1\cdots i_n}\epsilon^{j_1\cdots j_n}h_{i_1j_1}\cdots h_{i_nj_n}}{n!} \\
&= \left(\frac{1}{n!}\right)\epsilon^{i_1\cdots i_n}\epsilon^{j_1\cdots j_n}\left[l^{\mu}_{i_1}\cdots l^{\mu_n}_{i_n}\right]\left[l^{\nu_1}_{j_1}\cdots l^{\nu_n}_{j_n}\right]\chi_{\mu_1\nu_1}\cdots\chi_{\mu_n\nu_n} \\
&= \left(\frac{1}{n!}\right)\left(\frac{1}{\Delta}\right)^2\tilde{t}_\mu\tilde{t}_\nu\,\epsilon^{\mu\mu_1\cdots\mu_n}\epsilon^{\nu\nu_1\cdots\nu_n}\,\chi_{\mu_1\nu_1}\cdots\chi_{\mu_n\nu_n}
\end{aligned}
\tag{4.20}
$$

Using the above identities we process both J_{temp} and J_{space} as follows.

$$
\begin{aligned}
J_{temp} &= \sqrt{h}\,j^v\left(\frac{\epsilon_{i_1\cdots i_n}l^{i_1}_{\mu_1}\cdots l^{i_n}_{\mu_n}}{n!}\right)dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} \\
&= \sqrt{h}\Delta\,j^v\left(\frac{t^\mu\epsilon_{\mu\mu_1\cdots\mu_n}}{n!}\right)dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} \\
&= \left[\frac{\tilde{t}_\alpha\tilde{t}_\beta\,\epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\,\chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}}j^v\left(\frac{t^\mu\epsilon_{\mu\mu_1\cdots\mu_n}}{n!}\right)dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
J_{space} &= -\sqrt{h}\,j^k\left(\frac{\epsilon_{ki_2i_3\cdots i_n}}{(n-1)!}\right)\tilde{t}_\mu l^{i_2}_{\mu_2}\cdots l^{i_n}_{\mu_n}\,dx^\mu\wedge dx^{\mu_2}\cdots\wedge dx^{\mu_n} \\
&= -\sqrt{h}\,\left(j^kl^\nu_k\right)\left(\frac{\epsilon_{i_1i_2i_3\cdots i_n}}{(n-1)!}\right)\tilde{t}_\mu l^{i_1}_\nu l^{i_2}_{\mu_2}\cdots l^{i_n}_{\mu_n}\,dx^\mu\wedge dx^{\mu_2}\cdots\wedge dx^{\mu_n} \\
&= -\sqrt{h}\,\left(j^kl^\nu_k\right)\tilde{t}_\mu\left(\frac{\Delta}{(n-1)!}\right)t^\alpha\epsilon_{\alpha\nu\mu_2\cdots\mu_n}\,dx^\mu\wedge dx^{\mu_2}\cdots\wedge dx^{\mu_n} \\
&= -\sqrt{h}\left(\frac{\Delta}{n!}\right)\left(j^kl^\nu_k\right)\tilde{t}_\mu t^\alpha\left[\delta^\mu_\nu\epsilon_{\alpha\mu_1\mu_2\cdots\mu_n}-\delta^\mu_\alpha\epsilon_{\nu\mu_1\mu_2\cdots\mu_n}\right]dx^{\mu_1}\wedge dx^{\mu_2}\cdots\wedge dx^{\mu_n} \\
&= \left[\frac{\tilde{t}_\alpha\tilde{t}_\beta\,\epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\,\chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}}\left(\frac{(j^kl^\mu_k)\,\epsilon_{\mu\mu_1\cdots\mu_n}}{n!}\right)dx^{\mu_1}\wedge dx^{\mu_2}\cdots\wedge dx^{\mu_n}
\end{aligned}
\tag{4.22}
$$

So finally we have

$$
\begin{aligned}
&J_{temp}+J_{space} \\
&= \left[\frac{\tilde{t}_\alpha\tilde{t}_\beta\,\epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\,\chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}}\left[j^vt^\mu+(j^kl^\mu_k)\right]\frac{\epsilon_{\mu\mu_1\cdots\mu_n}}{n!}\,dx^{\mu_1}\wedge dx^{\mu_2}\cdots\wedge dx^{\mu_n}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow d(J_{temp}+J_{space}) \\
&= \partial_\mu\Bigg(\left[\frac{\tilde{t}_\alpha\tilde{t}_\beta\,\epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\,\chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}}\left[j^vt^\mu+(j^kl^\mu_k)\right]\Bigg)\left(\frac{\epsilon_{\nu_1\cdots\nu_{n+1}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n+1}}}{(n+1)!}\right)
\end{aligned}
\tag{4.23}
$$

Now from equation (4.18) we know the expression of $d(J_{temp} + J_{space})$ in terms of $\{v, \alpha^i\}$ co-ordinate system. If we rewrite the $(n+1)$ form that appears in equation (4.18) in terms of $\{x^\mu\}$ coordinates we get the following

$$\begin{aligned}
&\epsilon_{i_1\cdots i_n}\left(\frac{dv \wedge d\alpha^{i_1}\cdots d\alpha^{i_n}}{n!}\right)\\
&=\tilde{t}_\nu\, l^{i_1}_{\mu_1}\cdots l^{i_n}_{\mu_n}\epsilon_{i_1\cdots i_n}\left(\frac{dx^\nu \wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}}{n!}\right)\\
&=\Delta\,\tilde{t}_\nu\, t^\mu \epsilon_{\mu\mu_1\cdots\mu_n}\left(\frac{dx^\nu \wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}}{n!}\right)\\
&=\Delta\,\tilde{t}_\nu\, t^\mu \delta^\nu_\mu \epsilon_{\mu_1\mu_2\cdots\mu_{n+1}}\left(\frac{dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n+1}}}{(n+1)!}\right)\\
&=\Delta\,\epsilon_{\mu_1\mu_2\cdots\mu_{n+1}}\left(\frac{dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n+1}}}{(n+1)!}\right)
\end{aligned}\tag{4.24}$$

Substituting equation (4.24) in equation (4.18) and then comparing with equation (4.23) we find

$$\Delta\left[\partial_v\left(\sqrt{h}\, j^v\right)+\sqrt{h}\,\nabla_i j^i\right]=\partial_\mu\left(\left[\frac{\tilde{t}_\alpha \tilde{t}_\beta\, \epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\,\chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}}\left[j^v t^\mu+(j^k l^\mu_k)\right]\right)\tag{4.25}$$

As we have discussed before, in dynamical black holes, the expression $\left[\partial_v\left(\sqrt{h}\, j^v\right)+\sqrt{h}\,\nabla_i j^i\right]$ is identified with net entropy production in every infinitesimal subregion of the horizon and, up to the linear order in the amplitude of the dynamics, it must vanish (if it does not, then the same expression at linear order, will lead to both entropy production and destruction depending on the sign of the amplitude and thus violating the second law). Since Δ, the Jacobian of the coordinate transformation is non-vanishing everywhere, we conclude

$$\partial_\mu\left(\left[\frac{\tilde{t}_\alpha \tilde{t}_\beta\, \epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\,\chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}}\left[j^v t^\mu+(j^k l^\mu_k)\right]\right)=0 \quad \text{(up to terms nonlinear in amplitude)}$$

Now we can turn the above expression into a divergence of current covariant (i.e., in the form of equation (4.14)) with respect to the boundary metric, if we identify the boundary entropy current as

$$J^\mu=\frac{1}{\sqrt{g^{(b)}}}\left(\left[\frac{\tilde{t}_\alpha \tilde{t}_\beta\, \epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\,\chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}}\left[j^v t^\mu+(j^k l^\mu_k)\right]\right)\tag{4.26}$$

where $\left[g^{(b)}_{\mu\nu} = \lim_{r\to\infty}\left(\frac{\chi_{\mu\nu}}{r^2}\right)\right]$ is the boundary metric and $g^{(b)} = det[g^{(b)}_{\mu\nu}]$.

Equation (4.26) is one of our key results. Now a couple of comments about this formula.

- J^μ is a covariant vector in the boundary spacetime with boundary metric $g^{(b)}_{\mu\nu}$, provided we treat t^μ, l^μ_k and $\tilde{t}_\mu$ as independent upper and lower index vectors respectively.
- Though we have said that t^μ is the affinely parametrized null generator on the horizon expressed in terms of boundary coordinates, the analysis in this section nowhere used the affineness of the v parameter. So equation (4.26) is valid even when v is not an affine parameter, but it has to be a parameter along the null generator[2] .
- The expressions for j^v and j^i depend on the details of the equation of motion in higher derivative theory, which in turn depend on the affine parametrization of the null generators.
- j^v and j^i could be determined in terms of the functions appearing in metric (4.7) (i.e., X, ω_i and h_{ij}) and their appropriate derivatives. Using this horizon to boundary map, we could re-express j^v and j^i in terms of the fluid variables and the mapping vectors t^μ and l^μ_i.
- From the perspective of boundary fluid, t^μ, l^μ_k or $\tilde{t}_\mu$ are external variables. So the entropy current described in equation (4.26) would be a genuine fluid entropy current provided our mapping functions are such that the vectors t^μ, l^μ_k or $\tilde{t}_\mu$ are either constants or are determined entirely in terms of fluid variables.

4.3.2 Entropy current in boundary fluid dual to Einstein gravity

In Einstein gravity, the entropy on the horizon is simply given by the area of the spatial sections of the horizon. In our choice of horizon-adapted coordinate system, it is the square root of the

[2]For example, in [85] the null generators are parametrized using the boundary time-like coordinate v. This is not an affine parametrization, but still will we could apply our formula to recover the expression of entropy current derived in [85]. We have to use the following facts. In two derivative theories $j^v = 1, \quad j^i = 0$ and the choice of map in [85] is such that $\tilde{t}_\mu dx^\mu = dv, \quad t^\mu = \frac{n^\mu}{n^v}$. The boundary metric $g^{(b)}_{\mu\nu} = \eta_{\mu\nu}$.

determinant of h_{ij}. It follows

$$j^v_{(2)} = 1, \quad j^i_{(2)} = 0$$

where the subscript (2) denotes the fact that it is for a two derivative theory of gravity. Substituting it in equation (4.26), we get the following expression for the boundary entropy current for two derivative theory.

$$J^\mu_{(2)} = \frac{1}{\sqrt{g^{(b)}}}\left[\frac{\tilde{t}_\alpha \tilde{t}_\beta \; \epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n} \; \chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}} t^\mu$$

In the above expression, the vector fields $\tilde{t}_\mu$ and t^μ appear. They depend on our choice of mapping and naively, it seems that even in two derivative theories of gravity, the boundary entropy current might not admit a description in terms of fluid variables. But in this section, we would like to argue that this is not the case; all the factors that might not admit a derivative expansion or fluid description cancel between t^μ and $\tilde{t}_\mu$, and we could rewrite $J^\mu_{(2)}$ entirely in terms of fluid variables.

Note, $t_\mu = g^{(b)}_{\mu\nu} t^\nu$ and $\tilde{t}_\mu$ could be viewed as two vectors on the boundary with the following inner products with respect to the boundary metric

$$t_\mu \tilde{t}_\nu \; [g^{(b)}]^{\mu\nu} = 1$$

whereas $t^\mu = e^\phi G^{\mu r}$ is a time-like vector with respect to the boundary metric[3]. Define the unit vector along the direction of t^μ as follows

$$\hat{n}^\mu \equiv \frac{t^\mu}{||t||} = \frac{n^\mu}{||n||}, \quad \text{where} \quad ||t|| \equiv \sqrt{-t^\mu t^\nu g^{(b)}_{\mu\nu}}, \;\; ||n|| \equiv \sqrt{-n^\mu n^\nu g^{(b)}_{\mu\nu}} \;\; \text{and} \;\; n^\mu = G^{\mu r}(r=0)$$

We can always decompose the vector $\tilde{t}_\mu$ in the following way

$$\tilde{t}_\mu = \frac{\hat{n}_\mu}{||t||} + V_\mu = g^{(b)}_{\mu\nu}\left(\frac{\hat{n}^\nu}{||t||}\right) + V_\mu, \quad \text{such that} \;\; V_\nu t^\nu = 0 \tag{4.27}$$

Now $\chi_{\mu\nu}$ on the horizon satisfies the following identity $t^\mu \chi_{\mu\nu} = 0$. So both indices of $\chi_{\mu\nu}$ are in the directions perpendicular to t^μ or $\hat{n}^\mu$. It follows that in the tensor

$$A^{\alpha\beta} \equiv \left[\frac{\epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n} \; \chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]$$

[3] This is because at leading order in derivative expansion $t^\mu = e^\phi G^{\mu r} = e^\phi u^\mu + \mathcal{O}(\partial)$. Now u^μ is a time like vector and derivative corrections can never change the sign of the leading order result

all the indices $\{\alpha_i\}$ and $\{\beta_i\}$ in the Levi Cevita tensors are contracted with vectors perpendicular to $\hat{n}^\mu$. Hence, $A^{\alpha\beta}$ will be non-zero only when both of its free indices are projected along the direction of $\hat{n}$. In other words, $V_\alpha A^{\alpha\beta} = V_\beta A^{\alpha\beta} = 0$, where V_μ is defined in equation (4.27). Therefore

$$
\begin{aligned}
\tilde{t}_\alpha \tilde{t}_\beta A^{\alpha\beta} &= \frac{1}{||t||^2}\hat{n}_\alpha \hat{n}_\beta A^{\alpha\beta} \\
\Rightarrow \quad J^\mu_{(2)} &= \frac{1}{\sqrt{g^{(b)}}}\left[\frac{\tilde{t}_\alpha \tilde{t}_\beta\, \epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\, \chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}} t^\mu \\
&= \frac{1}{\sqrt{g^{(b)}}}\left[\frac{\hat{n}_\alpha \hat{n}_\beta\, \epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\, \chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}} \frac{t^\mu}{||t||} \\
&= \frac{1}{\sqrt{g^{(b)}}}\left[\frac{\hat{n}_\alpha \hat{n}_\beta\, \epsilon^{\alpha\alpha_1\cdots\alpha_n}\epsilon^{\beta\beta_1\cdots\beta_n}\, \chi_{\alpha_1\beta_1}\cdots\chi_{\alpha_n\beta_n}}{n!}\right]^{\frac{1}{2}} \hat{n}^\mu
\end{aligned}
\tag{4.28}
$$

Note $\hat{n}^\mu = \frac{n^\mu}{||n||}$ could be entirely expressed in terms of fluid variables and the boundary metric and therefore admit derivative expansion. Equation (4.28) is a manifestly covariant entropy current for the boundary fluid dual to Einstein gravity, which always admits a derivative expansion. After we know that the horizon current will translate into such a covariant 'hydro-like' expression for the boundary current, we are free to choose any kind of coordinates and mapping. Even if our choice breaks all the symmetries, the final result is guaranteed to be a covariant entropy current for the dual fluid theory.

4.3.3 Entropy current in higher derivative theories

In this subsection, we would like to contrast the previous description with the scenario in higher derivative theories. In higher derivative theories, j^v and j^i have non-trivial structures constructed out of the metric functions $(X,\ \omega_i,\ h_{ij})$ and their derivatives in the horizon-adapted coordinates. The details of these structures will depend on the particulars of the higher derivative equations of motion. As we have seen before, once translated to boundary coordinates, the entropy current, in general, will be a vector function of both the fluid variables and the mapping variables.

But unlike in two derivative theories where the entropy density (as given by $\sqrt{h}$) is invariant under any reparametrization of the null generators, here affine parametrization is crucial for the construction of j^v and j^i. This probably indicates that in a higher derivative theory, we would not be able to rearrange the formula for boundary entropy current to completely eliminate the dependence on the mapping like we have done in Einstein gravity. So, here the key question turns out to be whether there exists a choice of horizon to boundary map that allows us to express the final fluid entropy current entirely in terms fluid variables, without any explicit dependence on boundary coordinates (any arbitrary map, generically not compatible with derivative expansion will lead to such explicit dependence on boundary coordinates). Further, given the non-universality of the structures appearing in j^v and j^i it is unlikely that we would be able the answer this question in a universal way - a single map will not work for entropy current in all higher derivative theories. However, the following simplification could be predicted on a general ground.

- The final fluid entropy current J^μ will not have any free 'i' index (the spatial indices in the horizon adapted coordinates). Therefore, all the $l^\mu_i = \left(\frac{\partial x^\mu}{\partial \alpha^i}\right)$ must be contracted with the inverse mapping $l^i_\mu = \left(\frac{\partial \alpha^i}{\partial x^\mu}\right)$, which are the only sources of i indices in J^μ). Now

$$l^\mu_i l^i_\nu = \left(\frac{\partial x^\mu}{\partial \alpha^i}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right) = \delta^\mu_\nu - \left(\frac{\partial x^\mu}{\partial v}\right)\left(\frac{\partial v}{\partial x^\nu}\right) = \delta^\mu_\nu - t^\mu \tilde{t}_\nu$$

  So finally, all the dependence on the mapping functions could be transferred to the dependence on t^μ and $\tilde{t}_\mu$.

- t^μ could be written as $t^\mu = e^\phi n^\mu$, and it is the scalar function ϕ that does not admit a derivative expansion. So from the fluid point of view, the two scalar functions $\phi(x^\mu)$ and $v(x^\mu)$ could spoil the 'fluid nature' of the boundary entropy current.

- The variations of these scalars along the direction of n^μ are constrained.

$$(n \cdot \partial)\phi = \left(\frac{n^\mu n^\nu}{2}\right)\left[\partial_r \chi_{\mu\nu}\right]_{r=0}, \qquad (t \cdot \partial)v = 1 \;\Rightarrow\; (n \cdot \partial)v = e^{-\phi} \tag{4.29}$$

  Once we 'choose' these scalars on a given slice perpendicular to n^μ, these equations will fix their subsequent evolution along the n^μ directions.

- From the two equations in (4.29), we could solve v in terms of ϕ perturbatively using derivative expansion. This could be done as follows.

  Define $L \equiv e^{\phi} v$. Then the equation for L turns out to be

$$(n \cdot \partial) L - L (n \cdot \partial)\phi = 1$$

  Assume L admits a derivative expansion and could be expressed entirely in terms of fluid variables, with the leading terms having zero derivatives. Since we already know that $(n{\cdot}\partial)\phi$ starts from zeroth order, it follows that $(n{\cdot}\partial)L$ - the first term in the above equation is actually subleading in terms of derivative expansion. This allows us to solve the equation recursively generating the following infinite series

$$v \equiv e^{-\phi} L = e^{-\phi} \sum_{k=0}^{\infty} L_{(k)}, \text{ where } L_{(k)} = \left[\frac{(n \cdot \partial) L_{(k-1)}}{(n \cdot \partial)\phi}\right], \quad L_{(0)} = -\left(\frac{1}{(n \cdot \partial)\phi}\right) \tag{4.30}$$

  Note that this solution implies a very particular choice for the $v = 0$ slice of the horizon; it is the spatial slice where $\phi \to \infty$ Using equation (4.30) we could express $\tilde{t}_\mu$ in terms of $\partial_\mu \phi$.

$$\tilde{t}_\mu = e^{-\phi}\left(-L\, \partial_\mu \phi + \partial_\mu L\right) \tag{4.31}$$

- $\tilde{t}_\mu$ must satisfy the condition $l_i^\mu \tilde{t}_\mu = 0$ for every i index (coordinates along the spatial section of the horizon)

$$\begin{aligned} 0 = l_i^\mu \tilde{t}_\mu &= e^{-\phi}\left(L\, l_i^\mu \partial_\mu \phi + l_i^\mu \partial_\mu L\right) \\ \Rightarrow l_i^\mu \partial_\mu \phi &= \frac{l_i^\mu \partial_\mu L}{L} \end{aligned} \tag{4.32}$$

  Now we have seen that L satisfies derivative expansion with the leading term being zeroth order in derivatives. So from equation (4.32) we could infer that the variation of ϕ along the α^i directions also satisfies derivative expansion with the leading term being of first order.

Naively it seems that (4.32) is not consistent because $(t \cdot \partial)$ and $(l_i \cdot \partial)$ must commute; from equation (4.32), it follows that $(t \cdot \partial)(l_i \cdot \partial)\phi$ is a second order term whereas $(l_i \cdot \partial)(t \cdot \partial)\phi$

looks like a first order term since $(t \cdot \partial)\phi$ is of zeroth order. However, we could show that the first order piece in $(l_i \cdot \partial)(t \cdot \partial)\phi$ vanishes once we apply (4.32).

$$
\begin{aligned}
& e^{-\phi}\left(l_i^\mu \partial_\mu\right)(t \cdot \partial)\phi \\
& = e^{-\phi}\left(l_i^\mu \partial_\mu\right)\left(e^{\phi}\, n \cdot \partial\right)\phi \\
& = \left(l_i^\mu \partial_\mu \phi\right)(n \cdot \partial\phi) + \left(l_i^\mu \partial_\mu\right)(n \cdot \partial\phi) \\
& = \left(\frac{l_i^\mu \partial_\mu L_{(0)}}{L_{(0)}}\right)(n \cdot \partial\phi) + \left(l_i^\mu \partial_\mu\right)(n \cdot \partial\phi) + \mathcal{O}\left(\partial^2\right) \\
& = \mathcal{O}\left(\partial^2\right)
\end{aligned}
\tag{4.33}
$$

In the last line, we have used equation (4.30)for the expression of $L_{(0)}$.

- It turns out that the overall factors of e^{ϕ} finally get canceled between t^μ, $\tilde{t}_\mu$ and $\sqrt{h}$. We could see it as follows.

  The factors of e^{ϕ} in j^v or j^i are determined by their boost weight. Since j^v has zero boost weight, once translated into boundary coordinates, it will not have any factor of e^{ϕ}, whereas j^i having boost weight one, will carry a single factor of e^{ϕ}. We have already seen $\sqrt{h}$, expressed in terms of boundary coordinates, carries a factor of $e^{-\phi}$ from the $||t||$ factor in the denominator (see equation (4.28)). Hence in the expression $\sqrt{h}\left(j^v t^\mu + l_i^\mu j^i\right)$ all factors of overall e^{ϕ} cancel.

  Therefore, once we fix v in terms of ϕ using equation (4.30), the 'non-fluid' function remaining in our construction is the derivative of ϕ along the directions perpendicular to n^μ .

4.4 Entropy current in Einstein-Gauss-Bonnet theory

In this section, we shall specialize to Einstein-Gauss-Bonnet theory. The entropy density and the entropy current for black holes in Einstein-Gauss-Bonnet theory have been worked out in [37,38]. Using the horizon to boundary map, we shall rewrite the current in boundary coordinates. At this stage, we shall not use any derivative or amplitude expansion. We shall see that the final expressions will explicitly depend on the 'non-fluid' variables through $\tilde{t}_\mu$ and $\partial_\mu \phi$. Note that any term or factor that could be expressed as a product of metric components in boundary coordinates and their boundary derivatives are fluid variables. For example, the Christoffel symbols with respect to the bulk metric in boundary coordinates are always fluid variables.

In the end, we shall substitute the details of the bulk metric in Gauss-Bonnet theory dual to hydrodynamics. Since the metric is known up to the first order in derivative expansion, the boundary entropy current thus generated will also be correct only up to the first order. As mentioned before, up to this order, the entropy current will turn out to be trivial; it is simply equal to what it was for Einstein gravity. All the new terms generated by Gauss-Bonnet Action contribute to the boundary entropy current only in the second order.

4.4.1 j^v and j^i in terms of 'fluid' and 'non fluid' data

We shall first quote the expression for entropy density and the spatial entropy current for black holes in Gauss-Bonnet theory as given in [37, 38].

The final form of the entropy density and spatial entropy current density particular to Gauss-Bonnet theory is given as follows.

$$j^v = \sqrt{h}\left(1 + 2\alpha^2 \mathcal{R}\right), \quad j^i = -4\alpha^2 \left(\nabla_j K^{ij} - \nabla^i K\right) \tag{4.34}$$

where

$$\begin{aligned}
&h = \text{determinant of } h_{ij} \\
&\mathcal{R} = \text{ intrinsic curvature evaluated w.r.t the } h_{ij} \\
&\nabla_i = \text{covariant derivative with respect to } h_{ij} \\
&K_{ij} \equiv \frac{1}{2}\partial_v h_{ij}, \quad K \equiv h^{ij} K_{ij} \\
&\text{Lowering or raising of indices are done w.r.t } h_{ij} \text{ with } h^{ij} \text{ being the inverse}
\end{aligned} \tag{4.35}$$

One key simplifying factor here is that neither j^v nor j^i needs any information about how the horizon data changes as one moves away from the horizon or, more precisely, the r derivatives of the metric functions. This, in turn, implies that to evaluate the current, we need only the leading coefficients in the coordinate transformation as described in equation (4.12). In the previous subsection, we have already determined the expression for $\sqrt{h}$ in terms of fluid data. In this subsection, we shall compute $\nabla_k K_{ij}$ with appropriate index contractions for $j^i l_i^\mu$ and $\mathcal{R}$ for j^v

Extrinsic curvature and its covariant derivatives

The extrinsic curvature is defined as $K_{ij} = \frac{1}{2}\partial_v h_{ij}|_{r=0}$. On the horizon, the $r = 0$ hypersurface, h_{ij} is simply related to $\chi_{\mu\nu}$.

$$h_{ij} = l_i^\mu l_j^\nu \chi_{\mu\nu} \tag{4.36}$$

Here we have used the fact that $\left(\frac{\partial\rho}{\partial\alpha^i}\right)$ vanishes on the horizon. Now, using the fact that $\partial_v = t \cdot \partial$, we could determine K_{ij} as

$$K_{ij} = l_i^\mu l_j^\nu \mathcal{K}_{\mu\nu} \quad \text{where} \quad \mathcal{K}_{\mu\nu} = -t^\alpha \Gamma_{\alpha,\mu\nu} \tag{4.37}$$

Here we have used the fact that

$$(t \cdot \partial) l_i^\mu = (l_i \cdot \partial) t^\mu, \quad \text{and} \quad \chi_{\mu\nu}(l_i \cdot \partial) t^\mu = -t^\mu (l^i \cdot \partial)\chi_{\mu\nu}$$

Now we have to compute its covariant derivative. The following structure would prove useful for our computation. Note, for any boundary tensor with lower $\{\mu, \nu\}$ indices, we could define the following horizon tensor with $\{i, j\}$ indices

$$T_{i_1 i_2 \cdots i_n} = l_{i_1}^{\mu_1} l_{i_2}^{\mu_2} \cdots l_{i_n}^{\mu_n} \, \mathcal{T}_{\mu_1 \mu_2 \cdots \mu_n}$$

Now it turns out that the covariant derivative of the above tensor $\nabla_j T_{i_1 i_2 \cdots i_n}$ also has a similar expression in terms of $\{\mu, \nu\}$ indices of the boundary coordinates. We could write it in the following way

$$\nabla_j T_{i_1 i_2 \cdots i_n} = l_j^\nu l_{i_1}^{\mu_1} l_{i_2}^{\mu_2} \cdots l_{i_n}^{\mu_n} \, [\mathcal{D}_\nu \mathcal{T}_{\mu_1 \mu_2 \cdots \mu_n}] \tag{4.38}$$

where $\mathcal{D}_\nu$ is a new covariant derivative with its connection defined as

$$\begin{aligned}\tilde{\Gamma}^{\nu}_{\alpha\beta} &= \bar{\chi}^{\nu\theta}\Gamma_{\theta,\alpha\beta} + t^\nu \partial_\alpha \tilde{t}_\beta \\ \text{where } \tilde{t}_\mu &\equiv \left(\frac{\partial v}{\partial x^\mu}\right), \quad \bar{\chi}^{\mu\nu} = \Delta^\mu{}_\alpha \Delta^\nu{}_\beta\, \chi^{\alpha\beta}, \quad \Delta^\alpha{}_\beta \equiv \delta^\alpha_\beta - t^\alpha \tilde{t}_\beta\end{aligned} \tag{4.39}$$

One could easily show these structures by acting the covariant derivatives on vectors and recursively using the relations for higher indexed tensors. Note the new connection $\tilde{\Gamma}^{\mu}_{\alpha\beta}$ is also symmetric in its lower two indices. The other mixed tensor we defined here is actually a projector to constant v slices of the horizon because

$$t^\alpha \Delta^\beta{}_\alpha = \Delta^\alpha{}_\beta\, \tilde{t}_\alpha = 0$$

Using these structures, we could see that

$$\begin{aligned}\nabla_k K_{ij} &= l^\alpha_k l^\mu_i l^\nu_j \mathcal{D}_\alpha \mathcal{K}_{\mu\nu} \\ &= l^\alpha_k l^\mu_i l^\nu_j \left[\partial_\alpha \mathcal{K}_{\mu\nu} - \tilde{\Gamma}^\beta{}_{\alpha\mu}\mathcal{K}_{\beta\nu} - \tilde{\Gamma}^\beta{}_{\alpha\nu}\mathcal{K}_{\mu\beta}\right] \\ \text{where } \mathcal{K}_{\mu\nu} &= -t^\alpha \Gamma_{\alpha,\mu\nu}\end{aligned} \tag{4.40}$$

The spatial current on the horizon will add the following contribution to the boundary entropy current

$$J^\mu_{space} = \frac{1}{\sqrt{g^{(b)}}}\Delta\, l^\mu_a \left(\sqrt{h}\, j^a\right) = -4\alpha^2\, \frac{1}{\sqrt{g^{(b)}}}\Delta\, l^\mu_a \sqrt{h}\left(h^{ki}h^{ja} - h^{ka}h^{ij}\right)\left(\nabla_k K_{ij}\right)$$

Now using the identity $h^{ij} l^\mu_i l^\nu_j = \bar{\chi}^{\mu\nu}$, we finally get the following expression for the space part of the entropy current

$$J^\mu_{space} = -4\alpha^2 \frac{1}{\sqrt{g^{(b)}}}\sqrt{H}\left(\bar{\chi}^{\gamma\alpha}\bar{\chi}^{\mu\beta} - \bar{\chi}^{\gamma\mu}\bar{\chi}^{\alpha\beta}\right)\left(\mathcal{D}_\gamma \mathcal{K}_{\alpha\beta}\right) \text{ where } H \equiv \frac{\hat{n}_\mu \hat{n}_\nu \epsilon^{\mu\mu_1\cdots\mu_n}\epsilon^{\nu\nu_1\cdots\nu_n}\chi_{\mu_1\nu_1}\cdots\chi_{\mu_n\nu_n}}{n!} \tag{4.41}$$

Intrinsic Ricci scalar

For the temporal part of the entropy current, we need to compute the intrinsic Ricci scalar of the constant v slices of the horizon.

In this section, we note down the calculation for the Ricci scalar, $\mathcal{R}$, with respect to h_{ij}.

We start with the expression for the Riemann tensor

$$\mathcal{R}^a_{bcd} = \partial_c \Gamma^a_{bd} + \Gamma^a_{cm}\Gamma^m_{bd} - (c \leftrightarrow d) \tag{4.42}$$

Now we will process $\partial_c \Gamma^a_{bd}$ in the following way

$$\begin{aligned}\partial_c \Gamma^a_{bd} &= \partial_c \left(h^{ap}\Gamma_{p,bd}\right) \\ &= \partial_c h^{ap}\Gamma_{p,bd} + h^{ap}\partial_c\Gamma_{p,bd} \\ &= -h^{aq}\Gamma^p_{cq}\Gamma_{p,bd} - \Gamma^a_{cr}\Gamma^r_{bd} + h^{ap}\partial_c\Gamma_{p,bd}\end{aligned} \tag{4.43}$$

where in the last line, we have used

$$\begin{aligned}\partial_c h^{ap} &= -h^{aq}h^{pr}\partial_c h_{rq} \\ \partial_c h_{rq} &= \Gamma_{r,cq} + \Gamma_{q,cr}\end{aligned} \tag{4.44}$$

Hence, we have

$$\partial_c \Gamma^a_{bd} + \Gamma^a_{cr}\Gamma^r_{bd} = -h^{aq}\Gamma^p_{cq}\Gamma_{p,bd} + h^{ap}\partial_c\Gamma_{p,bd} \tag{4.45}$$

So, we can write the expression for the Riemann tensor in the following form

$$\mathcal{R}_{abcd} = \partial_c\Gamma_{a,bd} - \Gamma^p_{ca}\Gamma_{p,bd} - \left(\partial_d\Gamma_{a,bc} - \Gamma^p_{da}\Gamma_{p,bc}\right) \tag{4.46}$$

Now the expression for the $\Gamma_{k,ij}$ in the following

$$\Gamma_{k,ij} = l^\mu_i l^\nu_j l^\alpha_k \Gamma_{\alpha,\mu\nu} + \chi_{\mu\nu}\, l^\mu_k \left(l_i \cdot \partial l^\nu_j\right) \tag{4.47}$$

Then we can process $\partial_c\Gamma_{a,bd}$ in the following way

$$\partial_c\Gamma_{a,bd} = (l_c \cdot \partial)\left[l^\mu_b l^\nu_d l^\alpha_a \Gamma_{\alpha,\mu\nu} + \chi_{\mu\nu}\, l^\mu_a \left(l_b \cdot \partial l^\nu_d\right)\right] \tag{4.48}$$

And also,

$$\begin{aligned}&\Gamma^p_{ca}\Gamma_{p,bd} \\ &= h^{pm}\Gamma_{m,ca}\Gamma_{p,bd} \\ &= \bar{\chi}^{\alpha_1\alpha} l^\mu_c l^\nu_a l^{\mu_1}_b l^{\nu_1}_d \Gamma_{\alpha,\mu\nu}\Gamma_{\alpha_1,\mu_1\nu_1} + \Delta^\alpha_{\nu_1} l^\mu_c l^\nu_a \Gamma_{\alpha,\mu\nu}\left(l_b\cdot\partial l^{\nu_1}_d\right) + \Delta^{\alpha_1}_\nu l^{\mu_1}_b l^{\nu_1}_d \left(l_c\cdot\partial l^\nu_a\right)\Gamma_{\alpha_1,\mu_1\nu_1} \\ &\quad + \chi_{\nu\nu_1}\left(l_c\cdot\partial l^\nu_a\right)\left(l_b\cdot\partial l^{\nu_1}_d\right)\end{aligned} \tag{4.49}$$

where, we have defined $\bar{\chi}^{\alpha\beta} = l_i^\alpha l_j^\beta h^{ij}$ and used the fact that $\bar{\chi}^{\alpha\beta}\chi_{\beta\nu} = \Delta_\nu^\alpha$.

Then we have

$$
\begin{aligned}
&\partial_c \Gamma_{a,bd} - \Gamma^p_{ca}\Gamma_{p,bd} \\
=&\Gamma_{\alpha,\mu\nu}\left[l_d^\nu l_a^\alpha\left(l_c\cdot\partial\right)l_b^\mu + l_b^\mu l_a^\alpha\left(l_c\cdot\partial\right)l_d^\nu + l_c^\mu l_a^\alpha\left(l_b\cdot\partial\right)l_d^\nu\right] \\
&+ l_b^\mu l_d^\nu l_a^\alpha l_c^\beta\left[\partial_\beta\Gamma_{\alpha,\mu\nu} - \bar{\chi}^{\alpha_1\alpha_2}\Gamma_{\alpha_2,\beta\alpha}\Gamma_{\alpha_1,\mu\nu}\right] + \chi_{\mu\nu}l_a^\mu\left[\left(l_c\cdot\partial\right)\left(l_b\cdot\partial l_d^\nu\right)\right] \\
&+ t^\alpha\Gamma_{\alpha,\mu\nu}\left[l_c^\mu l_a^\nu \tilde{t}_{\nu_1}\left(l_b\cdot\partial l_d^{\nu_1}\right) + l_b^\mu l_d^\nu \tilde{t}_{\nu_1}\left(l_c\cdot\partial l_a^{\nu_1}\right)\right]
\end{aligned}
\tag{4.50}
$$

Hence, we have the expression for $\mathcal{R}_{abcd}$ as

$$
\begin{aligned}
\mathcal{R}_{abcd} =& \left[\partial_\beta\Gamma_{\alpha,\mu\nu} - \bar{\chi}^{\alpha_1\alpha_2}\Gamma_{\alpha_2,\beta\alpha}\Gamma_{\alpha_1,\mu\nu}\right]\left[l_b^\mu l_d^\nu l_a^\alpha l_c^\beta - l_b^\mu l_c^\nu l_a^\alpha l_d^\beta\right] \\
&+ t^\alpha\Gamma_{\alpha,\mu\nu}\partial_\delta\tilde{t}_{\nu_1}\left[l_b^\mu l_c^\nu l_d^\delta l_a^{\nu_1} + l_d^\mu l_a^\nu l_b^\delta l_c^{\nu_1} - l_b^\mu l_d^\nu l_c^\delta l_a^{\nu_1} - l_c^\mu l_a^\nu l_b^\delta l_d^{\nu_1}\right]
\end{aligned}
\tag{4.51}
$$

So, finally we have

$$
\begin{aligned}
\mathcal{R} =& h^{ac}h^{bd}\mathcal{R}_{abcd} \\
=& \left[\bar{\chi}^{\mu\nu}\bar{\chi}^{\alpha\beta} - \bar{\chi}^{\alpha\nu}\bar{\chi}^{\mu\beta}\right]\left[\partial_\beta\Gamma_{\alpha,\mu\nu} - \bar{\chi}^{\alpha_1\alpha_2}\Gamma_{\alpha_2,\beta\alpha}\Gamma_{\alpha_1,\mu\nu} - 2t^{\alpha_1}\Gamma_{\alpha_1,\mu\nu}\partial_\alpha\tilde{t}_\beta\right]
\end{aligned}
\tag{4.52}
$$

Hence, we finally get the following expression for the intrinsic Ricci scalar

$$
\mathcal{R} = \left(\bar{\chi}^{\mu_1\nu_1}\bar{\chi}^{\mu_2\nu_2} - \bar{\chi}^{\mu_1\nu_2}\bar{\chi}^{\mu_2\nu_1}\right)\left[\partial_{\mu_1}\Gamma_{\nu_1,\mu_2\nu_2} - \bar{\chi}^{\alpha_1\alpha_2}\Gamma_{\alpha_1,\mu_1\nu_1}\Gamma_{\alpha_2,\mu_2\nu_2} - 2t^\alpha\Gamma_{\alpha,\mu_1\nu_1}\left(\partial_{\mu_2}\tilde{t}_{\nu_2}\right)\right]
\tag{4.53}
$$

Separating 'fluid' and 'non-fluid' terms

The final form of the entropy current written in terms of boundary coordinates $\{x^\mu\}$ is

$$
\begin{aligned}
&J^\mu = J^\mu_{space} + J^\mu_{time} \\
\text{where}\ \ &J^\mu_{space} = -4\alpha^2\frac{1}{\sqrt{g^{(b)}}}\frac{\sqrt{H}}{||t||}\left(\bar{\chi}^{\gamma\alpha}\bar{\chi}^{\mu\beta} - \bar{\chi}^{\gamma\mu}\bar{\chi}^{\alpha\beta}\right)\left(\mathcal{D}_\gamma\mathcal{K}_{\alpha\beta}\right) \\
&J^\mu_{time} = \frac{1}{\sqrt{g^{(b)}}}\sqrt{H}\left(1 + 2\alpha^2\mathcal{R}\right)\hat{n}^\mu
\end{aligned}
\tag{4.54}
$$

$\mathcal{R}$ is given in equation (4.53).

In this expression of the current, most of the terms are 'fluid' terms in the sense that they depend solely on the metric components and their derivatives written in boundary coordinates. The exceptions are those terms where one has explicit $\tilde{t}_\mu$, e.g., in $\Delta^\mu_\nu = \delta^\mu_\nu - t^\mu \tilde{t}_\nu$. These terms could be further processed by expressing $\tilde{t}_\mu$ in terms of $\partial_\mu \phi$ using equation(4.31). The expressions turn out to be too big to be presented here. We have collected them in appendix (B.3).

In the final stage, we would like to evaluate this current on the hydrodynamic metric correctly up to first order in derivative expansion. However, just looking at equation (4.54), we could figure out that J^μ_{space} is of second order. This is because $\Gamma_{\alpha,\mu\nu}$ is always of first order in terms of derivative expansion and so is $\mathcal{K}_{\alpha\beta} \sim t^\mu \Gamma_{\mu,\alpha\beta}$. It follows that $J^\mu_{space} \sim \mathcal{D}_\gamma \mathcal{K}_{\alpha\beta} \sim \mathcal{O}(\partial^2)$. Using a similar argument, we could show that $\mathcal{R}$ is also of $\mathcal{O}(\partial^2)$, where we have used the fact that $\tilde{t}_\mu$ is of order $\mathcal{O}(1)$ in terms of derivative expansion. Therefore, up to first order in derivative expansion, there will not be any contribution to the entropy current from the Gauss-Bonnet correction. To have any non-trivial result, we need to go at least one higher order in derivative expansion, which we leave for future work.

4.5 Future Directions

If we follow our construction, the boundary entropy current will involve one 'non-fluid' function, the scalar field ϕ, whose exponential relates the two different parametrizations of the horizon null generator. But the fluid entropy current must not have any other field other than the fluid velocity and its local temperature. So the next natural question is whether we could use the non-uniqueness of the currents on both horizon and the boundary side to remove this unwanted ϕ dependence. Einstein-Gauss-Bonnet theory is the simplest well-studied example where such currents and ambiguities could be explicitly constructed and tested. But unfortunately to the order that we have worked here, no such fixing is required since up to this order all the non-trivial structures that have this 'non fluid' $\partial_\mu \phi$ factor just vanish. So our future goal would be to extend this calculation to order $\mathcal{O}(\partial^2)$. In this section we shall set up the stage for this future calculation.

4.5.1 Conditions of stationarity

As mentioned before, the entropy current and the entropy density in higher derivative theories work only for horizons where the amplitude (let's denote it as ϵ) of the dynamics is small and could be treated perturbatively. Moreover, the construction in [37, 38] works only up to the linear order in ϵ. So we should not expect the dual fluid entropy current to do any better. In other words, while applying formula (4.26), we should ignore all terms that are of $\mathcal{O}(\epsilon^2)$ or higher in $\chi_{\mu\nu}$, t^μ or $\tilde{t}_\mu$. Now derivative expansion is not the same as amplitude expansion. It is quite possible to have terms that are linear in ϵ but higher order in terms of derivative. So we need to have a clean prescription to identify fluid data that are linear in amplitude (but in principle, could have multiple derivatives).

A stationary fluid on the boundary (where both the boundary metric and the fluid configuration admit at least one Killing vector) should be dual to a stationary bulk metric with a Killing horizon. In other words, the Killing vector on the boundary could be extended to a bulk Killing vector, which on the horizon reduces to the Killing generator of the horizon. In terms of equations, what we mean is the following. Suppose $\xi = \xi^A \partial_A$ is the bulk Killing vector.

Since it reduces to the generator of the horizon (the $r = 0$ hypersurface in our choice of coordinates)

$$\lim_{r\to 0} \xi^r = 0, \quad \lim_{r\to 0} \xi^\mu \propto G^{\mu r}|_{r=0}$$

Further, $\xi^A \partial_A$ should reduce to the boundary Killing vector $\xi^\mu_{(b)} \partial_\mu$ in the limit $r \to \infty$

$$\lim_{r\to \infty} \xi^r = 0, \quad \lim_{r\to \infty} \xi^\mu = \xi^\mu_{(b)}$$

Now for our analysis, we shall assume[4] that

$$\lim_{r\to 0} \xi^\mu = G^{\mu r}|_{r=0} = \xi^\mu_{(b)}$$

[4] This assumption could be justified as follows. Let's choose a coordinate system where $\xi^A \partial_A = \partial_\tau$, i.e. τ is the parameter along the integral curve of ξ^A. The Killing coordinate is τ, and hence, the metric could be expressed such that all of its components are independent of τ. Since the boundary metric is just the boundary limit of the bulk metric, its components should also be independent of τ. The same should be true of the fluid variables like velocity and temperature as the bulk metric components are functions of these variables only. Therefore the same τ will also be a Killing coordinate from the perspective of the stationary boundary fluid.

The above condition will result in a set of constraints both on the fluid data and the horizon data (vanishing of some particular fluid/ horizon structures), respectively. Any violation of these constraints will be a departure from stationarity and, therefore, generically of order $\mathcal{O}(\epsilon)$ terms. We have a clean classification of such terms on the horizon side and using the map, we could translate them to the fluid side. The $\mathcal{O}(\epsilon)$ terms, thus derived on the fluid side, should be automatically compatible with constraints of stationarity (and departure from it) as expected from any stationary fluid configuration.

Product of two such order $\mathcal{O}(\epsilon)$ terms will be order $\mathcal{O}(\epsilon)^2$ and therefore neglected.

4.5.2 Choice of Fluid Frames

In section 4.2, we have presented the metric dual to boundary fluid dynamics (see equation (4.8)). This metric is written in terms of fluid velocity (u^μ) and temperature (T). But as one goes to higher order in derivative expansion, one has the freedom to redefine the velocity and the temperature of the fluid. This ambiguity is present in fluid dynamics itself and is usually fixed by a specific choice of fluid frames. Now fluid dynamics is about the dynamics of the stress tensor and other conserved charges of the system. So the fluid frames are also usually defined in terms of the stress tensor or the currents. For example, in 'Landau frame' the velocity of the energy flow is defined as u^μ. This implies that u^μ is the unique time-like eigenvector of the stress tensor (normalized). Once u^μ (and temperature) is unambiguously defined, the dual bulk metric is constructed. A given definition of the fluid frame amounts to a given boundary condition for the metric function while solving for the bulk metric.

In this section, we shall adopt a different choice of fluid frame which would be more suitable for our purpose, and in particular, for the description of equilibrium. We shall define our new velocity u^μ as

$$u^\mu \equiv \hat{n}^\mu, \quad \text{where} \quad n^\mu = G^{\mu r}|_{r=0} \quad \text{and} \quad \hat{n}^\mu = \frac{n^\mu}{\sqrt{-n^\mu n^\nu g^{(b)}_{\mu\nu}}}$$

For brevity, we shall denote this choice of velocity as 'Gravity frame'. One could choose this frame only if the fluid admits a gravity dual. Note that

$$u^\mu|_{\text{gravity frame}} = u^\mu|_{\text{Landau frame}} + \mathcal{O}\left(\partial\right)$$

So in zeroth order in derivative, these two definitions of velocity agree as they should. In fact, it turns out that even at first order in derivative expansion, these two velocities agree; the difference starts only at second order. However, since in this chapter, our computations are correct only up to first order in derivative expansion, this frame redefinition becomes particularly simple for us. Basically, it says there is no transformation at all up to first order in derivatives.

4.5.3 Metric Dual to Hydrodynamics in Gauss-Bonnet Theory in Gravity frame

The metric dual to hydrodynamic in Einstein-Gauss-Bonnet theory has been worked out in [83] up to first order in derivative expansion. However, in [83] the main concern was boundary hydrodynamics and therefore, the author has worked in a slightly different gauge than what is described in equation (4.10). In this subsection, we shall work out the same metric, but in the gauge most convenient for our purpose , i.e., using the gravity frame described in the previous section.

The action for the full "Einstein + Gauss-Bonnet" theory is given by [5]

$$\begin{aligned} S &= S_E + \alpha^2\, S_{GB} \\ S_E &= -\frac{1}{4\pi}\int d^5x\,\sqrt{-g}(R-2\Lambda) \\ S_{GB} &= -\frac{1}{4\pi}\int d^5x\,\sqrt{-g}(R^2 - 4R^{AB}R_{AB} + R^{ABCD}R_{ABCD}) \end{aligned} \tag{4.55}$$

We will parametrize Λ [6] as $\Lambda = -6\left(1-2\alpha^2\right)$.

[5]Here, we have used the convention $4G_5 = 1$ (where G_5 is the Newton's constant in five dimensions) to have only the horizon area term without any extra proportionality constants as the entropy of the Einstein theory. Accordingly, the proportionality constant in S_E and S_{GB} have been modified from those used in [83].

[6]In [83], to ensure the fact that the boundary metric is exactly equal to the Minkowski metric - $\eta_{\mu\nu}$, the author has to scale the boundary coordinates in an α dependent manner. As a result in the final covariant bulk metric the component $G_{\rho\mu}$ is no longer equal to $-u_\mu$, rather just proportional to it with an α dependent constant as proportionality factor. However, in our analysis we have crucially used the fact that $G_{\rho\mu} = -u_\mu$ and also the calculation simplifies if the boundary metric is just equal to $\eta_{\mu\nu}$. It turns out if we want to impose both these conditions on the bulk metric, we need to scale the consmological constant.

The equations of motion of the full theory are given by

$$\begin{aligned}E_{MN} =& \left(R_{MN} - \frac{1}{2}g_{MN}R + \Lambda g_{MN} - \frac{1}{2}\alpha^2\, g_{MN}(R^2 - 4R^{AB}R_{AB} + R^{ABCD}R_{ABCD})\right)\\ &+ \alpha^2\left(4R_{MPQL}R_N^{\ PQL} - 4R^{PQ}R_{MPNQ} - 4R_M^{\ P}R_{NQ} + 2R\,R_{MN}\right)\end{aligned} \tag{4.56}$$

The black-brane metric which is dual to a boundary fluid and solves these equations of motion up to first order in derivatives as well as in α^2 is given by

$$ds^2 = -2u_\mu dx^\mu dr + \chi_{\mu\nu}dx^\mu dx^\nu \tag{4.57}$$

Note that in this gauge, the boundary metric will be of the form $g^{(b)}_{\mu\nu} = \eta_{\mu\nu}$ and lowering and raising of the boundary indices have to be done w.r.t $g^{(b)}_{\mu\nu}$. $\chi_{\mu\nu}$ can be expressed as

$$\begin{aligned}\chi_{\mu\nu} =& -r_H^2\, f\left(\frac{r}{r_H}\right)\, u_\mu u_\nu + r_H^2\, K\left(\frac{r}{r_H}\right)\, P_{\mu\nu} + r_H\, F\left(\frac{r}{r_H}\right)\, \sigma_{\mu\nu} + r_H\, V\left(\frac{r}{r_H}\right)(u_\mu a_\nu + u_\nu a_\mu)\\ &+ \theta\left(r_H\, S_1\left(\frac{r}{r_H}\right)\, u_\mu u_\nu + r_H\, S_2\left(\frac{r}{r_H}\right)\, P_{\mu\nu}\right)\end{aligned} \tag{4.58}$$

As mentioned before, here, r_H is the scale associated with the black hole solution. The functions used in (4.58) are defined as

$$\begin{aligned}f(x) =& (1+x)^2\left[1 - \left(\frac{1}{1+x}\right)^4\right] - 2\,\alpha^2\frac{\left[(1+x)^4 - 1\right]}{(1+x)^6}\\ K(x) =& (1+x)^2\\ V(x) =& -x\\ S_1(x) =& \frac{2x}{3}\\ S_2(x) =& 0\\ F(x) =& F_0(x) + \alpha^2\, F_\alpha(x)\end{aligned} \tag{4.59}$$

with,

$$
\begin{aligned}
F_0(x) =& \frac{1}{2}(x+1)^2 \left(-4\log(x+1) + 2\log(x+2) + \log(x(x+2)+2) - 2\tan^{-1}(x+1) + \pi\right) \\
F_\alpha(x) =& \frac{1}{(1+x)^2}\Big[\pi(x+1)^4 - 4x(x(x+3)+3) - 4(x+1)^4\log(x+1) + 4\log(x+2) + 3\log(x(x+2) \\
& + x(x+2)(x(x+2)+2)(4\log(x+2) + 3\log(x(x+2)+2) - 2\log((x+2)(x(x+2)+2))) \\
& - 2\log((x+2)(x(x+2)+2)) - 2(x+1)^4\tan^{-1}(x+1) - 1\Big]
\end{aligned}
\tag{4.60}
$$

and the fluid variables θ and $\sigma_{\mu\nu}$ and the projector $P_{\mu\nu}$ are given by

$$
\begin{aligned}
P_{\mu\nu} &= g^{(b)}_{\mu\nu} + u_\mu u_\nu \\
\theta &= \partial \cdot u \\
\sigma_{\mu\nu} &= P^\alpha_\mu P^\beta_\nu \partial_{(\alpha} u_{\beta)}
\end{aligned}
\tag{4.61}
$$

4.5.4 Stationary solution in Gravity frame

In a stationary metric with horizon located at $r = 0$, the Killing vector is $\xi^\mu \propto G^{\mu r}|_{r=0}$. According to our assumption

$$
\xi^\mu_{(b)} \propto G^{\mu r}|_{r=0}, \quad \Rightarrow \quad \xi^\mu_{(b)} \propto u^\mu \quad \text{in Gravity frame}
$$

Now in a stationary situation $G^{\mu r}$ is proportional to the Killing vector, both for the Bulk and the boundary metric. Therefore, in case of stationary fluid, this particular choice of frame amounts to choosing the fluid velocity in the direction of the Killing vector for the boundary metric.

In this subsection, we shall start from the assumption that $\xi^A\partial_A = \xi^r\partial_r + F(r, x^\mu)u^\mu\partial_\mu$. Then we shall derive the conditions u^μ must satisfy so that $\xi^A\partial_A$ is a bulk Killing vector. We shall see that u^μ will turn out to be proportional to the boundary Killing vector as expected, with its shear tensor and expansion vanishing everywhere.

Now we will show that if we have a Killing vector proportional to the fluid velocity u^μ, then the expansion and shear tensor will vanish. We will also get constraints on the proportionality constant such that this condition is satisfied.

We will start by writing the fluid metric in a way such that the horizon is located at the origin of the radial coordinate.

$$ds^2 = -2u_\mu \, dx^\mu \, dr - r_H^2 \, f\left(r/r_H\right) \left(u_\mu dx^\mu\right)^2 + (r + r_H)^2 P_{\mu\nu} \, dx^\mu \, dx^\nu + \chi^{(1)}_{\mu\nu} \, dx^\mu \, dx^\nu \tag{4.62}$$

where, $\chi^{(1)}_{\mu\nu}$ contains terms first order in derivative of the fluid variables.
Then the killing vector will have the following form

$$\xi^A \partial_A \propto G^{\mu r}|_{r=0} = F \; u^\mu \partial_\mu \tag{4.63}$$

where, F is the proportionality constant.
In covariant form this becomes

$$\xi_A dx^A = F \; dr + F \left[r_H^2 \; f\left(r/r_H\right) \; u_\alpha + u^\mu \chi^{(1)}_{\mu\alpha}\right] \; dx^\alpha \tag{4.64}$$

Now we will solve for the Killing equation on this and write down the conditions it will give on F and $\chi^{(1)}$.
The Killing equation is

$$\nabla_A \xi_B + \nabla_B \xi_A = 0 \tag{4.65}$$

The (r, r) component of which will give the following condition

$$\partial_r F = 0 \tag{4.66}$$

The (r, μ) component will give

$$\partial_\mu F - F \; a_\mu = 0 \tag{4.67}$$

where, $a_\mu = (u \cdot \partial) \, u_\mu$.
The (μ, ν) component will give

$$\begin{aligned} &r_H^2 \; f\left(r/r_H\right) \; \Big[u_\mu \left(\partial_\nu F - F \; a_\nu\right) + u_\nu \left(\partial_\mu F - F \; a_\mu\right)\Big] + 2F(r + r_H)^2 \sigma_{\mu\nu} \\ &+ F \; r(r + r_H) \frac{2\theta}{D-2} P_{\mu\nu} + r_H \; F \left[2 r_H \; f\left(r/r_H\right) - r \; f'\left(r/r_H\right)\right] \frac{\theta}{D-2} u_\mu u_\nu = 0 \end{aligned} \tag{4.68}$$

where we have used the following identity and fluid constraint equation, $\partial_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} - u_\mu a_\nu + \frac{\theta}{D-2} P_{\mu\nu}$ and $\frac{(u\cdot\partial) r_H}{r_H} + \frac{\theta}{D-2} = 0$.

Now to be consistent with (4.67) we should have

$$\theta = 0, \quad \sigma_{\mu\nu} = 0 \tag{4.69}$$

Hence, we could show that with vanishing shear tensor and expansion, $F\, u^\mu$ is actually a Killing vector with F satisfying (4.66) and (4.67).

Note that $F = \frac{1}{r_H}$ is a solution to (4.66) and (4.67). Also note that in [128] the Killing vector $\xi^\alpha = \frac{c}{T}\, u^\alpha$ where, T is the local temperature $T = \left(\frac{D-1}{4\pi}\right) r_H$ and c is a constant. Hence, up to an overall constant, the two Killing vectors are equivalent.

Hence, these stationarity conditions are identical to the ones derived in [128] from the perspective of a stationary boundary fluid.

4.6 Conclusion

The construction of [37, 38] gives an expression of entropy density and entropy current on the dynamical black hole solution in the higher derivative theories of gravity. However, this construction works (i.e, it leads to entropy production) only when the amplitude of the dynamics is small, and all terms quadratic or higher order in the amplitude are neglected. Recently it has been extended to quadratic order in amplitude [39]. But clearly, this is not the most satisfying answer; the second law should hold for any dynamics irrespective of its amplitude. Our final goal is to extend the construction of [37, 38] to the nonlinear orders in amplitude.

In this chapter, we have used fluid-gravity correspondence to construct a dual entropy current in the boundary fluid by lifting the entropy current on the horizon via a horizon to boundary map. Since our horizon entropy current works only up to the linear order in the amplitude, we should not expect the fluid entropy current to do any better. So the entropy current constructed in this manner will have non negative divergence only up to the linear order in the dynamical fluid data.

However, in relativistic hydrodynamics we independently know how to extend a given entropy current that works only up to linear order in amplitude, to an entropy current where the amplitude is no longer a perturbation parameter [94]. So it is reasonable to hope that if we could construct the dual fluid entropy current nonperturbatively and use the horizon to boundary map in reverse, we might be able to say something about the entropy current in higher derivative theories of gravity in a similar nonperturbative manner.

With this goal in mind, in this chapter, we have taken the first baby step of constructing the fluid entropy current dual to the horizon entropy current [37,38] in dynamical black holes of Gauss-Bonnet gravity. The fluid entropy current thus constructed depends non trivially on the mapping functions that relate the boundary coordinates with the horizon coordinates. This dependence has complicated our construction since these mapping functions need not admit a derivative expansion like the fluid variables. The immediate future direction would be to search for a particular set of mapping functions so that the final fluid entropy current is expressible only in terms of fluid and fluid-like variables that admit derivative expansion in every stage.

In this chapter, we have made a couple of simplifications in this direction. Since both the horizon and the boundary are codimension-one hypersurfaces, naively, there could be $(D-1)$ such mapping functions, where D is the number of bulk dimensions. But using some symmetry and re-arrangement, we could reduce it to only one scalar 'non fluid' function, which could be $\phi(x^\mu)$ or $v(x^\mu)$. This scalar is also largely constrained in the sense that if it is specified on a given spatial slice, the consistency equation will fix it everywhere on the horizon (or boundary). So finally, the task of finding appropriate $(D-1)$ scalar 'mapping functions' has been reduced to the search for an appropriate equation, constraining a single scalar on a given spatial slice.

In this context, it might be useful to note that the horizon and also the entropy on it have symmetry under the reparametrization of the horizon generator. It has been explored in the case of Einstein-Gauss-Bonnet theory in [3, 39]. The discussion could be extended to include 'non-affine' reparametrization of the horizon generators, which might have some direct application for our analysis here.

Part II

Stability and Causality in theories of Relativistic Hydrodynamics

Chapter 5

Causality Criteria from Stability Analysis at Ultra-High Boost

This chapter is based on [96].

It has been established that the group velocity of the propagating mode exceeding the speed of light for some frequency range does not violate causality, as long as it is subluminal at the infinite frequency (wavenumber) limit [103, 104]. This necessary condition for causality is called the asymptotic causality condition which has been widely used to check the causal validity of a hydrodynamic theory [129–131]. But the conceptual anomaly with this approach is that the hydrodynamic gradient expansion has been tested to be a divergent series with factorial growth of large order corrections indicating a zero radius of convergence [132, 133]. Given the situation, an alternate definition of causality is imperative. On the other hand, the stability of a relativistic system has been known to behave distinctly depending upon the observer's frame of reference [56]. This issue has been recently addressed in [107, 134], where it has been argued that frame-invariant stability is possible only if the theory respects causality. The objective of this chapter is to employ the frame-invariance of the stability property of a theory to establish its causality constraints. The non-triviality again comes from the fact that checking linear stability at arbitrary reference frames to identify the invariantly stable parameter space can be a cumbersome job. In this chapter for two well-known stable-causal theories, we have demonstrated that the linear stability analysis in a reference frame boosted to a near luminal speed can alone provide the stability invariant parameter space at the spatially homogeneous limit of the theory and hence can be used to determine the causal domain of the theory as well. In [135], this identification has been observed from a kinetic theory derivation of a stable-causal first-order theory. Here, we show that one can solely use the low-wavenumber stability analysis to produce the exact results of asymptotic causality in the MIS and BDNK theories. The analysis presented here serves as a case study of two most well-known stable-causal theories to show that the causality of a theory can be probed without departing from the small-k domain. Since relativistic hydrodynamics is a low-energy effective theory, hence we believe this approach provides us with a more appropriate definition of causality.

5.1 Basic setup

In this chapter, hydrodynamic stability has been analyzed in a generalized Lorentz frame with an arbitrary boost velocity for both second-order Müller-Israel-Stewart (MIS) theory [60, 136, 137], and the recently proposed first-order stable-causal (BDNK) theory [61, 63, 138, 139]. We linearize the conservation equations for small perturbations of fluid variables around their hydrostatic equilibrium, $\psi(t,x) = \psi_0 + \delta\psi(t,x)$, with the fluctuations expressed in the plane wave solutions via a Fourier transformation $\delta\psi(t,x) \to e^{i(kx-\omega t)}\delta\psi(\omega,k)$, (subscript 0 indicates global equilibrium). The background fluid is considered to be boosted along the x-axis with a constant velocity $\mathbf{v}$, $u_0^\mu = \gamma(1,\mathbf{v},0,0)$ with $\gamma = 1/\sqrt{1-\mathbf{v}^2}$. The corresponding velocity fluctuation is $\delta u^\mu = (\gamma\mathbf{v}\delta u^x, \gamma\delta u^x, \delta u^y, \delta u^z)$ which gives $u_0^\mu \delta u_\mu = 0$ to maintain the velocity normalization. In the following analysis, we present the leading order stability analysis (at $k \to 0$ limit) for both the theories at conformal, charge less limit.

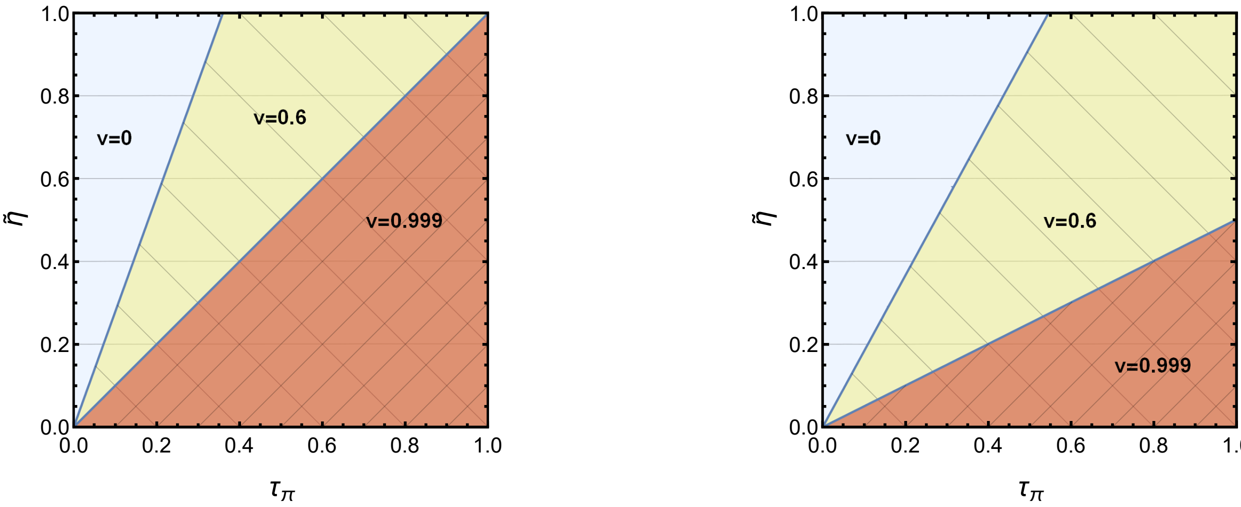
5.2 Conventions and notations

Throughout the manuscript, we have used natural unit ($\hbar = c = k_B = 1$) and flat space-time with mostly positive metric signature $\eta^{\mu\nu} = \text{diag}\,(-1,1,1,1)$. The used notations read, $D \equiv u^\mu\partial_\mu$, $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$, $\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\partial^\alpha u^\beta$ with $\Delta^{\mu\nu\alpha\beta} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta} + \frac{1}{2}\Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}$ and $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$, $\epsilon \equiv$ energy density, $P \equiv$ pressure, $u^\mu \equiv$ hydrodynamic four-velocity, $\tau_\pi \equiv$ relaxation time of shear-viscous flow, $\eta \equiv$ shear viscous coefficient, $\mathcal{E}, \theta$ are first order field correction coefficients of BDNK theory. From the constraints of the second law of thermodynamics, η should always be a positive number [140]. The scaling notation $\tilde{x}$ denotes $x/(\epsilon_0 + P_0)$.

5.3 Identifying stability invariant parameter space from ultra-high boost

First, we discuss the case of MIS theory where the energy-momentum tensor takes the form, $T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \pi^{\mu\nu}$. The conservation of energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$ and the relaxation equation of shear viscous flow $\pi^{\mu\nu} = -\tau_\pi \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} - 2\eta\sigma^{\mu\nu}$ together give us the equations of motion to be linearized. In the transverse or shear channel, the leading term of the frequency (ω) solution in wavenumber k-expansion is a single non-hydro non-propagating mode, $\omega^{\perp}_{\text{MIS}} = -i/\gamma(\tau_\pi - \tilde{\eta}\mathbf{v}^2)$. Now the demand that stability requires the imaginary part of the frequency to be negative renders the stability criteria $\tau_\pi/\tilde{\eta} > \mathbf{v}^2$ [104]. For sound channel, the leading order single non-propagating mode turns out to be, $\omega^{\parallel}_{\text{MIS}} = -i(1 - \frac{\mathbf{v}^2}{3})/\gamma[\tau_\pi(1 - \frac{\mathbf{v}^2}{3}) - \frac{4\tilde{\eta}}{3}\mathbf{v}^2]$. For the range of boost velocity $0 \leq \mathbf{v} < 1$, the stability condition becomes, $\tau_\pi/\tilde{\eta} > \frac{4}{3}\mathbf{v}^2/(1 - \frac{\mathbf{v}^2}{3})$. In both the channels, the right-hand sides of the inequalities for $\tau_\pi/\tilde{\eta}$ are monotonically increasing functions of $\mathbf{v}$ within the mentioned range that allow only positive values of τ_π and give the strictest bound for $\mathbf{v} \to 1$. So we infer that the allowed parameter space over the transport coefficients η and τ_π set by stability criteria at the spatially homogeneous limit ($k \to 0$) for any boost velocity $\mathbf{v}$, is always a subset of the same for any lower value of $\mathbf{v}$. Hence, we conclude here that the $\mathbf{v} \to 1$ bound ($\tau_\pi > \tilde{\eta}$ for shear channel and $\tau_\pi > 2\tilde{\eta}$ for sound channel) provides the necessary and sufficient region in the parameter space where the system is stable at the spatially homogeneous limit for all reference frames ($0 \leq \mathbf{v} < 1$). So here we see that for the MIS theory, checking stability alone in a reference frame with ultra-high boost ($\mathbf{v} \to 1$) is sufficient to identify the frame-invariantly stable parameter space at $k \to 0$ limit.

Next, we discuss the case of BDNK theory for which the energy-momentum tensor takes the form, $T^{\mu\nu} = (\epsilon + \epsilon_1)u^\mu u^\nu + (P + P_1)\Delta^{\mu\nu} + (u^\mu W^\nu + u^\nu W^\mu) + \pi^{\mu\nu}$, with the first order dissipative field corrections, $\epsilon_1 = \mathcal{E}\frac{D\epsilon}{\epsilon_0 + P_0} + \mathcal{E}(\partial \cdot u), P_1 = \frac{\mathcal{E}}{3}\frac{D\epsilon}{\epsilon_0 + P_0} + \frac{\mathcal{E}}{3}(\partial \cdot u), W^\mu = \theta[\frac{\nabla^\mu T}{T} + Du^\mu]$ and $\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$. The shear channel analysis is identical to that of MIS theory with the replacement $\tau_\pi = \theta/(\epsilon_0 + P_0)$ [63]. However, the situation becomes significantly more mathematically involved



(a) Shear Channel

(b) Sound Channel

Figure 5.1: Linearly stable parameter space for MIS Shear channel (Left) and MIS Sound channel (Right) for different $\mathbf{v}$ values

in the sound channel. The leading order ω solution in k-expansion gives rise to the quadratic dispersion relation $a\omega^2 + b\omega + c = 0$, with $a = \gamma^2[\tilde{\mathcal{E}}\tilde{\theta} - \frac{2}{3}\tilde{\mathcal{E}}(2\tilde{\eta} + \tilde{\theta})\mathbf{v}^2 + \frac{1}{9}\tilde{\theta}(\tilde{\mathcal{E}} - 4\tilde{\eta})\mathbf{v}^4]$, $b = i\gamma[(\tilde{\mathcal{E}} + \tilde{\theta}) - \frac{1}{3}(\tilde{\theta} + \tilde{\mathcal{E}} + 4\tilde{\eta})\mathbf{v}^2]$ and $c = (\mathbf{v}^2/3 - 1)$. This dispersion polynomial gives rise to two non-propagating, non-hydro modes whose stability has been analyzed using the Routh-Hurtwitz (R-H) stability test [102]. The stability criteria constrain the parameter space for BDNK sound channel through the two following inequalities,

$$\mathcal{E}\theta\left(1 - \frac{\mathbf{v}^2}{3}\right)^2 - \frac{4}{3}\eta\mathbf{v}^2\left(\mathcal{E} + \frac{\mathbf{v}^2}{3}\theta\right) > 0\,, \tag{5.1}$$

$$(\mathcal{E} + \theta)\left(1 - \frac{\mathbf{v}^2}{3}\right) - \frac{4}{3}\eta\mathbf{v}^2 > 0\,. \tag{5.2}$$

Eq.(5.1) and (5.2) together necessarily confine the parameter space within the region,

$$\frac{\theta}{\eta} > \frac{4}{3}\frac{\mathbf{v}^2}{(1 - \mathbf{v}^2/3)^2}\,, \qquad \frac{\mathcal{E}}{\eta} > \frac{4}{9}\frac{\mathbf{v}^4}{(1 - \mathbf{v}^2/3)^2}\,. \tag{5.3}$$

The right-hand sides of both the inequalities are monotonically increasing functions of $\mathbf{v}$ which allow only positive values of $\mathcal{E}$ and θ with lower bounds ranging from 0 to η and 0 to 3η respectively as $\mathbf{v}$ ranges from 0 to 1. Following these conditions, Fig.5.2 shows that the parameter space where the theory is stable at $\mathbf{v} \to 1$ is enclosed within the same for any lower value of $\mathbf{v}$. So, identical to the situation of MIS theory, for BDNK theory as well, the stability condition at $\mathbf{v} \to 1$, is a necessary and sufficient condition for stability to hold at the spatially homogeneous limit for all possible boost velocities $0 \leq \mathbf{v} < 1$.

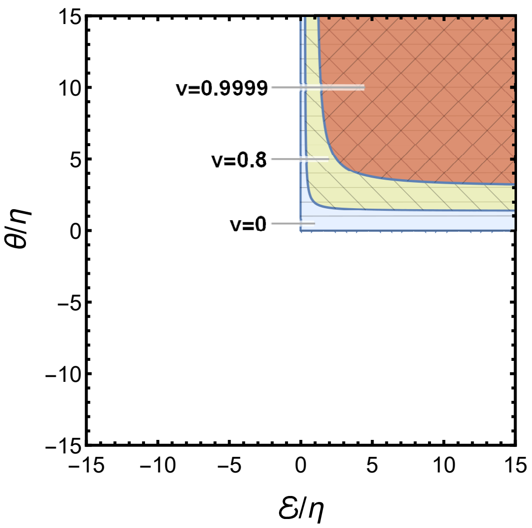


Figure 5.2: Linearly stable parameter space for BDNK sound channel for different $\mathbf{v}$ values.

Given the above analysis for MIS and BDNK theories, we establish our first key finding here. For relativistic dissipative hydrodynamic theories like BDNK and MIS, performing stability analysis at ultra-high boost velocity ($\mathbf{v} \to 1$) alone suffices to conclude the stability invariance of the theory. Stability analysis at any other boost velocity lacks this confirmation. The stable parameter space at $\mathbf{v} \to 1$ is a necessary and sufficient region of the theory for stability invariance to hold at the spatially homogeneous limit.

5.4 Causality from stability analysis

In this section, we will prove that only the stability criteria at $\mathbf{v} \to 1$ limit is enough to provide the region of parameter space over which each of these two theories is causal. The idea is that, since it has been proven for theories like MIS and BDNK that the stability conditions at $\mathbf{v} \to 1$ identify the region of parameter space where the system is frame invariantly stable, and since stability invariance requires the causality properties of the theory to be respected according to the arguments put forward in [107, 134], hence the stability constraints at ultra-high boost automatically lead us to the causal region of the parameter space. For MIS theory, the stability conditions at $\mathbf{v} \to 1$ limit for the shear and sound channels give us $\frac{\tau_\pi}{\tilde{\eta}} > 1$ and $\frac{\tau_\pi}{2\tilde{\eta}} > 1$ respectively. It can be shown that the expressions on the left-hand sides of the inequalities for both channels are functions of the square of respective asymptotic group velocities $v_g = \lim_{k\to\infty} \left|\frac{\partial \text{Re}(\omega)}{\partial k}\right|$, $(v_g^2)^\perp = \tilde{\eta}/\tau_\pi$ and $(v_g^2)^\parallel = \frac{4\tilde{\eta}}{3\tau_\pi} + \frac{1}{3}$. These expressions for both the channels finally reduce to $0 < v_g^2 < 1$, and therefore, the stability criteria at $\mathbf{v} \to 1$ boil down to the asymptotic causality condition $0 < v_g^2 < 1$ for the MIS theory in the parameter range $\eta, \tau_\pi > 0$.

For BDNK theory, the shear channel stability condition at $\mathbf{v} \to 1$ gives $\frac{\theta}{\eta} > 1$, which is again the asymptotic causality condition $0 < v_g^2 < 1$ where $v_g^2 = \frac{\eta}{\theta}$. Next, for the BDNK sound channel, we attempt to solve the inequalities (5.1) and (5.2) served as stability criteria in a boosted frame. Stability inequality (5.1) can be recast as,

$$\left\{\left(1/\mathbf{v}^2\right) - x_1\right\}\left\{\left(1/\mathbf{v}^2\right) - x_2\right\} > 0\,, \tag{5.4}$$

where x_1, x_2 are the roots of the equation,

$$(\mathcal{E}\theta)x^2 - \frac{2}{3}\mathcal{E}(2\eta + \theta)x + \frac{1}{9}\theta(\mathcal{E} - 4\eta) = 0\,. \tag{5.5}$$

Inequality (5.4) has two possible solutions $x_1, x_2 < \frac{1}{\mathbf{v}^2}$ or $x_1, x_2 > \frac{1}{\mathbf{v}^2}$. Since $|\mathbf{v}|$ ranges from 0 to 1 and hence $1/\mathbf{v}^2$ ranges from 1 to ∞, the second solution turns out to the unphysical. The first and only physically acceptable solution then gives us the strictest bound $x_1, x_2 < 1$ corresponding to the limit $\mathbf{v} \to 1$. Now, incorporation of the second stability inequality (5.2) restricts the allowed region to only positive values of $\mathcal{E}$ and θ. This restriction (along with $\eta > 0$) leads to a positive discriminant of (5.5), which restricts both the roots of x to be real, among which at least one root is always positive in our stable parameter space at $\mathbf{v} \to 1$. As it will be explicitly shown in the next section doing a large k analysis of the theory that the quadratic equation satisfied by v_g^2 for the BDNK sound channel is exactly identical to (5.5), the inequalities (5.1) and (5.2) condense down together to give $v_g^2 < 1$ with at least one $v_g^2 > 0$ that produces two subluminal propagating modes. So, our stability analysis at ultra-high boost independently identifies the causal parameter space of the MIS and BDNK theories, which exactly reproduces the results of asymptotic causality analysis for the respective theories without going to the large k limit.

5.5 Causality from large k analysis

Now, let us analyze the situation of causality in the high-k regime itself and compare how accurately the subluminal parameter space has been predicted by stability analysis at ultra-high boost. At the large k limit, an expansion of the form $\omega = v_g k + \sum_{n=0}^{\infty} c_n k^{-n}$ is used [130] as a solution of the dispersion equation from which a polynomial over the asymptotic group velocity v_g can be obtained. Next, we check the Schur stability of the polynomial [110] to check if the roots of these equations are subluminal and, if they are, then how the parameter space is constrained by them. Any polynomial $P(z)$ of degree d is called "Schur stable" if its roots lie within a unit disc around the origin of the complex plane. This can be tested by introducing a Möbius transformation $w = (z+1)/(z-1)$, which maps the unit disc about the origin of the complex plane into the left half plane, i.e., Re$(w) < 0$ if $|z| < 1$. So, $P(z)$ will be Schur stable if and only if the transformed polynomial of the same degree $Q(w) = (w-1)^d P\left(\frac{w+1}{w-1}\right)$ is Hurwitz stable. This method is extremely efficient, especially in cases where a direct extraction of roots from the polynomial is too complicated.

For the shear channels, the Schur stability conditions that can give rise to subluminal, propagating modes are $\tau_\pi - \tilde{\eta} > 0$ and $\tau_\pi + \tilde{\eta} > 0$ for MIS and $\theta - \eta > 0$ and $\theta + \eta > 0$ for BDNK. In both cases, the first conditions are identically the stability conditions obtained at $\mathbf{v} \to 1$ and the second conditions are obvious if the first ones are satisfied. For the propagating modes of MIS sound channel, the Schur stability conditions are given by $\tau_\pi - 2\tilde{\eta} > 0$ and $\tau_\pi + \tilde{\eta} > 0$. Again, the first one is the $\mathbf{v} \to 1$ stability criterion, and the rest is its obvious implication. So, we conclude that for both the shear channels and the MIS sound channel, the $\mathbf{v} \to 1$ stability region exactly reproduces the causal parameter space.

The situation in the BDNK sound channel is comparatively quite non-trivial. The v_g^2 values are to be extracted from the following quadratic polynomial with $z = v_g^2$,

$$P(z) = (\mathcal{E}\theta)z^2 - \frac{2}{3}\mathcal{E}(\theta + 2\eta)z + \frac{1}{9}\theta(\mathcal{E} - 4\eta) = 0 \,, \tag{5.6}$$

whose Schur stability needs to be checked to find the causal parameter space. Its Möbius transformation again turns out to be a quadratic polynomial,

$$Q(w) = \left(\frac{\mathcal{E}\theta}{3} - \mathcal{E}\eta - \frac{\eta\theta}{3}\right) w^2 + \frac{2}{3}\theta\left(\eta + 2\mathcal{E}\right) w + \left(\frac{4\mathcal{E}\theta}{3} + \mathcal{E}\eta - \frac{\eta\theta}{3}\right) = 0 \,, \tag{5.7}$$

whose Hurwitz stability requires all the three coefficients of Eq.(5.7) to be of the same sign, either positive or negative (along with a positive discriminant of $P(z)$ to ensure that all the non-real roots of v_g^2 on the complex plane are excluded).

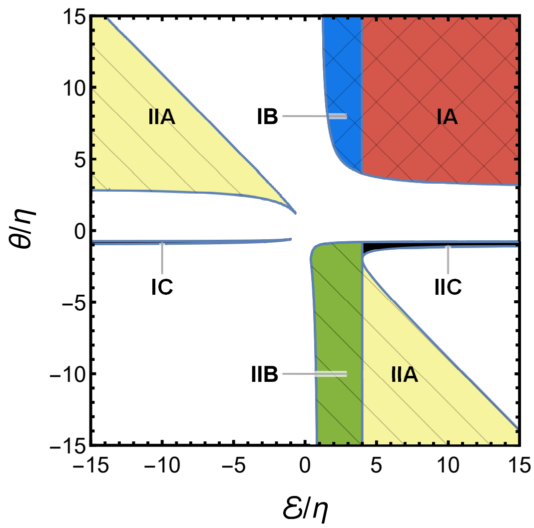


Figure 5.3: The subluminal parameter space for BDNK sound channel from Schur stability.

In Fig.5.3, the parameter space for which both the roots satisfy $\left|v_g^2\right| < 1$ are plotted for both the positive as well as negative conventions. The regions IA (red, crisscrossed), IB (blue, crisscrossed) and IC (black, solid-filled) are located within quadrants where both θ and $\mathcal{E}$ are of the same sign and indicate the regions of the parameter space where all the coefficients of (5.7) are positive. The regions IIA (yellow, striped), IIB (green, striped) and IIC (black, solid-filled) are located within quadrants with θ and $\mathcal{E}$ of opposite signs and denote the convention where all coefficients of (5.7) are negative. Together, all of these regions (IA-C, IIA-C) provide the full causal parameter space given by (5.6). Furthermore, the signs of the coefficients of (5.6) indicate that the regions IC and IIC bounded by $\mathcal{E} > 4\eta, \mathcal{E} < 0, -2\eta < \theta < 0$ give $-1 < v_g^2 < 0$ for both roots and hence, fail to generate any propagating mode. The rest of the regions (IA-B, IIA-B) correspond to at least one $0 < v_g^2 < 1$ and hence at least two subluminal propagating modes. The regions IA and IIA cover the parameter space with the additional constraints $\mathcal{E} < 0, \mathcal{E} > 4\eta, \theta > 0, \theta < -2\eta$, which give us both v_g^2 values between 0 and 1 and hence, four subluminal propagating modes. The remaining two regions, IB and IIB, belong to the parameter space constrained by $0 < \mathcal{E} < 4\eta$, which corresponds to $-1 < v_g^2 < 0$ for one root and $0 < v_g^2 < 1$ for the other, indicating the presence of two non-propagating modes besides the existence of the two subluminal propagating modes.

Now comes a crucial identification; we observe that the causal parameter space in the first quadrant covered by the regions IA and IB together exactly agrees with the stable region at $\mathbf{v} \to 1$ and hence, with the frame-invariantly stable parameter space as well. This can be readily checked by realizing that the Schur condition from (5.7), $-\frac{\eta\theta}{3} - \mathcal{E}\eta + \frac{\mathcal{E}\theta}{3} > 0$ is exactly identical to the stability constraint (5.1) at $\mathbf{v} \to 1$. The other two Schur conditions, $\theta(\eta + 2\mathcal{E}) > 0$ and $-\frac{\eta\theta}{3} + \mathcal{E}\eta + \frac{4\mathcal{E}\theta}{3} > 0$ along with a positive discriminant of (5.6), further restrict the region exclusively to within the $\theta > 0, \mathcal{E} > 0$ quadrant for propagating modes, which exactly resembles the role played by (5.2) with $\mathbf{v} \to 1$ to define the stable parameter space. So, the entire causal parameter space obtained from the asymptotic equation (5.6) (by Schur convention I, all coefficients > 0) is fully identified by the stable region at ultra-high boost depicted in Fig.5.2. In this context, we refer to the results obtained in [139], where the large wave-number causality constraint is given solely by region IA with four subluminal propagating modes. The analysis there lacks the region IB where two subluminal propagating modes are present along with two non-propagating modes. We duly point out that this lacking region is stable in every reference frame (Fig.5.2), which invariably identifies this region to respect causality since covariant stability is possible only for causal systems [107, 134]. So, we conclude that, because of the complexity involved, it is indeed difficult to analytically extract the full causal parameter space from the large-k dispersion polynomial. However, the method of stability analysis at $\mathbf{v} \rightarrow 1$ presented in this chapter is much more effective in pointing out the full stable and causal parameter space unambiguously.

We finally point out that for regions IIA and IIB, where θ and $\mathcal{E}$ are of opposite signs, the system is unstable in all reference frames. As mentioned in the stability arguments of [134], there could be other regions of the parameter space like IIA and IIB, where causality holds, but the system is invariantly unstable in all reference frames. The stability criteria at ultra-high boost strictly give us the parameter space where these two theories are causal as well as stable in all reference frames.

5.6 Conclusion

We have shown here, for the first time, for two well-known stable-causal hydrodynamic theories, viz. MIS and BDNK, an alternate way to derive the region of parameter space over which the theories are frame-invariantly stable at leading order in k and necessarily causal. Despite inherent differences in their construction, our analysis reveals that linearized stability analysis at ultra-high boost accurately leads us to the results of asymptotic causality conditions under which both the theories are frame-invariantly stable, without going to the large-k limit. Since the whole analysis is performed at a low-k limit, this approach liberates us from going to a non-perturbative high-k regime that seems outside the domain of validity of a low-energy effective theory like relativistic hydrodynamics. Moreover, in the presence of technical non-trivialities in solving the asymptotic causality equations, our method of stability check at $\mathbf{v} \rightarrow 1$ is more effective and simpler in detecting the causal parameter space. Although the current analysis has been carried out for a conformal, chargeless system, the results presented here do not lack in generality. In [135], a coarse-grained derivation of a non-conformal, charged, stable-causal first-order theory indeed shows that the monotonically decreasing stable parameter space becomes the strictest bound for $\mathbf{v} \to 1$ which singularly gives the causal parameter space as well.

The findings presented here heavily depend upon the monotonic behavior of the stable parameter space as a function of $\mathbf{v}$. The monotonic behavior that exists for these two most well-known stable-causal theories doesn't hold for the relativistic first-order Navier-Stokes theory. This indicates that this feature could be an important signature for pathology-free hydrodynamic theories. Further, the prediction of high-k results from the low-k domain using ultra-high boost, as observed here, indicates some possible connection between the two limiting k-regimes of the theories, which requires further investigation. In Appendix A, we have derived our results for a more general class of hydrodynamic problems and provided intuitive arguments in support of the current outcome. The causality criteria considered here are asymptotic causality criteria, which are necessary but not sufficient conditions [141]. A more rigorous study of causality requires a study of characteristics [108, 109], which will be explored in our future endeavors.

Chapter 6

Frame transformation and stable-causal hydrodynamic theory

This chapter is based on [98].

6.1 Introduction

6.1.1 Summary and discussion of our results

In this chapter, our goal is to rewrite the BDNK stress tensor in the Landau frame by redefining the velocity and the energy density (temperature). In some sense, the key result of this work is the relation between the fluid variables in BDNK formalism (denoted by u^μ and T respectively) and the velocity and the temperature field defined after frame transformation that are fixed through the Landau gauge condition (denoted as $\hat{u}^\mu$ and $\hat{T}$). We have explicitly worked out the relation for those fluid profiles that have small fluctuations around some global equilibrium. We have assumed that the amplitudes of the fluctuations are small enough so that a linearized treatment is justified. Further, in order to obtain an analytically tractable all-order theory, we have restricted our analysis only to conformal, uncharged fluids in BDNK formalism.

To state our results in terms of equations, let us first introduce a notation $u^\mu - \hat{u}^\mu = \delta u^\mu$ and $T - \hat{T} = \delta T$. We have found that the shift variables δu^μ and δT must satisfy the following differential equations up to terms that are linear in δT, δu^μ and their derivatives,

$$\frac{\delta T}{\hat{T}} + \tilde{\chi}\left[\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{\hat{\nabla}_\mu \hat{u}^\mu}{3}\right] + \tilde{\chi}\left[\frac{\hat{D}\delta T}{\hat{T}} + \frac{\hat{\nabla}_\mu \delta u^\mu}{3}\right] = 0\,,$$
$$\delta u^\mu + \tilde{\theta}\left[\hat{D}\hat{u}^\mu + \frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}}\right] + \tilde{\theta}\left[\hat{D}\delta u^\mu + \frac{\hat{\nabla}^\mu \delta T}{\hat{T}}\right] = 0\,. \tag{6.1}$$

Next, we develop a formal solution for the equations (6.1) using two different methods. In both cases, it is manifested that the solutions will have terms up to all orders in derivative expansion. Finally, we introduce a set of new tensorial 'non-fluid' variables (like the shear tensor in MIS theory) in order to recast the BDNK theory in an MIS-type formalism where the fluid variables like velocity and the temperature are defined through the Landau gauge condition.

In the first method, the equivalent system of equations will have an infinite number of 'non-fluid' variables with the following nested structure of the energy-momentum tensor $T^{\mu\nu}$:

$$
\begin{aligned}
&\partial_\mu T^{\mu\nu} = 0\,, \qquad T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu\hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} = -2\eta\hat{\sigma}^{\mu\nu} + \rho_1^{\mu\nu},\\
&(1+\tilde{\chi}\hat{D})\rho_1^{\mu\nu} = (-2\eta)(-\tilde{\theta})\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{T} + \rho_2^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\rho_2^{\mu\nu} = (-2\eta)(-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right)\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}\cdot\hat{u} + \rho_3^{\mu\nu},\\
&(1+\tilde{\chi}\hat{D})\rho_3^{\mu\nu} = (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{T} + \rho_4^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\rho_4^{\mu\nu} = (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)^2\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u} + \cdots\\
&\vdots
\end{aligned}
\tag{6.2}
$$

In the second method, we need to introduce only one 'shear tensor' type non-fluid variable, but its equation of motion turns out to be second order in spatial and third order in temporal derivatives,

$$
\begin{aligned}
&\partial_\mu T^{\mu\nu} = 0\,, \qquad T^{\mu\nu} = \hat{\varepsilon}\left(\hat{u}^\mu\hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right) + \hat{\pi}^{\mu\nu}\,,\\
&\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \tilde{\theta}\frac{\tilde{\chi}}{3}\,\hat{\nabla}^2\right]\left\{(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} + 2\eta\hat{\sigma}^{\mu\nu}\right\}\\
&\qquad = 2\eta\tilde{\theta}\left\{1+(\tilde{\theta}+\tilde{\chi})\hat{D}\right\}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}\,.
\end{aligned}
\tag{6.3}
$$

We have analyzed the spectrum of linearized fluctuations in both systems and found that all the hydrodynamic modes match those of the BDNK theory. This indicates that in the regime where fluid descriptions are applicable, all three systems of equations presented here are equivalent. However, equations (6.2) and equations (6.3) also have some extra non-hydrodynamic modes which are not there in the BDNK theory. The emergence of these new modes is possibly connected with the zero modes in the equations of the field redefinition (equations (6.1)) themselves.

Our equations are by no means more tractable than that of the BDNK. But here, the fluid variables have a clear and standard meaning, and since the velocity and temperature in BDNK theory could be precisely transformed to these variables (though we have derived it only at a linearized level), it attaches a similar definition to the BDNK fluid variables as well. Our analysis suggests that even in BDNK theory, there will be hidden non-fluid variables (or an infinite number of derivatives) if one would like to express the theory in terms of fluid variables only, which are locally defined through stress-energy tensor as we have in 'Landau frame'[1].

6.1.2 Convention and notations

Throughout the chapter, we have used natural unit ($\hbar = c = k_B = 1$) and flat space-time with mostly positive metric signature $g^{\mu\nu} = \text{diag}\,(-1,1,1,1)$. ε, T, P, u^μ are, respectively, energy density, temperature pressure and hydrodynamic four-velocity. The local rest frame is defined as $u^\mu = (1,0,0,0)$, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the space projection operator orthogonal to u^μ. $\Delta^{\mu\nu\alpha\beta} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta} + \frac{1}{2}\Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}$ is the traceless projection operator orthogonal to u_μ and $\Delta_{\mu\nu}$. Any rank-2, symmetric, traceless tensor is defined as, $A^{\langle\mu}B^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}A^\alpha B^\beta$. The used derivative operators read as: covariant time derivative $D = u^\mu\partial_\mu$, spatial gradient $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ and traceless, symmetric velocity gradient $\sigma^{\mu\nu} = \partial^{\langle\mu}u^{\nu\rangle}$. η is the shear viscous coefficient, τ_π is the relaxation time of shear-viscous flow $\pi^{\mu\nu}$ of MIS theory, χ, θ are the first order field correction coefficients of BDNK theory. From the constraints of the second law of thermodynamics, η should always be a positive number [140]. The scaling notation $\tilde{x}$ denotes $x/(\varepsilon + P)$. We linearize the conservation equations for small perturbations of fluid variables around their hydrostatic equilib-

[1]In this context, we should mention the analysis in [142]. Here also, the authors connect the MIS and the BDNK type formalism with field redefinition. However, the authors here tried to explain this field redefinition ambiguity more from a microscopic point of view. Whereas, in our analysis, we are completely agnostic about the microscopic descriptions or statistical interpretation of these field redefinitions. As a result, we could find more than one way (in fact, in principle, there should be just an infinite number of ways) of 'integrating in' the non-fluid variables for the same BDNK theory but recast in the Landau frame.

rium, $\psi(t,x) = \psi_0 + \delta\psi(t,x)$, with the fluctuations expressed in the plane wave solutions via a Fourier transformation $\delta\psi(t,x) \to e^{i(kx-\omega t)}\delta\psi(\omega,k)$, (subscript 0 indicates global equilibrium).

6.1.3 Outline of the rest of the chapter

This chapter is organized as follows. In section 6.2, we describe the MIS theory in its simplest form, and then we show how integrating out the extra 'non-fluid' variable results in a stress tensor with an infinite number of derivatives. This section will act as a warm-up for the techniques of infinite sum to be used in the next section. Also, it indicates how a causal theory in the Landau frame, if expressed only in terms of fluid variables, turns out to have an infinite number of derivatives. In the next section 6.3, we describe the BDNK theory and redefine the velocity and temperature variables (only at the linearized level) to bring them to the Landau frame. Redefinition involves generating an infinite number of derivatives. We can sum these infinite series in two different ways as described in two different subsections of section 6.3. These two different ways of summation lead to two different methods of 'integrating in' new 'non-fluid' variables, showing the non-uniqueness of the process of 'integrating in' new variables. In section 6.4, the dispersion relations and the corresponding spectra of these different systems of equations have been analyzed to check that our systems of equations are indeed equivalent to BDNK formalism, at least in the hydrodynamic regime. Finally, in section 6.5, we conclude.

6.2 MIS theory - an infinite order fluid formalism

The pathologies regarding superluminal signal propagation and thermodynamic stability of the long-established relativistic first-order theories [55, 143], have been first taken care by the higher order MIS theory [56, 111], where the dissipative field corrections are promoted to new degrees of freedom [59, 60, 136, 137]. Keeping up to the linear terms, the MIS equations of motion are given by [50],

$$\partial_\mu T^{\mu\nu} = 0\,, \quad T^{\mu\nu} = \varepsilon \left[u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu}\right] + \pi^{\mu\nu}\,, \tag{6.4}$$

$$\pi^{\mu\nu} + \tau_\pi D\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}\,. \tag{6.5}$$

Here, we attempt to derive the combined results of Eq.(6.4) and (6.5) without treating $\pi^{\mu\nu}$ as an independent degree of freedom. Instead of attributing an individual differential equation to $\pi^{\mu\nu}$ like Eq.(6.5), we express it as a sum of gradient corrections that includes all derivative orders in Eq.(6.4) itself such as,

$$\pi^{\mu\nu} = \sum_{n=1}^{\infty} \pi_n^{\mu\nu}\,,$$

$$\pi_1^{\mu\nu} = -2\eta\sigma^{\mu\nu}\,, \quad \pi_n^{\mu\nu} = -\tau_\pi D\pi_{n-1}^{\mu\nu}\,, n \geq 2\,. \tag{6.6}$$

This leads to the shear-stress tensor as the following,

$$\pi^{\mu\nu} = -\,2\eta \left\{\sum_{n=0}^{\infty} (-\tau_\pi D)^n\right\} \sigma^{\mu\nu} \tag{6.7}$$

$$= -\,2\eta\,(1 + \tau_\pi D)^{-1}\,\sigma^{\mu\nu}\,. \tag{6.8}$$

So, we conclude that if we want to write the MIS theory without introducing any additional degrees of freedom, this will lead to a stress tensor that is defined up to all orders of gradient correction. Any finite truncation of Eq.(6.7) fails to produce the relaxation operator like structure in the denominator of Eq.(6.8). However, it is to be noted that Eq.(6.5) is local in both time and space, whereas Eq.(6.8) becomes non-local in time since the frequency of the corresponding Fourier mode appears in the denominator. The details of the acausality of a truncated series in Eq.(6.7) can be found in [97].

6.3 BDNK theory and the transformation of the 'fluid frame'

In the last few years, a new study of the relativistic first-order stable-causal theory (BDNK theory) has been proposed by defining the out-of-equilibrium hydrodynamic variables in a general frame other than that is defined by Landau or Eckart, through their postulated constitutive relations that include spatial as well as temporal gradients [61,63,64,67,138,139] . In BDNK theory, if we further impose conformal symmetry and no conserved charges, the energy-momentum tensor ($T^{\mu\nu}$) takes the form,

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + \mathcal{A}) \left[u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3} \right] \\ &\quad + [u^\mu Q^\nu + u^\nu Q^\mu] - 2\eta\sigma^{\mu\nu}, \end{aligned} \tag{6.9}$$

with the first-order dissipative field corrections as,

$$\mathcal{A} = \chi \left[3\frac{DT}{T} + \nabla_\mu u^\mu \right], \quad Q^\mu = \theta \left[\frac{\nabla^\mu T}{T} + Du^\mu \right]. \tag{6.10}$$

We have used the identity $D\varepsilon/(\varepsilon + P) = 3DT/T$ for a conformal system where the energy density goes as $\varepsilon \sim T^4$, and it is connected with the pressure P as $(\varepsilon + P) = 4\varepsilon/3$ [2]. The dispersion relations resulting from Eq.(6.9) produce stable-causal modes only with non-zero values of θ and χ. The neatness of this method lies in not requiring any additional degrees of freedom other than the temperature and velocity to preserve causality and stability. Eq.(6.9) and (6.10) also show that the theory is local in fluid variables both spatially and temporally. However, as mentioned before, unlike the MIS theory, the definitions of the fluid velocity and the temperature are not fixed here in terms of stress tensor or any other microscopic operator. In this section, we would like to redefine the velocity and the temperature in a way so that the stress tensor, expressed in terms of these redefined fluid variables, satisfies the Landau frame condition. Our philosophy is as follows.

We shall assume that the one-point function of the microscopic stress tensor operator in a 'near thermal' state is given by the BDNK stress tensor (6.9). But it is expressed in terms of some 'velocity' and 'temperature' variables $\{u^\mu, T\}$, which agree with the traditional definitions of velocity and temperature in global equilibrium but deviate in a generic 'near equilibrium' state. On the other hand, we know that in the Landau frame, the velocity and the temperature fields are locally defined

[2] For simplicity, throughout this chapter, we shall restrict our analysis to conformal fluids, where temperature provides the only scale and the space-time dependence of all other dimensional variables like energy density is completely determined by that of the temperature. For example, $\varepsilon(x^\mu) = 3c\, T^4(x^\mu)$, $P(x^\mu) = c\, T^4(x^\mu)$, where c is some constant. Because of this, while discussing the space-time dependence of the fluid variables, we shall often use $\varepsilon(x^\mu)$, $P(x^\mu)$ or $T(x^\mu)$ interchangeably.

in terms of the one-point function of the Stress tensor $T^{\mu\nu}$ as the following,

$$T^{\mu}_{\nu}(\hat{T}, \hat{u}^{\mu})\, \hat{u}^{\nu} = -\hat{\varepsilon}\, \hat{u}^{\mu}\ . \tag{6.11}$$

We denote the notation $\hat{}$ to indicate fields at the Landau frame such as $\hat{u}^{\nu}$ is the velocity, $\hat{\varepsilon}$ is the local energy density and $\hat{T}$ is the temperature in the Landau frame. Transforming the BDNK stress tensor in the Landau frame involves two steps. First, we have to solve for $\hat{u}^{\mu}$ and $\hat{\varepsilon}$ by solving equation (6.11), where in place of T^{μ}_{ν} we shall use the BDNK stress tensor (6.9). The second step involves rewriting the BDNK stress tensor in terms of these new fluid variables $\hat{u}^{\mu}$ and $\hat{\varepsilon}$.

Generically, performing such a frame transformation in a non-perturbative manner is extremely cumbersome. But to make our analysis computationally tractable, we restrict it to linearized treatment. Physically, we are restricting our analysis only to those fluid states whose deviation from global equilibrium is of very small amplitude. Such perturbations are enough to decide the linear stability and the causality of the theory - the key motivation behind the BDNK formalism. Since all definitions of the fluid variables agree in global equilibrium (or at the level of 'ideal' fluid), field redefinition is needed only in 'non-equilibrium' fluid states. It follows that, if the deviation from equilibrium is of small amplitude such that a linearized treatment is allowed, the same should also be true for field redefinition. In other words, while redefining the velocity and the temperature, we can safely ignore terms that are nonlinear in the shift of the variables. In terms of equations, what we mean is the following.

We define that the velocity u^{μ} and the temperature T in the BDNK stress tensor are related to the Landau frame velocity $\hat{u}^{\mu}$ and temperature $\hat{T}$ in the following fashion,

$$u^{\mu} = \hat{u}^{\mu} + \delta u^{\mu}, \quad T = \hat{T} + \delta T\ , \tag{6.12}$$

where the shift variables δu^{μ} and δT are small enough to be treated only linearly. Note that both δu^{μ} and δT are non-trivial functions of $\hat{u}^{\mu}$ and $\hat{T}$. Once we impose the Landau gauge condition (6.11) after substituting (6.12) in the BDNK stress tensor (6.9), it reduces to the following set of coupled and linear partial differential equations (PDEs) for the shift variables,

$$\delta u^\mu + \tilde{\theta}\left[\hat{D}\hat{u}^\mu + \frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}}\right] + \tilde{\theta}\left[\hat{D}\delta u^\mu + \frac{\hat{\nabla}^\mu \delta T}{\hat{T}}\right] = 0, \tag{6.13}$$

$$\frac{\delta T}{\hat{T}} + \tilde{\chi}\left[\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{\hat{\nabla}_\mu \hat{u}^\mu}{3}\right] + \tilde{\chi}\left[\frac{\hat{D}\delta T}{\hat{T}} + \frac{\hat{\nabla}_\mu \delta u^\mu}{3}\right] = 0\,. \tag{6.14}$$

This linearization simplifies the analysis so that we can have an ‘all-order’ (in derivatives) formula for both the field redefinitions and the stress tensor in the new frame.

It turns out that the ‘MIS type nonlocality’ emerges here again, even in the BDNK theory, due to the infinite order field redefinition is needed to cast it in the Landau frame. At the linearized level, the field redefinition can be done in two different representations. In one case, we summed only the time derivatives up to the infinite order, leading to a set of equations that look nonlocal in time (with the time derivative appearing in the denominator) but local in space. In the second case, we summed both the time and the space derivatives, leading to a full nonlocal redefinition of the fluid variables. In either case, these nonlocalities (derivatives appearing in the denominator) could be absorbed by introducing new ‘non-fluid’ variables. These two different methods are described in the following two different subsections.

6.3.1 Method-1: Frame transformation order by order

In this subsection, we shall solve these PDEs (6.13) and (6.14) order by order in derivative expansion. We shall assume that $\delta u^\mu, \delta\varepsilon$ and δT admit the following infinite series expansion,

$$\delta u^\mu = \sum_{n=1}^{\infty} \delta u^\mu_n, \quad \delta\varepsilon = \sum_{n=1}^{\infty} \delta\varepsilon_n, \quad \delta T = \sum_{n=1}^{\infty} \delta T_n\,. \tag{6.15}$$

Here, the subscript (n) denotes the order in terms of derivative expansion. Substituting the expansion of (6.15) in the PDEs (6.13) and (6.14), one can easily find the solution in terms of the following recursive relations,

$$\delta T_1 = -\tilde{\chi}\left[\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{1}{3}\hat{\nabla}_\mu \hat{u}^\mu\right], \quad \delta u_1^\mu = -\tilde{\theta}\left[\frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}} + \hat{D}\hat{u}^\mu\right],$$
$$\delta T_n = -\tilde{\chi}\left[\frac{1}{\hat{T}}\hat{D}\delta T_{n-1} + \frac{1}{3}\hat{\nabla}_\mu \delta u^\mu_{n-1}\right] \quad \text{for } n \geq 2\,,$$
$$\delta u_n^\mu = -\tilde{\theta}\left[\frac{1}{\hat{T}}\hat{\nabla}^\mu \delta T_{n-1} + \hat{D}\delta u^\mu_{n-1}\right] \quad \text{for } n \geq 2\,. \tag{6.16}$$

Eq.(6.16) provides the successive field corrections up to any desired order.

The next step is to rewrite the energy-momentum tensor in terms of the new fluid variables. The energy-momentum tensor in this frame turns out to be,

$$\begin{aligned}T^{\mu\nu} = &\left[\hat{\varepsilon} + \sum_{n=1}^{\infty}\delta\varepsilon_n + \chi\left\{3\frac{\hat{D}\hat{T}}{\hat{T}} + \partial_\alpha \hat{u}^\alpha + \frac{3}{\hat{T}}\hat{D}\sum_{n=1}^{\infty}\delta T_n + \partial_\alpha \sum_{n=1}^{\infty}\delta u_n^\alpha\right\}\right]\left(\hat{u}^\mu\hat{u}^\mu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right)\\
&+\left[\frac{4}{3}\hat{\varepsilon}\sum_{n=1}^{\infty}\delta u_n^\nu + \theta\left\{\frac{\hat{\nabla}^\nu\hat{T}}{\hat{T}} + \hat{D}\hat{u}^\nu + \frac{1}{\hat{T}}\hat{\nabla}^\nu\sum_{n=1}^{\infty}\delta T_n + \hat{D}\sum_{n=1}^{\infty}\delta u_n^\nu\right\}\right]\hat{u}^\mu\\
&+\left[\frac{4}{3}\hat{\varepsilon}\sum_{n=1}^{\infty}\delta u_n^\mu + \theta\left\{\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}} + \hat{D}\hat{u}^\mu + \frac{1}{\hat{T}}\hat{\nabla}^\mu\sum_{n=1}^{\infty}\delta T_n + \hat{D}\sum_{n=1}^{\infty}\delta u_n^\mu\right\}\right]\hat{u}^\nu - 2\eta\left[\hat{\sigma}^{\mu\nu} + \sum_{n=1}^{\infty}\partial^{\langle\mu}\delta u_n^{\nu\rangle}\right].\end{aligned} \tag{6.17}$$

As mentioned before, only linearized terms are considered. The used notations (now defined in terms of Landau frame variable) read : $\hat{\Delta}^{\mu\nu} = g^{\mu\nu} + \hat{u}^\mu\hat{u}^\nu$, $\hat{D} = \hat{u}^\mu\partial_\mu$, $\hat{\nabla}^\mu = \hat{\Delta}^{\mu\nu}\partial_\nu$, $\hat{\sigma}^{\mu\nu} = \partial^{\langle\mu}\hat{u}^{\nu\rangle} = \hat{\Delta}^{\mu\nu}_{\alpha\beta}\partial^\alpha\hat{u}^\beta$. After substituting the recursive solution for δu_n^μ and δT_n as given in (6.16), the energy density correction and energy-flux or momentum flow vanish as expected in the Landau frame, and one finally has the following energy-momentum tensor upto all order,

$$T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu\hat{u}^\mu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] - 2\eta\left[\hat{\sigma}^{\mu\nu} + \sum_{n=1}^{\infty}\partial^{\langle\mu}\delta u_n^{\nu\rangle}\right]. \tag{6.18}$$

All order sum of the temporal derivatives

Once we explicitly evaluate δu_n^μ and δT_n for the first few orders, we observe that a very nice pattern emerges, which we could use to sum this infinite series to get an all-order expression.

In order to do so, first we list the velocity and temperature corrections up to first four orders obtained from the Landau matching conditions:

$$\delta u_1^\mu = -\tilde{\theta}\left[\hat{D}\hat{u}^\mu + \frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}}\right] , \tag{6.19}$$

$$\frac{\delta T_1}{\hat{T}} = -\tilde{\chi}\left[\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{1}{3}\left(\hat{\nabla}\cdot\hat{u}\right)\right] , \tag{6.20}$$

$$\delta u_2^\mu = \tilde{\theta}^2\hat{D}^2\hat{u}^\mu + \tilde{\theta}\left[\tilde{\theta}+\tilde{\chi}\right]\frac{1}{\hat{T}}\hat{D}\hat{\nabla}^\mu\hat{T} + \tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^\mu\left(\hat{\nabla}\cdot\hat{u}\right) , \tag{6.21}$$

$$\frac{\delta T_2}{\hat{T}} = \tilde{\chi}^2\frac{\hat{D}^2\hat{T}}{\hat{T}} + \frac{\tilde{\chi}}{3}\left[\tilde{\chi}+\tilde{\theta}\right]\hat{D}\left(\hat{\nabla}\cdot\hat{u}\right) + \frac{\tilde{\chi}}{3}\tilde{\theta}\frac{\hat{\nabla}^2\hat{T}}{\hat{T}} , \tag{6.22}$$

$$\delta u_3^\mu = -\tilde{\theta}^3\hat{D}^3\hat{u}^\mu - \tilde{\theta}\left[\tilde{\theta}^2+\tilde{\theta}\tilde{\chi}+\tilde{\chi}^2\right]\frac{\hat{D}^2\hat{\nabla}^\mu\hat{T}}{\hat{T}} - \tilde{\theta}\frac{\tilde{\chi}}{3}\left[2\tilde{\theta}+\tilde{\chi}\right]\hat{D}\hat{\nabla}^\mu\left(\hat{\nabla}\cdot\hat{u}\right) - \tilde{\theta}^2\frac{\tilde{\chi}}{3}\frac{\hat{\nabla}^2\hat{\nabla}^\mu\hat{T}}{\hat{T}}, \tag{6.23}$$

$$\frac{\delta T_3}{\hat{T}} = -\tilde{\chi}^3\frac{\hat{D}^3\hat{T}}{\hat{T}} - \frac{\tilde{\chi}}{3}\left[\tilde{\chi}^2+\tilde{\chi}\tilde{\theta}+\tilde{\theta}^2\right]\hat{D}^2\left(\hat{\nabla}\cdot\hat{u}\right) - \frac{\tilde{\chi}}{3}\tilde{\theta}\left[2\tilde{\chi}+\tilde{\theta}\right]\frac{\hat{D}\hat{\nabla}^2\hat{T}}{\hat{T}} - \frac{\tilde{\chi}^2}{9}\tilde{\theta}\hat{\nabla}^2\left(\hat{\nabla}\cdot\hat{u}\right) , \tag{6.24}$$

$$\begin{aligned}\delta u_4^\mu =& \tilde{\theta}^4\hat{D}^4\hat{u}^\mu + \tilde{\theta}\left[\tilde{\theta}^3+\tilde{\theta}^2\tilde{\chi}+\tilde{\theta}\tilde{\chi}^2+\tilde{\chi}^3\right]\frac{\hat{D}^3\hat{\nabla}^\mu\hat{T}}{\hat{T}} + \tilde{\theta}\frac{\tilde{\chi}}{3}\left[3\tilde{\theta}^2+2\tilde{\theta}\tilde{\chi}+\tilde{\chi}^2\right]\hat{D}^2\hat{\nabla}^\mu\left(\hat{\nabla}\cdot\hat{u}\right)\\ &+ \tilde{\theta}^2\frac{\tilde{\chi}}{3}\left[2\tilde{\theta}+2\tilde{\chi}\right]\hat{D}\frac{\hat{\nabla}^2\hat{\nabla}^\mu\hat{T}}{\hat{T}} + \tilde{\theta}^2\frac{\tilde{\chi}^2}{9}\hat{\nabla}^2\hat{\nabla}^\mu\left(\hat{\nabla}\cdot\hat{u}\right) ,\end{aligned} \tag{6.25}$$

$$\begin{aligned}\frac{\delta T_4}{\hat{T}} =& \tilde{\chi}^4\frac{\hat{D}^4\hat{T}}{\hat{T}} + \frac{\tilde{\chi}}{3}\left[\tilde{\chi}^3+\tilde{\chi}^2\tilde{\theta}+\tilde{\chi}\tilde{\theta}^2+\tilde{\theta}^3\right]\hat{D}^3\left(\hat{\nabla}\cdot\hat{u}\right) + \frac{\tilde{\chi}}{3}\tilde{\theta}\left[3\tilde{\chi}^2+2\tilde{\chi}\tilde{\theta}+\tilde{\theta}^2\right]\frac{\hat{D}^2\hat{\nabla}^2\hat{T}}{\hat{T}}\\ &+ \frac{\tilde{\chi}^2}{9}\tilde{\theta}\left[2\tilde{\chi}+2\tilde{\theta}\right]\hat{D}\hat{\nabla}^2\left(\hat{\nabla}\cdot\hat{u}\right) + \frac{\tilde{\chi}^2}{9}\tilde{\theta}^2\frac{\hat{\nabla}^4\hat{T}}{\hat{T}} ,\end{aligned} \tag{6.26}$$

$$\vdots \quad .$$

We see that, with increasing order n of derivative correction, the velocity correction terms δu_n^μ (as well as the temperature correction terms δT_n), include higher and higher orders of the spatial gradients on T and u^μ systematically. Moreover, the order of the temporal gradient on each such spatial gradient term also chronologically increases. This increase of temporal derivatives is observed to follow a particular pattern such that they can be clubbed together into products of infinite sums. Below, we write the fully summed (up to all orders) velocity and the temperature corrections such that this repetitive pattern in the temporal derivatives becomes manifest.

$$
\begin{aligned}
u^\mu = \hat{u}^\mu + \delta u_1^\mu + \delta u_2^\mu + \cdots = \left[1 + \left(-\tilde{\theta}\hat{D}\right) + \left(-\tilde{\theta}\hat{D}\right)^2 + \left(-\tilde{\theta}\hat{D}\right)^3 + \cdots\right]\hat{u}^\mu \\
+ (-\tilde{\theta})\left[1 + \left(-\tilde{\theta}\hat{D}\right) + \left(-\tilde{\theta}\hat{D}\right)^2 + \cdots\right]\left[1 + \left(-\tilde{\chi}\hat{D}\right) + \left(-\tilde{\chi}\hat{D}\right)^2 + \cdots\right]\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}} \\
+ (-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right))\left[1 + 2\left(-\tilde{\theta}\hat{D}\right) + 3\left(-\tilde{\theta}\hat{D}\right)^2 + \cdots\right]\left[1 + \left(-\tilde{\chi}\hat{D}\right) + \left(-\tilde{\chi}\hat{D}\right)^2 + \cdots\right]\hat{\nabla}^\mu\left(\hat{\nabla}\cdot\hat{u}\right) \\
+ (-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)\left[1 + 2\left(-\tilde{\theta}\hat{D}\right) + 3\left(-\tilde{\theta}\hat{D}\right)^2 + \cdots\right]\left[1 + 2\left(-\tilde{\chi}\hat{D}\right) + 3\left(-\tilde{\chi}\hat{D}\right)^2 + \cdots\right]\frac{\hat{\nabla}^2\hat{\nabla}^\mu\hat{T}}{\hat{T}} \\
+ (-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)^2\left[1 + 3\left(-\tilde{\theta}\hat{D}\right) + 6\left(-\tilde{\theta}\hat{D}\right)^2 + \cdots\right] \qquad (6.27) \\
\left[1 + 2\left(-\tilde{\chi}\hat{D}\right) + 3\left(-\tilde{\chi}\hat{D}\right)^2 + \cdots\right]\hat{\nabla}^2\hat{\nabla}^\mu\left(\hat{\nabla}\cdot\hat{u}\right) \\
+ \cdots \quad . \qquad (6.28)
\end{aligned}
$$

The infinite sums over the time derivative can be encompassed in a closed form following the relaxation operator-like terms to appear in the denominator of the thermodynamic quantities, giving rise to pole-like structures in the following manner,

$$
\begin{aligned}
u^\mu =& \hat{u}^\mu + \delta u_1^\mu + \delta u_2^\mu + \cdots \\
=& \frac{1}{(1+\tilde{\theta}\hat{D})}\hat{u}^\mu \\
&+ (-\tilde{\theta})\frac{1}{(1+\tilde{\theta}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}} \\
&+ (-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{(1+\tilde{\theta}\hat{D})^2}\frac{1}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^\mu\hat{\nabla}\cdot\hat{u} \\
&+ (-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{(1+\tilde{\theta}\hat{D})^2}\frac{1}{(1+\tilde{\chi}\hat{D})^2}\frac{\hat{\nabla}^2\hat{\nabla}^\mu\hat{T}}{\hat{T}} \\
&+ (-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)^2\frac{1}{(1+\tilde{\theta}\hat{D})^3}\frac{1}{(1+\tilde{\chi}\hat{D})^2}\hat{\nabla}^2\hat{\nabla}^\mu\hat{\nabla}\cdot\hat{u} \\
&+ \cdots \quad . \qquad (6.29)
\end{aligned}
$$

Similarly, for the temperature correction, we have the following derivative pattern :

$$
\begin{aligned}
&T = \hat{T} + \delta T_1 + \delta T_2 + \cdots = \left[1 + \left(-\tilde{\chi}\hat{D}\right) + \left(-\tilde{\chi}\hat{D}\right)^2 + \left(-\tilde{\chi}\hat{D}\right)^3 + \cdots\right]\hat{T} \\
&+ \hat{T}(-\frac{\tilde{\chi}}{3})\left[1 + \left(-\tilde{\chi}\hat{D}\right) + \left(-\tilde{\chi}\hat{D}\right)^2 + \cdots\right]\left[1 + \left(-\tilde{\theta}\hat{D}\right) + \left(-\tilde{\theta}\hat{D}\right)^2 + \cdots\right]\left(\hat{\nabla}\cdot\hat{u}\right) \\
&+ \left(-\frac{\tilde{\chi}}{3}\right)(-\tilde{\theta})\left[1 + 2\left(-\tilde{\chi}\hat{D}\right) + 3\left(-\tilde{\chi}\hat{D}\right)^2 + \cdots\right]\left[1 + \left(-\tilde{\theta}\hat{D}\right) + \left(-\tilde{\theta}\hat{D}\right)^2 + \cdots\right]\hat{\nabla}^2\hat{T} \\
&+ \hat{T}\left(-\frac{\tilde{\chi}}{3}\right)^2(-\tilde{\theta})\left[1 + 2\left(-\tilde{\chi}\hat{D}\right) + 3\left(-\tilde{\chi}\hat{D}\right)^2 + \cdots\right]\left[1 + 2\left(-\tilde{\theta}\hat{D}\right) + 3\left(-\tilde{\theta}\hat{D}\right)^2 + \cdots\right]\hat{\nabla}^2\left(\hat{\nabla}\cdot\hat{u}\right) \\
&+ \cdots \quad . \qquad (6.30)
\end{aligned}
$$

Just like the velocity variable, the above series can also be resummed as,

$$
\begin{aligned}
T =& \hat{T} + \delta T_1 + \delta T_2 + \cdots \\
=& \frac{1}{(1+\tilde{\chi}\hat{D})}\hat{T} \\
&+ \hat{T}\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{1}{(1+\tilde{\theta}\hat{D})}\left(\hat{\nabla}\cdot\hat{u}\right) \\
&+ \left(-\frac{\tilde{\chi}}{3}\right)(-\tilde{\theta})\frac{1}{(1+\tilde{\chi}\hat{D})^2}\frac{1}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^2\hat{T} \\
&+ \hat{T}\left(-\frac{\tilde{\chi}}{3}\right)^2(-\tilde{\theta})\frac{1}{(1+\tilde{\chi}\hat{D})^2}\frac{1}{(1+\tilde{\theta}\hat{D})^2}\hat{\nabla}^2\left(\hat{\nabla}\cdot\hat{u}\right) \\
&+ \cdots \quad . \qquad (6.31)
\end{aligned}
$$

Putting the velocity correction given by Eq.(6.29) in Eq.(6.18) we have the all order frame transformed BDNK stress tensor in Landau frame as,

$$
T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu\hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu} \, , \qquad (6.32)
$$

with the shear stress $\hat{\pi}^{\mu\nu} = -2\eta\left[\hat{\sigma}^{\mu\nu} + \sum_{n=1}^{\infty}\partial^{\langle\mu}\delta u_n^{\nu\rangle}\right]$ as the only dissipative contribution, now resummed under the all order frame transformation as the following,

$$\begin{aligned}\hat{\pi}^{\mu\nu} =& -2\eta\Bigg[\frac{\hat{\nabla}^{\langle\mu}\hat{u}^{\nu\rangle}}{(1+\tilde{\theta}\hat{D})} + \frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{(1+\tilde{\chi}\hat{D})} \\ &+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})^2}\frac{(-\frac{1}{3}\tilde{\chi})}{\left(1+\tilde{\chi}\hat{D}\right)}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u} \\ &+\frac{(-\tilde{\theta})^2}{(1+\tilde{\theta}\hat{D})^2}\frac{\left(-\frac{1}{3}\tilde{\chi}\right)}{(1+\tilde{\chi}\hat{D})^2}\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}^2\hat{T} \\ &+\frac{(-\tilde{\theta})^2}{(1+\tilde{\theta}\hat{D})^3}\frac{\left(-\frac{1}{3}\tilde{\chi}\right)^2}{(1+\tilde{\chi}\hat{D})^2}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u}+\cdots\Bigg].\end{aligned} \tag{6.33}$$

Note that, for each increasing spatial gradient, the temporal gradient resulting from the infinite sum also increases in the denominator, such that they exactly balance each other. This condition has been mentioned in [101] as a necessary condition of causality.

Both equations (6.29) and (6.31) are just formal solutions as they have derivatives in the denominator. Such an expression really makes sense in the space of frequencies rather than in real-time. However, what this indicates is a nonlocality in time (or integration over time). Just like in the MIS theory, such nonlocalities could be recast into a local set of equations by introducing new 'non-fluid' variables, which is the topic of the next subsection.

Introducing 'non-fluid' degrees of freedom to make BDNK a local theory in Landau frame

In section 6.3.1, Eq.(6.32) and (6.33) combinedly provide the energy-momentum tensor of a frame-transformed BDNK theory that is nonlocal in fluid variables. In this subsection, our goal is to introduce new 'non-fluid' degrees of freedom, ones that vanish at any state of global thermal equilibrium and, therefore, are not extensions of any conserved charges. This viewpoint also provides us some guidance as to how we should formulate the equations of motion for 'non-fluid' variables. Like $\pi^{\mu\nu}$ in MIS theory, any non-fluid variable should approach a vanishing value in a 'relaxation type' equation. The relaxation time scales are provided by the poles in the infinite sum of temporal derivatives we did in the previous subsection. However, unlike the MIS theory, here, after completing the infinite sum in the temporal derivatives, the degree of the pole increases ad infinitum along with more and more spatial derivatives in the numerator. This indicates an infinite number of non-fluid degrees of freedom in a nested series of 'relaxation type' equations.

We can make this intuition precise in the following set of infinitely many equations. This is a local theory both in space and time, equivalent to BDNK, at least with respect to linearized perturbations around equilibrium in the hydrodynamic regime (barring a few singular points in the frequency domain), but has an infinite number of degrees of freedom, (as we expected) in the following manner,

$$
\begin{aligned}
&\partial_\mu T^{\mu\nu} = 0\,, \qquad T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu\hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} = -2\eta\hat{\sigma}^{\mu\nu} + \rho_1^{\mu\nu},\\
&(1+\tilde{\chi}\hat{D})\rho_1^{\mu\nu} = (-2\eta)(-\tilde{\theta})\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{T} + \rho_2^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\rho_2^{\mu\nu} = (-2\eta)(-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right)\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}\cdot\hat{u} + \rho_3^{\mu\nu},\\
&(1+\tilde{\chi}\hat{D})\rho_3^{\mu\nu} = (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{T} + \rho_4^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\rho_4^{\mu\nu} = (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)^2\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u} + \cdots\\
&\vdots
\end{aligned}
\tag{6.34}
$$

Eq. (6.2) and so on set an infinite nested series of new degrees of freedom much in the same line as the conventional MIS theory given by Eq.(6.4) and (6.5). Eq.(6.2) combinedly boils down to Eq.(6.32) and (6.33) where each increasing spatial gradient term is now attributed to a new degree of freedom.

6.3.2 Method-2: Frame transformation in one go

In the previous section, we have solved the linearized frame transformation equations (6.13) and (6.14) using derivative expansion. Though the method of derivative expansion could be applied to solve even a nonlinear set of equations, we have heavily used linearization to simplify the solution further. In fact, the way we have summed the infinite series to generate temporal derivatives in the denominator is clearly a formal manipulation, and it makes sense only in the case of linearized treatment in Fourier space. It also indicates an integration over time, which is then made local by introducing new 'non-fluid' variables.

Now, while solving (6.13) and (6.14), if we eventually allow ourselves to have temporal derivatives ($\hat{D}$) in the denominator, there is no harm in having spatial derivatives as well (again makes sense only when viewed in Fourier space and indicates an infinite order of spatial derivatives or integration/nonlocality in space). In this subsection, we shall use this formal manipulation of having both spatial and temporal derivatives in the denominator. This will lead to solutions of the frame transformation equations (6.13) and (6.14) in one go.

The steps are as follows. First, we take the divergence of equation (6.13) and the following two coupled scalar equations will give the two scalar variables $(\hat{\nabla}\cdot\delta u)$ and $\delta T/\hat{T}$ as,

$$\left[1+\tilde{\theta}\hat{D}\right](\hat{\nabla}\cdot\delta u)+\tilde{\theta}\hat{\nabla}^2\frac{\delta T}{\hat{T}}+\tilde{\theta}\left[\frac{\hat{\nabla}^2\hat{T}}{\hat{T}}+\hat{D}\hat{\nabla}\cdot\hat{u}\right]=0\,, \tag{6.35}$$

$$\left[1+\tilde{\chi}\hat{D}\right]\frac{\delta T}{\hat{T}}+\frac{\tilde{\chi}}{3}(\hat{\nabla}\cdot\delta u)=0\,. \tag{6.36}$$

In Eq.(6.36) we have used the on shell identity $\frac{\hat{D}\hat{T}}{\hat{T}}+\frac{1}{3}\hat{\nabla}\cdot\hat{u}=0$ that always holds at linearized level under Landau frame condition. Now eliminating $(\hat{\nabla}\cdot\delta u)$ from the above two equations, first we find $\frac{\delta T}{\hat{T}}$. Then, substituting this solution in (6.13), we find the expression for δu^μ. The final solution (BDNK variables in terms of Landau frame variables) takes the following form:

$$u^\mu=(\hat{u}^\mu+\delta u^\mu) \tag{6.37}$$

$$=\left(\frac{\hat{u}^\mu}{1+\tilde{\theta}\hat{D}}\right) \tag{6.38}$$

$$-\left(1+\tilde{\theta}\hat{D}\right)^{-1}\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})-\tilde{\theta}\frac{\tilde{\chi}}{3}\,\hat{\nabla}^2\right]^{-1}\left[\tilde{\theta}\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}}-\frac{\tilde{\theta}}{3}\left(\tilde{\theta}+\tilde{\chi}\right)\hat{\nabla}^\mu(\hat{\nabla}\cdot\hat{u})\right], \tag{6.39}$$

$$T=\left(\hat{T}+\delta T\right)=\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})-\tilde{\theta}\frac{\tilde{\chi}}{3}\,\hat{\nabla}^2\right]^{-1}\left[(1+\tilde{\theta}\hat{D})\hat{T}-\frac{\tilde{\chi}}{3}\hat{T}\left(\hat{\nabla}\cdot\hat{u}\right)\right]. \tag{6.40}$$

In the Landau frame, the stress tensor will again have the structure of the form given in equation (6.18). After substituting the solutions (6.39) there, we finally get the following shear tensor,

$$\hat{\pi}^{\mu\nu} = -\left[\frac{2\eta}{1+\tilde{\theta}\hat{D}}\right]\hat{\sigma}^{\mu\nu} + \left[\frac{2\eta\tilde{\theta}}{1+\tilde{\theta}\hat{D}}\right]\left[\frac{\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}} - \frac{1}{3}(\tilde{\theta}+\tilde{\chi})\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\left(\hat{\nabla}\cdot\hat{u}\right)}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2}\right]. \tag{6.41}$$

Equation (6.41) could be further simplified using the fact that in Landau frame at the linearized level $\hat{\nabla}\cdot\hat{u}$ and $\frac{\hat{D}\hat{T}}{\hat{T}}$ are related as follows,

$$\hat{\nabla}\cdot\hat{u} + 3\left(\frac{\hat{D}\hat{T}}{\hat{T}}\right) = \text{terms nonlinear in fluctuations.} \tag{6.42}$$

Using identity (6.42), Eq.(6.41) becomes,

$$\begin{aligned}\hat{\pi}^{\mu\nu} = &-\left[\frac{2\eta}{1+\tilde{\theta}\hat{D}}\right]\hat{\sigma}^{\mu\nu} \\ &+\left[\frac{2\eta\tilde{\theta}}{1+\tilde{\theta}\hat{D}}\right]\left[\frac{\left\{1+(\tilde{\theta}+\tilde{\chi})\hat{D}\right\}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \frac{\tilde{\theta}\tilde{\chi}}{3}\hat{\nabla}^2}\right].\end{aligned} \tag{6.43}$$

The equations (6.39), (6.40), (6.41) and (6.43) are all very formal with spatial as well as temporal derivatives in the denominator. But following the strategy presented in the case of MIS theory, we can recast equation (6.43) as an inhomogeneous differential equation for the new 'nonfluid' degree of freedom $\pi^{\mu\nu}$ as follows,

$$\begin{aligned}&\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2\right]\left\{(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} + 2\eta\hat{\sigma}^{\mu\nu}\right\} \\ &\quad = 2\eta\tilde{\theta}\left\{1+(\tilde{\theta}+\tilde{\chi})\hat{D}\right\}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}.\end{aligned} \tag{6.44}$$

Here, just like in MIS theory, we are introducing only one new 'non-fluid' tensorial degree of freedom, but it follows a complicated inhomogeneous PDE, second order in spatial but third order in temporal derivatives[3].

[3]Note that in the limit $\tilde{\chi}\to 0$, the equation (6.44) becomes very similar to the corresponding equation in MIS theory with a slight modification as follows.

$$(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} = -2\eta\left[\hat{\sigma}^{\mu\nu} - \tilde{\theta}\left(\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}\right)\right]. \tag{6.45}$$

Comparison with the previous method with infinite 'non-fluid' variables

Generically, a nonlocal theory could be made local by introducing new degrees of freedom, but the process of 'integrating in' new degrees could have ambiguities. The two methods described in the previous two subsections could be one example of this ambiguity. Both methods attempt to write a system of coupled equations involving both fluid and 'non-fluid' variables that are equivalent to the equations in BDNK theory. However, the structure of the equations and also the extra 'non-fluid' variables are so widely different that in the first case, we need to introduce an infinite number of variables, whereas in the second case, we need just one. In this subsection, we would like to see how these two sets of equations are actually equivalent, at least in some regime of frequency and spatial momenta.

It turns out that the field redefinition we have used in the first method (see equations (6.29) and (6.31)) could be further rearranged in the following fashion. For the velocity redefinition, we have,

$$
\begin{aligned}
u^\mu &= \hat{u}^\mu + \delta u_1^\mu + \delta u_2^\mu + \cdots && (6.46)\\
&= \frac{1}{(1+\tilde{\theta}\hat{D})}\hat{u}^\mu + \frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})}\left[1+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{\left(-\frac{\tilde{\chi}}{3}\right)}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^2+\cdots\right]\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}}\\
&+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})^2}\frac{\left(-\frac{\tilde{\chi}}{3}\right)}{(1+\tilde{\chi}\hat{D})}\left[1+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{\left(-\frac{\tilde{\chi}}{3}\right)}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^2+\cdots\right]\hat{\nabla}^\mu\left(\hat{\nabla}\cdot\hat{u}\right). && (6.47)
\end{aligned}
$$

Similarly, for the temperature redefinition we have,

$$
\begin{aligned}
T &= \hat{T} + \delta T_1 + \delta T_2 + \cdots && (6.48)\\
&= \frac{1}{(1+\tilde{\chi}\hat{D})}\left[1+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{\left(-\frac{\tilde{\chi}}{3}\right)}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^2+\cdots\right]\hat{T}\\
&+\hat{T}\frac{\left(-\frac{\tilde{\chi}}{3}\right)}{(1+\tilde{\chi}\hat{D})}\frac{1}{(1+\tilde{\theta}\hat{D})}\left[1+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{\left(-\frac{\tilde{\chi}}{3}\right)}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^2+\cdots\right]\left(\hat{\nabla}\cdot\hat{u}\right). && (6.49)
\end{aligned}
$$

Substituting this rearranged field redefinition, the dissipative part of the stress tensor could also be rearranged as,

$$
\begin{aligned}
\hat{\pi}^{\alpha\beta} =& -2\eta\hat{\Delta}^{\alpha\beta}_{\mu\nu}\Bigg[\frac{1}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\mu}\hat{u}^{\nu} \\
&+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{1}{\hat{T}}\hat{\nabla}^{\mu}\hat{\nabla}^{\nu}\left\{1+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{(-\frac{1}{3}\tilde{\chi})}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^2+\cdots\right\}\hat{T} \\
&+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})^2}\frac{(-\frac{1}{3}\chi)}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^{\mu}\hat{\nabla}^{\nu}\left\{1+\frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{(-\frac{1}{3}\tilde{\chi})}{(1+\tilde{\chi}\hat{D})}\hat{\nabla}^2+\cdots\right\}\hat{\nabla}_{\rho}\hat{u}^{\rho}\Bigg] . \qquad (6.50)
\end{aligned}
$$

Now the infinite sum in powers of spatial derivative $\hat{\nabla}^2$ converges for those linearized perturbations where the operator satisfies the inequality,

$$
\left[\frac{\left(\frac{\tilde{\theta}\tilde{\chi}}{3}\right)\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right] < 1 . \qquad (6.51)
$$

Within this radius of convergence, we can again sum the spatial derivatives and get the following expression for the field redefinitions,

$$
\begin{aligned}
u^{\mu} =& \, \hat{u}^{\mu}+\delta u^{\mu}_1+\delta u^{\mu}_2+\cdots \qquad (6.52) \\
=&\frac{1}{(1+\tilde{\theta}\hat{D})}\hat{u}^{\mu} \\
&+(-\tilde{\theta})\frac{\frac{\hat{\nabla}^{\mu}\hat{T}}{\hat{T}}}{\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})-\tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2\right]}+\frac{(-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right)}{(1+\tilde{\theta}\hat{D})}\frac{\hat{\nabla}^{\mu}\left(\hat{\nabla}\cdot\hat{u}\right)}{\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})-\tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2\right]} , \\
=&\frac{1}{(1+\tilde{\theta}\hat{D})}\hat{u}^{\mu}+\frac{\left[-\tilde{\theta}\frac{\hat{\nabla}^{\mu}\hat{T}}{\hat{T}}+\frac{\tilde{\theta}}{3}\left(\tilde{\theta}+\tilde{\chi}\right)\hat{\nabla}^{\mu}(\hat{\nabla}\cdot\hat{u})\right]}{(1+\tilde{\theta}\hat{D})\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})-\tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2\right]} , \qquad (6.53)
\end{aligned}
$$

and,

$$
\begin{aligned}
T =& \, \hat{T}+\delta T_1+\delta T_2+\cdots \\
=&\frac{(1+\tilde{\theta}\hat{D})\hat{T}}{\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})-\tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2\right]}+\hat{T}\left(-\frac{\tilde{\chi}}{3}\right)\frac{\left(\hat{\nabla}\cdot\hat{u}\right)}{\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})-\tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2\right]} . \qquad (6.54)
\end{aligned}
$$

From (6.53) it is simple to estimate $\pi^{\mu\nu}$ as,

$$\begin{aligned}\hat{\pi}^{\mu\nu} =& -2\eta\frac{\tilde{\sigma}^{\mu\nu}}{(1+\tilde{\theta}\hat{D})} - 2\eta\frac{\left[-\tilde{\theta}\ \frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}} + \tilde{\theta}\frac{\tilde{\chi}}{3}\frac{1}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\left(\hat{\nabla}\cdot\hat{u}\right)\right]}{\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \tilde{\theta}\frac{\tilde{\chi}}{3}\hat{\nabla}^2\right]}\,,\\ =& -\left[\frac{2\eta}{1+\tilde{\theta}\hat{D}}\right]\sigma^{\mu\nu} + \left[\frac{2\eta\tilde{\theta}}{1+\tilde{\theta}\hat{D}}\right]\left[\frac{\left(1+(\tilde{\theta}+\tilde{\chi})\hat{D}\right)\left(\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}\right)}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \tilde{\theta}\frac{\tilde{\chi}}{3}\ \hat{\nabla}^2}\right]\,. \end{aligned} \tag{6.55}$$

It can be observed that Eq.(6.53), (6.54) and (6.55) are exactly identical as (6.39), (6.40) and (6.43) of the field correction at one go results. (In the second step of the derivation of (6.53) and (6.55), we have taken recourse to the identity (6.42). For detailed steps of the summation, the reader may refer to appendix D.1.) So, within the radius of convergence, both methods actually generate the same set of equations as expected.

At this stage, let us emphasize one point. This method of 'integrating in' new 'non-fluid' degrees of freedom with new equations of motion is highly non-unique, even at the linearized level. For example, we could have chosen δu^μ and δT themselves to be the new 'non-fluid' variables, satisfying the new equations as given in (6.13) and (6.14) and we could take a viewpoint that the u^μ and the T fields in the BDNK theory are actually the Landau frame fluid variables plus 'non-fluid' variables $\{\delta u^\mu, \delta T\}$. Note that though δu^μ and δT would look very much like velocity and temperature corrections, they are still 'non-fluid' variables in the Landau frame since they vanish in global equilibrium. Another choice of introducing infinitely many 'non-fluid' degrees of freedom would be to simply use δu^μ_n and δT_n (as defined in (6.15)) and then the recursive equations (6.16) would turn out to be the new equations of motion.

The two choices of new variables, discussed here in detail, are basically guided by our sense of mathematical aesthetics and an attempt to adhere to the philosophy of MIS theory where the new 'non-fluid' variable is a rank-2 symmetric tensor, structurally very similar to the energy-momentum tensor. At the moment, we do not have any further physical support behind our choice of variables.

6.4 Dispersion relation

As we have seen in the previous sections, a system of fluid equations with terms up to all orders in derivative expansion could be converted to PDEs with a finite number of derivatives, provided we introduce new 'non-fluid' degrees of freedom. The 'non-fluid' variables we introduced basically capture the effect of a formal infinite sum over derivatives, leading to pole-like structures in the momentum-frequency space.

Now, these infinite series in derivatives (or, more precisely, in the 4-momenta of the Fourier transform of linear fluctuations) could be summed only within their radius of convergence. Once we extend the summed-up theories beyond that radius, we often encounter 'non-hydrodynamic modes' that are not exactly the same as that of the BDNK theory[4]. However, in this section, we shall see that the hydrodynamic modes of the system of equations described in the previous two sections are both exactly the same as that of the BDNK theory at every order in k expansion. This is a consistency test of our claim that our system of equations is indeed equivalent to BDNK formalism, at least in the hydrodynamic regime.

[4]A similar situation arises in the case of the MIS theory as we have presented in section 6.2. In the hydrodynamic regime, the stress tensor must be described in a derivative expansion, which turns out to have an infinite number of terms (see equation(6.6)). Now, in the frequency space (ω), this infinite sum can be performed only within a radius of convergence, which in this case turns out to be

$$D \sim |\omega| \leq \frac{1}{\tau_\pi} \, .$$

Introducing new 'non-fluid' variables $\pi^{\mu\nu}$ essentially amounts to extending the theory beyond this radius of convergence. Now $\omega = -\frac{i}{\tau_\pi}$ is the new non-hydro mode that emerges in the process of integrating in $\pi^{\mu\nu}$ and this mode is exactly on the radius of convergence of the previous derivative expansion.

6.4.1 Method - 1

Here, the equivalent system is described by an infinite number of variables and, therefore, an infinite number of equations. For convenience, let us first quote the equations here again.

$$
\begin{aligned}
&\partial_\mu T^{\mu\nu} = 0\,, \qquad T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu \hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} = -2\eta\hat{\sigma}^{\mu\nu} + \rho_1^{\mu\nu},\\
&(1+\tilde{\chi}\hat{D})\rho_1^{\mu\nu} = (-2\eta)(-\tilde{\theta})\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{T} + \rho_2^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\rho_2^{\mu\nu} = (-2\eta)(-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right)\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}\cdot\hat{u} + \rho_3^{\mu\nu},\\
&(1+\tilde{\chi}\hat{D})\rho_3^{\mu\nu} = (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{T} + \rho_4^{\mu\nu},\\
&(1+\tilde{\theta}\hat{D})\rho_4^{\mu\nu} = (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)^2\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u} + \cdots\\
&\vdots
\end{aligned}
\tag{6.56}
$$

The 'non-fluid' variables are $\pi^{\mu\nu}$ and the infinite sequence of $\rho_n^{\mu\nu}$s, each satisfying a relaxation type of equation.

We parameterize the perturbation around static global equilibrium in the following fashion,

$$
\begin{aligned}
&\hat{T} = T_0 + \epsilon\,\delta T\, e^{iT_0(-\omega t+kx)}\\
&\hat{u}^\mu = \{1,0,0,0\} + \epsilon\,\{0,\beta_x,\beta_y,0\}\, e^{iT_0(-\omega t+kx)}\\
&\rho_n^{xx} = \epsilon\,\delta\rho_n^{xx}\, e^{iT_0(-\omega t+kx)} = -2\rho_n^{yy} = -2\rho_n^{zz} \quad \forall n\\
&\rho_n^{xy} = \epsilon\,\delta\rho_n^{xy}\, e^{iT_0(-\omega t+kx)} \quad \forall n\,,
\end{aligned}
\tag{6.57}
$$

all other components of $\rho_n^{\mu\nu}$ vanish for every n.

Here, ϵ is a book-keeping parameter for linearization. Any term quadratic or higher order in ϵ will be ignored. We have scaled the frequency and the spatial momenta with the equilibrium temperature T_0 so that both ω and k are dimensionless. Similarly, we introduce new dimensionless parameters of the theory $\tilde{\eta}_0$, $\tilde{\chi}_0$ and $\tilde{\theta}_0$ as follows,

$$
\tilde{\eta} \equiv \frac{\tilde{\eta}_0}{T_0}, \quad \tilde{\chi} \equiv \frac{\tilde{\chi}_0}{T_0}, \quad \tilde{\theta} \equiv \frac{\tilde{\theta}_0}{T_0}\,.
$$

If we substitute the fluctuations (6.57) in equations (6.56), we find the dispersion polynomial $\mathcal{P}(\omega, k)$ whose zeroes will give the modes where the fluctuations can have a nontrivial solution.

Now, in this case, it is difficult to express $\mathcal{P}(\omega, k)$ in a compact form since the equations involve an infinite number of variables. Instead, we shall determine the dispersion polynomial $\mathcal{P}_N(\omega, k)$ for the same system, truncated at some arbitrary but finite order $n = N$ recursively. The infinite N limit of $\mathcal{P}_N(\omega, k)$ will give the actual dispersion polynomial of the system. We have,

$$\mathcal{P}_N(\omega, k) = \mathcal{P}^{\text{shear}}(\omega, k)\, \mathcal{P}_N^{\text{sound}}(\omega, k)\ , \tag{6.58}$$

where,

$$\begin{aligned}
&\mathcal{P}^{\text{shear}}(\omega, k) = \tilde{\eta}_0\, k^2 - i\omega\, (1 - i\, \tilde{\theta}_0\, \omega)\ ,\\
&\mathcal{P}_N^{\text{sound}}(\omega, k) = (1 - i\, \tilde{\chi}_0\, \omega)^{\frac{N}{2}} (1 - i\, \tilde{\theta}_0\, \omega)^{\frac{N}{2}} P_N(\omega, k) \quad \text{When } N \text{ even},\\
&\mathcal{P}_N^{\text{sound}}(\omega, k) = (1 - i\, \tilde{\chi}_0\, \omega)^{\frac{N+1}{2}} (1 - i\, \tilde{\theta}_0\, \omega)^{\frac{N-1}{2}} P_N(\omega, k) \quad \text{When } N \text{ odd}\ .
\end{aligned} \tag{6.59}$$

Note the factor $\mathcal{P}^{\text{shear}}(\omega, k)$ is independent of N. We could further check that it has the same form as that of the dispersion polynomial in BDNK theory (see (6.9) and (6.10)) in the shear channel. For $P_N(\omega, k)$ we have a recursion relation as follows,

$$\begin{aligned}
P_{2m-1} &= (1 - i\, \tilde{\chi}_0\, \omega) P_{2m-2} - i \left(\frac{4\eta_0}{3^m}\right) \tilde{\theta}_0^m \tilde{\chi}_0^{m-1}\, (ik)^{2(m+1)} \quad \text{for odd } N = 2m-1,\ \ m \geq 1\ ,\\
P_{2m} &= (1 - i\, \tilde{\theta}_0\, \omega) P_{2m-1} - i \left(\frac{4\eta_0}{3^m}\right) \tilde{\theta}_0^m \tilde{\chi}_0^m\, (ik)^{2(m+1)}\, (-i\omega) \quad \text{for even } N = 2m,\ \ m > 0\ ,\\
P_0 &= 3i\omega^2 (1 - i\, \tilde{\chi}_0\, \omega) + k^2 (i + 4\tilde{\eta}_0 \omega + \tilde{\theta}_0 \omega)\ .
\end{aligned} \tag{6.60}$$

From equations (6.58), (6.59) and (6.60), we could see that the degree of the polynomial (and therefore the number of zeroes) in the sound channel increases as we include more and more $\rho_n^{\mu\nu}$s in our system of equations. In other words, with increasing N, we keep getting more and more modes. However, it is easy to take $k \to 0$ limit in these recursive equations, and one could see that in the sound channel, there are precisely two modes at $\omega = 0$, and all the rests are either at $\left[\omega = -\frac{i}{\tilde{\chi}_0}\right]$ or $\left[\omega = -\frac{i}{\tilde{\theta}_0}\right]$ similar to the BDNK theory at $k \to 0$ limit. According to our definitions, the modes with vanishing frequencies at $k \to 0$ limit are the hydro modes. So, this system of equations does have two hydro modes in the sound channel, as expected from the parent BDNK theory. Further, by explicit calculation, we can see that these hydrodynamic sound modes match with those of BDNK even at non-zero k, if we treat k perturbatively in a power series expansion [5]. So clearly, the hydro-modes in the equations described in these sections for both the sound and shear channel (in the shear channel, even the non-hydro modes match with BDNK) are the same as those of BDNK, justifying our claim that this system of equations is equivalent to the BDNK systems of equations in the hydrodynamic regime.

6.4.2 Method - 2

For convenience, let us first quote the system of equations that we would like to analyze,

$$\partial_\mu T^{\mu\nu} = 0\,, \quad T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu\hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu}\,, \tag{6.61}$$

$$\begin{aligned}\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \tilde{\theta}\frac{\tilde{\chi}}{3}\,\hat{\nabla}^2\right]&\left\{(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} + 2\eta\hat{\sigma}^{\mu\nu}\right\}\\ &= 2\eta\tilde{\theta}\left\{1+(\tilde{\theta}+\tilde{\chi})\hat{D}\right\}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}\,.\end{aligned} \tag{6.62}$$

As before, we parameterize the perturbation around static global equilibrium in the following fashion,

$$\begin{aligned}\hat{T} &= T_0 + \epsilon\,\delta T\, e^{iT_0(-\omega t + kx)}\,,\\ \hat{u}^\mu &= \{1,0,0,0\} + \epsilon\,\{0,\beta_x,\beta_y,0\}\, e^{iT_0(-\omega t+kx)}\,,\\ \pi^{xx} &= \epsilon\,\delta\pi^{xx}\, e^{iT_0(-\omega t+kx)} = -2\pi^{yy} = -2\pi^{zz}\,,\\ \pi^{xy} &= \epsilon\,\delta\pi^{xy}\, e^{iT_0(-\omega t+kx)}\,,\end{aligned} \tag{6.63}$$

[5] If we truncate the equations at $n = N$, then the frequency of the sound mode matches with that of BDNK upto order $\mathcal{O} \sim (k^{N+3})$. This we have checked in Mathematica for all $N \leq 10$.

with ϵ as a book-keeping parameter for linearization. Any term quadratic or higher order in ϵ will be ignored. Again, the frequency and the spatial momenta are scaled with the equilibrium temperature T_0 so that both ω and k are dimensionless. And also we have introduced new dimensionless parameters of the theory $\tilde{\eta}_0$, $\tilde{\chi}_0$ and $\tilde{\theta}_0$ as follows,

$$\tilde{\eta} \equiv \frac{\tilde{\eta}_0}{T_0}, \qquad \tilde{\chi} \equiv \frac{\tilde{\chi}_0}{T_0}, \qquad \tilde{\theta} \equiv \frac{\tilde{\theta}_0}{T_0} \; .$$

Substituting equation (6.63) in the system of equations ((6.61) and (6.62)), we find the following dispersion polynomial,

$$P(\omega, k) = (1 - i\tilde{\theta}_0\omega)\left[\left(\frac{\tilde{\chi}_0\tilde{\theta}_0}{3}\right)k^2 + (1 - i\tilde{\theta}_0\omega)(1 - i\tilde{\chi}_0\omega)\right]P_{\text{BDNK}}(\omega, k) \; , \tag{6.64}$$

where $P_{\text{BDNK}}(\omega, k)$ is the similar dispersion polynomial computed for the fluctuations around the static equilibrium solutions in BDNK systems of equations as given in (6.9) and (6.10) and given by,

$$P_{\text{BDNK}} = \left(\tilde{\eta}_0 k^2 - i\omega(1 - i\tilde{\theta}_0\omega)\right) \times \\ \left[\tilde{\chi}_0\tilde{\theta}_0\omega^4 + i\left(\tilde{\chi}_0 + \tilde{\theta}_0\right)\omega^3 - \left\{1 + \frac{2}{3}\tilde{\chi}_0\left(\tilde{\theta}_0 + 2\tilde{\eta}_0\right)k^2\right\}\omega^2 - \frac{i}{3}\left(\tilde{\chi}_0 + \tilde{\theta}_0 + 4\tilde{\eta}_0\right)\omega k^2 + \frac{k^2}{3} + \frac{\tilde{\theta}_0}{9}\,(\right. \tag{6.65}$$

In other words, the zeroes of $P_{\text{BDNK}}(\omega, k)$ are the hydro and non-hydro modes of the BDNK theory.

From equation (6.64), it is clear that all the modes of the BDNK system are already contained in the system of equations (6.61) and (6.62). However, they also contain some new modes, which are the zeros of the prefactor,

$$P_{\text{extra}}(\omega, k) \equiv \left(\frac{P(\omega, k)}{P_{\text{BDNK}}(\omega, k)}\right) = (1 - i\tilde{\theta}_0\omega)\left[\left(\frac{\tilde{\chi}_0\tilde{\theta}_0}{3}\right)k^2 + (1 - i\tilde{\theta}_0\omega)(1 - i\tilde{\chi}_0\omega)\right] \; . \tag{6.66}$$

Note that all these new modes are of non-hydro type. One could further check that they correspond to the zero modes of the linear PDEs that determine the shift of the velocity and the temperature field (δu^μ and δT respectively) under frame transformation (see equation (6.16)).

The existence of such zero modes implies that if we view δu^μ or δT as generated from a field redefinition (and not as new 'non-fluid' variables), then even after fixing the Landau frame condition, there are still some unfixed residual ambiguities (which exist only at some special form of $\omega(k)$) in the definition of the fluid variables. On the other hand, if we absorb these shift fields (δu^μ and δT) into new 'nonfluid' variables, the extra zeros of the prefactor $P_{\text{extra}}(\omega, k)$ do become the new modes of the theory. In some sense, the residual ambiguities in the field redefinition procedure translate to the non-uniqueness of the UV degrees of freedom beyond the hydrodynamic regime.

6.5 Conclusion

In this work, we rewrite the stress tensor of the BDNK hydrodynamic theory in the Landau frame at least for the part that will contribute to the spectrum of linearized perturbation around static equilibrium. Though the BDNK formalism has a finite number of derivatives, it turns out that in the Landau frame, it will have either an infinite number of derivatives or one has to introduce new non-fluid variables. There is no unique way to introduce these non-fluid variables. Here, motivated by the structure of the MIS formalism, we have presented two different ways of doing it, resulting in two completely different-looking sets of equations. However, both the sets have the same hydrodynamics modes as the BDNK theory. But in the process of 'integrating in' the non-fluid variables, new non-hydrodynamic modes are generated.

In both methods, we need to do a formal infinite sum over derivatives. We suspect that the convergence issues of these infinite sums, also related to the 'non-invertibility' of the zero modes of the linear operator involved in field redefinition, are responsible for these new non-hydrodynamic modes. However, this point needs further investigation.

More generally, it would be interesting to know if we can identify a part of the spectrum to be invariant under field redefinition and, therefore, truly physical. [6] In this context, the following observation seems useful. In BDNK theory, if we set viscosity (η) to zero (with nonzero χ and θ), then via field redefinition, the stress tensor could be made identical to that of an ideal fluid at the linearized level, though in the original 'BDNK' frame it will have nontrivial dispersion relation dependent on the values of χ and θ. This indicates that there might be some partial redundancy in the information contained in the spectrum of a fluid theory. It would be nice to have a more comprehensive understanding of this aspect of the spectrum.

Our work has set up a stage for comparison between the BDNK and MIS-type theories. At first glance, they look very different. However, the fluid variables like velocity and temperature used to express the BDNK stress tensor are not the same as the ones used in MIS theory. A comparison is meaningful only if the basic variables of the equations are the same. Once we have done the required transformation, it turns out that though there are differences in the details, the basic structure of nonlocality or 'non-fluid' variables is very similar in both theories. The advantage of the Landau frame is that the fluid variables are locally defined in terms of the one-point function of the stress tensor, but in this case, the causal equations turn out to have nonlocal terms or an infinite number of derivatives. Whereas in BDNK theories, the equations are local with a finite number of derivatives, but the fluid variables are related to the one point function of the stress tensor in a very non-trivial and nonlocal fashion.

However, there is more information in the BDNK formalism than what has just been stated above. It says that there exist causal fluid theories where the non-localities could be completely absorbed in a field redefinition, thereby generating causal but local fluid theory with a finite number of derivatives. Since the final equation we derived on the shear tensor $\pi^{\mu\nu}$ is different from what one conventionally has in the MIS theory, it also says that the non-localities of the MIS theory could possibly never be completely absorbed in the field redefinition.

It would be interesting to extend this analysis to full nonlinear order. Also, it would be very

[6] Building upon the work presented in this chapter, in [144, 145], it has been shown that it is indeed possible to do so using infinite order field redefinitions.

informative to know whether and, if yes, how the story changes as one adds higher order derivative corrections to the BDNK theory.

Finale: Conclusion, Future Directions, Bibliography

Chapter 7

Conclusion and Future Directions

In this thesis, we have analyzed different aspects of the dynamics of black holes and relativistic fluids in a linearized regime near equilibrium. In the first part of the thesis, we have looked closer into the effect of the horizon's null generators' reparametrization on the statement of the second law on a black hole horizon. In the second part, we have explored different aspects of two well-known stable-causal relativistic hydrodynamic theories in the conformal uncharged limit.

Both the works of Part 1 are focused on black hole solutions in the Gauss-Bonnet theory and the linearized regime in the dynamics of amplitude. The first work analyzes the effect of reparametrization of the horizon's null generators on the statement of local entropy production on the horizon. The results suggest that for the Gauss-Bonnet theory, although the entropy density and the spatial entropy current on the horizon transform non-trivially and non-covariantly, their combined divergence is covariant under the reparametrization up to linearized order in amplitude dynamics. Hence, local entropy production at each spacetime point on the horizon remains invariant under this affine-to-affine reparametrization of the null generators. In [2], we have extended this analysis for arbitrary higher-derivative gravity theories non-minimally coupled to matter fields. The transformation of the entropy density and spatial entropy current under such reparametrizations has been explicitly derived, and the effect of constructional ambiguities (Iyer-Wald ambiguities) can also be seen. In the second work, we have used the fluid-gravity duality to dualize this combination of entropy density and spatial entropy current on the horizon to an entropy current for a fluid residing on the boundary of an asymptotically AdS spacetime with a black brane in the bulk. For generic higher-derivative gravity theories and generic mapping functions between the horizon and the boundary, we find a non-trivial dependence of this entropy current on the mapping functions. This prevents them from being called "genuine fluid entropy current" as these mapping functions may or may not be expressible solely in terms of fluid data.

In the first work of Part 2, we derive the causality criteria for conformal uncharged MIS and BDNK theories, which are the two most well-known formulations of stable-causal relativistic hydrodynamic theories solely using linearized stability analysis. The derivation makes use of the fact that the causal parameter space of a theory is also stable at all frames connected by Lorentz boosts. Hence, the region of the parameter space, which stays stable in all the Lorentz-boosted frames, is identified to be the causal parameter space, and the corresponding inequalities give us the causality constraints on the parameters. The important result we find here is that the region of the parameter space stable at an ultra-high boost (boost velocity nearly equal to the speed of light) is the one that stays stable in all frames and, hence, is the causal parameter space. The corresponding stability criteria that one gets at an ultra-high boost are the causality criteria. In the second work of Part 2, we use the freedom to redefine hydrodynamic fields to rewrite the stress tensor of the BDNK theory written in a generalized hydro frame into the Landau frame. We find that, although the stress tensor in the BDNK theory as written in a generalized hydrodynamic frame is truncated at the first-order in derivative expansion, when written in the Landau frame, the stress tensor has an infinite number of derivative corrections. These infinite corrections can be recast as extra non-fluid degrees of freedom, as in the MIS theory. This procedure of incorporating new degrees of freedom in the theory is non-unique, and we show two of the possible ways to do the same, motivated by the form of the stress tensor in the MIS theory. We also find that the field redefinition leaves the hydrodynamic modes of the theory unchanged, but modifies the non-hydrodynamic spectrum of the theory, including leading to degenerate non-hydro modes.

The analyses performed in this thesis shed light on some possible avenues for future explorations. Broadly, the frameworks developed here can be used to understand entropy currents on a black hole's horizon in more detail, to understand the causality properties of a hydrodynamic theory without departing from the low-wavenumber regime, and to utilize the fluid-gravity correspondence in constructing stable-causal theories of hydrodynamics from some gravity dual spacetimes.

As indicated in [2], it might be an interesting open direction to use the Iyer-Wald ambiguities involved in the entropy current construction to generate such pieces in the entropy current that exactly cancel out the non-covariant terms in the affine-to-affine transformation of the currents. For example, in the Gauss-Bonnet theory, the entropy density and spatial entropy current would receive extra terms upon fixing the Iyer-Wald ambiguities, which might, in turn, contribute to the entropy current exactly canceling out the non-covariant terms in the entropy current's transformation. If this were possible, it would then be interesting to check if an algorithm can be developed to systematically cancel out such non-covariant pieces using appropriate ambiguity fixing and, therefore, develop a procedure to write entropy currents that transform covariantly under coordinate transformations on the horizon. The investigation can then be extended to test whether these entropy currents transform covariantly under non-affine coordinate transformations. This can then open up a variety of new directions as one can then try to extract an 'improved' boundary entropy current from here. These would be different from the ones derived in [95] as they'd have Iyer-Wald ambiguities fixed to certain values and might possibly be genuine fluid entropy currents. These 'genuine' fluid entropy currents originally valid for linear amplitude regime only can then be uplifted to construct entropy currents valid even in the nonlinear regime of dynamics, using algorithms developed in past works as [94]. These entropy currents can then be mapped back to the horizon to understand the second law for black holes in the regime of nonlinear dynamics.

One can also explore these entropy current structures and the associated coordinate transformations from the perspective of Carrollian symmetries. Since the horizon is a null hypersurface, it is endowed with an underlying Carrollian symmetry [121]. The reparametrizations on the horizon seem very similar to what has been described as Carrollian diffeomorphisms [146]. A possible first step towards understanding this connection can be to construct structures on the horizon that are covariant under Carrollian diffeomorphisms. This would help in examining the validity of the reparametrizations as Carrollian diffeomorphisms. One could then try to express the second law on the horizon in terms of Carroll covariant structures, possibly as some conservation equation for a current. A far shot would involve interpreting the entropy current as a Noether current, corresponding to some symmetry, possible symmetry under Carrollian diffeomorphisms.

The model-specific derivation of causality criteria for MIS and BDNK theories using stability analysis in [96] is limited to the conformal uncharged limit. Lifting these assumptions, it would be interesting to test whether stability criteria at ultra-high boost indeed give us the causality constraints in the theory. Some analyses indicate that it is indeed the case for non-conformal, uncharged, and charged conformal MIS theories [130], hinting towards the generality of the analysis. It would be useful to actually prove this identification of causality and near-luminal stability criteria for general dispersion polynomials. This would then allow for the causality of a theory to be analyzed without departing from the low-wavenumber regime of the theory, which is more suitable and conceptually appropriate for a derivative expandible effective theory like relativistic hydrodynamics. For hydrodynamic theories in general, and specifically in those derived from a gravity dual theory, this method of causality analysis can lead to a more proper way of deriving the constraints on the parameters.

The concept of introducing infinite order derivative corrections to a theory to maintain stability and causality, as studied in [98], can be extended to gravitational solutions as well with the motivation to construct stable-causal hydrodynamic theories from a gravitational dual. It would then be interesting to resum these infinite corrections into new degrees of freedom on the fluid side and find their interpretations on the gravity side. The authors in [68] have already constructed a gravity dual for the BDNK stress theory using an appropriate choice of the zero modes of the solutions. It would be interesting to compare these results with one obtained by adding an infinite no. of corrections to a uniform black brane solution in asymptotically AdS metric. One can also try to involve higher-derivative corrections to the gravitational solution and check whether a BDNK stress tensor can be constructed corresponding to these. It can further be checked whether the causality analysis for such a boundary fluid can constrain the coupling parameters of higher-derivative gravity corrections, as was done in [77–79]. Further, it would be interesting to explore the connections between the quasinormal modes of the bulk spacetime and those of the boundary dual fluid theory under the framework of BDNK hydrodynamics. In general, the BDNK theory gives an alternative formulation to test the general predictions obtained from a dual gravitational theory. One can also explore if any other alternative stable-causal hydrodynamic formulations are possible besides the MIS and BDNK and what their implications might be for a dual gravitational theory.

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Appendix A

(For Chapter - 3)

A.1 Notations, Conventions, and Definitions

In this appendix, we summarize our notation conventions and list the definitions of the various structures that we have used throughout our work.

- Indices: Uppercase Latin alphabets $A, B, C...$ will refer to full D space-time coordinates and Lowercase Latin alphabets $a, b, c...$ will refer to the $(D-2)$ dimensional spatial coordinates.

- Choice of coordinates:

$$\begin{aligned} X^A = \{r, v, x^a\},\;\; Y^A = \{\rho, \tau, y^a\} :\;& \text{The full space-time} \\ & \text{coordinates in } D \text{ dimensions} \\ r, \rho =\;& \text{The radial coordinates} \\ v, \tau =\;& \text{The Eddington-Finkelstein type time coordinates} \\ x^a, y^a =\;& \text{The } (D-2) \text{ spatial coordinates} \end{aligned}$$

- Choice of Space-time Metrics:

$$\begin{aligned} ds^2 &= 2\, dv\, dr - r^2\, X(r, v, x^a)\, dv^2 + 2\, r\, \omega_a(r, v, x^b)\, dv\, dx^a + h_{ab}(r, v, x^a)\, dx^a\, dx^b \\ &= G_{AB}(r, v, x^a)\, dX^A\, dX^B \\ &= 2\, d\tau\, d\rho - \rho^2\, \tilde{X}(\rho, \tau, y^a)\, d\tau^2 + 2\, \rho\, \tilde{\omega}_a\, (\rho, \tau, y^b)\, d\tau\, dy^a + \tilde{h}_{ab}(\rho, \tau, y^a)\, dy^a\, dy^b \\ &= g_{AB}(\rho, \tau, y^a)\, dY^A\, dY^B \end{aligned}$$

- Structures like spatial derivatives, curvature tensors, and metric components in the Y^A coordinate system will be represented with a $\tilde{}$ on their corresponding counterparts in the X^A coordinates. For example, $X, \omega_i, h_{ij}, \left(\partial_a = \frac{\partial}{\partial x^a}\right) \to \tilde{X}, \tilde{\omega}_i, \tilde{h}_{ij}, \left(\tilde{\partial}_a = \frac{\partial}{\partial y^a}\right)$

- Transformation of Coordinates and Derivatives on the Horizon:

$$
\begin{aligned}
r &= e^{-\zeta}\rho + \mathcal{O}(\rho^2) \\
\rho &= e^{\zeta} r + \mathcal{O}(r^2) \\
v &= e^{\zeta}\tau + \mathcal{O}(\rho) \\
\tau &= e^{-\zeta} v + \mathcal{O}(r) \\
x^a &= y^a + \mathcal{O}(\rho) \\
y^a &= x^a + \mathcal{O}(r) \\
\partial_r &= e^{\zeta}\left(\partial_\rho + \frac{1}{2}\tau^2\xi^2\partial_\tau + \tau\xi^a\tilde{\partial}_a\right) + \mathcal{O}(\rho) \\
\partial_v &= e^{-\zeta}\partial_\tau + \mathcal{O}(\rho) \\
\partial_a &= \tilde{\partial}_a - \tau\,\xi_a\partial_\tau + \mathcal{O}(\rho)
\end{aligned}
$$

where we've denoted $\partial_a\zeta = \tilde{\partial}_a\zeta$ by ξ_a.

- Definition of Curvature Tensors:

$$
K_{ab} = \frac{1}{2}\partial_v h_{ab} \qquad K = h^{ab}K_{ab} = \frac{1}{\sqrt{h}}\partial_v\sqrt{h}
$$

$$
\tilde{K}_{ab} = \frac{1}{2}\partial_\tau \tilde{h}_{ab} \qquad \tilde{K} = \tilde{h}^{ab}\tilde{K}_{ab} = \frac{1}{\sqrt{\tilde{h}}}\partial_\tau\sqrt{\tilde{h}}
$$

$$
R_{ABCD}, R_{AB}, R =
$$

Riemann tensor, Ricci tensor, Ricci scalar corresponding to full metric G or g

$$
\mathcal{R}_{abcd}, \mathcal{R}_{ab}, \mathcal{R} =
$$

Riemann tensor, Ricci tensor, Ricci scalar corresponding to intrinsic metric h or $\tilde{h}$

A.2 Detailed Expressions

In this appendix, we show the explicit calculations for the relation between quantities such as Christoffel connection, Ricci scalar, and the divergence of entropy current between X^A and Y^A coordinate systems.

- Expression for Christoffel connection in transformed coordinates

$$\begin{aligned}\Gamma_{a,bc} &= \frac{1}{2}\big(\partial_b h_{ac} + \partial_c h_{ab} - \partial_a h_{bc}\big)\\ &= \tilde{\Gamma}_{a,bc} - \frac{1}{2}\tau\partial_\tau\big(\xi_b h_{ac} + \xi_c h_{ab} - \xi_a h_{bc}\big)\\ &= \tilde{\Gamma}_{a,bc} - \tau\big(\xi_b \tilde{K}_{ac} + \xi_c \tilde{K}_{ab} - \xi_a \tilde{K}_{bc}\big)\end{aligned} \tag{A.1}$$

- Expressions for Riemann tensor and Ricci scalar

$$\mathcal{R}_{abcd} = -[\partial_d \Gamma_{a,bc} - \partial_c \Gamma_{a,bd} + h^{pq}\Gamma_{p,ac}\Gamma_{q,bd} - \Gamma_{p,ad}\Gamma_{q,bc}h^{pq}] \tag{A.2}$$

- Ricci Scalar in transformed coordinates

$$\begin{aligned}\mathcal{R} &= h^{ac}h^{bd}R_{abcd}\\ &= -h^{ac}h^{bd}\partial_d\Gamma_{a,bc} + h^{ad}h^{bc}\partial_d\Gamma_{a,bdc} + h^{ac}h^{bd}\Gamma_{p,ad}\Gamma_{q,bc}h^{pq} - h^{ad}h^{bc}\Gamma_{p,ad}\Gamma_{q,bc}h^{pq}\\ &= (h^{ad}h^{bc} - h^{ac}h^{bd})\big(\partial_d\Gamma_{a,bc} - h^{pq}\Gamma_{p,ad}\Gamma_{q,bc}\big)\end{aligned} \tag{A.3}$$

$$\begin{aligned}\partial_d\Gamma_{a,bc} =& \tilde{\partial}_d\Gamma_{a,bc} - \tau\xi_d\partial_\tau\Gamma_{a,bc}\\ &= \tilde{\partial}_d\big[\tilde{\Gamma}_{a,bc} - \tau(\xi_b\tilde{K}_{ac} + \xi_c\tilde{K}_{ab} - \xi_a\tilde{K}_{bc})\big] - \tau\xi_d\partial_\tau\big[\tilde{\Gamma}_{a,bc} - \tau(\xi_b\tilde{K}_{ac} + \xi_c\tilde{K}_{ab} - \xi_a\tilde{K}_{bc})\big]\\ &= \big[\tilde{\partial}_d\tilde{\Gamma}_{a,bc} - \tau(\xi_{bd}\tilde{K}_{ac} + \cancel{\xi_{cd}\tilde{K}_{ab}} - \xi_{ad}\tilde{K}_{bc}) - \tau(\xi_b\tilde{\partial}_d\tilde{K}_{ac} + \cancel{\xi_c\tilde{\partial}_d\tilde{K}_{ab}} - \xi_a\tilde{\partial}_d\tilde{K}_{bc})\big]\\ &+ \big[-\tau(\xi_d\tilde{\partial}_b\tilde{K}_{ac} + \cancel{\xi_d\tilde{\partial}_c\tilde{K}_{ab}} - \xi_d\tilde{\partial}_a\tilde{K}_{bc}) + \tau(\xi_d\xi_b\tilde{K}_{ac} + \cancel{\xi_d\xi_c\tilde{K}_{ab}} - \xi_a\xi_d\tilde{K}_{bc})\\ &+ \tau^2(\xi_d\xi_b\partial_\tau\tilde{K}_{ac} + \cancel{\xi_d\xi_c\partial_\tau\tilde{K}_{ab}} - \xi_d\xi_a\partial_\tau\tilde{K}_{bc})\big]\end{aligned} \tag{A.4}$$

The terms canceled in (A.4) due to the fact that terms symmetric in (c, d) will not contribute to the Ricci scalar as it has a prefactor of $(h^{ad}h^{bc} - h^{ac}h^{bd})$ which is antisymmetric in (c, d). Hence,

$$\begin{aligned}\partial_d\Gamma_{a,bc} = \tilde{\partial}_d\tilde{\Gamma}_{a,bc} - \tau\big[(\xi_{bd} + \xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b - \xi_d\xi_b)\tilde{K}_{ac} - (\xi_{ad} + \xi_a\tilde{\partial}_d + \xi_d\tilde{\partial}_a - \xi_a\xi_d)\tilde{K}_{bc}]\\ +\tau^2[\xi_d\xi_b\partial_\tau\tilde{K}_{ac} - \xi_d\xi_a\partial_\tau\tilde{K}_{bc}].\end{aligned} \tag{A.5}$$

From eqn. (A.3) and (A.1),

$$\begin{aligned} h^{pq}\Gamma_{p,ad}\Gamma_{q,bc} = h^{pq}\tilde{\Gamma}_{p,ad}\tilde{\Gamma}_{q,bc} - \tau h^{pq}\tilde{\Gamma}_{p,ad}(\xi_b\tilde{K}_{qc} + \xi_c\tilde{K}_{qb} - \xi_q\tilde{K}_{bc}) \\ - \tau h^{pq}\tilde{\Gamma}_{q,bc}(\xi_a\tilde{K}_{pd} + \xi_d\tilde{K}_{pa} - \xi_p\tilde{K}_{ad}) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{A.6}$$

Thus, from (A.3), (A.5), and (A.6)

$$\begin{aligned} &\mathcal{R} = \tilde{\mathcal{R}} + (h^{ad}h^{bc} - h^{ac}h^{bd}) \\ &[-2\tau\{\xi_{bd} + (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b) - \xi_d\xi_b\}\tilde{K}_{ac} \\ &+ 2\tau^2(\xi_d\xi_b\partial_\tau\tilde{K}_{ac}) + 2\tau h^{pq}\,\tilde{\Gamma}_{p,ad}(\xi_b\tilde{K}_{qc} + \xi_c\tilde{K}_{qb} - \xi_q\tilde{K}_{bc})] \\ &+ \mathcal{O}(\epsilon^2). \end{aligned} \tag{A.7}$$

- The divergence of entropy current in transformed coordinates

  The expression for entropy current for Gauss-Bonnet theory is given as

$$J^a = -4(\nabla_b K^{ba} - \nabla^a K_{cd}h^{cd}). \tag{A.8}$$

This implies, that the divergence of Entropy current is

$$\nabla_a J^a = -4(h^{ad}h^{bc} - h^{cd}h^{ab})\nabla_a\nabla_b K_{cd} = -4(h^{ad}h^{bc} - h^{ac}h^{bd})\nabla_b\nabla_d K_{ac}. \tag{A.9}$$

Let us define a three index object $M_{d,ac}$ such that

$$\begin{aligned} M_{d,ac} \equiv& \nabla_d K_{ac} = \nabla_d\left(e^{-\zeta}\tilde{K}_{ac}\right) = \partial_d\left(e^{-\zeta}\tilde{K}_{ac}\right) - \Gamma^p_{da}\left(e^{-\zeta}\tilde{K}_{pc}\right) - \Gamma^p_{dc}\left(e^{-\zeta}\tilde{K}_{ap}\right) \\ =&\{\tilde{\partial}_d - \xi_d\tau\partial_\tau\}\left(e^{-\zeta}\tilde{K}_{ac}\right) - \tilde{\Gamma}^p_{da}\left(e^{-\zeta}\tilde{K}_{pc}\right) - \tilde{\Gamma}^p_{dc}\left(e^{-\zeta}\tilde{K}_{ap}\right) + \mathcal{O}(\epsilon^2) \\ =&e^{-\zeta}\left[\tilde{\partial}_d\tilde{K}_{ac} - \xi_d\tilde{K}_{ac} - \xi_d\tau\partial_\tau\tilde{K}_{ac} - \tilde{\Gamma}^p_{da}\tilde{K}_{pc} - \tilde{\Gamma}^p_{dc}\tilde{K}_{ap}\right] + \mathcal{O}(\epsilon^2) \\ =&e^{-\zeta}\left(\tilde{\nabla}_d\tilde{K}_{ac} - \xi_d\tilde{K}_{ac} - \xi_d\tau\partial_\tau\tilde{K}_{ac}\right) + \mathcal{O}(\epsilon^2) \\ =&e^{-\zeta}\left(\tilde{M}_{d,ac} - (\delta\tilde{M})_{d,ac}\right) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{A.10}$$

$$(\delta\tilde{M})_{d,ac} = \xi_d\left(\tilde{K}_{ac} + \tau\partial_\tau\tilde{K}_{ac}\right) \tag{A.11}$$

Also, we define

$$
\begin{aligned}
& W_{abcd} \equiv \nabla_b M_{d,ac} \\
=& \partial_b M_{d,ac} - \Gamma^p_{bd} M_{p,ac} - \Gamma^p_{ba} M_{d,pc} - \Gamma^p_{bc} M_{d,ap} \\
=& \tilde{\partial}_b \left\{ e^{-\zeta} \left(\tilde{M}_{d,ac} - (\delta\tilde{M})_{d,ac} \right) \right\} - \xi_b e^{-\zeta} \tau \partial_\tau \left(\tilde{M}_{d,ac} - (\delta\tilde{M})_{d,ac} \right) \\
& - e^{-\zeta} \left[\tilde{\Gamma}^p_{bd} \left(\tilde{M}_{p,ac} - (\delta\tilde{M})_{p,ac} \right) + \tilde{\Gamma}^p_{ba} \left(\tilde{M}_{d,pc} - (\delta\tilde{M})_{d,pc} \right) + \tilde{\Gamma}^p_{bc} \left(\tilde{M}_{d,ap} - (\delta\tilde{M})_{d,ap} \right) \right] + \mathcal{O}(\epsilon^2) \\
=& e^{-\zeta} \left[\tilde{\nabla}_b \tilde{M}_{d,ac} - \xi_b (1 + \tau\partial_\tau) \left(\tilde{M}_{d,ac} - \delta\tilde{M}_{d,ac} \right) - \tilde{\nabla}_b \delta\tilde{M}_{d,ac} \right] \\
=& e^{-\zeta} \left[\tilde{\nabla}_b \tilde{\nabla}_d \tilde{K}_{ac} - \xi_b \tilde{M}_{d,ac} - \tilde{\nabla}_b \delta\tilde{M}_{d,ac} - \xi_b \tau\partial_\tau \tilde{M}_{d,ac} + \xi_b (1 + \tau\partial_\tau) \delta\tilde{M}_{d,ac} \right] \\
=& e^{-\zeta} \Bigg[\tilde{\nabla}_b \tilde{\nabla}_d \tilde{K}_{ac} \underbrace{- \xi_b \tilde{\nabla}_d \tilde{K}_{ac} - \tilde{\nabla}_b (\xi_d \tilde{K}_{ac})}_{\text{term 1}} - \underbrace{\tilde{\nabla}_b (\xi_d \tau\partial_\tau \tilde{K}_{ac}) - \xi_b \tau\partial_\tau \tilde{\nabla}_d \tilde{K}_{ac}}_{\text{term 2}} \\
& + \underbrace{\xi_b (1 + \tau\partial_\tau)(\xi_d \tilde{K}_{ac} + \xi_d \tau\partial_\tau \tilde{K}_{ac})}_{\text{term 3}} \Bigg] + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{A.12}
$$

Now,

$$
\begin{aligned}
\text{term 1} =& - \xi_b \tilde{\nabla}_d \tilde{K}_{ac} - \tilde{\nabla}_b (\xi_d \tilde{K}_{ac}) \\
=& - \xi_b \tilde{\partial}_d \tilde{K}_{ac} + \xi_b \tilde{\Gamma}^p_{da} \tilde{K}_{pc} + \xi_b \tilde{\Gamma}^p_{dc} \tilde{K}_{ap} - \xi_d \tilde{\partial}_b \tilde{K}_{ac} + \xi_d \tilde{\Gamma}^p_{ba} \tilde{K}_{pc} \\
& + \xi_d \tilde{\Gamma}^p_{bc} \tilde{K}_{ap} - \xi_{bd} \tilde{K}_{ac} + \tilde{\Gamma}^p_{bd} \xi_p \tilde{K}_{ac}.
\end{aligned}
\tag{A.13}
$$

From (A.9), we see that for calculation of the divergence of entropy current, the terms in (A.12) have to be contracted with $(h^{ad}h^{bc} - h^{ac}h^{bd})$, which is antisymmetric in (c, d) or (a, b). Now, in (A.13), the terms $\xi_b \tilde{\Gamma}^p_{dc} \tilde{K}_{ap}$ and $\xi_d \tilde{\Gamma}^p_{ba} \tilde{K}_{pc}$ are symmetric in (c, d) and (a, b) respectively. Hence, these can be dropped. In addition, we can perform some relabelling of indices and rewrite term 1 as

$$
\begin{aligned}
\text{term 1} =& - \xi_b \tilde{\partial}_d \tilde{K}_{ac} + \xi_b \tilde{\Gamma}^p_{da} \tilde{K}_{pc} - \xi_d \tilde{\partial}_b \tilde{K}_{ac} + \xi_c \tilde{\Gamma}^p_{ad} \tilde{K}_{bp} - \xi_{bd} \tilde{K}_{ac} - \tilde{\Gamma}^p_{ad} \xi_p \tilde{K}_{bc}. \\
=& - (\xi_b \tilde{\partial}_d + \xi_d \tilde{\partial}_b) \tilde{K}_{ac} - \xi_{bd} \tilde{K}_{ac} + \tilde{\Gamma}^p_{ad} (\xi_c \tilde{K}_{pb} + \xi_b \tilde{K}_{pc} - \xi_p \tilde{K}_{bc}).
\end{aligned}
\tag{A.14}
$$

In a similar fashion, we can express term 2 as

$$\begin{aligned}\text{term 2} =& -\xi_b\tau\partial_\tau\tilde{\nabla}_d\tilde{K}_{ac} - \tilde{\nabla}_b(\xi_d\tau\partial_\tau\tilde{K}_{ac})\\ =& -\tau\left[\xi_b\tilde{\nabla}_d(\partial_\tau\tilde{K}_{ac}) + \tilde{\nabla}_b(\xi_d\partial_\tau\tilde{K}_{ac})\right] + \mathcal{O}(\epsilon^2)\\ =& -\tau\left[(\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\partial_\tau\tilde{K}_{ac} + \xi_{bd}\partial_\tau\tilde{K}_{ac} - \tilde{\Gamma}^p_{ad}(\xi_c\partial_\tau\tilde{K}_{pb} + \xi_b\partial_\tau\tilde{K}_{pc} - \xi_p\partial_\tau\tilde{K}_{bc})\right] + \mathcal{O}(\epsilon^2).\end{aligned} \tag{A.15}$$

Now, evaluating term 3

$$\text{term 3} = \xi_b(1+\tau\partial_\tau)(\xi_d\tilde{K}_{ac} + \xi_d\tau\partial_\tau\tilde{K}_{ac}) = \xi_b\xi_d(\tilde{K}_{ac} + 3\tau\partial_\tau\tilde{K}_{ac} + \tau^2\partial_\tau^2\tilde{K}_{ac}). \tag{A.16}$$

Combining results from (A.12), (A.14), (A.15) and (A.16)

$$\begin{aligned}W_{abcd} = e^{-\zeta}\Big[&\tilde{\nabla}_b\tilde{\nabla}_d\tilde{K}_{ac} - \left(\xi_{bd}\tilde{K}_{ac}\right) - \left(\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b\right)\tilde{K}_{ac} + \tilde{\Gamma}^p_{ad}\left(\xi_b\tilde{K}_{pc} + \xi_c\tilde{K}_{pb} - \xi_p\tilde{K}_{bc}\right)\\ &-\tau\left\{\xi_{bd} + \left(\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b\right)\right\}(\partial_\tau\tilde{K}_{ac}) + \tau\tilde{\Gamma}^p_{ad}\left(\xi_b\partial_\tau\tilde{K}_{pc} + \xi_c\partial_\tau\tilde{K}_{pb} - \xi_p\partial_\tau\tilde{K}_{bc}\right)\\ &+\xi_b\xi_d\tilde{K}_{ac} + 3\tau\xi_b\xi_d\partial_\tau\tilde{K}_{ac} + \xi_b\xi_d\tau^2\partial_\tau^2\tilde{K}_{ac}\Big] + \mathcal{O}(\epsilon^2).\end{aligned} \tag{A.17}$$

Hence up to $\mathcal{O}(\epsilon)$, the divergence of the entropy current becomes

$$\begin{aligned}\nabla_a J^a = e^{-\zeta}\tilde{\nabla}_a\tilde{J}^a - 4e^{-\zeta}(h^{ad}h^{bc} - h^{ac}h^{bd})\Big[&-(\xi_{bd}\tilde{K}_{ac}) - (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\tilde{K}_{ac}\\ &+\tilde{\Gamma}^p_{ad}(\xi_b\tilde{K}_{pc} + \xi_c\tilde{K}_{pb} - \xi_p\tilde{K}_{bc}) - \tau\{\xi_{bd} + (\xi_b\tilde{\partial}_d + \xi_d\tilde{\partial}_b)\}(\partial_\tau\tilde{K}_{ac})\\ &+\tau\tilde{\Gamma}^p_{ad}(\xi_b\partial_\tau\tilde{K}_{pc} + \xi_c\partial_\tau\tilde{K}_{pb} - \xi_p\partial_\tau\tilde{K}_{bc}) + \xi_b\xi_d\tilde{K}_{ac} + 3\tau\xi_b\xi_d\partial_\tau\tilde{K}_{ac} + \xi_b\xi_d\tau^2\partial_\tau^2\tilde{K}_{ac}\Big].\end{aligned} \tag{A.18}$$

A.3 Action of Derivatives on some Specific Structures

In this appendix we'll see how the derivatives of certain boost weight 1 structures transform under the coordinate transformations. We'll see how these terms can be condensed into some particular forms that can help us manipulate them in simpler ways.

Any boost weight 1 term can be written in the form of ∂_v(some boost weight 0 structure, say $Q_{a_1a_2\ldots a_n}$). Transforming the ∂_v operator under the coordinate transformations as in (3.19), we can write it as $e^{-\zeta}(\partial_\tau Q_{a_1a_2\ldots a_n})$. Also since τ is analogous to the v coordinate itself, $(\partial_\tau Q_{a_1a_2\ldots a_n})$ itself is a boost weight 1 structure in the $\{\rho,\tau,y^a\}$ coordinate system. Now if we act with a ∇_{x^i} on this structure, we get

$$\begin{aligned}
&\nabla_i(\partial_v Q_{a_1a_2\ldots a_n}) = \nabla_i(e^{-\zeta}(\partial_\tau Q_{a_1a_2\ldots a_n}))\\
=&\partial_i(e^{-\zeta}(\partial_\tau Q_{a_1a_2\ldots a_n})) - e^{-\zeta}\Gamma^b_{ia_1}\partial_\tau Q_{ba_2\ldots a_n} - e^{-\zeta}\Gamma^b_{ia_2}\partial_\tau Q_{a_1b\ldots a_n}\ldots - e^{-\zeta}\Gamma^b_{ia_n}\partial_\tau Q_{a_1a_2\ldots b}
\end{aligned} \tag{A.19}$$

$$\begin{aligned}
&\partial_i(e^{-\zeta}(\partial_\tau Q_{a_1a_2\ldots a_n})) = (\tilde{\partial}_i - \xi_i\tau\partial_\tau)(e^{-\zeta}(\partial_\tau Q_{a_1a_2\ldots a_n}))\\
&= -\xi_i(e^{-\zeta}(\partial_\tau Q_{a_1a_2\ldots a_n})) - \xi_i\tau(e^{-\zeta}\partial_\tau(\partial_\tau Q_{a_1a_2\ldots a_n})) - e^{-\zeta}\tilde{\partial}_i(\partial_\tau Q_{a_1a_2\ldots a_n})\\
&= e^{-\zeta}\left[\tilde{\partial}_i(\partial_\tau Q_{a_1a_2\ldots a_n}) - \xi_i(1+\tau\partial_\tau)(\partial_\tau Q_{a_1a_2\ldots a_n})\right]
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
&\Gamma^b_{ia_m}(\partial_\tau Q_{a_1a_2..b..a_n}) = \left[\tilde{\Gamma}^b_{ia_m} - \tau(\xi\tilde{K}\ldots)\right](\partial_\tau Q_{a_1a_2..b..a_n}) = \tilde{\Gamma}^b_{ia_m}(\partial_\tau Q_{a_1a_2..b..a_n}) + \mathcal{O}(\epsilon^2)\\
\Rightarrow&\nabla_i(\partial_v Q_{a_1a_2\ldots a_n}) = e^{-\zeta}\left[\tilde{\nabla}_i - \xi_i(1+\tau\partial_\tau)\right]\partial_\tau Q_{a_1a_2\ldots a_n} + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{A.21}$$

This form becomes especially useful while calculating J^i and $\nabla_i J^i$.

One more structure that can appear in the calculations of the $\partial_v J^v$ is of the form $\partial_v(\tau Q)$ from the extra terms that are generated due to the coordinate transformation. This derivative can be arranged in the following form which makes it easier to manipulate.

$$\partial_v\left[\tau Q\right] = e^{-\zeta}\partial_\tau\left[\tau Q\right] = e^{-\zeta}(1+\tau\partial_\tau)Q \tag{A.22}$$

Appendix B

(For Chapter - 4)

B.1 Notations and Identities in 4

Here, unless explicitly mentioned, all identities and equations are valid only on the horizon, the null hypersurface at $r = 0$.

$$\begin{aligned} h_{ij} &= l_i^\mu l_j^\nu \chi_{\mu\nu} \\ \Gamma_{k,ij} &= l_i^\mu l_j^\nu l_k^\alpha \, \Gamma_{\alpha,\mu\nu} + \chi_{\mu\nu} \, l_k^\mu (l_i \cdot \partial) l_j^\nu \\ K_{ij} &= l_i^\mu l_j^\nu \mathcal{K}_{\mu\nu} \;\text{ where }\; \mathcal{K}_{\mu\nu} = -t^\alpha \Gamma_{\alpha,\mu\nu} \end{aligned} \tag{B.1}$$

Notation related to coordinate transformation

$$\begin{aligned} & t^\mu \equiv \frac{\partial x^\mu}{\partial v} \equiv e^\phi n^\mu \equiv e^\phi \, ||n|| \, \hat{n}^\mu \quad \text{where} \quad n^\mu \equiv G^{\mu r}, \quad ||n|| \equiv \sqrt{n^\mu n^\nu \eta_{\mu\nu}} \\ & \tilde{t}_\mu = \frac{\partial v}{\partial x^\mu}, \quad \tilde{t}_\mu t^\mu = 1, \quad t^\mu \chi_{\mu\nu} = 0, \quad t^\mu l_\mu^i = \tilde{t}_\mu l_i^\mu = 0, \quad l_i^\mu l_\mu^j = \delta_i^j, \quad l_i^\mu l_\nu^i + t^\mu \tilde{t}_\nu = \delta_\nu^\mu \\ & 0 = G^{\mu r} G_{rr} + G^{\mu\nu} G_{\nu r} = -\chi^{\mu\nu} u_\nu, \;\Rightarrow\; \chi^{\mu\nu} u_\mu = 0 \\ & 1 = G^{rr} G_{rr} + G^{r\mu} G_{\mu r} = -n^\mu u_\mu, \;\Rightarrow\; n^\mu u_\mu = -1 \end{aligned} \tag{B.2}$$

Proof for the first identity in equation (4.19)

Define $\Omega^\mu \equiv \epsilon^{\mu\mu_1\cdots\mu_n} l_{\mu_1}^{i_1} \cdots l_{\mu_n}^{i_n} \left(\frac{\epsilon_{i_1\cdots i_n}}{n!}\right)$ Now we could show that the expression $l_{\mu_1}^{i_1} \cdots l_{\mu_n}^{i_n} \epsilon_{i_1\cdots i_n}$ could be expressed as

$$\begin{aligned} l_{\mu_1}^{i_1} \cdots l_{\mu_n}^{i_n} \epsilon_{i_1\cdots i_n} &= \Omega^\mu \epsilon_{\mu\mu_1\cdots\mu_n} \\ \Omega^\mu \epsilon_{\mu\mu_1\cdots\mu_n} &= \epsilon_{\mu\mu_1\cdots\mu_n} \epsilon^{\mu\nu_1\cdots\nu_n} l_{\nu_1}^{i_1} \cdots l_{\nu_n}^{i_n} \left(\frac{\epsilon_{i_1\cdots i_n}}{n!}\right) \end{aligned} \tag{B.3}$$

Projectors and related identities

$$\Delta^\alpha{}_\mu \equiv \delta^\alpha_\mu - t^\alpha \tilde{t}_\mu, \quad \text{Note} \quad \tilde{t}_\alpha \Delta^\alpha{}_\mu = \Delta^\alpha{}_\mu t^\mu = 0$$
$$\bar{\chi}^{\alpha\beta} = \Delta^\alpha{}_\mu \, \chi^{\mu\nu} \, \Delta^\beta{}_\nu, \quad \bar{\chi}^{\mu\alpha}\chi_{\alpha\nu} = \Delta^\mu{}_\nu \tag{B.4}$$
$$\chi^{\mu\alpha}\chi_{\alpha\nu} = \delta^\mu_\nu + n^\mu u_\nu \quad \Rightarrow \quad u_\mu \, \chi^{\mu\alpha}\chi_{\alpha\nu} = \chi^{\mu\alpha}\chi_{\alpha\nu} \, n^\nu = 0$$

B.2 First few functions of the coordinate transformation

We shall determine $r_{(1)}$ and $x^\mu_{(1)}$ by processing the gauge conditions evaluated at $\rho = 0$. On the horizon, the gauge conditions impose the following constraints

$$-2u_\mu x^\mu_{(1)} r_{(1)} + x^\mu_{(1)} x^\nu_{(1)} \chi_{\mu\nu} = 0, \quad -u_\mu r_{(1)} t^\mu + t^\nu x^\mu_{(1)} \chi_{\mu\nu} = 1, \quad -u_\mu r_{(1)} l^\mu_i + l^\nu_i x^\mu_{(1)} \chi_{\mu\nu} = 0 \tag{B.5}$$

where

$$t^\mu \equiv \left(\frac{\partial x^\mu_{(0)}}{\partial v}\right), \quad l^\mu_i \equiv \left(\frac{\partial x^\mu_{(0)}}{\partial \alpha_i}\right)$$

From the second equation using the fact that $t^\mu (\chi_{\mu\nu})_{\rho=0} = 0$ we find

$$r_{(1)} = -(u_\mu t^\mu)^{-1} \tag{B.6}$$

To simplify the solution for $x^\mu_{(1)}$ we also need the relation between $\chi_{\mu\nu}$ and h_{ij} on the horizon.

$$h_{ij}(\rho = 0) = \left(\frac{\partial x^\mu}{\partial \alpha_i}\right)\left(\frac{\partial x^\nu}{\partial \alpha_j}\right)\chi_{\mu\nu}(r = 0) = l^\mu_i i^\nu_j \chi_{\mu\nu}|_{r=0}$$

$h^{ij}(\rho = 0) =$ Inverse of h_{ij} at horizon $= (ij)$ component of the inverse of the bulk metric on the horizon

$$\begin{aligned}
&= G^{\rho\mu}\left[\left(\frac{\partial \alpha^i}{\partial x^\mu}\right)\left(\frac{\partial \alpha^j}{\partial \rho}\right) + \left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial \rho}\right)\right] + G^{\rho\rho}\left(\frac{\partial \alpha^j}{\partial \rho}\right)\left(\frac{\partial \alpha^i}{\partial \rho}\right) \\
&\quad + G^{\mu\nu}\left[\left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right) + \left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right)\right] \\
&= t^\mu\left[\left(\frac{\partial \alpha^i}{\partial x^\mu}\right)\left(\frac{\partial \alpha^j}{\partial \rho}\right) + \left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial \rho}\right)\right] + G^{\mu\nu}\left[\left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right) + \left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right)\right] \\
&= \left(\frac{\partial x^\mu}{\partial v}\right)\left[\left(\frac{\partial \alpha^i}{\partial x^\mu}\right)\left(\frac{\partial \alpha^j}{\partial \rho}\right) + \left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial \rho}\right)\right] + G^{\mu\nu}\left[\left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right) + \left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right)\right] \\
&= \chi^{\mu\nu}\left[\left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right) + \left(\frac{\partial \alpha^j}{\partial x^\mu}\right)\left(\frac{\partial \alpha^i}{\partial x^\nu}\right)\right] = l^i_\mu l^j_\nu \chi^{\mu\nu}
\end{aligned} \tag{B.7}$$

In the third and the fourth lines, we have used the fact that

$$G^{\rho\mu}(\rho = 0) \propto t^\mu = \left(\frac{\partial x^\mu}{\partial v}\right) = \text{generator of the horizon}$$

also $G^{\rho\rho}(\rho = 0) = 0$ and $\chi^{\mu\nu} \equiv G^{\mu\nu} \neq$ Inverse of $\chi_{\mu\nu}$ (not defined on the horizon).

We also need the inverse of these relations i.e., $\chi_{\mu\nu}$ and $\chi^{\mu\nu}$ in terms of h_{ij} etc.

$$\begin{aligned}
\chi_{\mu\nu}(r = 0) &= l^i_\mu l^j_\nu h_{ij} \\
\chi^{\mu\nu}(r = 0) &= l^\mu_i l^\nu_j h^{ij} + \left[\left(\frac{\partial x^\mu}{\partial \rho}\right)\left(\frac{\partial x^\nu}{\partial \lambda}\right) + \left(\frac{\partial x^\nu}{\partial \rho}\right)\left(\frac{\partial x^\mu}{\partial \lambda}\right)\right] h^{\rho\lambda} \\
&= l^\mu_i l^\nu_j h^{ij} + \left[x^\mu_{(1)} t^\nu + x^\nu_{(1)} t^\mu\right]
\end{aligned} \tag{B.8}$$

Now we shall solve for $x^\mu_{(1)}$. For convenience, we shall express $x^\mu_{(1)}$ as

$$x^\mu_{(1)} = P\, t^\mu + P^i\, l^\mu_i \tag{B.9}$$

Substituting (B.9) and (B.6) in the third equation of (B.5) we find

$$\begin{aligned}
&\frac{u \cdot l_i}{u \cdot t} + P^j\, l^\mu_i l^\nu_j \chi_{\mu\nu} = 0 \;\Rightarrow\; P^i = -h^{ij}\left(\frac{u \cdot l_i}{u \cdot t}\right) \\
&\text{where} \quad h_{ij}(\rho = 0) = l^\mu_i l^\nu_j \chi_{\mu\nu}, \quad h^{ij} = \text{Inverse of } h_{ij}
\end{aligned} \tag{B.10}$$

Now we shall find P from the first equation of (B.5).

$$\begin{aligned}
&-2u_\mu x^\mu_{(1)} r_{(1)} + x^\mu_{(1)} x^\nu_{(1)} \chi_{\mu\nu} = 0 \\
&\Rightarrow 2P + 2P^i\left(\frac{l_i \cdot u}{u \cdot t}\right) + x^\mu_{(1)} x^\nu_{(1)} \chi_{\mu\nu} = 0
\end{aligned} \tag{B.11}$$

Solving this equation we find $x^\mu_{(1)}$.

$$x^\mu_{(1)} = \frac{1}{2} h^{ij}\left[\frac{(u \cdot l_i)(u \cdot l_j)}{(u \cdot t)^2}\right] t^\mu - h^{ij}\left[\frac{(u \cdot l_i) l^\mu_j}{(u \cdot t)}\right] \tag{B.12}$$

<u>Some Potentially useful identities for future works</u>

1. $x^\mu_{(1)}$ related

$$\begin{aligned}
x^\mu_{(1)} &= \frac{1}{2} h^{ij}\left[\frac{(u \cdot l_i)(u \cdot l_j)}{(u \cdot t)^2}\right] t^\mu - h^{ij}\left[\frac{(u \cdot l_i) l^\mu_j}{(u \cdot t)}\right] \\
x^\nu_{(1)} \chi_{\mu\nu} &= \tilde{t}_\mu - \frac{u_\mu}{(u \cdot t)}
\end{aligned} \tag{B.13}$$

Using the two identities

$$h^{ij} = l^i_\mu l^j_\nu \chi^{\mu\nu}, \quad l^i_\mu l^\nu_i = \delta^\mu_\nu - t^\mu \tilde{t}_\nu, \quad \chi^{\mu\nu} u_\nu = 0$$

we could further process the expression of $x^\mu_{(1)}$

$$\begin{aligned} h^{ij}(u\cdot l_i)(u\cdot l_j) &= (u\cdot t)^2 \left(\tilde{t}_\alpha \chi^{\alpha\beta} \tilde{t}_\beta\right) \\ h^{ij}(l_i\cdot u) l^\mu_j &= (u\cdot t)\left[-\tilde{t}_\nu \chi^{\mu\nu} + t^\mu \left(\tilde{t}_\alpha \chi^{\alpha\beta}\tilde{t}_\beta\right)\right] \\ \Rightarrow x^\mu_{(1)} &= -\frac{1}{2}\left(\tilde{t}_\alpha \chi^{\alpha\beta}\tilde{t}_\beta\right) t^\mu + \tilde{t}_\nu \chi^{\mu\nu} \end{aligned} \tag{B.14}$$

2. Metric related:

$$\begin{aligned} h_{ij}(\rho=0) &= l^\mu_i l^\nu_j \chi_{\mu\nu}(r=0) \\ h^{ij}(\rho=0) &= l^i_\mu l^j_\nu \chi^{\mu\nu}(r=0) \\ \chi^{\alpha\beta} &= l^\alpha_i l^\beta_j h^{ij} + x^\alpha_{(1)} t^\beta + x^\beta_{(1)} t^\alpha \end{aligned} \tag{B.15}$$

3. Geodesic related

$$t^A \nabla_A t_B|_{\rho=0} = 0 \;\; \Rightarrow \;\; t^\alpha t^\mu \Gamma_{\alpha,\mu\nu} = 0 \tag{B.16}$$

4. Extrinsic curvatures

$$K_{ij} = l^\mu_i l^\nu_j \mathcal{K}_{\mu\nu}, \quad \bar{K}_{ij} = l^\mu_i l^\nu_j \bar{\mathcal{K}}_{\mu\nu}$$

where

$$\begin{aligned} \mathcal{K}_{\mu\nu} &= -t^\alpha \Gamma_{\alpha,\mu\nu} \\ \bar{\mathcal{K}}_{\mu\nu} &= \left(\partial_\mu \tilde{t}_\nu + \partial_\nu \tilde{t}_\mu\right) - \left[\frac{\partial_\mu u_\nu + \partial_\nu u_\mu}{(u\cdot t)}\right] - \frac{\partial_r \chi_{\mu\nu}}{(u\cdot t)} - x^\alpha_{(1)}\Gamma_{\alpha,\mu\nu} \\ K_{ij}\bar{K}^{ij} &= -\left[\chi^{\mu_1\mu_2}\chi^{\nu_1\nu_2} - \left(\chi^{\mu_1\mu_2} x^{\nu_1}_{(1)} t^{\nu_2} + \chi^{\nu_1\nu_2} x^{\mu_1}_{(1)} t^{\mu_2}\right)\right]\mathcal{K}_{\mu_1\nu_1}\bar{\mathcal{K}}_{\mu_2\nu_2} \end{aligned} \tag{B.17}$$

B.3 Boundary current in terms of fluid variables and $\partial_\mu \phi$

Simplifying J^μ_{space}

We shall first show an identity $t^\mu \mathcal{K}_{\mu\nu} = 0$

$$\begin{aligned} t^\mu \mathcal{K}_{\mu\nu} =& - t^\mu t^\alpha \Gamma_{\alpha,\mu\nu} \\ =& - t^\mu t^\alpha \left[\partial_\mu \chi_{\nu\alpha} + \partial_\nu \chi_{\mu\alpha} - \partial_\alpha \chi_{\mu\nu}\right] \\ =& - t^\mu t^\alpha \partial_\nu \chi_{\mu\alpha} = -t^\mu \partial_\nu \left[t^\alpha \chi_{\mu\alpha}\right] + t^\mu \chi_{\mu\alpha} \left(\partial_\nu t^\alpha\right) = 0 \end{aligned} \tag{B.18}$$

Now expanding $\mathcal{D}_\alpha \mathcal{K}_{\mu\nu}$ we find

$$\mathcal{D}_\alpha \mathcal{K}_{\mu\nu} = \partial_\alpha \mathcal{K}_{\mu\nu} - \bar{\chi}^{\theta\phi} \left(\Gamma_{\phi,\alpha\mu} \mathcal{K}_{\theta\nu} + \Gamma_{\phi,\alpha\nu} \mathcal{K}_{\theta\mu}\right) + t^\theta \left(\mathcal{K}_{\theta\nu} \partial_\mu \tilde{t}_\alpha + \mathcal{K}_{\theta\mu} \partial_\nu \tilde{t}_\alpha\right) \tag{B.19}$$

The last term in the RHS of equation (B.19) will vanish as a consequence of the identity (B.18). The second term in the RHS of (B.19) could be further simplified using the expansion of $\bar{\chi}^{\theta\phi}$.

$$\begin{aligned} & \bar{\chi}^{\theta\phi} \left(\Gamma_{\phi,\alpha\mu} \mathcal{K}_{\theta\nu} + \Gamma_{\phi,\alpha\nu} \mathcal{K}_{\theta\mu}\right) \\ =& \chi^{\theta\phi} \left(\Gamma_{\phi,\alpha\mu} \mathcal{K}_{\theta\nu} + \Gamma_{\phi,\alpha\nu} \mathcal{K}_{\theta\mu}\right) \\ & - b^\phi t^\theta \left(\Gamma_{\phi,\alpha\mu} \mathcal{K}_{\theta\nu} + \Gamma_{\phi,\alpha\nu} \mathcal{K}_{\mu\theta}\right) \\ & + b^\theta \left(\mathcal{K}_{\alpha\mu} \mathcal{K}_{\theta\nu} + \mathcal{K}_{\alpha\nu} \mathcal{K}_{\mu\theta}\right) - B t^\theta \left(\mathcal{K}_{\alpha\mu} \mathcal{K}_{\theta\nu} + \mathcal{K}_{\alpha\nu} \mathcal{K}_{\mu\theta}\right) \\ \text{where} \quad & b^\mu \equiv \chi^{\mu\nu} \tilde{t}_\nu, \quad B \equiv \tilde{t}_\mu \tilde{t}_\nu \chi^{\mu\nu} \end{aligned} \tag{B.20}$$

Here the term $b^\theta \mathcal{K}_{\theta\nu} \mathcal{K}_{\alpha\mu}$ is quadratic in the amplitude of the dynamics and therefore is negligible within our approximation. The last two terms vanish if we apply the identity (B.18). Hence it follows

$$\mathcal{D}_\alpha \mathcal{K}_{\mu\nu} = \partial_\alpha \mathcal{K}_{\mu\nu} - \chi^{\theta\phi} \left(\Gamma_{\phi,\alpha\mu} \mathcal{K}_{\theta\nu} + \Gamma_{\phi,\alpha\nu} \mathcal{K}_{\theta\mu}\right) + \mathcal{O}\left(\epsilon^2\right)$$

From $\partial_\alpha \mathcal{K}_{\mu\nu}$ we can separate the fluid and non-fluid terms in the following way

$$\partial_\alpha \mathcal{K}_{\mu\nu} = -e^\phi \left\{\left(\partial_\alpha n^{\nu_1}\right) \Gamma_{\nu_1,\mu\nu} + n^{\nu_1} \partial_\alpha \Gamma_{\nu_1,\mu\nu}\right\} - e^\phi \left(\partial_\alpha \phi\right) n^{\nu_1} \Gamma_{\nu_1,\mu\nu} \tag{B.21}$$

Now for convenience we will write the expression for J^{μ}_{space} as a sum of two terms

$$J^{\mu}_{space} = T_1 + T_2 \tag{B.22}$$

with

$$\begin{aligned} T_1 &= -4\alpha^2 \frac{1}{\sqrt{g^{(b)}}} \frac{\sqrt{H}}{\sqrt{t^\alpha t^\beta g^{(b)}_{\alpha\beta}}} \left(\bar{\chi}^{\gamma\alpha}\bar{\chi}^{\mu\beta} - \bar{\chi}^{\gamma\mu}\bar{\chi}^{\alpha\beta}\right)\left(\partial_\gamma \mathcal{K}_{\alpha\beta}\right) \\ T_2 &= 4\alpha^2 \frac{1}{\sqrt{g^{(b)}}} \frac{\sqrt{H}}{\sqrt{t^\alpha t^\beta g^{(b)}_{\alpha\beta}}} \left(\bar{\chi}^{\gamma\alpha}\bar{\chi}^{\mu\beta} - \bar{\chi}^{\gamma\mu}\bar{\chi}^{\alpha\beta}\right)\chi^{\theta\phi}\left(\Gamma_{\phi,\gamma\alpha}\mathcal{K}_{\theta\beta} + \Gamma_{\phi,\gamma\beta}\mathcal{K}_{\theta\alpha}\right) \end{aligned} \tag{B.23}$$

Now we use the identity of (B.18) to simplify the terms and (4.31) to separate the terms

$$\begin{aligned} &[T_1]_{fluid} \\ =&4\alpha^2 \frac{1}{\sqrt{g^{(b)}}} \frac{\sqrt{H}}{\sqrt{n^\alpha n^\beta g^{(b)}_{\alpha\beta}}} \left\{\left(\partial_\gamma n^{\nu_1}\right)\Gamma_{\nu_1,\alpha\beta} + n^{\nu_1}\partial_\gamma \Gamma_{\nu_1,\alpha\beta}\right\} \\ &\Big[\left(\chi^{\alpha\alpha_1}\partial_{\alpha_1}L\right)\left(\chi^{\beta\beta_1}\partial_{\beta_1}L\right) n^\gamma n^\mu - \left(\chi^{\beta\beta_1}\partial_{\beta_1}L\right) n^\mu \chi^{\gamma\alpha} \\ &- \left(\chi^{\alpha\alpha_1}\partial_{\alpha_1}L\right)\left(\chi^{\gamma\gamma_1}\partial_{\gamma_1}L\right) n^\beta n^\mu + \left(\chi^{\gamma\gamma_1}\partial_{\gamma_1}L\right)\left(\chi^{\mu\mu_1}\partial_{\mu_1}L\right) n^\alpha n^\beta \\ &+ \left(\chi^{\gamma\gamma_1}\partial_{\gamma_1}L\right) n^\mu \chi^{\alpha\beta} + \left(\chi^{\alpha\alpha_1}\partial_{\alpha_1}L\right) n^\beta \chi^{\gamma\mu} \\ &- \chi^{\alpha\beta}\chi^{\gamma\mu} + \left(\chi^{\beta\beta_1}\partial_{\beta_1}L\right) n^\alpha \chi^{\gamma\mu} - \left(\chi^{\alpha\alpha_1}\partial_{\alpha_1}L\right) n^\gamma \chi^{\mu\beta} \\ &+ \chi^{\gamma\alpha}\chi^{\mu\beta} - \left(\chi^{\gamma\gamma_1}\partial_{\gamma_1}L\right) n^\alpha \chi^{\mu\beta} + \left(\chi^{\mu\mu_1}\partial_{\mu_1}L\right) n^\gamma \chi^{\alpha\beta} \\ &- \left(\chi^{\beta\beta_1}\partial_{\beta_1}L\right)\left(\chi^{\mu\mu_1}\partial_{\mu_1}L\right) n^\alpha n^\gamma - \left(\chi^{\mu\mu_1}\partial_{\mu_1}L\right) n^\beta \chi^{\gamma\alpha} \\ &+ \left(\chi^{\sigma_1\sigma_2}\partial_{\sigma_1}L\,\partial_{\sigma_2}L\right)\left(n^\beta n^\mu \chi^{\gamma\alpha} - n^\gamma n^\mu \chi^{\alpha\beta} - n^\alpha n^\beta \chi^{\gamma\mu} + n^\alpha n^\gamma \chi^{\mu\beta}\right)\Big] \end{aligned} \tag{B.24}$$

$$
\begin{aligned}
&[T_1]_{non-fluid} \\
&= 4\alpha^2 \frac{1}{\sqrt{g^{(b)}}} \frac{\sqrt{H}}{\sqrt{n^\alpha n^\beta g^{(b)}_{\alpha\beta}}} \left\{(\partial_\gamma \phi)\, n^{\nu_1} \Gamma_{\nu_1,\alpha\beta}\right\} \Big[\left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) n^\gamma n^\mu \\
&- \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) n^\mu \chi^{\gamma\alpha} - \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} L\right) n^\beta n^\mu + \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} L\right) n^\mu \chi^{\alpha\beta} + \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) n^\beta \chi^{\gamma\mu} \\
&- \chi^{\alpha\beta} \chi^{\gamma\mu} + \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) n^\alpha \chi^{\gamma\mu} - \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) n^\gamma \chi^{\mu\beta} + \chi^{\gamma\alpha} \chi^{\mu\beta} - \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} L\right) n^\alpha \chi^{\mu\beta} \\
&+ \left(\chi^{\mu\mu_1} \partial_{\mu_1} L\right) n^\gamma \chi^{\alpha\beta} - \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) \left(\chi^{\mu\mu_1} \partial_{\mu_1} L\right) n^\alpha n^\gamma - \left(\chi^{\mu\mu_1} \partial_{\mu_1} L\right) n^\beta \chi^{\gamma\alpha} \\
&+ \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} L\right) \left(\chi^{\mu\mu_1} \partial_{\mu_1} L\right) n^\alpha n^\beta + \left(\chi^{\sigma_1\sigma_2} \partial_{\sigma_1} L\, \partial_{\sigma_2} L\right) \left(n^\beta n^\mu \chi^{\gamma\alpha} - n^\gamma n^\mu \chi^{\alpha\beta} - n^\alpha n^\beta \chi^{\gamma\mu} + n^\alpha n^\gamma \chi^{\mu\beta}\right) \Big] \\
&+ 4\alpha^2 \frac{1}{\sqrt{g^{(b)}}} \frac{\sqrt{H}}{\sqrt{n^\alpha n^\beta g^{(b)}_{\alpha\beta}}} \left\{(\partial_\gamma \phi)\, n^{\nu_1} \Gamma_{\nu_1,\alpha\beta} + (\partial_\gamma n^{\nu_1})\, \Gamma_{\nu_1,\alpha\beta} + n^{\nu_1} \partial_\gamma \Gamma_{\nu_1,\alpha\beta}\right\} \Big[\Big(- L\, \partial_{\alpha_1} L\, \partial_{\beta_1} \phi \\
&- L\, \partial_{\beta_1} L\, \partial_{\alpha_1} \phi + L^2\, \partial_{\alpha_1} \phi\, \partial_{\beta_1} \phi \Big) n^\gamma n^\mu \chi^{\alpha\alpha_1} \chi^{\beta\beta_1} + L \left(\chi^{\beta\beta_1} \partial_{\beta_1} \phi\right) n^\mu \chi^{\gamma\alpha} - \Big(- L\, \partial_{\alpha_1} L\, \partial_{\gamma_1} \phi \\
&- L\, \partial_{\gamma_1} L\, \partial_{\alpha_1} \phi + L^2\, \partial_{\alpha_1} \phi\, \partial_{\gamma_1} \phi \Big) n^\beta n^\mu \chi^{\alpha\alpha_1} \chi^{\gamma\gamma_1} - L \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} \phi\right) n^\mu \chi^{\alpha\beta} - L \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} \phi\right) n^\beta \chi^{\gamma\mu} \\
&- L \left(\chi^{\beta\beta_1} \partial_{\beta_1} \phi\right) n^\alpha \chi^{\gamma\mu} + L \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} \phi\right) n^\gamma \chi^{\mu\beta} + L \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} \phi\right) n^\alpha \chi^{\mu\beta} - L \left(\chi^{\mu\mu_1} \partial_{\mu_1} \phi\right) n^\gamma \chi^{\alpha\beta} \\
&- \Big(- L\, \partial_{\beta_1} L\, \partial_{\mu_1} \phi - L\, \partial_{\mu_1} L\, \partial_{\beta_1} \phi + L^2\, \partial_{\beta_1} \phi\, \partial_{\mu_1} \phi \Big) n^\alpha n^\gamma \chi^{\beta\beta_1} \chi^{\mu\mu_1} + L \left(\chi^{\mu\mu_1} \partial_{\mu_1} \phi\right) n^\beta \chi^{\gamma\alpha} \\
&+ \Big(- L\, \partial_{\gamma_1} L\, \partial_{\mu_1} \phi - L\, \partial_{\gamma_1} \phi\, \partial_{\mu_1} L + L^2\, \partial_{\gamma_1} \phi\, \partial_{\mu_1} \phi \Big) n^\alpha n^\beta \chi^{\gamma\gamma_1} \chi^{\mu\mu_1} - \Big(L\, \partial_{\sigma_1} L\, \partial_{\sigma_2} \phi \\
&+ L\, \partial_{\sigma_1} \phi\, \partial_{\sigma_2} L - L^2\, \partial_{\sigma_1} \phi\, \partial_{\sigma_2} \phi \Big) \chi^{\sigma_1\sigma_2} \left(n^\beta n^\mu \chi^{\gamma\alpha} - n^\gamma n^\mu \chi^{\alpha\beta} - n^\alpha n^\beta \chi^{\gamma\mu} + n^\alpha n^\gamma \chi^{\mu\beta}\right) \Big]
\end{aligned}
\tag{B.25}
$$

$$
\begin{aligned}
&[T_2]_{fluid} \\
&= 4\alpha^2 \frac{1}{\sqrt{g^{(b)}}} \frac{\sqrt{H}}{\sqrt{n^\alpha n^\beta g^{(b)}_{\alpha\beta}}} \chi^{\theta\phi} \Big[\left(n^{\sigma_3} \Gamma_{\sigma_3,\theta\beta} \Gamma_{\phi,\gamma\alpha} + n^{\sigma_4} \Gamma_{\sigma_4,\theta\alpha} \Gamma_{\phi,\gamma\beta}\right) \Big\{ - \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) n^\mu n^\gamma \\
&+ \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) n^\mu \chi^{\gamma\alpha} - \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} L\right) n^\mu \chi^{\alpha\beta} + \chi^{\alpha\beta} \chi^{\gamma\mu} + \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) n^\gamma \chi^{\mu\beta} - \chi^{\gamma\alpha} \chi^{\mu\beta} \\
&- \left(\chi^{\mu\mu_1} \partial_{\mu_1} L\right) n^\gamma \chi^{\alpha\beta} + \left(\chi^{\sigma_1\sigma_2} \partial_{\sigma_1} L\, \partial_{\sigma_2} L\right) n^\gamma n^\mu \chi^{\alpha\beta} \Big\} \\
&+ \left(n^{\sigma_4} \Gamma_{\sigma_4,\theta\alpha}\right) \Gamma_{\phi,\gamma\beta} \Big\{ \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} L\right) n^\beta n^\mu - \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1} L\right) n^\beta \chi^{\gamma\mu} + \left(\chi^{\mu\mu_1} \partial_{\mu_1} L\right) n^\beta \chi^{\gamma\alpha} \\
&- \left(\chi^{\sigma_1\sigma_2} \partial_{\sigma_1} L\, \partial_{\sigma_2} L\right) n^\beta n^\mu \chi^{\gamma\alpha} \Big\} + \left(n^{\sigma_3} \Gamma_{\sigma_3,\theta\beta}\right) \Gamma_{\phi,\gamma\alpha} \Big\{ - \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) n^\alpha \chi^{\gamma\mu} + \left(\chi^{\gamma\gamma_1} \partial_{\gamma_1} L\right) n^\alpha \chi^{\mu\beta} \\
&+ \left(\chi^{\mu\mu_1} \partial_{\mu_1} L\right) \left(\chi^{\beta\beta_1} \partial_{\beta_1} L\right) n^\alpha n^\gamma - \left(\chi^{\sigma_1\sigma_2} \partial_{\sigma_1} L\, \partial_{\sigma_2} L\right) n^\alpha n^\gamma \chi^{\mu\beta} \Big\} \Big]
\end{aligned}
\tag{B.26}
$$

$$
\begin{aligned}
&[T_2]_{non-fluid} \\
=&4\alpha^2 \frac{1}{\sqrt{g^{(b)}}} \frac{\sqrt{H}}{\sqrt{n^\alpha n^\beta g^{(b)}_{\alpha\beta}}} \chi^{\theta\phi} \Bigg[\Bigg\{ \left(L\, \partial_{\alpha_1} L\, \partial_{\beta_1}\phi + L\, \partial_{\alpha_1}\phi\, \partial_{\beta_1} L - L^2\, \partial_{\alpha_1}\phi\, \partial_{\beta_1}\phi \right) n^\mu n^\gamma \chi^{\alpha\alpha_1} \chi^{\beta\beta_1} \\
&+ L\left(\chi^{\gamma\gamma_1} \partial_{\gamma_1}\phi\, \chi^{\alpha\beta} - \chi^{\beta\beta_1} \partial_{\beta_1}\phi\, \chi^{\gamma\alpha} \right) n^\mu - L\left(\chi^{\alpha\alpha_1} \partial_{\alpha_1}\phi\, \chi^{\mu\beta} \right) n^\gamma \\
&+ L\left(\chi^{\mu\mu_1} \partial_{\mu_1}\phi\, \chi^{\alpha\beta} \right) n^\gamma \Bigg\} \left(n^{\sigma_3} \Gamma_{\sigma_3,\theta\beta} \Gamma_{\phi,\gamma\alpha} + n^{\sigma_4} \Gamma_{\sigma_4,\theta\alpha} \Gamma_{\phi,\gamma\beta} \right) \\
&- n^\beta n^\mu \left(n^{\sigma_4} \Gamma_{\sigma_4,\theta\alpha} \Gamma_{\phi,\gamma\beta} \right) \chi^{\alpha\alpha_1} \chi^{\beta\beta_1} \left(L\, \partial_{\alpha_1} L\, \partial_{\gamma_1}\phi + L\, \partial_{\alpha_1}\phi\, \partial_{\gamma_1} L - L^2\, \partial_{\alpha_1}\phi\, \partial_{\gamma_1}\phi \right) \\
&+ L\left(n^{\sigma_4} \Gamma_{\sigma_4,\theta\alpha} \right) \Gamma_{\phi,\gamma\beta} \left\{ \left(\chi^{\alpha\alpha_1} \partial_{\alpha_1}\phi \right)\, n^\beta \chi^{\gamma\mu} - \left(\chi^{\mu\mu_1} \partial_{\mu_1}\phi \right) n^\beta \chi^{\gamma\alpha} \right\} \\
&+ L\left(n^{\sigma_3} \Gamma_{\sigma_3,\theta\beta} \right) \Gamma_{\phi,\gamma\alpha} \left\{ \chi^{\beta\beta_1} \partial_{\beta_1}\phi\, n^\alpha \chi^{\gamma\mu} - \chi^{\gamma\gamma_1} \partial_{\gamma_1}\phi\, n^\alpha \chi^{\mu\beta} \right\} \\
&- \left(L\, \partial_{\beta_1} L\, \partial_{\mu_1}\phi + L\, \partial_{\beta_1}\phi\, \partial_{\mu_1} L - L^2\, \partial_{\beta_1}\phi\, \partial_{\mu_1}\phi \right) n^\alpha n^\gamma \chi^{\mu\mu_1} \chi^{\beta\beta_1} \left(n^{\sigma_3} \Gamma_{\sigma_3,\theta\beta} \right) \Gamma_{\phi,\gamma\alpha} \\
&- \chi^{\sigma_1\sigma_2} \left(L\, \partial_{\sigma_1} L\, \partial_{\sigma_2}\phi + L\, \partial_{\sigma_1}\phi\, \partial_{\sigma_2} L - L^2\, \partial_{\sigma_1}\phi\, \partial_{\sigma_2}\phi \right) \Bigg\{ - \left(n^{\sigma_4} \Gamma_{\sigma_4,\theta\alpha} \right) \Gamma_{\phi,\gamma\beta} n^\beta n^\mu \chi^{\gamma\alpha} \\
&- \left(n^{\sigma_3} \Gamma_{\sigma_3,\theta\beta} \right) \Gamma_{\phi,\gamma\alpha} n^\alpha n^\gamma \chi^{\mu\beta} + \left(n^{\sigma_3} \Gamma_{\sigma_3,\theta\beta} \Gamma_{\phi,\gamma\alpha} + n^{\sigma_4} \Gamma_{\sigma_4,\theta\alpha} \Gamma_{\phi,\gamma\beta} \right) n^\gamma n^\mu \chi^{\alpha\beta} \Bigg\} \Bigg]
\end{aligned}
\tag{B.27}
$$

Simplifying J^μ_{time}

In this section we will write down the intrinsic Ricci scalar as a sum of 'fluid' and 'non-fluid' terms. Using the definition of $\mathcal{K}$ and ignoring the terms quadratic in the amplitude of dynamics, we can write

$$
\begin{aligned}
\mathcal{R} =& \left(\bar{\chi}^{\mu_1\nu_1} \bar{\chi}^{\mu_2\nu_2} - \bar{\chi}^{\mu_1\nu_2} \bar{\chi}^{\mu_2\nu_1} \right) \Big[\partial_{\mu_1} \Gamma_{\nu_1,\mu_2\nu_2} - \chi^{\alpha_1\alpha_2} \Gamma_{\alpha_1,\mu_1\nu_1} \Gamma_{\alpha_2,\mu_2\nu_2} \\
&- b^{\alpha_1} \Gamma_{\alpha_1,\mu_1\nu_1} \mathcal{K}_{\mu_2\nu_2} - b^{\alpha_2} \Gamma_{\alpha_2,\mu_2\nu_2} \mathcal{K}_{\mu_1\nu_1} + 2\mathcal{K}_{\mu_1\nu_1} \left(\partial_{\mu_2} \tilde{t}_{\nu_2} \right) \Big] \\
=& T_1 + T_2 + T_3 + T_4 + T_5
\end{aligned}
\tag{B.28}
$$

Now we use the identity of (B.18) to simplify the terms and (4.31) to separate the terms

$$\begin{aligned}
&[T_1+T_2]_{fluid}\\
&=\Big[\partial_{\mu_1}\Gamma_{\nu_1,\mu_2\nu_2}-\chi^{\alpha_1\alpha_2}\Gamma_{\alpha_1,\mu_1\nu_1}\Gamma_{\alpha_2,\mu_2\nu_2}\Big]\Big[-\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\nu_2}\chi^{\mu_1\nu_1}+\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\nu_1}\chi^{\mu_1\nu_2}\\
&\quad+\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\nu_2}\chi^{\mu_2\nu_1}-\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\nu_1}\chi^{\mu_2\nu_2}-\chi^{\mu_1\nu_2}\chi^{\mu_2\nu_1}+\chi^{\mu_1\nu_1}\chi^{\mu_2\nu_2}\\
&\quad+\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta L\right)\left(n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}-n^{\mu_2}n^{\nu_1}\chi^{\mu_1\nu_2}-n^{\mu_1}n^{\nu_2}\chi^{\mu_2\nu_1}+n^{\mu_1}n^{\nu_1}\chi^{\mu_2\nu_2}\right)\\
&\quad-\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}n^{\nu_2}+\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\mu_1}n^{\nu_2}\\
&\quad+\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}\chi^{\mu_1\nu_2}-\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_1}\chi^{\mu_2\nu_2}+\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}L\right)\Big\{\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\mu_2}n^{\nu_1}\\
&\quad-\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\mu_1}n^{\nu_1}-\chi^{\mu_1\nu_1}n^{\mu_2}+\chi^{\mu_2\nu_1}n^{\mu_1}\Big\}\Big]
\end{aligned}\tag{B.29}$$

$$\begin{aligned}
&[T_1+T_2]_{non-fluid}\\
&=\Big[\partial_{\mu_1}\Gamma_{\nu_1,\mu_2\nu_2}-\chi^{\alpha_1\alpha_2}\Gamma_{\alpha_1,\mu_1\nu_1}\Gamma_{\alpha_2,\mu_2\nu_2}\Big]\Big[L\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)\left(n^{\nu_2}\chi^{\mu_1\nu_1}-n^{\nu_1}\chi^{\mu_1\nu_2}\right)\\
&\quad+L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)\left(n^{\nu_1}\chi^{\mu_2\nu_2}-n^{\nu_2}\chi^{\mu_2\nu_1}\right)+L\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)\left(n^{\mu_1}\chi^{\mu_2\nu_2}-n^{\mu_2}\chi^{\mu_1\nu_2}\right)\\
&\quad+\{n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}-n^{\mu_2}n^{\nu_1}\chi^{\mu_1\nu_2}-n^{\mu_1}n^{\nu_2}\chi^{\mu_2\nu_1}+n^{\mu_1}n^{\nu_1}\chi^{\mu_2\nu_2}\}\Big\{-L\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta\phi\right)\\
&\quad-L\left(\chi^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta L\right)+L^2\left(\chi^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\right)\Big\}+L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}n^{\nu_2}\\
&\quad+L\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\mu_2}n^{\nu_2}-L^2\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_2}n^{\nu_2}\\
&\quad-L\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_1}n^{\nu_2}-L\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\mu_1}n^{\nu_2}\\
&\quad+L^2\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)n^{\mu_1}n^{\nu_2}-L\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}\phi\right)\Big\{-n^{\mu_2}\chi^{\mu_1\nu_1}+n^{\mu_1}\chi^{\mu_2\nu_1}\\
&\quad+\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\mu_2}n^{\nu_1}-\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\mu_1}n^{\nu_1}-L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)n^{\mu_2}n^{\nu_1}\\
&\quad+L\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)n^{\mu_1}n^{\nu_1}\Big\}\Big]
\end{aligned}\tag{B.30}$$

$$\begin{aligned}
[T_3]_{fluid}=&\left(\chi^{\alpha_1\theta_6}\partial_{\theta_6}L\right)\left(n^{\sigma_3}\Gamma_{\sigma_3,\mu_2\nu_2}\right)\Gamma_{\alpha_1,\mu_1\nu_1}\Big[\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\nu_1}\chi^{\mu_1\nu_2}-\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\nu_1}\chi^{\mu_2\nu_2}\\
&-\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_1}\chi^{\mu_2\nu_2}+\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}L\right)n^{\mu_1}\chi^{\mu_2\nu_1}+\chi^{\mu_1\nu_1}\chi^{\mu_2\nu_2}-\chi^{\mu_1\nu_2}\chi^{\mu_2\nu_1}\\
&+\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta L\right)n^{\mu_1}n^{\nu_1}\chi^{\mu_2\nu_2}-\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}L\right)n^{\mu_1}n^{\nu_1}\Big]
\end{aligned}\tag{B.31}$$

$$
\begin{aligned}
&[T_3]_{non-fluid} \\
&= \left(\chi^{\alpha_1\theta_6}\partial_{\theta_6}L\right)\left(n^{\sigma_3}\Gamma_{\sigma_3,\mu_2\nu_2}\right)\Gamma_{\alpha_1,\mu_1\nu_1}\Big[-L\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)n^{\nu_1}\chi^{\mu_1\nu_2}+L\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_1}\chi^{\mu_2\nu_2} \\
&\quad +L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)n^{\nu_1}\chi^{\mu_2\nu_2}-L\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}\phi\right)n^{\mu_1}\chi^{\mu_2\nu_1}-\big(L\,\partial_\alpha L\,\partial_\beta\phi+L\,\partial_\alpha\phi\,\partial_\beta L \\
&\quad -L^2\,\partial_\alpha\phi\,\partial_\beta\phi\big)\chi^{\alpha\beta}n^{\mu_1}n^{\nu_1}\chi^{\mu_2\nu_2}\Big] \\
&\quad -L\left(\chi^{\alpha_1\theta_6}\partial_{\theta_6}\phi\right)\left(n^{\sigma_3}\Gamma_{\sigma_3,\mu_2\nu_2}\right)\Gamma_{\alpha_1,\mu_1\nu_1}\Big[-L\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)n^{\nu_1}\chi^{\mu_1\nu_2}+L\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_1}\chi^{\mu_2\nu_2} \\
&\quad +L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)n^{\nu_1}\chi^{\mu_2\nu_2}-L\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}\phi\right)n^{\mu_1}\chi^{\mu_2\nu_1}-\big(L\,\partial_\alpha L\,\partial_\beta\phi+L\,\partial_\alpha\phi\,\partial_\beta L \\
&\quad -L^2\,\partial_\alpha\phi\,\partial_\beta\phi\big)\chi^{\alpha\beta}n^{\mu_1}n^{\nu_1}\chi^{\mu_2\nu_2}+\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\nu_1}\chi^{\mu_1\nu_2}-\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\nu_1}\chi^{\mu_2\nu_2} \\
&\quad -\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_1}\chi^{\mu_2\nu_2}+\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}L\right)n^{\mu_1}\chi^{\mu_2\nu_1}+\chi^{\mu_1\nu_1}\chi^{\mu_2\nu_2}-\chi^{\mu_1\nu_2}\chi^{\mu_2\nu_1} \\
&\quad +\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta L\right)n^{\mu_1}n^{\nu_1}\chi^{\mu_2\nu_2}-\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}L\right)n^{\mu_1}n^{\nu_1}\Big]
\end{aligned}
\tag{B.32}
$$

$$
\begin{aligned}
&[T_4+T_5]_{fluid} \\
&= n^{\sigma_4}\Gamma_{\sigma_4,\mu_1\nu_1}\left(2\partial_{\mu_2}\partial_{\nu_2}L-\chi^{\alpha_2\theta_7}\partial_{\theta_7}L\,\Gamma_{\alpha_2,\mu_2\nu_2}\right)\Big[-\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta L\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1} \\
&\quad +\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\nu_2}\chi^{\mu_1\nu_1}-\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\nu_2}\chi^{\mu_2\nu_1}-\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}\chi^{\mu_1\nu_2} \\
&\quad +\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}L\right)n^{\mu_2}\chi^{\mu_1\nu_1}+\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}n^{\nu_2}+\chi^{\mu_1\nu_2}\chi^{\mu_2\nu_1}-\chi^{\mu_1\nu_1}\chi^{\mu_2\nu_2}\Big]
\end{aligned}
\tag{B.33}
$$

$$
\begin{aligned}
&[T_4+T_5]_{non-fluid}\\
=&n^{\sigma_4}\Gamma_{\sigma_4,\mu_1\nu_1}\left(2\partial_{\mu_2}\partial_{\nu_2}L-\chi^{\alpha_2\theta_7}\partial_{\theta_7}L\,\Gamma_{\alpha_2,\mu_2\nu_2}\right)\Big[L\left(\chi^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta L\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}\\
&-L^2\left(\chi^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}+L\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta\phi\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}\\
&-L\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)n^{\nu_2}\chi^{\mu_1\nu_1}+L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)n^{\nu_2}\chi^{\mu_2\nu_1}+L\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_2}\chi^{\mu_1\nu_2}\\
&-L\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}\phi\right)n^{\mu_2}\chi^{\mu_1\nu_1}-L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}n^{\nu_2}\\
&-L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_2}n^{\nu_2}+L^2\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_2}n^{\nu_2}\Big]\\
&+n^{\sigma_4}\Gamma_{\sigma_4,\mu_1\nu_1}\bigg(-2\partial_{\mu_2}\phi\,\partial_{\nu_2}L-2\partial_{\mu_2}L\,\partial_{\nu_2}\phi-2L\,\partial_{\mu_2}\partial_{\nu_2}\phi+2\partial_{\mu_2}\phi\,\partial_{\nu_2}\phi\\
&+L\chi^{\alpha_2\theta_7}\partial_{\theta_7}\phi\,\Gamma_{\alpha_2,\mu_2\nu_2}\bigg)\Big[-\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta L\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}+\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}L\right)n^{\nu_2}\chi^{\mu_1\nu_1}\\
&-\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)n^{\nu_2}\chi^{\mu_2\nu_1}-\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}\chi^{\mu_1\nu_2}+\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}L\right)n^{\mu_2}\chi^{\mu_1\nu_1}\\
&+\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}n^{\nu_2}+\chi^{\mu_1\nu_2}\chi^{\mu_2\nu_1}-\chi^{\mu_1\nu_1}\chi^{\mu_2\nu_2}\\
&+L\left(\chi^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta L\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}-L^2\left(\chi^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}\\
&+L\left(\chi^{\alpha\beta}\partial_\alpha L\,\partial_\beta\phi\right)n^{\mu_2}n^{\nu_2}\chi^{\mu_1\nu_1}-L\left(\chi^{\mu_2\theta_2}\partial_{\theta_2}\phi\right)n^{\nu_2}\chi^{\mu_1\nu_1}+L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)n^{\nu_2}\chi^{\mu_2\nu_1}\\
&+L\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_2}\chi^{\mu_1\nu_2}-L\left(\chi^{\nu_2\theta_5}\partial_{\theta_5}\phi\right)n^{\mu_2}\chi^{\mu_1\nu_1}\\
&-L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}L\right)n^{\mu_2}n^{\nu_2}-L\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}L\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_2}n^{\nu_2}\\
&+L^2\left(\chi^{\mu_1\theta_3}\partial_{\theta_3}\phi\right)\left(\chi^{\nu_1\theta_4}\partial_{\theta_4}\phi\right)n^{\mu_2}n^{\nu_2}\Big]
\end{aligned}
\tag{B.34}
$$

Appendix C

(For Chapter - 5)

C.1 Causality criteria from near-luminal stability for a general hydrodynamic theory

C.1.1 Monotonic behavior of stable parameter space with boost velocity

In the current analysis, the conservation equations (giving rise to hydrodynamic evolution equations) are linearized for small perturbations of fluid variables around their hydrostatic equilibrium. The method gives the dispersion polynomial in the frequency $(\omega, \boldsymbol{k})$ plane as $F(\omega, \boldsymbol{k}) = 0$, whose solution provides the dispersion relation $\omega = \omega(\boldsymbol{k})$ that is required for the stability analysis. Here, we are deriving our results for a general hydrodynamic dispersion polynomial (irrespective of shear or sound channel), which obeys just two assumptions guided by generic physics requirements. The assumptions are motivated by the conservation rules (of the number of fluid modes) and the symmetry requirements and do not compromise the generality of our method.

Assumption 1 : The total power of any term that contains $\boldsymbol{k}$ (it can be a term that contains only $\boldsymbol{k}$ or an admixture of ω and $\boldsymbol{k}$) must not exceed the largest power of a pure ω term. Following this criteria, a most general dispersion polynomial must obey,

$$\mathcal{O}_\omega[F(\omega, \boldsymbol{k} \neq 0)] = \mathcal{O}_{|\boldsymbol{k}|}[F(\omega = a|\boldsymbol{k}|, \boldsymbol{k} = \mathbf{b}|\boldsymbol{k}|] \; , \tag{C.1}$$

with a as a nonzero real scalar constant, $\mathbf{b}$ as a real unit vector and $\mathcal{O}_x$ denoting the order of the polynomial in the variable x.

In Ref. [101], Eq.(C.1) has been mentioned as a condition for causality. We are justifying this assumption from the point of Lorentz invariance of the number of modes in a theory. If the right-hand side of (C.1) has a larger order than the left-hand side, then a Lorentz boost of the background fluid with a velocity $\mathbf{v}$ always produces spurious modes, modes that never appeared in the local rest frame analysis [129, 131]. Given that the number of modes changes with the arbitrary choice of equilibrium state, it is indicative that the equations of motion that lead to such a polynomial cannot constitute a viable theory of viscous hydrodynamics. Moreover, the solution of these new modes will be inversely proportional to some powers of $\mathbf{v}$ (in the boosted frame the polynomial variable changes from ω to $\mathbf{v}\omega$), that diverges as $\mathbf{v} \to 0$ and hence are unphysical. With this chain of arguments, below we are writing the most general form of the dispersion polynomial (of order M) for any arbitrary hydrodynamic theory in the local rest frame of the fluid:

$$a_M\omega^M + a_{M-1}\omega^{M-1} + \cdots + a_2\omega^2 + a_1\omega + a_0 = 0\ ,$$

with,

$$\begin{aligned}
a_0 &= a_0^0 + a_0^1 k + \cdots + a_0^{M-2}k^{M-2} + a_0^{M-1}k^{M-1} + a_0^M k^M,\\
a_1 &= a_1^0 + a_1^1 k + \cdots + a_1^{M-2}k^{M-2} + a_1^{M-1}k^{M-1}\ ,\\
a_2 &= a_2^0 + a_2^1 k + \cdots + a_2^{M-2}k^{M-2}\ ,\\
&\vdots\\
a_{M-2} &= a_{M-2}^0 + a_{M-2}^1 k + a_{M-2}^2 k^2\ ,\\
a_{M-1} &= a_{M-1}^0 + a_{M-1}^1 k\ ,\\
a_M &= a_M^0\ ,
\end{aligned} \tag{C.2}$$

which in a consolidated form can be written as,

$$\sum_{n=0}^{M} a_n(k)\omega^n = 0, \quad a_n(k) = \sum_{m=0}^{M-n} a_n^m k^m\ . \tag{C.3}$$

The coefficients a_n^m (the subscript n denotes the power of ω and the superscript m denotes the power of k of the term it is associated with) are functions of transport coefficients of the underlying coarse-grained system that set the parameter space of the theory. We are putting no constraint on the a_n^m values. They can be both real and imaginary and can have positive or negative values or even become zero depending upon the construction of a particular hydrodynamic theory.

Our next step is to boost Eq.(C.2) with velocity $\mathbf{v}$ and extract the stability criteria of that boosted polynomial at the spatial homogeneous limit ($k \to 0$). At $k \to 0$, the boosted form of Eq.(C.2) becomes,

$$\begin{aligned}
&(\gamma\omega)^M \Big[a_M^0(-\mathbf{v})^0 + a_{M-1}^1(-\mathbf{v})^1 + a_{M-2}^2(-\mathbf{v})^2 + \cdots \\
&\qquad + a_2^{M-2}(-\mathbf{v})^{M-2} + a_1^{M-1}(-\mathbf{v})^{M-1} + a_0^M(-\mathbf{v})^M\Big] \\
&+ (\gamma\omega)^{M-1} \Big[a_{M-1}^0(-\mathbf{v})^0 + a_{M-2}^1(-\mathbf{v})^1 + \cdots \\
&\qquad + a_1^{M-2}(-\mathbf{v})^{M-2} + a_0^{M-1}(-\mathbf{v})^{M-1}\Big] + \cdots \\
&+ (\gamma\omega)^1 \Big[a_1^0(-\mathbf{v})^0 + a_0^1(-\mathbf{v})^1\Big] + (\gamma\omega)^0 \Big[a_0^0(-\mathbf{v})^0\Big] = 0\,,
\end{aligned} \tag{C.4}$$

with $\gamma = 1/\sqrt{1-\mathbf{v}^2}$. Eq.(C.4) can again be expressed in a general form as,

$$\sum_{n=0}^{M} A_n(\gamma\omega)^n = 0, \quad A_n = \sum_{m=0}^{n} a_m^{n-m}(-\mathbf{v})^{n-m}\,. \tag{C.5}$$

Since an analytical solution of Eq.(C.4) is beyond the scope, in order to check its stability we take recourse of Routh-Hurtwitz (R-H) stability test [102]. The stability condition requires the elements belonging to the first column of the Routh array (includes the coefficients of $(\gamma\omega)^M$, $(\gamma\omega)^{M-1}$ and determinants involving other coefficients of (C.4)) to be of identical sign, either positive or negative. This leads us to $M+1$ number of inequalities which say that, in order to have a stable theory, all these elements are either greater or lesser than zero. So if these elements are expressed as $f_i\left(\{a_n^m\}, \mathbf{v}\right)$, for all roots of ω to be stable, we must have either

$$f_i\left(\{a_n^m\}, \mathbf{v}\right) > / < 0\,, \tag{C.6}$$

for all $i \in \{1, M+1\}$. At this point, we state our second assumption.

Assumption 2 : The local rest frame dispersion polynomial (C.2) only allows even power of $|\boldsymbol{k}| = \sqrt{\boldsymbol{k}^2}$, making it $F(\omega, \boldsymbol{k}^2) = 0$, i.e, the coefficients a_n^m with an odd m are zero [147]. This can be simply understood from the fact that $\boldsymbol{k}$ being a vector, only the powers of $\boldsymbol{k}^2$ are allowed in the scalar dispersion polynomial (C.2). As a consequence, the boosted polynomial (C.4) contains only even power of $|\mathbf{v}| = \sqrt{\mathbf{v}^2}$ (also required since $\mathbf{v}$ is a vector as well).

These two assumptions lead to the fact that the R-H stability criteria of (C.4) boil down to a set of inequalities where a power series over $\mathbf{v}^2$ is greater or lesser than zero. To demonstrate the situation we are writing here the condition over the first element of the first column of the Routh array,

$$a_M^0(\mathbf{v})^0 + a_{M-2}^2(\mathbf{v})^2 + \cdots + a_2^{M-2}(\mathbf{v})^{M-2} + a_0^M(\mathbf{v})^M > 0\,. \tag{C.7}$$

Here, M is considered to be even (odd M conditions can be similarly extracted where the power of the last term would be $M-1$) and we illustrate the result for the "all positive" possibility. Now, the left-hand side of inequality (C.7) can be decomposed as,

$$(\mathbf{v}^2 - x_1)(\mathbf{v}^2 - x_2)\cdots(\mathbf{v}^2 - x_{M/2}) > 0\,, \tag{C.8}$$

where x_l are the roots of the polynomial,

$$a_M^0 + a_{M-2}^2 x + \cdots + a_2^{M-2} x^{M/2-1} + a_0^M x^{M/2} = 0\,, \tag{C.9}$$

and are functions of the a_n^m coefficients only (i.e., $x_l \equiv x_l(a_n^m)$), which are again functions of the transport coefficients of the system. So, to hold inequality (C.8), each factor $(\mathbf{v}^2 - x_l)$ has to be positive or negative accordingly. So finally, the R-H criteria boil down to a set of inequalities such that,

$$(\mathbf{v}^2 - x_l) > / < 0 \quad \Rightarrow \quad x_l(a_n^m) > / < \mathbf{v}^2\,. \tag{C.10}$$

So, from (C.10), we can see that the stability criteria of any theory reduces to a set of inequalities where a function of the fluid parameters is greater or lesser than $\mathbf{v}^2$. Clearly, this indicates a monotonic behavior of the parameter space on $\mathbf{v}^2$, and consequently, at spatial homogeneous limit $(k \to 0)$, the stable parameter space must monotonically decrease from $\mathbf{v} = 0$ to 1 or from $\mathbf{v} \to 1$

to 0 respectively. So, if we follow the 'greater than' possibility ($x_l(a_n^m) > \mathbf{v}^2$) of (C.10), the stability region of parameter space for $\mathbf{v} \to 1$ includes the same for any lower value of $\mathbf{v}$ turning the stability condition at $\mathbf{v} \to 1$ a necessary and sufficient condition for stability to hold at the spatially homogeneous limit for all possible boost velocities $0 \leq \mathbf{v} < 1$. Conversely, following the 'lesser than' possibility, the direction of monotonicity reverses.

Now, the sign of the inequality in (C.10) (that leads to the direction of monotonicity) suffers from ambiguity. The reason is that since Eq.(C.2) describes the dispersion polynomial of a possible most general theory, the signs of the a_n^m coefficients are completely unknown and arbitrary. To resolve this ambiguity, we investigate Eq.(C.4) at different boost velocities and provide the following line of arguments.

At $v = 0$, we observe that for each n, only the coefficients a_n^m with $m = 0$ are contributing to the stability analysis. For a non-zero value of $\mathbf{v}$, all the a_n^m coefficients with even m are contributing. If we have a look at Eq.(C.2), we can see that the stability conditions at non-zero $\mathbf{v}$ constrain a much larger number of elements of the parameter space, making the system of inequalities more restrictive than the ones at $\mathbf{v} = 0$. In other words, the conditions at $\mathbf{v} \neq 0$ lay a stricter bound on the entire parameter space than those at $\mathbf{v} = 0$. So, it is indicative that the monotonicity over $\mathbf{v}^2$ that has been discussed so far is uniformly restricting the parameter space from $\mathbf{v} = 0$ to $\mathbf{v} \to 1$. This turns the parameter space, which is stable at near-luminal boost velocity, a necessary and sufficient region for frame-invariant stability to hold (at the spatially homogeneous limit), and consequently, identify the causal parameter space as well [134].

In support of the above discussion, here we are writing the polynomial equation for asymptotic group velocity v_g at $k \to \infty$ resulting from (C.2) for even M :

$$a_M^0 (v_g^2)^{\frac{M}{2}} + a_{M-2}^2 (v_g^2)^{\frac{M}{2}-1} + \cdots + a_2^{M-2} v_g^2 + a_0^M = 0\,. \tag{C.11}$$

In order to have a causal, propagating mode, (C.11) must have real, positive, subluminal roots of v_g^2, which are the functions of the a_n^m coefficients of (C.11). From Eq.(C.2), we see that these a_n^m are the coefficients of the largest k power for each a_n term with n even. Clearly, the conditions for subluminal roots will involve constraints on these coefficients. Here, we see that stability conditions for $\mathbf{v} = 0$ only include the a^0_M among these coefficients and can not able to identify the causal parameter space because of this nominal overlap. On the other hand, the stability constraints with nonzero $\mathbf{v}$ include all the coefficients of Eq.(C.11). So, the monotonicity over $\mathbf{v}^2$ leaves us with the choice that stability at $\mathbf{v} \to 1$ demarcates the causal parameter space.

C.1.2 Connection between stable parameter space at $k \to 0$, $\mathbf{v} \to 1$ and causal parameter space at large k

A mathematical explanation regarding this connection can be followed here. For that, say for an even M case we divide Eq.(C.7) by $(\mathbf{v}^2)^{M/2}$. Being a positive quantity, it will not alter the sign of inequality and converts (C.7) into,

$$a^0_M \left(\frac{1}{\mathbf{v}^2}\right)^{\frac{M}{2}} + a^2_{M-2}\left(\frac{1}{\mathbf{v}^2}\right)^{\frac{M}{2}-1} + \cdots + a^{M-2}_2\left(\frac{1}{\mathbf{v}^2}\right) + a^M_0 > 0\,, \tag{C.12}$$

which can be decomposed as,

$$\left(\frac{1}{\mathbf{v}^2} - y_1\right)\left(\frac{1}{\mathbf{v}^2} - y_2\right)\cdots\left(\frac{1}{\mathbf{v}^2} - y_{M/2}\right) > 0\,, \tag{C.13}$$

with y_l being roots of,

$$a^0_M y^{\frac{M}{2}} + a^2_{M-2} y^{\frac{M}{2}-1} + \cdots + a^{M-2}_2 y + a^M_0 = 0\,. \tag{C.14}$$

Now, in order to hold inequality (C.13), each bracketed quantity on the left-hand side has to be individually positive or negative. The only physical choice is the positive convention, which for each y_l leads to,

$$\left(\frac{1}{\mathbf{v}^2} - y_l\right) > 0\,, \quad y_l < \frac{1}{\mathbf{v}^2}\,, \tag{C.15}$$

which gives the strictest bound at $\mathbf{v} \to 1$ such that $y_l < 1$. Here we make an important observation. We notice that Eq.(C.14) and the polynomial for asymptotic group velocity (C.11) are identical.

Consequently, the y_l are the solutions for v_g^2 itself. So from (C.15), we can see that the stability conditions at $\mathbf{v} \to 1$ are indeed related to the causality criteria of the theory ($v_g^2 < 1$). It is to be noted here that (C.7) is not the only stability condition (it is the first one of them; there are M more). In the theories that we have studied in our work - MIS and BDNK - the other conditions basically set the convention for the direction of inequalities that cancels any choice of y_l other than (C.15). Nevertheless, the structural similarity of (C.11) and (C.12) is enough to indicate the connection between the near luminal ($\mathbf{v} \to 1$) stability conditions at the spatial homogeneous limit ($k \to 0$) and the causality criteria predicted at the asymptotic causality limit ($k \to \infty$).

Appendix D

(For Chapter - 6)

D.1 Detailed calculations of Method-1

In this section, we will derive the form of the frame transformations in an infinite-order derivative expansion. To begin with, we'll rewrite the transformations of T and u^μ under frame redefinitions

$$T - \hat{T} = \delta T = \sum_{n=1}^{\infty} \delta T_n, \quad u^\mu - \hat{u}^\mu = \delta u^\mu = \sum_{n=1}^{\infty} \hat{u}^\mu_n$$

Substituting this into the expression of the stress-tensor and using the Landau-frame condition, the following expressions are obtained for δT_n and δu^μ_n.

$$\begin{aligned}
\delta T_1 &= -\tilde{\chi} \left(\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{1}{3} \hat{\nabla} \cdot \hat{u} \right) \\
\delta T_{n\geq 2} &= -\tilde{\chi} \left(\frac{\hat{D}\delta T_{n-1}}{\hat{T}} + \frac{1}{3} \hat{\nabla} \cdot \delta u_{n-1} \right) \\
\delta u^\mu_1 &= -\tilde{\theta} \left(\hat{D}\hat{u}^\mu + \hat{\nabla}^\mu \hat{T} \right) \\
\delta u^\mu_{n\geq 2} &= -\tilde{\theta} \left(\hat{D}\delta u^\mu_{n-1} + \hat{\nabla}^\mu \delta T_{n-1} \right)
\end{aligned} \tag{D.1}$$

D.1.1 Transformation of velocity

Using the forms given above, we can try to express δu^μ_n in terms of the lower order δTs and δu^μ_ns as,

$$
\begin{aligned}
\delta u^\mu_n &= (-\tilde\theta)\left[\hat D\delta u^\mu_{n-1} + \nabla^\mu \frac{\delta T_{n-1}}{\hat T}\right] \\
&= \left[(-\tilde\theta)^2\hat D^2\delta^m u_\nu + \frac{\tilde\chi\tilde\theta}{3}\hat\nabla_\nu\hat\nabla^\mu\right]\delta u^\nu_{n-2} + (-\tilde\theta)(-\tilde\theta-\tilde\chi)\hat D\frac{\nabla^\mu\delta T_{n-2}}{\hat T} \\
&= \left[(-\tilde\theta)^3\hat D^3\delta^\mu_\nu + \frac{\tilde\chi\tilde\theta}{3}(-2\tilde\theta-\tilde\chi)\hat D\hat\nabla_\nu\hat\nabla^\mu\right]\delta u^n u_{n-3} \\
&+ (-\tilde\theta)\left[(\tilde\theta^2+\tilde\chi\tilde\theta+\tilde\chi^2)\hat D^2 + \frac{\tilde\chi\tilde\theta}{3}\hat\nabla^2\right]\hat\nabla^\mu\frac{\delta T_{n-3}}{\hat T} \\
&= \left[(-\tilde\theta)^4\hat D^4\delta^\mu_\nu + \frac{\tilde\chi\tilde\theta}{3}(3\tilde\theta^2+2\tilde\chi\tilde\theta+\tilde\chi^2)\hat D^2\hat\nabla_\nu\hat\nabla^\mu + \left(\frac{\tilde\chi\tilde\theta}{3}\right)^2\hat\nabla^2\hat\nabla_\nu\hat\nabla^\mu\right]\delta u^\nu_{n-4} \\
&+ (-\tilde\theta)\left[-(\tilde\theta^3+\tilde\chi\tilde\theta^2+\tilde\chi^2\tilde\theta+\tilde\chi^3)\hat D^3 + \frac{\tilde\chi\tilde\theta}{3}2(-\tilde\theta-\tilde\chi)\hat D\hat\nabla^2\right]\hat\nabla^\mu\frac{\delta T_{n-4}}{\hat T} \\
&= \left[(-\tilde\theta)^5\hat D^5\delta^\mu_\nu - \frac{\tilde\chi\tilde\theta}{3}(4\tilde\theta^3+3\tilde\theta^2\tilde\chi+2\tilde\chi^2\tilde\theta+\tilde\chi^3)\hat D^3\hat\nabla_\nu\hat\nabla^\mu + \left(\frac{\tilde\chi\tilde\theta}{3}\right)^2(-3\tilde\theta-2\tilde\chi)\hat D\hat\nabla^2\hat\nabla_\nu\hat\nabla^\mu\right]\delta u^\nu_{n-5} \\
&+ (-\tilde\theta)\Bigg[(\tilde\theta^4+\tilde\theta^3\tilde\chi+\tilde\theta^2\tilde\chi^2+\tilde\theta\tilde\chi^3+\tilde\chi^4)\hat D^4 \\
&+ \frac{\tilde\chi\tilde\theta}{3}(3\tilde\theta^2+4\tilde\chi\tilde\theta+3\tilde\chi^2)\hat D^2\hat\nabla^2 + \left(\frac{\tilde\chi\tilde\theta}{3}\right)^2(\hat\nabla^2)^2\Bigg]\hat\nabla^\mu\frac{\delta T_{n-5}}{\hat T}
\end{aligned}
\tag{D.2}
$$

In this way, continuing the sequence, δu^μ_n can be expressed in terms of $\hat T$ and $\hat u$ as,

$$
\delta u^\mu_n = (-\tilde\theta\hat D)^n\hat u^\mu + (-\tilde\theta)\sum_{m=0}^{n-1} c_m\left(\frac{\tilde\chi\tilde\theta}{3}\hat\nabla^2\right)^m \hat D^{n-1-2m}\frac{\hat\nabla^\mu\hat T}{\hat T} \tag{D.3}
$$

$$
+\sum_{m=0}^{n-2}\frac{\tilde\chi\tilde\theta}{3}d_m\left(\frac{\tilde\chi\tilde\theta}{3}\hat\nabla^2\right)^m\hat D^{n-2-2m}\hat\nabla^\mu\hat\nabla\cdot\hat u\,, \tag{D.4}
$$

where,

$$
c_{mn} = \frac{1}{(m!)^2}\left(-\frac{\partial}{\partial\tilde\theta}\right)^m\left(-\frac{\partial}{\partial\tilde\chi}\right)^m\sum_{l=0}^{n-1}(-\tilde\theta)^l(-\tilde\chi)^{n-1-l}, \tag{D.5}
$$

$$
d_{mn} = \frac{1}{(m+1)}\left(-\frac{\partial}{\partial\tilde\theta}\right)c_{mn}\,. \tag{D.6}
$$

The expressions in (6.19)-(6.27) can be reproduced from this form in (D.3). To find δu^μ, we need to sum over all the δu^μ_ns from $n = 1$ to ∞ as follows,

$$\delta u^\mu = \sum_{n=1}^{\infty} \delta u^\mu_n = \left(\sum_{n=1}^{\infty} (-\tilde{\theta}\hat{D})^n \hat{u}^\mu\right) + (-\tilde{\theta}) \left(\sum_{n=1}^{\infty}\sum_{m=0}^{n-1} c_m \left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m \hat{D}^{n-1-2m}\right) \frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}} + \left(\sum_{n=1}^{\infty}\sum_{m=0}^{n-2} \frac{\tilde{\chi}\tilde{\theta}}{3} d_m \left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m \hat{D}^{n-2-2m}\right) \hat{\nabla}^\mu \hat{\nabla}\cdot\hat{u}\,. \tag{D.7}$$

Considering the first summation of (D.7), we find that it is an infinite summation of the form

$$\sum_{n=1}^{\infty} x^n = x \sum_{n=0}^{\infty} x^n = \frac{x}{1-x}\,. \tag{D.8}$$

Hence, from the first summation, we get

$$\left(\sum_{n=1}^{\infty} (-\tilde{\theta}\hat{D})^n\right) \hat{u}^\mu = \frac{(-\tilde{\theta}\hat{D})}{(1+\tilde{\theta}\hat{D})}\hat{u}^\mu\,. \tag{D.9}$$

The second summation in (D.7) is actually a nested summation of three different indices as,

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \frac{1}{(m!)^2} \left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m \hat{D}^{n-1-2m} \left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m \left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m \left(\sum_{l=0}^{n-1} (-\tilde{\theta})^l (-\tilde{\chi})^{n-1-l}\right)\,. \tag{D.10}$$

Replacing the index n by $N = n-1$, (D.10) becomes,

$$\sum_{N=0}^{\infty}\sum_{m=0}^{N} \frac{1}{(m!)^2} \left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m \hat{D}^{N-2m} \left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m \left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m \left(\sum_{l=0}^{N} (-\tilde{\theta})^l (-\tilde{\chi})^{N-l}\right)\,. \tag{D.11}$$

For values $m > N$, we see that $\left(\frac{\partial}{\partial\tilde{\theta}}\right)^m$ or $\left(\frac{\partial}{\partial\tilde{\chi}}\right)^m$ acting on the summation over l gives 0 as the highest power of $\tilde{\theta}$ or $\tilde{\chi}$ in the series is N only. So, we can add an infinite no. of such zeros and extend the summation over m to ∞ instead of N.

$$\sum_{N=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{(m!\hat{D})^2} \left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m \hat{D}^{N} \left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m \left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m \left(\sum_{l=0}^{N} (-\tilde{\theta})^l (-\tilde{\chi})^{N-l}\right) \tag{D.12}$$

The summations over m and N now have independent limits; hence, their order can be interchanged, and we can rewrite the summation as

$$\sum_{m=0}^{\infty} \frac{1}{(m!\hat{D})^2} \left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m \left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m \left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m \sum_{N=0}^{\infty} \hat{D}^{N} \left(\sum_{l=0}^{N} (-\tilde{\theta})^l (-\tilde{\chi})^{N-l}\right) \tag{D.13}$$

The summations over N and l can then be interchanged using the Cauchy product formula

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{m=0}^{\infty} b_l\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} a_l b_{n-m}\right)$$

and (D.10) can now be expressed as

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{1}{(m!\hat{D})^2}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\left(\sum_{N=0}^{\infty}(-\tilde{\chi}\hat{D})^N\right)\left(\sum_{l=0}^{\infty}(-\tilde{\theta}\hat{D})^l\right)\\
=&\sum_{m=0}^{\infty}\frac{1}{(m!\hat{D})^2}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\left(\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{1}{(1+\tilde{\theta}\hat{D})}\right)\\
=&\sum_{m=0}^{\infty}\frac{1}{(m!\hat{D})^2}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(\frac{m!\hat{D}^m}{(1+\tilde{\chi}\hat{D})^{m+1}}\frac{m!\hat{D}^m}{(1+\tilde{\theta}\hat{D})^{m+1}}\right)\\
=&\sum_{m=0}^{\infty}\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{1}{(1+\tilde{\theta}\hat{D})}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D})}\right)^m\\
=&\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{1}{(1+\tilde{\theta}\hat{D})}\frac{1}{1-\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D})}}\\
=&\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D})-\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}
\end{aligned} \tag{D.14}$$

Now, let us consider the third summation

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-2}\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(m!)^2}\frac{1}{(m+1)}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^{m+1}\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\hat{D}^{n-2-2m}\left(\sum_{l=0}^{n-1}(-\tilde{\theta})^l(-\tilde{\chi})^{n-1-l}\right) \tag{D.15}$$

Here, we see that for $m = n-1$, the no. of $\frac{\partial}{\partial\tilde{\theta}}$ derivatives becomes more than the highest power of $\tilde{\theta}$ present in the series over l, thus making the term corresponding to $m = n-1$ zero. We can add this zero term, and then our sum becomes

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(m!)^2}\frac{1}{(m+1)}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^{m+1}\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\hat{D}^{n-2-2m}\left(\sum_{l=0}^{n-1}(-\tilde{\theta})^l(-\tilde{\chi})^{n-1-l}\right) \tag{D.16}$$

Using $N = n-1$ as before,

$$\sum_{N=0}^{\infty}\sum_{m=0}^{N}\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(m!\hat{D})^2}\frac{1}{(m+1)}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\hat{D}^{-1}\left(-\frac{\partial}{\partial\tilde{\theta}}\right)\left(\sum_{l=0}^{N}(-\tilde{\theta})^l(-\tilde{\chi})^{N-l}\hat{D}^N\right) \tag{D.17}$$

Again, extending the sum over m up to ∞ and interchanging summations like the previous case, we get

$$\sum_{m=0}^{\infty}\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(m!\hat{D})^2}\frac{1}{(m+1)}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\hat{D}^{-1}\left(-\frac{\partial}{\partial\tilde{\theta}}\right)\left(\sum_{N=0}^{\infty}\sum_{l=0}^{N}(-\tilde{\theta})^l(-\tilde{\chi})^{N-l}\hat{D}^N\right)$$
$$=\sum_{m=0}^{\infty}\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(m!\hat{D})^2}\frac{1}{(m+1)}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\hat{D}^{-1}\left(-\frac{\partial}{\partial\tilde{\theta}}\right)\left(\frac{1}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)$$
$$=\sum_{m=0}^{\infty}\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(m!\hat{D})^2}\frac{1}{(m+1)}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(-\frac{\partial}{\partial\tilde{\theta}}\right)^m\left(-\frac{\partial}{\partial\tilde{\chi}}\right)^m\hat{D}^{-1}\left(\frac{\hat{D}}{(1+\tilde{\theta}\hat{D})^2}\frac{1}{(1+\tilde{\chi}\hat{D})}\right)$$
$$=\sum_{m=0}^{\infty}\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(m!\hat{D})^2}\frac{1}{(m+1)}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(\frac{(m+1)!\hat{D}^m}{(1+\tilde{\theta}\hat{D})^{m+2}}\frac{m!\hat{D}^{m+1}}{(1+\tilde{\chi}\hat{D})^m}\right)$$
$$=\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(1+\tilde{\theta}\hat{D})^2}\frac{1}{(1+\tilde{\chi}\hat{D})}\sum_{m=0}^{\infty}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(\frac{1}{(1+\tilde{\theta}\hat{D})^m}\frac{1}{(1+\tilde{\chi}\hat{D})^m}\right)$$
$$=\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(1+\tilde{\theta}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D})-\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2} \tag{D.18}$$

So, putting all these results together, (D.7) becomes

$$\delta u^\mu = \frac{(-\tilde{\theta}\hat{D})}{(1+\tilde{\theta}\hat{D})}\hat{u}^\mu + (-\tilde{\theta})\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D})-\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}}$$
$$+\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(1+\tilde{\theta}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D})-\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}\hat{\nabla}^\mu\hat{\nabla}\cdot\hat{u}$$
$$\Rightarrow \delta u^\mu = \frac{(-\tilde{\theta}\hat{D})}{(1+\tilde{\theta}\hat{D})}\hat{u}^\mu + \frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D})-\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}\left(-\tilde{\theta}\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}}+\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^\mu\hat{\nabla}\cdot\hat{u}\right) \tag{D.19}$$

which we see is identical to the δu^μ calculated in (6.39).

Also worth noticing is the point that, had we not summed over m in the second and third summations, then δu^μ would have been left in the form of an infinite series of the form

$$\delta u^\mu = \frac{(-\tilde{\theta}\hat{D})}{(1+\tilde{\theta}\hat{D})}\hat{u}^\mu - \frac{1}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\sum_{m=0}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m \left\{\tilde{\theta}\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}} - \frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^\mu\hat{\nabla}\cdot\hat{u}\right\}$$

$$u^\mu = \hat{u}^\mu + \delta u^\mu = \frac{1}{(1+\tilde{\theta}\hat{D})}\hat{u}^\mu + \frac{1}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\sum_{m=0}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m \left\{-\tilde{\theta}\frac{\hat{\nabla}^\mu\hat{T}}{\hat{T}} + \frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^\mu\hat{\nabla}\cdot\hat{u}\right\}$$

$$\sigma^{\mu\nu} = \frac{1}{(1+\tilde{\theta}\hat{D})} - 2\eta\hat{\sigma}^{\mu\nu} - 2\eta\frac{1}{(1+\tilde{\theta}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})}\sum_{m=0}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m \left\{-\tilde{\theta}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}} + \frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}\right\} \tag{D.20}$$

We can recast this form of $\sigma^{\mu\nu}$ into the form of a relaxation equation given by

$$(1+\tilde{\theta}\hat{D})\sigma^{\mu\nu} = -2\eta\Bigg[\hat{\sigma}^{\mu\nu} + \frac{1}{(1+\tilde{\chi}\hat{D})}\sum_{m=0}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m \left\{-\tilde{\theta}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}} + \frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}\right\}\Bigg] = -2\eta\hat{\sigma}^{\mu\nu} + \rho_1^{\langle\mu\nu\rangle} \tag{D.21}$$

where $\rho_1^{\langle\mu\nu\rangle}$ is given by

$$\rho_1^{\langle\mu\nu\rangle} = \left[-2\eta\frac{1}{(1+\tilde{\chi}\hat{D})}\sum_{m=0}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m \left\{-\tilde{\theta}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}} + \frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}\right\}\right] \tag{D.22}$$

It can again be recast into a relaxation equation as

$$\Rightarrow (1+\tilde{\chi}\hat{D})\rho_1^{\langle\mu\nu\rangle} = \left[-2\eta\sum_{m=0}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m \left\{-\tilde{\theta}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}} + \frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}\right\}\right] = -2\eta(-\tilde{\theta})\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}} + \rho_2^{\langle\mu\nu\rangle} \tag{D.23}$$

with $\rho_2^{\langle\mu\nu\rangle}$ defined and associated with another relaxation equation as

$$\begin{aligned}
\rho_2^{\langle\mu\nu\rangle} &= \frac{-2\eta\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u} \\
&- 2\eta\sum_{m=1}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m\left\{-\tilde{\theta}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}+\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}\right\} \\
(1+\tilde{\theta}\hat{D})\rho_2^{\langle\mu\nu\rangle} &= -2\eta\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}+\rho_3^{\langle\mu\nu\rangle}
\end{aligned} \tag{D.24}$$

where again $\rho_3^{\langle\mu\nu\rangle}$ contains the infinite series. In this way, the sequence would continue, and any general term would be given by (for $n \geq 0$)

$$\begin{aligned}
(1+\tilde{\chi}\hat{D})\rho_{2n+1}^{\langle\mu\nu\rangle} &= (-2\eta)(-\tilde{\theta})\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^n\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}+\rho_{2n+2}^{\langle\mu\nu\rangle} \\
(1+\tilde{\theta}\hat{D})\rho_{2n+2}^{\langle\mu\nu\rangle} &= (-2\eta)\frac{\tilde{\chi}\tilde{\theta}}{3}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^n\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}+\rho_{2n+3}^{\langle\mu\nu\rangle} \\
\rho_{2n+1}^{\langle\mu\nu\rangle} &= \frac{-2\eta}{(1+\tilde{\chi}\hat{D})}\sum_{m=n}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m\left\{-\tilde{\theta}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}+\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}\right\} \\
\rho_{2n+2}^{\langle\mu\nu\rangle} &= \frac{-2\eta}{(1+\tilde{\theta}\hat{D})}\frac{\tilde{\chi}\tilde{\theta}}{3}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^n\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u} \\
&+ \sum_{m=n+1}^{\infty}\left(\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}{(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D})}\right)^m\left\{-\tilde{\theta}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}+\frac{\frac{\tilde{\chi}\tilde{\theta}}{3}}{(1+\tilde{\theta}\hat{D})}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}\right\}
\end{aligned} \tag{D.25}$$

These are the general forms of the $\rho_n^{\langle\mu\nu\rangle}$s given in (6.2).

D.1.2 Transformation of temperature

As it was done in the previous subsection, the expression for $\frac{\delta T_n}{\hat{T}}$ can be written as,

$$\begin{aligned}
\frac{\delta T_n}{\hat{T}} &= \frac{(-\tilde{\chi}\hat{D})^n\hat{T}}{\hat{T}}+(-\tilde{\chi})\sum_{m=0}^{n-1}c_{mn}\hat{D}^{n-1-2m}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\left(\frac{\hat{\nabla}\cdot\hat{u}}{3}\right) \\
&+\frac{\tilde{\chi}\tilde{\theta}}{3}\sum_{m=0}^{n-2}f_{mn}\hat{D}^{n-2-2m}\left(\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\right)^m\frac{\hat{\nabla}^2\hat{T}}{\hat{T}}
\end{aligned} \tag{D.26}$$

where c_{mn} is defined the same way as in δu_n^μ and f_{mn} is defined in terms of c_{mn} as

$$f_{mn} = \frac{1}{m+1}\left(-\frac{\partial}{\partial\tilde{\chi}}\right)c_{mn} \tag{D.27}$$

Similar to the case of δu^μ, we again take an infinite summation over n to obtain $\frac{\delta T}{\hat{T}}$ as

$$\begin{aligned}\frac{\delta T}{\hat{T}} = \sum_{n=1}^{\infty}\frac{\delta T_n}{\hat{T}} =& -\frac{\tilde{\chi}}{(1+\tilde{\chi}\hat{D})}\frac{\hat{D}\hat{T}}{\hat{T}} + (-\tilde{\chi})\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D}) - \frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}\left(\frac{\hat{\nabla}\cdot\hat{u}}{3}\right)\\ &+\frac{\tilde{\chi}\tilde{\theta}}{3}\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D}) - \frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}\frac{\hat{\nabla}^2\hat{T}}{\hat{T}}\end{aligned} \tag{D.28}$$

and from there, obtain the same $T = \hat{T} + \delta T$ as in (6.40)

$$\begin{aligned}T =& \frac{1}{(1+\tilde{\chi}\hat{D})}\hat{T} - \hat{T}\tilde{\chi}\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D}) - \frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}\left(\frac{\hat{\nabla}\cdot\hat{u}}{3}\right)\\ &+\frac{1}{(1+\tilde{\chi}\hat{D})}\frac{1}{(1+\tilde{\chi}\hat{D})(1+\tilde{\theta}\hat{D}) - \frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2}\frac{\tilde{\chi}\tilde{\theta}}{3}\hat{\nabla}^2\hat{T}\\ &= \frac{1}{(1+\tilde{\theta}D)(1+\tilde{\chi}D) - \frac{\tilde{\chi}\tilde{\theta}}{3}\nabla^2}\left[(1+\tilde{\theta}\hat{D})\hat{T} - \hat{T}\tilde{\chi}\frac{\hat{\nabla}\cdot\hat{u}}{3}\right]\end{aligned} \tag{D.29}$$